Critical point and duality in planar lattice models

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Abstract

These lecture notes describe the content of a six-hours course given by the two authors at the 2012 probability summer school in Saint-Petersburg. The goal is to provide a derivation of several critical parameters of classical planar models such as Bernoulli and Fortuin-Kasteleyn percolation as well as the Ising and Potts models.

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Motivation

Pierre Curie showed experimentally in 1895 that a ferromagnet exhibits a phase transition by loosing its magnetization when heated above a critical temperature, called the Curie temperature. Many other examples of phase transition have been found in real systems coming from physics. During more than one century, mathematicians and physicists have been developing mathematical models to understand such phase transitions. Beyond the original motivation, these models have turned out to be very interesting objects, leading to rich mathematical theories and challenging questions.

In these lecture notes, we focus on a few classical models of statistical physics. The most famous one is probably the Ising model, which was introduced by Lenz [60] in order to understand ferromagnetic phase transitions and the existence of the Curie temperature. Historically, this model has been central in the development of statistical physics. Another model which has the favor of mathematicians is Bernoulli percolation. Defined as a product measure, it is a natural playground for developing new mathematical theories.

A third model, called the Potts model, is a model of random coloring of the vertices of a lattice and has been the object of much interest in the past half-century. These seemingly different models share many properties, and our goal is to develop the theory of the FK percolation model (also called random-cluster model), a model which creates a connection between all these models.

The models above can be defined on any lattice but for simplicity we will mostly deal with the square lattice $\mathbb{Z}^2$. In dimension two, lattice models often exhibit very specific features: one of the most important ones is *duality*, and another one is *integrability*. A recurrent theme of these notes is the study of the consequences of these properties. In particular, they will be used to compute the critical parameters of the model.

The next section presents briefly each of the models we will be interested in, with an emphasis on their common features. Most importantly, we define the FK percolation and show the connection with Bernoulli percolation, Ising and Potts models. Section 3 shows the existence of a phase transition for the FK percolation. This result implies the existence of a phase transition for the three other models. Section 4 contains the computation of the critical parameter for FK percolation using duality. Section 5 describes an alternative computation of the critical parameter based on exact integrability. Section 6 contains a number of conjectures and a non-rigorous description of the phase transition.

Notation The lattice $\mathbb{Z}^2$ is constructed by considering vertices (m,n), with $n,m \in \mathbb{Z}$, and edges between nearest neighbors. Consider the graph $(\mathbb{Z}^2)^*$ defined as a translate of $\mathbb{Z}^2$ by the vector $(1/2,1/2)$. For any edge e of $\mathbb{Z}^2$, there exists a corresponding edge e^* of $(\mathbb{Z}^2)^*$ intersecting e in its midpoint. In our context, a graph G will always be a connected subgraph of the square lattice; it is given by its set of vertices V_G and its set of edges E_G. For a graph G, the dual graph G^* is given by the subgraph of $(\mathbb{Z}^2)^*$ whose vertices correspond to inner faces of G, and edges connect nearest neighbors. Finally, we will denote by ∂G be the set of vertices in G connected by an edge to a vertex in $V_{\mathbb{Z}^2} \smallsetminus V_G$.

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1 "One model to rule them all"

1.1 Bernoulli percolation

Percolation is probably the simplest model of statistical physics. It was introduced by Broadbent and Hammersley in 1957 as a model for a fluid in a porous medium [13]. The medium contains a network of randomly arranged microscopic pores through which fluid can flow.

Let us define the model formally. A *configuration* ω on G is an element of $\{0,1\}^{E_G}$. It can be seen as a subgraph of G, composed of the same sites and a subset of its edges which are those for which $\omega(e) = 1$. The edges belonging to ω are called *open*, the others *closed*. The Bernoulli percolation measure of parameter $p \in [0,1]$ is the product measure $\mathbb{P}_p$ on $\{0,1\}^{E_G}$ such that $\mathbb{P}_p(\omega(e) = 1) = p$ for any $e \in E_G$.

We are interested in connectivity properties of ω when $G = \mathbb{Z}^2$. A *path* is a sequence of neighboring vertices $v_1, \dots, v_n$. The path is said to be *open* if each of the edges $[v_1 v_2], \dots, [v_{n-1} v_n]$ is open. Write $a \leftrightarrow b$ if a and b are connected by an open path. We allow the notation $a \leftrightarrow \infty$ to denote the fact that a belongs to an infinite open path.

Bernoulli percolation on $\mathbb{Z}^2$ undergoes a phase transition in the following sense.

Theorem 1.1 *There exists $0 < p_c < 1$ such that $\mathbb{P}_p(0 \leftrightarrow \infty) > 0$ for $p > p_c$ and $\mathbb{P}_p(0 \leftrightarrow \infty) = 0$ for $p < p_c$.*

Proof: Our goal here is to emphasize several classical arguments allowing for an easy proof of this theorem.

Consider independent uniform random variables U_e on $[0,1]$ indexed by edges. For any $p \in [0,1]$, let $\omega_p \in \{0,1\}^{E_{\mathbb{Z}^2}}$ be the configuration given by $\omega_p(e) = \mathbf{1}_{U_e \leq p}$ for any $e \in E_{\mathbb{Z}^2}$. Obviously, the law of ω_p is $\mathbb{P}_p$. This construction is known as the *standard coupling*, and it gives an increasing coupling of the measures $\mathbb{P}_p$ for different values of p in the following sense: $\omega_p \leq \omega_{p'}$ for any $p \leq p'$ (here and below, we consider the natural ordering on configurations given by the partial ordering on $\{0,1\}^{E^2_{\mathbb{Z}}}$).

Since $\omega_p \leq \omega_{p'}$ for any $p \leq p'$, this construction of configurations $(\omega_p : p \in [0,1])$ allows us to define

$$p_c = \inf\{p \in [0,1] : \mathbb{P}_p(0 \leftrightarrow \infty) > 0\} = \sup\{p \in [0,1] : \mathbb{P}_p(0 \leftrightarrow \infty) = 0\}.$$

At this point, p_c could possibly be equal to 0 or 1. We now prove that $p_c > 0$. Fix $n > 0$ and let Ω_n be the set of paths starting from the origin with n edges. If there exists an infinite open path, there must exist an open path in Ω_n. We deduce that

$$\mathbb{P}_p(0 \leftrightarrow \infty) \leq \mathbb{P}_p\big(\exists \gamma \in \Omega_n \text{ which is open}\big) \leq \sum_{\gamma \in \Omega_n} \mathbb{P}_p\big(\gamma \text{ is open}\big) \leq |\Omega_n| p^n.$$

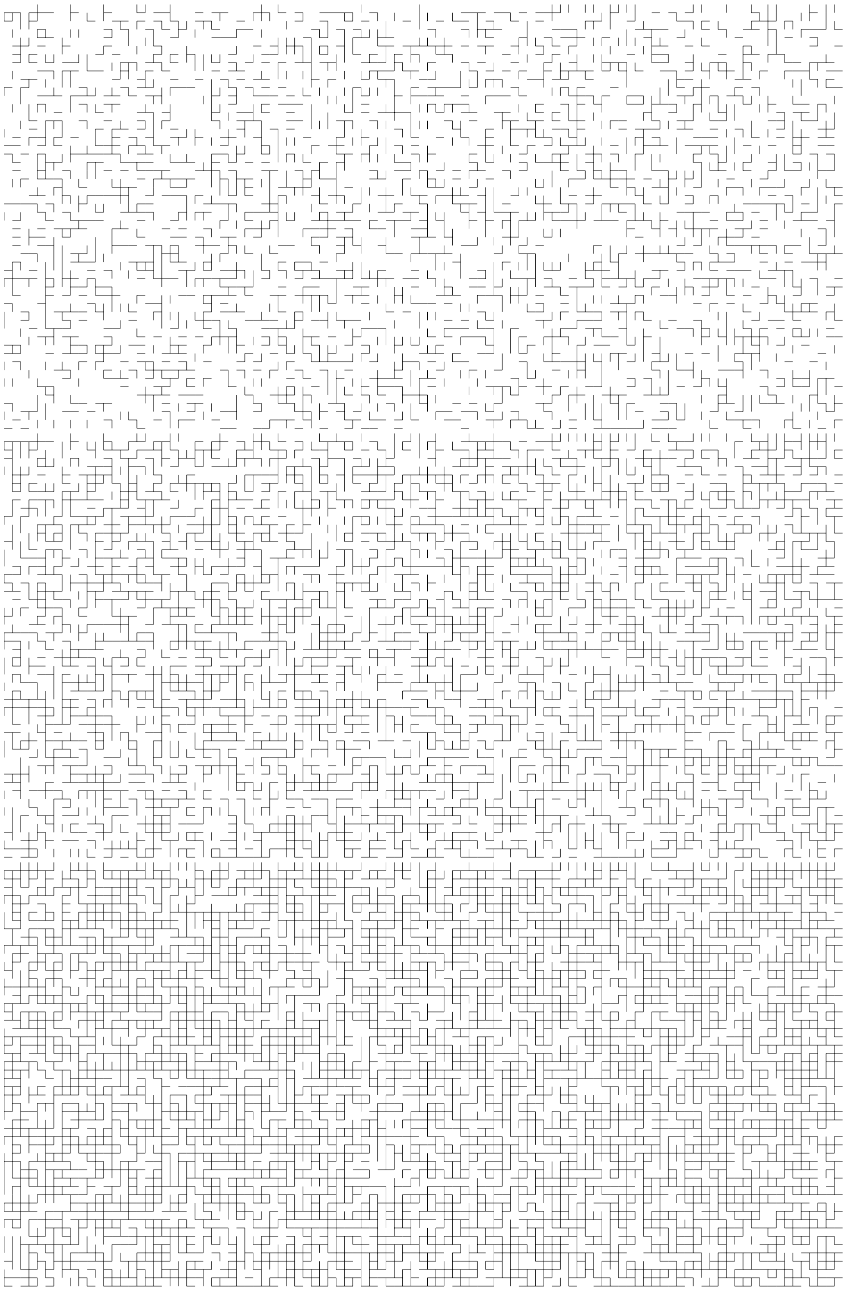


Figure 1: Three percolation configurations for three different parameters, respectively sub-critical, critical and super-critical.

Since $|\Omega_n| \le 4^n$, the quantity on the right tends to 0 when $p < 1/4$. Therefore, we deduce that $p_c \ge 1/4$. This argument is called a *Peierls argument.*

It only remains to prove the inequality $p_c < 1$. The proof is based on the concept of *duality* that we present now. For any configuration $\omega \in \{0,1\}^{E_{\mathbb{Z}^2}}$, define the configuration $\omega^* \in \{0,1\}^{E_{(\mathbb{Z}^2)^*}}$ by $\omega^*(e^*) = 1 - \omega(e)$. In words, a dual edge is open if the corresponding edge on $\mathbb{Z}^2$ is closed, and vice versa. The configuration ω^* is then distributed according to a Bernoulli percolation of parameter $p^* := 1 - p$.

With this dual configuration in our possession, we can now prove $p_c < 1$. In order for the origin not to be connected to infinity, ω^* must contain an open path of dual-edges surrounding the origin.

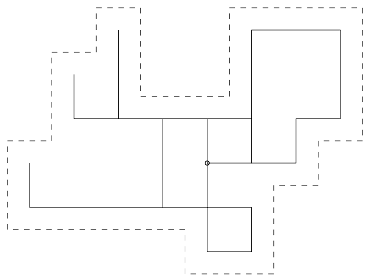


Figure 2: One realization of a finite connected component containing the origin, and the associated surrounding dual circuit.

Let $\Omega_{m,n}$ be the set of dual circuits of length m surrounding the origin and passing by $(n + \frac{1}{2}, 0)$. We deduce that

$$\mathbb{P}_p(0 \nleftrightarrow \infty) = \mathbb{P}_p\left(\exists \gamma \in \bigcup_{m,n} \Omega_{m,n} \text{ which is open}\right) \le \sum_{n\ge1}\sum_{m\ge 2n+4} |\Omega_{m,n}|(1-p)^m$$
$$\le \sum_{n\ge1}\sum_{m\ge 2n+4} (4-4p)^m = \frac{(4-4p)^6}{(4p-3)(32p-15-16p^2)}.$$

The second line is based on the observation that a dual circuit passing by $(n+\frac{1}{2},0)$ and surrounding the origin contains more than $2n+4$ edges. The last expression has no real physical meaning, but it shows that when p is close enough to 1, the double sum is strictly smaller than 1 (one could alternatively derive this from dominated convergence). Equivalently, $\mathbb{P}_p(0 \leftrightarrow \infty) > 0$ and therefore $p_c \le p < 1$. □

> The increasing coupling, the Peierls argument as well as the duality between $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^*$ are not specific to Bernoulli percolation. In particular, one goal of these notes is to generalize these concepts to other models and to investigate the possible consequences.

The critical point of percolation on $\mathbb{Z}^2$ was proved by Kesten to be equal to 1/2 in 1980. Even though the proof is quite involved, predicting that $p_c = 1/2$ is fairly natural. Indeed, $p = 1/2$ is the point for which percolation on $\mathbb{Z}^2$ and $(\mathbb{Z}^2)^*$ play symmetric roles, meaning that they are both Bernoulli percolations of parameter 1/2. It is the unique point for which p^* is equal to p. We call this point the *self-dual point.*

Theorem 1.2 (Kesten [51]) *The critical point of Bernoulli percolation on $\mathbb{Z}^2$ equals its self-dual point,* i.e. $p_c = 1/2$.

1.2 The Ising model

The Ising model is one of the simplest models in statistical physics to exhibit an order-disorder transition. It was introduced by Lenz in [60] and studied by his student Ising in his thesis [46] as an attempt to explain the existence of a Curie temperature for ferromagnets. In the Ising model, the ferromagnet is modeled as a collection of atoms with fixed positions on a crystalline lattice. Each atom has a magnetic "spin", pointing in one of two possible directions +1 or −1.

The formal definition is slightly more intricate than for percolation. Let G be a finite graph. The Ising model is a random assignment $\sigma \in \{-1,1\}^{V_G}$, where σ_x denotes the *spin* at site x. The *Hamiltonian* of the model is defined by

$$H_G(\sigma) := -\sum_{[xy]\in E_G} \sigma_x\sigma_y. \tag{1}$$

The *partition function* of the model is

$$Z^{\text{free}}_{\beta,G} = \sum_{\sigma\in\{-1,1\}^{V_G}} \exp\left[-\beta H_G(\sigma)\right], \tag{2}$$

where β is the inverse temperature of the model. The probability of a configuration σ is then equal to

$$\mu^{\text{free}}_{\beta,G}(\sigma) = \frac{1}{Z^{\text{free}}_{\beta,G}} \exp\left[-\beta H_G(\sigma)\right]. \tag{3}$$

The measure thus obtained is called the Ising measure at inverse temperature β on G with free boundary conditions. The appearance of boundary conditions comes from the fact that the Ising model is not a model with independence such as Bernoulli percolation. Let us construct other boundary conditions. Assume that G is a finite subgraph of $\mathbb{Z}^2$, and fix a configuration $b \in \{-1,1\}^{V_{\mathbb{Z}^2}}$ on the whole lattice. In such a case, it is natural to extend the definition of E_G to include bonds connecting a vertex of G to one in its complement; if $[xy]$ is such a bond, with $x \in V_G$, simply add the term $\sigma_x b_y$ to the right-hand side of (1). The definition in (2) then includes an interaction part along the boundary of G. Let $\mu^+_{\beta,G}$ be the measure obtained for boundary conditions $b \equiv 1$, and $Z^+_{\beta,G}$ be the corresponding partition function. In such case, we speak of plus boundary conditions. Similarly, one can define minus boundary conditions and the associated measure and partition function.

Defining the Ising model in infinite volume is not completely trivial. For plus and minus boundary conditions, one can use correlation inequalities to take weak limits $\mu^\pm_{\beta,\mathbb{Z}^2}$ of the measures $\mu^\pm_{\beta,[-n,n]^2}$ – here and later, convergence is always taken to mean weak convergence: for any function f depending on a finite number of spins, *i.e.*

$$\mu^\pm_{\beta,[-n,n]^2}(f) \longrightarrow \mu^\pm_{\beta,\mathbb{Z}^2}(f).$$

The justification of the existence of this measure will be given later.

The Ising model in infinite volume exhibits a phase transition at some critical inverse temperature β_c, above which a spontaneous magnetization appears:

Theorem 1.3 *There exists $\beta_c \in (0,\infty)$ such that $\mu^+_{\beta,\mathbb{Z}^2}(\sigma_0) = 0$ for any $\beta < \beta_c$, and $\mu^+_{\beta,\mathbb{Z}^2}(\sigma_0) > 0$ for any $\beta > \beta_c$ (σ_0 denotes the spin at the origin).*

The proof of this theorem bears similarities with the proof of Theorem 1.1. Nevertheless, some difficulties arise quickly, for instance when justifying the existence of β_c. Indeed, in the case of percolation, the existence of p_c is straightforward thanks to the increasing coupling, while no useful coupling between Ising measures at different temperatures allows to justify the existence of two well separated phases (one could imagine that $\mu^+_{\beta,\mathbb{Z}^2}(\sigma_0)$ alternates from 0 to positive several times).

Proof: Let us prove that β_c exists. First fix G and differentiate $\mu^+_{\beta,G}(\sigma_0)$ in β to find

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\Big(\mu^+_{\beta,G}(\sigma_0)\Big) = \sum_{e=[xy]\in E_G} \mu^+_{\beta,G}(\sigma_0\sigma_x\sigma_y) - \mu^+_{\beta,G}(\sigma_0)\mu^+_{\beta,G}(\sigma_x\sigma_y).$$

The GKS inequality [37, 49] shows that

$$\mu^+_{\beta,G}(\sigma_0\sigma_x\sigma_y) - \mu^+_{\beta,G}(\sigma_0)\mu^+_{\beta,G}(\sigma_x\sigma_y) \geq 0$$

for any x,y, which implies that $\mu^+_{\beta,G}(\sigma_0)$ is increasing. Since the previous claim is valid for any $[-n,n]^2$, $\mu^+_{\beta,\mathbb{Z}^2}(\sigma_0) = \lim_{n\to\infty}\mu^+_{\beta,[-n,n]^2}(\sigma_0)$ is increasing and $\beta_c \in [0,\infty]$ exists.

Now that β_c exists, let us show that it lies strictly between 0 and infinity. The proof of this fact relies on a combinatorial argument, similar to the one first introduced by Peierls. Historically, this argument [63] changed the face of statistical physics since it invalidated a conjecture of Ising on the absence of phase transition for the Ising model. The argument is very similar to the one described above for percolation, and is in fact anterior to it. Peierls' argument is best understood via the low and high temperature expansions of the planar Ising model.

