Empirical evaluations of Feynman integrals as L-functions of modular forms with weight > 2

David Broadhurst[a,b]

IHES, Bures, 19 February 2014

Summary: There is growing evidence, from both massless and massive Feynman diagrams, that modular forms first arise in quantum field theory when polylogarithms no longer suffice. In this talk, I aim to
(1) introduce a variety of modular forms,
(2) link enumerations of modular forms and multiple zeta values,
(3) identify modular forms that obstruct evaluations to polylogs,
(4) give a modular form that controls massive and massless diagrams,
(5) give empirical evaluations of Feynman diagrams as L-functions of modular forms, with weights > 2, inside their critical strips,
(6) indicate how algebraic geometry inspired these discoveries.

[a] Department of Physical Sciences, Open University, Milton Keynes MK7 6AA, UK
[b] Institut für Mathematik und Institut für Physik, Humboldt-Universität zu Berlin

1 Dramatis personæ

For $|q| < 1$, let

$$\eta(q) \equiv q^{1/24} \prod_{n>0} (1 - q^n) = \sum_{n\in\mathbf{Z}} (-1)^n q^{(6n+1)^2/24}$$

then for $\Im z > 0$,

$$\eta(\exp(2\pi \mathrm{i} z)) = (\mathrm{i}/z)^{1/2} \eta(\exp(-2\pi \mathrm{i}/z)).$$

If $f(z) = (\sqrt{-N}/z)^w f(-N/z)$, we say that f is a modular form of weight w and level N.

$$\Delta(q) \equiv \eta(q)^{24} = \sum_{n>0} A(n) q^n = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + \ldots$$

is a modular form of weight 12 and level 1. Moreover its Fourier coefficients $A(n)$ are multiplicative, with $A(mn) = A(m)A(n)$ when $\gcd(m, n) = 1$. Finally for prime p there is a simple rule for obtaining $A(p^n)$ from $A(p)$:

$$L(s) \equiv \sum_{n>0} \frac{A(n)}{n^s} = \prod_p \frac{1}{1 - A(p)p^{-s} + p^{11-2s}}.$$

Note that $-1472 = A(4) = 24^2 - 2^{11}$ and $-6048 = A(6) = -24 \times 252$. This product form leads to the analytic continuation

$$\Lambda(s) \equiv \frac{\Gamma(s)}{(2\pi)^s} L(s) = \sum_{n>0} A(n) \int_1^\infty \mathrm{d}x \left(x^{s-1} + x^{11-s}\right) \exp(-2\pi x) = \Lambda(12 - s)$$

with an easy integral for integer $s \in [1, 11]$. Only two of these 11 L-values are independent, since $1620\Lambda(3) = 691\Lambda(1)$, $14\Lambda(5) = 9\Lambda(3)$, $48\Lambda(4) = 25\Lambda(2)$ and $5\Lambda(6) = 4\Lambda(4)$.

1.1 Multiplicative modular forms with level 1

To count these, consider the Eisenstein series defined, for $n > 0$, by

$$E_{2n}(q) \equiv 1 - \frac{4n}{B_{2n}} \sum_{k>0} \frac{k^{2n-1}q^k}{1-q^k}$$

where the Bernoulli numbers yield $-4n/B_{2n} = -24, 240, -504, 480, -264, 65520/691$, for $n = 1 \ldots 6$. Then E_2 is not modular, since $(q\mathrm{d}/\mathrm{d}q)\Delta = E_2\Delta$ is not. For $n > 1$, E_{2n} is modular, with weight $2n$, but is not multiplicative, since it does not vanish at $q = 0$.

$$\Delta = \frac{E_4^3 - E_6^2}{1728}$$

is both modular and multiplicative. Note that $1728 = 3 \times 240 + 2 \times 504$. For $n > 3$, E_{2n} is a rational polynomial of E_4 and E_6. For example: $E_8 = E_4^2$, $E_{10} = E_4 E_6$,

$$E_{12} = \frac{441E_4^3 + 250E_6^2}{691}, \quad E_{14} = E_4^2 E_6, \quad E_{16} = \frac{1617E_4^4 + 2000E_4E_6^2}{3617}.$$

Let M_w be the number of multiplicative modular forms with weight w and level 1. Then

$$\sum_{w>0} M_w x^w = \frac{x^{12}}{(1-x^4)(1-x^6)} = x^{12} + x^{16} + x^{18} + x^{20} + x^{22} + 2x^{24} \ldots$$

with Δ at $w = 12$, none at $w = 14$, $E_4\Delta$ at $w = 16$, $E_6\Delta$ at $w = 18$, $E_4^2\Delta$ at $w = 20$, $E_4E_6\Delta$ at $w = 22$ and two independent modular forms at $w = 24$, namely $E_4^3\Delta$ and $E_6^2\Delta$.

1.2 Relations between eta values

For brevity, let $\eta_n(q) \equiv \eta(q^n)$. Then η_1 and η_2 are algebraically independent. Yet

$$\eta_2^{24} = \eta_1^8 \eta_4^8 (\eta_1^8 + 4\eta_4^8)$$

relates $\{\eta_1, \eta_2, \eta_4\}$ and is the basis for the process

$$(a_{n+1},\, b_{n+1}) = \left(\frac{a_n + b_n}{2},\, \sqrt{a_n b_n}\right)$$

of the arithmetic–geometric mean (AGM) devised by Gauss for rapid computation of

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{\mathrm{d}\theta}{(a_0 \sin^2\theta + b_0 \cos^2\theta)^{1/2}} = \frac{1}{\mathrm{agm}(a_0,\, b_0)} = \frac{1}{a_\infty} = \frac{1}{b_\infty}.$$

There is a more ornate relation between $\{\eta_1, \eta_2, \eta_3\}$. Let

$$F_2(x) \equiv \frac{(x+8)(x-1)^2}{x}, \qquad F_3(y) \equiv \frac{(y^2+6y-3)^2}{y}.$$

Then, by a method to be explained later, one may obtain the algebraic relation,

$$2^6 F_2\left(2^9(\eta_2/\eta_1)^{24}\right) = 3^3 F_3\left(3^5(\eta_3/\eta_1)^{12}\right).$$

Replacing q by q^2, one may relate $\{\eta_2, \eta_4, \eta_6\}$ and eliminate η_4 in favour of $\{\eta_1, \eta_2\}$. Thus there are two algebraic relations between $\{\eta_1, \eta_2, \eta_3, \eta_6\}$. We shall see that these are perfectly tuned to allow evaluation of a massive Feynman diagram.

