The Craig Interpolation Theorem for Schematic Systems

A. CARBONE *

Institut für Algebra und Diskrete Mathematik
Technische Universität Wien
e-mail: ale@logic.tuwien.ac.at

Abstract. The notion of Schematic System has been introduced by Parikh in the early seventies. It is a metamathematical notion describing the concept of deduction system and the operation of substitution of terms and formulas in it. We show a generalization of the Craig Interpolation Theorem for a natural class of schematic systems while we determine sufficient conditions for a schematic system to enjoy Interpolation. These conditions are much weaker than the usual conditions of symmetricity that are satisfied by the logics usually studied. The proof of the Craig Interpolation Theorem that we propose is a refinement of Maehara's construction and it is based on the idea of tracing the flow of formulas in a proof.

1 Introduction

For two given formulas A and B such that $A \to B$ is provable, there is a formula C (called *interpolant*) expressed in the language common to A, B (roughly all symbols in C appear in both A and B) and such that $A \to C$ and $C \to B$ are provable.

This is the statement of the Interpolation Theorem proved for classical logic by Craig in 1957 ([?]). Later, several authors showed that the statement holds indeed for many other logical systems such as intuitionistic logic, infinitary logics, certain multiple-valued logics, fragments of Linear Logic, and so forth. We will prove a generalization of Craig's Theorem for a class of logical systems (called *regular schematic systems*) and show that the theorem is properly a statement about the structure of derivations and it is not based on the validity of the calculus. For instance the cut-free system of classical logic extended with a pair of rules of the form

$$\frac{\Gamma \to \Delta, P}{\Gamma \to \Delta, P \wedge \bot} \qquad\qquad \frac{P, \Gamma \to \Delta}{P \wedge \bot, \Gamma \to \Delta}$$

(with Γ, Δ being two arbitrary collections of formulas) enjoys the Interpolation Property even though the left rule is *not* valid.

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We will see that the logical form of the rules allowed in regular schematic systems does not require the well-known *subformula property* (i.e. roughly speaking all formulas occurring in the antecedent of a rule should appear as (sub)formulas in the consequent) to hold. It will just require a weaker 'embedding' condition (as discussed in section **??**).

The class of logical systems we consider will be defined through the metamathematical notion of *schematic system* introduced by Parikh in the early seventies. Section **??** will be devoted to a discussion of schematic systems and regular schematic systems. In section **??** we introduce logical graphs tracing the flow of occurrences of formulas in a proof. Our definition is a variant of the notion of *logical flow graph* introduced in [**?**]. Similar notions appeared also in [**?**]. In section **??** we use the graph theoretic tool to prove Craig's Theorem for regular schematic systems. As a corollary of the result, we derive the interpolation property for several systems known in the literature. Section **??** contains a discussion about further development of this metamathematical approach to logical systems.

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2 Schematic Systems

Axiom schemes and rules of inference are generally explained in the literature with the help of formula variables, so in order to define the notion of *schematic system* one can do the obvious, namely, expand the notation of the predicate calculus to include metamathematical symbols and emphasize *substitution* of terms and formulas as the central idea. The notion of schematic system was first introduced by Parikh in [**?**]. The notion has been used in further work in [**?**], [**?**].

We will use Gentzen style sequent schematic systems where we can easily formalize proofs as trees of sequents. We will think of each sequent as used only once in the proof.

2.1 The language

To introduce the concepts we follow Parikh's presentation ([**?**]). The language $\mathcal{L}$ will include

1. variables $x, y, z, \ldots$;
2. constants $c, c_1, c_2, \ldots$;
3. first order function symbols $f, g, \ldots$;
4. first order n-ary (for $n > 0$) predicate constants $F(x), G(x, y), \ldots$ (possibly the binary equality symbol $=$);
5. special predicate constants $\top, \bot, \ldots$;
6. logical symbols $\wedge, \vee, \supset, \neg, \exists, \forall, \ldots$;

7. other symbols $(,), [,], \ldots$.

We do not assume that the whole apparatus mentioned above is present in our language; on the other hand, we allow *extra* symbols to occur, for instance new types of connectives or higher order variables [2].

The language $\mathcal{L}$ includes as metamathematical symbols

1. metavariables $u, v, w, u_1, \ldots$;
2. term variables $r, s, t, r_1, \ldots$;
3. n-ary predicate (formula) variables (for $n \geq 0$) $P, Q(u), R(u, v), \ldots$.

Terms will be formed from variables, constants, metavariables and term variables, by means of function symbols. Terms not containing metanotation will be called *regular terms*. Atomic formulas will be formed from predicate variables and predicate constants by suitably placing terms as their arguments. Formulas will be formed from atomic formulas, special predicate constants, connectives and quantifiers in the usual way. Formulas not containing metanotation will be called *regular formulas*. Terms and formulas will be denoted by Greek and script letters respectively, Roman if they are regular.

Since we will work with a Gentzen style formulation of a calculus, we say that a *schematic sequent* is a sequent $\mathcal{A}_1, \ldots, \mathcal{A}_l \rightarrow \mathcal{A}_{l+1}, \ldots, \mathcal{A}_k$ of formulas in $\mathcal{L}$. The shorthand notation $\mathcal{S}[\mathcal{A}_1, \ldots, \mathcal{A}_n]$ denotes a schematic sequent containing the formula occurrences $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and maybe other formulas as well. The position (i.e. left or right) of the formulas $\mathcal{A}_1, \ldots, \mathcal{A}_n$ in $\mathcal{S}[\mathcal{A}_1, \ldots, \mathcal{A}_n]$ is intended to be fixed. Namely, everytime we write $\mathcal{S}[\mathcal{A}_1, \ldots, \mathcal{A}_n]$ we really mean to write $\mathcal{S}[\mathcal{A}_1^{i_1}, \ldots, \mathcal{A}_n^{i_n}]$ (with $i_j \in \{1, 2\}$ and $j = 1 \ldots n$) where $\mathcal{A}_j^1$ denotes the formula A_j occurring in the left hand side of the sequent arrow, and $\mathcal{A}_j^2$ denotes A_j occurring in the right hand side.