Let us start with the *low temperature expansion*. It is a graphical representation on the dual lattice. Fix a spin configuration σ for the Ising model on G with + boundary conditions. The *collection of contours* $\omega(\sigma)$ of a spin configuration σ is the set of interfaces separating +1 and −1 connected components. They are naturally defined on the dual graph G^*. In a collection of contours, an even number of dual edges emanate from each dual vertex. Reciprocally, any family of dual edges with an even number of edges emanating from each dual vertex is the collection of contours of exactly one spin configuration (since we fix + boundary conditions). The probability of σ can be restated in terms of the corresponding $\omega(\sigma)$ in the following way. Let $\mathscr{E}^{\text{low}}_{G^*}$ be the set of possible collections of contours, and let $|\omega|$ be the number of edges of a collection of contours ω. One can easily check that

$$\mu^+_{\beta,G}(\sigma) = \frac{e^{-2\beta|\omega(\sigma)|}}{\sum_{\omega\in\mathscr{E}^{\text{low}}_{G^*}} e^{-2\beta|\omega|}}.$$

As a consequence,

$$\mu^+_{\beta,G}(\sigma_0=-1) \ \leq \ \frac{\displaystyle\sum_{\omega\in\mathscr{E}^{\text{low}}_{G^*}\text{ with one loop surrounding }0} e^{-2\beta|\omega|}}{\displaystyle\sum_{\omega\in\mathscr{E}^{\text{low}}_{G^*}} e^{-2\beta|\omega|}}.$$

We used the fact that $\{\sigma_0=-1\}$ is included in the event that there exists a circuit in $\omega(\sigma)$ surrounding 0. Since

$$\mu^+_{\beta,G}(\sigma_0) = 1 - 2\mu^+_{\beta,G}(\sigma_0=-1),$$

it is sufficient to show that $\mu^+_{\beta,G}[\sigma_0 = -1] < 1/2$ uniformly in the subgraph G of $\mathbb{Z}^2$ provided that β is large enough. Using the previous expression in terms of the low temperature expansion, we easily obtain that

$$\mu^+_{\beta,G}[\sigma_0 = -1] \;\le\; \sum_{\gamma \text{ loop surrounding } 0} e^{-2\beta|\gamma|} \;\le\; \sum_{n=1}^{\infty} n4^n e^{-2\beta n},$$

where the sum goes over circuits γ surrounding 0. For β large enough, this term is smaller than $1/2$. In the first inequality, we used the fact that

$$\sum_{\omega\in\mathscr{E}^{\text{low}}_{G^*} \text{ with a loop surrounding } 0} e^{-2\beta|\omega|} \le \Bigg(\sum_{\gamma \text{ loop surrounding } 0} e^{-2\beta|\gamma|}\Bigg)\Bigg(\sum_{\omega\in\mathscr{E}^{\text{low}}_{G^*}} e^{-2\beta|\omega|}\Bigg).$$

In the second inequality, we bounded the number of self-avoiding loops of length n surrounding the origin by $n4^n$.

The *high temperature expansion* is a graphical representation on the primal lattice itself [73]. It is not a geometric representation in the sense that a spin configuration σ cannot be mapped to a subset of configurations in the graphical representation. Yet, it is a rather convenient way to represent correlations between spins using statistics of contours. It is based on the fact that spins take only two values, which translates into the following identity:

$$e^{\beta\sigma_x\sigma_y} \;=\; \cosh(\beta) + \sigma_x\sigma_y\sinh(\beta) = \cosh(\beta)\left[1+\tanh(\beta)\sigma_x\sigma_y\right]. \tag{4}$$

Let us start by expressing the partition function with + boundary conditions. We know

$$\begin{aligned}
Z^+_{\beta,G} &= \sum_\sigma \prod_{[xy]\in E_G} e^{\beta\sigma_x\sigma_y}\\
&= \cosh(\beta)^{|E_G|}\sum_\sigma \prod_{[xy]\in E}\left[1+\tanh(\beta)\sigma_x\sigma_y\right]\\
&= \cosh(\beta)^{|E_G|}\sum_\sigma\sum_{\omega\subset E}\tanh(\beta)^{|\omega|}\prod_{e=[xy]\in\omega}\sigma_x\sigma_y\\
&= \cosh(\beta)^{|E_G|}\sum_{\omega\subset E}\tanh(\beta)^{|\omega|}\sum_\sigma\prod_{e=[xy]\in\omega}\sigma_x\sigma_y,
\end{aligned}$$

where we used (4) in the second equality. Let $\mathscr{E}^{\text{high}}_G$ be the set of families of edges of G such that an even number of edges emanate from each vertex in G (possibly including edges from a vertex in G to one in $\mathbb{Z}^2\setminus G$). Notice that $\sum_\sigma\prod_{e=[xy]\in\omega}\sigma_x\sigma_y$ equals $2^{|V_G|}$ if ω is in $\mathscr{E}^{\text{high}}_G$, and 0 otherwise, hence proving that

$$Z^+_{\beta,G} = \;2^{|V_G|}\cosh(\beta)^{|E_G|}\sum_{\omega\in\mathscr{E}^{\text{high}}_G}\tanh(\beta)^{|\omega|}.$$

A similar computation can be performed with $\sum_\sigma \sigma_0 e^{-\beta H(\sigma)}$. We obtain the following identity

$$\sum_\sigma \sigma_0 e^{-\beta H(\sigma)} = \cosh(\beta)^{|E_G|}\sum_{\omega\in\mathscr{E}^{\text{high}}_G(0)}\tanh(\beta)^{|\omega|},$$

where $\mathscr{E}^{\text{high}}_G(0)$ is the set of families of edges of G such that an even number of edges emanate from each vertex, except at 0 and possibly at some of the vertices of ∂G, where an odd number of edges emanates. Note that an element of $\mathscr{E}_G^{\text{high}}(0)$ is the union of an element in $\mathscr{E}_G^{\text{high}}$ and a path from 0 to the boundary. Taking the ratio of the previous quantity and the partition function, one obtains

$$\mu_{\beta,G}^{+}(\sigma_0) = \frac{\sum_{\omega\in\mathscr{E}_G^{\text{high}}(0)}\tanh(\beta)^{|\omega|}}{\sum_{\omega\in\mathscr{E}_G^{\text{high}}}\tanh(\beta)^{|\omega|}}.$$

Then, a argument similar to Peierls' argument described above shows that whenever $\tanh(\beta)$ is small enough, the above quantity tends to 0 as G tends to the whole lattice (this is due to the existence of the additional path from 0 to ∂G, we leave it as an exercise for the reader). This implies that $\beta < \beta_c$ for β small enough. □

Some will have noticed the similarity between the low and high temperature expansions, since they both involve sets of edges with similar structures. In fact, the low and high temperature expansions can be extended to free boundary conditions. When performing the high temperature expansion of the Ising model on G^*, one obtains that the set of contours is exactly $\mathscr{E}_{G^*}^{\text{low}}$. In [53, 54], Kramers and Wannier used this observation to relate properties of the Ising model on $\mathbb{Z}^2$ at β and the Ising model on $(\mathbb{Z}^2)^*$ at β^* given by $\tanh(\beta^*) = e^{-2\beta}$. This relation is a duality relation since $(\beta^*)^* = \beta$. They then used this duality to identify (albeit non rigorously) the critical point as being the unique point such that $\beta_c = \beta_c^*$. This was later proved formally by Onsager:

Theorem 1.4 (Onsager [62]) *For the Ising model on $\mathbb{Z}^2$, the critical inverse temperature β_c equals $\frac{1}{2}\log(1+\sqrt{2})$.*

> The study of the Ising model and Bernoulli percolation share similarities. A Peierls argument shows that $\beta_c \in (0,\infty)$ and exactly as in the case of Bernoulli percolation, the critical parameter corresponds to the self-dual point of some duality relation (here the so-called Kramers-Wannier duality). Nevertheless, the proofs are already more involved than in the percolation case, and some ingredients do not apply (for instance the increasing coupling).

1.3 The Potts models

The Potts model with q colors is a random q-coloring of a graph G. On a finite graph, the energy of a configuration σ is given by

$$H_{q,G}(\sigma) \;:=\; -2\sum_{x\sim y}1_{\sigma_x=\sigma_y}$$

and the probability at inverse temperature β by

$$\mu_{q,\beta,G}^{\text{free}}(\sigma) = \frac{e^{-\beta H_{q,G}(\sigma)}}{\sum_{\tilde\sigma}e^{-\beta H_{q,G}(\tilde\sigma)}},$$

where the summation is over any q-coloring of G. This measure is called the Potts measure with free boundary conditions at inverse temperature β. As for the Ising model, one can fix vertices in ∂G to have color 1. The conditioned measure, denoted $\mu_{q,\beta,G}^{1}$, is called Potts measure with q colors and monochromatic boundary conditions 1. Alternatively, one can also, as in the case of the Ising model, fix a configuration b on the whole lattice and use it

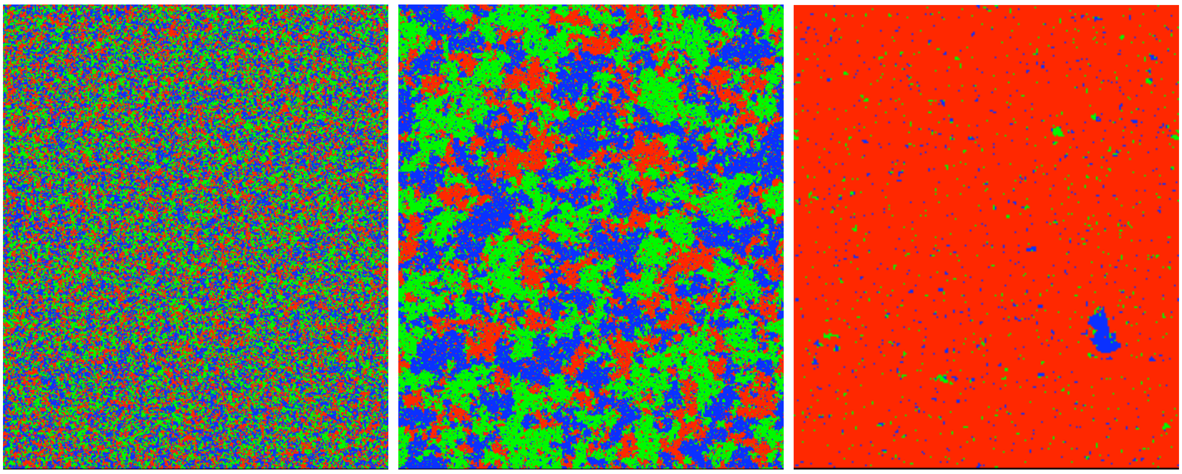


Figure 3: Three Potts configurations with $q = 3$ for $\beta < \beta_c(3)$, $\beta = \beta_c(3)$ and $\beta > \beta_c(3)$. The color 1 (corresponding to the boundary condition) is red.

to add an interaction term along the boundary of G. The $q = 2$ case corresponds exactly to the Ising model — the factor 2 was introduced to make the values of β match.

A phase transition also occurs for Potts models. Namely, there exists a critical inverse temperature $\beta_c(q)$ such that

- for $\beta < \beta_c(q)$, $\lim_{n\to\infty} \mu^1_{q,\beta,[-n,n]^2}(\sigma_0 = 1) = \frac{1}{q}$,
- for $\beta > \beta_c$, $\lim_{n\to\infty} \mu^1_{q,\beta,[-n,n]^2}(\sigma_0 = 1) > \frac{1}{q}$.

The first phase is called *disordered*, and the second *ordered*. The former corresponds to infinite range memory. It is sometimes called *symmetry breaking*, since the color 1 is favored.

Theorem 1.5 (Beffara, Duminil-Copin [6]) *For the Potts model with q colors, the critical inverse temperature on $\mathbb{Z}^2$ equals $\beta_c(q) = \frac{1}{2}\log(1 + \sqrt{q})$.*

As mentioned earlier, the case $q = 2$ is exactly Onsager's result about the Ising model. Pirogov-Sinai theory [52] shows that the large-q regime exhibits a first-order phase transition, which was used to derive the value of β_c in that case as well — see below after the corresponding statement about the FK percolation for a more detailed discussion.

The existence of the phase transition is much harder to comprehend in this case. In particular, there is no equivalent of the GKS inequality. In addition, the low and high temperature expansions do not work easily in this case *(but it can be done — we suggest it as an exercise for the serious reader of these notes)*. Last but not least, no direct duality relation can be exhibited. Even though the tools used in the case of percolation and the Ising model are not at our disposal for this model, the critical inverse temperature still exists and still has a simple expression, but we will need to change our point of view to determine it.

At this point of the study, we have presented three seemingly different models, all having very explicit critical parameters. Furthermore, in two cases the value of this critical parameter can be guessed via a duality relation. It becomes natural to ask whether these three models can be seen as belonging to a more general family, which would make a unified approach possible. This family of models does exist, and we present it now.

1.4 Fortuin-Kasteleyn percolation

The *FK percolation* or *random-cluster model* is a model of dependent bond-percolation on graphs. It was introduced by Fortuin and Kasteleyn in [31]. The space of configuration is $\{0,1\}^{E_G}$, and similarly to Bernoulli percolation, a *configuration* ω can be seen as a subgraph of G, composed of the same sites and a subset of its edges. The edges belonging to ω are called *open*, the others *closed*. Similarly to the case of Bernoulli percolation, a path is said to be *open* if all its edges are open. Two sites a and b are still said to be *connected* if there is an *open path* connecting them (this event will be denoted by $a \leftrightarrow b$). The maximal connected components will be called *clusters* (they can be isolated vertices).

Let $o(\omega)$ be the number of open edges in ω, $c(\omega)$ be the number of closed edges and $k(\omega)$ be the number of clusters. The probability measure $\phi_{p,q,G}$ of the FK percolation on a finite graph G with parameters $p \in [0,1]$ and $q > 0$ is defined by

$$\phi_{p,q,G}(\{\omega\}) := \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega)}}{Z_{p,q,G}}$$

for every configuration ω on G, where $Z_{p,q,G}$ is a normalizing constant referred to as the *partition function*.

The case $q = 1$ corresponds to Bernoulli percolation. In this case, the topological term $q^{k(\omega)}$ is identically equal to 1, and the model is independent. For general values of q, the topological term introduces some long range dependency.

We will extend the definition above in the following way. *Boundary conditions* ξ are given as a partition of ∂G. The graph obtained from the configuration ω by identifying (or *wiring*) the vertices in ∂G that belong to the same component of ξ is denoted by $\omega \cup \xi$. Boundary conditions should be understood as encoding how sites are connected outside of G. Let $o(\omega)$ (resp. $c(\omega)$) denote the number of open (resp. closed) edges of ω and $k(\omega,\xi)$ the number of connected components of $\omega \cup \xi$.

Definition 1.6 *The probability measure $\phi^{\xi}_{p,q,G}$ of the FK percolation on G with parameters p and q and boundary conditions ξ is defined by*

$$\phi^{\xi}_{p,q,G}(\{\omega\}) := \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega,\xi)}}{Z^{\xi}_{p,q,G}} \tag{5}$$

for every configuration ω on G, where $Z^{\xi}_{p,q,G}$ is the partition function.

The case presented above corresponds to the absence of wiring between boundary vertices. In such case, we say that we are in presence of free boundary conditions and we set $\xi = 0$. The somewhat opposite example is given by wired boundary conditions, corresponding to all the boundary vertices being wired together. In such case, we set $\xi = 1$.

This model also satisfies a duality relation. In two dimensions, one can associate with any FK percolation on a graph G a dual model on G^*. Given a subgraph configuration ω, construct a model on G^* by declaring any edge of the dual graph to be open (resp. closed) if the corresponding edge of the primal lattice is closed (resp. open) for the initial configuration. The new configuration is called the *dual configuration* of ω and is denoted ω^*. So far, this is simply a version (on finite graphs) of the construction for Bernoulli percolation and we did not use any property of the measure. The "miracle" of this duality is that the dual configuration is also an FK configuration, although with different parameters.

Proposition 1.7 (Planar duality) *The dual model of the FK percolation on G with parameters (p,q) and wired boundary conditions is the FK percolation with parameters (p^*,q^*) and free boundary conditions on G^*, where*

$$\frac{pp^*}{(1-p)(1-p^*)} = q \quad \textit{and} \quad q^* = q.$$

When $q = 1$, $p^* = 1-p$ and we recover the duality in Bernoulli percolation.

Proof: Note that the state of edges between two sites of ∂G is not relevant when boundary conditions are wired. Indeed, sites on the boundary are connected via boundary conditions anyway, so that the state of each boundary edge does not alter the connectivity properties of the subgraph. For this reason, forget about edges between boundary sites and consider only inner edges. These edges correspond to edges of G^* (remember that G^* was constructed from inner faces of G). In this proof, $o(\omega)$ and $c(\omega)$ denote the number of open and closed inner edges.

From the definition of the dual configuration ω^* of ω, we have $o(\omega^*) + o(\omega) = |E_{G^*}|$, where $o(\omega^*)$ is the number of open dual edges. Moreover, connected components of ω^* correspond exactly to faces of ω, so that $f(\omega) = k(\omega^*)$, where $f(\omega)$ is the number of faces (counting the infinite face). Using Euler's formula

$$o(\omega) \ + \ k(\omega) \ + \ 1 \ = \ |V_G| \ + \ f(\omega),$$

which is valid for any planar graph (here we applied it to ω), we obtain

$$k(\omega) \ = \ |V_G| - 1 + f(\omega) - o(\omega) \ = \ |V_G| - 1 + k(\omega^*) - |E_{G^*}| + o(\omega^*).$$

The probability of ω^* is equal to the probability of ω under $\phi^1_{p,q,G}$, i.e.

$$\begin{aligned}
\phi^1_{p,q,G}(\omega) &= \frac{1}{Z^1_{p,q,G}} p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega)} \\
&= \frac{(1-p)^{|E_{G^*}|}}{Z^1_{p,q,G}} [p/(1-p)]^{o(\omega)} q^{k(\omega)} \\
&= \frac{(1-p)^{|E_{G^*}|}}{Z^1_{p,q,G}} [p/(1-p)]^{|E_{G^*}|-o(\omega^*)} q^{|V_G|-1-|E_{G^*}|+k(\omega^*)+o(\omega^*)} \\
&= \frac{p^{|E_{G^*}|} q^{|V_G|-1-|E_{G^*}|}}{Z^1_{p,q,G}} [q(1-p)/p]^{o(\omega^*)} q^{k(\omega^*)} \\
&= \frac{p^{|E_{G^*}|} q^{|V_G|-1-|E_{G^*}|}}{Z^1_{p,q,G}} [p^*/(1-p^*)]^{o(\omega^*)} q^{k(\omega^*)} \\
&= \frac{p^{|E_{G^*}|} q^{|V_G|-1-|E_{G^*}|}}{(1-p^*)^{|E_{G^*}|} Z^1_{p,q,G}} (p^*)^{o(\omega^*)} (1-p^*)^{c(\omega^*)} q^{k(\omega^*)} \\
&= \phi^0_{p^*,q,G^*}(\omega^*).
\end{aligned}$$

In the third and sixth lines, we used the relation $o(\omega^*) + o(\omega) = |E_{G^*}|$. The Euler formula was harnessed in the third line, and the relation between p and p^* in the fifth. □

Let us introduce the self-dual point $p_{sd}(q)$, given by the unique solution of the equation $p^*(p_{sd}, q) = p_{sd}$, *i.e.*

$$p_{sd}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}. \tag{6}$$

For fixed $q \geq 1$, the FK percolation shares many properties with Bernoulli percolation. In particular, the definition, which required G to be finite, can be extended to infinite volume and there exists a natural ordering (in p) between measures. We will describe these properties in the following section. In fact, the main object of these lecture notes is to show that the self-dual point is indeed the critical point for all FK percolation with $q \geq 1$.

Theorem 1.8 (Beffara, Duminil-Copin [6]) *Let $q \geq 1$. The critical point $p_c = p_c(q)$ for the FK percolation with cluster-weight q on the square lattice satisfies*

$$p_c = p_{sd}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

The proof of this theorem will be the core of this article. It is postpone to the next sections. A rigorous derivation of the critical point was previously known in three cases. For $q = 1$, the model is Bernoulli percolation, proved by Kesten in 1980 [51] to be critical at $p_c(1) = 1/2$. For $q = 2$, the model can be related to the Ising model, as we will see very soon, and the computation of Baxter and Yang allows for a computation of $p_c(2)$. Finally, for sufficiently large q, a proof is known based on the fact that the FK percolation exhibits a first order phase transition (see [55, 56], the proofs are valid for q larger than 25.72). Let us mention that physicists derived the critical temperature for the Potts models with $q \geq 4$ in 1978, using non-geometric arguments based on analytic properties of the Hamiltonian [42].

