SHARPNESS OF PROPER AND COCOMPACT ACTIONS ON REDUCTIVE HOMOGENEOUS SPACES

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Abstract. We prove that if G is a noncompact connected real reductive linear Lie group, then any discrete subgroup of G acting properly discontinuously and cocompactly on some homogeneous space G/H of G is quasi-isometrically embedded and sharp for G/H, i.e. satisfies a strong, quantitative form of proper discontinuity. For noncompact reductive H, this was known as the Sharpness Conjecture, with applications to spectral analysis on pseudo-Riemannian locally symmetric spaces developed in [KK2]. For G/H rational of real corank one, we use sharpness to fully characterize properly discontinuous and cocompact actions on G/H in terms of Anosov representations. This enables us to show that in real corank one, acting properly discontinuously and cocompactly on G/H is an open property, and also to prove that a number of homogeneous spaces do not admit compact quotients, such as $\mathrm{SL}(n+1,\mathbb{K})/\mathrm{SL}(n,\mathbb{K})$ for $n>1$ and $\mathbb{K}=\mathbb{R}$, $\mathbb{C}$, or the quaternions.

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1. Introduction

1.1. **Compact quotients of reductive homogeneous spaces.** Let X be a manifold equipped with a faithful and transitive action of a connected real linear Lie group G. Then X identifies with the right quotient G/H, where H is the stabilizer of a basepoint in X. A *compact quotient* of X (sometimes also called a *compact Clifford–Klein form* of X) is a closed orbifold of the form $\Gamma\backslash X$, where Γ is a discrete subgroup of G acting properly discontinuously and cocompactly on X. Up to passing to a finite-index subgroup, one

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can furthermore assume that the action of Γ is free, so that the quotient $\Gamma\backslash X$ is a closed manifold.

Compact quotients of homogeneous spaces are thus (up to a finite orbifold cover) manifolds whose universal covering has a "geometric incarnation"; they were both a motivation and a source of examples for Poincaré's invention of the notion of fundamental group.

Three fundamental questions drive the research on compact quotients of homogeneous spaces X:

Question 1.1. Does X admit compact quotients?

Question 1.2. If X admits compact quotients, are these quotients rigid?

Question 1.3. If X admits compact quotients, what is their topology/geometry?

These questions have been thoroughly investigated when the isotropy group H is compact. In this case, G preserves a Riemannian metric on X, any discrete subgroup Γ of G automatically acts properly discontinuously on X, and this action is cocompact if and only if Γ is a *uniform lattice* in G. When G is simple, Question 1.1 is thus answered positively by Borel's existence theorem for uniform lattices [Bo2], while Calabi–Weil's local rigidity theorem [We] states that any uniform lattice in G is rigid except when G is isogenous to $\mathrm{PSL}(2,\mathbb{R})$, answering Question 1.2; if K is a maximal compact subgroup of G, then $\Gamma\backslash X$ fibers over $\Gamma\backslash G/K$, which is a closed nonpositively-curved Riemannian (hence aspherical) orbifold. This already gives very strong information on the inexhaustible Question 1.3.

In contrast, the case where H is noncompact is far from understood. In this case, not all discrete subgroups of G act properly discontinuously on G/H (e.g. infinite discrete subgroups of H do not, as they have a global fixed point in G/H); determining when the action of a discrete subgroup Γ can be simultaneously properly discontinuous (which requires Γ to be "not too large") and cocompact (which requires Γ to be "not too small") is a difficult problem, which has given rise to a very rich literature (see e.g. the surveys [KY, Co] or the introduction of [Ka1]). Various obstructions have been given to the existence of compact quotients of certain homogeneous spaces G/H, but Question 1.1 remains open in general. For instance, it is known from [CM, Wo, Ko1] that if $\mathrm{rank}_{\mathbb{R}}(G) = \mathrm{rank}_{\mathbb{R}}(H)$, then only finite groups can act properly discontinuously on the reductive homogeneous space G/H (*Calabi–Markus phenomenon*). In some cases where $\mathrm{rank}_{\mathbb{R}}(G) > \mathrm{rank}_{\mathbb{R}}(H)$, only virtually cyclic or virtually abelian groups can acts properly discontinuously on G/H (see e.g. [Be1, Ka2]), and these are too small to give rise to compact quotients. In other cases certain nonabelian free groups can act properly discontinuously, but still G/H does not admit any compact quotients. The known constructions of compact quotients are often rigid, but several infinite families of homogeneous spaces admitting nonrigid compact quotients have also been found [Go, Ko4, Ka3, LM, MST]. Finally, very little is known about the topology of compact quotients. For instance, it is not known whether the fundamental group of a compact quotient of the affine space must be virtually solvable (this is the famous Auslander conjecture, see e.g. [DDGS]).

In this paper, we will focus on the case that G is reductive (and H noncompact). We will prove a general "sharpness" property of compact quotients of G/H (Theorem 1.5), which has strong applications to Questions 1.1, 1.2, and 1.3 above. Let us start by explaining what sharpness is.

1.2. **Sharpness.** A general and systematic study of compact quotients of homogeneous spaces G/H for reductive G and noncompact H was initiated by Kobayashi in the late 1980s. He and Benoist independently formulated a criterion for a subgroup Γ of G to act properly discontinuously on G/H. Let $\mathfrak{a}$ be a Cartan subspace of the Lie algebra $\mathfrak{g}$ of G, let $\mathfrak{a}^+ \subset \mathfrak{a}$ be a choice of closed positive Weyl chamber, and let $\mu : G \to \mathfrak{a}^+$ be the corresponding Cartan projection (see Section 2.1). Let $d_{\mathfrak{a}}$ be the distance function on $\mathfrak{a}$ associated to some Euclidean norm $\|\cdot\|$.

Benoist–Kobayashi's properness criterion ([Be1, Ko3])**.** *Let G be a connected real linear reductive Lie group, with Cartan projection $\mu : G \to \mathfrak{a}^+$, and let H and Γ be two closed subgroups of G. Then Γ acts properly discontinuously on G/H if and only if, for every $R > 0$, the set*

$$\{\gamma \in \Gamma \mid d_{\mathfrak{a}}(\mu(\gamma), \mu(H)) \leq R\}$$

is finite.

In her thesis [Ka1], the first-named author proved a stronger property for properly discontinuous actions on the 3-dimensional anti-de Sitter space $\mathrm{AdS}^3 = G/H = \mathrm{PO}(2,2)/\mathrm{O}(2,1) \simeq (\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}))/\mathrm{Diag}(\mathrm{PSL}(2,\mathbb{R}))$: namely, in this case, if a discrete subgroup Γ of G acts properly discontinuously and cocompactly on G/H, then the image of Γ under the Cartan projection μ escapes *linearly* from the set $\mu(H)$. This led to the introduction of the notion of *sharpness*, which should be thought of as a strong, quantitative version of proper discontinuity.

Definition 1.4 ([KK2])**.** Let G be a connected real linear reductive Lie group and H a closed subgroup of G. A discrete subgroup Γ of G is called *sharp for G/H* (and the associated quotient $\Gamma\backslash G/H$ is called *sharp*) if there exist $c, c' > 0$ such that

$$d_{\mathfrak{a}}(\mu(\gamma), \mu(H)) \geq c\,\|\mu(\gamma)\| - c' \qquad \text{for all } \gamma \in \Gamma.$$

Equivalently, Γ is sharp for G/H if the *limit cones* of Γ and H (see Section 4.1) intersect only at 0.

If Γ is sharp for G/H, then it acts properly discontinuously on G/H by the Benoist–Kobayashi properness criterion. The converse is false in general (see [GK, § 10.1]). However, the Sharpness Conjecture [KK2, Conj. 4.12] states that if Γ acts properly discontinuously *and cocompactly* on G/H, then it should be sharp for G/H. In the current paper, we prove this conjecture.

Theorem 1.5. *Let G be a connected real linear reductive Lie group and H a closed subgroup of G. Then the Sharpness Conjecture holds for G/H. More precisely, any discrete subgroup Γ of G acting properly discontinuously and cocompactly on G/H is finitely generated and* sharply embedded *in G with respect to H.*

Here we use the following terminology.

Definition 1.6. Let G be a connected real linear reductive Lie group, with Cartan projection $\mu : G \to \mathfrak{a}^+$, and let H be a closed subgroup of G. A closed subgroup Γ of G, with a compact generating subset S and associated word length function $|\cdot|_S : \Gamma \to \mathbb{N}$, is called *sharply embedded in G with respect to H* if there exist $C, C' > 0$ such that $d_{\mathfrak{a}}(\mu(\gamma), \mu(H)) \geq C\,|\gamma|_S - C'$ for all $\gamma \in \Gamma$.

Note that changing the compact set S may change the constants C, C', but not their existence, so the notion is well defined independently of S (see Section 2.6). This notion is equivalent to Γ being sharp for G/H and quasi-isometrically embedded in G, see Remark 2.8.

Theorem 1.5 was previously proven for $G/H = (G_0 \times G_0)/\mathrm{Diag}(G_0)$ with $G_0 = \mathrm{PSL}(2,\mathbb{R})$ in [Ka1] (this is the 3-dimensional anti-de Sitter space), with $G_0 = \mathrm{SO}(n,1)$ in [GK], and with G_0 a general semisimple linear group of real rank one in [GGKW1]. Note that [Ka1, GK, GGKW1] actually have weaker assumptions than Γ being cocompact. In particular, for $G_0 = \mathrm{PSL}(2,\mathbb{R})$, proper actions are sharp as soon as Γ is finitely generated. On the other hand, there are some infinitely generated groups acting properly discontinuously but not sharply on the 3-dimensional anti-de Sitter space [GK].

1.3. **Application 1: spectral analysis on pseudo-Riemannian locally symmetric spaces.** Throughout the paper, we denote by K_H a maximal compact subgroup of H and by K a maximal compact subgroup of G containing K_H.

Suppose that the homogeneous space G/H is of reductive type, i.e. the closed subgroup H has finitely many connected components and is stable under a Cartan involution of the reductive Lie group G. Then the space G/H carries a G-invariant pseudo-riemannian metric of signature $(\mathtt{d}_+, \mathtt{d}_-)$, where

$$\mathtt{d}_+ = \dim(G/K) - \dim(H/K_H) \quad \text{and} \quad \mathtt{d}_- = \dim(K_G/K_H). \tag{1.1}$$

(For semisimple G, such a metric is induced by the Killing form on the Lie algebra of G.) This pseudo-Riemannian metric descends to any quotient manifold $\Gamma\backslash G/H$ and defines a Laplacian on the quotient. We note that the principal symbol of this differential operator is the pseudo-Riemannian metric, hence it is not elliptic when H is noncompact.

In [KK1, KK2], Kobayashi and the first-named author initiated the study of the spectral theory of the Laplacian on $\Gamma\backslash G/H$. In particular, assuming G/H is symmetric, they proved [KK2, Th. 1.8] that if the discrete spectrum of the Laplacian on G/H is infinite and if Γ is sharp for G/H, then the discrete spectrum of the Laplacian on $\Gamma\backslash G/H$ is infinite; this was done by constructing generalized Poincaré series. The fact that the discrete spectrum of the Laplacian on G/H is infinite is known [F, MO] to be equivalent to the rank condition $\mathrm{rank}(G/H) = \mathrm{rank}(K/K_H)$; see [KK2, § 2] for examples. By Theorem 1.5, the fact that Γ is sharp for G/H is satisfied whenever the quotient $\Gamma\backslash G/H$ is compact.

1.4. **Application 2: openness for proper and cocompact actions.** It is expected that properly discontinuous and cocompact actions on homogeneous spaces G/H of reductive type should always be stable under small deformations.

Conjecture 1.7. Let G/H be a homogeneous space of reductive type and Γ a discrete subgroup of G acting properly discontinuously and cocompactly on G/H. Then there is a neighborhood $\mathcal{U} \subset \mathrm{Hom}(\Gamma, G)$ of the natural inclusion such that every $\rho \in \mathcal{U}$ is discrete and faithful and satisfies that the action of Γ on G/H via ρ is properly discontinuous and cocompact.

Remark 1.8. By the Ehresmann–Thurston principle, small deformations of the inclusion $\iota : \Gamma \to G$ are holonomies of $(G, G/H)$-structures on a closed manifold $\Gamma\backslash G/H$. Conjecture 1.7 would thus hold if we knew that closed manifolds locally modelled on G/H are

covered by G/H, which is a reductive version of the *Markus completeness conjecture* for affine manifolds (see [Ca, Kl]).

Conjecture 1.7 has been proven when Γ is a lattice in a connected Lie subgroup L of G [Ka3] (the so-called *standard quotients*), and also when $G/H = (G_0 \times G_0)/\mathrm{Diag}(G_0)$, with G_0 a semisimple Lie group of real rank 1 [Ka1, GK, GGKW1]. Here we use Theorem 1.5 to prove the following, where $W \coloneqq N_G(\mathfrak{a})/Z_G(\mathfrak{a})$ is the restricted Weyl group of G.

Theorem 1.9. *Conjecture 1.7 holds whenever* $\mathrm{rank}_{\mathbb{R}}(G) - \mathrm{rank}_{\mathbb{R}}(H) = 1$.

More generally, let G be a connected real linear reductive Lie group and H any closed subgroup of G such that $\mu(H)$ is the intersection of $\mathfrak{a}^+$ with the W-orbit of a hyperplane of $\mathfrak{a}$. Then for any discrete subgroup Γ of G acting properly discontinuously and cocompactly on G/H, there is a neighborhood $\mathcal{U} \subset \mathrm{Hom}(\Gamma, G)$ of the natural inclusion such that every $\rho \in \mathcal{U}$ is discrete and faithful and satisfies that the action of Γ on G/H via ρ is properly discontinuous and cocompact.

Theorem 1.9 applies in particular to some homogeneous spaces G/H where H is not reductive, and not even cocompact in a reductive subgroup of G, such as the exotic homogeneous spaces G/H with $G = \mathrm{SO}(2, 2n)$ constructed by Oh–Witte [OW1].

In order to establish Theorem 1.9, we develop further the connection between sharpness and *Anosov representations* which was initiated in [GGKW1].

1.5. **Sharpness and Anosov representations.** Anosov representations are representations of word hyperbolic groups into noncompact real semisimple (or reductive) Lie groups which have finite kernel, discrete image, and some strong dynamical properties. They were first introduced by Labourie [L], and further studied by Guichard–Wienhard [GuWi] and many others. They now play an important role in higher Teichmüller theory and in recent developments on discrete subgroups of Lie groups (see e.g. [Ka4]).

Anosov representations with values in G are defined with respect to the choice of a conjugacy class of proper parabolic subgroups of G, which is equivalent to the choice of a nonempty subset of the simple restricted roots defining the Weyl chamber $\mathfrak{a}^+$.

1.5.1. *The case where $\mu(H)$ is a union of walls.* Here is an application of Theorem 1.5 in the case that $\mu(H)$ is a union of walls of $\mathfrak{a}^+$.

Theorem 1.10. *Let G be a connected real linear reductive Lie group, $\mu : G \to \mathfrak{a}^+$ a Cartan projection, and H a closed subgroup of G such that $\mu(H) = \mathfrak{a}^+ \cap \bigcup_{\alpha\in\theta} \mathrm{Ker}(\alpha)$ for some subset $\theta \subset \Delta$ of the simple restricted roots of G. Then for any discrete subgroup Γ of G, the following conditions are equivalent:*

(1) Γ acts properly discontinuously and cocompactly on G/H,
(2) Γ is word hyperbolic, its Gromov boundary has covering dimension $\dim(G/H) - \dim(K/K_H) - 1$, and the natural inclusion $\Gamma \hookrightarrow G$ is a P_θ-Anosov representation.

Here we denote by P_θ a parabolic subgroup of type θ (with the convention that P_Δ is a minimal parabolic subgroup of G, see (4.1)).

The implication (1) $\Rightarrow$ (2) follows from Theorem 1.5, from the main result of [KLP2] (see Fact 4.2), and from Facts 1.15 and 4.3 below. The implication (2) $\Rightarrow$ (1) follows from the

Benoist–Kobayashi properness criterion and from Facts 1.15 and 4.3; it was first observed in [GGKW1].

One important property [L] of Anosov representations is that they form an open subset of $\mathrm{Hom}(\Gamma, G)$. Theorem 1.10 and this property immediately imply Conjecture 1.7 for reductive homogeneous spaces G/H such that $\mu(H)$ is a union of walls of $\mathfrak{a}^+$.

	G	H	restr. root sys. for G	θ
(i)	$\mathrm{SO}(p,q)$	$\mathrm{O}(p,q-1)$	B_q	$\{\alpha_q\}$
(ii)	$\mathrm{SU}(p,q)$	$\mathrm{U}(p,q-1)$	$(BC)_q$ (if $p>q$) or C_q	$\{\alpha_q\}$
(iii)	$\mathrm{Sp}(p,q)$	$\mathrm{Sp}(p,q-1)\times\mathrm{Sp}(1)$	$(BC)_q$ (if $p>q$) or C_q	$\{\alpha_q\}$
(iv)	$\mathrm{SO}(2p+k,2)$	$\mathrm{U}(p,1)$	B_2 (if $p=1=k+1$) or $A_1\times A_1$	$\{\alpha_1\}$
(v)	$\mathrm{SU}(2p+k,2)$	$\mathrm{Sp}(p,1)$	$(BC)_2$ (if $p=1=k+1$) or C_2	$\{\alpha_1\}$

TABLE 1. Some triples (G,H,θ) to which Theorem 1.10 applies. In example (i) we assume $p>q>1$, in examples (ii)–(iii) we assume $p\geq q>1$, and in examples (iv)–(v) we assume $p\geq 1$ and $k\geq 0$.

Theorem 1.10 applies to the examples in Table 1. In these examples, the set $\mu(H)$ is a single wall $\mathrm{Ker}(\alpha)$ of $\mathfrak{a}^+$, for some simple restricted root α of G, and $\theta=\{\alpha\}$. Discrete subgroups Γ of G acting properly discontinuously and cocompactly on G/H are known to exist in case (i) for $(p,q)\in(2\mathbb{N}^*,2)$ or $(4\mathbb{N}^*,4)$ or $(8,8)$, in case (ii) for $(p,q)\in(2\mathbb{N}^*,2)$, and in cases (iv) and (v) for $p\in\mathbb{N}^*$ and $k=0$.

We refer to Proposition 4.4 below for a version of Theorem 1.10 which holds for discrete subgroups Γ of G which are sharp for G/H but for which the quotient $\Gamma\backslash G/H$ is not necessarily compact. In that broader setting, all values of p,q,k may arise in examples (i), (ii), (iii), (iv), (v) of Table 1.

1.5.2. *Geometric aspects in one example.* A particularly interesting example is $G/H=\mathrm{SO}(2p,2)/\mathrm{U}(p,1)$. This semisimple symmetric space has a G-invariant pseudo-Riemannian metric of signature $(\mathtt{d}_+,\mathtt{d}_-)=(2p,p(p-1))$, but also a G-invariant indefinite Kähler structure (see [KK2, § 10.3]). Here $G/\{\pm\mathrm{I}\}=\mathrm{PO}(2p,2)$ is the isometry group of the $(2p+1)$-dimensional *anti-de Sitter space* $\mathrm{AdS}^{2p+1}=\mathbb{H}^{2p,1}=\mathrm{PO}(2p,2)/\mathrm{O}(2p,1)$, which is a model space for Lorentzian manifolds of constant negative sectional curvature. We denote by $P_1=P_{\{\alpha_1\}}$ the stabilizer in $G=\mathrm{SO}(2p,2)$ of an isotropic line of $\mathbb{R}^{2p,2}$.

Theorem 1.11. *Let Γ be a discrete subgroup of $\mathrm{SO}(2p,2)$, where $p\geq 1$. Then the following are equivalent:*

(1) Γ acts properly discontinuously and cocompactly on $\mathrm{SO}(2p,2)/\mathrm{U}(p,1)$,
(2) Γ is word hyperbolic, its Gromov boundary has covering dimension $2p-1$, and the natural inclusion $\Gamma\hookrightarrow\mathrm{SO}(2p,2)$ is P_1-Anosov,
(2') Γ is word hyperbolic, its Gromov boundary is a $(2p-1)$-dimensional sphere, and the natural inclusion $\Gamma\hookrightarrow\mathrm{SO}(2p,2)$ is P_1-Anosov,
(3) Γ preserves and acts cocompactly on a convex spacelike hypersurface in $\mathrm{AdS}^{2p+1}=\mathbb{H}^{2p,1}$,
(4) Γ is $\mathbb{H}^{2p,1}$-convex cocompact in the sense of [DGK1, DGK2] *and* $\mathrm{vcd}(\Gamma)=2p$.

Condition (3) states that Γ is the holonomy of a GHMC (globally hyperbolic maximal spatially compact) $(2p+1)$-dimensional Lorentzian manifold. The equivalence (2') $\Leftrightarrow$ (3) was established in [BM]. The equivalences (2) $\Leftrightarrow$ (2') $\Leftrightarrow$ (4) were established in [DGK1, DGK2]. The implication (2) $\Rightarrow$ (1) was first observed in [GGKW1]. Theorem 1.10 applied to $\mathrm{SO}(2p,2)/\mathrm{U}(p,1)$ gives the converse implication (1) $\Rightarrow$ (2).

