ON ENDOMORPHISMS OF AFFINE SPACES AND THE JACOBIAN PROBLEM

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ABSTRACT. Let p be a prime. We provide examples which show that étale endomorphisms of affine planes in characteristic p can have fibers of arbitrary finite cardinal and hence are not always surjective. Let $(l, m) \in \mathbb{N} \times \mathbb{N}^*$. We provide examples of such étale endomorphisms whose images have complements of cardinality l and whose geometric degrees are pm. Several conjectures are disproved, and in particular we provide a positive characteristic analogue of Kulikov's counterexample to the 'Generalized Jacobian Conjecture' of Bass for surfaces. If $e : X \to X$ is an endomorphism of a variety over an algebraically closed field, then we show that there exists $n \in \mathbb{N}$ such that $\mathrm{Im}(e^n) = \mathrm{Im}(e^{n+1})$ provided either (i) e is quasi-finite or (ii) $\dim(X) \leq 2$ and $X \setminus \mathrm{Im}(e)$ is finite. We provide examples of endomorphisms of the affine planes of dimensions at least 3 whose images have complements of cardinality l and for all $n \in \mathbb{N}$ we have $\mathrm{Im}(e^n) \neq \mathrm{Im}(e^{n+1})$. We prove that all affine moduli schemes of étale endomorphisms of affine spaces with Jacobian matrices of determinant 1 over a field are connected and classify all those that are smooth. We prove that the classical Jacobian Conjecture and Adjamagbo's analogue of it in characteristic p hold for étale endomorphisms of affine spaces that are composites $g \circ f$, where f is quasi-finite and a locally closed embedding in codimension 1 outside a specific finite subset and g is a projection that omits one coordinate.

KEY WORDS: affine space, automorphism, complete intersection, endomorphism, étale, field, hypersurface, invariant, Jacobian Conjecture, matrix, morphism, prime number, quasi-finite, parametrization, p-bases, ring, scheme, smooth, surface, surjective, torsor, variety, vector bundle, zeta function.

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1. Introduction

The classical Jacobian Conjecture states that for an integer $n \geq 1$ if a collection of n complex polynomials in n indeterminates has a non-zero constant Jacobian determinant, then the corresponding polynomial map, usually called a (complex) Keller map, has a polynomial inverse. In other words, every locally analytically invertible polynomial self-map of an affine space over $\mathbb{C}$ is globally invertible, by a polynomial map. This conjecture goes back to the 1939 paper of Keller ([Kel]) and has received considerable attention, especially for $n = 2$, the first non-trivial case. In fact, it is known to be a very tricky problem that has generated a large number of incorrect proofs.

On its face, the conjecture seems to be solely about characteristic 0, and, indeed, many attempts to solve it use analytic methods. However, it is also known that if it is false for a given n, one can construct a counterexample over the integers that involves more indeterminates than n (see [CvdD], Subsect. 4.3). It has also been well-known that if a polynomial map in question is injective then it must be surjective and, moreover, its inverse must also be a polynomial map. That injectivity implies surjectivity is a particular case of Ax–Grothendieck's Theorem (see [Ax], Thm., [Gro4], Prop. (10.4.11); see also [Bore1], Thm. 2.3 and Sect. 3), whose original proof is by *reduction to positive characteristic*. Due to this, the Jacobian Conjecture can be viewed as a strengthening of the following conjecture.

Conjecture 1.1 (Surjectivity Conjecture over $\mathbb{C}$)**.** *Each complex Keller map is surjective.*

Let p be a prime number. If taken verbatim, the Jacobian Conjecture in characteristic p is false already in dimension $n=1$, as the polynomial $x+x^p$ has derivative 1 but the function $x \mapsto x+x^p$ is not injective; however, it is surjective over algebraically closed fields. In fact, for $n=1$ the Surjectivity Conjecture is clearly true over all algebraically closed fields.

Generalizing the above example, one can consider in characteristic p a natural class of Keller maps, that we call *basic endomorphisms*:

$$e(x_1,\ldots,x_n) = \bigl(x_1 + f_1(x_1^p,\ldots,x_n^p),\ldots,x_n + f_n(x_1^p,\ldots,x_n^p)\bigr).$$

It is natural to ask if over algebraically closed fields they are surjective.

Finding a preimage by e of the point $(0,\ldots,0)$ and, by a shift, of any other point, leads to a system of equations of a particular kind, that we call a Frobenius system:

$$x_i + f_i(x_1^p,\ldots,x_n^p) = 0, \quad i \in \{1,\ldots,n\}. \tag{1}$$

Based on the case when the f_is have degrees at most 1 (see [V], Thm. 2.4.1(b)) and other explicit examples, the third author had conjectured that Frobenius systems over algebraically closed fields are always consistent[1]. His persistent effort to prove it was the original impetus and the starting point of our investigation.

Incidentally, if the p-th power is applied to the indeterminates of the first terms rather than to the second terms of the left hand side of System (1), the system

$$x_i^p + f_i(x_1,\ldots,x_n) = 0, \quad i \in \{1,\ldots,n\}$$

is consistent if p is large enough (say greater than the degree of f_i for each $i \in \{1,\ldots,n\}$). Moreover, if the field is algebraically closed of characteristic p and a dominant map $(x_1,\ldots,x_n) \mapsto \bigl(f_1(x_1,\ldots,x_n),\ldots,f_n(x_1,\ldots,x_n)\bigr)$ is fixed, then the union indexed by $q \in \mathbb{N}^*$ of the set of solutions of the system

$$x_i^{p^q} + f_i(x_1,\ldots,x_n) = 0, \quad i \in \{1,\ldots,n\}$$

is Zariski dense in the affine space (e.g., see [BS], Thm. 1.4 for the particular case of an algebraic closure of $\mathbb{F}_p$ and see [SV], Thm. 1.3 for the general case[2]). This added to the hope that Frobenius systems are consistent, though the first author considered this problem to be about as hard as the Jacobian Conjecture.

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[1] Talk at the 60th Anniversary of Gerd Faltings, MPI, Bonn, Germany on June 13, 2014.

[2] The general case follows from the particular case via a standard specialization argument as for $q >> 0$ such systems define finite flat endomorphisms of affine spaces.