1.3 Multiplicative modular forms and eta products

For levels $N < 16$, there are precisely 15 multiplicative modular forms that are products of eta values. Here they are listed with notes on quantum field theory (QFT):

form	weight	level	QFT
$\eta_1^2\eta_{11}^2$	2	11	
$\eta_1\eta_2\eta_7\eta_{14}$	2	14	
$\eta_1\eta_3\eta_5\eta_{15}$	2	15	
$\eta_1^3\eta_7^3$	3	7	BS
$\eta_1^2\eta_2\eta_4\eta_8^2$	3	8	BS
$\eta_2^3\eta_6^3$	3	12	BS + BBBG + BV : Sections 3 and 4
$\eta_1^4\eta_5^4$	4	5	BS
$\eta_1^2\eta_2^2\eta_3^2\eta_6^2$	4	6	BS + BB : Sections 3 and 5
$\eta_2^4\eta_4^4$	4	8	
η_3^8	4	9	
$\eta_1^4\eta_2^2\eta_4^4$	5	4	BS
$\eta_1^6\eta_3^6$	6	3	BS
η_2^{12}	6	4	BS
$\eta_1^8\eta_2^8$	8	2	BS
η_1^{24}	12	1	BK : Section 2

with 10 of these 15 already exposed as participants in QFT, thanks to work by Brown and Schnetz (BS), Bailey, Borwein, Broadhurst and Glasser (BBBG), Bloch and Vanhove (BV), Broadhurst and Brown (BB), Broadhurst and Kreimer (BK). The absence of weight-2 examples is remarkable: does QFT avoid Birch and Swinnerton–Dyer?

1.4 Multiplicative modular forms and lattice sums

Moreover, QFT has links to a pair of multiplicative modular forms that involve lattice sums. With ingenuity one may reduce these to combinations of eta products or quotients.

From BBBG and BB, we find a multiplicative modular form with $w = 3$ and $N = 15$:

$$f_{3,15} \equiv \eta_1\eta_3\eta_5\eta_{15} \sum_{j,k\in\mathbf{Z}} q^{j^2+jk+4k^2} = (\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3$$

with a remarkable evaluation as a sum of cubes of eta products.

At $w = 6$ and $N = 6$, QFT led me to a multiplicative modular form

$$f_{6,6} \equiv (\eta_1\eta_2\eta_3\eta_6)^2 \sum_{j,k,l,m\in\mathbf{Z}} q^{j^2+jk+k^2} q^{2(l^2+lm+m^2)}$$

with a lattice sum that factorizes. This too may be written as a sum of cubes:

$$f_{6,6} = \left(\frac{\eta_2^3\eta_3^3}{\eta_1\eta_6}\right)^3 + \left(\frac{\eta_1^3\eta_6^3}{\eta_2\eta_3}\right)^3 .$$

These two modular forms will be used in Section 5 to evaluate Feynman integrals.

2 Modular forms and multiple zeta values

In 1996, Dirk Kreimer and I (BK) arrived at a conjectural enumeration of irreducible multiple zeta values (MZVs), graded by weight and depth. Let $D_{w,d}$ be the number of MZVs with weight w and depth d that are not reduced by the double–shuffle algebra to MZVs of lesser weight and depth and their products. From extensive data with $d < 4$, and sparser data at higher depths, we conjectured that

$$\prod_{w>2}\prod_{d>0}(1 - x^w y^d)^{D_{w,d}} = 1 - y\frac{x^3}{1-x^2} + y^2(1-y^2)\frac{x^{12}}{(1-x^4)(1-x^6)}.$$

It is now proven, at the motivic level, that any difference between the left and right hand sides must be of order y^4. Moreover it must vanish at $y = 1$, where Brown and Zagier (BZ) have proven an enumeration that is blind to depth. Blümlein, Vermaseren and I (BBV) have checked the conjecture for depths $d < 9$ and weights $w < 27$, by laborious methods. Francis Brown indicates that further checking may be done more efficiently.

If the BK conjecture be true, it sets a fine puzzle. Why should a count of modular forms

$$\sum_{w>0} M_w x^w = \frac{x^{12}}{(1-x^4)(1-x^6)} = x^{12} + x^{16} + x^{18} + x^{20} + x^{22} + 2x^{24}\ldots$$

furnish the bizarre final term of our empirical Ansatz? Is this coincidence significant?

3 Polylogs and modular forms in ϕ^4 theory

The standard model of particle physics involves all three of the interactions that are renormalizable, yet not trivially super-renormalizable, in $D = 4$ spacetime dimensions:

1. ϕ^4 self-coupling of the Higgs boson,
2. Yukawa couplings of the Higgs boson to fermions,
3. gauge couplings of vector bosons to the Higgs boson and to fermions.

In 1985, I studied a 6-loop diagram (see blackboard) that contributes to the 4-point amplitude for Higgs scattering. Its counterterm contributes to the running of the Higgs self-coupling. It is the first 6-loop entry in the recent census by Schnetz (S). I conjectured that the relevant period (B) is:

$$P_{6,1} = 168\zeta(9)$$

and Natalia Ussyukina (U) proved this in 1991. In 2012, BS proved a BK conjecture for all such zigzag diagrams.