These groups have a rich deformation theory; Barbot [Ba] proved that the space of P_1-Anosov representations $\rho : \Gamma \to \mathrm{SO}(2p,2)$ is not only open in $\mathrm{Hom}(\Gamma, \mathrm{SO}(2p,2))$, but also closed. Exotic examples (which are not deformations of $\mathrm{SO}(2p,1)$-lattices) were constructed by Lee–Marquis [LM] and by Monclair, Schlenker and the second-named author [MST]. It is proved in [MST] that if Condition (3) of Theorem 1.11 holds, then there is a natural Γ-equivariant fibration of $\mathrm{SO}(2p,2)/\mathrm{U}(p,1)$ over a strongly convex Γ-invariant spacelike hypersurface in AdS^{2p+1}. Combining this result with our Theorem 1.10 (which gives the implication (1) $\Rightarrow$ (2) of Theorem 1.11), we obtain the following.

Corollary 1.12. *Let Γ be a discrete subgroup of $G = \mathrm{SO}(2p,2)$ acting freely, properly discontinuously and cocompactly on $G/H = \mathrm{SO}(2p,2)/\mathrm{U}(p,1)$. Then $\Gamma\backslash G/H$ admits a smooth fibration over a closed aspherical manifold of dimension $\mathfrak{d}_+ = 2p$ whose fibers are left translates of K/K_H.*

1.5.3. *The general corank-one case.* In the corank-one case, even if $\mu(H)$ is not a union of walls of $\mathfrak{a}^+$, we can still prove that proper and cocompact actions on G/H yield Anosov representations in G, using Theorem 1.5, [Ka2], and [KLP2].

Proposition 1.13. *Let G be a connected real linear reductive Lie group, $\mu : G \to \mathfrak{a}^+$ a Cartan projection, and H a closed subgroup of G such that $\mu(H)$ is the intersection of $\mathfrak{a}^+$ with the W-orbit of a hyperplane in $\mathfrak{a}$ (for instance, G/H could be any homogeneous space of reductive type with $\mathrm{rank}_{\mathbb{R}}(G) - \mathrm{rank}_{\mathbb{R}}(H) = 1$). Let Γ be a discrete subgroup of G acting properly discontinuously and cocompactly on G/H. Then Γ is word hyperbolic, its Gromov boundary has covering dimension $\dim(G/H) - \dim(K/K_H) - 1$, and the natural inclusion $\Gamma \hookrightarrow G$ is a P-Anosov representation, for some proper parabolic subgroup P of G.*

Proposition 1.13 applies in particular to the cases in Table 2, where G/H is a semisimple symmetric space with $\mathrm{rank}_{\mathbb{R}}(G) - \mathrm{rank}_{\mathbb{R}}(H) = 1$. (This table is obtained from the classification of semisimple symmetric spaces of split rank one given e.g. in [OS, Table II].) Proposition 1.13 was previously proved in [Ka1, GK, GGKW1] for $G/H = (G_0 \times G_0)/\mathrm{Diag}(G_0)$ with $\mathrm{rank}_{\mathbb{R}}(G_0) = 1$ (case III_i^d of Table 2).

In general, proper and cocompact actions are not necessarily fully characterized by an Anosov property in G. However, we will see (Proposition 6.2 and Corollary 6.3) that they are essentially characterized by an Anosov property *after composing with some appropriate finite-dimensional linear representation of G*. This will enable us to prove Theorem 1.9.

Remark 1.14. When G/H has real corank one, the sharp embedding condition in G with respect to H is equivalent to the *ω-undistorted* condition introduced independently by Davalo and Riestenberg in [DR], where ω is a linear form on $\mathfrak{a}$ such that $\mu(H) = W \cdot \mathrm{Ker}(\omega) \cap \mathfrak{a}^+$. These authors also make a link with Anosov representations, and mention that as a consequence of their work ω-undistortedness is an open condition. We also prove this openness in the present paper (see Corollary 6.3), with a completely different approach.

	G	H	restr. root sys. for G	θ
I_1	$\mathrm{SO}(p,q)$	$\mathrm{O}(p,q-1)$	B_q	$\{\alpha_q\}$
I'_1	$\mathrm{SO}(p,p)$	$\mathrm{O}(p,p-1)$	D_p	$\{\alpha_{p-1}\}$ or $\{\alpha_q\}$
I_2	$\mathrm{SU}(p,q)$	$\mathrm{U}(p,q-1)$	$(BC)_q$ (if $p>q$) or C_q	$\{\alpha_q\}$
I_3	$\mathrm{Sp}(p,q)$	$\mathrm{Sp}(p,q-1)\times\mathrm{Sp}(1)$	$(BC)_q$ (if $p>q$) or C_q	$\{\alpha_q\}$
III_1	$\mathrm{SO}(2p,\mathbb{C})$	$\mathrm{SO}(2p-1,\mathbb{C})$	D_m	$\{\alpha_{p-1}\}$ or $\{\alpha_p\}$
III_i^d	$G_0\times G_0$	$\mathrm{Diag}(G_0)$	$A_1\times A_1$	$\{\alpha_1\}$ or $\{\alpha_2\}$
IV_1	$\mathrm{SO}^*(4p)$	$\mathrm{SO}^*(4p-2)\times\mathrm{SO}^*(2)$	D_m	$\{\alpha_{p-1}\}$ or $\{\alpha_p\}$
IV_1^d	$\mathrm{SO}(2p,2)$	$\mathrm{U}(p,1)$	B_2 (if $p>1$) or $A_1\times A_1$	$\{\alpha_1\}$
IV_2^d	$\mathrm{SU}(2p,2)$	$\mathrm{Sp}(p,1)$	$(BC)_2$ (if $p>1$) or C_2	$\{\alpha_1\}$
IV_3^d	$E_{6(-14)}$	$F_{4(-20)}$	$(BC)_2$	$\{\alpha_2\}$

TABLE 2. List of all semisimple symmetric spaces G/H of split rank 1 with H noncompact and $\mathrm{rank}_{\mathbb{R}}(G)-\mathrm{rank}_{\mathbb{R}}(H)=1$, up to coverings and compact factors. In case I_1 we assume $p>q\geq 1$. In case I'_1 we assume $p\geq 3$. In cases I_2, I_3, III_1, IV_1, IV_1^d, IV_2^d we assume $p\geq q\geq 1$. In case III_i^d we assume $\mathrm{rank}_{\mathbb{R}}(G_0)=1$. The last column gives a subset θ of the simple roots Δ such that Proposition 1.13 applies with $P=P_\theta$.

1.6. **Application 3: nonexistence of compact quotients.** As mentioned in Section 1.1, the existence problem for compact quotients of reductive homogeneous spaces has a long history and has given rise to a very rich literature.

In order to act properly discontinuously and cocompactly on G/H, a discrete subgroup Γ of G must have sufficiently high virtual cohomological dimension, as described by the following classical fact. Recall that by the Cartan–Iwasawa–Malcev theorem (see [Bo1]), every connected Lie group (and indeed every connected locally compact group) admits maximal compact subgroups and they are all conjugate to one another.

Fact 1.15. *Let G be a real connected Lie group and H a closed connected subgroup of G. Let K be a maximal compact subgroup of G and K_H a maximal compact subgroup of H, with $K_H\subset K$. Then for any torsion-free discrete subgroup Γ of G acting properly discontinuously on G/H, the cohomological dimension of Γ satisfies*

$$\mathrm{cd}(\Gamma)\leq \dim(G/H)-\dim(K/K_H)\ ,$$

with equality if and only if the action of Γ on G/H is cocompact.

We refer to Section 5.1 below for the notion of (virtual) cohomological dimension.

Fact 1.15 is proved in [Ko1, Cor. 5.5] for G/H of reductive type, and in [M1, Lem. 2.2] in general. We give a short proof in Section 5.1 for the reader's convenience.

On the other hand, when Γ is the image of an Anosov representation to G, the existence of a transverse boundary map from $\partial_\infty\Gamma$ into some flag variety of G gives an upper bound on the covering dimension of $\partial_\infty\Gamma$, hence on the virtual cohomological dimension of Γ (see Section 5.2). In some cases, this bound prevents Γ from acting cocompactly on G/H, and Theorem 1.10 or Proposition 1.13 then implies that G/H does not admit any compact quotients.

Corollary 1.16. *The homogeneous spaces G/H given in Table 3 do not admit any compact quotients for $k \geq 2$.*

	G	H	$\dim(G/K) - \dim(H/K_H)$
(i)	$\mathrm{SL}(2\ell, \mathbb{K})$	$\mathrm{SL}(2\ell-1, \mathbb{K})$	2ℓ ($\mathbb{K}=\mathbb{R}$), $4\ell-1$ ($\mathbb{K}=\mathbb{C}$), $8\ell-3$ ($\mathbb{K}=\mathbb{H}$)
(ii)	$\mathrm{SO}(2, 2k+k')$	$\mathrm{U}(1,k)$	$2(k+k')$
(iii)	$\mathrm{SU}(2, 2k+k')$	$\mathrm{Sp}(1,k)$	$4(k+k')$
(iv)	$\mathrm{SO}(\ell+1, \ell)$	$\mathrm{S}(\mathrm{O}(2)\times\mathrm{O}(\ell-1,\ell))$	2ℓ
(v)	$\mathrm{Sp}(2\ell, \mathbb{K})$	$\mathrm{Sp}(2\ell-2, \mathbb{K})$	2ℓ ($\mathbb{K}=\mathbb{R}$), $4\ell-1$ ($\mathbb{K}=\mathbb{C}$)
(vi)	$\mathrm{Spin}(5,3)$	$G_{2(2)}$	7
(vii)	$\mathrm{Spin}(5,4)$	$\mathrm{Spin}(4,3)$	8

TABLE 3. Some homogeneous spaces G/H that do not admit any compact quotients for $k, k' \geq 1$ and $\ell \geq 2$ (Corollary 1.16). Here $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$, or (in case (i)) the ring of quaternions. In (vii) we consider the composition of the spinor embedding $\mathrm{Spin}(4,3) \hookrightarrow \mathrm{Spin}(4,4)$ with the standard embedding $\mathrm{Spin}(4,4) \hookrightarrow \mathrm{Spin}(5,4)$.

Case (iv) was already treated in [Ko2] by a different method (note that $\mathrm{SO}(3,2)/\mathrm{S}(\mathrm{O}(2)\times\mathrm{O}(1,2)) \simeq \mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(1,1)$), but it seems that none of the other cases was previously known.

In particular, Corollary 1.16 states that $\mathrm{SO}(2,2k+1)/\mathrm{U}(1,k)$ does not admit any compact quotients for $k \geq 1$. This had been conjectured by Oh–Witte [OW2, Conj. 1.8], and implies, by [OW2, Th. 1.9], that the only connected closed subgroups H of $G = \mathrm{SO}(2,2k+1)$ for which G/H admits compact quotients are those groups H that are either compact or cocompact in G.

Corollary 1.16 also states that $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n-1,\mathbb{R})$ does not admit compact quotients for even $n \geq 4$. Note that it had been an open conjecture since the early 1990s that $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R})$ does not admit compact quotients for $n > m > 1$. The case $n > \lceil 3m/2 \rceil$ was treated by Kobayashi [Ko2] in 1992 using a maximality principle, the case $n \geq m+3 \geq 5$ was treated in a series of papers in the 1990s by Labourie, Mozes and Zimmer [Zi$_{\mathrm{R}}$, LMZ, LZ] using cocycle rigidity, the case $n = m+1$ odd was treated by Benoist [Be1] in 1996 using Lie theory, the case $(n,m) = (4,2)$ by Shalom [Sh] in 2000 using unitary representations, and the case m even by Morita [M1] and the second-named author [T2] in the mid-2010s using topological considerations. Thus the remaining open cases prior to the present paper were $n = m+1$ even and $n = m+2$ odd. Corollary 1.16 treats the case of $n = m+1$ even. It also applies to $\mathrm{SL}(n,\mathbb{K})/\mathrm{SL}(m,\mathbb{K})$ for $\mathbb{K} = \mathbb{C}$ or the quaternions.

Remark 1.17. In an independent paper with Yosuke Morita [KMT], we give a different obstruction to the existence of compact quotients, which is of a topological nature and does not involve sharpness. This enables us to prove in [KMT] that $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(m,\mathbb{R})$ does not admit compact quotients for all values of (n,m) except $(4,3)$ and $(8,7)$. The cases of $\mathrm{SL}(4,\mathbb{R})/\mathrm{SL}(3,\mathbb{R})$ and $\mathrm{SL}(8,\mathbb{R})/\mathrm{SL}(7,\mathbb{R})$ are covered by Corollary 1.16 of the present

paper; sharpness seems to be the only known obstruction to the existence of compact quotients in these cases.

We note that several cases in Table 3 are interesting because they give subgeometries without compact quotients of geometries that do admit compact quotients (see Table 4). Namely, in these cases the group G embeds into a larger simple Lie group G' and acts transitively on a G'-homogeneous space G'/H' with isotropy group $G \cap H' = H$; there exist subgroups of G' acting properly discontinuously and cocompactly on G'/H', but Corollary 1.16 shows that such subgroups cannot be contained in G. We refer to [KY] for details on the compact quotients of G'/H'. Note that G'/H' is the double cover of a pseudo-Riemannian hyperbolic space in cases (i) and (v) with $\mathbb{K} = \mathbb{R}$ and in case (vii), while it is a complex sphere in cases (i) and (v) with $\mathbb{K} = \mathbb{C}$ and in case (vi).

	G	H	G'	H'	
(i), $\mathbb{K} = \mathbb{R}$	$\mathrm{SL}(2\ell, \mathbb{R})$	$\mathrm{SL}(2\ell - 1, \mathbb{R})$	$\mathrm{SO}(2\ell, 2\ell)$	$\mathrm{SO}(2\ell, 2\ell - 1)$	$\ell = 2, 4$
(i), $\mathbb{K} = \mathbb{C}$	$\mathrm{SL}(2\ell, \mathbb{C})$	$\mathrm{SL}(2\ell - 1, \mathbb{C})$	$\mathrm{SO}(4\ell, \mathbb{C})$	$\mathrm{SO}(4\ell - 1, \mathbb{C})$	$\ell = 2$
(ii)	$\mathrm{SO}(2, 2k + 1)$	$\mathrm{U}(1, k)$	$\mathrm{SO}(2, 2k + 2)$	$\mathrm{U}(1, k + 1)$	all $k \geq 1$
(iii)	$\mathrm{SU}(2, 2k + 1)$	$\mathrm{Sp}(1, k)$	$\mathrm{SU}(2, 2k + 2)$	$\mathrm{Sp}(1, k + 1)$	all $k \geq 1$
(v), $\mathbb{K} = \mathbb{R}$	$\mathrm{Sp}(2\ell, \mathbb{R})$	$\mathrm{Sp}(2\ell - 2, \mathbb{R})$	$\mathrm{SO}(2\ell, 2\ell)$	$\mathrm{SO}(2\ell, 2\ell - 1)$	$\ell = 2, 4$
(v), $\mathbb{K} = \mathbb{C}$	$\mathrm{Sp}(2\ell, \mathbb{C})$	$\mathrm{Sp}(2\ell - 2, \mathbb{C})$	$\mathrm{SO}(4\ell, \mathbb{C})$	$\mathrm{SO}(4\ell - 1, \mathbb{C})$	$\ell = 2$
(vi)	$\mathrm{Spin}(5, 3)$	$G_{2(2)}$	$\mathrm{SO}(8, \mathbb{C})$	$\mathrm{SO}(7, \mathbb{C})$	
(vii)	$\mathrm{Spin}(5, 4)$	$\mathrm{Spin}(4, 3)$	$\mathrm{SO}(8, 8)$	$\mathrm{SO}(8, 7)$	

TABLE 4. Some quadruples (G, H, G', H') such that G embeds into G' and acts transitively on a G'-homogeneous space G'/H' with isotropy group $H = G \cap H'$. These G/H are taken from Table 3 and Corollary 1.16 states that they do not admit compact quotients. However, compact quotients of G'/H' are known to exist for the values of k, ℓ indicated in the last column.

Note that we actually prove a version of Theorem 1.5 for all *compactly generated* (not necessarily discrete) locally compact subgroups Γ of G (see Theorem 3.1 below). It has the following consequence.

Theorem 1.18. *Let G be a connected real linear reductive Lie group and H a closed subgroup of G. If G/H admits compact quotients, then H must be compactly generated and quasi-isometrically embedded in G.*

This also provides an obstruction to the existence of compact quotients of G/H for certain nonreductive subgroups H, see Section 5.4.

1.7. **Application 4: compact quotients of group manifolds of type A_2.** A *group manifold* is a semisimple Lie group G_0 seen as a homogeneous space under the action of $G_0 \times G_0$ by left and right multiplication:

$$(g, h) \cdot y = gyh^{-1} \ .$$

Pointed at the identity, it is isomorphic to the quotient G/H where $G = G_0 \times G_0$ and $H = \mathrm{Diag}(G_0)$, the diagonal embedding of G_0.

Compact quotients of group manifolds of rank 1 where the first to be thoroughly studied and well-understood (see [Ka1, GK, GGKW1]). We hope that the sharpness theorem will help understand the higher-rank case. To illustrate this, we classify compact quotients of some group manifolds of rank 2:

Theorem 1.19. *Let G_0 be a real simple Lie group with a restricted root system of type A_2 (i.e. G_0 is locally isomorphic to $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{SL}(3,\mathbb{C})$, $\mathrm{SL}(3,\mathbb{H})$, or $E_{6(-26)}$). Let Γ be a discrete subgroup of $G_0 \times G_0$ acting properly discontinuously and cocompactly on $(G_0 \times G_0)/\mathrm{Diag}(G_0)$. Then, up to finite index and to switching the two factors of $G_0 \times G_0$, we have $\Gamma = \Gamma_0 \times \{1_{G_0}\}$ where Γ_0 is a cocompact lattice in G_0.*

Note that, unlike in the rank-1 case, these compact quotients are rigid by Margulis superrigidity.

Our proof will rely crucially on the fact that, when the restricted root system is of type A_2, the opposition involution is nontrivial. For real simple Lie groups of rank 2 with trivial opposition involution, there may exist other discrete subgroups of $G_0 \times G_0$ acting properly discontinuously and cocompactly on $(G_0 \times G_0)/\mathrm{Diag}(G_0)$ (see [KK2, § 2.2]), including some that are not rigid. For instance, for $G_0 = \mathrm{SO}(2n, 2)$ we can take $\Gamma = \Gamma_1 \times \Gamma_2$ where Γ_1 is a cocompact lattice of $\mathrm{U}(n, 1)$ and Γ_2 acts properly discontinuously and cocompactly on $\mathrm{SO}(2n, 2)/\mathrm{U}(n, 1)$; such Γ_2 can be deformed e.g. by bending (see [Ka3]). This example leads us to formulate the following general conjecture; we refer to Section 4.1 for the notion of limit cone.

Conjecture 1.20. Let G_0 be any real linear simple Lie group of real rank ≥ 2 and let Γ be any discrete subgroup of $G_0 \times G_0$ acting properly discontinuously and cocompactly on $(G_0 \times G_0)/\mathrm{Diag}(G_0)$. Then, up to replacing Γ by a finite-index subgroup, we have $\Gamma = \Gamma_1 \times \Gamma_2$ where Γ_1 and Γ_2 are discrete subgroups of G_0 with disjoint limit cones and

$$\mathrm{vcd}(\Gamma_1) + \mathrm{vcd}(\Gamma_2) = \dim(G_0/K) \ .$$

InConjecture 1.20 we allow the case that one of the groups Γ_1 or Γ_2 is trivial (hence with empty limit cone); this is the so-called *standard* case appearing in Proposition 1.19.

The proof of Proposition 1.19 includes some steps towards a proof of Conjecture 1.20. In general, we are not able for the moment to rule out the possibility that both projections of Γ are faithful with dense image in G_0.

1.8. **Organization of the paper.** Section 2 is devoted to some reminders about the Cartan decomposition of a reductive Lie group G, the polar decomposition of G with respect to a reductive subgroup H, and compactly generated groups. In Section 3 we prove (a slightly more general version of) the Sharpness Theorem 1.5. In Section 4 we relate sharp embeddings to Anosov representations, and prove Theorem 1.10 and Proposition 1.13. In Section 5 we prove Corollary 1.16 about the nonexistence of compact quotients for the homogeneous spaces G/H of Table 3. In Section 6 we study further the links between sharp embeddings and Anosov representations, via linear representations of G, and deduce Theorem 1.9 stating the openness of proper and cocompact actions in the corank-one case. In Section 7 we prove a general preliminary result on discrete groups acting properly discontinuously and cocompactly on group manifolds. We use it in Section 8 to prove Theorem 1.19 about compact quotients of group manifolds of type A_2.

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2. Reminders: Cartan and polar decompositions

In the whole paper, we fix a noncompact connected real linear reductive Lie group G, with Lie algebra $\mathfrak{g}$.

2.1. **Cartan decomposition.** Let K be a maximal compact subgroup of G; it is the set of fixed points of some Cartan involution σ of G. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \mathrm{Lie}(K)$ and $\mathfrak{p}$ are respectively the fixed points and anti-fixed points of the involution $\mathrm{d}\sigma$ on $\mathfrak{g}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ (also known as a Cartan subspace of $\mathfrak{g}$), and let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the set of restricted roots of $\mathfrak{a}$ in $\mathfrak{g}$, namely the set of linear forms $\alpha \in \mathfrak{a}^*$ such that $\mathfrak{g}_\alpha := \{v \in \mathfrak{g} \mid [a, v] = \alpha(a)\, v \text{ for all } a \in \mathfrak{a}\}$ is nonzero. We choose a set Δ of simple restricted roots of $\mathfrak{a}$ in $\mathfrak{g}$, i.e. a subset of Σ such that any element of Σ can be written as a linear combination of elements of Δ with all coefficients nonnegative or all coefficients nonpositive. We denote by Σ^+ the set of those elements of Σ that can be written as a linear combination of elements of Δ with all coefficients nonnegative. The choice of Δ defines a closed positive Weyl chamber

$$\mathfrak{a}^+ := \{a \in \mathfrak{a} \mid \alpha(a) \geq 0 \text{ for all } \alpha \in \Delta\},$$

which is a closed convex cone in $\mathfrak{a}$, and a fundamental domain for the action of the restricted Weyl group $W = N_G(\mathfrak{a})/Z_G(\mathfrak{a})$ on $\mathfrak{a}$.