Although we will not analyse specific logical systems here, let us describe briefly how a specific system can be derived by substitution from a schematic system. A *substitution* δ will be an assignment of variables $x_1, \ldots, x_n$ to certain metavariables $u_1, \ldots u_n$, of regular terms $\alpha_1, \ldots, \alpha_m$ to term variables $t_1, \ldots, t_m$ and of regular formulas $A_1, A_2, \ldots$ to certain predicate variables $P, Q(u), \ldots$. Substitutions will be informal objects used to study formal systems and variables will always be such as to avoid conflicts.

The substitution δ induces a map that we will also call δ, from certain terms to regular terms uniquely defined by

$$\delta(u_i) = x_i, \quad \delta(t_j) = \alpha_j, \qquad i = 1 \ldots, n, \quad j = 1 \ldots m$$

and

$$\delta(f(\sigma_1, \ldots \sigma_k)) = f(\delta(\sigma_1), \ldots, \delta(\sigma_k))$$

provided that the right hand side is defined and that k is the arity of the function symbol f.

The substitution δ also induces a map on formulas $\mathcal{F}$ defined by

[2] Variables of order $n+1$ are intended to range over the set of objects of order n; in particular, variables have no arguments and the only atomic expression in which $(n+1)$-th order variables $\mathcal{X}$ can occur is $X \in \mathcal{X}$ where X is of order n.

1. $\delta(\mathcal{F}) = F(\delta(\sigma_1), \ldots, \delta(\sigma_k))$ when $\mathcal{F}$ is the atomic formula $F(\sigma_1, \ldots, \sigma_k)$ with predicate constant F and the right hand side is defined;
2. $\delta(\mathcal{F}) = A[\delta(\sigma_1), \ldots, \delta(\sigma_k)]$ when $\mathcal{F}$ is the atomic formula $P(\sigma_1, \ldots, \sigma_k)$ with predicate variable P such that $\delta(P(u_1, \ldots u_k)) = A[x_1, \ldots x_n]$ (where the symbol $A[x_1, \ldots x_n]$ means that A is a regular formula containing the variables $x_1, \ldots x_n$) and the right hand side is defined;
3. $\delta(\mathcal{F} \wedge \mathcal{G}) = \delta(\mathcal{F}) \wedge \delta(\mathcal{G})$, etc.;
4. $\delta((\forall x)\mathcal{F}) = (\forall x)\delta(\mathcal{F})$, etc.;
5. $\delta((\forall u)\mathcal{F}) = (\forall \delta(u))\delta(\mathcal{F})$, etc.

Finally a substitution δ applied to a sequent $\mathcal{A}_1, \ldots, \mathcal{A}_l \to \mathcal{A}_{l+1}, \ldots, \mathcal{A}_k$ is defined as the sequent $\delta(\mathcal{A}_1), \ldots, \delta(\mathcal{A}_l) \to \delta(\mathcal{A}_{l+1}), \ldots, \delta(\mathcal{A}_k)$.

A *restriction* is called *admissible* if it is of the form 'provided σ is free for u (for x) in P ($\mathcal{S}$)' or 'provided u (x) is not free in P ($\mathcal{S}$)' or 'provided u (x) does not occur in P ($\sigma, \mathcal{S}$)' where u is a metavariable, x is a variable, σ is a term, P a predicate variable and $\mathcal{S}$ is a schematic sequent.

It is clear what we intend by saying that a substitution δ obeys a restriction R, e.g. δ obeys 'provided σ is free for u in P' iff $\delta(\sigma)$ is free for $\delta(u)$ in $\delta(P)$. Whenever we will refer to restrictions we will intend them to be admissible restrictions.

We are now ready to define what are a schematic axiom and a schematic rule. A *schematic axiom* is a pair $(\mathcal{S}, R)$ where R is a finite set of restrictions and $\mathcal{S}$ is a schematic sequent of the form

$$\mathcal{A}_1, \Gamma \to \Delta, \mathcal{A}_2$$

where $\mathcal{A}_1$ and $\mathcal{A}_2$ are called *distinguished* occurrences in the axiom and Γ, Δ denote arbitrary collections (i.e. sets that possibly contain multiple occurrences of the same formula) of formulas called *side* formulas of the axiom. We ask either that all (non special) predicate constants and predicate variables occurring in $\mathcal{A}_1$ also occur in $\mathcal{A}_2$ or the other way around. We allow a schematic axiom to have no distinguished occurrences in one of the sides of the sequent arrow. A schematic axiom has to have at least one distinguished occurrence though. We assume that sequents of the form $\mathcal{A}, \Gamma \to \Delta, \mathcal{A}$, $\Gamma \to \Delta, \top$ and $\bot, \Gamma \to \Delta$ are schematic axioms.