1.5 The relation between the Potts models and FK percolation

As emphasized earlier, it would be very useful if the critical parameters of spin models (the Ising and Potts models) could be understood as the solution of some duality relation. This is indeed the case. FK percolations and Potts models (and therefore also the Ising model) are all related via the following coupling, which allows to reinterpret the critical parameters as self-dual points for the duality of FK percolation models. Since the Ising model is simply the Potts model with 2 color, we treat the case of Potts models directly.

Let G be a finite graph and let ω be a configuration of open and closed edges on G. A coloring $\sigma \in \{1,\dots,q\}^{V_G}$ can be constructed on the graph G by assigning independently to each cluster of ω not touching the boundary a color uniformly among $\{1,\dots,q\}$ (in this construction all the sites of a cluster in ω receive the same color), and by assigning color 1 to any site connected to the boundary by an open path.

Proposition 1.9 (Coupling) *Let $q \in \{2,3,\dots\}$. Let $p \in (0,1)$ and G a finite graph. If the configuration ω is distributed according to an FK percolation measure with parameters (p,q) and wired boundary conditions, then the coloring σ is distributed according to a Potts measure with inverse temperature $\beta(q) = -\frac{1}{2}\ln(1-p)$ and monochromatic boundary conditions equal to 1.*

When $q = 2$, we obtain a coupling between the FK percolation with cluster-weight $q = 2$ and the Ising model, called the Edwards-Sokal coupling [30].

Proof: Consider a finite graph G, and let $p \in (0,1)$. Consider a measure P on pairs (ω,σ), where ω is an FK configuration with wired boundary conditions and σ is the corresponding random coloring, constructed as explained above. Then, for (ω,σ), we have:

$$P[(\omega,\sigma)] \;=\; \frac{1}{Z^1_{p,q,G}}p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega)}\cdot q^{-k(\omega)} \;=\; \frac{1}{Z^1_{p,q,G}}p^{o(\omega)}(1-p)^{c(\omega)}.$$

Now, we construct another measure $\tilde P$ on pairs of percolation configurations and colorings as follows. Let $\tilde\sigma$ be a coloring distributed according to a Potts model with inverse temperature β satisfying $e^{-2\beta} = 1-p$ and monochromatic boundary conditions equal to 1. We deduce $\tilde\omega$ from $\tilde\sigma$ by closing all edges between neighboring sites with different colorings, and by independently opening edges between neighboring sites with same colors with probability p. Then, for any $(\tilde\omega,\tilde\sigma)$,

$$\tilde P[(\tilde\omega,\tilde\sigma)] = \frac{e^{-2\beta r(\tilde\sigma)}p^{o(\tilde\omega)}(1-p)^{|E_G|-o(\tilde\omega)-r(\tilde\sigma)}}{Z} = \frac{p^{o(\tilde\omega)}(1-p)^{c(\tilde\omega)}}{Z}$$

where $r(\tilde\sigma)$ is the number of edges between sites with different colors.

Note that the two previous measures are in fact defined on the same set of "compatible" pairs of configurations: if σ has been obtained from ω, then ω can be obtained from σ via the second procedure described above, and the same is true in the reverse direction for $\tilde\omega$ and $\tilde\sigma$. This implies that $P = \tilde P$ and the marginals of P are the FK percolation with parameters (p,q) and the Potts model at inverse temperature β, which is the claim. □

The coupling gives a randomized procedure to obtain a Potts configuration from an FK configuration (it suffices to assign random colors). The proof also provides a randomized procedure to obtain an FK configuration from a Potts configuration: simply open with probability p edges between two sites with same color.

If one considers free boundary conditions for the FK percolation, the coupling provides us with a Potts configuration with free boundary conditions by coloring uniformly any cluster (even those touching the boundary). We omit the details, since the generalization is straightforward.

The coupling provides us with a "dictionary" between properties of the Potts models and properties of the percolation models. For instance, Potts correlations and FK connectivity probabilities can be related since two sites which are connected in the FK configuration must have the same color, while sites which are not have independent colors. As an illustration, let us prove the following proposition.

Proposition 1.10 *For $p \in (0,1)$, $q \in \mathbb{N} \setminus \{0,1\}$, and $\beta(q) = -\frac{1}{2}\ln(1-p)$,*

$$\mu^1_{\beta,G}[\sigma_x = 1] \quad = \quad \frac{1}{q} + \Big(1 - \frac{1}{q}\Big)\phi^1_{p,q,G}(x \leftrightarrow \partial G).$$

Proof: Using the coupling P,

$$\begin{aligned}\mu^1_{\beta,G}[\sigma_x = 1] &= P(\sigma_x = 1) = P(\sigma_x = 1 \text{ and } x \leftrightarrow \partial G) + P(\sigma_x = 1 \text{ and } x \not\leftrightarrow \partial G)\\ &= P(x \leftrightarrow \partial G) + \frac{1}{q}P(x \not\leftrightarrow \partial G) = \frac{1}{q} + \Big(1 - \frac{1}{q}\Big)\phi^1_{p,q,G}(x \leftrightarrow \partial G).\end{aligned}$$

In the second line, we used the fact that $\sigma_x = 1$ if $x \leftrightarrow \partial G$, and that the color is chosen uniformly if $x \not\leftrightarrow \partial G$. □

As a consequence,

$$\lim_{n\to\infty} \mu^1_{q,\beta,[-n,n]^2}[\sigma_0 = 1] > \frac{1}{q}$$

if and only if

$$\lim_{n\to\infty} \phi^1_{p,q,[-n,n]^2}(0 \leftrightarrow \partial[-n,n]^2) = \phi^1_{p,q,\mathbb{Z}^2}(0 \leftrightarrow \infty) > 0$$

for $p = 1 - e^{-2\beta}$. In particular, $\beta_c(q) = \frac{1}{2}\log(1-p_c(q))$. This observation implies that Theorems 1.4 and 1.5 are consequences of Theorem 1.8. Furthermore, recall that Theorem 1.2 is a special case of Theorem 1.8.

Taking into account the previous discussion, we are now facing the following situation. At this point, we successfully constructed a family of models unifying the Ising, Potts and Bernoulli percolation. A duality relation allows to identify non rigorously the critical point as the unique self-dual point. The rest of these notes is devoted to the proof of existence of a phase transition, and to the rigorous justification of the identification of the critical parameter. We now focus on the FK percolation.

Let us mention that the previous discussion required the measures in infinite volume to be defined properly. We will introduce infinite-volume measure for FK percolation rigorously in the next section. Note that the coupling between FK percolation and Potts models can be extended to the infinite volume, and this can be used to justify the definition of infinite-volume measures for Ising and Potts models.

2 Existence of a critical point for FK percolation with $q \geq 1$

In this section, we develop the required technology to prove that FK percolation measures exist in infinite volume (i.e., on the whole lattice $\mathbb{Z}^2$). We also describe a natural ordering between FK measures with the same cluster-weight q. Together, these two facts imply the existence of a phase transition.

2.1 Strong positive association when $q \geq 1$

The object of this section is to compare FK measures with different p but same q and to derive correlation inequalities generalizing the Harris inequality for Bernoulli percolation.

Let G be a finite graph. The space $\Omega = \{0,1\}^{E_G}$ is (partially) ordered in the standard way. Probability measures are always assumed to be *positive*, in the sense that the probability of ω is strictly positive for any $\omega \in \{0,1\}^{E_G}$. An event A is *increasing* if for any $\omega' \geq \omega$, $\omega \in A$ implies $\omega' \in A$. A probability measure μ_1 on $\Omega = \{0,1\}^{E_G}$ *stochastically dominates* μ_2 if $\mu_1(A) \geq \mu_2(A)$ for every increasing event A (in such case, μ_2 is said to be *stochastically dominated* by μ_1).

Holley criterion Let us start by discussing stochastic ordering and correlation inequalities for general probability measures. Define $\omega_1 \vee \omega_2$ and $\omega_1 \wedge \omega_2$ by the formulas

$$(\omega_1 \vee \omega_2)(e) = \max\{\omega_1(e), \omega_2(e)\} \text{ and } (\omega_1 \wedge \omega_2)(e) = \min\{\omega_1(e), \omega_2(e)\}.$$

Theorem 2.1 (Holley inequality [43]) *Let μ_1, μ_2 be two positive measures such that*

$$\mu_1(\omega_1 \vee \omega_2)\mu_2(\omega_1 \wedge \omega_2) \;\geq\; \mu_1(\omega_1)\mu_2(\omega_2), \quad \omega_1, \omega_2 \in \Omega, \tag{7}$$

then $\mu_1(A) \geq \mu_2(A)$ for any increasing event A.

The proof of this statement is a construction via Markov chains of a coupling (ω_1, ω_2) between the two measures (the law of ω_1 is μ_1, while the law of ω_2 is μ_2), in a way that $\omega_2 \leq \omega_1$ almost surely. See [38, Theorem (2.1)] for the complete proof.

Theorem 2.1 possesses an elegant simplification: (7) does not need to be checked for every configurations ω_1, ω_2. Define ω^e (resp. ω_e) to be the configurations coinciding with ω on $E \smallsetminus \{e\}$, and with e open (resp. e closed). Define ω_f^e (resp. ω_e^f, ω^{ef} and ω_{ef}) to be the configurations coinciding with ω on $E \smallsetminus \{e,f\}$ and with e open and f closed (resp. e closed and f open, e,f open and e,f closed). Then, it is sufficient to check that for any ω and e,f,

$$\mu_1(\omega^e)\mu_2(\omega_e) \;\geq\; \mu_1(\omega_e)\mu_2(\omega^e) \tag{8}$$

$$\text{and} \quad \mu_1(\omega^{ef})\mu_2(\omega_{ef}) \;\geq\; \mu_1(\omega_e^f)\mu_2(\omega_f^e). \tag{9}$$

The Holley criterion is particularly suitable to prove the Fortuin-Kasteleyn-Ginibre inequality [32] (FKG inequality for short). First proved by Harris in the case of product measures (in this case, it is called Harris inequality), the inequality relates the probability of the intersection of two events to the product of the probabilities.

Theorem 2.2 (FKG lattice condition) *Let $G = (V, E)$ be a finite graph and μ be a positive measure on Ω. If for any configuration ω and $e, f \in E$*

$$\mu(\omega^{ef})\mu(\omega_{ef}) \;\geq\; \mu(\omega_e^f)\mu(\omega_f^e), \tag{10}$$

then for any increasing events A, B,

$$\mu(A \cap B) \;\geq\; \mu(A)\mu(B). \tag{11}$$

Proof: Equation (11) can be understood as $\mu(\cdot|B)$ stochastically dominating μ. Let us check Holley inequalities (8) and (9). We do it only for (9) ((8) is even easier). Fix ω as well as e and f. We obtain

$$\mathbf{1}_{\omega^{ef}\in B}\; \mu(\omega^{ef})\mu(\omega_{ef}) \;\geq\; \mathbf{1}_{\omega_e^f\in B}\; \mu(\omega_e^f)\mu(\omega_f^e).$$

by noticing that the indicator function on the left is equal to 1 if the one on the right is equal to 1 (we use that B is increasing). Dividing by $\mu(B)$, we get

$$\mu(\omega^{ef}|B)\mu(\omega_{ef}) \;\geq\; \mu(\omega_e^f|B)\mu(\omega_f^e).$$

□

By taking complements, the inequality $\mu(A \cap B) \leq \mu(A)\mu(B)$ holds for decreasing events. Similarly, if A is increasing and B is decreasing, then $\mu(A \cap B) \leq \mu(A)\mu(B)$. The theorem above also implies that $\mu(XY) \geq \mu(X)\mu(Y)$ for any two increasing (resp. decreasing) random variables X, Y.

Corollary 2.3 (FKG inequality) *Let $p \in [0,1]$ and $q \geq 1$, and consider boundary conditions ξ. For any two increasing events A and B,*

$$\phi^\xi_{p,q,G}(A \cap B) \;\geq\; \phi^\xi_{p,q,G}(A)\phi^\xi_{p,q,G}(B). \tag{12}$$

Beware of the fact that q is required to be larger or equal to 1: in fact, the result is false when $q < 1$. For instance, uniform spanning trees can be obtained as limits of FK percolation with q going to 0: since they are negatively correlated, it is in fact natural to conjecture that negative association would hold whenever $q < 1$.

Proof: Let us check (10). Fix a configuration ω and two edges e, f. We need to prove

$$p^{o(\omega^{ef})+o(\omega_{ef})}(1-p)^{o(\omega^{ef})+o(\omega_{ef})}q^{k(\omega^{ef})+k(\omega_{ef})}$$
$$\geq p^{o(\omega_e^f)+o(\omega_f^e)}(1-p)^{o(\omega_e^f)+o(\omega_f^e)}q^{k(\omega_e^f)+k(\omega_f^e)}.$$

The terms involving p and $(1-p)$ do not create any difficulty since $o(\omega^{ef}) + o(\omega_{ef}) = o(\omega_e^f) + o(\omega_f^e)$ and $c(\omega^{ef}) + c(\omega_{ef}) = c(\omega_e^f) + c(\omega_f^e)$. Recalling that $q \geq 1$, we only need to check that $k(\omega^{ef}) + k(\omega_{ef}) \geq k(\omega_e^f) + k(\omega_f^e)$. This inequality follows by studying whether both end-points of f are already connected or not in $\omega_{|G\setminus\{e,f\}}$. □

Corollary 2.4 (Comparison in p) *Fix boundary conditions ξ and $q \geq 1$. For any $p_1 \leq p_2$ and any increasing event A,*

$$\phi^\xi_{p_1,q,G}(A) \;\leq\; \phi^\xi_{p_2,q,G}(A). \tag{13}$$

This corollary affirms that $\phi^\xi_{p_1,q,G}$ is stochastically dominated by $\phi^\xi_{p_2,q,G}$. It legitimates the intuition that the larger p is, the more edges are open.

Proof: For a random variable X, an easy computation implies

$$\phi^{\xi}_{p_2,q,G}(X) = \phi^{\xi}_{p_1,q,G}(XY)/K$$

where K is a normalizing constant and

$$Y(\omega) \;=\; \left(\frac{p_2/(1-p_2)}{p_1/(1-p_1)}\right)^{o(\omega)}.$$

Plugging $X = 1$, we find $K = \phi^{\xi}_{p_1,q,G}(Y)$. Now, X and Y being increasing (recall that $p_1 \le p_2$), the FKG inequality implies

$$\phi^{\xi}_{p_2,q,G}(X) \;=\; \phi^{\xi}_{p_1,q,G}(XY)/\phi^{\xi}_{p_1,q,G}(Y) \;\ge\; \phi^{\xi}_{p_1,q,G}(X).$$

□

Corollary 2.5 (Comparison between boundary conditions) *Fix $p \in [0,1]$ and $q \ge 1$. For any boundary conditions $\xi \le \psi$ (meaning that sites wired in ξ are also wired in ψ) and any increasing event A,*

$$\phi^{\xi}_{p,q,G}(A) \;\le\; \phi^{\psi}_{p,q,G}(A). \tag{14}$$

For stochastic ordering, the free and the wired boundary conditions are thus extremal: for any increasing event A and any boundary conditions ξ,

$$\phi^{0}_{p,q,G}(A) \;\le\; \phi^{\xi}_{p,q,G}(A) \;\le\; \phi^{1}_{p,q,G}(A). \tag{15}$$

In order to prove this inequality, an additional property is required. This property will be crucial in the next sections. Consider a subset F of the edges of G. The following proposition presents how the influence of the configuration outside F on the measure within F can be encoded using appropriate boundary conditions ξ. This property is the analog of the Dobrushin-Lanford-Ruelle conditions for Gibbs measures [35].

Proposition 2.6 (Domain Markov Property) *Let $p \in [0,1]$, $q > 0$ and ξ boundary conditions. Fix $F \subset E_G$. Let X be a random variable measurable in terms of edges in F (call $\mathscr{F}_{E_G\setminus F}$ the σ-algebra generated by edges of $E_G \setminus F$). Then,*

$$\phi^{\xi}_{p,q,G}(X|\mathscr{F}_{E_G\setminus F})(\psi) = \phi^{\xi\cup\psi}_{p,q,F}(X),$$

where ψ is a configuration outside F and $\xi \cup \psi$ is the wiring inherited from ξ and the edges in ψ.

Proof: Let us deal with the case $F = E_G \setminus \{e\}$. Let ω be a configuration on F. For any configuration ω,

$$\begin{aligned}
\phi^{\xi}_{p,q,G}(\omega|\mathscr{F}_{\{e\}})(\omega(e)=1) &:= \phi^{\xi}_{p,q,G}(\omega|\omega(e)=1)\\
&= \frac{\phi^{\xi}_{p,q,G}(\omega^e)}{\phi^{\xi}_{p,q,G}(\omega(e)=1)}\\
&= \frac{\frac{p^{o(\omega)+1}(1-p)^{c(\omega)}q^{k(\omega^e,\xi)}}{\sum_{\tilde\omega} p^{o(\tilde\omega)}(1-p)^{c(\tilde\omega)}q^{k(\tilde\omega,\xi)}}}{\frac{\sum_{\tilde\omega(e)=1} p^{o(\tilde\omega)}(1-p)^{c(\tilde\omega)}q^{k(\tilde\omega,\xi)}}{\sum_{\tilde\omega} p^{o(\tilde\omega)}(1-p)^{c(\tilde\omega)}q^{k(\tilde\omega,\xi)}}}\\
&= \frac{p^{o(\omega)+1}(1-p)^{c(\omega)}q^{k(\omega,\psi)}}{\sum_{\tilde\omega_{|F}} p^{o(\tilde\omega_{|F})+1}(1-p)^{c(\tilde\omega_{|F})}q^{k(\tilde\omega_{|F},\psi)}}\\
&= \phi^{\psi}_{p,q,G}(\omega)
\end{aligned}$$

where ψ is given by the boundary conditions ξ with the two end-points of e wired together. In the third line, the sum on $\tilde{\omega}$ ranges over all configurations on E_G. Similarly

$$\phi^{\xi}_{p,q,G}(\omega|\mathscr{F}_{\{e\}})(\omega(e)=0)=\phi^{\xi}_{G\setminus e,p,q}(\omega)$$

and the claim follows easily for $F=E_G\setminus\{e\}$. The result can be deduced for every random variable X by linearity. Now, one can repeat the previous reasoning recursively and the result follows for any arbitrary subset of edges F. $\square$

Proof of Corollary 2.5: Consider ξ as being the partition $(V_1,\dots,V_k)$ of boundary vertices and construct a new graph by adding edges between vertices of V_i for every i. Call this new graph G_0 and E_0 the set of additional edges. Now, the domain Markov property implies

$$\begin{aligned}\phi^{\xi}_{p,q,G}(\cdot) &= \phi^{\xi}_{p,q,G_0}(\cdot|\text{ all the edges of } E_0 \text{ are closed})\\ \phi^{\psi}_{p,q,G}(\cdot) &= \phi^{\xi}_{p,q,G_0}(\cdot|\text{ all the edges of } E_0 \text{ are open}).\end{aligned}$$

Using the FKG inequality twice, we obtain

$$\phi^{\xi}_{p,q,G}(A)\ \le\ \phi^{\xi}_{p,q,G_0}(A)\ \le\ \phi^{\psi}_{p,q,G}(A)$$

for any increasing event A depending on edges in G. $\square$

2.2 FK measures on $\mathbb{Z}^2$ and the phase transition

The definition of an FK measure on $\mathbb{Z}^2$ is not direct. Indeed, one cannot invoke the number of open or closed edges on $\mathbb{Z}^2$ since they would typically be infinite. We thus define FK measures on $\mathbb{Z}^2$ indirectly by taking (weak) limits of FK measures on finite graphs. The set of configurations is equipped with the product σ-field.