The Cartan decomposition $G = K \exp(\mathfrak{a}^+) K$ holds: any element $g \in G$ can be written $g = k \exp(a) k'$ for some $k, k' \in K$ and a unique $a \in \mathfrak{a}^+$. Setting $\mu(g) := a$ defines a map $\mu : G \to \mathfrak{a}^+$, called the *Cartan projection* associated to the Cartan decomposition $G = K \exp(\mathfrak{a}^+) K$. It is continuous, proper, and surjective.

Let w_0 be the longest element of the Weyl group (i.e. $w_0 \cdot \mathfrak{a}_+ = -\mathfrak{a}_+$. Then $\iota := -w_0$ is the *opposition involution* of G. For every $g \in G$, we have $\mu(g^{-1}) = \iota(\mu(g))$.

Note that μ is "strongly subadditive" in the following sense (see e.g. [Ka2, Lem. 2.1]): for any $g, g_1, g_2 \in G$,

$$\|\mu(g_1 g g_2) - \mu(g)\| \leq \|\mu(g_1)\| + \|\mu(g_2)\|. \tag{2.1}$$

2.2. **Vector-valued distance function on G/K.** Let $(\cdot, \cdot)$ be a G-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$ which is positive definite on $\mathfrak{p}$, negative definite on $\mathfrak{k}$, and for which $\mathfrak{p}$ and $\mathfrak{k}$ are orthogonal. (If G is semisimple, we can take $(\cdot, \cdot)$ to be the Killing

form of $\mathfrak{g}$.) It defines a G-invariant Riemannian metric on the symmetric space G/K. Let $\|\cdot\|$ be the Euclidean norm on $\mathfrak{p}$ associated to $(\cdot,\cdot)$.

The Cartan projection $\mu : G \to \mathfrak{a}^+$ of Section 2.1 is by construction invariant under left and right multiplication by K, and therefore defines a "vector-valued distance function" $\overrightarrow{d} : G/K \times G/K \to \mathfrak{a}^+$ on the symmetric space G/K, given by

$$\overrightarrow{d}(g \cdot x_0, g' \cdot x_0) = \mu(g^{-1}g')$$

for all $g, g' \in G$. It satisfies the following properties:

(1) $\overrightarrow{d}$ is G-invariant: $\overrightarrow{d}(gx, gx') = \overrightarrow{d}(x, x')$ for all $x, x' \in G/K$ and all $g \in G$,
(2) $\overrightarrow{d}(x', x) = \iota(\overrightarrow{d}(x, x'))$, where $\iota : \mathfrak{a}^+ \to \mathfrak{a}^+$ is the opposition involution,
(3) G acts 2-transitively on $(G/K, \overrightarrow{d})$: for $x_1, x_2, x_1', x_2' \in G/K$, we have $\overrightarrow{d}(x_1, x_2) = \overrightarrow{d}(x_1', x_2')$ if and only if there exists $g \in G$ such that $x_1' = gx_1$ and $x_2' = gx_2$,
(4) $\|\overrightarrow{d}\|$ is the symmetric Euclidean distance: for any $x, x' \in G/K$,

$$d_{G/K}(x, x') = \|\overrightarrow{d}(x, x')\|.$$

Recall that the opposition involution $\iota : \mathfrak{a} \to \mathfrak{a}$ is given by $\iota(a) = -w_0 \cdot a$ for all $a \in \mathfrak{a}$, where w_0 is the longest element of the Weyl group W.

2.3. **Cartan decomposition for H.** From now on (until the end of this Section 2), we fix a closed connected subgroup H of G which is stable under the Cartan involution σ of G. Then H is itself a real linear reductive Lie group, with Cartan involution the restriction of σ, with maximal compact subgroup $K_H := K \cap H$, and the Lie algebra $\mathfrak{h}$ admits the Cartan decomposition $\mathfrak{h} = (\mathfrak{k} \cap \mathfrak{h}) + (\mathfrak{p} \cap \mathfrak{h})$. The pair (G, H) is called a *reductive pair* and the homogeneous space G/H is said to be *of reductive type*.

Let $\mathfrak{a}_H$ be a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{h}$. We may and shall choose the maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ from Section 2.1 to contain $\mathfrak{a}_H$. The Cartan decomposition of H states that every $h \in H$ can be written $h = k \exp(a) k'$ for some $k, k' \in K_H$ and some a in (some H-Weyl chamber of) $\mathfrak{a}_H$; while the vector a need not be contained in $\mathfrak{a}^+$, there exists an element $w \in W$ such $w \cdot a \in \mathfrak{a}^+$ and we have $\mu(h) = w \cdot a$. Thus the image of H under the Cartan projection $\mu : G \to \mathfrak{a}^+$ is

$$\mu(H) = \mathfrak{a}^+ \cap \bigcup_{w \in W} w \cdot \mathfrak{a}_H, \tag{2.2}$$

i.e. $\mu(H)$ is the intersection of $\mathfrak{a}^+$ with the finitely many linear subspaces $w \cdot \mathfrak{a}_H$ of $\mathfrak{a}$.

2.4. **A duality between G/K and G/H.** Throughout this section, we will denote points in the Riemannian symmetric space G/K with the letter x and points in the homogeneous space G/H with the letter y. In particular,

- x_0 denotes the class of $\mathbf{1}_G$ in G/K, i.e. the point fixed by K,
- y_0 denotes the class of $\mathbf{1}_G$ in G/H, i.e. the point fixed by H.

As mentioned in Section 2.2, the G-invariant nondegenerate symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$, restricted to $\mathfrak{p} = \mathfrak{k}^\perp \simeq T_{x_0} G/K$, extends to a G-invariant Riemannian metric on G/K which is *symmetric* in the sense of Cartan and has nonpositive sectional curvature.

We denote by $d_{G/K}$ the associated distance function. On the other hand, the restriction of $(\cdot,\cdot)$ to $\mathfrak{q} \coloneqq \mathfrak{h}^\perp \simeq T_{y_0}G/H$, extends to a G-invariant *pseudo-Riemannian* metric $g_{G/H}$ on G/H, of signature $(\mathtt{d}_+, \mathtt{d}_-)$ as in (1.1).

The K-orbit $\mathcal{K}_0 \coloneqq K \cdot y_0$ in G/H is a totally geodesic timelike subspace of dimension $\mathtt{d}_-$ isomorphic to K/K_H. Its stabilizer in G is the group K, hence we can identify G/K with the space of G-translates of K/K_H. We call these translates *maximal timelike subspaces*. given $x = g \cdot x_0 \in G/K$, we denote by $\mathcal{K}_x \coloneqq g \cdot \mathcal{K}_0$ the corresponding maximal timelike subspace.

Dually, the H-orbit $\mathcal{H}_0 \coloneqq H \cdot x_0$ in G/K is a totally geodesic subspace of G/K isomorphic to H/K_H, and the space G/H can be identified with the space of translates of this subspace. Given $y = g \cdot y_0 \in G/H$, we denote by $\mathcal{H}_y \coloneqq g \cdot \mathcal{H}_0$ the corresponding subspace. For $x \in G/K$ and $y \in G/H$, we then have the following equivalence:

$$x \in \mathcal{H}_y \iff y \in \mathcal{K}_x \ . \tag{2.3}$$

2.5. **Polar decomposition with respect to H.** As above, let $\mathfrak{p}$ and $\mathfrak{q}$ be the orthogonal complements in $\mathfrak{g}$ of $\mathfrak{k}$ and $\mathfrak{h}$, respectively, with respect to the G-invariant nondegenerate symmetric bilinear form $(\cdot,\cdot)$. Then $\mathfrak{p} \cap \mathfrak{q}$ can be identified with both :

- the orthogonal of $T_{x_0}\mathcal{H}_0$ in $T_{x_0}G/K$,
- the orthogonal of $T_{y_0}\mathcal{K}_0$ in $T_{y_0}G/H$.

The following polar decomposition is well known when G/H is symmetric; we refer to [Ko1, Lem. 2.7] for a proof in the general case.

Fact 2.1. *Any $g \in G$ can be written as $g = k\exp(b)h$ for some $k \in K$, some $b \in \mathfrak{p} \cap \mathfrak{q}$, and some $h \in H$. Moreover, this decomposition is unique up to K_H: if $k\exp(b)h = k'\exp(b')h'$ where $k, k' \in K$, $b, b' \in \mathfrak{p} \cap \mathfrak{q}$, and $h, h' \in H$, then there exists $\ell \in K_H$ such that $k' = k\ell^{-1}$, $b' = \operatorname{Ad}\ell(b)$, and $h' = \ell h$.*

If $g = k\exp(b)h$ as above, we denote by $\nu(g)$ the class of b in $K_H\backslash(\mathfrak{p} \cap \mathfrak{q})$ (which is well-defined by the second part of Fact 2.1) and by $\|\nu(g)\|$ its norm. By construction, $\|\nu(\cdot)\|$ is invariant under left multiplication by K and right multiplication by H.

Remark 2.2. If $H = K$, then $\mathfrak{q} = \mathfrak{p}$ and $K_H\backslash\mathfrak{p} = \mathfrak{a}^+$, and $\mu = \nu$. More generally, suppose G/H is *symmetric*, i.e. H is the group of fixed points of some involution τ of G (which commutes with our chosen Cartan involution σ). Then the space $K_H\backslash(\mathfrak{p} \cap \mathfrak{q})$ of K_H-orbits of $\mathfrak{p} \cap \mathfrak{q}$ identifies with a closed Weyl chamber $\mathfrak{b}^+$ in a Cartan subspace $\mathfrak{b}$ of the Lie subalgebra $\mathfrak{g}^{\sigma\tau}$ of $\mathfrak{g}$ consisting of those vectors fixed by the derivative of $\sigma\tau$ at $\mathbf{1}_G$.

We define a function $D : G/K \times G/H \to \mathbb{R}_{\geq 0}$ by setting, for any $x = g \cdot x_0 \in G/K$ and any $y = g' \cdot y_0 \in G/H$,

$$D(x,y) = \|\nu(g^{-1}g')\|. \tag{2.4}$$

The function D is invariant under the diagonal action of G. It admits the following dual interpretations. Consider again $x = g \cdot x_0 \in G/K$ and $y = g' \cdot y_0 \in G/H$, and (using Fact 2.1) write $g^{-1}g' = k\exp(b)h$ where $k \in K$, $b \in \mathfrak{p} \cap \mathfrak{q}$ and $h \in H$.

Riemannian interpretation of D: The curve $c = (\exp(-tb) \cdot x_0)_{t\in[0,1]}$ is a geodesic segment in G/K starting at x_0 and ending at $\exp(-b) \cdot x_0$, orthogonal to the symmetric subspace

$\mathcal{H}_0$, of length $\|b\| = \|\nu(g^{-1}g')\|$. Because of the relation

$$g'h^{-1} = gk\exp(b)\ ,$$

we get that $g'h^{-1}\cdot c$ is a geodesic segment of length $\|\nu(g^{-1}g')\|$ between

$$g'h^{-1}\exp(-b)\cdot x_0 = x$$

and

$$g'h^{-1}\cdot x_0 \in g'\mathcal{H}_0 = \mathcal{H}_y$$

which is orthogonal to $\mathcal{H}_y$ at its endpoint. Hence $g'h^{-1}\cdot x_0$ is the orthogonal projection of x to $\mathcal{H}_y$ and

$$D(x,y) = d_{G/K}(x,\mathcal{H}_y)\ .$$

Pseudo-Riemannian interpretation of D: Dually, the curve $c' = (\exp(tb)\cdot y_0)_{t\in[0,1]}$ is a spacelike geodesic segment in G/H starting at y_0, ending at $\exp(b)\cdot y_0$ and orthogonal to the maximal timelike subspace $\mathcal{K}_0$ at y_0. Hence the curve $gk\cdot c'$ is a spacelike geodesic segment between

$$gk\cdot y_0 \in g\cdot\mathcal{K}_0 = \mathcal{K}_x$$

and

$$gk\exp(b)\cdot y_0 = g'h^{-1}\cdot y_0 = y\ ,$$

orthogonal to $\mathcal{K}_x$ at $gk\cdot y_0$. The point $gk\cdot y_0$ is thus the orthogonal projection $\pi_{\mathcal{K}_x}(y)$ of y to $\mathcal{K}_x$, and $D(x,y)$ is the length of the spacelike segment between y and $\pi_{\mathcal{K}_x}(y)$.

Remark 2.3. The projection $\pi_{\mathcal{K}_x}(y)$ is well-defined because k is defined up to right multiplication in K_H. However it does not have an obvious metric interpretation as in the Riemannian case. In fact, among the points of $\mathcal{K}_x$ which are connected to y by a spacelike geodesic segment, $\pi_{\mathcal{K}_x}(y)$ *maximizes* the length of this segment.

2.6. **Compactly generated groups.** Here we recall that the notions of word length and quasi-isometric embedding can be defined, not only for finitely generated groups, but more generally for any compactly generated locally compact group.

A locally compact topological group Γ is *compactly generated* if it is generated by a compact subset. Given a compact generating subset S, which we assume symmetric (i.e. $S = S^{-1}$) and containing a neighborhood of the identity element of Γ, one can define a word length function and a left-invariant distance function on Γ by setting, as usual,

$$|\gamma|_S \coloneqq \inf\{n\in\mathbb{N} \mid \gamma\in S^n\}$$

and

$$d_S(\gamma,\gamma') \coloneqq |\gamma^{-1}\gamma'|_S$$

for all $\gamma,\gamma'\in\Gamma$. (Here S^n denotes the set of all possible products of n elements of S, and we use the convention $S^0 = \{1\}$.)

If S' is another compact generating subset of Γ, then S' is contained in $\bigcup_{n\in\mathbb{N}} S^n$ and, since S^n is contained in the interior of $\S^{n+1}$, there exists an integer n such that $S'\subset S^n$ by the Borel–Lebesgue property. Conversely (assuming S' is also symmetric and contains a neighborhood of the identity), there exists an integer m such that $S\subset S'^m$. It follows that

$$\frac{1}{n}d_{S'} \le d_S \le m d_{S'}\ .$$

Thus, the quasi-isometric class of (Γ, d_S) does not depend on the choice of S and it makes sense to talk about quasi-isometric embeddings of Γ without referring to an explicit compact generating set.

When Γ is a discrete group, a compact generating set S is finite and we recover the usual word length and distance functions associated to the Cayley graph $\mathrm{Cay}(\Gamma, S)$. If Γ is not discrete, then the distance function d_S is not compatible with the topology of Γ (since every nontrivial element of Γ is at distance at least 1 from the identity element of Γ). This is not an issue, as d_S is meant to capture the "large scale geometry" of Γ.

The Švarc–Milnor lemma generalizes to the setting of compactly generated groups:

Fact 2.4. *Let Γ be a compactly generated locally compact topological group acting properly and cocompactly by isometries on a proper geodesic metric space X. Then the map*

$$\begin{array}{ccc} \Gamma & \to & X \\ \gamma & \mapsto & \gamma \cdot x \end{array}$$

is a quasi-isometry for any basepoint $x \in X$.

In particular, if Γ is a connected Lie group equipped with a left-invariant Riemannian metric g with associated distance function d_g, then Γ acts properly and cocompactly on itself by isometries, and we deduce the following.

Corollary 2.5. *Let H be a connected Lie group, S a symmetric compact generating subset of H, and g a left-invariant Riemannian metric on H. Then (H, d_S) is quasi-isometric to (H, d_g).*

Another consequence of the Švarc–Milnor lemma is the following.

Corollary 2.6. *Let H be a connected Lie group and H' a Lie subgroup such that $H' \backslash H$ is compact. Then H' is quasi-isometric to H.*

In particular, a noncompact connected real linear reductive Lie group is quasi-isometric to any of its parabolic subgroups. For instance, the affine group of the real line is quasi-isometric to $\mathrm{PSL}(2, \mathbb{R})$, hence to the hyperbolic plane.

We shall use the following, which is a consequence of the compactness of K and of the interpretation $\|\mu(g)\| = d_{G/K}(x_0, g \cdot x_0)$ (see Section 2.2).

Fact 2.7. *Let G be a noncompact connected real linear reductive Lie group, and let $\mu : G \to \mathfrak{a}^+$ be a Cartan projection and $\|\cdot\|$ a norm on $\mathfrak{a}$ as in Sections 2.1–2.2. Then for any closed subgroup Γ of G with a compact generating subset S, the following are equivalent:*

- *the natural inclusion $\Gamma \hookrightarrow G$ is a quasi-isometric embedding;*
- *there exist $C, C' > 0$ such that $\|\mu(\gamma)\| \geq C\,|\gamma|_S - C'$ for all $\gamma \in \Gamma$.*

Note that, by the interpretation $\|\mu(g)\| = d_{G/K}(x_0, g \cdot x_0)$ and the triangle inequality, if Γ is a closed subgroup of G with a compact generating subset S, then

$$\|\mu(\gamma)\| \leq \kappa\,|\gamma|_S$$

for all $\gamma \in \Gamma$, where $\kappa \coloneqq \max_{s \in S} \|\mu(s)\|$. From this and Fact 2.7 we deduce the following:

Remark 2.8. Let Γ be a closed subgroup of G with a compact generating subset S, and let H be another closed subgroup of G. Then Γ is sharply embedded in G with respect to H

(Definition 1.6) if and only if Γ is sharp for G/H (Definition 1.4) and quasi-isometrically embedded in G.

3. Proof of the Sharpness Theorem

In this section we prove the following generalization of Theorem 1.5, where Γ does not need to be discrete in G. We refer to Definition 1.6 and Section 2.6 for the notion of sharp embedding.

Theorem 3.1. *Let G be a connected real linear reductive Lie group, and H, Γ two closed subgroups of G. If Γ acts properly and cocompactly on G/H, then Γ is compactly generated and sharply embedded in G with respect to H.*

It is sufficient to prove Theorem 3.1 for H connected and stable under some Cartan involution of G (i.e. G/H of reductive type). Indeed, for general H, we use the fact that Γ acts properly and cocompactly on G/H if and only if $\Gamma \times H$ acts properly and cocompactly on $(G \times G)/\mathrm{Diag}(G)$, which is of reductive type. We can apply Theorem 3.1 to $(G \times G, \mathrm{Diag}(G), \Gamma \times H)$ and conclude thanks to the following elementary fact.

Lemma 3.2. *Let Γ and H be two closed subgroups of G. Then $\Gamma \times H$ is sharply embedded in $G \times G$ with respect to $\mathrm{Diag}(G)$ if and only if H is quasi-isometrically embedded in G and Γ is sharply embedded in G with respect to H.*

Proof. Choosing a compact generating subset S of $\Gamma \times H$ of the form $S' \times S''$. Then

$$|(\gamma, h)|_S = |\gamma|_{S'} + |h|_{S''}$$

for all $(\gamma, h) \in \Gamma \times H$. It follows that $\Gamma \times H$ is quasi-isometrically embedded in $G \times G$ if and only if Γ and H are both quasi-isometrically embedded in G.

Moreover, the limit cone $\mathcal{L}_{\Gamma \times H}$ of $\Gamma \times H$ in $\mathfrak{a}_+ \times \mathfrak{a}_+$ (see Section 4.1) is the product $\mathcal{L}_\Gamma \times \mathcal{L}_H$, which avoids $\mathrm{Diag}(\mathfrak{a}_+) = \mu(\mathrm{Diag}(G))$ if and only if $\mathcal{L}_\Gamma$ avoids $\mathcal{L}_H$. □

For the rest of this section, we assume H connected and stable under a Cartan involution σ of G. We use the notation of Section 2.

3.1. Spacelike gap between maximal compact subspaces.

Let us introduce the function $\delta : G/K \times G/K \to \mathbb{R}_{\geq 0}$ given by

$$\delta(x, x') \coloneqq d_{\mathfrak{a}}\Big(\overrightarrow{d}(x, x'), \mu(H)\Big) \tag{3.1}$$

for all $x, x' \in G/K$, where $\overrightarrow{d} : G/K \times G/K \to \mathfrak{a}^+$ is the vector-valued distance function of Section 2.2 and

$$d_{\mathfrak{a}}(u, \mu(H)) = \inf_{h \in H} \|u - \mu(h)\| = \min_{w \in W} d_{\mathfrak{a}}(u, w \cdot \mathfrak{a}_H).$$

Note that δ is G-invariant since $\overrightarrow{d}$ is. Moreover, we have $\delta(x, x') = \delta(x', x)$ for all $x, x' \in G/K$ because $\overrightarrow{d}(x', x) = \iota(\overrightarrow{d}(x, x'))$ and the opposition involution $\iota : \mathfrak{a}^+ \to \mathfrak{a}^+$ is an isometry for $d_{\mathfrak{a}}$ and preserves $\mu(H)$.

Our goal is to prove that if Γ acts properly and cocompactly on G/H, then Γ is compactly generated and there exist $C, C' > 0$ such that

$$\delta(\gamma_1 \cdot o, \gamma_2 \cdot o) \geq C\, d_S(\gamma_1, \gamma_2) - C'$$

for all $\gamma_1, \gamma_2 \in \Gamma$, where $d_S : (\gamma_1, \gamma_2) \mapsto |\gamma_2^{-1}\gamma_1|_S$ is the distance function on Γ associated to the choice of a compact generating subset S.