3.1 Counterterms reducible to polylogs

In 1995, Dirk Kreimer and I (BK) identified all periods for ϕ^4 primitive divergences up to 6 loops. At 7 loops we lacked three evaluations. Since then I have determined two of these, as follows.

$$
\begin{aligned}
P_{7,8} &= \frac{22383}{20}\zeta(11) - \frac{4572}{5}[\zeta(3)\zeta(5,3) - \zeta(3,5,3)] - 700\zeta(3)^2\zeta(5)\\
&\quad + 1792\zeta(3)\left[\frac{27}{80}\zeta(5,3) + \frac{45}{64}\zeta(5)\zeta(3) - \frac{261}{320}\zeta(8)\right]\\
P_{7,9} &= \frac{92943}{160}\zeta(11) - \frac{3381}{20}[\zeta(3)\zeta(5,3) - \zeta(3,5,3)] - \frac{1155}{4}\zeta(3)^2\zeta(5)\\
&\quad + 896\zeta(3)\left[\frac{27}{80}\zeta(5,3) + \frac{45}{64}\zeta(5)\zeta(3) - \frac{261}{320}\zeta(8)\right]
\end{aligned}
$$

with indices of MZVs written in the order adopted by Zagier, by Borwein, Bradley, Broadhurst and Lisonek (BBBL), and in the extensive MZV datamine (BBV):

$$\zeta(5,3) \equiv \sum_{m=2}^{\infty} \frac{1}{m^5} \sum_{n=1}^{m-1} \frac{1}{n^3}.$$

In these two case, the methods of BS allowed the possibility that the periods might involve alternating sums. In fact they do not. One sheep remains lost: the period $P_{7,11}$ in the census has not yet been reduced to MZVs. BS suggest that it might eventually be reduced to polylogs of weight 11 at sixth roots of unity. Such polylogs result from massive diagrams at lesser weights (B).

3.2 Panzer's reductions to MZVs

To calculate counterterms at L loops, it is usually sufficient to obtain the ε expansions of two-point diagrams, at $L-1$ loops in $D = 4 - 2\varepsilon$ dimensions, up to weight $2L-3$.

In May 2013, Erik Panzer (P) showed that the dressing (see blackboard) of many two-point diagrams by propagator sub-divergences does not take one beyond the realm of MZVs. As a concrete example, consider the 3-loop non-planar two-point diagram, whose ε-expansion was previously known to weight 7. Now it is known up to weight 9:

$$\begin{aligned} N_3(\epsilon) &= 20\zeta_5 + \left(\frac{80}{7}\zeta_2^3 + 68\zeta_3^2\right)\varepsilon + \left(\frac{408}{5}\zeta_3\zeta_2^2 + 450\zeta_7\right)\varepsilon^2 + \left(\frac{102228}{125}\zeta_2^4 - 2448\zeta_3\zeta_5\right. \\ &\left. -\frac{9072}{5}\zeta_{5,3}\right)\varepsilon^3 + \left(\frac{88036}{9}\zeta_9 - \frac{4640}{3}\zeta_3^3 - \frac{10336}{7}\zeta_2^3\zeta_3 + \frac{19872}{5}\zeta_2^2\zeta_5\right)\varepsilon^4 + \ldots \end{aligned}$$

with $\zeta_{5,3} \equiv \sum_{m>n>0} 1/(m^5 n^3)$ appearing at weight 8. Even more impressively, he has shown that at 3 loops no dressing of internal lines by subdivergences can modify the polylogarithmic character of the ε-expansion. Specifically, he proves that the only non-MZV terms that might occur would be alternating Euler sums. As in BS cases at weight 11, no such alternating sum has yet emerged from a massless two-point diagram.

3.3 Brown–Schnetz modular obstructions

In April 2013, Francis Brown and Oliver Schnetz (BS) announced results of a fascinating study that classifies obstructions to polylogarithmic evaluations of ϕ^4 counterterms at 8, 9 and 10 loops. In 16 cases they were able to exhibit a modular form, inferred from study of the Symanzik polynomial, modulo a selection of primes. In 9 cases, listed in Section 1, the modular form was both multiplicative and reducible to an eta product. Here I select for particular attention ϕ^4 diagrams (see blackboard) that led BS to these modular forms

$$f_{3,12} \equiv (\eta_2\eta_6)^3 \quad \text{and} \quad f_{4,6} \equiv (\eta_1\eta_2\eta_3\eta_6)^2.$$

4 Sunrise in two spacetime dimensions

Here I consider the two-loop massive sunrise diagram in $D = 2$ spacetime dimensions:

$$I(p^2, m_1, m_2, m_3) \equiv \frac{1}{\pi^2} \left(\prod_{k=1}^{3} \int \frac{\mathrm{d}^2 q_k}{q_k^2 - m_k^2 + \mathrm{i}\epsilon} \right) \delta^{(2)}(p - q_1 - q_2 - q_3)$$

with a Minkowski metric: $p^2 \equiv p_0^2 - p_1^2$ where p_0 is the energy and p_1 is the momentum.

4.1 A Bessel moment in configuration space

For $0 < w < m_1 + m_2 + m_3$, configuration space yields BBBG's Bessel moment:

$$I(w^2, m_1, m_2, m_3) = 4 \int_0^\infty I_0(wy) K_0(m_1 y) K_0(m_2 y) K_0(m_3 y) y \mathrm{d}y.$$

4.2 Algebraic geometry in Schwinger parameter space

Algebraic geometers prefer Feynman integrals in parameter space, where Schwinger gives

$$I(w^2, m_1, m_2, m_3) = \int_0^\infty \int_0^\infty \frac{\mathrm{d}x \, \mathrm{d}y}{P(x, y, 1)}$$

with momentum conservation achieved by setting, for example, $z = 1$ in

$$P(x, y, z) = (m_1^2 x + m_2^2 y + m_3^2 z)(xy + yz + zx) - w^2 xyz.$$

I shall not use this representation, here. Yet I respect Spencer Bloch's preference for it.

4.3 Cut constructibility in momentum space

Following BBBG, we obtain an efficient result from the imaginary part on the cut:

$$I(w^2, m_1, m_2, m_3) = 8\pi \int_{m_1+m_2+m_3}^{\infty} \frac{A(x) x \mathrm{d}x}{x^2 - w^2}$$

where

$$A(w) = \frac{1}{\mathrm{agm}\left(\sqrt{F(w)}, \sqrt{F(w) - F(-w)}\right)}$$

is the reciprocal of an AGM governed by the quartic

$$F(w) = (w + m_1 + m_2 + m_3)(w + m_1 - m_2 - m_3)(w - m_1 + m_2 - m_3)(w - m_1 - m_2 + m_3)$$

studied by Davydychev and Delbourgo (DD) and conveniently satisfying

$$F(w) = F(-w) + 16 m_1 m_2 m_3 w.$$

4.4 Wronskian from Legendre

The sunrise integral satisfies an inhomogeneous second–order differential equation whose homogeneous form is satisfied by $A(w)$. The complementary solution is

$$B(w) = \frac{1}{\mathrm{agm}\left(\sqrt{F(w)}, \sqrt{F(-w)}\right)}.$$

$$W(w) = A'(w)B(w) - A(w)B'(w) = \frac{N_1(w) + N_2(w) + N_3(w)}{\pi w F(w) F(-w)}$$

is the Wronskian of the homogeneous equation, easily found by using Legendre's relation between elliptic integrals. The Wronskian of $F(w)$ with $F(-w)$ yields

$$N_1(w) = (w^2 - m_1^2)^2 - (m_2^2 - m_3^2)^2$$

with $N_2(w)$ and $N_3(w)$ obtained by cyclic permutation of masses. Müller–Stach, Weinzierl and Zayadeh (MWZ) have determined the inhomogeneous term.