The actual answer was opposite to the initial expectation. Indeed, we have the following particular case of Example 12.3 for $m = 1$ and $f(y) = 1 - y + y^p$.

Example 1.2. Let $i \in \{0, 1, \ldots, p-1\}$. The Frobenius system

$$\begin{cases} x = \big(1+x^p(1-y+y^p)^p\big)\big[1-x\big(1-x(1-y+y^p)+x^p(1-y+y^p)^p\big)^{p-1}\big((x(1-y+y^p))^{p-1}-1\big)\big]^p \\ y = i + x^p(1-y+y^p)^p \end{cases}$$

has no solutions in any field of characteristic p.

To show that the assumption that it has a solution leads to a contradiction, let $z := x(1-y+y^p)$. Hence $y = i + z^p$ and the first equation can be rewritten as

$$x = (1+z^p)\big[1 - x^p(1-y+y^p)^{p-1}\big((y-i)^{p-1}-1\big)\big]$$

(note that $i^p = i$ in fields of characteristic p). Multiplying this by $1 - y + y^p$, we get $z = (1+z^p)\big[1-y+y^p-(y-i)\big((y-i)^{p-1}-1\big)\big]$, which simplifies to $z = 1 + z^p$. Thus $1 - z + z^p = 0$. Taking this to the power p, we get $1 - y + y^p = 0$. Hence $y = i$ by the second equation, which contradicts $1 - y + y^p = 0$.

This example implies that, in positive characteristic, the naive analog of the Surjectivity Conjecture is false, even for basic endomorphisms in dimension 2.

An important work on the Jacobian Conjecture in positive characteristic was done by Adjamagbo. In particular, he proposed the following conjecture as the natural analog of the Jacobian Conjecture in characteristic p.

Conjecture 1.3 (Jacobian Conjecture in characteristic p, [Ad], Conj. 3.1)**.** *Every Keller map over an algebraically closed field of characteristic p is an automorphism if its geometric degree is not a multiple of p. [The geometric degree is the cardinality of the preimage of a generic point or the degree of the corresponding field extension.]*

While this conjecture seems to be weaker than the Jacobian Conjecture, it is stronger in the following sense: its validity for infinitely many primes implies the classical Jacobian Conjecture. At this time we cannot prove or disprove it. However, we do disprove another conjecture by Adjamagbo ([Ad], Conj. 3.4, see Theorem 2.7), that was meant to be an analog of the 'Generalized Jacobian Conjecture' of Bass in characteristic 0 (see [Ba2], Rm. after Prop. (4.3)) that was disproved by Kulikov in dimension 2 (see [Kul], Sect. 3; the explicit equations of loc. cit. work over any algebraically closed field of characteristic different from 2 and 3).

The main goal of this paper is to initiate a systematic study of several natural invariants of basic endomorphisms and general Keller maps (étale endomorphisms) over algebraically closed fields of characteristic p. For the sake of generality and for including also the characteristic 0 case whenever possible, we often work with larger classes of endomorphisms of affine spaces over arbitrary algebraically closed fields, such as quasi-finite or even only dominant endomorphisms. Example 1.2 above is a particular case of one of several series of interesting étale endomorphisms that we were able to construct, sometimes with the help of other smooth affine hypersurfaces that contain affine spaces as open dense subvarieties. Note that while most of our theorems are stated as existential results, our proofs are ultimately constructive, even if it is not always possible to write down explicit formulas.

In the next section we give precise definitions and describe our main results, that are proven in the subsequent sections. Finally, in the last section we propose a number of open problems, questions and conjectures that naturally arose from our work. Perhaps, the most interesting open problems and questions are the following.

(1) Can some of our examples (resp. methods) be tweaked to disprove (resp. prove) Conjecture 1.3? Note that a disproof of Conjecture 1.3 would not necessarily disprove the classical Jacobian Conjecture.

(2) Can some of our examples and methods be used to prove or disprove the Surjectivity Conjecture over $\mathbb{C}$?

(3) Classify all non-decreasing eventually constant sequences $(a_n)_{n\in\mathbb{N}}$ of natural numbers with the properties that $a_0 = 0$, the first order difference sequence $(a_{n+1} - a_n)_{n\in\mathbb{N}}$ is non-increasing, and there exists an étale endomorphism e of an affine plane over an algebraically closed field of characteristic p such that for each $n \in \mathbb{N}$, the complement of the image of e^n has a_n points.

2. Notation, definitions and description of main results

Let K be an arbitrary algebraically closed field and let $\mathrm{char}(K)$ be its characteristic. For a polynomial $f(t) \in K[t] \setminus \{0\}$, let $\mathrm{z}(f) \in \mathbb{N}$ be the number of roots of f in K. Recall that p is a prime number. Let k be an algebraically closed field with $\mathrm{char}(k) = p$. For $l, m \in \mathbb{N}$, let

$$[[l, m]] := \{i \in \mathbb{N} | l \leq i \leq m\}$$

(so $[[l, m]] = \emptyset$ if $l > m$). Let R be a commutative $\mathbb{Z}$-algebra. Let R^* be the multiplicative group of units of R. Let $n \in \mathbb{N}^*$. For indeterminates $x_1, \ldots, x_n$, the R-algebra $R[x_1, \ldots, x_n]$ is $\mathbb{N}$-graded by $\deg(1) = 0$ and $\deg(x_i) = 1$ for $i \in [[1, n]]$.

All the standard schemes or group schemes

$$\mathbb{A}^n,\ \mathbf{SL}_2,\ \mathbf{GL}_2,\ \mathbb{G}_a,\ \mathbb{G}_m, \text{ etc.}$$

are over $\mathrm{Spec}\,\mathbb{Z}$ and we use their pullbacks

$$\mathbb{A}^n_R,\ \mathbf{SL}_{2,R},\ \mathbf{GL}_{2,R},\ \mathbb{G}_{a,R},\ \mathbb{G}_{m,R}, \text{ etc.}$$

to $\mathrm{Spec}\,R$. Thus

$$\mathbb{A}^n = \mathrm{Spec}\,(\mathbb{Z}[x_1, \ldots, x_n]),\ \mathbf{SL}_2 = \mathrm{Spec}\,\big(\mathbb{Z}[w, x, y, z]/(wz - xy - 1)\big),$$

$$\mathbf{GL}_2 = \mathrm{Spec}\,\big(\mathbb{Z}[w, x, y, z, (wz - xy)^{-1}]\big),\ \mathbb{G}_a = \mathrm{Spec}\,(\mathbb{Z}[t]),$$

$$\mathbb{G}_m = \mathrm{Spec}\,(\mathbb{Z}[t, t^{-1}]),\ \mathbb{A}^n_R = \mathrm{Spec}\,(R[x_1, \ldots, x_n]), \text{ etc.}$$

Let $\mathrm{End}_n(R)$ be the monoid of R-endomorphisms of $\mathbb{A}^n_R$ under the composition (denoted multiplicatively) and let $\mathrm{GA}_n(R)$ be its subgroup of invertible elements, i.e., of R-automorphisms of $\mathbb{A}^n_R$.