Remark 3.3. When H is compact, $\mu(H) = \{0\}$ and δ is simply the distance function $d_{G/K}$. The group Γ acts properly discontinuously and cocompactly on G/H if and only if it is a uniform lattice in G, and Theorem 1.5 in this particular case simply states that a uniform lattice is quasi-isometrically embedded in G, which follows from the Švarc–Milnor lemma.

Our first key lemma is the following interpretation of δ as a *spacelike gap* between maximal compact subspaces on G/H. Recall the function $D : G/K \times G/H \to \mathbb{R}_{\geq 0}$ from Section 2.5.

Lemma 3.4. *Let $x = g \cdot x_0$ and $x' = g' \cdot x_0$ be two points in G/K. Then*

$$\delta(x, x') = \inf_{k \in K} d_{G/K}(x, g'k \cdot \mathcal{H}_0) = \inf_{y \in \mathcal{K}_{x'}} D(x, y).$$

Proof. To simplify notation, we set $d_{G/K}(x, g'K \cdot \mathcal{H}_0) \coloneqq \inf_{k \in K} d_{G/K}(x, g'k \cdot \mathcal{H}_0)$.

Let us prove the first equality. Since δ and $d_{G/K}$ are both G-invariant, we may assume $g' = \mathbf{1}_G$, hence $x' = x_0$. For any $h \in H$, we have

$$\begin{aligned} d_{G/K}(g \cdot x_0, K \cdot \mathcal{H}_0) &\leq d_{G/K}(g \cdot x_0, Kh \cdot x_0) = d_{G/K}(\exp(\mu(g)) \cdot x_0, K \exp(\mu(h)) \cdot x_0) \\ &\leq d_{G/K}(\exp(\mu(g)) \cdot x_0, \exp(\mu(h)) \cdot x_0) = d_{\mathfrak{a}}(\mu(g), \mu(h)). \end{aligned}$$

Taking the infimum over all $h \in H$, we get

$$d_{G/K}(g \cdot x_0, K \cdot \mathcal{H}_0) \leq d_{\mathfrak{a}}(\mu(g), \mu(H)) = \delta(x, x_0)\ .$$

On the other hand, for any $g, g' \in G$ we have $d_{\mathfrak{a}}(\mu(g), \mu(g')) \leq d_{G/K}(g \cdot x_0, g' \cdot x_0)$ (see (2.1)). In particular, for any $h \in H$ and any $k \in K$, we have $d_{\mathfrak{a}}(\mu(g), \mu(h)) \leq d_{G/K}(g \cdot x_0, kh \cdot x_0)$, and so $\delta(x, x_0) = d_{\mathfrak{a}}(\mu(g), \mu(H)) \leq d_{G/K}(x, K \cdot \mathcal{H}_0)$.

We now prove the second equality. The set $g'K \cdot H_0$ is the union of all G-translates of $\mathcal{H}_0$ passing through x'. By (2.3), these are exactly the $\mathcal{H}_y$ with $y \in \mathcal{K}_{x'}$. We deduce that

$$\delta(x, x') = \inf_{y \in \mathcal{K}_{x'}} d_{G/K}(x, \mathcal{H}_y) = \inf_{y \in \mathcal{K}_{x'}} D(x, y). \qquad \square$$

Remark 3.5. Using the second interpretation of $D(x, y)$ in Section 2.5 as a "spacelike distance" between y and $\mathcal{K}_x$, the second equality interprets $\delta(x, x')$ as the infimum of spacelike distances from a point $y \in \mathcal{K}_{x'}$ to $\mathcal{K}_x$. In this sense, $\delta(x, x')$ measures a "spacelike gap" between $\mathcal{K}_x$ and $\mathcal{K}_{x'}$. In particular, we have

$$\delta(x, x') = 0 \iff \mathcal{K}_x \cap \mathcal{K}_{x'} \neq \emptyset.$$

3.2. **The function Δ_Γ.** We now pull back the function δ to a Γ-invariant function Δ_Γ on G/H. For this, we remark that the set of maximal timelike subspaces passing through a point x is contractible, so we can chose such a space for every x in a continuous and Γ-equivariant way. Let us give more details.

We identify G/K_H with the space of pairs $(x, y) \in G/K \times G/H$ such that $y \in \mathcal{K}_x$. The projections to the first and second factor are respectively given by the natural G-equivariant fibrations $\pi_H : G/K_H \to G/H$ with fiber H/K_H, and $\pi_K : G/K_H \to G/K$ with fiber K/K_H.

Let Γ be a discrete subgroup of G acting freely, properly discontinuously and cocompactly on G/H. The fibrations above induce fibrations $\pi_H : \Gamma\backslash G/K_H \to \Gamma\backslash G/H$ and $\pi_K : \Gamma\backslash G/K_H \to \Gamma\backslash G/K$. Since the fibers H/K_H of π_H are contractible, we can find a smooth section $\Gamma\backslash G/H \to \Gamma\backslash G/K_H$ of π_H, which lifts to a smooth Γ-equivariant map $G/H \to G/K_H$. Composing with π_K yields a Γ-equivariant map $\varphi : G/H \to G/K$ with the property that for any $y \in G/H$ we have $y \in \mathcal{K}_{\varphi(y)}$, or equivalently $\varphi(y) \in \mathcal{H}_y$. We use φ to define a function $\Delta_\Gamma : G/H \times G/H \to \mathbb{R}_{\geq 0}$ by

$$\Delta_\Gamma(y_1, y_2) \coloneqq \delta(\varphi(y_1), \varphi(y_2)). \tag{3.2}$$

Since δ is G-invariant and symmetric in its arguments, the function Δ_Γ is Γ-invariant and satisfies $\Delta_\Gamma(y_1, y_2) = \Delta_\Gamma(y_2, y_1)$ for all $y_1, y_2 \in G/H$.

We will prove that the function Δ_Γ is coarsely lower bounded by any Γ-invariant distance on G/H, and conclude the proof using the Svarč–Milnor lemma. The key properties of Δ_Γ that make this approach possible are given by the following Lemmas 3.6 and 3.7.

Lemma 3.6. *For any compact set $\mathcal{C}$ of G/H, the set*

$$\{y \in G/H \mid \exists y' \in \mathcal{C},\ \Delta_\Gamma(y, y') \leq 1\}$$

is compact.

Proof. Let $\mathcal{C}$ be a compact subset of G/H. Let us set

$$\mathcal{C}' \coloneqq \bigcup_{x \in \mathcal{C}} \{y \in G/H \mid D(\varphi(x), y) \leq 1\} \quad \text{and} \quad \mathcal{C}'' = \bigcup_{x \in \mathcal{C}} \{y \in G/H \mid \Delta_\Gamma(x, y) \leq 1\}\ .$$

In order to prove the compactness of $\mathcal{C}''$ stated by the lemma, we first prove the compactness of $\mathcal{C}'$.

Compactness of $\mathcal{C}'$: Let (y_n) be a sequence in $\mathcal{C}'$ and $(y'_n) \in \mathcal{C}^{\mathbb{N}}$ be such that $D(\varphi(y'_n), y_n) \leq 1$. By Fact 2.1, we can write $y_n = k_n \exp(b_n) \cdot y_0$ with $b_n \in \mathfrak{p} \cap \mathfrak{q}$ and we have $\|b_n\| = D(x_0, y_n)$. Using the Riemannian interpretation of δ, we get

$$\begin{aligned} \delta(x_0, y_n) &= d_{G/K}(x_0, \mathcal{H}_{y_n}) \\ &\leq d_{G/K}(\varphi(y'_n), \mathcal{H}_{y_n}) + d_{G/K}(o, \varphi(y'_n)) \\ &\leq 1 + \sup_{y' \in \mathcal{C}} d_{G/K}(x_0, \varphi(y)) \end{aligned}$$

which is finite because $\mathcal{C}$ is compact and φ is continuous. Hence $\|u_n\|$ is bounded and y_n has an accumulation point, proving that $\mathcal{C}'$ is compact.

Compactness of $\mathcal{C}''$: Because Γ acts cocompactly on G/H, up to enlarging $\mathcal{C}$, we can assume that

$$\bigcup_{\gamma \in \Gamma} \gamma \cdot \mathcal{C} = G/H\ .$$

By properness of the action of Γ on G/H, the set $S \coloneqq \{\gamma \in \Gamma \mid \gamma \cdot \mathcal{C}' \cap \mathcal{C}' \neq \emptyset\}$ is compact. Note that $\mathcal{C}'$ contains $\bigcup_{y' \in \mathcal{C}} \mathcal{K}_{\varphi(y')}$.

Let y be a point in $\mathcal{C}''$. By Lemma 3.4, there exists $y' \in \mathcal{K}_{\varphi(y)}$ such that $y' \in \mathcal{C}'$. Let $\gamma \in \Gamma$ be such that $\gamma \cdot y \in \mathcal{C}$. Then

$$\gamma \cdot y' \in \mathcal{K}_{\gamma \cdot \varphi(y)} = \mathcal{K}_{\varphi(\gamma \cdot y)} \subset \mathcal{C}' \ ,$$

hence γ belongs to the compact set S. We conclude that $\mathcal{C}''$ is contained in $\bigcup_{\gamma \in S} \gamma^{-1} \cdot \mathcal{C}$, which is compact. □

Lemma 3.7. *For any $y, y' \in G/H$ and any $t \in [0,1]$, there exists $z_t \in G/H$ such that $\Delta_\Gamma(y, z_t) \leq t\, \Delta_\Gamma(y, y')$ and $\Delta_\Gamma(z_t, y') \leq (1-t)\, \Delta_\Gamma(y, y')$.*

Proof. Let us choose $g, g' \in G$ such that $\varphi(y) = g \cdot x_0$ and $\varphi(y') = g' \cdot x_0$ respectively. By definition (3.1) of δ, there exists $w \in W$ such that

$$\Delta_\Gamma(y, y') = \delta(\varphi(y), \varphi(y')) = d_{\mathfrak{a}}\big(w \cdot \mu(g^{-1}g'), \mathfrak{a}_H\big) \ .$$

We can thus write

$$w \cdot \mu(g^{-1}g') = b + a \ ,$$

with $a \in \mathfrak{a}_H$ and $b \in \mathfrak{a} \cap \mathfrak{a}_H^\perp \subset \mathfrak{p} \cap \mathfrak{q}$ satisfying $\|b\| = \Delta_\Gamma(y, y')$.

Consider $k, k' \in K$ such that $g^{-1}g' = k \exp(w \cdot \mu(g^{-1}g'))\, k'$, and let us set

$$z_t \coloneqq gk \exp(t\mu(g^{-1}g')) \cdot y_0 = gk \exp(tb) \cdot y_0 \ .$$

Note that $z_0 \in gK \cdot y_0 = \mathcal{K}_{\varphi(y)}$ and $z_1 \in g'K \cdot y_0 = \mathcal{K}_{\varphi(y')}$. Moreover, since b belongs to $\mathfrak{p} \cap \mathfrak{q}$, we have

$$D(\varphi(y), z_1) = \|b\| = \delta(\varphi(y), \varphi(y')) = \inf_{z \in \mathcal{K}_{y'}} D(\varphi(y), z) \ .$$

In other words, z_1 is the point of $\mathcal{K}_{y'}$ minimizing its spacelike distance to $\mathcal{K}_y$, the point z_0 is its projection to $\mathcal{K}_y$, and $(z_t)_{t \in [0,1]}$ is the spacelike geodesic segment between them.

From the expression

$$z_t = gk \exp(tb) \cdot y_0$$

with $b \in \mathfrak{p} \cap \mathfrak{q}$ we deduce that

$$D(\varphi(y), z_t) = t\|b\| = t\Delta_\Gamma(y, y') \ .$$

Hence

$$\Delta_\Gamma(y, z_t) = \delta(\varphi(y), \varphi(z_t)) = \inf_{z \in \mathcal{K}_{\varphi(z_t)}} D(y, z) \leq D(y, z_t) \leq t\Delta_\Gamma(y, y'),$$

where the second equality follows from Lemma 3.4.

On the other hand, we can rewrite

$$\begin{aligned} z_t &= gk_1 \exp(tb) \cdot y_0 \\ &= gk_1 \exp(b + a) k_2 k_2^{-1} \exp((t-1)b - a) \cdot y_0 \\ &= g'k_2^{-1} \exp((t-1)b) \cdot y_0 \ , \end{aligned}$$

from which we deduce that

$$D(\varphi(y'), z_t) = (1-t)\|b\| = (1-t)\Delta_\Gamma(y, y') \ .$$

We conclude as above that $\Delta_\Gamma(y', z_t) \leq (1-t)\Delta_\Gamma(y, y')$. □

3.3. **A Švarc–Milnor lemma for Δ_Γ.** We can finally prove that Δ_Γ grows at least linearly with some Γ-invariant distance function on G/H and conclude the proof of Theorem 3.1 (hence of Theorem 1.5).

Proposition 3.8. *For any continuous Γ-invariant distance function $d_{G/H}$ on G/H, there exists $R \geq 1$ such that $\Delta_\Gamma(x,y) \geq R^{-1}\, d_{G/H}(x,y) - 1$ for all $x, y \in G/H$.*

Proof. Let $\mathcal{C}$ be a compact subset of G/H with $\Gamma \cdot \mathcal{C} = G/H$. By Lemma 3.6, there exists $R > 0$ such that $d_{G/H}(x,y) \leq R$ for all $x \in \mathcal{C}$ and all y with $\Delta_\Gamma(x,y) \leq 1$. By Γ-invariance of $d_{G/H}$ and Δ_Γ this inequality remains valid for every x, y with $\Delta_\Gamma(x,y) \leq 1$.

Consider $x, y \in G/H$, and let $N \in \mathbb{N}$ be the integer part of $\Delta_\Gamma(x,y)$. We set $x_0 = x$ and $x_{N+1} = y$. By Lemma 3.7, there exists $x_1 \in G/H$ such that $\Delta_\Gamma(x_0, x_1) \leq 1$ and $\Delta_\Gamma(x_1, x_{N+1}) \leq \Delta_\Gamma(x_0, x_{N+1}) - 1$. Applying Lemma 3.7 again, we find, by induction, points $x_2, \dots, x_N \in G/H$ such that $\Delta_\Gamma(x_i, x_{i+1}) \leq 1$ and $\Delta_\Gamma(x_{i+1}, x_{N+1}) \leq \Delta_\Gamma(x_i, x_{N+1}) - 1 \leq \Delta_\Gamma(x_0, x_{N+1}) - (i+1)$ for all i. In particular, $\Delta_\Gamma(x_N, x_{N+1}) \leq 1$, and so $\Delta_\Gamma(x_i, x_{i+1}) \leq 1$ for all $0 \leq i \leq N$. We then have $d_{G/H}(x_i, x_{i+1}) \leq R$ for all i, hence, by triangle inequality,

$$d_{G/H}(x,y) = d_{G/H}(x_0, x_{N+1}) \leq (N+1)R \leq \Delta_\Gamma(x,y)\, R + R\ .$$

□

Proof of Theorem 3.1. We can assume without loss of generality that the map φ satisfies $\varphi(y_0) = x_0$. We thus have

$$\Delta_\Gamma(y_0, \gamma \cdot y_0) = \delta(x_0, \gamma \cdot x_0) = d_{\mathfrak{a}}(\mu(\gamma), \mu(H))$$

for all $\gamma \in \Gamma$. (If we do not assume $\varphi(y_0) = x_0$, the difference between the first and last term is uniformy bounded by $2d_{G/K}(x_0, \varphi(y_0))$.

Let $d_{G/H}$ be the distance function associated to a Γ-invariant Riemannian metric on G/H. By Proposition 3.8, there exists $R_1 \geq 1$ such that

$$\Delta_\Gamma(y_0, \gamma \cdot y_0) \geq R_1^{-1} d_{G/H}(y_0, \gamma \cdot y_0) - R_1$$

for all $\gamma \in \Gamma$.

On the other hand, by Fact 2.4, the orbital map $\gamma \mapsto \gamma \cdot y_0$ is a quasi-isometry from Γ to $(G/H, d_{G/H})$. In particular, there exists $R_2 \geq 1$ such that $d_{G/H}(y_0, \gamma y_0) \geq R_2^{-1}\, |\gamma|_S - R_2$ for all $\gamma \in \Gamma$. Combining the two inequalities gives

$$d_{\mathfrak{a}}(\mu(\gamma), \mu(H)) \geq R_3^{-1} |\gamma|_S - R_3\ ,$$

concluding the proof of Theorem 3.1. □

4. Sharpness implies Anosov

In this section we prove Theorem 1.10 and Proposition 1.13. We start with some reminders.

4.1. **Reminders: limit cones.** As above, let G be a connected real linear reductive Lie group, with Cartan projection $\mu : G \to \mathfrak{a}^+$ as in Section 2.1. Given a subgroup Γ of G, the

limit cone of Γ is defined as

$$\mathcal{L}_\Gamma = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{\gamma\in\Gamma,\ \|\mu(\gamma)\|\geq n} \mathbb{R}_{\geq 0}\,\mu(\gamma)}.$$

It is a closed cone of $\mathfrak{a}^+$, invariant under the opposition involution ι. It is nonempty as soon as Γ is unbounded in G.

The Jordan projection $\lambda : G \to \mathfrak{a}^+$ of G can be defined as

$$\lambda(g) \coloneqq \lim_{n\to+\infty} \frac{1}{n}\mu(g^n)$$

for all $g \in G$. It is a continuous function which is invariant under conjugation. Given a subgroup Γ of G, the *Jordan limit cone* of Γ is defined as

$$\mathcal{L}^\lambda_\Gamma = \overline{\bigcup_{\gamma\in\Gamma} \mathbb{R}_{\geq 0}\,\lambda(\gamma)}.$$

It is also a closed of $\mathfrak{a}^+$, invariant under the opposition involution ι.

We always have the inclusion $\mathcal{L}^\lambda_\Gamma \subset \mathcal{L}_\Gamma$. Benoist [Be2] proved that the two cones actually coincide when the Zariski closure of Γ is reductive. Moreover, when Γ is Zariski dense in$G$, he proved that Λ_Γ is convex with nonempty interior.

Not that the sharpness condition (Definition 1.4) can be reformulated in terms of limit cones: given a closed subgroup H of G, a discrete subgroup Γ of G is sharp for G/H if and only if

$$\mathcal{L}_\Gamma \cap \mathcal{L}_H = \{0\}.$$

4.2. **Background on Anosov representations.** As mentioned in Section 1.5, Anosov representations are representations of infinite word hyperbolic groups Γ to noncompact real reductive Lie groups, which have finite kernel and discrete image, and which are defined by strong dynamical properties. These dynamical properties are very similar to those satisfied by convex cocompact representations of word hyperbolic groups into simple Lie groups of real rank one (see e.g. [L, GuWi, KLP1, GGKW1, BPS]). There is in fact a geometric way to see Anosov representations as convex cocompact representations, in the world of convex projective geometry [DGK1, DGK2, Zi_A].

Given a reductive Lie group G, there are several possible types of Anosov representations into G, depending on the choice of a proper parabolic subgroup P of G up to conjugation (or equivalently on the choice of a flag variety G/P of G). Fix such a parabolic subgroup P of G, and let P^* be an opposite parabolic subgroup. Recall that there are several possible transversality positions between a point x of G/P and a point y of G/P^*; we shall say that x and y are *compatible* (resp. *transverse*) if they correspond to parabolic subgroups of G whose intersection is parabolic (resp. reductive).

Example 4.1. For $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, let $G = \mathrm{SL}(d,\mathbb{K})$ and, for $1 \leq i \leq d-1$, let $P = P_i$ be the stabilizer in G of an i-dimensional linear subspace of $\mathbb{K}^d$. Then $G/P = \mathrm{Gr}_i(\mathbb{K}^d)$ is the Grassmannian of i-planes of $\mathbb{K}^d$, and $G/P^* = \mathrm{Gr}_{d-i}(\mathbb{K}^d)$ is the Grassmannian of $(d-i)$-planes of $\mathbb{K}^d$. Two points $x \in G/P$ and $y \in G/P^*$ are compatible (resp. transverse) if the i-plane x is contained in (resp. is transverse to) the $(d-i)$-plane y.

The original definition of Anosov representations is due to Labourie [L]. Let Γ be an infinite word hyperbolic group with Gromov boundary $\partial_\infty\Gamma$. A representation $\rho : \Gamma \to G$ is *P-Anosov* if there exists two continuous, ρ-equivariant maps (called *boundary maps*) $\xi : \partial_\infty\Gamma \to G/P$ and $\xi^* : \partial_\infty\Gamma \to G/P^*$ which

(1) are *compatible* (i.e. $\xi(\eta)$ and $\xi^*(\eta)$ are compatible for all η in $\partial_\infty\Gamma$),
(2) are *transverse* (i.e. $\xi(\eta)$ and $\xi^*(\eta')$ are transverse for all $\eta \neq \eta'$ in $\partial_\infty\Gamma$),
(3) have an associated flow with some uniform contraction/expansion properties described in [L, GuWi].

We do not state condition (3) precisely, but refer instead to the original papers [L, GuWi], and to the survey [Ka4] for various characterizations established by several authors.

In the current paper, we shall use the following characterization due to Kapovich, Leeb, and Porti [KLP2], for which an alternative proof was later given by Bochi, Potrie, and Sambarino [BPS].