4.5 Bloch–Vanhove q-series in the equal–mass case

From now on, we assume that $m_1 = m_2 = m_3 = 1$. Then $F(w) = (w+3)(w-1)^3$ and the Wronskian is $W(w) = 3/(\pi w(w^2-1)(w^2-9))$. We define $q(w) \equiv \exp(-\pi B(w)/A(w))$, which is the nome of the elliptic integral resulting from the Dalitz plot (in this case a Dalitz line). Then the inhomogeneous differential equation, found with Jochem Fleischer and Oleg Tarasov (BFT) in 1993, may be written as

$$-\left(\frac{q(w)}{q'(w)}\frac{\mathrm{d}}{\mathrm{d}w}\right)^2 \left(\frac{I(w^2,1,1,1)}{24\sqrt{3}A(w)}\right) = \frac{w^2(w^2-1)(w^2-9)A(w)^3}{9\sqrt{3}}.$$

At a seminar on 6 June 2013, in Berlin, Spencer Bloch announced the stunning result that he and Pierre Vanhove (BV) had solved this BFT equation, using q-series. I now show how to recover the BV result, without using the algebraic geometry that inspired it.

Regarding w and $A(w)$ as functions of q, we obtain from Maier (M) the parametric solution

$$\frac{w}{3} = \left(\frac{\eta_3}{\eta_1}\right)^4 \left(\frac{\eta_2}{\eta_6}\right)^2, \quad 4\sqrt{3}A = \frac{\eta_1^6\eta_6}{\eta_2^3\eta_3^2}.$$

Moreover, the two algebraic relations between $\{\eta_1, \eta_2, \eta_3, \eta_6\}$ may be written as

$$\frac{w^2-1}{8} = \left(\frac{\eta_2}{\eta_1}\right)^9 \left(\frac{\eta_3}{\eta_6}\right)^3, \quad \frac{w^2-9}{72} = \left(\frac{\eta_6}{\eta_1}\right)^5 \frac{\eta_2}{\eta_3},$$

whose resultant w.r.t. η_6 was given in Section 1. Hence the BFT equation reduces to

$$-\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 \left(\frac{I}{24\sqrt{3}A}\right) = \frac{w}{3} f_{3,12} = \left(\frac{\eta_3^3}{\eta_1}\right)^3 + \left(\frac{\eta_6^3}{\eta_2}\right)^3$$

where $f_{3,12} \equiv (\eta_2\eta_6)^3$ is the weight-3 level-12 modular form found in ϕ^4 theory by BS and the sum of cubes yields Lambert q-series given by Borwein and Borwein (B&B) in 1991.

Now define a character with $\chi(n) = \pm 1$ for $n = \pm 1 \bmod 6$ and $\chi(n) = 0$ otherwise. Then

$$-\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 \left(\frac{I}{24\sqrt{3}A}\right) = \sum_{n>0} \frac{n^2(q^n - q^{5n})}{1-q^{6n}} = \sum_{n>0}\sum_{k>0} n^2\chi(k)q^{nk}.$$

Integrating twice and using the known imaginary part on the cut, I recover the BV result

$$\frac{I(w^2,1,1,1)}{4A(w)} = C(-1) - C\left(\mathrm{e}^{-\pi B(w)/A(w)}\right), \quad C(q) = \pi\log(-q) + \sum_{k>0} \frac{6\sqrt{3}\chi(k)q^k}{k^2(1-q^k)},$$

where the Clausen value $C(-1) = -5\,\mathrm{Cl}_2(\pi/3)$ makes $I(1,1,1,1)$ finite. So we are done.

5 Modular forms and higher–loop on–shell sunrises

In two dimensions, equal–mass on–shell sunrise diagrams and massive vacuum banana diagrams (see blackboard) are examples of Bessel moments studied by BBBG:

$$S_{N,L} \equiv 2^L \int_0^\infty I_0(y)^{N-L-1} K_0(y)^{L+1} y \mathrm{d}y$$

where N is the total number of Bessel functions and L is the number of loops. For convergence, we require that $L < N \leq 2L+2$. With $N = 2L+2$ we require that $L > 1$. BBBG proved that:

$$S_{1,0} = S_{2,1} = 1, \quad S_{3,1} = \frac{2\pi}{3\sqrt{3}}, \quad S_{3,2} = \frac{4\,\mathrm{Cl}_2(\pi/3)}{\sqrt{3}}, \quad S_{4,2} = \frac{\pi^2}{4}, \quad S_{4,3} = 7\zeta(3),$$

$$S_{5,2} = \frac{\pi^2}{8}\left(\sqrt{15}-\sqrt{3}\right)\left(\sum_{n\in\mathbf{Z}} \mathrm{e}^{-\sqrt{15}\pi n^2}\right)^4 = \frac{\sqrt{3}}{120\pi}\prod_{k=0}^{3}\Gamma\left(\frac{2^k}{15}\right)$$

where the final product of Gamma values results from the Chowla–Selberg theorem. We also conjectured (and checked to 1000 digits) that

$$S_{5,3} = \frac{4\pi}{\sqrt{15}} S_{5,2}, \quad S_{6,4} = \frac{4\pi^2}{3} S_{6,2}, \quad S_{8,5} = \frac{18\pi^2}{7} S_{8,3}.$$

5.1 Sunrise at 3 loops from a modular form of weight 3

Let $L_{3,15}(s)$ be the Dirichlet L-function defined by the multiplicative modular form

$$f_{3,15} = (\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3$$

with weight 3 and level 15. Then I conjecture (and have checked to 1000 digits) that

$$S_{5,2} = 3L_{3,15}(2), \quad S_{5,3} = \frac{8\pi^2}{15}L_{3,15}(1),$$

where $S_{5,3}$ is the 5-Bessel moment giving the on-shell 3-loop sunrise diagram.