If $m \in \mathbb{N}^*$ and $(g_1, \ldots, g_m) \in R[x_1, \ldots, x_n]^m$, by the degree of the m-tuple $(g_1, \ldots, g_m)$ or of the system $\mathcal{V}$ of m polynomial equations in n indeterminates

$$g_i(x_1, \ldots, x_n) = 0, \quad i \in [[1, m]]$$

it defines, we mean

$$\pi(g_1, \ldots, g_m) = \pi(\mathcal{V}) := \max\{\deg(g_i) | i \in [[1, m]]\} \in \mathbb{N} \cup \{-\infty\}.[3]$$

If $m = n$, by the endomorphism of $\mathbb{A}^n_R$ over $\mathrm{Spec}\,R$ defined by $(g_1, \ldots, g_n)$ we mean the endomorphism $\mathrm{e}(g_1, \ldots, g_n) \in \mathrm{End}_n(R)$ defined by the R-algebra endomorphism of $R[x_1, \ldots, x_n]$ that maps x_i to g_i for each $i \in [[1, n]]$.

We use the following formalism of *invariants* for such endomorphisms.

[3]By convention, the degree of the zero polynomial is $-\infty$.

Definition 2.1. *Let M be a subset of $\mathrm{End}_n(R)$ stable under left and right translations by elements of $\mathrm{GA}_n(R)$. Let $\mathcal{N}$ be an arbitrary set. A function $\Im : M \to \mathcal{N}$ is called an $\mathcal{N}$-valued:*

(1) *inner invariant for M if $\Im(aea^{-1}) = \Im(e)$ for each pair $(a,e) \in \mathrm{GA}_n(R) \times M$;*

(2) *left (resp. right) invariant for M if $\Im(ae) = \Im(e)$ (resp. $\Im(ea) = \Im(e)$) for each pair $(a,e) \in \mathrm{GA}_n(R) \times M$;*

(3) *invariant for M if $\Im(aeb) = \Im(e)$ for each triple $(a,b,e) \in \mathrm{GA}_n(R)^2 \times M$.*

In practice, M is a set in the chain of inclusions

$$\mathrm{GA}_n(R) \subset \cdots \subset \mathrm{EE}_n(R) \subset \mathrm{QFE}_n(R) \subset \mathrm{QF}_n(R) \subset \mathrm{D}_n(R) \subset \mathrm{End}_n(R),$$

where $\mathrm{D}_n(R)$ is the submonoid of all endomorphisms that are fiberwise dominant, $\mathrm{QF}_n(R)$ is the submonoid of all quasi-finite endomorphisms of $\mathbb{A}^n_R$, $\mathrm{QFE}_n(R)$ is the submonoid of all quasi-finite endomorphisms of $\mathbb{A}^n_R$ that are fiberwise generically étale, $\mathrm{EE}_n(R)$ is the submonoid of all étale endomorphisms (also called Keller maps) of $\mathbb{A}^n_R$, and the dots stand for extra sets to be introduced in the case when R is an $\mathbb{F}_p$-algebra. Often $\mathcal{N}$ is a subset of either $\mathbb{N} \cup \{-\infty, \infty\}$ (ordered as a subset of the extended real line) or $\mathcal{P}_f(\mathbb{N})$ (the set of finite subsets of $\mathbb{N}$).

Though our emphasis is on the case when $n \leq 3$ and $R = k$, whenever possible we work with an arbitrary n and $R = K$.

Definition 2.2. *Assume that R is an $\mathbb{F}_p$-algebra.*

(1) *By an F_n-system (Frobenius system in n indeterminates) $\mathcal{S}$ over R we mean a system of n polynomial equations in n indeterminates of the form*

$$x_i + f_i(x_1^p, \ldots, x_n^p) = 0, \quad i \in [[1,n]],$$

where $f_1, \ldots, f_n \in R[x_1, \ldots, x_n]$. If S is an $\mathbb{F}_p$-subalgebra of R, we say that $\mathcal{S}$ is defined over S if we have $f_i \in S[x_1, \ldots, x_n]$ for all $i \in [[1,n]]$. By the endomorphism $e_{\mathcal{S}} \in \mathrm{End}_n(R)$ defined by $\mathcal{S}$ we mean the endomorphism $e\big(x_1 + f_1(x_1^p, \ldots, x_n^p), \ldots, x_n + f_n(x_1^p, \ldots, x_n^p)\big)$. If $R = k$, let $\mathrm{s}(\mathcal{S}) \in \mathbb{N}$ be the number of solutions of $\mathcal{S}$ in k^n.

(2) *By a basic endomorphism of $\mathbb{A}^n_R$ we mean an endomorphism $e \in \mathrm{End}_n(R)$ such that an element of $\mathrm{GA}_n(R)e\mathrm{GA}_n(R)$ is defined by an F_n-system over R.*

From the very definitions, each basic endomorphism of $\mathbb{A}^n_R$ is étale as, referring to Definition 2.2(1), for the $R[y_1, \ldots, y_n]$-algebra

$$R[x_1, y_1, \ldots, x_n, y_n]/\big(y_i - x_i - f_i(x_1^p, \ldots, x_n^p) | i \in [[1,n]]\big) = R[x_1, \ldots, x_n],$$

the $2n$ differentials

$$dy_i, d\big(y_i - x_i - f_i(x_1^p, \ldots, x_n^p)\big) = dy_i - dx_i \in \Omega_{R[x_1,y_1,\ldots,x_n,y_n]/R} \text{ with } i \in [[1,n]]$$

are linearly independent even at all points of $\mathrm{Spec}\,(R[x_1, y_1, \ldots, x_n, y_n]) \cong \mathbb{A}^{2n}_R$. Thus for the set $\mathrm{BE}_n(R)$ of basic endomorphisms of $\mathbb{A}^n_R$ we have inclusions

$$\mathrm{GA}_n(R) \subset \mathrm{BE}_n(R) \subset \mathrm{EE}_n(R).$$

Clearly, $\mathrm{BE}_n(R)$ is stable under left and right translations by elements of $\mathrm{GA}_n(R)$.