Theorem 2.7 *There exist two infinite measures $\phi^0_{p,q,\mathbb{Z}^2}$ and $\phi^1_{p,q,\mathbb{Z}^2}$, called the infinite-volume FK percolation measures with free and wired boundary conditions respectively, such that for any event A depending on a finite number of edges,*

$$\begin{aligned}&\phi^1_{p,q,[-n,n]^2}(A)\longrightarrow\phi^1_{p,q,\mathbb{Z}^2}(A)\ \text{ as } n\to\infty \text{ and}\\ &\phi^0_{p,q,[-n,n]^2}(A)\longrightarrow\phi^0_{p,q,\mathbb{Z}^2}(A)\ \text{ as } n\to\infty.\end{aligned}$$

Proof: We treat the case of the wired boundary conditions, the free boundary conditions being handled similarly. Let A be an increasing event depending on a finite number of edges, say on $[-N,N]^2$. For $n\ge N$, we get

$$\begin{aligned}\phi^1_{p,q,[-(n+1),n+1]^2}(A) &= \phi^1_{p,q,[-(n+1),n+1]^2}[\phi^1_{p,q,[-(n+1),n+1]^2}(A|\mathscr{F}_{E_{[-(n+1),n+1]^2}\setminus E_{[-n,n]^2}})]\\ &\le \phi^1_{p,q,[-(n+1),n+1]^2}[\phi^1_{p,q,[-n,n]^2}(A)]=\phi^1_{p,q,[-n,n]^2}(A).\end{aligned}$$

In the second line, we used the Domain Markov Property and the comparison between boundary conditions (since the wired boundary conditions dominate arbitrary boundary conditions). We deduce that $\phi^1_{p,q,[-n,n]^2}(A)$ is decreasing for $n\ge N$ and therefore converges to a limit denoted $\phi^1_{p,q,\mathbb{Z}^2}(A)$. One can check compatibility relations between limits for different events. Since $\mathscr{F}$ is generated by events depending on a finite number of edges, these limits define a measure denoted $\phi^1_{p,q,\mathbb{Z}^2}$. □

There are a priori many possible constructions of measures on $\mathbb{Z}^2$ and we described only two of them. Some measures could also not be limits of measures in finite volume. Nevertheless, let us mention without justification (see [38, Theorem (4.19)]) that FK measures on $\mathbb{Z}^2$ are all stochastically dominated by $\phi^1_{p,q,\mathbb{Z}^2}$ and all dominate $\phi^0_{p,q,\mathbb{Z}^2}$.

We do not aim for a description, or even a formal definition of FK measures on $\mathbb{Z}^2$ and we refer to [38, Chapter 4] for a complete discussion. Let us mention that the description of these measures is very difficult in general, but the following powerful theorem (we omit the proof here) shows that the set of p for which several FK measures exist, or equivalently, for which $\phi^0_{p,q,\mathbb{Z}^2} \neq \phi^1_{p,q,\mathbb{Z}^2}$, is fairly small.

Theorem 2.8 (Theorem (4.60) in [38]) *For $q \geq 1$, the set $\mathscr{D}_q$ of edge-weights p for which uniqueness fails is at most countable.*

We are now in a position to discuss the phase transition of the FK percolation.

Theorem 2.9 *There exists a* critical point $p_c \in [0,1]$ *such that:*

- *For $p < p_c$, any FK measure on $\mathbb{Z}^2$ a.s. has no infinite cluster;*
- *For $p > p_c$, any FK measure on $\mathbb{Z}^2$ a.s. has an infinite cluster.*

Proof: Fix $q \geq 1$. For any $N > 0$,

$$\phi^1_{p,q,\mathbb{Z}^2}(0 \leftrightarrow \partial[-N,N]^2) = \lim_{n\to\infty} \phi^1_{p,q,[-n,n]^2}(0 \leftrightarrow \partial[-N,N]^2)$$

and the quantity on the right is increasing in p (thanks to Corollary 2.4), we deduce that $\phi^1_{p,q,\mathbb{Z}^2}(0 \leftrightarrow \partial[-N,N]^2)$ and therefore $\phi^1_{p,q,\mathbb{Z}^2}(0 \leftrightarrow \infty)$ are increasing. We can define

$$p_c^1 = \inf\{p \in [0,1] : \phi^1_{p,q,\mathbb{Z}^2}(0 \leftrightarrow \infty) > 0\}$$

and

$$p_c^0 = \inf\{p \in [0,1] : \phi^0_{p,q,\mathbb{Z}^2}(0 \leftrightarrow \infty) > 0\}.$$

Theorem 2.8 implies that these two quantities must be equal. Let p_c be the common value $p_c^0 = p_c^1$.

Now, the existence of an infinite cluster is a translationally invariant event. The measure $\phi^0_{p,q,\mathbb{Z}^2}$ can be proved to be ergodic (the proof is easy but technical and we therefore refer to [38, Corollary (4.23)]), which forces the existence of an infinite cluster almost surely.

Since any FK measure ϕ on $\mathbb{Z}^2$ with parameters p and q is stochastically dominated by $\phi^1_{p,q,\mathbb{Z}^2}$ and dominates $\phi^0_{p,q,\mathbb{Z}^2}$, we find

$$\phi(0 \leftrightarrow \infty) \leq \phi^1_{p,q,\mathbb{Z}^2}(0 \leftrightarrow \infty) = 0,$$

which immediately implies that there is no infinite cluster almost surely. Similarly, when $p > p_c$,

$$\phi(\exists \text{ infinite cluster}) \geq \phi^0_{p,q,\mathbb{Z}^2}(\exists \text{ infinite cluster}) = 1,$$

which concludes the proof. □

The fact that p_c lies strictly between 0 and 1 is not completely obvious, but can be proved by a counting argument similar to Peierls's proof. We do not present this argument since we will directly compute the critical value in the next section, but it is an interesting exercise that we recommend to the reader.

At this point, the previous theorem implies the existence of a critical $\beta_c(q)$ for any integer $q \geq 2$. This provides an alternative proof of the existence of a phase transition for the Ising model, and a first justification of the existence of a phase transition for Potts models.

3 A first computation of p_c based on duality

Fix $q \geq 1$. We know from the last section that the critical parameter $p_c = p_c(q)$ exists. We now aim to exploit the duality of FK percolation to compute the critical point. Recall that

$$p_{sd} = p_{sd}(q) = \sqrt{q}/(1 + \sqrt{q}).$$

3.1 The inequality $p_c \geq p_{sd}$

The proof of this inequality is fairly easy and uses a classical construction known as *Zhang's argument*. This argument was first used in the case of percolation, but extends easily to the FK percolation, and works as follows. If one assumes that $p_c < p_{sd}$, the configuration at p_{sd} must contain one infinite open cluster and one infinite dual open cluster (since the dual FK percolation is then supercritical as well). Intuition indicates that such coexistence would imply that there is more than one infinite open cluster, which would be in contradiction with the following important fact.

Lemma 3.1 (Uniqueness of the infinite cluster) *For any $p \in [0,1]$, the number of infinite cluster is equal to 0 or 1 almost surely. Moreover, for any extremal Gibbs measure, it is either equal to* 0 *almost surely or equal to* 1 *almost surely.*

This property is not specific to $\mathbb{Z}^2$. On any amenable infinite transitive graph, the infinite cluster is unique when it exists. This uniqueness can fail when considering FK percolations on more general graphs such as non-amenable Cayley graphs. We refer to the argument of Burton and Keane [14] for the case of percolation and to [38, Theorem (6.17)] for full detail on the FK percolation. We still give a sketch of the argument.

Proof: We prove the (slightly stronger) second version of the statement. Let N be the number of infinite components in the configuration (it is a random variable). First, ergodicity of the extremal measures under translation shows that for each of them, N takes an almost sure value $n \in \mathbb{Z}_+ \cup \{\infty\}$. We need to show that in fact $n \in \{0,1\}$.

Assume first that $n \in \mathbb{N} \setminus \setminus \{0,1\}$. There exists L such that, with probability at least $1/2$, the box of size L at the origin meets all n infinite components. If that is the case, opening all the bonds in that box would create a configuration with exactly one infinite component. The domain Markov property shows that given the configuration, the conditional probability that all the bonds inside the box are indeed open is strictly positive. This implies that $P[N = 1] > 0$, which is a contradiction.

Now assume that $n = \infty$. Again, there exists L such that with probability $1/2$, the box of size L at the origin meets at least 3 infinite components. Again, one can modify the states of the (finitely many) bonds within that box to create a configuration where the origin is a *trifurcation*, in the sense that it is on an infinite connected component but that removing it from its component separates it into three disjoint infinite components. Hence, the probability η that the origin is a trifurcation is strictly positive.

Consider a box of size $M > 0$, and look at the number X_M of trifurcations in that box. The expectation of X_M is equal to η times the number of vertices in the box, i.e. of order ηM^2. On the other hand, it is easy to check by induction on X_M that the number of infinite components in the complement of the box (vertices in Λ_m do not count) which touch the boundary of the box is at least equal to X_M+2 whenever $X_M > 0$ (indeed, adding a trifurcation on an existing cluster forces the existence of one more infinite component outside the box). This gives a deterministic upper bound on X_M which is the number of vertices on the boundary of the box, of order M. This leads again to a contradiction when taking M to be large enough.

The only remaining cases are $n = 0$ and $n = 1$, which ends the proof. □

We are now in a position to present Zhang's argument.

Proposition 3.2 (Lower bound $p_c \geq p_{sd}$) *For $q \geq 1$, there exists almost surely no infinite cluster for $\phi^0_{p_{sd},q,\mathbb{Z}^2}$. As a consequence, $p_c \geq p_{sd}$.*

Proof: Let $\varepsilon \ll 1$. Assume that $\phi^0_{p_{sd},q,\mathbb{Z}^2}(0 \leftrightarrow \infty) > 0$ and choose N large enough that

$$\phi^0_{p_{sd},q,\mathbb{Z}^2}([-N,N]^2 \leftrightarrow \infty) > 1 - \varepsilon.$$

The integer N exists since the infinite cluster exists almost surely, which implies that the quantity on the left tends to 1 as N tends to infinity.

Fix $n \geq N$. Let A_{left} (resp. A_{right}, A_{top} and A_{bottom} be the events that $\{-n\} \times [-n,n]$ (resp. $\{n\} \times [-n,n]$, $[-n,n] \times \{n\}$ and $[-n,n] \times \{-n\}$) are connected to infinity in $\mathbb{Z}^2 \setminus [-n,n]^2$. By symmetry,

$$\phi^0_{p_{sd},q,\mathbb{Z}^2}(A_{\text{left}} \cup A_{\text{right}}) = \phi^0_{p_{sd},q,\mathbb{Z}^2}(A_{\text{top}} \cup A_{\text{bottom}}).$$

We also find that

$$\phi^0_{p_{sd},q,\mathbb{Z}^2}(A_{\text{left}} \cup A_{\text{right}} \cup A_{\text{top}} \cup A_{\text{bottom}}) = \phi^0_{p_{sd},q,\mathbb{Z}^2}([-n,n]^2 \leftrightarrow \infty) > 1 - \varepsilon.$$

A naive bound would say that $\phi^0_{p_{sd},q,\mathbb{Z}^2}(A_{\text{left}} \cup A_{\text{right}}) \geq \frac{1-\varepsilon}{2}$. Nevertheless, we are dealing with increasing events and we can invoke the FKG inequality to improve this naive lower bound. Indeed, a very simple computation based on the FKG inequality shows that

$$\phi^\xi_{p,q,G}(A \cup B) \geq 1 - (1 - \phi^\xi_{p,q,G}(A))(1 - \phi^\xi_{p,q,G}(B))$$

for any two increasing events A and B. If A and B have same probability, we find

$$\phi^\xi_{p,q,G}(A) \geq 1 - (1 - \phi^\xi_{p,q,G}(A \cup B))^{1/2}.$$

This argument is called the *square-root trick*. Applying this trick twice, we deduce that

$$\phi^0_{p_{sd},q,\mathbb{Z}^2}(A_{\text{left}}) \geq 1 - \varepsilon^{1/4}.$$

As a consequence,

$$\phi^0_{p_{sd},q,\mathbb{Z}^2}(A_{\text{left}} \cap A_{\text{right}}) \geq 1 - 2\varepsilon^{1/4}.$$

Since $\phi^0_{p_{sd},q,\mathbb{Z}^2}$ is stochastically dominated by $\phi^1_{p_{sd},q,\mathbb{Z}^2}$, we also obtain

$$\phi^1_{p_{sd},q,\mathbb{Z}^2}(A_{\text{top}} \cap A_{\text{bottom}}) \geq 1 - 2\varepsilon^{1/4}.$$

We now use that $p^*_{sd} = p_{sd}$. Note that the dual measure of $\phi^0_{p_{sd},q,\mathbb{Z}^2}$ is $\phi^1_{p_{sd},q,(\mathbb{Z}^2)^*}$. Indeed, one can easily see that the duality relation extends to infinite volume by looking at measures on $[-n,n]^2$ and letting n go to infinity. For $n \geq N+1$,

$$\phi^0_{p_{sd},q,\mathbb{Z}^2}(A_{\text{left}} \cap A_{\text{right}} \cap A^*_{\text{top}} \cap A^*_{\text{bottom}}) > 1 - 4\varepsilon^{1/4},$$

where A^*_{top} and A^*_{bottom} are the events that $(\frac{1}{2},\frac{1}{2}) + [1-n,n-1] \times \{n-1\}$ and $(\frac{1}{2},\frac{1}{2}) + [1-n,n-1] \times \{1-n\}$ are connected to infinity in the dual configuration restricted to the outside of $(\frac{1}{2},\frac{1}{2}) + [1-n,n-1]^2$.

Now, Let B be the event that every dual edge in $(\frac{1}{2},\frac{1}{2}) + [1-n,n-1]^2$ is dual open (the corresponding edge is closed in the FK model on $\mathbb{Z}^2$). Note that the events B and $A_{\text{left}} \cap A_{\text{right}} \cap A^*_{\text{top}} \cap A^*_{\text{bottom}}$ depend on disjoint sets. The domain Markov property shows that

$$\phi^0_{p_{sd},q,\mathbb{Z}^2}(B | A_{\text{left}} \cap A_{\text{right}} \cap A^*_{\text{top}} \cap A^*_{\text{bottom}}) > 0 \tag{16}$$

which immediately implies that

$$\phi^0_{p_{sd},q,\mathbb{Z}^2}(B \cap A_{\text{left}} \cap A_{\text{right}} \cap A^*_{\text{top}} \cap A^*_{\text{bottom}}) > 0.$$

But this last event is contained in the event that there are two disjoint infinite clusters, which we excluded, thus leading to a contradiction. □

Let us mention that the fact that the Domain Markov property implies (16) can be generalized. In fact, the Domain Markov property immediately implies that the probability that an edge e is open conditionally to a configuration on $E_G \setminus \{e\}$ is bounded away from 0 and 1 uniformly on the configuration. This property is called *finite energy property*, and is a very important feature of FK percolation.

It is a good time to mention that on $\mathbb{Z}^2$, Theorem 2.8 can be improved. Indeed, it can be shown that the FK measure on $\mathbb{Z}^2$ is unique for $p < p_c$. By duality, this implies that the FK measure is unique for $p > p^*_c$ as well. Since $p_c \geq p_{sd}$, this implies that the unique point for which uniqueness can possibly fail is the self-dual point. The previous argument uses duality in a strong way.

3.2 Interlude

We now aim to prove the much harder bound $p_c \leq p_{sd}$. To get this inequality, one needs to prove that $\phi^1_{p,q,\mathbb{Z}^2}(0 \leftrightarrow \infty) > 0$ for any $p > p_{sd}$. This is done by estimating the probability that there exists an open path crossing a rectangle, and then by combining paths in different rectangles to create an infinite path from 0 to infinity. More precisely, the proof is based on three main ingredients.

The first one is an estimate on crossing probabilities at the self-dual point $p = p_{sd}$: the probability of crossing a rectangle with aspect ratio $(\alpha, 1)$ — meaning that the ratio between the width and the height is of order α — in the horizontal direction is bounded away from 0 and 1 uniformly in the size of the box. For a rectangle R, let $\mathscr{C}_h(R)$ denote the event that there exists a path between the left and the right sides which stays inside the rectangle. Such a path is called a *horizontal (open) crossing* of the rectangle. If there exists a vertical open crossing of the rectangle, this rectangle is said to be *crossed horizontally*. Similarly, we denote $\mathscr{C}_v(R)$ for the existence of a vertical crossing, i.e. an open path from top to bottom. The targeted result would be something of the following kind:

$$\phi^1_{p_{sd},q,\mathbb{Z}^2}(\mathscr{C}_h([0,2n]\times[0,n])\geq c>0$$

uniformly in n.

The second ingredient is a sharp threshold theorem which was originally introduced for product measures. In our case, it may be used to estimate the derivative of the probability of crossings. In particular, the probability of crossings goes to 1 when $p>p_{sd}$ and the speed of convergence can be bounded from below. Roughly speaking, we aim for a claim of the following kind: for any $p>p_{sd}$, there exists $c>0$ such that

$$\phi^1_{p,q,\mathbb{Z}^2}(\mathscr{C}_h([0,2n]\times[0,n])\geq 1-n^{-c}$$

for any $n\geq 0$.

The last ingredient is a construction combining different crossings together to construct an infinite cluster. More precisely, define $R_n=[0,2^n]\times[0,2^{n+1}]$ if n is odd and $R_n=[0,2^{n+1}]\times[0,2^n]$ if n is even. Let $\mathscr{C}(R_n)$ be the event that R_n is crossed in the long direction (meaning vertically if n is odd and horizontally if n is even). Let B be the event that edges in $[0,2]^2$ are open. The FKG inequality then implies

$$\begin{aligned}\phi^1_{p,q,\mathbb{Z}^2}(0\leftrightarrow\infty)&\geq\phi^1_{p,q,\mathbb{Z}^2}\big(B\cap(\cap_{n\geq1}\mathscr{C}(R_n))\big)\\&\geq\phi^1_{p,q,\mathbb{Z}^2}(B)\prod_{n\geq1}\phi^1_{p,q,\mathbb{Z}^2}(\mathscr{C}(R_n))\\&\geq\phi^1_{p,q,\mathbb{Z}^2}(B)\prod_{n\geq1}(1-2^{-cn})>0.\end{aligned}$$

There are many details to account for in the approach sketched above. The main difficulty is that the second step requires (see Section 3.4) to work with a torus. Let us describe this in more details. For $m\geq1$, the torus of size m (meaning of volume m^2) can be seen as the box $[0,m]^2$ with the boundary condition obtained by imposing that $(i,0)$ is wired to (i,m) for every $i\in[0,m]$ and that $(0,j)$ is wired to (m,j) for every $j\in[0,m]$. The FK percolation measure on the torus of size m will be denoted by $\phi^{\mathrm{p}}_{p,q,[0,m]^2}$ or more concisely $\phi^{\mathrm{p}}_{p,q,m}$. Note that this realization of the torus provides us with a natural embedding in the plane.

The precise theorem corresponding to the first step is the following generalization of the celebrated Russo-Seymour-Welsh theorem [65, 69] for percolation.

Theorem 3.3 *Let $\alpha>1$ and $q\geq1$. There exists $c(\alpha)>0$ such that for every $m>\alpha n>0$,*

$$\phi^{\mathrm{p}}_{p_{sd},q,m}\big(\mathscr{C}_h([0,\alpha n]\times[0,n])\big)\geq c(\alpha).\tag{17}$$

With this theorem at our disposition, the precise target of the second step will be the following result asserting that crossing probabilities converge to 1 fairly fast when $p>p_{sd}$.

Theorem 3.4 *Let $q \geq 1$ and $p > p_{sd}$. There exists $c'(p) > 0$ such that for every $n > 0$,*

$$\phi^{\mathrm{p}}_{p,q,8n}\big(\mathscr{C}_h([0,4n] \times [0,n])\big) \geq 1 - n^{-c'(p)}. \tag{18}$$

The last step then consists in using Theorem 3.4 to construct an infinite path with positive probability. As one can see, the boundary conditions raise new difficulties and the construction presented above does not quite work.