To avoid any confusion on side formulas, let us remark that two schematic axioms $\mathcal{A}_1, \Gamma \to \Delta, \mathcal{A}_2$ and $\mathcal{A}_1, \Gamma' \to \Delta', \mathcal{A}_2$ are assumed to be the same schematic axiom. An axiom $A_1, \Theta \to \Lambda, A_2$ is *derived* from a schematic axiom $\mathcal{A}_1, \Gamma \to \Delta, \mathcal{A}_2$ if there is a substitution map δ such that $\delta(\mathcal{A}_1) = A_1$ and $\delta(\mathcal{A}_2) = A_2$, and Θ, Λ are regular formulas.

Similarly, a *schematic rule* of *arity* k is a sequence

$$(\mathcal{S}_0[\mathcal{A}_0], \mathcal{S}_1[\mathcal{A}_{1,1}, \ldots, \mathcal{A}_{1,n_1}], \ldots, \mathcal{S}_k[\mathcal{A}_{k,1}, \ldots, \mathcal{A}_{k,n_k}], R)$$

where R is a finite set of restrictions and $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_k$ are schematic sequents. A schematic rule will be usually written as

$$\frac{\mathcal{S}_1[\mathcal{A}_{1,1}, \ldots, \mathcal{A}_{1,n_1}], \ldots, \mathcal{S}_k[\mathcal{A}_{k,1}, \ldots, \mathcal{A}_{k,n_k}]}{\mathcal{S}_0[\mathcal{A}_0]}$$

where $\mathcal{A}_{i,1}, \ldots, \mathcal{A}_{i,n_i}$ are the *auxiliary* occurrences in the i-th sequent of the rule and $\mathcal{A}_0$ is the *main* formula of the rule. All formulas occurring in the sequents $\mathcal{S}_i$ which are neither an auxiliary formula (i.e. $A_{i,j}$, for some $i = 1 \ldots k$, $j = 1 \ldots n_i$) nor a main formula (i.e. A_0), are called *side* formulas of the rule. If the condition 'provided u (x) does not occur in $\mathcal{S}_0$' is in R then we call the variable u (x) *eigenvariable*.

We require that

1. $\mathcal{L}(\mathcal{A}_{i,j}) \subset^* \mathcal{L}(\mathcal{A}_0)$ for all $i = 1 \ldots k$ and $j = 1 \ldots n_i$ (where the symbol $\mathcal{L}(\mathcal{A})$ denotes the set of predicate constants and predicate variables occurring in $\mathcal{A}$, and the symbol $\mathcal{L} \subset^* \mathcal{L}'$ denotes $\mathcal{L} \subset \mathcal{L}' \cup \{\bot, \top, \ldots\}$), and
2. if a formula appears as a side formula in a premise of a rule, then it should appear in the consequent of the rule (and in the same side of the sequent arrow). Any side formula in $\mathcal{S}_0$ should also appear as a side formula in some premise; there is no restriction on the number of side formulas and on their logical form.

From condition 1 it follows that $\mathcal{L}(\mathcal{A}_0) \supset^* \bigcup_{i=1\ldots n, j=1\ldots n_i} \mathcal{L}(\mathcal{A}_{i,j})$. It is worthwhile to stress here that we do not demand that $\mathcal{A}_0$ contain the $\mathcal{A}_i$'s as proper subformulas. We only require $\mathcal{A}_0$ to satisfy condition 1 of the definition of schematic rule (i.e. the predicate symbols in $\mathcal{A}_i$ should appear in $\mathcal{A}_0$). We will refer to condition 1 as *weak embeddability* condition because it is essentially a weaker form of the well-known subformula property.

Rules admitted by the definition are for instance those rules where the auxiliary formulas appear (maybe more than once) as subformulas of A_0. Since $\bot, \top$ are considered as special symbols in the language, we can express moreover Schutte's rule of substitution as a schematic rule. Note that the script letters $\mathcal{A}_0$ and $\mathcal{A}_{i,j}$ we use might denote formulas of any logical form as well as formulas with a prefixed logical form. Examples of schematic rules are

$$\frac{\Gamma \to \Delta, (P \wedge Q) \vee R}{\Gamma \to \Delta, (P \vee R) \wedge (Q \vee R)} \qquad \frac{\Gamma \to \Delta, P \quad \Gamma \to \Delta, Q}{\Gamma \to \Delta, (P \vee R) \wedge (Q \vee R)}$$

In the example on the left the auxiliary formula of the rule is expected to have a certain logical form while in the example on the right the form is arbitrary since any substitution for P and Q is allowed.

Condition 2 expresses minimal requirements for the logical links existing between side formulas lying in the upper and lower sequents. It allows one to distinguish between the rules

$$\frac{\Gamma \to \Delta, P \quad \Theta \to \Lambda, Q}{\Gamma, \Theta \to \Delta, \Lambda, P \wedge Q} \qquad \frac{\Gamma \to \Delta, P \quad \Gamma \to \Delta, Q}{\Gamma \to \Delta, P \wedge Q}$$

where in the left rule there is an intended logical link between all formulas in $\Gamma, \Theta, \Delta, \Lambda$ in the upper sequents and the corresponding copy in the lower sequent; in the right rule, each occurrence in Γ, Δ in the lower sequent is linked to the pair of corresponding occurrences in the upper sequents.

The notion of *step of inference* derived from a schematic rule is defined in the obvious way, similar to the notion of derived axiom. As for schematic axioms we intend collections of side formulas in a schematic rule to be *arbitrary.* Therefore one can derive from the same schematic rule steps of inference which are different exactly because of different side formulas.

A *schematic system* is defined to be a finite set of schematic axioms and schematic rules. The vast majority of systems studied in the literature fall under the notion of schematic system in the sense defined above.