For a scheme W, let W_{red} be its associated reduced scheme.

We study the following list of invariants or inner invariants of suitable submonoids of $\mathrm{End}_n(K)$, formed by endomorphisms whose domain and codomain are denoted also as $\mathbb{A}^n_{K,\mathrm{s}} = \mathrm{Spec}\,(K_{\mathrm{s}}[x_1,\ldots,x_n])$ and $\mathbb{A}^n_{K,\mathrm{t}} = \mathrm{Spec}\,(K_{\mathrm{t}}[x_1,\ldots,x_n])$ (respectively), with the indexes s and t standing for source and target (respectively).

(i) The *algebraic degree invariant*

$$\pi = \pi_{n,K} : \mathrm{End}_n(K) \to \mathbb{N} \cup \{-\infty\}$$

is defined by the rule: $\pi(e)$ is the minimum of the set of all $\pi(g_1,\ldots,g_n)$ with $(g_1,\ldots,g_n) \in K_{\mathrm{s}}[x_1,\ldots,x_n]^n$ varying subject to the condition that it defines an endomorphism in the double coset $\mathrm{GA}_n(K)e\mathrm{GA}_n(K)$. We similarly define the inner, left or right algebraic degree invariants by

$$\pi_{\mathrm{i}} = \pi_{\mathrm{inner},n,K} : \mathrm{End}_n(K) \to \mathbb{N} \cup \{-\infty\},$$
$$\pi_{\mathrm{l}} = \pi_{\mathrm{left},n,K} : \mathrm{End}_n(K) \to \mathbb{N} \cup \{-\infty\},$$
$$\pi_{\mathrm{r}} = \pi_{\mathrm{right},n,K} : \mathrm{End}_n(K) \to \mathbb{N} \cup \{-\infty\}$$

(respectively) via the rules: $\pi_{\mathrm{i}}(e)$, $\pi_{\mathrm{l}}(e)$ or $\pi_{\mathrm{r}}(e)$ is the minimum of the set of all $\pi(g_1,\ldots,g_n)$ with $(g_1,\ldots,g_n) \in K_{\mathrm{s}}[x_1,\ldots,x_n]^n$ varying subject to the condition that it defines an endomorphism in the conjugacy class $\{aea^{-1}|a \in \mathrm{GA}_n(K)\}$, in the left coset $\mathrm{GA}_n(K)e$ or in the right coset $e\mathrm{GA}_n(K)$ (respectively).

These types of algebraic degrees measure how efficient the n-tuple $\pi(g_1,\ldots,g_n)$ defines suitable cosets of e from the perspective of computing other invariants.

(ii) The *geometric degree invariant*

$$\deg = \deg_{n,K} : \mathrm{D}_n(K) \to \mathbb{N}^*$$

is a multiplicative invariant defined by the rule: $\deg(e)$ is the separable degree of the finite field extension $K_{\mathrm{t}}(x_1,\ldots,x_n) \to K_{\mathrm{s}}(x_1,\ldots,x_n)$ induced by e; so, if $e \in \mathrm{QFE}_n(K)$, then $\deg(e)$ is the degree of the mentioned field extension.

(iii) The *non-generic fibers set invariant*

$$\mathcal{F} = \mathcal{F}_{n,K} : \mathrm{QF}_n(K) \to \mathcal{P}_f(\mathbb{N})$$

is defined by

$$\mathcal{F}(e) := \{s \in [[0, \deg(e)-1]] | \exists P \in \mathbb{A}^n_{K,\mathrm{t}}(K) \text{ such that } \big(e^{-1}(P)\big)_{\mathrm{red}} \cong \mathrm{Spec}\,(K^s)\}.$$

We recall that if $e \in \mathrm{EE}_n(K)$, then e is finite iff the set $\mathcal{F}(e)$ is empty by [Gro5], Cor. (18.2.9).

In all that follows by a variety X over K we mean a reduced scheme of finite type over $\mathrm{Spec}\,K$. The identity automorphism of X is denoted as 1_X. For a non-empty variety, let $\mathrm{Irr}(X)$ be its set of irreducible components, let $\mathrm{Sing}(X)$ be its singular locus, let $\mathrm{Reg}(X) := X \setminus \mathrm{Sing}(X)$ be its regular locus, and let $\dim(X)$ be its dimension; if X is affine, by its embedding dimension $\mathrm{edim}(X)$ we mean the smallest $n \in \mathbb{N}$ such that X is isomorphic to a closed subvariety of $\mathbb{A}^n_K$. If X is regular, let T_X be the tangent bundle over X. If X is equidimensional of dimension 1 (resp. 2), we call X a curve (resp. surface) over K.

(iv) The *iterate inner invariant*

$$\iota = \iota_{n,K} : \mathrm{End}_n(K) \to \mathbb{N} \cup \{\infty\}$$

can be defined for each endomorphism $e : \mathcal{B} \to \mathcal{B}$ of a set $\mathcal{B}$: if there exists $m \in \mathbb{N}$ such that $\mathrm{Im}(e^m) = \mathrm{Im}(e^{m+n})$ for all $n \in \mathbb{N}$ (equivalently, for $n = 1$), then $\iota(e)$ is the smallest such m; if no such m exists, then $\iota(e)$ is ∞. By the *complement* of e we mean the subset $\Gamma_e := \mathcal{B} \setminus \mathrm{Im}(e)$. If $d : X \to X$ is an endomorphism of a variety

X over K, then $\iota(d) = \iota\big(d(K)\big)$ and its complement Γ_d is endowed with its reduced subscheme structure if it is locally closed.

Lemma 6.1(2) recalls the well-known fact that for $e \in QF_2(K)$ we have $\varphi(e) \in \mathbb{N}$. Note that for $s \in \mathbb{N}^*$ we have $\iota(e) \leq s\iota(e^s)$.