The rest of this section is devoted to the presentation of these three steps. They are ordered by difficulty, starting by the third one and finishing by the first step.

3.3 A new construction for the infinite path

As explained in the previous subsection, Theorem 3.3 can be used to construct a path from 0 to infinity when $p > p_{sd}$ with positive probability. There is a major difficulty in doing such a construction: one needs to transform estimates in the torus into estimates in the whole plane. One solution is to replace the periodic boundary conditions with wired boundary conditions. The path construction is a little tricky since it must propagate wired boundary conditions through the construction.

Let $n \geq 1$; define the annulus

$$A_n := [-2^{n+1}, 2^{n+1}]^2 \setminus [-2^n, 2^n]^2.$$

An *open circuit* in an annulus is an open path that surrounds the origin. Denote by $\mathscr{A}_n$ the event that there exists an open circuit in A_n surrounding the origin, together with an open path from this circuit to the boundary of $[-2^{n+2}, 2^{n+2}]^2$, see Fig. 4. Note that the intersection of the following events is included in $\mathscr{A}$:

$$\begin{array}{ll} \mathscr{C}_h([-2^{n+1}, 2^{n+1}] \times [2^n, 2^{n+1}]), & \mathscr{C}_h([-2^{n+1}, 2^{n+1}] \times [-2^{n+1}, 2^n]), \\ \mathscr{C}_v([2^n, 2^{n+1}] \times [-2^{n+1}, 2^n]), & \mathscr{C}_v([2^n, 2^{n+1}] \times [2^n, 2^{n+1}]), \\ \mathscr{C}_v([-2^n, 2^n] \times [0, 2^{n+2}]), & \end{array}$$

where $\mathscr{C}_v$ means that there exists an open path from bottom to top. As a consequence, for any graph G containing $[-2^{n+2}, 2^{n+2}]^2$,

$$\phi^{\xi}_{p,q,G}(\mathscr{A}_n) \geq \phi^{\xi}_{p,q,G}(\mathscr{C}_h([0, 2^{n+2}] \times [0, 2^n])^5. \tag{19}$$

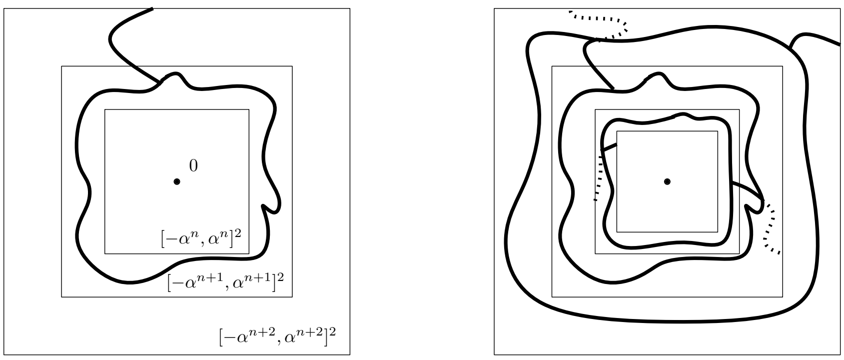


Figure 4: Left: The event $\mathscr{A}_n$. **Right:** The combination of events $\mathscr{A}_n$: it indeed constructs a path from the origin to infinity.

By observing that

$$\phi^1_{p,q,\mathbb{Z}^2}(0\leftrightarrow\infty)\ge\phi^1_{p,q,\mathbb{Z}^2}\left(\bigcap_{k\ge1}\mathscr{A}_k\right)=\lim_{n\to\infty}\phi^1_{p,q,\mathbb{Z}^2}\left(\bigcap_{k=1}^{n}\mathscr{A}_k\right)>0,$$

it becomes clear that it is sufficient to bound $\phi^1_{p,q,\mathbb{Z}^2}\left(\bigcap_{k=1}^{n}\mathscr{A}_k\right)$ from below, uniformly in n. For every $n\ge1$,

$$\phi^1_{p,q,\mathbb{Z}^2}\left(\bigcap_{k=1}^{n}\mathscr{A}_k\right)=\phi^1_{p,q,\mathbb{Z}^2}(\mathscr{A}_n)\prod_{k=1}^{n-1}\phi^1_{p,q,\mathbb{Z}^2}(\mathscr{A}_k|\mathscr{A}_j,k<j\le n).\tag{20}$$

On $\bigcap_{j>k}\mathscr{A}_j$, consider the exterior-most open circuit Γ in A_{k+1}. Conditionally on $\Gamma=\gamma$, where γ is a possible realization of Γ, the configuration within the interior of γ follows the law of a FK configuration with wired boundary condition (indeed, the Domain Markov property allows to encode the law of the configuration inside in terms of the boundary conditions, which are wired since the path γ is open). In particular, the conditional probability that there exists a circuit in A_k connected by an open path to γ is greater than the probability that there exists a circuit in A_k connected to the boundary of $[-2^{k+2},2^{k+2}]^2$ with wired boundary conditions. Therefore, we obtain that almost surely

$$\begin{aligned}\phi^1_{p,q,\mathbb{Z}^2}(\mathscr{A}_k|\mathscr{A}_j,k+1\le j\le n)&=\phi^1_{p,q,\mathbb{Z}^2}\big(\phi_{p,q,\mathbb{Z}^2}(\mathscr{A}_k|\Gamma)\big)\\&\ge\phi^1_{p,q,[-2^{k+2},2^{k+2}]^2}(\mathscr{A}_k)\\&\ge\phi^{\mathrm{p}}_{p,q,2^{k+3}}(\mathscr{A}_k)\\&\ge\phi^{\mathrm{p}}_{p,q,2^{k+3}}(\mathscr{C}_h([0,2^{k+2}]\times[0,2^k])^5\\&\ge(1-2^{-c'(p)k})^5\end{aligned}$$

In the first inequality, we used the uniform lower bound on the conditional probability. In the second, we used the comparison between boundary conditions between $[-2^{k+2},2^{k+2}]^2$ with wired boundary conditions and periodic boundary conditions on the torus of size 2^{k+3}. The third is due to (19). The last inequality follows from Theorem 3.4 (in particular $c'(p)>0$ is the universal constant given by the theorem).

Plugging the previous estimate into (20), we obtain

$$\phi^1_{p,q,\mathbb{Z}^2}\left(\bigcap_{k=1}^{n}\mathscr{A}_k\right)\ge\phi^1_{p,q,\mathbb{Z}^2}(\mathscr{A}_n)\prod_{k=1}^{n-1}(1-2^{-c'(p)k})^5\ge C\phi^1_{p,q,\mathbb{Z}^2}(\mathscr{A}_n)$$

Letting n go to infinity will conclude the proof if $\phi^1_{p,q,\mathbb{Z}^2}(\mathscr{A}_n)$ remains bounded away from 0 uniformly in n, or equivalently if $\phi^1_{p,q,\mathbb{Z}^2}(\mathscr{C}_h([0,4n]\times[0,n]))$ remains bounded away from 0 uniformly in n (thanks to (19)). Yet, for m large enough,

$$\begin{aligned}\phi^1_{p,q,[-\frac{m}{2},\frac{m}{2}]^2}\big[\mathscr{C}_h\big([0,4n]\times[0,n]\big)\big]&\ge\phi^{\mathrm{p}}_{p,q,m}\big[\mathscr{C}_h\big([0,4n]\times[0,n]\big)\big]\\&\ge\phi^{\mathrm{p}}_{p_{sd},q,m}\big[\mathscr{C}_h\big([0,4n]\times[0,n]\big)\big]\ge c(4).\end{aligned}$$

In the second line, we used Theorem 3.3. As m goes to infinity, the left hand side converges to the probability in infinite volume and

$$\phi^1_{p,q,\mathbb{Z}^2}(\mathscr{C}_h\big([0,4n]\times[0,n]\big))\ge c(4).$$

The proof is therefore finished.

3.4 A sharp threshold theorem for crossing probabilities

The aim of this section is to prove Theorem 3.4 with the help of Theorem 3.3. In order to do so, one needs to understand the behavior of the function $p \mapsto \phi^{\xi}_{p,q,n}(A)$ for the increasing event $A := \mathscr{C}_h([0,\alpha n]\times[0,n])$. This increasing function is equal to 0 at $p=0$ and to 1 at $p=1$, and we are interested in the range of p for which its value is between ε and $1-\varepsilon$ for $\varepsilon \in (0,1/2)$ (this range is usually referred to as a *window*). The study does not restrict to the event $\mathscr{C}_h([0,\alpha n]\times[0,n])$ but extends to a large class of increasing events. For this reason, we do not focus on the event $\mathscr{C}_h([0,\alpha n]\times[0,n])$ only and work with arbitrary increasing events A.

Let us start by deriving lower bounds on the derivative of increasing events. The case of the Bernoulli percolation is the most classical and we begin by discussing it. Given a configuration ω, an edge e is said to be *pivotal* for the event A if $\omega^e \in A$ and $\omega_e \notin A$.

Proposition 3.5 (Russo's formula [66]) *For any increasing event A depending on a finite set of edges E,*

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) = \sum_{e\in E}\mathbb{P}_p(e \text{ is pivotal for } A).$$

Proof: Assume that A depends on edges $e_1,\dots,e_n$ only. We reuse the increasing coupling for Bernoulli percolation. Let $U(e_1),\dots,U(e_n)$ be independent uniform variables on $[0,1]$ and let $\mathbf{P}$ be the law of the collection $(U(e_1),\dots,U(e_n))$. For any $\bar p = (p_1,\dots,p_n)$, define the inhomogeneous percolation configuration

$$\omega_{\bar p} = (\mathbf{1}_{U(e_1)\le p_1},\dots,\mathbf{1}_{U(e_n)\le p_n}).$$

Let $j\in\{1,\dots,n\}$ and let $\bar p'$ be defined by $p'_j > p_j$ and $p'_i = p_i$ for any $i\neq j$. Then

$$\begin{aligned}\mathbf{P}(\omega_{\bar p'}\in A)-\mathbf{P}(\omega_{\bar p}\in A) &= \mathbf{P}(\omega_{\bar p'}\in A \text{ and } \omega_{\bar p}\notin A)\\ &= \mathbf{P}(e_j \text{ is pivotal for } A \text{ and } \omega_{\bar p'}(e_j)=1 \text{ and } \omega_{\bar p}(e_j)=0)\\ &= \mathbf{P}(e_j \text{ is pivotal for } A \text{ and } U(e_j)\in(p_j,p'_j))\\ &= (p'_j-p_j)\mathbf{P}(e_j \text{ is pivotal for } A \text{ and } \omega_{\bar p})\end{aligned}$$

In the third line, we used the fact that e_j is the only edge that differs between $\omega_{\bar p}$ and $\omega_{\bar p'}$. In the fourth, we used the fact that the state of e_j is independent of the fact that it is pivotal. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}p_j}\mathbf{P}(\omega_{\bar p}\in A) = \mathbf{P}(e_j \text{ is pivotal for } A).$$

Summing on j for $\bar p = (p,\dots,p)$, we deduce the result. □

For general cluster weights, the previous coupling is not available but Russo's formula still extends (as an inequality) to this context, with appropriate modifications.

Proposition 3.6 *Let $q\ge 1$ and $\varepsilon > 0$. There exists $c = c(q,\varepsilon) > 0$ such that for any $p\in[\varepsilon,1-\varepsilon]$ and any increasing event A,*

$$\frac{\mathrm{d}}{\mathrm{d}p}\phi^{\xi}_{p,q,G}(A) \ge c\sum_{e\in E} I_A(e),$$

where $I_A(e) := \phi^{\xi}_{p,q,G}(A|\omega(e)=1)-\phi^{\xi}_{p,q,G}(A|\omega(e)=0)$ is the (conditional) influence *of the edge e on A.*

When $q=1$, $I_A(e) = \mathbb{P}_p(e \text{ is pivotal for } A)$ and we find Russo's formula.

Proof: By differentiating with respect to p (for details of the computation, see of [38, Theorem (2.46)]), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}p}\phi^{\xi}_{p,q,G}(A) = \frac{1}{p(1-p)}\sum_{e\in E}\left[\phi^{\xi}_{p,q,G}(\mathbf{1}_A\omega(e)) - \phi^{\xi}_{p,q,G}(\omega(e))\phi^{\xi}_{p,q,G}(A)\right]. \tag{21}$$

By definition of $I_A(e)$, the summation term on the right is equal to

$$I_A(e)\phi^{\xi}_{p,q,G}(\omega(e))(1-\phi^{\xi}_{p,q,G}(\omega(e))),$$

so that (21) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}\phi^{\xi}_{p,q,G}(A) &= \frac{1}{p(1-p)}\sum_{e\in E}\phi^{\xi}_{p,q,G}(\omega(e))\big(1-\phi^{\xi}_{p,q,G}(\omega(e))\big)I_A(e)\\ &= \sum_{e\in E}\frac{\phi^{\xi}_{p,q,G}(\omega(e))\big(1-\phi^{\xi}_{p,q,G}(\omega(e))\big)}{p(1-p)}I_A(e)\end{aligned}$$

from which the claim follows since the term

$$\frac{\phi^{\xi}_{p,q,G}(\omega(e))\big(1-\phi^{\xi}_{p,q,G}(\omega(e))\big)}{p(1-p)}$$

is bounded away from 0 uniformly in $p\in[\varepsilon,1-\varepsilon]$ and $e\in E$ when q is fixed. □

There has been an extensive study of the largest influence for an arbitrary event A. It was initiated for the uniform measure (equivalently the Bernoulli percolation with parameter 1/2) in [47] (see also [33, 34, 48] for related theorems) and was later extended to product measures in [12] and to FK percolation in [36].

Theorem 3.7 ([36]) *Let $q\geq 1$ and $\varepsilon>0$. There exists a constant $c=c(q,\varepsilon)\in(0,\infty)$ such that the following holds. For every $p\in[\varepsilon,1-\varepsilon]$ and every increasing event A,*

$$\max_{e\in E_G} I_A(e) \;\geq\; c\,\phi^{\xi}_{p,q,G}(A)\big(1-\phi^{\xi}_{p,q,G}(A)\big)\frac{\log|E_G|}{|E_G|}.$$

In general, the maximum influence provides little information on the sum of influences over all edges, which is the quantity we are ultimately interested in. For instance, one may think of the event that a prescribed edge e_0 is open, for which the influence of the edge e_0 equals 1, while the one of edges far away is very small (observe that except for Bernoulli percolation, this influence is still non zero for edges different from e_0). For this example, the graph of $p\mapsto\phi^{\xi}_{p,q,G}(e_0)$ does not have a small window between ε and $1-\varepsilon$.

In the case of a translation-invariant events, this problem does not arise since horizontal (resp. vertical) edges play a symmetric role, so that the influence is the same for all the edges of a given orientation. With this observation in mind, Proposition 3.6 together with Theorem 3.7 provide us with the following differential inequality.

Theorem 3.8 *Let $q\geq 1$ and $\varepsilon>0$. There exists a constant $c=c(q,\varepsilon)\in(0,\infty)$ such that the following holds. For any $p\in[\varepsilon,1-\varepsilon]$ and for any translation-invariant increasing event A on the torus of size n,*

$$\frac{\mathrm{d}}{\mathrm{d}p}\phi^{\mathrm{p}}_{p,q,n}(A) \;\geq\; c\,\phi^{\mathrm{p}}_{p,q,n}(A)\big(1-\phi^{\mathrm{p}}_{p,q,n}(A)\big)\log n.$$

For a translational invariant and increasing event A, the previous inequality can be integrated between two parameters $p_1 < p_2$. Recognizing the derivative of $\log(\frac{x}{1-x})$, one obtains

$$\frac{1-\phi^{\mathrm{p}}_{p_1,q,n}(A)}{\phi^{\mathrm{p}}_{p_1,q,n}(A)} \geq \frac{1-\phi^{\mathrm{p}}_{p_2,q,n}(A)}{\phi^{\mathrm{p}}_{p_2,q,n}(A)} n^{c(p_2-p_1)}.$$

If $\phi^{\xi}_{p_1,q,n}(A)$ is assumed to stay bounded away from 0 uniformly in $n \geq 1$, there exists $c' > 0$ such that

$$\phi^{\mathrm{p}}_{p_2,q,n}(A) \geq 1 - c' n^{-c(p_2-p_1)}.$$

Our goal is to apply this result to the event $\mathscr{C}_h([0,4n]\times[0,n])$ which unfortunately is not invariant under translations. In [10, 11], Bollobás and Riordan applied a brilliant strategy to relate the probability of the previous event to the probability of a certain translational invariant event. We follow the same idea here.

Proof of Theorem 3.4: Let B be the event that there exists a vertical crossing of a rectangle with dimensions $(n/2, 8n)$ in the torus of size $8n$. This event is invariant under translations and satisfies

$$\phi^{\mathrm{p}}_{p_{sd},q,8n}(B) \geq \phi^{\mathrm{p}}_{p_{sd},q,8n}\big(\mathscr{C}_h([0,8n]\times[0,n/2])\big) \geq c(16)$$

uniformly in n.

Let $p > p_{sd}$. Since B is increasing and invariant under translation, Theorem 3.8 and the discussion following it imply the existence of $\varepsilon = \varepsilon(p,q)$ and $c = c(p,q)$ such that

$$\phi^{\mathrm{p}}_{p,q,8n}(B) \geq 1 - cn^{-\varepsilon}. \tag{22}$$

If B holds, one of the 32 rectangles

$$[j4n,(j+1)4n]\times[in/2, in/2+n], \quad (i,j)\in\{0,1\}\times\{0,\dots,15\}$$

must be crossed from top to bottom. Denote these events by A_{ij} (the way we index rectangles here is obvious). These events are translates of $\mathscr{C}_h([0,4n]\times[0,n])$. Using the FKG inequality in the second line (this is another instance of the square-root trick mentioned earlier), we find

$$\begin{aligned}\phi^{\mathrm{p}}_{p,q,8n}(B) &= 1-\phi^{\mathrm{p}}_{p,q,8n}(B^c) \leq 1-\phi^{\mathrm{p}}_{p,q,8n}(\cap_{i,j}A^c_{ij})\\ &\leq 1-\prod_{i,j}\phi^{\mathrm{p}}_{p,q,8n}(A^c_{ij}) = 1-\Big[1-\phi^{\mathrm{p}}_{p,q,8n}\big(\mathscr{C}_h([0,4n]\times[0,n])\big)\Big]^{32}.\end{aligned}$$

Plugging (22) into the previous inequality, we deduce

$$\phi^{\mathrm{p}}_{p,q,8n}\left(\mathscr{C}_h([0,4n]\times[0,n])\right) \geq 1-(cn^{-\varepsilon})^{1/32}.$$

□

3.5 Crossing probabilities for rectangles at the self-dual point

We are now getting close to a complete proof of $p_c \le p_{sd}$. The third and second steps of our program have been implemented in the previous sections and we now focus on the first one, i.e. the proof of Theorem 3.3. The inherent difficulty of this theorem is two-fold. The first difficulty comes from boundary conditions. The result would be false with arbitrary boundary conditions, such as free boundary conditions. Indeed, for q large enough, the probability of crossing a rectangle at the self-dual point is known to decay exponentially fast for free boundary conditions. The second difficulty comes from the lack of independence. There are many ways to prove this result in the case of percolation but they always involve independence in a crucial way. In our case, no independence is available and we overcome this difficulty by using self-duality in a very strong fashion.

We work on the torus of size m. For technical reasons, it will be convenient to *rotate the lattice in this torus by* $\pi/4$ In such case, the graph $[0,\alpha n]\times[0,n]$ is then the intersection of the rotated lattice with the rectangle $[0,\alpha n]\times[0,n]$. We will prove the following result.