A *proof* in a schematic system $\mathcal{Z}$ is a tree of sequents of *regular* formulas. Each sequent must be derived either by a schematic axiom (in this case the sequent is labelling a leaf of the tree) or by one of the schematic rules of inference in $\mathcal{Z}$ (the sequent is a label for an internal node of the tree). Every occurrence of a sequent in a proof other than the end-sequent is used exactly once as a premise of an inference. The end-sequent of the proof is labelling the root of the tree.

Remark. From our analysis it will turn out that the construction of an interpolant for a sequent is inherently linked to the *logical* rules used in the proof of such a sequent. For this reason without loss of generality we assume to work with schematic systems with built-in permutation and weakening rules. For, we assumed sequents to have the form $\Gamma \rightarrow \Delta$ where Γ, Δ are *collections* of formulas (this assumption avoids explicit permutations), and axioms to admit side formulas both on the right and left hand side of the sequent arrow (this assumption avoids explicit weakenings).

Proposition 1. (Addition of weak occurrences) *If $\Pi : \Gamma \rightarrow \Delta$ is a proof of k lines in a schematic system $\mathcal{Z}$ and Θ, Λ are regular formulas then there is a proof $\Pi' : \Gamma, \Theta \rightarrow \Delta, \Lambda$ of k lines in $\mathcal{Z}$.*

Proof. By induction on the height of a proof. Add suitably Θ, Λ as side occurrences in the axioms of Π.

2.2 Regular Schematic Systems

The Craig Interpolation Theorem will be proved for schematic systems containing a kernel of schematic rules closed under a condition of *regularity.* This condition is natural, and indeed most of the well-known logical systems are exactly defined by a set of rules closed under regularity (i.e. the kernel is the system itself). Given a proof for a certain sequent, an interpolant will be built from distinguished occurrences in the axioms of the proof out of regular rules only. Since the kernel of rules closed under regularity may be *properly* included in the set of rules of the system, the exclusive use of regular rules for building the interpolant implies that the usual symmetry between the logical form of the interpolant and the logical connectives introduced in the proof need not hold when we look at regular schematic systems. It is indeed just a coincidence that the phenomenon holds for specific cases.

A pair of schematic rules is said to be *k-regular* (for $k > 1$) if one has arity k, the other has arity 1, their form is

$$\frac{\mathcal{S}_1[\mathcal{A}_1],\ldots,\mathcal{S}_k[\mathcal{A}_k]}{\mathcal{S}_0[\mathcal{A}_0]} \qquad\qquad \frac{\mathcal{S}[\mathcal{A}_1',\ldots,\mathcal{A}_k']}{\mathcal{S}[\mathcal{A}_0]}$$

and the following conditions are satisfied

1. the $\mathcal{A}_i$'s and $\mathcal{A}_i'$'s are atomic formulas formed from predicate variables, variables and metavariables only (for $i = 1 \ldots k$), and
2. $\mathcal{A}_i$ is a variant of $\mathcal{A}_i'$ for all $i = 1 \ldots k$ (i.e. $\mathcal{A}_i$ can be obtained from $\mathcal{A}_i'$ by replacing some of the variables and metavariables in $\mathcal{A}_i$ with different ones), and
3. $\mathcal{L}(\mathcal{A}_0) \subset^* \bigcup_{i=1\ldots k} \mathcal{L}(\mathcal{A}_i)$ (note that $\mathcal{L}(\mathcal{A}_i) = \mathcal{L}(\mathcal{A}_i')$ by condition 2), and
4. the formula $\mathcal{A}_i$ lies on the right (left) hand side of $\mathcal{S}_i$ whenever $\mathcal{A}_i'$ lies on the left (right) hand side of $\mathcal{S}$ (for $i = 1 \ldots k$); the formula $\mathcal{A}_0$ lies on the right (left) hand side of $\mathcal{S}_0$ whenever $\mathcal{A}_0$ lies on the left (right) hand side of $\mathcal{S}$, and
5. if $\mathcal{A}_0$ does not contain quantifiers then there are no restrictions on the pair of rules; otherwise (i.e. $\mathcal{A}_0$ contains quantifiers) restrictions on eigenvariables (i.e. variables only occurring in $\mathcal{A}_1, \ldots, \mathcal{A}_k$ and $\mathcal{A}_1', \ldots, \mathcal{A}_k'$ but not occurring in the side formulas of $\mathcal{S}_0, \mathcal{S}$ respectively) are admitted, and each $\mathcal{A}_i$ (for $i = 1 \ldots k$) is obtained from $\mathcal{A}_i'$ precisely by changing some of the variables and metavariables into eigenvariables, and eigenvariables into variables and metavariables.

The k-ary rule is called the *dual* of the unary rule, and vice versa. Rules which belong to a regular pair will be called *regular* rules. It is important to notice that auxiliary formulas of regular rules never have a specific logical form because of condition 1. Condition 3 together with condition 1 in the definition of schematic rule implies that for regular rules $\mathcal{L}(\mathcal{A}_0) =^* \bigcup_{i=1\ldots k} \mathcal{L}(\mathcal{A}_i)$.

A rule is *$\mathcal{Z}$-definable* if either it is a regular rule which belongs to a schematic system $\mathcal{Z}$ or it is obtained by successive applications of regular rules in $\mathcal{Z}$. For instance we say that the rule

$$\frac{\Gamma \to \Delta, P \quad \Gamma \to \Delta, Q}{\neg(P \wedge Q), \Gamma \to \Delta}$$

is LK-definable because obtained by a $\wedge$:*right* rule followed by a $\neg$:*left* in LK.