(v) The *gap (or missed points) invariant*

$$\varphi = \varphi_{n,K} : \mathrm{QF}_n(K) \to \mathbb{N} \cup \{\infty\}$$

can be also defined for each endomorphism $e : \mathcal{B} \to \mathcal{B}$ of a set $\mathcal{B}$: if Γ_e is finite then $\varphi(e)$ is the cardinality of Γ_e, and if φ_e is infinite then $\varphi(e) = \infty$.

If $\iota(e), \varphi(e) \in \mathbb{N}$, then $\varphi\big(e^{\iota(e)}\big)$ is the smallest cardinality of a subset $\mathcal{C} \subset \mathcal{B}$ with the property that e induces a surjective endomorphism of $\mathcal{B} \setminus \mathcal{C}$.

To define the last invariant we first need a definition.

Definition 2.3. *By a Jacobian variety over K of dimension $n \geq 2$ we mean an endomorphism $\psi : X \to X$ of a normal connected variety X over K of dimension n such that $\psi(X)$ is an open subvariety of X, the induced morphism $X \to \psi(X)$ is finite, and $\psi|\psi(X)$ is étale. The Jacobian variety ψ is called étale (resp. regular or affine) if ψ (resp. X) is so. Let $\rho(\psi)$ be the number of irreducible components of Γ_ψ (i.e., the cardinality of $\mathrm{Irr}_{\Gamma_\psi}$). Let $\rho_{\text{ét}}(\psi)$ be the number of irreducible components of Γ_ψ whose generic points are in the étale locus $X_e^{\text{ét}}$ of ψ. By an isomorphism (resp. a quasi-isomorphism) between two Jacobian varieties $\psi : X \to X$ and $\psi_1 : X_1 \to X_1$ over K we mean an isomorphism $\tilde{\imath} : X \to X_1$ (resp. a pair $(\tilde{\imath}_{\mathrm{s}}, \tilde{\imath}_{\mathrm{t}})$ of isomorphisms $\tilde{\imath}_{\mathrm{s}}, \tilde{\imath}_{\mathrm{t}} : X \to X_1$) over K such that $\psi_1 \circ \tilde{\imath} = \tilde{\imath} \circ \psi$ (resp. $\psi_1 \circ \tilde{\imath}_{\mathrm{s}} = \tilde{\imath}_{\mathrm{t}} \circ \psi$ and $\tilde{\imath}_{\mathrm{s}}\big(\mathrm{Im}(\psi)\big) = \mathrm{Im}(\psi_1)$); we identify $\tilde{\imath}$ with the quasi-isomorphism $(\tilde{\imath}, \tilde{\imath})$. Let $\mathbb{J}_n(K)$ (resp. $\mathbb{J}_n^{\mathrm{q}}(K)$) be the set of isomorphism (resp. quasi-isomorphism) classes $[\psi]$ of Jacobian varieties ψ over K of dimension n.*

(vi) For $n \geq 2$, the *Jacobian variety inner invariant*

$$\Psi = \Psi_K : \mathrm{EE}_n(K) \to \mathbb{J}_n(K)$$

and its associated *class rank* (resp. *étale class rank*) *invariant*

$$\rho = \rho_{K,n} : \mathrm{EE}_n(K) \to \mathbb{N} \;\; (\text{resp. } \rho_{\text{ét}} = \rho_{\text{ét},K,n} : \mathrm{EE}_n(K) \to \mathbb{N})$$

are defined as follows. Let X_e be the normalization of $\mathbb{A}^n_{K,\mathrm{t}}$ in the field of fractions of $\mathbb{A}^n_{K,\mathrm{s}}$; we have an open embedding $\imath_e : \mathbb{A}^n_{K,\mathrm{s}} \to X_e$ by Zariski's Main Theorem. Let ψ_e be the composite of the finite morphism $X_e \to \mathbb{A}^n_{K,\mathrm{t}}$ with $\imath_e : \mathbb{A}^n_{K,\mathrm{t}} = \mathbb{A}^n_{K,\mathrm{s}} \to X_e$; so $\rho(e) := \rho(\psi_e)$ is the number of irreducible components (they have dimension $n-1$, see [SGA2], Exp. V, Ex. 3.4) of the complement $X_e \setminus \mathrm{Im}(\imath_e) = \Gamma_{\psi_e}$ and we call it the class rank of e and $\rho_{\text{ét}}(e) := \rho_{\text{ét}}(\psi_e)$ is the number of such irreducible components whose generic points are contained in the étale locus of ψ_e and we call it the étale class rank of e.[4] We call $\psi_e : X_e \to X_e$ as the Jacobian variety associated to e and define $\Psi(e) := [\psi_e]$. If $\rho(e) > 0$ (i.e., $e \neq \psi_e$), then $\iota(\psi_e) = \iota(e) + 1$.

Note that e and ψ_e determine each other and the image of Ψ is in natural bijection to the quotient set of the conjugacy relation on $\mathrm{EE}_n(K)$ defined by the conjugacy action of $\mathrm{GA}_n(K)$ on it. Similarly, the image of the *Jacobian variety invariant*

$$\Psi^{\mathrm{q}} = \Psi^{\mathrm{q}}_K : \mathrm{EE}_n(K) \to \mathbb{J}^{\mathrm{q}}_n(K)$$

defined by the same rule $\Psi^{\mathrm{q}}(e) := [\psi_e]$ is in natural bijection to the set of double cosets $\mathrm{GA}_n(K) \backslash \mathrm{EE}_n(K) / \mathrm{GA}_n(K)$.

[4]It is easy to see that the divisor class group $Cl(X_e)$ is isomorphic to $\mathbb{Z}^{\rho(e)}$.

Note that ψ_e is finite flat in codimension at most 2 (see [SGA2], Exp. X, Lem. 3.5). Thus, if $n=2$, then ψ_e is finite flat and we call ψ_e a Jacobian surface.

The first goal of the paper is to study extensively the main six (inner) invariants introduced above for $e \in \mathrm{EE}_n(k)$; other invariants that help in or complement the study of the main six ones are introduced when required. Our basic results for $n=2$ are three explicit examples (Examples 12.3, 13.2 and 14.1) whose existential essences are combined here as one theorem.