Proposition 3.9 *Let $q \ge 1$. There exists $c > 0$ such that for every $m > \frac{3}{2}n > 0$,*

$$\phi^{\mathrm{p}}_{p_{sd},q,m}\big(\mathscr{C}_h([0,\tfrac{3}{2}n]\times[0,n])\big) \ge c. \tag{23}$$

Let us explain how this result implies Theorem 3.3.

Proof of Theorem 3.3: We use Proposition 3.9. Let us emphasize once again that the lattice is rotated by $\pi/4$. Let $R = T([0,n]\times[0,\alpha n])$, where T is the composition of the rotation of angle $\pi/4$ and the translation of vector $(\frac{n}{2},0)$, see Figure 5. Define the following rectangles:

$$R_j^v = [j\tfrac{n}{2},(j+1)\tfrac{n}{2}]\times[j\tfrac{n}{2},(j+\tfrac{3}{2})\tfrac{n}{2}] \text{ and } R_j^h = [j\tfrac{n}{2},(j+\tfrac{3}{2})\tfrac{n}{2}]\times[(j+\tfrac{1}{2})\tfrac{n}{2},(j+\tfrac{3}{2})\tfrac{n}{2}]$$

for $j \in [0,\lfloor 2\alpha\rfloor - 1]$, where $\lfloor x\rfloor$ denotes the integral part of x. If every rectangle R_j^h is crossed horizontally, and every rectangle R_j^v is crossed vertically, then $T([0,\alpha n]\times[0,n])$ is crossed in the long direction. The FKG inequality and Proposition 3.9 provide us with a lower bound on the crossing probability. Now, the graph $T([0,\alpha n]\times[0,n])$ is isomorphic to the rectangle $[0,\alpha n]\times[0,n]$ when the lattice is not rotated, which concludes the proof. □

Let us now turn to the proof of Proposition 3.9. Proving lower bounds on the probability of crossing rectangles $[0,\frac{3}{2}n]\times[0,n]$ is much harder than bounding the probability that $[0,n]^2$ is crossed. Indeed, in the case of squares, the proof relies entirely on duality, as explained in the next lemma.

Lemma 3.10 *Let $q \ge 1$, there exists $c > 0$ such that for every $m > n \ge 1$,*

$$\phi^{\mathrm{p}}_{p_{sd},q,m}\big(\mathscr{C}_h([0,n]^2)\big) \ge c.$$

Proof: We work on the torus T_m of size m, which has m^2 vertices and $2m^2$ edges. The proof uses duality once again. The case of periodic boundary conditions is a little more involved that free and wired boundary conditions (which we recall was treated in Proposition 1.7). Indeed, its dual is *not* a FK percolation; yet it is not very different

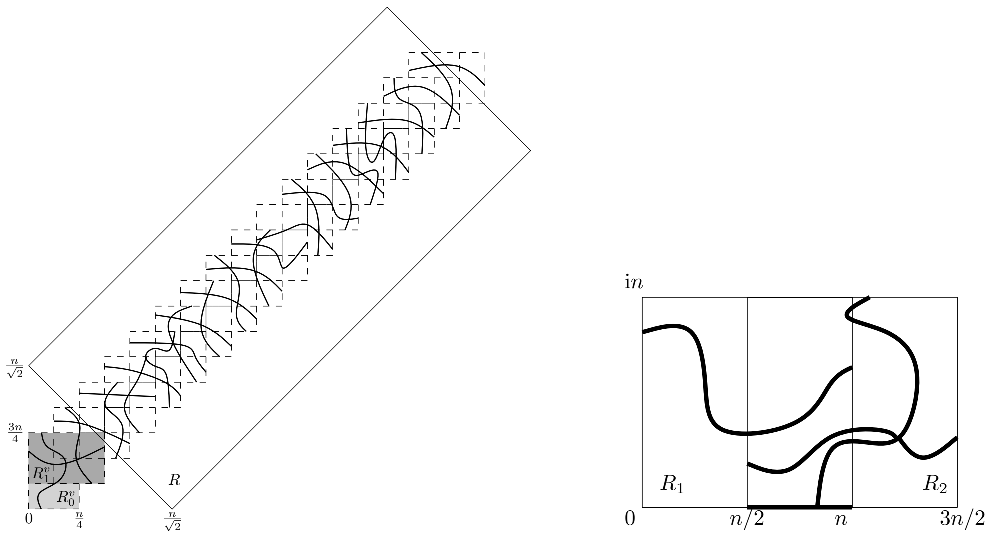


Figure 5: Left: A combination of crossings in smaller rectangles creating a crossing in the rectangle $T([0,\alpha n]\times[0,n])$. **Right:** The rectangles R, R_1 and R_2 and the event A.

from one. In order to state duality in this case, additional notation is required. Define an additional parameter $\delta(\omega)$ as follows. Call a (maximal) connected component of ω a *net* if it contains two non-contractible simple loops of different homotopy classes, and a *cycle* if it is non-contractible but is not a net. Notice that every configuration ω can be of one of three types:

- One of the clusters of ω is a net. Then no other cluster of ω can be a net or a cycle. In that case, let $\delta(\omega)=2$;
- One of the clusters of ω is a cycle. Then no other cluster can be a net, but other clusters can be cycles as well (in which case all the involved, simple loops are in the same homotopy class). Then let $\delta(\omega)=1$;
- None of the clusters of ω is a net or a cycle. Let $\delta(\omega)=0$.

With this additional notation, Euler's formula becomes

$$|V_{T_m}| - o(\omega) + f(\omega) = k(\omega) + 1 - \delta(\omega). \tag{24}$$

Besides, these terms transform in a simple way under duality: $o(\omega)+o(\omega^*)=|E_{T_m}|$, $f(\omega)=k(\omega^*)$ and $\delta(\omega)=2-\delta(\omega^*)$. The same proof as that of Proposition 1.7, taking the additional topology into account, then leads to the relation

$$(\phi^{\mathrm{p}}_{p,q,n})(\{\omega\}) = \frac{q^{1-\delta(\omega)}}{Z}\phi^{\mathrm{p}}_{p^*,q,n}(\{\omega^*\}), \tag{25}$$

where Z is a normalizing constant. This means that even though the dual model of the periodic boundary conditions FK percolation is not exactly a FK percolation at the dual parameter, it is absolutely continuous with respect to it and the Radon-Nikodym derivative is bounded above and below by constants depending only on q.

The dual of the subgraph $[0,n]^2$ is $[0,n]^2$ (meaning the sites of the dual torus inside $[0,n]^2$). If there is no crossing from left to right in $[0,n]^2$, there exists necessarily a crossing in the dual configuration from top to bottom. Hence, the complement of $\mathscr{C}_h([0,n]^2)$

is the event $\mathscr{C}_v^*([0,n]^2)$ that $[0,n]^2$ is crossed vertically in the dual configuration, thus yielding

$$\phi^{\mathrm{p}}_{p_{sd},q,m}\big(\mathscr{C}_h([0,n]^2)\big) + \phi^{\mathrm{p}}_{p_{sd},q,m}\big(\mathscr{C}_v^*([0,n]^2)\big) = 1.$$

Using the duality property for periodic boundary conditions and the symmetry of the lattice, the probability $\phi^{\mathrm{p}}_{p_{sd},q,m}(\mathscr{C}_v^*([0,n]^2))$ is larger than $c\phi^{\mathrm{p}}_{p_{sd},q,m}(\mathscr{C}_h([0,n]^2))$ (for some constant c only depending on q), giving

$$1 \le (1+c)\phi^{\mathrm{p}}_{p_{sd},q,m}\big(\mathscr{C}_h([0,n]^2)\big),$$

which concludes the proof. □

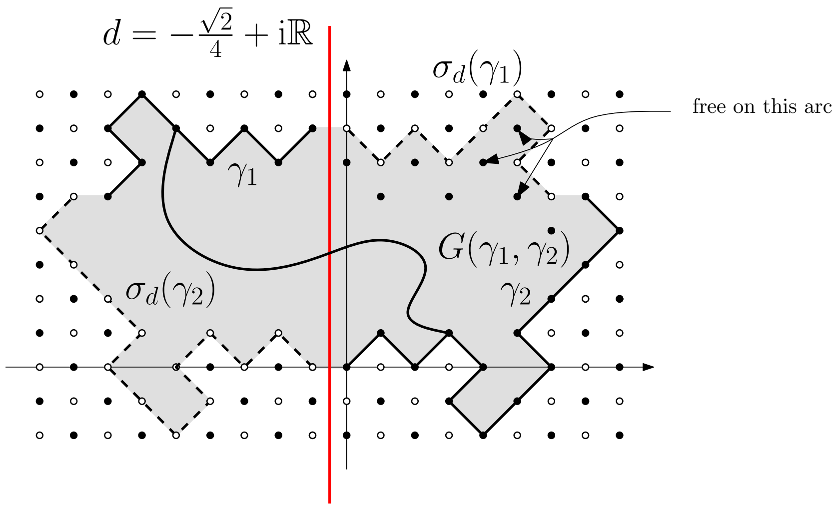


Figure 6: Two paths γ_1 and γ_2 satisfying Hypothesis ($\star$) and the graph $G(\gamma_1,\gamma_2)$.

One can in fact go much further with duality. We extend crossing estimates to a very large family of symmetric domains. We refer to Figure 6 for a picture of what is a symmetric domain. Such a domain is given by two paths γ_1 and γ_2 satisfying a certain Hypothesis ($\star$) (see below) and is denoted $G(\gamma_1,\gamma_2)$. The *mixed boundary conditions* on this graph are wired on γ_1 (all the edges are pairwise connected), wired on γ_2, and free elsewhere. The measure on $G(\gamma_1,\gamma_2)$ with parameters (p_{sd},q) and mixed boundary conditions is denoted by $\phi_{p_{sd},q,\gamma_1,\gamma_2}$ or more simply ϕ_{γ_1,γ_2}.

More formally, define the line $d := -\sqrt{2}/4 + \mathrm{i}\mathbb{R}$. The orthogonal symmetry σ_d with respect to this line maps $\mathbb{Z}^2$ to $(\mathbb{Z}^2)^*$. Let γ_1 and γ_2 be two paths satisfying the following Hypothesis ($\star$):

- γ_1 remains on the left of d and γ_2 remains on the right;
- γ_2 begins at 0 and γ_1 begins on a site of $\mathbb{Z}^2 \cap (-\sqrt{2}/2 + \mathrm{i}\mathbb{R}_+)$;
- γ_1 and $\sigma_d(\gamma_2)$ do not intersect (as curves in the plane);
- γ_1 and $\sigma_d(\gamma_2)$ end at two sites (one primal and one dual) which are at distance $\sqrt{2}/2$ from each other.

The definition extends trivially via translation, so that the pair (γ_1, γ_2) is said to satisfy Hypothesis $(\star)$ if one of its translates does. When following the paths in counter-clockwise order, one can create a circuit by linking the end points of γ_1 and $\sigma_d(\gamma_2)$ by a straight line, the start points of $\sigma_d(\gamma_2)$ and γ_2, the end points of γ_2 and $\sigma_d(\gamma_1)$, and the start points of $\sigma_d(\gamma_1)$ and γ_1. The circuit $(\gamma_1, \sigma_d(\gamma_2), \gamma_2, \sigma_d(\gamma_1))$ surrounds a set of vertices of $\mathbb{Z}^2$. Define the graph $G(\gamma_1, \gamma_2)$ composed of sites of $\mathbb{Z}^2$ that are surrounded by the circuit $(\gamma_1, \sigma_d(\gamma_2), \gamma_2, \sigma_d(\gamma_1))$, and of edges of $\mathbb{Z}^2$ that remain entirely within the circuit (boundary included).

Lemma 3.11 *For any pair (γ_1, γ_2) satisfying Hypothesis $(\star)$, the following estimate holds:*

$$\phi_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2) \geq \frac{1}{1+q^2}.$$

Proof: If γ_1 and γ_2 are not connected, $\sigma_d(\gamma_1)$ and $\sigma_d(\gamma_2)$ must be connected by a path in the dual configuration (event corresponding to $\sigma_d(\gamma_1) \leftrightarrow \sigma_d(\gamma_2)$ in the dual model). Hence,

$$1 = \phi_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2) + \sigma_d * \phi^*_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2), \tag{26}$$

where $\sigma_d * (\phi^*_{\gamma_1,\gamma_2})$ denotes the image under σ_d of the dual measure of ϕ_{γ_1,γ_2}. This measure lies on $G(\gamma_1, \gamma_2)$ as well and has parameters (p_{sd}, q).

When looking at the dual measure of a FK percolation, the boundary conditions are transposed into new boundary conditions for the dual measure. Here, the boundary conditions become wired on $\gamma_1 \cup \gamma_2$ and free elsewhere (once again, this is easy to check using Euler's formula).

It is very important to notice that the boundary conditions are *not* exactly the mixed one, since γ_1 and γ_2 are wired *together*. Nevertheless, the Radon-Nikodym derivative of $\sigma_d * \phi^*_{\gamma_1,\gamma_2}$ with respect to ϕ_{γ_1,γ_2} is easy to bound. Indeed, for any configuration ω, the number of cluster can differ by at most 1 when counted for $\sigma_d * \phi^*_{\gamma_1,\gamma_2}$ or ϕ_{γ_1,γ_2} so that the ratio of partition functions belongs to $[1/q, q]$. Therefore, the ratio of probabilities of the configuration ω remains between $1/q^2$ and q^2. This estimate extends to events by summing over all configurations. Therefore,

$$\sigma_d * \phi^*_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2) \leq q^2 \phi_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2).$$

When plugging this inequality into (26), we obtain

$$\phi_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2) + q^2 \phi_{\gamma_1,\gamma_2}(\gamma_1 \leftrightarrow \gamma_2) \geq 1$$

which implies the claim. □

Proof of Proposition 3.9: The proof goes as follows: we start with creating two paths crossing square boxes, and we then prove that they are connected with good probability.

Setting of the proof. Consider the rectangle $R = [0, 3n/2] \times [0, n]$ which is the union of the rectangles $R_1 = [0, n]^2$ and $R_2 = [n/2, 3n/2] \times [0, n]$, see Figure 5. Let A be the event defined by the following conditions:

- R_1 and R_2 are both crossed horizontally (these events have probability at least c to occur, using Lemma 3.10);

- $[n/2,n]\times\{0\}$ is connected inside R_2 to the top side of R_2 (which has probability greater than $c/2$ to occur, by symmetry and Lemma 3.10).

Employing the FKG inequality, we deduce that

$$\phi^{\mathrm{p}}_{p_{sd},q,m}(A)\geq\frac{c^3}{2}. \tag{27}$$

When A occurs, define Γ_1 to be the top-most horizontal crossing of R_1, and Γ_2 the right-most vertical crossing of R_2 from $[n/2,n]\times\{0\}$ to the top side. Note that this path is automatically connected to the right-hand side of R_2 — which is the same as the right-most side of R. If Γ_1 and Γ_2 are connected, then there exists a horizontal crossing of R. In the following, Γ_1 and Γ_2 are shown to be connected with good probability.

Exploration of the paths Γ_1 and Γ_2. There is a standard way of exploring R in order to discover Γ_1 and Γ_2. Start an exploration from the top-left corner of R that leaves open edges on its right, closed edges on its left and remains in R_1. If A occurs, this exploration will touch the right-hand side of R_1 before its bottom side; stop it the first time it does. Note that the exploration process "slides" between open edges of the primal lattice and dual open edges of the dual (formally, this exploration process is defined on the medial lattice, see later). The open edges that are adjacent to the exploration form the top-most horizontal crossing of R_1, *i.e.* Γ_1. At the end of the exploration, the process has *a priori* discovered a set of edges which lies "above" Γ_1, so that the remaining part of R_1 is undiscovered.

By starting an exploration at point $(n,0)$, leaving open edges on its left and closed edges on its right, the rectangle R_2 can be explored entirely. If A holds, the exploration ends on the top side of R_2. The open edges adjacent to the exploration constitute the path Γ_2 and the set of edges already discovered lies "to the right" of Γ_2.

The reflection argument. Assume first that $\Gamma_1=\gamma_1$ and $\Gamma_2=\gamma_2$, and that they do not intersect. Let x be the end-point of γ_1, *i.e.* its unique point on the right-hand side of R_1. We wish to define a set $G_0(\gamma_1,\gamma_2)$ similar to those considered in Lemma 3.11. Apply the following "surgical procedure," see Figure 7:

- First, define the symmetric paths $\sigma_d(\gamma_1)$ and $\sigma_d(\gamma_2)$ of γ_1 and γ_2 with respect to the line $d:=(n-\sqrt{2}/4)+\mathrm{i}\mathbb{R}$;

- Then, parametrize the path $\sigma_d(\gamma_1)$ by the distance (along the path) to its starting point $\sigma_d(x)$ and define $\tilde\gamma_1\subset\gamma_1$ so that $\sigma_d(\tilde\gamma_1)$ is the part of $\sigma_d(\gamma_1)$ between the start of the path and the first time it intersects γ_2. As before, the paths are considered as curves of the plane; denote the intersection point of the two curves by z. Note that γ_1 and γ_2 are not intersecting, which forces $\sigma_d(\gamma_1)$ and γ_2 to be;

- From this, parametrize the path γ_2 by the distance to its starting point $(n,0)$ and set y to be the last visited site in $\mathbb{Z}^2$ before the intersection z. Define $\tilde\gamma_2$ to be the part of γ_2 between the last point intersecting $n+\mathrm{i}\mathbb{R}$ before y and y itself;

- Paths $\tilde\gamma_1$ and $\tilde\gamma_2$ satisfy Hypothesis $(\star)$ so that the graph $G(\tilde\gamma_1,\tilde\gamma_2)$ can be defined;

- Construct a sub-graph $G_0(\gamma_1,\gamma_2)$ of $G(\tilde\gamma_1,\tilde\gamma_2)$ as follows: the edges are given by the edges of $\mathbb{Z}^2$ included in the connected component of $G(\tilde\gamma_1,\tilde\gamma_2)\setminus(\gamma_1\cup\gamma_2)$ (*i.e.*

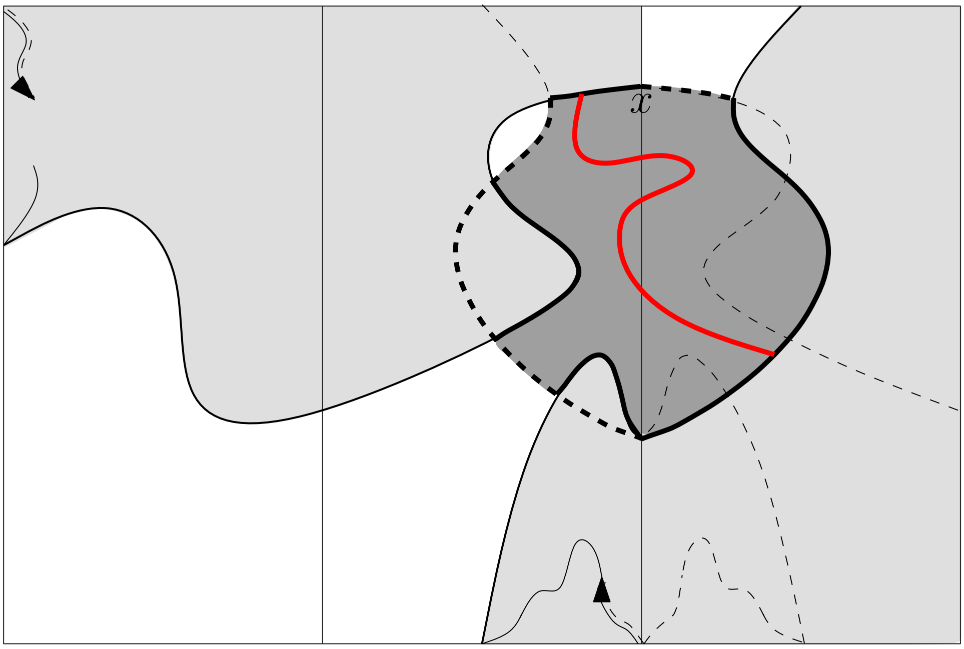


Figure 7: The light gray area is the part of R that is *a priori* discovered by the exploration processes (note that this area can be much smaller). The dark gray is the domain $G_0(\tilde{\gamma}_1, \tilde{\gamma}_2)$. All the paths involved in the construction are depicted. Note that dashed curves are "virtual paths" of the dual lattice obtained by the reflection σ_d: they are not necessarily dual open.