Let $\mathcal{Z}$ be a schematic system consisting of rules of arity $k_1, \ldots, k_n$. The system $\mathcal{Z}$ is called *regular* if for all $k \in \{k_1, \ldots, k_n\}$ and all choices of $i_0, i_1, \ldots, i_k$ (with $i_j \in \{1, 2\}$), there is a pair of $\mathcal{Z}$-definable k-regular rules such that the k-ary rule of the pair has auxiliary formulas $\mathcal{A}_j^{i_j}$ defined out of predicate variables of arity 0 only. If a system admits restrictions on eigenvariables for some of its rules, we say that it is a *regular* system whenever the above condition is satisfied and moreover for all $n > 0$ it contains pairs of $\mathcal{Z}$-definable unary regular rules with auxiliary formulas $\mathcal{A}_1, \mathcal{A}_1'$ defined out of predicate variables of arity n, and main formula $\mathcal{A}_0$ where none of the variables and metavariables occurring in $\mathcal{A}_1, \mathcal{A}_1'$ appear free.

As emphasised at the beginning of this section, notice that a system closed under regular rules might also contain rules that are not regular.

3 Logical graphs

To trace the logical relations between formula occurrences in a proof Π (i.e. a tree of sequents of regular formulas as defined in section **??**) we will be using the notion of *logical edge* introduced in [**?**] and [**?**]. We will let positive and negative atomic occurrences in a sequent be logically linked as a rule requires (this is a crucial difference with the definition of logical edges allowed in [**?**] and [**?**] where positive occurrences are required to be linked to positive occurrences, and negative to negative). By doing this, we respect those logical conditions on the structure of the rules which are necessary to ensure the construction of an interpolant. To keep track of these logical conditions, even though at the end we will consider only graphs based on logical edges between *atomics*, we will define logical graphs also on those logical links that are induced by schematic rules and schematic axioms (notice that atomic formulas in schematic axioms and schematic rules may lead to non-atomic formulas after substitution).

An *s-formula* is an *occurrence* of a subformula of a formula (here, '*s*-' stands for '*semi-*' or '*sub-*'). It has to be emphasized that an *s*-formula is an *occurrence* of a subformula (in the proof) as opposed to the subformula itself which may occur many times (in the proof). A formula $\mathcal{A}$ is a *variant* of $\mathcal{B}$ if $\mathcal{A}$ can be obtained from $\mathcal{B}$ by replacing some of the terms and metaterms in $\mathcal{B}$ with different ones. If δ is a substitution map and $\delta(\mathcal{A}), \delta(\mathcal{B})$ are defined, it is easy to check that $\delta(\mathcal{A})$ is a variant of $\delta(\mathcal{B})$ whenever $\mathcal{A}$ is a variant of $\mathcal{B}$.

Two *s*-formulas in a proof will be connected by an edge only if they are variants of each other; any two *s*-formulas connected by an edge will occur respectively in an upper and lower sequent of some rule, or will both be in an axiom on opposite sides of the sequent arrow. More precisely, they will be defined as follows.

First, an axiom in Π is derived with some substitution δ from a schematic axiom containing two distinguished formulas $\mathcal{A}_1, \mathcal{A}_2$. By definition of schematic axiom either all atomic *s*-formulas $\mathcal{B}$ (which are not special constants) in $\mathcal{A}_1$ have variants $\mathcal{B}_1, \ldots \mathcal{B}_n$ (for some $n \geq 1$) in $\mathcal{A}_2$ (i.e. $\mathcal{L}(\mathcal{A}_1) \subset^* \mathcal{L}(\mathcal{A}_2)$) or the other way around (i.e. $\mathcal{L}(\mathcal{A}_2) \subset^* \mathcal{L}(\mathcal{A}_1)$). Define an edge between each formula $\delta(\mathcal{B})$ and some of its variants $\delta(\mathcal{B}_i)$ (maybe all but at least one). If an axiom is derived from a schematic axiom containing only one distinguished occurrence, there will be no edges defined.

Second, in any schematic rule all atomic *s*-formulas $\mathcal{B}$ in auxiliary formulas must have variants $\mathcal{B}_1, \ldots \mathcal{B}_n$ (for some $n \geq 1$) in the main formula, except for special constants. Suppose a step of inference in Π is derived with a substitution δ from some schematic rule. We define an edge between each *s*-formula $\delta(\mathcal{B})$ and some of its variants $\delta(\mathcal{B}_i)$ (maybe all but at least one; we do not ask that *all* possible variants in the main formula be linked to $\delta(\mathcal{B})$).

Third, for any side formula $\mathcal{A}$ in the upper sequent(s) of a schematic rule, by condition 2 (in the definition of schematic rule) there is a side formula $\mathcal{A}$ in the lower sequent. Suppose a rule is derived with some substitution δ from a schematic rule. We define an edge between each side formula $\delta(\mathcal{A})$ occurring in the *upper* sequent(s) and the side formula $\delta(\mathcal{A})$ occurring in the *lower* sequent.

Fourth, suppose there is an edge from an s-formula A_1 to A_2 in Π (A_1, A_2 are regular formulas because they occur in Π) and suppose B_1 is an atomic subformula of A_1. Since A_1 and A_2 are variants, there is a subformula B_2 of A_2 which correspond to the subformula B_1 of A_1; the s-formulas B_1 and B_2 are of course variants. Define an edge between B_1 and B_2.