5.2 Sunrise at 4 loops from a modular form of weight 4

Let $L_{4,6}(s)$ be the Dirichlet L-function defined by the multiplicative modular form

$$f_{4,6} = (\eta_1\eta_2\eta_3\eta_6)^2$$

with weight 4 and level 6. Then I conjecture (and have checked to 1000 digits) that

$$S_{6,2} = 6L_{4,6}(2), \quad S_{6,3} = 12L_{4,6}(3), \quad S_{6,4} = 8\pi^2 L_{4,6}(2),$$

where $S_{6,4}$ is the 6-Bessel moment giving the on-shell 4-loop sunrise diagram.

5.3 Almost sunrise at 6 loops from a modular form of weight 6

Let $L_{6,6}(s)$ be the Dirichlet L-function defined by the multiplicative modular form

$$f_{6,6} = \left(\frac{\eta_2^3\eta_3^3}{\eta_1\eta_6}\right)^3 + \left(\frac{\eta_1^3\eta_6^3}{\eta_2\eta_3}\right)^3$$

with weight 6 and level 6. Then I conjecture (and have checked to 1000 digits) that

$$S_{8,3} = 8L_{6,6}(3), \quad S_{8,4} = 36L_{4,6}(4), \quad S_{8,5} = 216L_{4,6}(5),$$

but lack a result for $S_{8,6}$, the 8-Bessel moment giving the on-shell 6-loop sunrise diagram.

6 Massive bananas

6.1 Schwinger's bananas

Let A be the diagonal $N \times N$ matrix with entries $A_{i,j} = \delta_{i,j}\alpha_i$. Let U be the column vector of length N with unit entries, $U_i = 1$. Then $B = U\widetilde{U}$ is the $N \times N$ matrix with unit entries, $B_{i,j} = 1$. The banana diagram with $N+1$ edges of unit mass, in two space-time dimensions, may be evaluated by Schwinger's trick as a multiple of the N-dimensional integral

$$\overline{V}_{N+1} = \int_{\alpha_i > 0} \frac{\mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_N}{\mathrm{Det}(A+B)(\mathrm{Tr}(A)+1)} \tag{1}$$

where

$$\mathrm{Det}(A+B)=\sum_{i=0}^{N}\frac{1}{\alpha_i}\prod_{j=0}^{N}\alpha_j$$

is the first Symanzik polynomial, with $\alpha_0=1$ fixed by momentum conservation, and the second Symanzik polynomial

$$\mathrm{Tr}(A)+1=\sum_{i=0}^{N}\alpha_i$$

results from the fact that the $N+1$ edges are propagators with unit mass.

6.2 Bessels's bananas

We may also evaluate banana diagrams in x-space, since the two-dimensional Fourier transform of the p-space Euclidean propagator $1/(p^2+m^2)$, with $p^2=p_0^2+p_1^2$, yields the Bessel function $K_0(mx)$, with $x^2=x_0^2+x_1^2$. The normalization in (1) corresponds to

$$\overline{V}_{N+1}=2^N\int_0^\infty [K_0(t)]^{N+1}t\,\mathrm{d}t \tag{2}$$

which differs by a power of 2 from the Bessel moments that I studied with Bailey, Borwein and Glasser.

Hence I but a bar over V and use the subscript $N+1$ to indicate the number of Bessel functions.

6.3 Known bananas

$$\begin{aligned}
\overline{V}_1 &= 1\\
\overline{V}_2 &= 1\\
\overline{V}_3 &= 3L_{-3}(2)\\
\overline{V}_4 &= 7\zeta(3)
\end{aligned}$$

where

$$L_{-3}(s) = \sum_{n\geq 0}\left(\frac{1}{(3n+1)^s} - \frac{1}{(3n+2)^s}\right)$$

is the Dirichlet L function with conductor -3.

6.4 Unknown banana

The next diagram has 5 edges and hence 4 loops. After an easy first integration, we obtain

$$\overline{V}_5 = \int_0^\infty\int_0^\infty\int_0^\infty \frac{M(a,b,c)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{(ab+a+b)c^2+(ab+a+b)(a+b)c+(a+b)ab}$$

with

$$M(a,b,c) = \log(a+b+c+1) + \log\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

But then integration over c will produce complicated dilogarithms with arguments involving the square root of the discriminant

$$D(a,b) = (ab + a + b)(a + b)(ab(a + b) + (a - b)^2)$$

of the quadratic in c. The result will have the form

$$\overline{V}_5 = \int_0^\infty \int_0^\infty \frac{L_2(a,b)\mathrm{d}a\,\mathrm{d}b}{\sqrt{D(a,b)}}$$

with undisclosed dilogs in $L_2(a,b)$. Integration by parts, to reduce the dilogs to logs, would require us to introduce an elliptic function, since $D(a,b)$ is a quartic in b.

We know nothing about the number theory of $\overline{V}_5$. Its value is known to 1000 decimal places.

6.5 Cut bananas

For $N > 2$ we may cut an edge in $\overline{V}_N$ and set the two external half edges on the unit mass shell, which is at $p^2 = -1$. I call the result $\overline{S}_N$. It has $N - 1$ internal edges and hence $N - 2$ loops.

6.6 Schwinger's cut bananas

At N loops, the integral over Schwinger parameters is

$$\overline{S}_{N+2} = \int_{\alpha_i>0} \frac{\mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_N}{\mathrm{Det}(A+B)\mathrm{Tr}(A) + \widetilde{U}CU}. \tag{3}$$

where C is the adjoint of $A+B$, with

$$(A+B)C = \mathrm{Det}(A+B)I$$

where I is the unit matrix with $I_{i,j} = \delta_{i,j}$. The denominator in (3) is the second Symanzik polynomial.

6.7 Bessels's cut bananas

In x-space, cutting an edge and putting it on the mass shell corresponds to replacing one instance of the Bessel function $K_0(t)$ by $I_0(t)$, to obtain

$$\overline{S}_{N+2} = 2^N \int_0^\infty I_0(t)[K_0(t)]^{N+1} t \, \mathrm{d}t \tag{4}$$

at N loops.