Fact 4.2 ([KLP2])**.** *Let G be a noncompact real reductive Lie group, $\theta \subset \Delta$ a nonempty subset of the simple restricted oots of G, and Γ an infinite word hyperbolic group. For any representation $\rho : \Gamma \to G$, the following are equivalent:*

(1) Γ is word hyperbolic and $\rho : \Gamma \to G$ is P_θ-Anosov,
(2) there exist $C, C' > 0$ such that $\langle \alpha, \mu(\rho(\gamma))\rangle \geq C\,|\gamma| - C'$ for all $\gamma \in \Gamma$ and all $\alpha \in \theta$.

Here P_θ is the standard parabolic subgroup of G with Lie algebra

$$\mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha\in\Sigma^+\cap \mathrm{span}(\Delta\smallsetminus\theta)} \mathfrak{g}_{-\alpha}, \tag{4.1}$$

where for each $\alpha \in \mathfrak{a}^*$ we set

$$\mathfrak{g}_\alpha \coloneqq \{v \in \mathfrak{g} \mid \mathrm{ad}(a)\, v = \alpha(a)\, v \quad \forall a \in \mathfrak{a}\}.$$

Note that condition (2) is equivalent to ρ being a quasi-isometrically embedding and the limit cone $\mathcal{L}_{\rho(\Gamma)}$ not meeting the walls $\mathrm{Ker}(\alpha)$, $\alpha \in \theta$ outside of 0.

4.3. **Sharpness and Anosov representations when $\mu(H)$ is a union of walls.** Theorem 1.10 is an immediate consequence of Theorem 1.5, Fact 1.15, and of the following fact and proposition.

Fact 4.3 (Bestvina–Mess, [BM])**.** *Let Γ be a word hyperbolic group. Then the covering dimension of the Gromov boundary of Γ is its virtual cohomological dimension minus* 1*.*

Proposition 4.4. *Let G/H be a reductive homogeneous space such that $\mu(H) = \mathfrak{a}^+ \cap \bigcup_{\alpha\in\theta} \mathrm{Ker}(\alpha)$ for some subset $\theta \subset \Delta$ of the simple roots of G. Then for any finitely generated discrete subgroup Γ of G, the following conditions are equivalent:*

(1) Γ is sharply embedded in G with respect to H (Definition 1.6),
(2) Γ is word hyperbolic and the natural inclusion $\Gamma \hookrightarrow G$ is a P_θ-Anosov representation.

In this case, there is a neighborhood $\mathcal{U} \subset \mathrm{Hom}(\Gamma, G)$ of the natural inclusion consisting entirely of injective and discrete representations ρ for which $\rho(\Gamma)$ is sharply embedded in G with respect to H.

Proof of Proposition 4.4. If Γ is sharply embedded in G with respect to H, then there exists $C > 0$ such that $\langle \alpha, \mu(\gamma) \rangle \geq C^{-1} |\gamma| - C$ for all $\alpha \in \theta$ and $\gamma \in \Gamma$; therefore Γ is word hyperbolic and the natural inclusion is P_θ-Anosov by Fact 4.2. Conversely (see [GGKW1, Cor. 1.9]), if Γ is word hyperbolic and the natural inclusion $\Gamma \hookrightarrow G$ is P_θ-Anosov, then one easily checks that Γ is quasi-isometrically embedded in G and that the limit cone $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ of Γ does not meet the walls $\mathrm{Ker}(\alpha)$, $\alpha \in \theta$, outside of 0, and so Γ is sharply embedded in G with respect to H (see Section 4.1); in particular the action of Γ on G/H is properly discontinuous by the Benoist–Kobayashi properness criterion (see Section 1.2). To conclude, we use the fact that being Anosov is an open property [L]. □

Proposition 4.4 applies in particular to the triples (G, H, θ) in Table 1. For these triples, compactifications of locally homogeneous spaces $\Gamma \backslash G/H$ were described in [GGKW2] when Γ is word hyperbolic and $\Gamma \hookrightarrow G$ is P_θ-Anosov. In the case that $\Gamma \backslash G/H$ is already compact, using Theorem 1.5 and Proposition 4.4, we obtain the following.

Corollary 4.5. *Let G/H be as in cases (i), (ii), (iii) (resp. (iv), (v)) of Table 1, and let $Q = P_{\{\alpha_1\}}$ (resp. $Q = P_{\{\alpha_2\}}$). Then for any discrete subgroup Γ of G acting properly discontinuously and cocompactly on G/H, there exists a continuous Γ-equivariant fibration of $G/Q \to \partial_\infty \Gamma$ whose fibers are maximal isotropic subspaces (resp. subsets of the form*

$$\mathcal{L}_x = \{ y \in G/P_{\{\alpha_2\}} \subset \mathrm{Gr}_2(\mathbb{K}^{2p,2}) \,|\, x \in y \}$$

for $x \in G/P_{\{\alpha_1\}}$).

Proof. Let (G, H, θ) be as in Table 1, and let Γ be a discrete subgroup of G acting properly discontinuously and cocompactly on G/H. By Proposition 4.4, the group Γ is word hyperbolic and the natural inclusion $\Gamma \hookrightarrow G$ is P_θ-Anosov. In particular, there is a continuous Γ-equivariant boundary map $\xi : \partial_\infty \Gamma \to G/P_\theta$ which is *transverse* in the sense that for any $\eta \neq \eta'$ in $\partial_\infty \Gamma$, the points $\xi(\eta), \xi(\eta') \in G/P$ are in generic position.

Let Q be $P_{\{\alpha_1\}}$ in cases (i), (ii), (iii) of Table 1, and $P_{\{\alpha_2\}}$ in cases (iv), (v). For any $\eta \in \partial_\infty \Gamma$, let $\mathcal{L}_{\xi(\eta)} \subset G/Q$ be the set of points in G/Q which are *not* transverse to $\xi(\eta)$. In cases (i), (ii), (iii), the points in G/Q are isotropic lines, the points in G/P_θ are maximal isotropic subspaces, and so $\mathcal{L}_\eta \subset G/Q \subset \mathbb{P}(V)$ is the projectivization of a maximal isotropic subspace. In cases (iv), (v), the points in G/Q are isotropic 2-planes, the points in G/P_θ are isotropic lines, and $\mathcal{L}_{\xi(\eta)}$ is the set of points in $G/Q \subset \mathrm{Gr}_2(\mathbb{K}^{2p,2})$ which contain $\xi(\eta)$.

By [GGKW2], we can compactify $\Gamma \backslash G/H$ by adding, at infinity, the set $G/Q \smallsetminus \bigcup_{\eta \in \partial \Gamma} \mathcal{L}_{\xi(\eta)}$. Since $\Gamma \backslash G/H$ is already compact, we obtain

$$\bigcup_{\eta \in \partial_\infty \Gamma} \mathcal{L}_{\xi(\eta)} = G/Q. \tag{4.2}$$

This union is disjoint because the Anosov boundary map $\xi : \partial_\infty \Gamma \to G/P_\theta$ is transverse. Moreover, $\mathcal{L}_{\xi(\gamma \cdot \eta)} = \gamma \cdot \mathcal{L}_{\xi(\eta)}$ for all $\gamma \in \Gamma$ and $\eta \in \partial_\infty \Gamma$ since ξ is Γ-equivariant. □

4.4. **Sharpness implies Anosov in the general corank-one case.** Proposition 1.13 is an immediate consequence of Theorem 1.5, Fact 1.15, Fact 4.3, and of the following proposition.

Proposition 4.6. *Let G be a connected real linear reductive Lie group and H a closed subgroup of G such that $\mu(H)$ is the intersection of $\mathfrak{a}^+$ with the W-orbit of a hyperplane.*

Let Γ be a finitely generated discrete subgroup of G which is sharply embedded in G with respect to H. Suppose that Γ is not virtually cyclic. Then Γ is word hyperbolic and the natural inclusion $\Gamma \hookrightarrow G$ is a P-Anosov representation, for some proper parabolic subgroup P of G.

Remark 4.7. The assumption that Γ be not virtually cyclic is necessary in Proposition 4.6: for instance, if $G/H = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(2,\mathbb{R})$ and if Γ is the cyclic group generated by $\mathrm{diag}(2,2,1/4)$, then Γ is sharply embedded in G with respect to H but it is not Anosov in G. However, even for virtually cyclic groups Γ, the fact that Γ is sharply embedded in G with respect to H still implies that it satisfies some Anosov property *after composing with some linear representation of G*: see Corollary 6.3 below.

Remark 4.8. Let $\varphi \in \mathfrak{a}^*$ and let $H_\varphi := \exp(\mathrm{Ker}(\varphi)) \subset A$. Then for any finitely generated discrete subgroup Γ of G, the following are equivalent:

(i) Γ is sharply embedded in G with respect to H_φ for all $\varphi \in \mathcal{F}$;
(ii) there exist $C, C' > 0$ such that $d(\mu(\gamma), \bigcup_{w\in W} \mathrm{Ker}(w\cdot\varphi)) \geq C\,|\gamma| - C'$.

After this work was completed, we learnt that condition (ii) was also considered by Davalo–Riestenberg [DR], who named it *φ-distortion* and also independently proved [DR, Prop. 3.3] that it implies Anosovness in G.

In order to prove Proposition 4.6, consider a W-invariant scalar product $(\cdot,\cdot)$ on $\mathfrak{a}^*$. For any simple root $\alpha \in \Delta$, we denote by $\omega_\alpha \in \mathfrak{a}^*$ the fundamental weight associated with α, defined by

$$2\frac{(\omega_\alpha,\beta)}{(\beta,\beta)} = \delta_{\alpha,\beta} \quad \text{for all } \beta \in \Delta \tag{4.3}$$

where $\delta_{\cdot,\cdot}$ is the Kronecker symbol. We also denote by $s_\alpha \in W$ the reflection associated with α, which acts on $\mathfrak{a}^*$ by

$$s_\alpha \cdot \varphi := \varphi - 2\frac{(\varphi,\alpha)}{(\alpha,\alpha)}\,\alpha. \tag{4.4}$$

Lemma 4.9. *Consider $\varphi \in \mathfrak{a}^*$ and $\alpha \in \Delta$ such that $(\varphi,\alpha) > 0$. Let $\mathcal{C}$ be a connected component of $\mathfrak{a}^+ \smallsetminus (\mathrm{Ker}(\varphi) \cup \mathrm{Ker}(s_\alpha\cdot\varphi))$ on which $\varphi > 0$, and such that $\overline{\mathcal{C}} \cap \mathrm{Ker}(\varphi) \neq \{0\}$. Then $\mathcal{C} \cap \mathrm{Ker}(\alpha) = \emptyset$.*

Proof. By assumption we have $(\varphi,\alpha) > 0$, hence $s_\alpha\cdot\varphi < 0$ on $\mathrm{Ker}(\varphi)$ (see (4.4)). Since $\overline{\mathcal{C}} \cap \mathrm{Ker}(\varphi) \neq \{0\}$, there exists $x \in \mathcal{C}$ such that $(s_\alpha\cdot\varphi)(x) < 0$. Since $\mathcal{C}$ is connected and misses $\mathrm{Ker}(s_\alpha\cdot\varphi)$, we have $s_\alpha\cdot\varphi < 0$ on $\mathcal{C}$. Thus $\varphi - s_\alpha\cdot\varphi > 0$ on $\mathcal{C}$. But $\varphi - s_\alpha\cdot\varphi$ is a positive multiple of α (see (4.4)), hence $\alpha > 0$ on $\mathcal{C}$. □

We also use the following result from [Ka2]. We denote by $\iota : \mathfrak{a} \to \mathfrak{a}$ the opposition involution, sending any x to $-w_0\cdot x$ where w_0 is the longest word of the Weyl group W.

Fact 4.10 ([Ka2, Th. 1.1])**.** *Let G be a connected real linear reductive Lie group and H a closed subgroup of G such that $\mu(H)$ is the intersection of $\mathfrak{a}^+$ with finitely many hyperplanes of $\mathfrak{a}$. Let Γ be a discrete subgroup of G acting properly discontinuously on G/H. Then there is a connected component $\mathcal{C}$ of $\mathfrak{a}^+ \smallsetminus \mu(H)$ such that $\mu(\gamma) \in \mathcal{C} \cup \iota(\mathcal{C})$ for all but finitely many $\gamma \in \Gamma$. Moreover, if Γ is* not *virtually cyclic, then $\mathcal{C} = \iota(\mathcal{C})$.*

Proof of Proposition 4.6. By Fact 4.10, since Γ acts properly discontinuously on G/H and is not virtually cyclic, there is a connected component $\mathcal{C}$ of $\mathfrak{a}^+ \smallsetminus \mu(H)$ such that $\mu(\gamma) \in \mathcal{C}$ for all but finitely many $\gamma \in \Gamma$. By Lemma 4.9, there exists $\alpha \in \Delta$ such that $\mathcal{C} \cap \mathrm{Ker}(\alpha) = \emptyset$. Since Γ is sharply embedded in G with respect to H, there exist $C, C' > 0$ such that $\langle \alpha, \mu(\gamma) \rangle \geq C\,|\gamma| - C'$ for all $\alpha \in \theta$ and all $\gamma \in \Gamma$; therefore Γ is word hyperbolic and the natural inclusion is P_θ-Anosov by Fact 4.2. □

5. Nonexistence of compact quotients

The goal of this section is to prove Corollary 1.16. This is done in Section 5.3, after recalling some bounds on cohomological dimension in Sections 5.1 and 5.2. We also discuss Theorem 1.18 in Section 5.4.

5.1. **A bound on the cohomological dimension for proper and cocompact actions.** Recall that the *cohomological dimension* $\mathrm{cd}(\Gamma)$ of a group Γ is the largest integer $n \in \mathbb{N}$ for which there exists a $\mathbb{Z}[\Gamma]$-module M with $H^n(\Gamma, M)$ nonzero. If Γ admits a finite-index subgroup which is torsion-free (this is the case e.g. if Γ is a finitely generated linear group, by Selberg's lemma), then all torsion-free finite-index subgroups of Γ have the same cohomological dimension, called the *virtual cohomological dimension* of Γ, denoted by $\mathrm{vcd}(\Gamma)$.

Here we recall the proof of Fact 1.15, which is due to Kobayashi [Ko1] for reductive H and was generalized by Morita in [M1].

Proof of Fact 1.15. Let Γ be a torsion-free discrete subgroup of G acting properly discontinuously on G/H. Recall that the manifold $\Gamma\backslash G/K_H$ admits two Ehresmann fibrations

$$\pi_H : \Gamma\backslash G/K_H \longrightarrow \Gamma\backslash G/H \quad \text{and} \quad \pi_K : \Gamma\backslash G/K_H \longrightarrow \Gamma\backslash G/K.$$

By the Cartan–Iwasawa–Malcev theorem (see [Bo1]), the spaces G/K and H/K_H are contractible. This has the following two consequences:

- The space $\Gamma\backslash G/K$ is a classifying space for Γ, hence
$$\mathrm{cd}(\Gamma) = \mathrm{cd}(\Gamma\backslash G/K).$$
- The fibration π_H has contractible fibers and is thus a homotopy equivalence, hence
$$\mathrm{cd}(\Gamma\backslash G/K_H) = \mathrm{cd}(\Gamma\backslash G/H).$$

The fibration π_K, on the other hand, is a fibration with compact fibers K/K_H over a classifying space for Γ. A classical consequence of the Leray–Serre spectral sequence for fibrations then gives

$$\mathrm{cd}(\Gamma\backslash G/K_H) = \mathrm{cd}(\Gamma\backslash G/K) + \dim(K/K_H)\ .$$

Gathering these identities, we get

$$\mathrm{cd}(\Gamma) = \mathrm{cd}(\Gamma\backslash G/H) - \dim(K/K_H)\ .$$

Finally, $\Gamma\backslash G/H$ is a manifold, hence

$$\mathrm{cd}(\Gamma\backslash G/H) \leq \dim(G/H)\ ,$$

with equality if and only if $\Gamma\backslash G/H$ is closed. Hence

$$\mathrm{cd}(\Gamma) \leq \dim(G/H) - \dim(K/K_H) = \dim(G/K) - \dim(H/K_H)\ ,$$

with equality if and only if the action of Γ on G/H is cocompact. □

In Table 3 we computed all values of $\dim(G/K) - \dim(H/K_H)$ from the following.

G	$\dim(G/K)$
$\mathrm{SL}(n,\mathbb{R})$	$(n-1)(n+2)/2$
$\mathrm{SL}(n,\mathbb{C})$	n^2-1
$\mathrm{SL}(n,\mathbb{H})$	$(n-1)(2n+1)$
$\mathrm{SO}(p,q)$	pq
$\mathrm{SU}(p,q)$	$2pq$
$\mathrm{Sp}(p,q)$	$4pq$
$\mathrm{Sp}(2\ell,\mathbb{R})$	$\ell(\ell+1)$
$\mathrm{Sp}(2\ell,\mathbb{C})$	$\ell(2\ell+1)$
$G_{2(2)}$	8

TABLE 5. Dimensions of certain Riemannian symmetric spaces

5.2. **Bounds on the cohomological dimension for Anosov representations.** Recall that the cohomological dimension of a Gromov hyperbolic group is related to the covering dimension of its boundary by Fact 4.3 due to Bestvina and Mess. This can be used to give bounds on the cohomological dimension of Anosov subgroups. Here we will use a rather elementary bound. We refer for instance to the work of Canary and Tsouvalas [CT] for more refined bounds.

We start by stating the bound for $G = \mathrm{SL}(2\ell,\mathbb{K})$. Let $P_\ell = P_{\{\alpha_\ell\}}$ denote the stabilizer in G of an ℓ-dimensional linear subspace of $\mathbb{K}^{2\ell}$.

Lemma 5.1. *Let $G = \mathrm{SL}(2\ell,\mathbb{K})$ where $\ell \geq 1$ and $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{C}$, resp. $\mathbb{H}$). If a word hyperbolic group Γ admits a P_ℓ-Anosov representation to G, then $\mathrm{vcd}(\Gamma)$ is at most $\ell+1$ (resp. $2\ell+1$, resp. $4\ell+1$).*

Proof. Let $\rho : \Gamma \to G$ be a P_ℓ-Anosov representation, with boundary map $\xi : \partial_\infty\Gamma \to G/P_\ell = \mathrm{Gr}_\ell(\mathbb{K}^{2\ell})$ (see Example 4.1). We can see each $\xi(\eta)$, for $\eta \in \partial_\infty\Gamma$, as an $(\ell-1)$-dimensional projective subspace of $\mathbb{P}(\mathbb{K}^{2\ell})$; these $(\ell-1)$-dimensional projective subspaces are pairwise disjoint by transversality of ξ (see Section 4.2). Therefore the subset $\bigcup_{\eta\in\partial_\infty\Gamma}\xi(\eta)$ of $\mathbb{P}(\mathbb{K}^{2\ell})$ fibers over $\partial_\infty\Gamma$ with fiber $\mathbb{P}(\mathbb{K}^\ell)$. We deduce that the covering dimension of $\partial_\infty\Gamma$ is at most $\ell\dim(\mathbb{K})$, and conclude using Fact 4.3. □

More generally, the following holds with a similar proof.

Lemma 5.2. *Let G be a connected real reductive algebraic group and P, Q two proper parabolic subgroups of G. Suppose that the G-orbit of (P,Q) is closed in $G/P \times G/Q$ and that the complement consists of a single open orbit. If a word hyperbolic group Γ admits a P-Anosov representation to G, then $\mathrm{vcd}(\Gamma) \leq \dim(G) - \dim(\mathrm{Lie}(P) + \mathrm{Lie}(Q)) + 1$.*

Proof. Let $\mathcal{O} \subset G/P \times G/Q$ be the G-orbit of (P,Q); it is a closed algebraic subset of $G/P \times G/Q$, of dimension $\dim(G) - \dim(P \cap Q)$.

Let $\rho : \Gamma \to G$ be a P-Anosov representation, with boundary map $\xi : \partial_\infty\Gamma \to G/P$. For each $\eta \in \partial_\infty\Gamma$, the set $\mathcal{N}_{\xi(\eta)} \coloneqq \{z \in G/Q \,|\, (\xi(\eta), z) \in \mathcal{O}\}$ is a closed algebraic subspace of G/Q of dimension $\dim(\mathcal{O}) - \dim(G/P)$; these sets $\mathcal{N}_{\xi(\eta)}$ are pairwise disjoint by transversality of ξ (see e.g. [ST, Lem. 3.27]). Therefore the subset $\bigcup_{\eta\in\partial_\infty\Gamma} \mathcal{N}_{\xi(\eta)}$ of G/Q fibers over $\partial_\infty\Gamma$ with fibers of dimension $\dim(\mathcal{O}) - \dim(G/P)$. We deduce that the covering dimension of $\partial_\infty\Gamma$ is at most

$$\begin{aligned}\dim(G/Q) - \dim(\mathcal{O}) + \dim(G/P) &= \dim(G) - \dim(Q) + \dim(P \cap Q) - \dim(P)\\ &= \dim(G) - \dim\big(\mathrm{Lie}(P) + \mathrm{Lie}(Q)\big).\end{aligned}$$

We conclude using Fact 4.3. □

If we take standard parabolic subgroups $P = P_\theta$ and $Q = P_{\theta'}$ as in (4.1), where $\theta, \theta' \subset \Delta$ are subsets of the simple restricted roots of G, then the integer $\dim(G) - \dim(P \cap Q)$ is equal to the sum of the dimensions of the root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \Sigma^+ \smallsetminus (\mathrm{span}(\Delta \smallsetminus \theta) \cup \mathrm{span}(\Delta \smallsetminus \theta'))$.

As a special case, we recover the following fact, where $P_\ell = P_{\{\alpha_\ell\}}$ is the stabilizer in G of an ℓ-dimensional linear subspace of V which is totally isotropic for b.