The *logical graph* of a proof is a graph obtained by tracing the edges between the atomic s-formulas in the proof as described above (and simply forgetting afterwards all those edges connecting formulas which are not atomic).

Note that more than one s-formula in the auxiliary formula(s) of a rule might have an edge to the same variant in the main formula and more than one s-formula in the main formula might have edges to the same s-formula in the antecedent(s). Therefore a connected subgraph of the logical graph for a schematic proof is not necessarily a tree.

It should be stressed that no rules having logical edges defined between formulas in upper sequents (e.g. the cut rule), belong to schematic systems we consider.

A path in a proof is defined as a sequence of consecutive edges in the logical graph of the proof such that there are no two edges of the path crossing the same axiom or the same rule of inference. A path is *full* when it is maximal in the logical graph of a proof where it lies. A path is *direct* if it does not cross any axiom.

Proposition 2. *Let Π be a proof of $A \to B$ and C be an atomic s-formula (which is not a special predicate constant) in $A \to B$. All full paths from C in the logical graph of Π are either direct or they go back to a variant in $A \to B$.*

Proof. A full path from C either crosses an axiom (in which case it goes back to the end-sequent) or it does not (in which case it is a direct path).

Proposition 3. *Let Π be a proof of $A \to B$ and C be a formula occurrence in Π. Then either all direct paths from atomic s-formulas (which are not special predicate constants) in C down to the end-sequent go to A or they all go to B.*

Proof. The main point is that while an atomic occurrence in the upper sequent of a rule may be linked to more than one atomic occurrence in the lower sequent, all its linked occurrences in the lower sequent lie in the same formula.

If all direct paths from atomic s-formulas of C in Π go to A (B) we say that the formula *goes* to A (B).

Notice that Proposition **??** does *not* imply the well known *subformula property* ([?]). Clearly, any schematic system that satisfies the subformula property is a schematic system satisfying Proposition **??**. As we will see in the conclusion, our proof of the Craig Interpolation Theorem does not need the subformula property to hold.

4 The Statement and the Proof

We are now ready to formalize the statement of Craig's Theorem for regular schematic systems.

Theorem 4. (Craig's Interpolation Theorem for Regular Schematic Systems) *Let $\mathcal{Z}$ be a regular schematic system. Let A and B be two regular formulas such that the sequent $A \to B$ is provable in $\mathcal{Z}$, and that A and B have at least one predicate constant in common. Then there exists a formula C (called the interpolant of $A \to B$) containing only those predicate constants that occur in both A and B and possibly special predicate constants, such that $A \to C$ and $C \to B$ are provable in $\mathcal{Z}$.* [3]

The interpolant will be built by steps on the complexity of the subproofs using some easy considerations on the logical flows of formulas in the proof. The method brings to light the logical relations between the s-formulas occurring in the interpolant and the ones occurring in A and B. Regular rules will be the only rules used to build the interpolant C at each stage of the construction.

The technique was used to prove the theorem for LK and LJ in [**?**].

4.1 Proof of the Interpolation Theorem

Let Π_* be a proof of $A \to B$. The interpolant C_* for $A \to B$ associated to Π_* together with the proofs $\Pi_*^A : A \to C_*$ and $\Pi_*^B : C_* \to B$ will be built in stages from the subproofs of Π_*. The idea consists of 'extracting' from the proof Π_*, two proofs associated to A and B respectively. As a consequence, only part of the original proof Π_* that 'contributes' to the construction of the formula A (B respectively) will appear in the proof Π_*^A (Π_*^B respectively).

Proof. (Theorem **??**) We will build a formula C_* and proofs $\Pi_*^A : A \to C_*, \Pi_*^B : C_* \to B$ in $\mathcal{Z}$. To indicate that the language of C_* (i.e. the set of symbols occurring in C_*) is contained in the intersection of the language of A with the language of B extended with $\top, \bot$, we use the symbol $\mathcal{L}(C_*) \subset^* \mathcal{L}(A) \cap \mathcal{L}(B)$.

The proof is by induction on the height of the subproofs Π of Π_*. For each subproof Π of Π_* with end-sequent $\Gamma \to \Delta$ we show that

1. there exist subcollections Γ^A, Δ^A and Γ^B, Δ^B of Γ, Δ such that
   a. each formula occurrence in Γ^A, Δ^A (Γ^B, Δ^B, respectively) goes to A (B, respectively), and
   b. Γ is the collection obtained by combining Γ^A, Γ^B; similarly Δ is defined by combining Δ^A, Δ^B;
2. there exists a formula C_Π such that
   a. $\mathcal{L}(C_\Pi) \subset^* \mathcal{L}(A) \cap \mathcal{L}(B)$, and

[3] A stronger statement where we require that variables, function symbols and constants in C must appear in both A and B, can be given. But in this case some extra conditions on terms appearing in rules are needed.

b. $\Gamma^A \to \Delta^A, C_\Pi$ and $C_\Pi, \Gamma^B \to \Delta^B$ are provable sequents.

Notice that if Π is Π_* then Γ^A is A, Δ^A is $\emptyset$, Γ^B is $\emptyset$, Δ^B is B and $A \to C_*$, $C_* \to B$ turn out to be provable sequents (following the notation, C_* is precisely C_{Π_*}).