$G(\tilde{\gamma}_1, \tilde{\gamma}_2)$ minus the set $\gamma_1 \cup \gamma_2$) containing d (it is the connected component which contains $x - \varepsilon \mathrm{i}$, where ε is a very small number), and the sites are given by their endpoints.

The graph $G_0(\gamma_1, \gamma_2)$ has a very useful property: none of its edges has been discovered by the previous exploration paths. Indeed, $\sigma_d(\gamma_1)$ and $\sigma_d(x)$ are included in the unexplored connected component of $R \smallsetminus R_1$, and so does $G_0(\gamma_1, \gamma_2) \cap (R \smallsetminus R_1)$. Edges of $G_0(\gamma_1, \gamma_2)$ in R_1 are in the same connected component of $R \smallsetminus (\gamma_1 \cup \gamma_2)$ as $x - \epsilon \mathrm{i}$, and thus lie 'below' γ_1.

Conditional probability estimate. Still assuming that γ_1 and γ_2 do not intersect, we would like to estimate the probability of γ_1 and γ_2 being connected by a path knowing that $\Gamma_1 = \gamma_1$ and $\Gamma_2 = \gamma_2$. Following the exploration procedure described above, γ_1 and γ_2 can be discovered without touching any edge in the interior of $G_0(\gamma_1, \gamma_2)$. Therefore, the process in the domain is a FK percolation with specific boundary conditions.

The boundary of $G_0(\gamma_1, \gamma_2)$ can be split into several sub-arcs of various types (see Figure 7): some are sub-arcs of γ_1 or γ_2, while the others are (adjacent to) sub-arcs of their symmetric images $\sigma_d(\gamma_1)$ and $\sigma_d(\gamma_2)$. The conditioning on $\Gamma_1 = \gamma_1$ and $\Gamma_2 = \gamma_2$ ensures that the edges along the sub-arcs of the first type are open; the connections along the others depend on the exact explored configuration in a much more intricate way, but in any case the boundary conditions imposed on the configuration inside $G(\tilde{\gamma}_1, \tilde{\gamma}_2)$ are larger than the mixed boundary conditions. Notice that *any* boundary condition dominates the free one and that $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are two sub-arcs of the first type (they are then wired). Thus, the measure restricted to $G_0(\tilde{\gamma}_1, \tilde{\gamma}_2)$ stochastically dominates the restriction of $\phi_{\tilde\gamma_1,\tilde\gamma_2}$ to $G_0(\tilde\gamma_1,\tilde\gamma_2)$.

From these observations, we deduce that for any increasing event B *depending only on edges in* $G_0(\gamma_1,\gamma_2)$,

$$\phi^{\mathrm{p}}_{p_{sd},q,m}(B|\Gamma_1=\gamma_1,\Gamma_2=\gamma_2) \geq \phi_{\tilde\gamma_1,\tilde\gamma_2}(B). \tag{28}$$

In particular, this inequality can be applied to $\{\gamma_1 \leftrightarrow \gamma_2 \text{ in } G_0(\gamma_1,\gamma_2)\}$. Note that if $\tilde\gamma_1$ and $\tilde\gamma_2$ are connected in $G(\tilde\gamma_1,\tilde\gamma_2)$, γ_1 and γ_2 are connected in $G_0(\tilde\gamma_1,\tilde\gamma_2)$. The first event is of $\phi_{\tilde\gamma_1,\tilde\gamma_2}$-probability at least $1/(1+q^2)$, implying

$$\begin{aligned}\phi^{\mathrm{p}}_{p_{sd},q,m}(\gamma_1 \leftrightarrow \gamma_2|\Gamma_1=\gamma_1,\Gamma_2=\gamma_2) &\geq \phi_{\tilde\gamma_1,\tilde\gamma_2}(\gamma_1 \leftrightarrow \gamma_2 \text{ in } G_0(\gamma_1,\gamma_2))\\ &\geq \phi_{\tilde\gamma_1,\tilde\gamma_2}(\tilde\gamma_1 \leftrightarrow \tilde\gamma_2) \geq \frac{1}{1+q^2}.\end{aligned} \tag{29}$$

Conclusion of the proof. Note the following obvious fact: if γ_1 and γ_2 intersect, the conditional probability that Γ_1 and Γ_2 intersect, knowing $\Gamma_1=\gamma_1$ and $\Gamma_2=\gamma_2$ is equal to 1 — in particular, it is greater than $1/(1+q^2)$. Now,

$$\begin{aligned}\phi^{\mathrm{p}}_{p_{sd},q,m}(\mathscr{C}_h(R)) &\geq \phi^{\mathrm{p}}_{p_{sd},q,m}(\mathscr{C}_h(R)\cap A)\\ &\geq \phi^{\mathrm{p}}_{p_{sd},q,m}(\{\Gamma_1 \leftrightarrow \Gamma_2\}\cap A)\\ &= \phi^{\mathrm{p}}_{p_{sd},q,m}\Big(\phi^{\mathrm{p}}_{p_{sd},q,m}(\Gamma_1 \leftrightarrow \Gamma_2|\Gamma_1,\Gamma_2)\mathbf{1}_A\Big)\\ &\geq \frac{1}{1+q^2}\phi^{\mathrm{p}}_{p_{sd},q,m}(A) \geq \frac{c^3}{2(1+q)^2}\end{aligned}$$

where the first two inequalities are due to inclusion of events, the third one to the definition of conditional expectation, and the fourth and fifth ones, to (29) and (27). □

We have therefore finished the proof that $p_c=\sqrt{q}/(1+\sqrt{q})$. The duality was at the heart of the proof and was used extensively. We now sketch an alternative proof based on another property of the FK percolation, called integrability.

From now on, we go back to the unrotated lattice.

4 A second computation of p_c based on integrability

4.1 Exact integrability

The historical approach to statistical physics was not based on couplings or duality but rather on exact integrability. In the physical approach to statistical models, studying the growth of the partition function is the first step towards a deep understanding of the model. This discussion is voluntarily kept informal, since we will adopt an alternative point of view on integrability in the next section.

Take the fundamental example of the Ising model. Consider the cylinder C_{mn} of size $n\times m$ and let $Z(C_{mn},\beta)$ be the partition function of the Ising on C_{mn} at inverse temperature β. The *free energy* is defined by

$$f(\beta)=\lim_{m,n\to\infty}\frac{1}{nm}\log Z(C_{mn},\beta).$$

The first computation of the free energy for the square lattice Ising model is due to Onsager [62]. In few words, the proof is based on the so-called transfer matrix method,

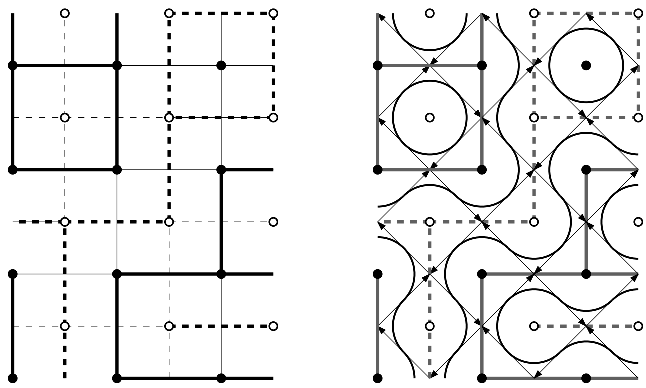


Figure 8: Left. The configuration ω with its dual configuration ω^*. **Right.** The loop representation associated to ω.

and is very algebraic. The main idea is to express the partition function in C_{mn} as the trace of a product of matrices. By co-diagonalizing all these matrices, the rate of growth of the partition function can be identified in terms of their spectral properties. Once computed, the free energy enables to identify the critical point (and in fact much more things) as being the unique singularity of the free energy seen as an analytic function in β.

The fact that the matrices involved in the previous program can be co-diagonalized is somewhat of a "miracle" and there is no reason that this should be doable in general: for arbitrary models, the program cannot always be implemented. Nevertheless, for specific planar models at their critical point, the free energy may be expressed in a relatively simple way. These models satisfy special algebraic properties, and are usually referred to as *integrable models*. We do not wish to describe the theory of integrable models in details, and we refer to [4] for an exposition (in particular the connection to the Yang-Baxter equation).

Of great interest for us is the fact that the transfer matrix method has been implemented for FK percolation at the self-dual point by Baxter [4]. As a corollary, he computed the free energy. Unfortunately, the computation works only at the self-dual point, and is therefore not sufficient to identify the self-dual point as being a singular point for the free energy, and therefore the critical point.

4.2 Loop representation of the FK percolation

When computing the free energy, physicists work with a different representation of the FK percolation, called loop representation of the model, or fully packed $O(n)$-model. Let us describe this representation now.

For $G \subset \mathbb{Z}^2$, define $G^\diamond$ to be the graph with vertex set given by center of edges in E_G, and edges between nearest neighbors. This graph is called the *medial graph*. For instance, $(\mathbb{Z}^2)^\diamond$ is a rescaled and rotated version of $\mathbb{Z}^2$.

Consider a configuration ω. It defines clusters in G and dual clusters in G^*. Through every face of the medial graph passes either an open edge of G or a dual open edge of G^*. Therefore, there is a unique way to draw Eulerian (*i.e.* using every edge exactly once) loops on the medial graph — *interfaces*, separating clusters from dual clusters. Namely, a loop arriving at a vertex of the medial lattice always makes a $\pm\pi/2$ turn so as not to cross the open or dual open bond through this vertex; see Figure 8. This provides us with a bijection between FK configurations on G and Eulerian loop configurations on $G^\diamond$. This bijection is called the *loop representation* of the FK percolation. We orientate loops by orienting edges of $G^\diamond$ counter-clockwise around faces corresponding to vertices in V_G.

Proposition 4.1 *Let $p \in (0,1)$ and $q > 0$, then for any configuration ω,*

$$\phi^1_{p,q,G}(\omega) \;=\; \frac{1}{Z}\, x^{o(\omega)} \sqrt{q}^{\ell(\omega)} \tag{30}$$

where $x = p/[\sqrt{q}(1-p)]$, $\ell(\omega)$ is the number of loops in the loop configuration associated to ω, $o(\omega)$ is the number of open edges, and Z is the partition function.

When $p = p_{sd}$, one obtains that $\phi^1_{p,q,G}(\omega)$ is proportional to $\sqrt{q}^{\ell(\omega)}$ and the FK percolation has been rephrased in terms of a loop model.

Proof: Recall that

$$\phi^1_{p,q,G}(\omega) \;=\; \frac{1}{Z^1_{p,q,G}}[p/(1-p)]^{o(\omega)} q^{k(\omega)}.$$

The dual of $\phi^1_{p,q,G}$ is $\phi^0_{G^*,p^*,q}$. With ω^* being the dual configuration of ω, we find

$$\begin{aligned}
\phi^1_{p,q,G}(\omega) &= \sqrt{\phi^1_{p,q,G}(\omega)\ \phi^0_{G^*,p^*,q}(\omega^*)}\\
&= \frac{\sqrt{p/(1-p)}^{\,o(\omega)}\sqrt{q}^{\,k(\omega)}\sqrt{p^*/(1-p^*)}^{\,o(\omega^*)}\sqrt{q}^{\,k(\omega^*)}}{\sqrt{Z^1_{p,q,G}Z^0_{p^*,q,G^*}}}\\
&= \frac{\sqrt{p^*/(1-p^*)}^{\,o(\omega^*)+o(\omega)}}{\sqrt{Z^1_{p,q,G}Z^0_{p^*,q,G^*}}}\sqrt{q}^{\,k(\omega)+k(\omega^*)}\sqrt{\frac{p(1-p^*)}{(1-p)p^*}}^{\,o(\omega)}\\
&= \frac{\sqrt{p^*/(1-p^*)}^{\,o(\omega)+o(\omega^*)}}{\sqrt{Z^1_{p,q,G}Z^0_{p^*,q,G^*}}}\ \sqrt{q}^{\,k(\omega)+k(\omega^*)}x^{o(\omega)}
\end{aligned}$$

where the definition of p^* was used to prove that $\frac{p(1-p^*)}{(1-p)p^*} = x^2$. Note that $\ell(\omega) = k(\omega) + k(\omega^*) - 1$ and $o(\omega) + o(\omega^*) = |E_G|$. We deduce the result since

$$Z = \frac{\sqrt{Z^1_{p,q,G}Z^0_{p^*,q,G^*}}}{\sqrt{q}\sqrt{p^*/(1-p^*)}^{|E_G|}}$$

does not depend on the configuration. □

4.3 Parafermionic observables

Integrability cannot be reduced to the fact that the free energy can be computed explicitly. In recent years, another point of view on planar models has been developed. The idea is that integrability can be related to the existence of *observables* which are discrete holomorphic at the self-dual point, in the sense that they satisfy discrete analogues of

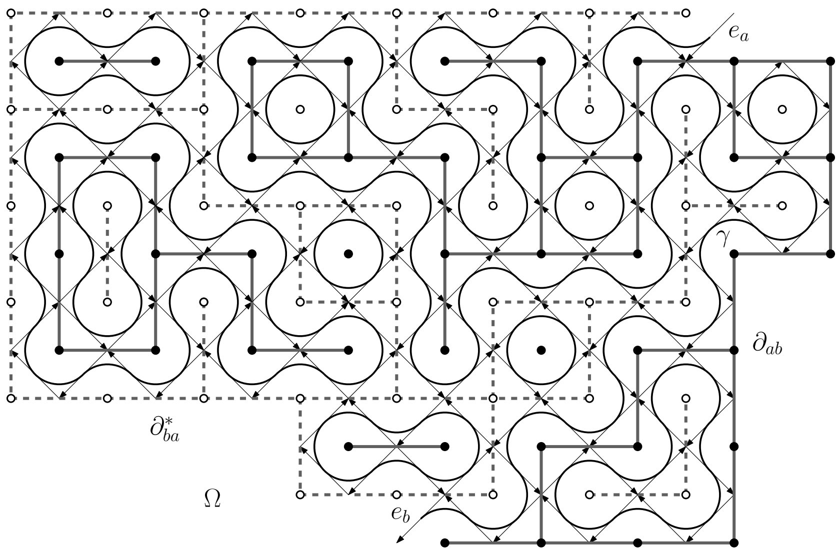


Figure 9: A loop representation in a Dobrushin domain.

Cauchy-Riemann equations. For a comprehensive discussion on discrete holomorphicity and its relation to discrete models, see [28].

In the case of FK percolation, these observables were first introduced by Smirnov in [71] as holomorphic parafermions of fractional spin $\sigma \in [0,1]$, given by certain vertex operators (in [71], the observable is defined for $q \in [0,4]$ only but it was extended later to every $q \geq 0$). The goal of this section is to define these observables and to briefly explain how they can be used to compute critical points. We only explain the overall structure of the proof, and most of the technical details will not be described. We refer to the literature for further details.

The observable is defined in terms of the loop representation of FK percolation with specific boundary conditions, called *Dobrushin boundary conditions* (due to its similarity in spirit with the original $+/-$ Dobrushin boundary condition for the Ising model). We define it now. Assume that ∂G is a self-avoiding polygon in $\mathbb{Z}^2$. Let a and b be two sites of ∂G. The triplet (G,a,b) is called a *Dobrushin domain*. Orienting its boundary counterclockwise defines two oriented boundary arcs ∂_{ab} and ∂_{ba}. The Dobrushin boundary conditions are defined to be free on ∂_{ab} (there are no wirings between boundary sites) and wired on ∂_{ba} (all the boundary sites are pairwise connected). The measure associated to these boundary conditions will be denoted by $\phi^{a,b}_{p,q,G}$.

Let us now look at the loop representation in this context. Besides loops, the configuration contains a single curve joining the edges adjacent to a and b, which are the medial edges e_a and e_b between ∂_{ab} and ∂^*_{ba}. This curve is called the *exploration path*; it is denoted by γ. The *winding* $\mathrm{W}_\gamma(e,e_b)$ of γ is the total rotation (in radians) that the curve makes from the center of the medial edge e to the center of the medial edge e_b. It can also be seen as $\pi/2$ times the number of left turns minus the number of right turns that the curve makes between e and e_b.

Following [71], we define the observable F for any medial edge $e \in E_{G^\diamond}$ as

$$F(e) = \phi^{a,b}_{p,q,G}\left(\mathrm{e}^{i\sigma \mathrm{W}_\gamma(e,e_b)}\mathbf{1}_{e\in\gamma}\right),$$

where σ is given by the relation

$$\sin\left(\sigma\frac{\pi}{2}\right) = \frac{\sqrt{q}}{2}.$$

The observable F is called the *parafermionic observable.* An interesting feature of this observable is the following local relations.

Proposition 4.2 *Consider a medial vertex v with four adjacent edges in $E_{G^\diamond}$. The two edges pointing towards v are indexed by A and C, and the other two by B and D in such a way that A, B, C and D are found in clockwise order. Then,*

$$F(B) - F(D) \;=\; \frac{\mathrm{e}^{\mathrm{i}\sigma\pi/2} + x}{\mathrm{e}^{\mathrm{i}\sigma\pi/2}x + 1}\, i[F(A) - F(C)]. \tag{31}$$

Before discussing the consequences of this relation, we provide a proof, which turns out to be completely elementary.

Proof: Consider the involution s on the space of configurations which switches the state (open or closed) of the edge of G passing through v. Let e be an edge of the medial graph and denote by $e_\omega = \mathrm{e}^{i\sigma W_\gamma(e,e_b)}\mathbf{1}_{e\in\gamma}p(\omega)$ the contribution of ω to $F(e)$ (here $p(\omega)$ is the probability of the configuration ω). Since s is an involution, the following relation holds:

$$F(e) = \sum_\omega e_\omega = \tfrac{1}{2}\sum_\omega \left[e_\omega + e_{s(\omega)}\right].$$

In order to prove (31), it suffices to prove the following for any configuration ω:

$$B_\omega + B_{s(\omega)} - D_\omega - D_{s(\omega)} = \frac{\mathrm{e}^{\mathrm{i}\sigma\pi/2} + x}{\mathrm{e}^{\mathrm{i}\sigma\pi/2}x + 1}\, i\left[A_\omega + A_{s(\omega)} - C_\omega - C_{s(\omega)}\right]. \tag{32}$$

When $\gamma(\omega)$ does not go through any of the edges incident to v, neither does $\gamma(s(\omega))$. All the contributions then vanish and identity (32) trivially holds. Thus we may assume that $\gamma(\omega)$ passes through at least one edge incident to v. The interface enters through either A or C and leaves through B or D. Without loss of generality, we assume that it enters first through A and leaves last through D; the other cases are treated similarly.

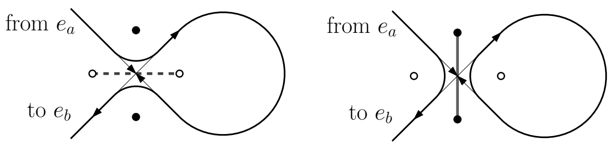


Figure 10: Two associated configurations ω and $s(\omega)$

Two cases can occur (see Figure 10): either the exploration curve, after arriving through A, leaves through B and then returns a second time through C, leaving through D; or the exploration curve arrives through A and leaves through D, with B and C belonging to a loop. Since the involution exchanges the two cases, we can assume that ω corresponds to the first case. Knowing the term A_ω, it is possible to compute the contributions of ω and $s(\omega)$ to all of the edges incident to v. Indeed,

- the probability of $s(\omega)$ is equal to $x\sqrt{q}$ times the probability of ω (due to the fact that there is one additional loop, and the primal edge passing through v is open);

- windings of the curve can be expressed using the winding of the edge A. For instance, the winding of B in the configuration ω is equal to the winding of the edge A plus an additional $-\frac{\pi}{2}$ turn.