Theorem 2.4. **(1)** *For each $l \in \mathbb{N}$, there exists an F_2-system $\mathcal{U}$ over k with* $\mathrm{s}(\mathcal{U}) = l < \deg(e_{\mathcal{U}})$ *(i.e., there exists $e \in \mathrm{BE}_2(k)$ such that $l \in \mathcal{F}(e)$).*

(2) *Let $(m,q,s) \in (\mathbb{N}^*)^2 \times \mathbb{N}$ be such that $p^q \geq s+1$. Then there exists $e \in \mathrm{EE}_2(k)$ defined over $\mathbb{F}_{p^q}$ and with $\deg(e) = pm$, $\varphi(e) = s$, ψ_e étale, $\rho_{\text{ét}}(e) = \rho(e) = (pm-1)(s+1)$, and $\pi(e) \leq \pi_{\mathrm{l}}(e) = \pi_{\mathrm{r}}(e) = (3s+2)(pm-1)-1$. Moreover, $\mathcal{F}(e)$ is $\{1\}$ if $s=0$ and $pm=2$, is $\mathcal{F}(e) = \{0,1\}$ if $s \geq 1$ and $pm = 2$, is $\{1,2,pm-1\}$ if $s=0$ and $pm>2$, and is $\{0,1,2,pm-1\}$ if $s \geq 1$ and $pm > 2$.*

(3) *Let $(m,q) \in (\mathbb{N}^*)^2$ be such that p divides mq. Then there exists $e \in \mathrm{EE}_2(k)$ defined over $\mathbb{F}_p$ with invariants $\deg(e) = mq$, $\varphi(e) = mq-1$, $\mathcal{F}(e) = \{0,1,mq-1\}$, $\pi(e) \leq \pi_{\mathrm{l}}(e) = \pi_{\mathrm{r}}(e) = mq + \max\{(mq-1)q, m\} \geq 2mq$, and ψ_e is regular. If $mq > 2$ we have $\iota(e) = 1$, $\rho_{\text{ét}}(e) = mq-1$, $\rho(e) = mq$ and ψ_e is non-étale, and if $mq = 2$ (so $p = 2$) we have $\iota(e) = 2$, $\rho_{\text{ét}}(e) = \rho(e) = 2$ and ψ_e is étale. Moreover, if p divides both m and q then $e \in \mathrm{BE}_2(k)$ and if $q = 1$ then $\pi_{\mathrm{i}}(e) = \pi_{\mathrm{l}}(e) = \pi_{\mathrm{r}}(e) = 2m$.*

A key novelty of Theorem 2.4(1) is the case $l=0$. To compute the number of solutions in k^2 of an F_2-system $x + f^p = 0 = y + g^p$ over k, so $(f,g) \in k[x,y]^2$, one looks at the surjective étale (first projection) morphism

$$\mathrm{Spec}\left(k[x,y]/(y+g^p)\right) \to \mathrm{Spec}\left(k[x]\right)$$

and to get examples of inconsistent F_2-systems over k one needs to choose (f,g) so that $x + f^p + (y+g^p) \in k[x,y]/(y+g^p)$ is a unit.

First, one is led to identify integral curves C on $\mathrm{Spec}\left(k[x,y]\left[\frac{1}{x+y^p}\right]\right)$ whose normalizations have open subvarieties that map étale surjectively into the affine line $\mathrm{Spec}\left(k[x]\right)$ via the natural first projection. There exist many Cs including the ones given by equations $y^l(1+y^{pm}) = 1$ with $(l,m) \in (\mathbb{N} \setminus p\mathbb{N}) \times \mathbb{N}^*$ (cf. Example 22.5). The simplest such curve C is defined by the equation $y(x+y^p) = 1$ and is isomorphic to $\mathrm{Spec}\left(k[t,t^{-1}]\right)$ via the finite étale morphism

$$\Sigma : \mathbb{G}_{m,k} \to \mathbb{A}^1_k = \mathrm{Spec}\left(k[x_1]\right)$$

over $\mathrm{Spec}\, k$ defined at the level of k-algebras by the rule: $x_1 \mapsto t^{-1} - t^p$. Second, one is led to find ways to dominate such a 'good' curve C via a morphism $\mathrm{Spec}\left(k[x,y]/(y+g^p)\right) \to C$ that commutes with the first projections.

Theorem 2.4(2) gives that for all $m \in \mathbb{N}^*$ we have an identity

$$\{\varphi(e) | e \in \mathrm{EE}_2(k), \deg(e) = pm, \psi_e \text{ is étale}\} = \mathbb{N}.$$

Theorem 17.6 proves the identity $\{\varphi(e) | e \in \mathrm{EE}_2(k), \psi_e \text{ is non-étale}\} = \mathbb{N}$. Directly from either one of these identities we get (see Section 18) the following result.

Corollary 2.5. *Let X be a non-empty affine variety over k and let $l := \mathrm{edim}(X)$. Let $q,n \in \mathbb{N}^*$ with $n \geq l+2$. Then there exist $e \in \mathrm{EE}_n(k)$ with $\iota(e) \in \mathbb{N}^*$ and a sequence $q = q_1 \leq q_2 \leq \cdots \leq q_{\iota(e)}$ of integers such that for each $i \in [[1,\iota(e)]]$ the complement Γ_{e^i} is a closed subvariety of $\mathbb{A}^n_{k,\mathrm{t}}$ isomorphic to q_i copies of X.*

E.g., if $X = \mathbb{A}^l_k$, then the inequality $n \geq l+2$ of Corollary 2.5 cannot be improved by Lemma 6.1(2); moreover, if $n = l+2$ then one can exceptionally take e to be a cartesian product $d \times 1_X$ with $d \in \mathrm{EE}_2(k)$ such that $\varphi(d) = q$.

If κ is a field of characteristic p and $e \in \mathrm{End}_n(\kappa)$ is defined by an n-tuple $(g_1, \ldots, g_n) \in \kappa[x_1, \ldots, x_n]$, then it is well-known that the following three statements are equivalent (see [No], Thm. (6.2); see also [Ni], Thm. 1.4).

(N1) We have $e \in \mathrm{EE}_n(\kappa)$.

(N2) We have $\kappa[x_1, \ldots, x_n] = \kappa[x_1^p, \ldots, x_n^p, g_1, \ldots, g_n]$.