Let us start first by considering subproofs Π of Π_*, of height 1 of the form $D, \Theta \to \Lambda, D'$. There are four cases we should consider:

1. the occurrence D goes to A in Π_*, while D' goes to B; if $\mathcal{L}(D) \subset^* \mathcal{L}(D')$ then let C_Π be D, the proof Π^A be $D, \Theta^A \to \Lambda^A, D$, the proof Π^B be $D, \Theta^B \to \Lambda^B, D'$ and the subcollections Γ^A, Δ^A, Γ^B, Δ^B be the pairs D, Θ^A, Λ^A, D, D, Θ^B, Λ^B, D respectively (in the following we will not mention anymore how subcollections are defined; they will be clear from the context). If $\mathcal{L}(D') \subset^* \mathcal{L}(D)$ then let C_Π be D', the proof Π^A be $D, \Theta^A \to \Lambda^A, D'$ and Π^B be $D', \Theta^B \to \Lambda^B, D'$.
2. the occurrence D goes to B, while D' goes to A; if $\mathcal{L}(D) \subset^* \mathcal{L}(D')$ then let C_Π be D, take Π^A to be $D, \Theta^A \to \Lambda^A, D'$, and Π^B to be $D, \Theta^B \to \Lambda^B, D$; if $\mathcal{L}(D') \subset^* \mathcal{L}(D)$ then let C_Π be D', take Π^A to be $D', \Theta^A \to \Lambda^A, D'$, and and Π^B to be $D, \Theta^B \to \Lambda^B, D'$.
3. both occurrences D, D' go to A; let C_Π be $\bot$, take Π^A to be $D, \Theta^A \to \Lambda^A, D', \bot$ and Π^B to be $\bot, \Theta^B \to \Lambda^B$.
4. both occurrences D, D' go to B; let C_Π be $\top$, take Π^A to be $\Theta^A \to \Lambda^A, \top$ and Π^B to be $\top, D, \Theta^B \to \Lambda^B, D'$.

In case a schematic axiom contains only one distinguished occurrence we proceed in a similar spirit and say that if the schematic axiom has the form $\Theta \to \Lambda, D'$ with D' going to B (A) then Π^A is $\Theta^A \to \Lambda^A, \top$, the proof Π^B is $\top, \Theta^B \to \Lambda^B, D'$ and C_Π is $\top$. Similarly if the schematic axiom has the form $D, \Theta \to \Lambda$ with D going to A (B).

For subproofs Π of height greater than 1 we will examine the last rule of inference R. Suppose R is a k-ary (for $k \geq 1$) rule applied to subproofs $\Pi_1 \ldots \Pi_k$

$$\frac{\begin{matrix}\Pi_1 \\ \mathcal{S}_1[A_{1,1}, \ldots, A_{1,n_1}]\end{matrix} \quad \ldots \quad \begin{matrix}\Pi_k \\ \mathcal{S}_k[A_{k,1}, \ldots, A_{k,n_k}]\end{matrix}}{\mathcal{S}_0[A_0]}$$

with eigenvariables $b_1 \ldots b_r$ (with $r \geq 0$) of Π. By induction hypothesis there are k pairs of proofs Π_i^A and Π_i^B for $i = 1 \ldots k$ and k formulas $C_{\Pi_1}, \ldots, C_{\Pi_k}$ such that $\mathcal{L}(C_{\Pi_i}) \subset^* \mathcal{L}(A) \cap \mathcal{L}(B)$ for $i = 1 \ldots k$. We will use this information to build the formula C_Π and, $\Pi^A : \mathcal{S}^A[C_\Pi]$ and $\Pi^B : \mathcal{S}^B[C_\Pi]$. There are two cases:

1. the main formula A_0 goes to A. Let C_Π be C_0 and define Π^A to be

$$\cfrac{\cfrac{\begin{matrix}\Pi_{1,*}^A \\ \mathcal{S}_1^A[A_{1,1}, \ldots, A_{1,n_1}, C'_{\Pi_1}]\end{matrix} \quad \ldots \quad \begin{matrix}\Pi_{k,*}^A \\ \mathcal{S}_k^A[A_{k,1}, \ldots, A_{k,n_k}, C'_{\Pi_k}]\end{matrix}}{\mathcal{S}^A[A_0, C'_{\Pi_1}, \ldots, C'_{\Pi_k}]}}{\mathcal{S}^A[A_0, C_0]}$$

where $\Pi^A_{1,*} \ldots \Pi^A_{k,*}$ are subproofs obtained from $\Pi^A_1 \ldots \Pi^A_k$ by applying a regular unary rule to $C_{\Pi_1} \ldots C_{\Pi_k}$ to eliminate all eigenvariables, for all $i = 1 \ldots k$. If C_{Π_i} does not contain eigenvariables, $\Pi^A_{i,*}$ is defined to be Π^A_i. Notice that the regular unary rule applied to C_{Π_i} (if needed) exists (to be precise, it is $\mathcal{Z}$-definable) because the system is assumed to be regular.
We assume it is clear from the context what are the formula occurrences which are auxiliary and/or main formulas of each rule application. In the figure we explicitly displayed all those formula occurrences relevant to the construction and not just the ones relevant to a single rule.
Suppose $\Pi^A_{i,*}$ is different than Π^A_i for some $i = 1 \ldots k$. Let Π^B be

$$\frac{\begin{array}{c}\Pi^B_{i,*}\\ \mathcal{S}^B_{i,*}[C'_{\Pi_1}, C'_{\Pi_2} \ldots C'_{\Pi_k}]\end{array}}{\mathcal{S}^B[C_0]}$$

where $\Pi^B_{i,*}$ is obtained by applying Proposition **??** to Π^B_i so to add to the sequent $\mathcal{S}^B_i$ as weak occurrences, the formulas C'_{Π_j}'s and all formulas in the end-sequents $\mathcal{S}^B_j$ of Π^B_j except the main formulas C_{Π_j}'s (for $j = 1 \ldots i-1, i+1 \ldots k$), and by applying afterwards a unary regular rule (dual to the one used in Π^A_i) to C_{Π_i} to eliminate all eigenvariables occurring in it. Take C_Π to be C_0. Clearly, $\mathcal{L}(C_0) \subset^* \mathcal{L}(A) \cap \mathcal{L}(B)$.
If no eigenvariables appear in C_{Π_i} of Π^A_i, Π^B_i for all $i = 1 \ldots k$, notice that C'_Π is C_Π. The construction of Π^B is the obvious simplification of the one just discussed.