Fact 5.3 ([GuWi, Prop. 8.3])**.** *For $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, let V be a $\mathbb{K}$-vector space and $b : V \otimes_{\mathbb{R}} V \to \mathbb{K}$ a nondegenerate $\mathbb{R}$-bilinear form which is symmetric, antisymmetric, Hermitian, or anti-Hermitian over $\mathbb{K}$, with $G \coloneqq \mathrm{Aut}_{\mathbb{K}}(b)$ noncompact. Let $r \coloneqq \mathrm{rank}_{\mathbb{R}}(G) \geq 1$. If a word hyperbolic group Γ admits a P_1-Anosov representation or a P_r-Anosov representation to G, then* $\mathrm{vcd}(\Gamma) \leq (\dim_{\mathbb{K}}(V) - r)\, \dim(\mathbb{K})$.

5.3. **Proof of Corollary 1.16.** We treat separately the various cases of Table 3.

Proof of Corollary 1.16 in case (i). Write $G = \mathrm{SL}(2\ell, \mathbb{K})$ where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The restricted root system of G is of type $A_{2\ell-1}$. We can identify $\mathfrak{a}$ with the hyperplane of equation $t_1 + \cdots + t_{2\ell} = 0$ in $\mathbb{R}^{2\ell}$, and take

$$\mathfrak{a}^+ = \{(t_1, \dots, t_{2\ell}) \in \mathfrak{a} \,|\, t_1 \geq \cdots \geq t_{2\ell}\}.$$

We have $\mu(H) = \bigcup_{2\leq j\leq 2\ell-1} \mathrm{Ker}(e_j^*)$, and so $\mathfrak{a}^+ \smallsetminus \mu(H)$ has $2\ell - 1$ connected components, namely $\mathcal{C}_1, \dots, \mathcal{C}_{2\ell-1}$ where

$$\mathcal{C}_i \coloneqq \{(t_1, \dots, t_{2\ell}) \in \mathfrak{a}^+ \,|\, t_1 \geq \cdots \geq t_i > 0 > t_{i+1} \geq \cdots \geq t_{2\ell}\}.$$

The opposition involution $\iota : \mathfrak{a} \to \mathfrak{a}$ maps $(t_1, \dots, t_{2\ell})$ to $(-t_{2\ell}, \dots, -t_1)$, hence switches $\mathcal{C}_i$ and $\mathcal{C}_{2\ell-i}$ for all $1 \leq i \leq 2\ell - 1$. In particular, there is a unique connected component of $\mathfrak{a}^+ \smallsetminus \mu(H)$ which is invariant under ι, namely $\mathcal{C}_\ell$.

Let Γ be a discrete subgroup of G acting properly discontinuously and cocompactly on G/H. By Theorem 1.5, the group Γ is sharp for G/H. By Fact 4.10, the set $\mu(\Gamma)$ is contained in $\mathcal{C}_\ell$ up to finitely many points. Together with the sharpness and Fact 4.2, this implies that the group Γ is word hyperbolic and that the natural inclusion $\Gamma \hookrightarrow G$ is P_ℓ-Anosov. Lemma 5.1 then states that $\mathrm{vcd}(\Gamma)$ is at most $\ell + 1$ (resp. $2\ell + 1$, resp. $4\ell + 1$) if $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{C}$, resp. $\mathbb{H}$). Using the last column of Table 3, we see that this is strictly less than $\dim(G/K) - \dim(H/K_H)$ in all cases. This contradicts Fact 1.15, which states that we must have $\mathrm{vcd}(\Gamma) = \dim(G/K) - \dim(H/K_H)$. □

Proof of Corollary 1.16 in cases (ii)–(iii). The restricted root system of G is of type B_2 in case (ii), and of type $(BC)_2$ in case (iii). In both cases $\mu(H)$ is the wall $\mathrm{Ker}(\alpha_1)$.

Let Γ be a discrete subgroup of G acting properly discontinuously and cocompactly on G/H. By Theorem 1.10, the group Γ is word hyperbolic and the natural inclusion $\Gamma \hookrightarrow G$ is $P_{\{\alpha_1\}}$-Anosov. Fact 5.3 then implies that $\mathrm{vcd}(\Gamma)$ is at most $2k + k'$ in case (ii), and $4k + 2k'$ in case (iii). Using the last column of Table 3, we see that this is strictly less than $\dim(G/K) - \dim(H/K_H)$ for $k' \geq 1$. This contradicts Fact 1.15, which states that we must have $\mathrm{vcd}(\Gamma) = \dim(G/K) - \dim(H/K_H)$. □

Proof of Corollary 1.16 in cases (iv)–(v). The restricted root system of G is of type B_ℓ in case (iv), and of type C_ℓ in case (v). In both cases $\mu(H)$ is the wall $\mathrm{Ker}(\alpha_\ell)$.

Let Γ be a discrete subgroup of G acting properly discontinuously and cocompactly on G/H. By Theorem 1.10, the group Γ is word hyperbolic and the natural inclusion $\Gamma \hookrightarrow G$ is $P_{\{\alpha_\ell\}}$-Anosov. Fact 5.3 then implies that $\mathrm{vcd}(\Gamma)$ is at most $\ell + 1$ in case (iv), and $\ell \dim(\mathbb{K})$ in case (v). Using the last column of Table 3, we see that this is strictly less than $\dim(G/K) - \dim(H/K_H)$. This contradicts Fact 1.15, which states that we must have $\mathrm{vcd}(\Gamma) = \dim(G/K) - \dim(H/K_H)$. □

Proof of Corollary 1.16 in case (vi). The restricted root system of G is of type B_3, and $\mu(H) = \mathfrak{a}^+ \cap \mathrm{Ker}(\alpha_1 - \alpha_3)$. The set $\mathfrak{a}^+ \smallsetminus \mu(H)$ has two connected components, one whose closure does not meet the wall $\mathrm{Ker}(\alpha_1)$ and the other one whose closure does not meet the wall $\mathrm{Ker}(\alpha_3)$.

Let Γ be a discrete subgroup of G acting properly discontinuously and cocompactly on G/H. By Fact 4.10, there is a connected component $\mathcal{C}$ of $\mathfrak{a}^+ \smallsetminus \mu(H)$ such that $\mu(\gamma) \in \mathcal{C}$ for all but finitely many $\gamma \in \Gamma$. Let $i \in \{1, 3\}$ be such that the closure of $\mathcal{C}$ does not meet the wall $\mathrm{Ker}(\alpha_i)$. By Theorem 1.5, the group Γ is sharply embedded in G with respect to H. Therefore there exist $C, C' > 0$ such that $\langle \alpha_i, \mu(\gamma) \rangle \geq C\,|\gamma| - C'$ for all $\gamma \in \Gamma$. By Fact 4.2, the group Γ is word hyperbolic and the natural inclusion is $P_{\{\alpha_i\}}$-Anosov. Fact 5.3 then implies $\mathrm{vcd}(\Gamma) \leq 5$, which is strictly less than $\dim(G/K) - \dim(H/K_H) = 7$. This contradicts Fact 1.15, which states that we must have $\mathrm{vcd}(\Gamma) = \dim(G/K) - \dim(H/K_H)$. □

Proof of Corollary 1.16 in case (vii). The restricted root system of G is of type B_4, and $\mu(H) = \mathfrak{a}^+ \cap \mathrm{Ker}(\alpha_1 - \alpha_3)$. The set $\mathfrak{a}^+ \smallsetminus \mu(H)$ has two connected components, one whose closure does not meet the wall $\mathrm{Ker}(\alpha_1)$ and the other one whose closure does not meet the wall $\mathrm{Ker}(\alpha_3)$.

Let Γ be a discrete subgroup of G acting properly discontinuously and cocompactly on G/H. By Fact 4.10, there is a connected component $\mathcal{C}$ of $\mathfrak{a}^+ \smallsetminus \mu(H)$ such that $\mu(\gamma) \in \mathcal{C}$ for all but finitely many $\gamma \in \Gamma$. Let $i \in \{1, 3\}$ be such that the closure of $\mathcal{C}$ does not meet the wall $\mathrm{Ker}(\alpha_i)$. By Theorem 1.5, the group Γ is sharply embedded in G with respect to H. Therefore there exist $C, C' > 0$ such that $\langle \alpha_i, \mu(\gamma) \rangle \geq C\,|\gamma| - C'$ for all $\gamma \in \Gamma$. By Fact 4.2, the group Γ is word hyperbolic and the natural inclusion is $P_{\{\alpha_i\}}$-Anosov. Fact 5.3 (for $i = 1$) and Lemma 5.2 (for $i = 3$, with $P = P_{\{\alpha_3\}}$ and $Q = P_{\{\alpha_1\}}$) then imply $\mathrm{vcd}(\Gamma) \leq 6$, which is strictly less than $\dim(G/K) - \dim(H/K_H) = 8$. This contradicts Fact 1.15, which states that we must have $\mathrm{vcd}(\Gamma) = \dim(G/K) - \dim(H/K_H)$. □

5.4. **The group H must be quasi-isometrically embedded in G.**

Proof of Theorem 1.18. Suppose that there is a discrete subgroup Γ of G acting properly discontinuously and cocompactly on G/H. Then $\Gamma \times H$ acts properly and cocompactly on $(G \times G)/\mathrm{Diag}(G)$. By Theorem 3.1, the group $\Gamma \times H$ is compactly generated and quasi-isometrically embedded in $G \times G$. In particular, H is quasi-isometrically embedded in G. □

For reductive pairs (G, H), the subgroup H is always quasi-isometrically embedded in G. Indeed, finding a Cartan involution of G preserving H amounts to finding an H-equivariant geodesic embedding of $H/K_H \hookrightarrow G/K$.

On the other hand, Theorem 1.18 provides obstructions to the existence of compact quotients of some homogeneous spaces G/H with nonreductive H. For instance, we obtain a new proof of [M2, Example 7.1], namely that if U is a unipotent subgroup of G, then G/U does not admit any compact quotients.

A more general criterion for quasi-isometric embeddedness is given by Abels and Alperin in [AA]. Let H be a connected subgroup of G. Let N_H be the unipotent radical of H and L a Levi factor. Let A_L be a Cartan subgroup of L and N_L a maximal unipotent subgroup of L normalized by A_L. Finally, let U_H be the nilpotent subgroup $N_L \ltimes N_H$, which is still normalized by A_L. Note that $A_L \ltimes U_H$ is cocompact in H, hence H is quasi-isometrically embedded in G if and only if $A_L \ltimes U_H$ is.

Fact 5.4 ([AA, Th. 3.2 & Rem. 3.3])**.** *Let G be a connected real linear reductive Lie group and H a closed connected subgroup of G. Then with the notations above, H is quasi-isometrically embedded in G if and only if the action of A_L on $U_H/[U_H, U_H]$ does not have a nontrivial global fixed point.*

Compact quotients of G/H for nonreductive H have not been as thoroughly studied as for reductive H, but we can mention the work of Oh–Witte [OW1, OW2] exhibiting subtle examples of compact quotients of G/H for $G = \mathrm{PO}(2n, 2)$ and H non reductive, as well as the work of Morita [M2, M3].

6. Openness for proper and cocompact actions

We introduce the following terminology, where P_1 is the stabilizer in $\mathrm{SL}(\mathbb{R}^d)$ of a line of $\mathbb{R}^d$.

Definition 6.1. Let G be a connected real linear reductive Lie group, $\tau : G \to \mathrm{SL}(\mathbb{R}^d)$ a finite-dimensional linear representation of G, and Γ a word hyperbolic group. A representation $\rho : \Gamma \to G$ is *τ-projective Anosov* if $\tau \circ \rho : \Gamma \to \mathrm{SL}(\mathbb{R}^d)$ is P_1-Anosov.

The goal of this section is to establish the following proposition and corollary. For any open subset $\mathcal{C}$ of $\mathfrak{a}^+$ and any group Γ, we denote by $\mathrm{Hom}_{\mathcal{C}}(\Gamma, G)$ the set of representations $\rho : \Gamma \to G$ such that the limit cone of $\rho(\Gamma)$ (see Section 4.1) meets $\mathcal{C}$.

Proposition 6.2. *Let G be a connected real linear reductive Lie group and H a closed subgroup of G such that $\mu(H)$ is the intersection of $\mathfrak{a}^+$ with finitely many rational hyperplanes of $\mathfrak{a}$. Let $\mathcal{C}$ be a connected component of $\mathfrak{a}^+ \smallsetminus \mu(H)$. Then there is a finite-dimensional real linear representation (τ, V) of G such that for any finitely generated, not virtually cyclic group Γ and any representation $\rho \in \mathrm{Hom}_{\mathcal{C}}(\Gamma, G)$, the following are equivalent:*

(1) ρ is a quasi-isometric embedding and $\rho(\Gamma)$ is sharp for G/H;

(2) Γ *is word hyperbolic and* ρ *is* τ*-projectively Anosov.*

For fixed Γ*, the set of representations* $\rho \in \mathrm{Hom}_{\mathcal{C}}(\Gamma, G)$ *satisfying these conditions is open in* $\mathrm{Hom}(\Gamma, G)$*.*

Corollary 6.3. *Let* G *be a connected real linear reductive Lie group and* H *a closed subgroup of* G *such that* $\mu(H)$ *is the intersection of* $\mathfrak{a}^+$ *with finitely many (not necessarily rational) hyperplanes of* $\mathfrak{a}$*. Then for any finitely generated group* Γ*,*

(1) if there exists a quasi-isometric embedding $\rho : \Gamma \to G$ *such that* $\rho(\Gamma)$ *is sharp for* G/H*, then* Γ *is word hyperbolic and* ρ *is* τ*-projective Anosov for some finite-dimensional real linear representation* (τ, V) *of* G*;*

(2) the set of quasi-isometric embeddings $\rho : \Gamma \to G$ *such that* $\rho(\Gamma)$ *is sharp for* G/H *is open in* $\mathrm{Hom}(\Gamma, G)$*.*

Theorem 1.9 is an immediate consequence of Theorem 1.5 and Corollary 6.3.

6.1. **Reminders: restricted weights of linear representations of** G**.** As in Section 4.4, we denote by $(\cdot, \cdot)$ a W-invariant scalar product on $\mathfrak{a}^*$. Any finite-dimensional real linear representation (τ, V) of G decomposes under the action of $\mathfrak{a}$; the joint eigenvalues (elements of $\mathfrak{a}^*$) are called the *restricted weights* of (τ, V). The union of the restricted weights of all such representations is the set

$$\Phi := \left\{ \alpha \in \mathfrak{a}^* \;\middle|\; 2\frac{(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z} \quad \forall \beta \in \Sigma \right\}.$$

Let $\mathfrak{a}_s^0 \subset \mathfrak{a}^*$ be the annihilator of $\mathfrak{a}_s$, i.e. the linear subspace of $\mathfrak{a}^*$ consisting of those linear forms on $\mathfrak{a}$ that vanish on $\mathfrak{a}_s$; it identifies with the dual $\mathfrak{z}(\mathfrak{g})^*$ of $\mathfrak{z}(\mathfrak{g})$. Then

$$\Phi = \mathfrak{a}_s^0 + \sum_{\alpha \in \Delta} \mathbb{Z}\, \omega_\alpha,$$

where $\omega_\alpha \in \mathfrak{a}^*$ is the fundamental weight associated with α (see (4.3)), and $(\omega_\alpha)_{\alpha \in \Delta}$ projects to a basis of $\mathfrak{a}_s^*$. The set of *dominant* weights is the semigroup $\Phi^+ = \mathfrak{a}_s^0 + \sum_{\alpha \in \Delta} \mathbb{N}\, \omega_\alpha$. The cone generated by the positive roots (or, equivalently, by the simple roots) determines a partial ordering on $\mathfrak{a}^*$, given by

$$\nu \leq \nu' \quad \Longleftrightarrow \quad \nu' - \nu \in \sum_{\alpha \in \Sigma^+} \mathbb{R}_{\geq 0}\, \alpha = \sum_{\alpha \in \Delta} \mathbb{R}_{\geq 0}\, \alpha.$$

Given an *irreducible* linear representation (τ, V) of G, the set of restricted weights of τ admits, for that ordering, a unique maximal element (see e.g. [GoWa, Cor. 3.2.3]), which is a dominant weight called the *highest weight* of τ.

Lemma 6.4. *(1) (*[GGKW1, Lem. 3.2 & 3.7]*) There exists* $N \in \mathbb{N}^*$ *such that for any* $\chi \in N\Phi^+$*, there is an irreducible representation* V_χ *which has highest weight* $N\chi$ *and which is proximal in* G/P_α *for all* $\alpha \in \Delta$ *with* $(\alpha, \chi) \neq 0$*; we have*

$$\langle \varepsilon_1 - \varepsilon_2, \mu_{V_\chi}(g) \rangle = \min_{\alpha \in \Delta,\, (\alpha,\chi) \neq 0} \langle \alpha, \mu(g) \rangle \quad \textit{for all } g \in G.$$

(2) (see [GGKW1, Lem. 7.11]*) For any dominant weights* $\chi, \chi' \in \mathfrak{a}^*$ *and any* $g \in G$*, if* $\langle \chi, \mu(g) \rangle \geq \langle \chi', \mu(g) \rangle$*, then*

$$\langle \varepsilon_1 - \varepsilon_2, \mu_{V_\chi \oplus V_{\chi'}}(g) \rangle = \min \big(\langle \chi - \chi', \mu(g) \rangle, \langle \varepsilon_1 - \varepsilon_2, \mu_{V_\chi}(g) \rangle \big).$$

(3) If $V^{(1)}, \dots, V^{(m)}$ are linear representations of G, then

$$\langle \varepsilon_1 - \varepsilon_2, \mu_{V^{(1)} \otimes \dots \otimes V^{(m)}}(g) \rangle = \min_{1 \leq \ell \leq m} \langle \varepsilon_1 - \varepsilon_2, \mu_{V^{(\ell)}}(g) \rangle \quad \textit{for all } g \in G.$$

6.2. **Proof of Proposition 6.2.** Suppose $\mathfrak{a}_H$ is the kernel of some linear form $\varphi_0 = \sum_{\alpha \in \Delta} t_\alpha \, \omega_\alpha$ with all t_α rational. We have $\mu(H) = \bigcup_{w \in W} \mathrm{Ker}(w \cdot \varphi_0)$. Up to multiplying all t_α by a common integer, we may assume that $w \cdot \varphi_0 = \sum_{\alpha \in \Delta} t_{w,\alpha} \, \omega_\alpha$ with $t_{w,\alpha} \in \mathbb{Z}$ for all $w \in W$ and all $\alpha \in \Delta$. Indeed, since W is generated by the reflections in the simple roots, it is sufficient to check this for $w = s_\alpha$: it is an immediate consequence of (4.4).

The connected component $\mathcal{C}$ of $\mathfrak{a}^+ \smallsetminus \mu(H)$ can be written as

$$\mathcal{C} = \mathfrak{a}^+ \cap \bigcap_{\varphi \in \mathcal{F}_\mathcal{C}} \{\varphi > 0\}$$

for some $\mathcal{F}_\mathcal{C} \subset W \cdot \varphi_0 \cup W \cdot (-\varphi_0)$ such that $\overline{\mathcal{C}} \cap \mathrm{Ker}(\varphi) \neq \{0\}$ for all $\varphi \in \mathcal{F}_\mathcal{C}$.

For each $\varphi \in \mathcal{F}_\mathcal{C}$, let Δ_φ^+ be the set of simple roots $\alpha \in \Delta$ such that $(\varphi, \alpha) > 0$; it is nonempty since $\varphi > 0$ on $\mathcal{C}$. We can write $\varphi = \varphi^+ - \varphi^-$ where

$$\varphi^+ \coloneqq \sum_{\alpha \in \Delta_\varphi^+} \frac{(\varphi, \alpha)}{(\alpha, \alpha)} \, \omega_\alpha \qquad \text{and} \qquad \varphi^- \coloneqq \varphi^+ - \varphi$$

are both ≥ 0 on $\mathfrak{a}^+$.

Let $N \in \mathbb{N}^*$ be given by Lemma 6.4.(1). Consider the linear representation

$$(\tau, V) \coloneqq \bigotimes_{\varphi \in \mathcal{F}_\mathcal{C}} \left(V_{N\varphi^+} \oplus V_{N\varphi^-} \right). \tag{6.1}$$

Let us check that it satisfies the conclusion of Proposition 6.2.

We first observe that by Lemma 6.4, for any $g \in G$ with $\mu(g) \in \mathcal{C}$ we have

$$\begin{aligned} \langle \varepsilon_1 - \varepsilon_2, \mu_V(g) \rangle &= \min_{\varphi \in \mathcal{F}_\mathcal{C}} \langle \varepsilon_1 - \varepsilon_2, \mu_{V_{\varphi^+} \oplus V_{\varphi^-}}(g) \rangle \\ &= \min_{\varphi \in \mathcal{F}_\mathcal{C}} \min \left(\langle \varphi^+ - \varphi^-, \mu(g) \rangle, \langle \varepsilon_1 - \varepsilon_2, \mu_{V_{\varphi^+}}(g) \rangle \right) \\ &= \min_{\varphi \in \mathcal{F}_\mathcal{C}} \min \left(\langle \varphi, \mu(g) \rangle, \min_{\alpha \in \Delta_\varphi^+} \langle \alpha, \mu(g) \rangle \right) \end{aligned} \tag{6.2}$$

(2) $\Rightarrow$ (1): Suppose ρ is τ-projective Anosov (Definition 6.1). Then there exist $C, C' > 0$ such that $\langle \varepsilon_1 - \varepsilon_2, \mu_V(\rho(\gamma)) \rangle \geq C \, |\gamma|_\Gamma - C'$ for all $\gamma \in \Gamma$. By (6.2) we have $|\langle \varphi, \mu(\rho(\gamma)) \rangle| \geq C \, |\gamma|_\Gamma - C'$ for all $\varphi \in W \cdot \varphi_0$ and all $\gamma \in \Gamma$, hence $d(\mu(\rho(\gamma)), \mu(H)) \geq C \, |\gamma|_\Gamma - C'$ for all $\gamma \in \Gamma$. Thus ρ is a quasi-isometric embedding and $\rho(\Gamma)$ is sharp for G/H.