(N3) The n-tuple $(g_1, \ldots, g_n)$ is a p-basis of $\kappa[x_1, \ldots, x_n]$, i.e., its associated family $\{\prod_{i=1}^n g_i^{m_i}; m_i \in [[0, p-1]]\ \forall\ i \in [[1, n]]\}$ is a basis of the $\kappa[x_1^p, \ldots, x_n^p]$-module $\kappa[x_1, \ldots, x_n]$.

Based on the equivalence $(N1) \Leftrightarrow (N3)$ we have the following interpretation of the case $(m, q, s) = (p, 1, 1)$ of Theorem 2.4(2) or of the case $mq = p$ of Theorem 2.4(3) in terms of p-bases (see Subsection 14.1 for the case $mq = p$ that involves smaller algebraic degrees).

Corollary 2.6. *Let κ be an arbitrary field of characteristic p, let $n \geq 2$ be an integer and let $R := \kappa[x_1, \ldots, x_n]$. Then there exists a p-basis $(g_1, \ldots, g_n)$ of R such that the degree of the finite field extension $\kappa(g_1, \ldots, g_n) \to \kappa(x_1, \ldots, x_n)$ is p and we have an equality $(g_1, \ldots, g_n) = (x_1, \ldots, x_n)$ of ideals of R (resp. and $\sum_{i=1}^n Rg_i = R$, i.e., the n-tuple $(g_1, \ldots, g_n) \in R^n$ is unimodular)*

Conjecture [Ad], Conj. 3.4 is not properly formulated but it was meant to be the positive characteristic analogue of the 'Generalized Jacobian Conjecture' of Bass in characteristic 0 (see [Ba2], Rm. after Prop. (4.3)) which was disproved by Kulikov in dimension 2 (see [Kul], Sect. 3). The following theorem is a 3- and 2-dimensional non-units analogues of the étale morphism Σ that disproves [Ad], Conj. 3.4 (in all formulations), that is an output of the geometry of $\mathbf{SL}_{2,k}$, that is closely related to [Kr], Sect. 2, Ex. 1, i.e., the hypersurface of $\mathrm{Spec}\,(\mathbb{C}[t, x, y, z])$ defined by the equation $x + x^2y + z^2 + t^2 = 0$ equipped with a $\mathbb{G}_{m,\mathbb{C}}$-action, and that refines [Kam], Thm. in the particular cases of simply connected semisimple groups $\mathbf{SL}_{2,\mathbb{F}_{p^q}}$ with $q \in \mathbb{N}$ and affine irreducible varieties $\mathbb{A}^3_k$ and $\mathbb{A}^2_k$.

Theorem 2.7. *The following properties hold for $n \in \{2, 3\}$.*

(1) *Let $q \in \mathbb{N}^*$. There exists a finite Galois cover*

$$\mathcal{M}_n : \mathbb{X}^n_k \to \mathbb{A}^n_k$$

of Galois group $\mathbf{SL}_2(\mathbb{F}_{p^q})$ between rational affine varieties over k with group of units equal to k^* which is defined over $\mathbb{F}_{p^q}$. Hence $\mathcal{M}_n$ factors through finite étale morphisms

$$\mathbb{Y}^n_k \to \mathbb{A}^n_k \text{ and } \mathbb{W}^n_k \to \mathbb{A}^n_k$$

of degree $p^{2q} - 1$ and $p^q + 1$ (respectively) between rational affine varieties over k with group of units equal to k^* which are defined over $\mathbb{F}_{p^q}$.

(2) *If $p = 2$ and $q = 1$, then $\mathcal{M}_n$ factors through a finite Galois cover $\mathbb{V}^n_k \to \mathbb{A}^n_k$ of degree 2 defined over $\mathbb{F}_2$, where $\mathbb{V}^n_k$ is rational, has group of units equal to k^*, and has a finite Galois cover $\mathbb{X}^n_k \to \mathbb{V}^n_k$ of degree 3 (hence its prime-to-2 fundamental group is non-trivial).*

(3) *If $p=3$ and $q=1$, then $\mathcal{M}_n$ factors through a finite Galois cover $\mathbb{V}_k^n \to \mathbb{A}_k^n$ of degree 3 defined over $\mathbb{F}_3$, where $\mathbb{V}_k^n$ is rational, has group of units equal to k^*, and has a finite Galois cover $\mathbb{X}_k^n \to \mathbb{V}_k^n$ with Galois group isomorphic to the quaternion group Q_8 of order 8 (hence its prime-to-3 fundamental group is non-trivial).*

(4) *In dimension 2, the morphisms of (1) to (3) are obtained from the analogue ones in dimension 3 via taking quotients through a left $\mathbb{G}_{a,k}$-action defined over $\mathbb{F}_{p^q}$ (i.e., we have $\mathbb{X}_k^2 = \mathbb{G}_{a,k}\backslash\mathbb{X}_k^3$, $\mathcal{M}_2 = \mathbb{G}_{a,k}\backslash\mathcal{M}_3$, etc.).*

The proof of Theorem 2.7 uses Nori's idea (see [Kam]) to take $\mathcal{M}_3$ as the pullback of Lang $\mathbf{SL}_2(\mathbb{F}_{p^q})$-torsor $\mathbf{SL}_{2,k} \to \mathbf{SL}_{2,k}$ via a 'good' morphism $\mathbb{A}_k^3 \to \mathbf{SL}_{2,k}$. The finite Galois cover $\mathbb{V}_k^n \to \mathbb{A}_k^n$ of Theorem 2.7(2) (resp. 2.7(3)) is of Artin–Schreier type by either [Mi1], Thm. 1.1 or Fact 25.1; as $\mathbb{V}_k^n$ has group of units equal to k^*, the existence of a non-trivial cyclic cover of it of degree 3 (resp. 2) implies by Kummer theory that it is non-factorial.