2. the main formula A_0 goes to B. Suppose C_{Π_i} contains an eigenvariable of Π. Let C_Π be C_0 and define Π^A to be

$$\frac{\begin{array}{c}\Pi^A_{i,*}\\ \mathcal{S}^A_{i,*}[C'_{\Pi_1}, C'_{\Pi_2} \ldots C'_{\Pi_k}]\end{array}}{\mathcal{S}^A[C_0]}$$

where $\Pi^A_{i,*}$ is obtained by applying in a suitable way Proposition **??** to Π^A_i and a unary regular rule to eliminate all eigenvariables from C_{Π_i}. Let Π^B be

$$\frac{\dfrac{\begin{array}{c}\Pi^B_{1,*}\\ \mathcal{S}^B_1[A_{1,1}, \ldots, A_{1,n_1}, C'_{\Pi_1}]\end{array} \quad \ldots \quad \begin{array}{c}\Pi^B_{k,*}\\ \mathcal{S}^B_k[A_{k,1}, \ldots, A_{k,n_k}, C'_{\Pi_k}]\end{array}}{\mathcal{S}^B[A_0, C'_{\Pi_1}, \ldots, C'_{\Pi_k}]}}{\mathcal{S}^B[A_0, C_0]}$$

where $\Pi^B_{1,*} \ldots \Pi^B_{k,*}$ are defined similarly to case 1. Notice that the regular rule applied to Π^B_i should be the dual of the one applied to Π^A_i.

This concludes the proof.

Remark. Our definition of regularity requires regular pairs to satisfy a *minimal* set of conditions sufficient to prove Theorem **??**. Because of this generality, the number of regular pairs required to belong to such a set of rules might be quite large. Theorem **??** can be carried out for a restricted number of regular pairs whenever constraints on the rules used in a proof are given. For instance if we can prove $A \rightarrow B$ with rules whose auxiliary formulas and main formulas both belong either to the right or to the left hand side of a sequent, then we can ask the formulas C_{Π} in Π_1^A (Π_1^B) to always belong to the right (left) hand side of a sequent, respectively. Therefore we would only need to consider regular rules whose arguments and values lie both either in the left or in the right hand side of the sequents.

In the sequent calculus for classical logic LK (for a reference see [**?**]) all pairs of rules introducing a connective to the right and to left hand side of the sequent arrow, form a regular pair. Because of Theorem **??** this means that LK enjoys the interpolation property. For the same reason also the LK-fragments $[\neg]$, $[\wedge, \vee, \neg, \supset]$, $[\wedge]$, $[\vee]$, $[\wedge, \vee]$ satisfy the property (notice that for the last three fragments we should apply the discussion in our latter remark). For the fragments $[\wedge, \neg]$ and $[\vee, \neg]$ we should observe that regularity follows from $\mathcal{Z}$-definability, for $\mathcal{Z}$ being $[\wedge, \neg]$ and $[\vee, \neg]$ respectively. Last, let us remark that Schutte system for the treatment of formula substitutions enjoys the property as well as the fragments of linear logic LL of type I (i.e. elementary fragments without linear negation and without multiplicative implication) and II (i.e. elementary fragments without negation or without multiplicative implication together with 0 (falsum)). This was proved also in [**?**]. This list of calculi is of course not exhaustive.

Remark. There is no assumption on the number of formulas occurring on the right and left hand side of a schematic sequent. Notice that the intuitionistic calculus LJ is formalized by assuming that sequents contain *at most* one formula occurrence on the right of a sequent. Our result can be easily modified for regular systems satisfying such requirement. The central idea consists in building the interpolant by suitably switching its position in the sequent during the construction. To do this, one needs some extra conditions to the notion of regular schematic system which we will not discuss here.

5 Conclusions

Let us briefly point out some directions for further inquiry.

As a result of our analysis we now have a number of conditions sufficient for a logical system to enjoy Interpolation. Hopefully these conditions will provide new insights for the automatic analysis of properties of logical systems (work in this direction was done in [**?**] for multi-valued logics).

Theorem **??** shows that a logical system need not satisfy the *subformula property* for its rules in order to enjoy Interpolation, but simply some weak form of *embedding* of the antecedents of a rule into its consequent. We want to remark here that our definition of embedding can be weakened so to interpret logical systems formalizing proofs with analytic cuts (i.e. cuts on formulas appearing as subformulas of the endsequent of the proof) which are known to enjoy Interpolation (the reader can refer to [**?**] for results in this direction).

The notion of schematic systems we use together with notions like weak embeddability and *$\mathcal{Z}$-definability* of rules of inference seem useful to characterize *similarity* of proofs, i.e. similar proofs of similar conclusions. There are different approaches to the question of similarity already in the literature (see [**?**], [**?**], [**?**]) and a definite mathematical concept capturing the intuition of *similarity* has not been obtained yet.

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