Contributions are computed in the following table.

configuration	A	B	C	D
ω	A_ω	$e^{i\sigma\frac{\pi}{2}}A_\omega$	$e^{-i\sigma\pi}A_\omega$	$e^{-i\sigma\frac{\pi}{2}}A_\omega$
$s(\omega)$	$x\sqrt{q}A_\omega$	0	0	$e^{-i\sigma\frac{\pi}{2}}x\sqrt{q}A_\omega$

Using the identity $e^{i\sigma\frac{\pi}{2}} - e^{-i\sigma\frac{\pi}{2}} = i\sqrt{q}$, we deduce (32) by summing the contributions of all the edges incident to v. □

When $p = p_{sd}$, the previous relation becomes

$$F(B) - F(D) = i[F(A) - F(C)]$$

which is a discrete version of the Cauchy-Riemann equation. Nevertheless, there are in general no reason to be able to extract information from the relations above. Indeed, these relations do not determine the observable since we get one complex relation per vertex, while there is one complex value of the observable per edge. There are still some special cases that are easier to study.

The case $q = 2$ corresponds to $\sigma = \frac{1}{2}$. For this value of q, the complex argument modulo π of the observable can easily be checked to depend only on the orientation θ of the edge (the winding takes value in $\theta + 2\pi i\mathbb{Z}$ and therefore $e^{i\frac{1}{2}W_\gamma(e,e_b)}$ equals $e^{i\theta/2}$ or $-e^{i\theta/2}$). At $p = p_{sd}$, this specificity implies a strong notion of discrete holomorphicity for the observable, called s-holomorphicity; see [71, 72]. The observable can then be used to compute the critical parameter [7]. Roughly speaking, the observable can be computed explicitly, and can also be related to probabilities of connections. This allows to prove directly that there exists an infinite cluster almost surely when $p > p_{sd}$, and no infinite cluster almost surely when $p < p_{sd}$. In fact, the observable can be used to understand many other properties on the model, including conformal invariance of the observable [21, 71] and loops [17, 45], correlations [18, 19, 44] and crossing probabilities [9, 16, 26]. The s-holomorphicity has been the subject of an extensive study, and we refer to [28] and references therein for mode details.

Another case for which the complex argument of the observable is known is the case $q \geq 4$. Indeed, the spin σ takes values in $1 + i\mathbb{R}$ and once again the complex argument (modulo 2π this time) of the observable depends on the orientation of the edge only. This feature enables us to prove $p_c = p_{sd}$ using the observable. We refer to [8] for details of the proof.

Interestingly, even though the complex argument of the observable cannot be determined when $q \in (0,4) \setminus \{2\}$, the parafermionic observable can still be used to show non-trivial facts on the model. For instance, the divergence of the correlation length when p approaches the critical point and $1 \leq q \leq 4$ was proved in [25]. Let us mention that this is not an isolated fact. In the self-avoiding walk model (a model that can be related to FK percolation), similar observables were introduced. Once again, only partial information is available but it remains nonetheless sufficient to compute several critical parameters of the model (e.g. the connective constant [29] or critical fugacity [5]).

Even though we did not describe in details how parafermionic observables can be used to compute the critical point, we emphasize that they satisfy local relations which are reminiscent of the underlying integrability of the models.

4.4 Other lattices and universality

A general question in statistical physics is the understanding of universal behavior, i.e. the behavior of a certain model, for instance the planar FK percolation, on different graphs. The class of graphs on which we want to consider the model is a priori unrestricted, but several mild conditions should hold in order to be able to develop a theory of universality. In other words, one needs to consider a large enough class of interesting graphs, but small enough that tools are still available for our study.

A large class of such graphs, which appeared to be central in different domains of planar statistical physics, is the class of isoradial graphs. We describe these graphs below and explain what is known about FK percolation on them.

An *isoradial graph* is a planar graph admitting an embedding in the plane in such a way that every face is inscribed in a circle of radius one. We will say that such an embedding is isoradial.

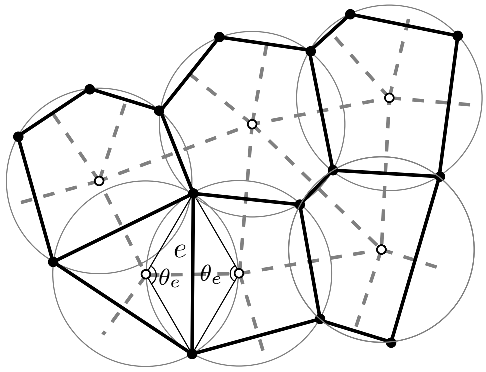


Figure 11: The black graph is the isoradial graph. White vertices are the vertices of the dual graph. All faces can be inscribed in a circle of radius one. Dual vertices have been drawn in such a way that they are the centers of these circles.

Isoradial graphs were introduced by Duffin in [23] in the context of discrete complex analysis. The author noticed that isoradial embeddings form a large class of embeddings for which a discrete notion of the Cauchy-Riemann equations is available. Isoradial graphs first appeared in the physics literature in the work of Baxter [3], where they are called Z-invariant graphs. In this work, they are obtained as intersections of lines in the plane, and are not embedded in the isoradial way. The so-called star-triangle transformation was then used to relate the Ising free energy on different Z-invariant graphs. The term isoradial was only coined later by Kenyon, who studied discrete complex analysis and the graph structure of these graphs [50]. Since then, isoradial graphs were used extensively, and we refer to [20, 41, 50, 61] for literature on the subject.

We will consider the probability measure $\phi^{\xi}_{\mathbf{p},q,G}$ of the FK percolation on G with particular parameters $\mathbf{p} = (p_e : e \in E_G)$, $q \in (0,\infty)$ and boundary conditions ξ is defined by

$$\phi^{\xi}_{\mathbf{p},q,G}(\{\omega\}) = \frac{\prod_{e\in\omega} p_e \cdot \prod_{e\notin\omega}(1-p_e)\cdot q^{k(\omega,\xi)}}{Z^{\xi}_{\mathbf{p},q,G}}. \tag{33}$$

In order to study the phase transition, we parametrize FK measures with cluster-weight $q \geq 4$ with the help of an additional parameter $\beta > 0$. For $\beta > 0$, define the edge-weight $p_e(\beta) \in [0,1]$ for $e \in E_G$ by the formula

$$\frac{p_e(\beta)}{[1-p_e(\beta)]\sqrt{q}} = \beta\,\frac{\sinh[\frac{i(1-\sigma)(\pi-\theta_e)}{2}]}{\sinh[\frac{i(1-\sigma)\theta_e}{2}]}.$$

The dual of an isoradial graph G is an isoradial graph. Furthermore, the dual of the FK measure on G with parameter β is a FK measure with parameter $1/\beta$ on G^*. In this sense, $\beta = 1$ is self-dual.

Isoradial graphs are not required *a priori* to be invariant under translations. We will still make this technical assumption on every graph we will consider here — more technically, for any isoradial graph G, we will assume the existence of an action of $\mathbb{Z}^2$ on G with finitely many orbits. As for $\mathbb{Z}^2$, infinite volume measures can be defined in this case by taking limits of measures on finite graphs. The infinite-volume measure on G with cluster-weight $q \geq 4$, edge-weights $(p_e(\beta) : e \in E_G)$ and free (resp. wired) boundary conditions is denoted by $\phi^0_{\beta,q,G}$ (resp. $\phi^1_{\beta,q,G}$).

Theorem 4.3 *Let $q \geq 4$ and $\theta > 0$. For any infinite periodic isoradial graph G and any boundary condition $*$,*

- *For $\beta < 1$, there is $\phi^*_{\lambda,q,G}$-almost surely no infinite-cluster.*
- *For $\beta > 1$, there is $\phi^*_{\lambda,q,G}$-almost surely an infinite-cluster.*

The main ingredient of the proof is the parafermionic observable, which can be defined naturally on isoradial graphs. The result holds only for $q \geq 4$ because we use the fact that the complex argument of the observable depends only on the orientation of the edges. The observable has been used in the case $q = 2$ in order to prove conformal invariance on isoradial graphs [21]. This result implies the previous theorem in the case $q = 2$. For $q = 1$, Manolescu and Grimmett [39, 40, 41] obtained the previous theorem together with further results via another route.

The previous theorem has an interesting corollary. Inhomogeneous FK percolations on the square, the triangular and the hexagonal lattices can be seen as FK percolations on isoradial graphs, which leads to the following corollary.

Corollary 4.4 *For cluster-weight $q \geq 4$, the FK percolation on the square, triangular and hexagonal lattices $\mathbb{Z}^2$, $\mathbb{T}$ and $\mathbb{H}$ have the following critical surfaces:*

$$\text{on } \mathbb{Z}^2 \quad \frac{p_1}{1-p_1}\frac{p_2}{1-p_2} = q,$$

$$\text{on } \mathbb{T} \quad \frac{p_1}{1-p_1}\frac{p_2}{1-p_2}\frac{p_3}{1-p_3} + \frac{p_1}{1-p_1}\frac{p_2}{1-p_2} + \frac{p_1}{1-p_1}\frac{p_3}{1-p_3} + \frac{p_2}{1-p_2}\frac{p_3}{1-p_3} = q$$

$$\text{on } \mathbb{H} \quad \frac{p_1}{1-p_1}\frac{p_2}{1-p_2}\frac{p_3}{1-p_3} = q\frac{p_1}{1-p_1} + q\frac{p_2}{1-p_2} + q\frac{p_3}{1-p_3} + q^2,$$

where p_1, p_2 (resp. p_1, p_2, p_3) are the edge-weights of edges in the different directions.

Let us mention that the isotropic case ($p_1 = p_2 = p_3$) on the hexagonal and triangular lattices was treated for any $q \geq 1$ in [7] following the strategy of Section 3.

5 The phase diagram of FK percolation on $\mathbb{Z}^2$

5.1 Determination of the critical point

On the one hand, we proved in Section 3 that $p_c(q) = p_{sd}(q) = \sqrt{q}/(1+\sqrt{q})$ for any $q \geq 1$. The main ingredient was duality, but another crucial tool was the positive association of the model. This positive association is very specific to $q \geq 1$. As mentioned before, it fails whenever $q < 1$. On the other hand, the strategy of Section 4 does not a priori use the FKG inequality. The fact that parafermionic observables are defined for any $q > 0$ leads to the following question:

Question. Use the parafermionic observable to (prove the existence and to) compute the critical point on isoradial graphs (or simply on $\mathbb{Z}^2$) for any $q \in (0,4)$?

5.2 The subcritical and supercritical phases

The FK percolation undergoes a sharp transition in the following sense.

Theorem 5.1 ([6]) *Let $q \geq 1$. For any $p < p_c(q)$, there exists $c > 0$ such that for any $x \in \mathbb{Z}^2$,*

$$\phi_{p,q}(0 \leftrightarrow \partial[-n,n]^2) \leq \mathrm{e}^{-cn}. \tag{34}$$

Theorem 5.1 follows from a refinement of the techniques in Section 4. We refer to [6] for details. This theorem is crucial in the description of the subcritical phase $p < p_c$. Several conditional results (the assumption being the exponential decay of correlations) can now be stated as theorems; e.g. Ornstein-Zernike estimates [15], exponential decay in volume [38, Theorem (5.86)], mixing properties [1, 2], or classifications of Gibbs states for Potts models [22]. Duality then allows to describe the supercritical phase $p > p_c$. Let us mention that these phases are now very well understood. We refer to [38] and references therein for a discussion of these two phases.

5.3 The critical phase

Let us now describe the critical phase $(p_c(q), q)$. The behavior is very different depending on the value of q. In physics, the order of a phase transition is defined as the lowest derivative of the free energy which is discontinuous at the phase transition. For FK percolation, Baxter computation of the free energy [4] predicts (non rigorously) that the phase transition is of first order when $q > 4$, and of second when $q < 4$.

Interestingly, this transition corresponds, for the definition of the parafermionic observable, to the transition between $\sigma \in [0,1]$ and $\sigma \in 1 + i\mathbb{R}$. Since the spin behaves differently when $q < 4$ and $q > 4$, it is natural to expect that the observable can provide information on the order of the phase transition (see discussion below).

5.3.1 The case $q > 4$

As mentioned above, the phase transition is conjectured to be of first order, which in our case means that there are multiple infinite-volume measures at criticality. In particular, $\phi^0_{p_c,q,\mathbb{Z}^2}$ and $\phi^1_{p_c,q,\mathbb{Z}^2}$ would be different. In fact, the following stronger statement is expected:

- $\phi^1_{p_c,q,\mathbb{Z}^2}(0 \leftrightarrow \infty) > 0$,
- there exists $c > 0$ such that $\phi^0_{p_c,q,\mathbb{Z}^2}(0 \leftrightarrow \partial[-n,n]^2) \le \exp(-cn)$ for every n.

This result is known only for $q \ge 25.72$, see [55, 56]. The proof is based on the Pirogov-Sinai theory, which combines combinatorics and Peierls's argument.

Question. Use the parafermionic observable to show that the phase transition is of first order whenever $q > 4$?

5.3.2 The case $q < 4$

The model is conjectured to be conformally invariant in the scaling limit. We refer to [29] for a discussion of conformal invariance, its implication and the associated physics background. Here, we will mention only one conjecture.

Consider a simply connected domain (Ω, a, b) of $\mathbb{R}^2$ with two points on its boundary. Consider the Dobrushin domain $(\Omega_\delta, a_\delta, b_\delta)$ where $\Omega_\delta = \delta\mathbb{Z}^2 \cap \Omega$ and a_δ and b_δ are the points on the boundary of Ω_δ closest to the boundary. We consider the critical FK percolation in $(\Omega_\delta, a_\delta, b_\delta)$ with Dobrushin boundary conditions. Let γ_δ be the exploration curve in Ω_δ.

Conjecture 1 (Schramm, [68]) *Fix $q \in [0,4]$. The law of γ_δ converges to the Schramm-Loewner Evolution with parameter $\kappa = \frac{4\pi}{\arccos(-\sqrt{q}/2)}$ as δ tends to zero.*

Schramm introduced the *Schramm-Loewner Evolution* in [67] (SLE for short). For $\kappa > 0$, the SLE(κ) is the random Loewner Evolution with driving process $\sqrt{\kappa}B_t$, where (B_t) is a standard one-dimensional Brownian motion. We refer to [64, 58, 57] for a description of the fundamental fractal properties of SLEs. Proving convergence of the discrete interfaces to SLE is crucial since the path properties of SLEs can then be related to the fractal properties of the critical models. This is probably one of the most challenging questions in rigorous mathematical physics; see [68, 70, 71] for a collection of problems.

Conjecture 1 was proved by Lawler, Schramm and Werner [59] for $q = 0$: they showed that the perimeter curve of the uniform spanning tree (UST for short) converges to SLE(8). Note that the loop representation with Dobrushin boundary conditions still makes sense for $q = 0$ (more precisely for the model obtained by letting $q \to 0$ and $q/p \to 0$). In fact, configurations have no loops, just a curve running from a to b (which then necessarily passes through all the edges), with all configurations being equally probable. The $q = 2$ case was proved in [17]. For values of $q \in [0,4] \setminus \{0,2\}$, Conjecture 1 is open. The $q = 1$ case is particularly interesting, since it corresponds to bond percolation on the square lattice.

Without entering into details, let us say that the previous conjecture follows from the following one, which emphasizes the importance of the parafermionic observable (see [28, 25] for details on the connection between the two conjectures). Parafermionic observables can be defined on medial vertices by the formula

$$F(v) = \frac{1}{2}\sum_{e\sim v} F(e)$$

where the summation is over medial edges emanating from v. Let F_δ be the parafermionic observable on $(\Omega_\delta, a_\delta, b_\delta)$.

Conjecture 2 (Smirnov [70]) *Fix $q \in [0,4]$ and (Ω, a, b) be a simply connected domain with two points on its boundary. For every $z \in \Omega$,*

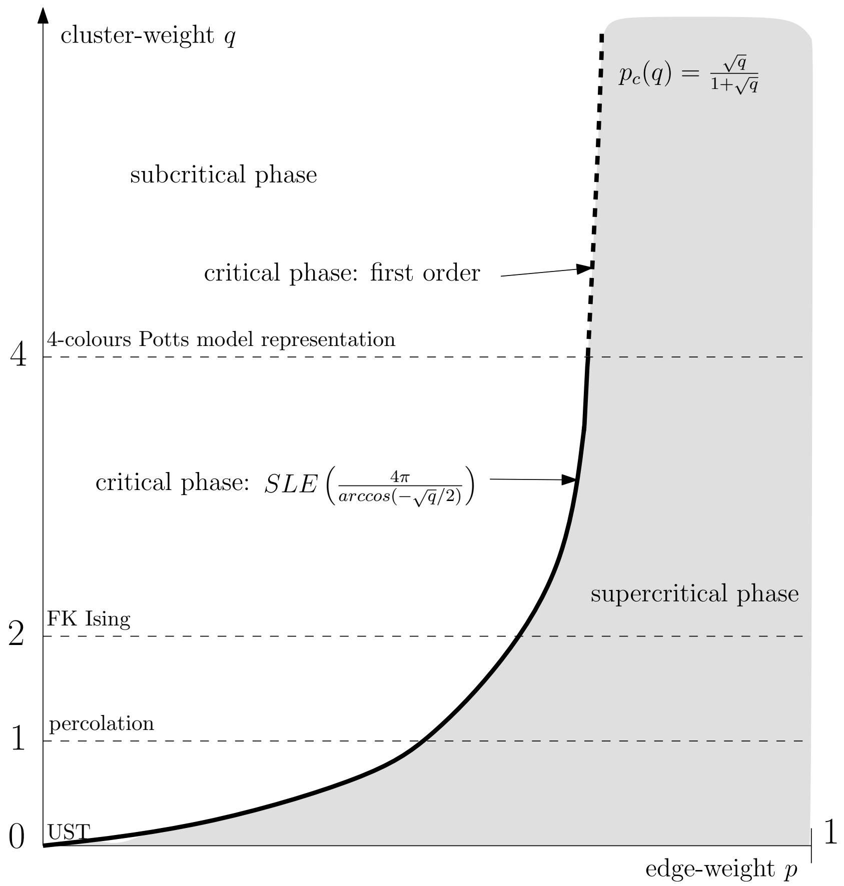
$$\frac{1}{(2\delta)^\sigma} F_\delta(z) \longrightarrow \phi'(z)^\sigma \quad \text{when } \delta \to 0, \tag{35}$$

where $\sigma = 1 - \frac{2}{\pi}\arccos(\sqrt{q}/2)$ and ϕ is any conformal map from Ω to $\mathbb{R} \times (0,1)$ sending a to $-\infty$ and b to ∞.

As mentioned earlier, F_δ is not determined by the collection of relations (31) for general q (the number of variables exceeds the number of equations) and a proof of this conjecture is still missing. Let us mention a very important exception. For $q = 2$, we mentioned that Smirnov in [71] proved a strong version of discrete holomorphicity thanks to the fact that the spin takes the special value $\sigma = 1/2$. This discrete holomorphicity implies the convergence in this special case. This step is the main step towards the proof in [17] of Conjecture 1 in the $q = 2$ case.

As mentioned before, the parafermionic observable was used to show the divergence of the correlation length when p approaches the critical point and $1 \le q \le 4$ [25]. Since this course was given, the previous result was proved to imply the uniqueness of the FK percolation measure on $\mathbb{Z}^2$. We refer to [27, 24] for details.

In conclusion, we can draw the following phase diagram for the FK percolation (though most of the items included in it remain conjectural).



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