6.8 Known cut bananas

$$\begin{aligned}\overline{S}_3 &= 2L_{-3}(1) = \frac{2\pi}{3\sqrt{3}}\\ \overline{S}_4 &= \mathrm{Li}_2(1) - \mathrm{Li}_2(-1) = \frac{\pi^2}{4}\end{aligned}$$

and it is conjectured that

$$\overline{S}_5 \stackrel{?}{=} \frac{1}{30\sqrt{5}}\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right) \tag{5}$$

which holds to at least 1000 decimal places.

6.9 Cut banana at the 15th singular value

At three loops, we have

$$\overline{S}_5 = \int_0^\infty \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{P(a,b,c)}$$

where

$$P(a,b,c) = (abc + ab + bc + ca)(a + b + c) + (ab + bc + ca)$$

with the final term, $(ab + bc + ca)$, resulting from the adjoint matrix. Grouping powers of c, we see that

$$P(a,b,c) = (ab + a + b)c^2 + (ab + a + b)(a + b + 1)c + (a + b + 1)ab$$

yields a discriminant

$$\Delta(a,b) = (ab+a+b)(a+b+1)((ab+a+b)(a+b+1) - 4ab)$$

and the integral over c gives

$$\overline{S}_5 = \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{\Delta(a,b)}} \log\left(\frac{1+X(a,b)}{1-X(a,b)}\right)$$

with

$$X(a,b) = \sqrt{1 - \frac{4ab}{(ab+a+b)(a+b+1)}}\,.$$

The conjecture for $\overline{S}_5$ was stimulated by a proven result for

$$\overline{T}_5 \equiv 4\int_0^\infty [I_0(t)]^2[K_0(t)]^3 t\,\mathrm{d}t = \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{\Delta(a,b)}}$$

namely

$$\overline{T}_5 = \frac{\sqrt{3}}{120\pi}\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right) \tag{6}$$

obtained at the 15th singular value.

7 L-series of a K3 surface from 5 Bessels

For $s > 2$ let

$$L(s) = \prod_p \frac{1}{1 - \frac{A_p}{p^s} + \left(\frac{p}{15}\right)\frac{p^2}{p^{2s}}}$$

where $\left(\frac{\cdot}{15}\right)$ is a Kronecker symbol, the product is over all primes p, and

$$\begin{aligned}
A_3 &= -3, \\
A_5 &= 5, \\
A_p &= 0, \text{ for } \left(\frac{p}{15}\right) = -1, \\
A_p &= 2x^2 + 2xy - 7y^2, \text{ for } x^2 + xy + 4y^2 = p \equiv 1,\, 4 \bmod 15, && (7)\\
A_p &= x^2 + 8xy + y^2, \text{ for } 2x^2 + xy + 2y^2 = p \equiv 2,\, 8 \bmod 15, && (8)
\end{aligned}$$

with pairs of integers (x, y) defined, for $x > 0$, by the quadratic forms in (7,8).

The L-series

$$L(s) = \sum_{n>0} \frac{A_n}{n^s}$$

is generated by the weight-3 modular form

$$f_3(q) = \eta(q)\eta(q^3)\eta(q^5)\eta(q^{15}) \sum_{m,n\in\mathbf{Z}} q^{m^2+mn+4n^2} = \sum_{n>0} A_n q^n \tag{9}$$

I now describe how I was able to evaluate 20000 good digits of the conditionally convergent series $L(2)=\sum_{n>0}A_n/n^2$. Let

$$\Lambda(s)=\frac{\Gamma(s)}{c^s}L(s),\ \text{ with } c=\frac{2\pi}{\sqrt{15}}.$$

Then we have the functional equation $\Lambda(s)=\Lambda(3-s)$, which may be used to extend the Mellin transform

$$\Lambda(s)=\sum_{n>0}A_n\int_0^\infty\frac{\mathrm{d}x}{x}x^s\exp(-cnx) \tag{10}$$

throughout the complex s-plane, as follows

$$\Lambda(s)=\sum_{n>0}A_n\left(\frac{\Gamma(s,cn\lambda)}{(cn)^s}+\frac{\Gamma(3-s,cn/\lambda)}{(cn)^{3-s}}\right) \tag{11}$$

where

$$\Gamma(s,y)=\int_y^\infty\frac{\mathrm{d}x}{x}x^s\exp(-x)$$

is the incomplete Γ function and $\lambda\geq 0$ is an arbitrary real parameter. To establish (11), I remark that it agrees with (10), at $\lambda=0$, and that its derivative with respect to λ vanishes by virtue of the inversion symmetry

$$M(\lambda)\equiv\lambda^{3/2}\sum_{n>0}A_n\exp(-cn\lambda)=M(1/\lambda).$$

Optimal convergence is achieved at $\lambda=1$. Setting $s=2$, we obtain

$$L(2) \equiv \sum_{n>0} \frac{A_n}{n^2} = \sum_{n>0} \frac{A_n}{n^2} \left(1 + \frac{4\pi n}{\sqrt{15}}\right) \exp\left(-\frac{2\pi n}{\sqrt{15}}\right) \tag{12}$$

from which I obtained more than 20000 good digits in less than a minute, by computing the first 30000 terms, with the aid of (7,8). The result is consistent with the conjecture

$$\begin{aligned}
3L(2) &\stackrel{?}{=} \overline{T}_5 && (13)\\
&\equiv 4\int_0^\infty [I_0(t)]^2 [K_0(t)]^3 t\,\mathrm{d}t && (14)\\
&= \int_0^\infty \int_0^\infty \frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{(ab+a+b)(a+b+1)((ab+a+b)(a+b+1)-4ab)}} && (15)\\
&= \frac{\pi^2}{8}\left(\sqrt{15}-\sqrt{3}\right)\left(1+2\sum_{n>0}\exp\left(-\sqrt{15}\pi n^2\right)\right)^4 && (16)\\
&= \frac{\sqrt{3}}{120\pi}\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right) && (17)\\
&\stackrel{?}{=} \frac{\sqrt{15}}{4\pi}\,\overline{S}_5 && (18)
\end{aligned}$$