(1) $\Rightarrow$ (2): Suppose that ρ is a quasi-isometric embedding and that $\rho(\Gamma)$ is sharp for G/H: there exist $C, C' > 0$ such that $d(\mu(\rho(\gamma)), \mu(H)) \geq C \, |\gamma|_\Gamma - C'$ for all $\gamma \in \Gamma$. By Fact 4.10, since Γ is *not* virtually cyclic, we have $\mu(\gamma) \in \mathcal{C}$ for all but finitely many $\gamma \in \Gamma$. Using Lemma 4.9 and (6.2) we deduce that there exist $C'', C''' > 0$ such that $\langle \varepsilon_1 - \varepsilon_2, \mu_V(\rho(\gamma)) \rangle \geq C'' \, |\gamma|_\Gamma - C'''$ for all $\gamma \in \Gamma$. Therefore Γ is word hyperbolic and ρ is τ-projective Anosov by Fact 4.2.

Suppose $\rho \in \mathrm{Hom}_\mathcal{C}(\Gamma, G)$ satisfies conditions (1) and (2). By [L], there is a neighborhood $\mathcal{U}$ of ρ in $\mathrm{Hom}(\Gamma, G)$ such that any $\rho' \in \mathcal{U}$ still satisfies (2). We claim that, up to making $\mathcal{U}$ smaller, it is contained in $\mathrm{Hom}_\mathcal{C}(\Gamma, G)$. Indeed, let $\gamma \in \Gamma$ be an infinite-order element.

Then $\lambda(\gamma) \neq 0$, and $\lambda(\rho(\gamma)) \in \mathcal{C}$ by sharpness. By continuity of λ, up to making $\mathcal{U}$ smaller, we have $\lambda(\rho'(\gamma)) \in \mathcal{C}$ for all $\rho' \in \mathcal{U}$. This completes the proof of Proposition 6.2.

6.3. **Proof of Corollary 6.3.** We first assume that Γ is *not* virtually cyclic. Let $\rho : \Gamma \to G$ be a quasi-isometric embedding such that $\rho(\Gamma)$ is sharp for G/H. By Fact 4.10, the limit cone of $\rho(\Gamma)$ (see Section 4.1) is entirely contained in one connected component $\mathcal{C}$ of $\mathfrak{a}^+ \smallsetminus \mu(H)$. By compactness of the projectivization of the limit cone, there is a finite subset $\mathcal{F}$ of $\mathfrak{a}^*$, consisting entirely of rational linear forms, such that

$$\mathcal{C}' := \mathfrak{a}^+ \cap \bigcup_{\varphi \in \mathcal{F}} \{\varphi > 0\} \subset \mathfrak{a}^+ \cap \bigcup_{\varphi \in \mathcal{F}} \{\varphi \geq 0\} \subset \mathcal{C}$$

and the limit cone of $\rho(\Gamma)$ is contained in $\mathcal{C}'$. Then ρ belongs to $\mathrm{Hom}_{\mathcal{C}'}(\Gamma, G)$. By Proposition 6.2, the group Γ is word hyperbolic and there is a finite-dimensional real linear representation (τ, V) of G such that ρ is τ-projective Anosov. Since being Anosov is an open condition (see [L]), there is a neighborhood of ρ in $\mathrm{Hom}(\Gamma, G)$ consisting entirely of τ-projective Anosov representations; by Proposition 6.2 again, these representations ρ' are quasi-isometric embeddings such that $\rho'(\Gamma)$ is sharp for G/H.

We now assume that Γ has a finite-index subgroup which is cyclic, generated by an element γ of infinite order. Then for any representation $\rho : \Gamma \to G$, the following are equivalent (where $\lambda : G \to \mathfrak{a}^+$ denotes the Jordan projection):

- ρ is a quasi-isometric embedding and $\rho(\Gamma)$ is sharp for G/H,
- $\lambda(\rho(\gamma)) \in \mathfrak{a}^+ \smallsetminus \mu(H)$.

The second condition is clearly open, and so the first one is open two. Suppose that $\rho : \Gamma \to G$ satisfies these conditions. Let $\varphi \in \mathfrak{a}^*$ be any linear form such that $\lambda(\rho(\gamma)) \notin \mathrm{Ker}(w \cdot \varphi)$ for all $w \in W$. Then $\rho(\Gamma)$ is sharp for G/H_φ where $H_\varphi := \exp(\mathrm{Ker}(\varphi)) \subset A$ as in Remark 4.8. By Proposition 6.2, the group Γ is word hyperbolic and there is a finite-dimensional real linear representation (τ, V) of G such that ρ is τ-projective Anosov.

This completes the proof of Corollary 6.3.

7. Manifolds locally modeled on big cells of G

We include here a result which will be useful in Section 8. It is a higher-rank variation on a result proven by the first-named author for Lie groups of rank one [T1].

Let G be a connected real linear reductive Lie group, P a proper parabolic subgroup of G, and P^* an opposite parabolic subgroup. Let $\Omega = PP^*$ be the open $(P \times P^*)$-orbit in G; it identifies with $(P \times P^*)/\mathrm{Diag}(P \cap P^*)$. Let ω denote the biinvariant volume form on G.

Following Thurston, given a manifold M, we call $(P \times P^*, \Omega)$-*structure* on M the data of a pair $(\mathrm{dev}, \mathrm{hol})$, where hol is a representation of $\pi_1(M)$ to $P \times P^*$ and $\mathrm{dev} : \widetilde{M} \to \Omega$ is a hol-equivariant local diffeomorphism; we call it *complete* if dev is a covering map.

Theorem 7.1. *A closed manifold cannot admit a $(P \times P^*, \Omega)$-structure.*

A manifold cannot admit a complete $(P \times P^*, \Omega)$-structure which is of finite volume with respect to ω.

In particular, a discrete subgroup of $P \times P^*$ cannot act properly discontinuously and cocompactly on G by left and right multiplication (or equivalently on $(G \times G)/\mathrm{Diag}(G)$).

Proof. Let a be a vector in $\mathrm{Lie}(G)$ such that ad_a is diagonalisable over $\mathbb{R}$ and such that $\mathrm{Lie}(P)$ and $\mathrm{Lie}(P^*)$ are respectively spanned by the nonnegative and nonpositive eigenspaces of ad_a. The stabilizer of $\mathbf{1}_G$ in Ω is $P \cap P^* = L$, whose adjoint action centralizes a. Hence a (seen as a tangent vector at $\mathbf{1}_G$) extends to a $P \times P^*$-invariant vector field X_a on Ω.

At every point $g = pp'$ in Ω, we have

$$X_a = L_p R_{p'}(a) = R_g(\mathrm{Ad}_p(a)) \ .$$

Since $\mathrm{Ad}_p(a)$ belongs to $\mathrm{Lie}(P)$, and since the left action of G is spanned by right-invariant vector fields, we get that X_a is tangent to the left P-orbit at every $g \in \Omega$. Moreover, along this P orbit, X_a is left-invariant. Hence X_a generates a complete flow $(\Phi_t)_{t\in\mathbb{R}}$.

The Lie derivative of ω along X_a can be written as $\mathrm{div}(X_a)\omega$, where $\mathrm{div}(X_a)$ is a constant by $P \times P^*$ homogeneity of both X_a and ω. We claim that $\mathrm{div}(X_a) < 0$.

To compute this, let $(e_1, \dots, e_d)$ denote a diagonalization basis of ad_a and $(\lambda_1, \dots, \lambda_d)$ the corresponding eigenvalues. Let us extend $(e_1, \dots, e_d)$ to a left invariant frame $(E_1, \dots, E_d)$ of TG and let $(E_1^*, \dots, E_d^*)$ denote the dual frame of T^*G. By left invariance of ω, we can assume (up to scaling) that $\omega = E_1^* \wedge \ldots \wedge E_d^*$.

Let us write $X_a = \sum_{i=1}^d \alpha_i E_i$. Then the Cartan formula gives that

$$\mathrm{div}(X_a) = \sum_{i=1}^d d\alpha_i(E_i) \ ,$$

which we now compute at the point $\mathbf{1}_G$.

Fix some $i \in \{1, \dots, d\}$. If $\lambda_i \geq 0$, then e_i belongs to $\mathrm{Lie}(P)$. Since X_a is left-invariant along P, the α_j are constant along P-orbits and in particular $d\alpha_i(e_i) = 0$. On the other hand, if $\lambda_i < 0$, then e_i belongs to $\mathrm{Lie}(P^*)$ and we have

$$\begin{aligned} X_a(\exp(te_i)) &= R_{\exp(te_i)} a \\ &= L_{\exp(te_i)} \mathrm{Ad}_{\exp(-te_i)}(a) \\ &= L_{\exp(te_i)} \exp(-t \,\mathrm{ad}_{e_i})(a) \\ &= L_{\exp(te_i)}(a + \lambda_i t e_i) \ . \end{aligned}$$

Taking the derivative at $t = 0$ and using the left invariance of the vector fields E_j, we get that

$$d\alpha_i(e_i) = \lambda_i \ .$$

We conclude that $\mathrm{div}(X_a)$ is the sum of all the negative eigenvalues of ad_a, hence $\mathrm{div}(X_a) < 0$.

Assume now that M is a manifold equipped with a $(P \times P^*, \Omega)$-structure $(\mathrm{dev}, \mathrm{hol})$. Then, pulling back ω and X_a by dev and quotienting by $\pi_1(M)$, we obtain a volume form $\bar{\omega}$ and a vector field $\bar{X}_a$ on M, which still satisfies

$$\bar{X}_a \cdot \bar{\omega} = \mathrm{div}(X_a)\bar{\omega} \ .$$

If M is compact, then $\bar{X}_a$ generates a complete flow Φ_t. If M is not assumed compact but the $(P \times P^*, \Omega)$-structure is complete, then completeness of the flow of X_a on Ω also guaranties that $\bar{X}_a$ generates a complete flow Φ_t.

Now, the change of variable formula gives that

$$\int_M \Phi_t^* \bar{\omega} = \int_M \bar{\omega} \ .$$

On the other hand, we have

$$\Phi_t^* \omega = \exp(t\mathrm{div}(X_a))\omega \ ,$$

and by the previous computation, $\exp(t\mathrm{div}(X_a)) < 1$. Together, these two identities contradict the finiteness of $\int_M \omega$, proving the first two claims of the theorem.

Finally, assume by contradiction $\Gamma \subset P \times P^*$ acts properly discontinuously and cocompactly on G. Up to taking a finite index subgroup we can assume that the action is also torsion free. Then

$$\Gamma\backslash\Omega \subset \Gamma\backslash G$$

is a manifold equipped with a complete $(P\times P^*, \Omega)$-structure of finite volume, contradicting the second statement of the theorem. □

8. Quotients of group manifolds of type A_2

The goal of this section is to prove Theorem 1.19, namely that for $G_0 = \mathrm{SL}(3,\mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, or for $G_0 = E_{6(-26)}$, a discrete subgroup Γ of $G_0 \times G_0$ acting properly and cocompactly on the group manifold G_0 is virtually a lattice of G_0 embedded in one factor.

8.1. **The Zariski dense case.** The key fact here is that the opposition involution of the root system of type A_2 is nontrivial and has a fixed locus of dimension 1. To explain how we use this property, let us start by sketching the proof of Theorem 1.19 assuming Γ is Zariski dense in $G_0 \times G_0$.

Let $\mathfrak{a}^+$ denote a Weyl chamber of a Cartan subspace of G_0, $\iota : \mathfrak{a}^+ \to \mathfrak{a}^+$ the opposition involution and v_0 a nonzero vector fixed by ι. Then the fixed locus of ι is $\mathbb{R}_+ v_0$. We denote respectively by p_1 and p_2 the projections of $G_0 \times G_0$ on its first and second factor.

Proposition 8.1. *Let G_0 be a simple linear Lie group with restricted root system of type A_2. Let Γ be a Zariski dense discrete subgroup of $G_0 \times G_0$. If Γ is sharp for $(G_0 \times G_0)/\mathrm{Diag}(G_0)$, then either p_1 or p_2 restricted to Γ has discrete image and finite kernel.*

Proof. Assume by contradiction that none of the projections is discrete with finite kernel. Then there exist sequences (γ_n) and (γ'_n) in Γ such that $p_2(\gamma_n)$ and $p_1(\gamma'_n)$ remain bounded while $p_1(\gamma_n)$ and $p_2(\gamma'_n)$ leave every compact set. Taking Cartan projections, we deduce that the limit cone $\mathcal{L}_\Gamma \subset \mathfrak{a}^+ \times \mathfrak{a}^+$ of Γ (see Section 4.1) intersects $\mathfrak{a}^+ \times \{0\}$ and $\{0\} \times \mathfrak{a}^+$.

Let $v, w \in \mathfrak{a}^+ \smallsetminus \{0\}$ be such that $(v, 0)$ and $(0, w)$ belong to $\mathcal{L}_\Gamma$. Since Γ is assumed Zariski dense, his limit cone is convex and invariant by the opposition involution. Hence $(v + \iota(v), 0)$ and $(0, w + \iota(w))$ also belong to $\mathcal{L}_\Gamma$. Finally, since the fixed locus of ι is one-dimensional, both $v + \iota(v)$ and $w + \iota(w)$ span $\mathbb{R}_+ v_0$. Thus $(v_0, 0)$ and $(0, v_0)$ belong to $\mathcal{L}_\Gamma$, and so does (v_0, v_0). This contradicts the fact that Γ is sharp for $G_0 \times G_0/\mathrm{Diag}(G_0)$. □

Now, once we know that one of the projections has discrete image and finite kernel, we can conclude using cohomological dimension and Margulis's classification of higher-rank lattice representations.

Proposition 8.2. *Let G_0 be a simple linear Lie group of real rank at least 2. Let Γ be a discrete subgroup of $G_0 \times G_0$ acting properly discontinuously and cocompactly on the group space G_0. Assume the restriction of p_1 to Γ has discrete image and finite kernel. Then Γ is virtually isomorphic to $\Gamma_0 \times \{\mathbf{1}_{G_0}\}$, where Γ_0 is a uniform lattice in G_0.*

Proof. Up to taking a finite-index subgroup, we can assume that $p_{1|\Gamma}$ is discrete and faithful. Hence $\Gamma = \{(\gamma, \rho(\gamma)) \mid \gamma \in \Gamma_0\}$, where Γ_0 is a discrete subgroup of G_0 and $\rho : \Gamma_0 \to G_0$.

The groups Γ and Γ_0 being isomorphic, they have the same virtual cohomological dimension, and since Γ acts properly discontinuously and cocompactly on G_0, so does Γ_0 by Fact 1.15. Hence Γ_0 is a uniform lattice in G_0.

Now, the superrigidity theorem of Margulis states that either ρ has bounded image or there exists a ρ-equivariant isometry of the Riemannian symmetric space of G_0. In the latter case, we have $\mu(\rho(\gamma)) = \mu(\gamma)$ for all $\gamma \in \Gamma_0$ and Γ does not act properly on $(G_0 \times G_0)/\mathrm{Diag}(G_0)$.

Hence ρ has bounded image. In that case, a well-known consequence of Margulis's arithmetic superrigidity implies that ρ either has finite image or the closure of its image is isogenous to a compact real form of the complexification of G_0. But G_0 cannot contain a compact real form of its complexification (since it is, itself, a noncompact real form). Hence ρ has finite image and is trivial on a finite-index subgroup of Γ_0. □

Combining Propositions 8.1 and 8.2 we conclude that a discrete subgroup of $G_0 \times G_0$ acting properly and cocompactly on G_0 is never Zariski dense. In order to complete the proof of Theorem 1.19 we need to deal with all possible Zariski closures. Unfortunately, we could not find a synthetic way to do that and, though we tried to use as much as possible general arguments that might be used in a different context, the end of the proof will be rather technical and not so far from a case-by-case study.

8.2. **Nonreductive Zariski closure.** In order to exploit Cartan projections and limit cones, it would be helpful to reduce to the situation where the Zariski closure of Γ is reductive. In this direction, we will prove a (midly satisfactory) general result.

Let G be a real reductive linear group and Γ a discrete subgroup of G. The Zariski closure $\overline{\Gamma}^Z$ of Γ in G can be written as the semidirect product $L \ltimes U$, where U is its unipotent radical and L a Levi factor. We denote by $Z(L)$ the center of L and by L_0 the derived subgroup of L. Recall that L_0 is semisimple and L is isogenous to $L_0 \times Z(L)$. The normal subgroup $Z(L) \ltimes U$ is the solvable radical of $\overline{\Gamma}^Z$. We denote by $\pi_{red} : L \ltimes U \to L$ the projection to L and by $\pi_{ss} : L \ltimes U \to L/Z(L)$ the projection modulo $Z(L) \ltimes U$.

Lemma 8.3. *Assume that the restriction of π_{red} to Γ is discrete and faithful. Let H be a closed subgroup of G. If Γ acts properly discontinuously and cocompactly on G/H, then so does $\pi_{red}(\Gamma)$.*

Proof. By sharpness, the limit cone $\mathcal{L}_\Gamma$ of Γ (see Section 4.1) meets the limit cone $\mathcal{L}_H$ only in 0, and so does the Jordan limit cone $\mathcal{L}^\lambda_\Gamma$ (since it is contained in $\mathcal{L}_\Gamma$). Now, the projection to the Levi factor preserves Jordan projections, hence $\mathcal{L}^\lambda_{\pi_{red}(\Gamma)} = \mathcal{L}^\lambda_\Gamma$. Finally, since the Zariski closure of $\pi_{red}(\Gamma)$ is reductive, we have $\mathcal{L}_{\pi_{red}(\Gamma)} = \mathcal{L}^\lambda_{\pi_{red}(\Gamma)}$, hence $\mathcal{L}_{\pi_{red}(\Gamma)}$ meets $\mathcal{L}_H$ only in 0 and $\pi_{red}(\Gamma)$ thus acts properly on G/H. Finally, since $\pi_{red}(\Gamma)$ is isomorphic

to Γ, they have the same cohomological dimension, hence $\pi_{red}(\Gamma)$ also acts cocompactly by Fact 1.15. □

It is tempting to conjecture that the condition of Lemma 8.3 is always satisfied for compact quotients.

Conjecture 8.4. Let G be a connected real linear reductive Lie group and H a reductive subgroup of G. If Γ is a discrete subgroup of G acting properly discontinuously and cocompactly on G/H, then the projection π_{red} restricted to Γ is discrete and faithful.

Note that we have the following related result of Raghunathan, which will be of some use in the proof.

Fact 8.5 (Raghunathan, [R, Cor. 8.27])**.** *The projection $\pi_{ss}(\Gamma)$ is discrete in $L/Z(L)$.*

Theorem 7.1 will also be useful in order to eliminate many nonreductive Zariski closures.

8.3. **Proof of Theorem 1.19.** We keep the notation of the previous subsections, where Γ is a discrete subgroup of $G_0 \times G_0$. From now on, G_0 is a connected simple linear Lie group with restricted root system of type A_2.

Lemma 8.6. *Let Γ be a discrete subgroup of $G_0 \times G_0$ acting properly discontinuously and cocompactly on $(G_0 \times G_0)/\mathrm{Diag}(G_0)$. Let Γ_U be a normal unipotent subgroup of Γ. Then the projection of Γ_U to one of the factors is trivial.*

Proof. Write the set of simple restricted roots of G_0 as $\Delta = \{\alpha_1, \alpha_2\}$, so that $\Sigma^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. There are two conjugacy classes of (nontrivial) unipotent elements of G_0: the class of $\exp(v_1)$ and the class of $\exp(v_1 + v_2)$ where $v_i \in (\mathfrak{g}_0)_{\alpha_i} \smallsetminus \{0\}$.

Assume by contradiction that both $p_1(\Gamma_U)$ and $p_2(\Gamma_U)$ are nontrivial.

Suppose first that $p_i(\Gamma_U)$ is conjugate to a subgroup of $\exp((\mathfrak{g}_0)_{\alpha_1} + (\mathfrak{g}_0)_{\alpha_1+\alpha_2})$ for both $i = 1, 2$. Then all nontrivial elements of $p_1(\Gamma_U)$ and all nontrivial elements of $p_2(\Gamma_U)$ are conjugate to the same element $\exp(v_1)$ with $v_1 \in (\mathfrak{g}_0)_{\alpha_1} \smallsetminus \{0\}$. This contradicts the proper discontinuity of the action of Γ on $(G_0 \times G_0)/\mathrm{Diag}(G_0)$.

Suppose now there exists $i \in \{1, 2\}$ such that $p_i(\Gamma_U)$ is *not* conjugate to a subgroup of $\exp((\mathfrak{g}_0)_{\alpha_1} + (\mathfrak{g}_0)_{\alpha_1+\alpha_2})$. Then the normalizer of $p_i(\Gamma_U)$ in G_0 is contained in a minimal parabolic subgroup of G_0, and so $p_i(\Gamma)$ is contained in a minimal parabolic subgroup of G_0. On the other hand, $p_{3-i}(\Gamma)$ is contained in a proper (not necessarily minimal) parabolic subgroup of G_0. Hence, up to conjugation, Γ is contained in $P \times P^*$ for some proper parabolic subgroup P of G_0. This contradicts Theorem 7.1. □

Let Γ be a discrete subgroup of $G_0 \times G_0$ and let $i = 1$ or 2.