8 L-series for 6 Bessel functions

We are interested in relating Bessel moments of the form

$$\overline{V}_N = 2^{N-1}\int_0^\infty [K_0(t)]^N t\,\mathrm{d}t, \text{ for } N>0, \tag{19}$$

$$\overline{S}_N = 2^{N-2}\int_0^\infty I_0(t)[K_0(t)]^{N-1} t\,\mathrm{d}t,\ \text{for } N>2, \tag{20}$$

$$\overline{T}_N = 2^{N-3}\int_0^\infty I_0^2(t)[K_0(t)]^{N-2} t\,\mathrm{d}t,\ \text{for } N>4, \tag{21}$$

$$\overline{U}_N = 2^{N-4}\int_0^\infty I_0^3(t)[K_0(t)]^{N-3} t\,\mathrm{d}t,\ \text{for } N\geq 6, \tag{22}$$

$$\overline{W}_N = 2^{N-5}\int_0^\infty I_0^4(t)[K_0(t)]^{N-4} t\,\mathrm{d}t,\ \text{for } N\geq 8, \tag{23}$$

to L-series derived from modular forms. To high precision, we have the conjectural relations

$$\overline{S}_5 \stackrel{?}{=} \frac{4\pi}{\sqrt{15}}\,\overline{T}_5 \tag{24}$$

$$\overline{S}_6 \stackrel{?}{=} \frac{4\pi^2}{3}\,\overline{U}_6 \tag{25}$$

$$\overline{T}_8 \stackrel{?}{=} \frac{18\pi^2}{7}\,\overline{W}_8 \tag{26}$$

with a notable appearance of 7 in the denominator on the right hand side of (26).

Francis Brown suggested that the weight-4 modular form

$$f_4(q) = \Big[\eta(q)\eta(q^2)\eta(q^3)\eta(q^6)\Big]^2 = \sum_{n>0} A_{4,n} q^n \tag{27}$$

might yield an L-series

$$L_4(s) = \sum_{n>0}\frac{A_{4,n}}{n^s} = \frac{1}{1+2^{1-s}}\,\frac{1}{1+3^{1-s}}\prod_{p>3}\frac{1}{1-\frac{A_{4,p}}{p^s}+\frac{p^3}{p^{2s}}}$$

with values related to the problem with 6 Bessel functions. Note that $A_{4,1} = 1$, since $2(1+2+3+6) = 24$.

Then, at $s = 2$ and $s = 3$, I obtained the very convenient formulas

$$L_4(2) = \sum_{n>0} \frac{A_{4,n}}{n^2} \left(2 + \frac{4\pi n}{\sqrt{6}}\right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right) \tag{28}$$

$$L_4(3) = \sum_{n>0} \frac{A_{4,n}}{n^3} \left(1 + \frac{2\pi n}{\sqrt{6}} + \frac{2\pi^2 n^2}{3}\right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right) \tag{29}$$

and hence computed 20000 good digits of (28,29) in less than 100 seconds. Then the conjectural evaluations

$$\begin{aligned}\overline{S}_6 &\stackrel{?}{=} 48\zeta(2)L_4(2)\\ \overline{T}_6 &\stackrel{?}{=} 12L_4(3)\\ \overline{U}_6 &\stackrel{?}{=} 6L_4(2)\end{aligned}$$

were discovered and checked at 1000-digit precision.

9 L-series for 8 Bessel functions

Next, Francis Brown provided the first 100 Fourier coefficients of a weight-6 modular form $\sum_{n>0} A_{6,n} q^n$, whose L-series

$$L_6(s) = \sum_{n>0} \frac{A_{6,n}}{n^s} = \frac{1}{1-2^{2-s}} \frac{1}{1+3^{2-s}} \prod_{p>3} \frac{1}{1 - \frac{A_{6,p}}{p^s} + \frac{p^5}{p^{2s}}}$$

was expected to yield values related to the problem with 8 Bessel functions. His data may be condensed down to the values

```
-66, 176, -60, -658, -414, 956, 600, 5574, -3592, -8458, 19194, 13316, -19680,
-31266, 26340, -31090, -16804, 6120, -25558, 74408, -6468, -32742, 166082
```

of $A_{6,p}$ for the primes $p = 5, 7, \ldots, 97$. From this, I inferred a modular form

$$\sum_{n>0} A_{6,n} q^n = g(q) g(q^2) \tag{30}$$

$$g(q) = \left[\eta(q)\eta(q^3)\right]^2 \sum_{m,n\in\mathbf{Z}} q^{m^2+mn+n^2} \tag{31}$$

Proceeding along the lines of the 6-Bessel problem I accelerated the convergence of

$$L_6(3) = \sum_{n>0} \frac{A_{6,n}}{n^3} \left(2 + \frac{4\pi n}{\sqrt{6}} + \frac{2\pi^2 n^2}{3}\right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right), \tag{32}$$

$$L_6(4) = \sum_{n>0} \frac{A_{6,n}}{n^4} \left(1 + \frac{2\pi n}{\sqrt{6}} + \frac{4\pi^2 n^2}{9} + \frac{4\pi^3 n^3}{9\sqrt{6}}\right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right), \tag{33}$$

$$L_6(5) = \sum_{n>0} \frac{A_{6,n}}{n^5} \left(1 + \frac{2\pi n}{\sqrt{6}} + \frac{\pi^2 n^2}{3} + \frac{2\pi^3 n^3}{9\sqrt{6}} + \frac{\pi^4 n^4}{27}\right) \exp\left(-\frac{2\pi n}{\sqrt{6}}\right). \tag{34}$$

The resulting fits

$$\overline{T}_8 \stackrel{?}{=} 216 L_6(5) \tag{35}$$
$$\overline{U}_8 \stackrel{?}{=} 36 L_6(4) \tag{36}$$
$$\overline{W}_8 \stackrel{?}{=} 8 L_6(3) \tag{37}$$

are rather satisfying. Moreover, I discovered that

$$L_6(5) \stackrel{?}{=} \frac{4}{7} \zeta(2) L_6(3) \tag{38}$$

10 The problem of 7 Bessel functions

It would be neat if we could find a weight-5 modular form $f_5(q) = \sum_{n>0} A_{5,n} q^n$, whose L-series, $L_5(s) = \sum_{n>0} A_{5,n}/n^s$, might yield a rational multiple of $\overline{U}_7$, at $s = 3$, and a rational multiple of $\overline{T}_7$, at $s = 4$.

My first guess was modelled on the results with 5 and 8 Bessel functions, which involved the lattice sums.

Consider the weight-5 modular form

$$f_5(q) = \sum_{n>0} A_{5,n} q^n = \left[\eta(q)\eta(q^7)\right]^3 \left(\sum_{m,n\in\mathbf{Z}} q^{m^2+mn+2n^2}\right)^2 \tag{39}$$

with $A_{5,1} = 1$, since $3(1+7) = 24$. Then convergence of the L-series

$$L_5(s) = \sum_{n>0} \frac{A_{5,n}}{n^s} = \frac{1}{1-7^{2-s}} \prod_{p\neq 7} \frac{1}{1 - \frac{A_{5,p}}{p^s} + \left(\frac{p}{7}\right)\frac{p^4}{p^{2s}}}$$

may be accelerated, as before, to compute

$$L_5(3) = \sum_{n>0} \frac{A_{5,n}}{n^3}\left(1 + \frac{3\pi n}{\sqrt{7}} + \frac{4\pi^2 n^2}{7}\right)\exp\left(-\frac{2\pi n}{\sqrt{7}}\right) \tag{40}$$

$$L_5(4) = \sum_{n>0} \frac{A_{5,n}}{n^4}\left(1 + \frac{2\pi n}{\sqrt{7}} + \frac{2\pi^2 n^2}{7} + \frac{8\pi^3 n^3}{21\sqrt{7}}\right)\exp\left(-\frac{2\pi n}{\sqrt{7}}\right) \tag{41}$$

to high precision.

These numbers do not appear to be rational multiples of $\overline{U}_7$ and $\overline{T}_7$.

So what are these numbers? Empirically, they are given by

$$L_5(4) \stackrel{?}{=} \frac{\sqrt{7}\pi}{8} L_5(3) \stackrel{?}{=} \frac{\pi^4}{96}\left(1 + 2\sum_{n>0}\exp\left(-\sqrt{7}\pi n^2\right)\right)^8 = \frac{1}{42}\left(\frac{\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{2}{7}\right)\Gamma\left(\frac{4}{7}\right)}{4\pi}\right)^4. \tag{42}$$

Thus the modular form of weight 5 in (39) defines an L-series that evaluates at $s = 4$ as a rational multiple of the 4th power of a complete elliptic integral

$$L_5(4) \stackrel{?}{=} \frac{1}{6}\left[\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - k_7^2 \sin^2\theta}}\right]^4 \tag{43}$$

at the 7th singular value, $k_7 = \sqrt{2}(3 - \sqrt{7})/8$.

This is quite amusing, but seems to be unrelated to the issue of moments of 7 Bessel functions, for which no relation to the 7th singular value had been found, despite considerable effort.

However there is a relation to the square of an L-series of the Brown-Schnetz K3 surface:

$$L_5(4) \stackrel{?}{=} \frac{14}{3}\left[\overline{L}(2)\right]^2$$

.

Work is in hand to try relate the 7-Bessel problem to the symmetric square of weight-3 newform in $\Gamma_0(525)$ with eigenfield $\mathbf{Q}(\sqrt{-1}, \sqrt{6}, \sqrt{14})$.

Conclusions

1. Point counting in finite fields may give clues about analytical structure.

2. It suggests that a unique 7-loop graph in ϕ^4 theory may not reduce to MZVs, but rather to polylogs of the sixth root of unity.

3. It has motivated numerically successful guesses for massive Feynman integrals involving 5, 6 and 8 Bessel functions.

4. The problem with 7 Bessel functions seems to be tougher. Not needed for QED, which conserves fermion number.

5. For $g-2$ at 4 loops we expect to encounter

$$\overline{S}_6 \stackrel{?}{=} 48\zeta(2)L_4(2)$$

   where the integer 48 was discovered at few-digit precision and then gives thousands of digits in a few seconds.

References

[BV] Bloch and Vanhove: seminar by Bloch, Berlin, 6 June 2013

[BS] Brown and Schnetz:
http://arxiv.org/pdf/1304.5342v2.pdf
http://arxiv.org/pdf/1208.1890.pdf
http://arxiv.org/pdf/1006.4064.pdf

[BZ] Brown and Zagier:
http://arxiv.org/pdf/1301.3053v1.pdf
http://people.mpim-bonn.mpg.de/zagier/files/doi/10.4007/annals.2012.175.2.11
http://arxiv.org/pdf/1102.1312.pdf
http://people.mpim-bonn.mpg.de/zagier/files/doi/10.1112/S0010437X0500182X

[BK] Broadhurst and Kreimer:
http://arxiv.org/pdf/hep-ph/9504352.pdf
http://arxiv.org/pdf/hep-th/9609128.pdf

[BBBG] Bailey, Borwein, Broadhurst and Glasser:
http://arxiv.org/pdf/0801.0891.pdf
http://arxiv.org/pdf/0801.4813.pdf

[BBV] Blümlein, Broadhurst and Vermaseren:
http://arxiv.org/pdf/0907.2557.pdf

[B&B] Borwein and Borwein: http://www.jstor.org/stable/pdfplus/2001551.pdf

[BBBL] Borwein, Bradley, Broadhurst, Lisonek:
http://arxiv.org/pdf/math/9910045.pdf
http://arxiv.org/pdf/math/9812020.pdf

[B] Broadhurst:
http://inspirehep.net/record/219836?ln=en
http://arxiv.org/pdf/hep-th/9803091.pdf

[BB] Broadhurst and Brown: conference talks; chapter to appear in *Computer Algebra in Quantum Field Theory*, Springer, 2013

[BFT] Broadhurst, Fleischer and Tarasov:
http://arxiv.org/pdf/hep-ph/9304303v1.pdf

[DD] Davydychev and Delbourgo: http://arxiv.org/pdf/hep-th/0311075v1.pdf

[M] Maeir: http://arxiv.org/pdf/math/0611041v4.pdf

[MWZ] Müller–Stach, Weinzierl and Zayadeh:
http://arxiv.org/pdf/1112.4360v2.pdf

[P] Panzer: http://arxiv.org/pdf/1305.2161.pdf

[S] Schnetz: http://arxiv.org/pdf/0801.2856.pdf

[U] Ussyukina:
http://www.sciencedirect.com/science/article/pii/037026939190950U