- We say that Γ satisfies *property D_i* if the projection p_i restricted to Γ has finite kernel and discrete image.
- We say that Γ satisfies *property L_i* if for every $\varepsilon > 0$, there exists $\gamma \in \Gamma$ such that

$$\|\lambda_{3-i}(\gamma)\| \geq 1 \ ,$$
$$\|\lambda_i(\gamma)\| < \varepsilon \|\lambda_{3-i}(\gamma)\|$$

and

$$\|\lambda_{3-i}(\gamma) - \iota(\lambda_{3-i}(\gamma))\| \leq \varepsilon \|\lambda_{3-i}(\gamma) \ .$$

- We say that Γ satisfies *property* C_i if there exists $(\gamma_n) \in \Gamma^{\mathbb{N}}$ such that
$$\|\mu_{3-i}(\gamma)\| = n + o(n) \ ,$$
$$\|\mu_i(\gamma)\| = o(n)$$
and
$$\|\mu_{3-i}(\gamma) - \iota(\mu_{3-i}(\gamma))\| = o(n) \ .$$

Property L_1 is equivalent to asking that $\mathcal{L}^\lambda_\Gamma$ contains $\{0\} \times \mathbb{R}_{\geq 0} v_0$, where $v_0 \in \mathfrak{a}_+$ is as in Section 8.1, and Property C_1 implies the same for $\mathcal{L}_\Gamma$. Our goal is to prove the following lemma, from which we will conclude as in the proof of Proposition 8.1.

Lemma 8.7. *Let Γ be a discrete subgroup of $G_0 \times G_0$ acting properly discontinuously and cocompactly on G_0. Then for each $i = 1, 2$, the group Γ satisfies properties D_i or C_i.*

The proof of this lemma will occupy some time, so it seems useful to postpone it and start by explaining how it implies Theorem 1.19.

Proof of Theorem 1.19 assuming Lemma 8.7. If Γ satisfies property D_1 or D_2, then Γ is virtually equal to $\Gamma_0 \times \{\mathbf{1}_{G_0}\}$ or $\{\mathbf{1}_{G_0}\} \times \Gamma_0$ by Proposition 8.2.

Otherwise, by Lemma 8.7, the group Γ must satisfy both properties C_1 and C_2. Let (γ_n) and (γ'_n) be as in the definition of properties C_1 and C_2 respectively. Recall that for all $g, h \in G$,
$$\|\mu(gh) - \mu(g)\| \leq \|\mu(h)\|$$
(see (2.1)). Since the restricted root system of G is of type A_2, the locus of fixed points of the opposition involution in $\mathfrak{a}^+$ is a ray, and the conditions on $\mu(p_2(\gamma_n))$ and $\mu(p_1(\eta_n))$ imply
$$\mu(p_2(\gamma_n)) = \mu(p_1(\gamma'_n)) + o(n) \ .$$
Together with the conditions on $\mu(p_1(\gamma_n))$ and $\mu(p_2(\gamma'_n))$, and using that $\|\mu(gh) - \mu(g)\| \leq \|\mu(h)\|$ (again by (2.1)), we obtain that
$$\mu(p_1(\gamma_n\gamma'_n)) = \mu(p_2(\gamma_n\gamma'_n)) + o(\|\mu(p_2(\gamma_n\gamma'_n))\|) \ ,$$
contradicting the sharpness of Γ for $G_0 \times G_0/\mathrm{Diag}(G_0)$ given by Theorem 1.5. □

To prepare for the proof of Lemma 8.7, let us start by giving some sufficient conditions for satisfying property C_i:

Proposition 8.8. *If one of the following properties hold:*

(*i*) *Γ satisfies property L_i,*
(*ii*) *the kernel of p_i contains a nontrivial unipotent element,*
(*iii*) *the kernel of p_i is abelian and its reductive projection is not discrete and faithful,*
(*iv*) *the kernel of p_i is abelian and not virtually cyclic,*

then Γ satisfies property C_i.

Here, we call reductive projection of a discrete group Λ the projection to a Levi factor of its Zariski closure.

Proof. Assume (i): Let γ_k be such that
$$\|\lambda_{3-i}(\gamma_k)\| \geq 1 \ ,$$

$$\|\lambda_i(\gamma_k)\| < \frac{1}{k}\|\lambda_{3-i}(\gamma)\|$$

and

$$\|\lambda_{3-i}(\gamma_k) - \iota(\lambda_{3-i}(\gamma_k))\| \leq \frac{1}{k}\|\lambda_{3-i}(\gamma_k)\ .$$

Set $\lambda_k = \|\lambda_{3-i}(\gamma_k)\|$. We have

$$\frac{1}{n}\mu_{3-i}(\gamma_k^n) \underset{n\to+\infty}{\longrightarrow} \lambda_{3-i}(\gamma_k)\ ,$$

hence

$$\mu_{3-i}(\gamma_k^{\lfloor n/\lambda_k\rfloor}) = n + o(n)\ .$$

Moreover, for n large enough we have

$$\left\|\mu_i(\gamma_k^{\lfloor n/\lambda_k\rfloor})\right\| \leq \frac{2n}{k}$$

and

$$\left\|\mu_{3-i}(\gamma_k^{\lfloor n\lambda_k\rfloor}) - \iota\mu_{3-i}(\gamma_k^{\lfloor n/\lambda_k\rfloor})\right\| \leq \frac{2n}{k}\ .$$

Set k_n to be the supremum of n and all the indices k such that

$$(1-\frac{1}{k})n \leq \left\|\mu_{3-i}(\gamma_k^{\lfloor n/\lambda_k\rfloor})\right\| \leq (1+\frac{1}{k})n\ ,$$

$$\|\mu_i(\gamma_k^{\lfloor n/\lambda_k\rfloor})\| \leq \frac{2n}{k}$$

and

$$\left\|\mu_{3-i}(\gamma_k^{\lfloor n/\lambda_k\rfloor}) - \iota\mu_{3-i}(\gamma_k^{\lfloor n/\lambda_k\rfloor})\right\| \leq \frac{2n}{k}\ .$$

Since, for a given k, these properties are satisfied for n sufficiently large, we get that the sequence k_n diverges to $+\infty$. Setting

$$\gamma'_n = \gamma_{k_n}^{\lfloor n/\lambda_{k_n}\rfloor}\ ,$$

$$\|\mu_{3-i}(\gamma'_n)\| = n + o(n)\ ,$$

$$\|\mu_i(\gamma'_n)\| = o(n)$$

and

$$\|\mu_{3-i}(\gamma'_n) - \iota(\mu_{3-i}(\gamma'_n))\| = o(n)\ .$$

Hence Γ satisfies C_i.

Assume (ii): Let γ be a nontrivial unipotent element in the kernel of p_i. Then one has

$$\mu_{3-i}(\gamma^n) \simeq \log(n)\, v\ ,$$

where $v \neq 0$ depends on the conjugacy class of γ. Moreover, since γ and γ^{-1} are conjugate, we have $\iota(v) = v$. Then

$$\gamma'_n = \gamma^{\left\lfloor e^{\frac{n}{\|v\|}}\right\rfloor}$$

satisfies the required properties.

Assume (iii): Let $\{\mathbf{1}_{G_0}\}\times\Lambda$ be the kernel of p_1 which is assumed to be abelian. Let us write the Zariski closure of Λ in G_0 as $A'_0U'_0$ where U'_0 is the unipotent radical of $\overline{\Lambda}^Z$ and

A_0' its Levi factor. As a general property of abelian Lie groups, the following holds: either the projection of Λ to A_0' is discrete and faithful, or there exists $u \in U_0'$ and a compact set K such that $\langle u \rangle \subset \Lambda K$. Let $\gamma_n \in \Lambda$ be such that

$$u^{\left\lfloor e^{\frac{n}{\|v\|}} \right\rfloor} \in p_{3-i}(\gamma_n) K \ .$$

Then the Cartan projection of γ_n remains at bounded distance from that of $u^{\left\lfloor e^{\frac{n}{\|v\|}} \right\rfloor}$, proving property C_i as in case (ii).

Assume (iv): Let again $\{\mathbf{1}_{G_0}\} \times \Lambda$ be the (abelian) kernel of p_1, with Zariski closure $\{\mathbf{1}_{G_0}\} \times A_0'U_0'$ as above. If the projection of Λ to A_0' is not discrete and faithful then we are in case (iii). Otherwise, this projection contains a discrete copy of $\mathbb{Z}^2$. Since G_0 has real rank 2, we get that the Jordan projection of A_0' is the whole Weyl chamber, and that the projection of Λ is a lattice in A_0'. There thus exist elements of Λ whose Jordan projection are at uniformly bounded distance from any vector in $\{0\} \times \mathfrak{a}_0^+$. Therefore Γ satisfies property L_i, hence property C_i. □

We will also need the following result:

Proposition 8.9. *Let Γ be a discrete subgroup of $G_0 \times G_0$ which is sharp for $(G_0 \times G_0)/\mathrm{Diag}(G_0)$. Suppose that there exists a central element of Γ of the form $\gamma = (1_{G_0}, \gamma_0)$, where $\gamma_0 \in G_0$ is semisimple and generates an infinite discrete cyclic subgroup $\langle \gamma_0 \rangle$ of G_0. Then the action of Γ on $(G_0 \times G_0)/\mathrm{Diag}(G_0)$ is not cocompact.*

Proof. Let $Z_0 \coloneqq Z_{G_0}(\gamma_0)$ be the centralizer of γ_0 in G_0: it is a reductive subgroup of G_0, of real rank 2, contained in a Levi factor L_0 of a proper parabolic subgroup of G_0. The center of Z_0 contains the infinite discrete subgroup $\langle \gamma_0 \rangle$, hence it contains a one-parameter subgroup A_0' containing $\langle \gamma_0 \rangle$. By construction, $\langle \gamma_0 \rangle$ is a uniform lattice in A_0', and the centralizer of A_0' in G_0 is exactly Z_0. The group Z_0/A_0' is reductive of real rank 1.

We can see G_0/A_0' as the homogeneous space G'/H', where $G' = G_0 \times (Z_0/A_0')$ (a reductive group of real rank 3) and H' is the subgroup of G' which is the image of the diagonal embedding of Z_0 (a reductive group of real rank 2). Thus G'/H' has real corank 1.

The group $\Gamma/\langle \gamma \rangle \coloneqq \Gamma' \subset G'$ acts properly discontinuously and cocompactly on G'/H'. By Proposition 1.13, this implies that the group Γ' is word hyperbolic and that the natural inclusion $\Gamma' \hookrightarrow G' = G_0 \times (Z_0/A_0')$ is Anosov with respect to some proper parabolic subgroup of G', which means that either the first projection of Γ' to G_0 or the second projection of Γ' to Z_0/A_0' is Anosov with respect to some proper parabolic subgroup.

Suppose the first projection of Γ' to G_0 is Anosov with respect to some proper parabolic subgroup P_0 of G_0. Then $\partial_\infty \Gamma'$ embeds into the flag manifold G_0/P_0. This bounds the covering dimension of $\partial_\infty \Gamma'$ by $\dim(G_0/P_0) \leq \dim((\mathfrak{g}_0)_{\alpha_1}) + \dim((\mathfrak{g}_0)_{\alpha_1+\alpha_2})$. Using Fact 4.3, we obtain

$$\mathrm{vcd}(\Gamma) = \mathrm{vcd}(\Gamma') + 1 = \dim \partial_\infty \Gamma' + 2 \leq \dim(\mathfrak{g}_0)_{\alpha_1} + \dim(\mathfrak{g}_0)_{\alpha_1+\alpha_2} + 2.$$

Now suppose the second projection of Γ' to Z_0/A_0' is Anosov; in particular, it has finite kernel and discrete image. The group Γ' acts properly discontinuously on the Riemannian symmetric space of Z_0/A_0', and so Fact 1.15 implies that $\mathrm{vcd}(\Gamma')$ is bounded by the dimension of this symmetric space, which is the dimension of the symmetric space of Z_0 minus

one. But we saw above that Z_0 is contained in a Levi subgroup L_0 of a proper parabolic subgroup of G_0, hence the dimension of the symmetric space of Z_0 is bounded by that of L_0, which is $1 + \dim(\mathfrak{g}_0)_{\alpha_1}$. Using Fact 4.3, we obtain

$$\mathrm{vcd}(\Gamma) = \mathrm{vcd}(\Gamma') + 1 \leq \dim(\mathfrak{g}_0)_{\alpha_1}.$$

In either case, we see that $\mathrm{vcd}(\Gamma)$ is strictly less than $\dim(\mathfrak{g}_0)_{\alpha_1} + (\mathfrak{g}_0)_{\alpha_2} + \dim(\mathfrak{g}_0)_{\alpha_1+\alpha_2} + 2$, which is the dimension of the Riemannian symmetric space of G_0. By Fact 1.15, this means that the action of Γ on $(G_0 \times G_0)/\mathrm{Diag}(G_0)$ cannot be cocompact. □

Lemma 8.10. *Let Γ be a discrete subgroup of $G_0 \times G_0$. Assume that the Zariski closure of $p_2(\Gamma)$ is reductive with compact center. Then Γ satisfies property D_1 or L_1.*

This lemma follows from the following result, which is a consequence of a theorem of Abels–Margulis–Soifer [AMS], as developed by Benoist [Be2]: see [GGKW1, Th. 4.12].

Fact 8.11. *Let G be a connected real linear reductive Lie group and Γ a Zariski dense subgroup of G. Then there exists a finite subset F of Γ and a constant $C > 0$ such that:*

(1) for every $\gamma \in \Gamma$, there exists $f \in F$ such that

$$\|\lambda(\gamma f) - \mu(\gamma)\| \leq C,$$

(2) for every $\gamma, \gamma' \in \Gamma$, there exists $f \in F$ such that

$$\|\mu(\gamma f \gamma') - \mu(\gamma) - \mu(\gamma')\| \leq C.$$

Proof of Lemma 8.10. Suppose that the Zariski closure L_0 of $p_2(\Gamma)$ in G_0 is reductive with compact center. Let us denote by $\mathfrak{a}_L^+$ a Weyl chamber in a Cartan subspace of L_0 and let μ_L, λ_L and ι_L denote the Cartan projection, Jordan projection and, opposition involution of L_0, respectively.

Assume Γ does not satisfy property D_1. Then for every $\varepsilon > 0$, there exists $\gamma \in \Gamma$ such that

$$\|\mu(p_1(\gamma))\| \leq \varepsilon \text{ and } \|\mu_L(p_2(\gamma))\| \geq \frac{1}{\varepsilon}\ .$$

Since $p_2(\Gamma)$ is Zariski dense in L, we can apply Fact 8.11 and find f, f' in (a finite subset F of) Γ and a constant C (independent of ε) such that

$$\|\mu_L(p_2(\gamma f \gamma^{-1})) - \mu_L(p_2((\gamma)) - \iota_L(\mu_L(p_2(\gamma)))\| \leq C$$

and

$$\|\lambda_L(p_2(\gamma f \gamma^{-1} f')) - \mu_L(p_2(\gamma f \gamma^{-1}))\| \leq C.$$

By the triangle inequality, we obtain

$$\|\lambda_L(p_2(\gamma f \gamma^{-1} f')) - \mu_L(p_2(\gamma)) - \iota_L(\mu_L(p_2(\gamma)))\| \leq 2C.$$

Since L has compact center, the cone $\mathfrak{a}_+^L$ is acute and $\mu_L(p_2(\gamma)) + \iota_L\mu_L(p_2(\gamma))$ is a non-zero vector fixed by the opposition involution. Then $\gamma f \gamma^{-1} f'$ satisfies the conditions of property L_1 (up to replacing ε by $2C\varepsilon$). □

We finally turn to the proof of Lemma 8.7, which will complete the proof of Theorem 1.19.

Proof of Lemma 8.7. We prove that Γ satisfies property D_1 or C_1; the proof that it satisfies D_2 or C_2 is analogous. Let us write the Zariski closure of Γ as $L \ltimes U$, where L is a Levi factor and U the unipotent radical. Then the commutator group $[\Gamma, \Gamma]$ is Zariski dense in

$[L,L] \ltimes U'$, where U' is a subgroup of U and $[L,L]$ is semisimple. Let $\pi : L \ltimes U \to L$ denote the projection to the Levi factor. By Fact 8.5, the group $\pi([\Gamma,\Gamma])$ is discrete. If $\pi([\Gamma,\Gamma])$ satisfies property L_1, then so does $[\Gamma,\Gamma]$ (because π preserves Jordan projections), and so does Γ (the property is obviously stable by enlarging the group). Hence Γ satisfies Property C_1 by Proposition 8.8 (i) and we are done.

So we now assume that $\pi([\Gamma,\Gamma])$ does *not* satisfy Property L_1, which implies, by Lemma 8.10, that $\pi([\Gamma,\Gamma])$ satisfies Property D_1: the projection p_1 restricted to $\pi([\Gamma,\Gamma])$ is discrete with finite kernel. By Lemma 8.6, the kernel of $\pi_{|[\Gamma,\Gamma]}$ is contained in $G_0 \times \{\mathbf{1}_{G_0}\}$ or in $\{\mathbf{1}_{G_0}\} \times G_0$. In particular, its first projection is discrete. If its second projection is nontrivial, then the kernel of p_1 contains a nontrivial unipotent element, Γ satisfies property C_1 by Proposition 8.8 (iii), and we are done. Otherwise, p_1 restricted to $[\Gamma,\Gamma]$ is discrete with finite kernel and, up to passing to a finite-index subgroup, we can assume that p_1 restricted to $[\Gamma,\Gamma]$ is discrete and faithful, which we do from now on.

Recall that we have assumed that Γ does not satisfy Property D_1. Therefore, $p_1(\Gamma)$ is not discrete or $p_{1|\Gamma}$ has discrete image but infinite kernel.

First case: $p_1(\Gamma)$ non discrete. Let A_0' be the connected component of the identity in the topological closure of $p_1(\Gamma)$. Fixing some $\gamma \in \Gamma$, for every $\eta \in \Gamma$ such that $p_1(\eta)$ is sufficiently close to $\mathbf{1}_{G_0}$, we have that $p_1([\gamma,\eta]) \in p_1([\Gamma,\Gamma])$ is close to $\mathbf{1}_{G_0}$. Since $p_1([\Gamma,\Gamma])$ is discrete, we get that γ commutes with every such η and deduce that A_0' is centralized by $p_1(\Gamma)$. The fact that $p_1(\Gamma)$ centralizes a 1-parameter subgroup implies that $p_1(\Gamma)$ is either contained in a minimal parabolic subgroup or in a Levi factor of a proper parabolic subgroup of G_0. Hence $p_2(\Gamma)$ cannot be contained in a proper parabolic subgroup of G_0 (otherwise Γ would be conjugate to a subgroup of $P_1 \times P_2$ or $P_2 \times P_1$, contradicting Theorem 7.1). Similarly, no finite index subgroup of $p_2(\Gamma)$ can be contained in a proper parabolic subgroup of G_0, hence the Zariski closure of $p_2(\Gamma)$ is reductive with compact center and we conclude that Γ satisfies property C_1 by Lemma 8.10.

Second case: $p_{1|\Gamma}$ has discrete image and infinite kernel. Since this kernel does not intersect $[\Gamma,\Gamma]$, it is central in Γ. By Proposition 8.8 (iii)-(iv), we may assume that this kernel is virtually infinite cyclic, and that its reductive projection is discrete and faithful, for otherwise Γ satisfies Property C_1 and we are done.

To summarize, we now assume that $p_{1|\Gamma}$ has discrete image and a cyclic central kernel generated by an element $\gamma = (\mathbf{1}_{G_0}, \gamma_0)$ whose semisimple part span an infinite discrete group, and we want to derive a contradiction. Let us write the Zariski closure of $\langle \gamma_0 \rangle$ as $A_0' \times U_0'$, where U_0' is the unipotent radical and A_0' is a Levi factor. Note that $\{\mathbf{1}_{G_0}\} \times U_0'$ is a central (hence normal) unipotent subgroup of $L \ltimes U$, hence $\{\mathbf{1}_{G_0}\} \times U_0' \subset U$. Up to conjugating the Levi factor L of $\overline{\Gamma}^Z$, we can assume that $\{\mathbf{1}_{G_0}\} \times A_0' \subset L$. Then $\pi_{|\langle\gamma\rangle}$ is the projection to the $\{\mathbf{1}_{G_0}\} \times A_0'$ factor, which is discrete by our assumption on γ.

Since $p_2(\Gamma)$ centralizes an infinite cyclic subgroup, arguing as above, we get that $p_1(\Gamma)$ cannot be contained in a proper parabolic subgroup of G_0 (otherwise we would derive a contradiction from Theorem 7.1). Thus $p_1(U) = \{\mathbf{1}_{G_0}\}$ (for otherwise $p_1(\Gamma)$ would normalize a non-trivial unipotent subgroup), and $p_1 \circ \pi(\Gamma) = p_1(\Gamma)$ is discrete. Hence the

preimage under π of a small neighborhood of the identity is contained in the kernel of p_1. Since $\pi_{|\langle\gamma\rangle}$ is discrete and faithful, we conclude that $\pi_{|\Gamma}$ is discrete and faithful.

Now, $\pi(\Gamma)$ must still act properly discontinuously and cocompactly on G_0 by Lemma 8.3 while, on the other hand, $\pi(\Gamma)$ centralizes the cyclic subgroup $\langle\pi(\gamma)\rangle$ generated by a semisimple element. This contradicts Proposition 8.9. □

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