The classification of Jacobian varieties ψ_e with $e \in \mathrm{EE}_n(K)$ would implicitly classify all finite covers of $\mathbb{A}_K^n$ of the form X_e with $e \in \mathrm{EE}_n(K)$. If $e \in \mathrm{EE}_n(K)$ is such that ψ_e is regular and non-étale, then X_e is not a complete intersection (see Theorem 7.3(5)) and it is not clear even how to begin the classification of such X_es. Thus, to begin with we first consider hypersurfaces in $\mathbb{A}_K^{n+1}$. Let

$$\mathrm{IntRat}_n(K) := \{(f,g) \in K[x_1,\ldots,x_n]^2 | g \neq 0,\ \text{g.c.d.}(f,g)=1\}.$$

For $(f,g) \in \mathrm{IntRat}_n(K)$, the zero locus

$$f(x_1,\ldots,x_n) + g(x_1,\ldots,x_n)x_{n+1} = 0 \tag{2}$$

is an integral rational hypersurface $\mathbb{H}^n_{f,g,K}$ in $\mathbb{A}_K^{n+1}$ but in general it is neither smooth nor contains an open subvariety isomorphic to $\mathbb{A}_K^n$. Thus we are let to consider the three subsets

$$\mathrm{NorIntRat}_n(K) := \{(f,g) \in \mathrm{IntRat}_n(K) | \mathbb{H}^n_{f,g,K} \text{ is normal}\},$$
$$\mathrm{SmoIntRat}_n(K) := \{(f,g) \in \mathrm{IntRat}_n(K) | \mathbb{H}^n_{f,g,K} \text{ is smooth}\},$$
$$\mathrm{AffIntRat}_n(K) := \{(f,g) \in \mathrm{IntRat}_n(K) | \exists \text{ an open embedding } \mathbb{A}_K^n \to \mathbb{H}^n_{f,g,K}\}.$$

The hypersurfaces $\mathbb{H}^n_{f,g,K}$ with $(f,g) \in \mathrm{IntRat}_n(K)$ such that $g = \prod_{i=1}^n x_i^{m_i}$ with $(m_1,\ldots,m_n) \in \mathbb{N}^n$ (i.e., g is a primitive monomial) and with f of certain types have been extensively studied in the literature from many points of views, including automorphisms, group actions, invariants, cancellation problems, isomorphism classes (more general types of fs are considered in [Du2] and [GG], e.g., in [GG], Thm. B, f is a monic polynomial in one of the variables).

The main object of interest is the intersection $\mathrm{NorIntRat}_n(K) \cap \mathrm{AffIntRat}_n(K)$. One would like to classify (up to isomorphisms) all open embeddings $\mathbb{A}_K^n \to \mathbb{H}^n_{f,g,K}$ with $(f,g) \in \mathrm{NorIntRat}_n(K) \cap \mathrm{AffIntRat}_n(K)$ and, when $\mathrm{char}(K)=p$, all finite étale covers $\mathbb{H}^n_{f,g,K} \to \mathbb{A}_K^n$ with $(f,g) \in \mathrm{SmoIntRat}_n(K) \cap \mathrm{AffIntRat}_n(K)$.

It does not suffice to study $\mathbb{J}_n(K)$ intrinsically via hypersurfaces. We note that $\mathrm{NorIntRat}_n(K) \cap \mathrm{AffIntRat}_n(K)$ can be used to study complete intersections in $\mathbb{A}_K^n$ isomorphic to certain X_es with $e \in \mathrm{EE}_n(K)$ (e.g., see Examples 22.12 and 22.13).

In what follows we introduce a two polynomial parameters family of finite étale covers of $\mathbb{A}_k^2$. For the sake of generality, we introduce them over $\mathrm{Spec}\, R$ as follows.

We say that a monic polynomial $f(t) \in R[t]$ is separable if its reduction modulo each maximal ideal of R is so. Let

$$\Theta_R := \{(f,g) \in (R[t])^2 | f \text{ is monic separable}, g \text{ is monic}, f(0) \in R^*\};$$

e.g., for $n \geq 2$ we have $(t^{n-1}-1,1) \in \Theta_{\mathbb{Z}[\frac{1}{n-1}]}$. For $(f,g) \in \Theta_R$ we consider the affine smooth surface

$$\mathbb{S}^2_{f,g,R} := \mathrm{Spec}\,\big(R[x,y,z]/(xf(x)+yg(y)z)\big)$$

over $\mathrm{Spec}\,R$; so $\mathbb{S}^2_{1,1,\mathbb{Z}} \cong \mathbb{A}^2$ and for $n \geq 2$ we single out the affine smooth surface

$$\mathbb{S}_n := \mathbb{S}_{t^{n-1}-1,1,\mathbb{Z}[\frac{1}{n-1}]} = \mathrm{Spec}\,\big(R[x,y,z]/(x^n-x+yz)\big)$$

over $\mathbb{Z}[\frac{1}{n-1}]$. We call

$$\mathbb{S}^2 := \mathbb{S}^2_2 = \mathbb{S}^2_{t-1,1,\mathbb{Z}} = \mathrm{Spec}\,\big(\mathbb{Z}[x,y,z]/(x^2-x+yz)\big)$$

the arithmetic 2-dimensional sphere (over $\mathrm{Spec}\,\mathbb{Z}$) as we have isomorphisms $\mathbb{S}^2_{\mathbb{Z}[\frac{1}{2}]} \cong \mathrm{Spec}\,\big(\mathbb{Z}[\frac{1}{2}][x,y,z]/(x^2+y^2-z^2-1)\big)$ and, with $i \in \mathbb{C}$, $i^2=-1$,

$$\mathbb{S}^2_{\mathbb{Z}[i][\frac{1}{2}]} \cong \mathrm{Spec}\,\big(\mathbb{Z}[i][\frac{1}{2}][x,y,z]/(x^2+y^2+z^2-1)\big).$$

Over K, groups of automorphisms, group actions, and isomorphism classes related to certain subclasses of the surfaces $\mathbb{S}_{f,g,K}$ have been extensively studied in the literature starting with [GD], especially when $\mathrm{char}(K)=0$. For instance, $\mathbb{S}^2_K$ is isomorphic to $\mathbb{P}^1_K \times_{\mathrm{Spec}\,K} \mathbb{P}^1_K \setminus \Delta_{\mathbb{P}^1_K}$, where $\Delta_{\mathbb{P}^1_K}$ is the image of the diagonal embedding $\mathbb{P}^1_K \to \mathbb{P}^1_K \times_{\mathrm{Spec}\,K} \mathbb{P}^1_K$ (e.g., see [BvS], Lem. 2.1); the ind-group scheme $\mathrm{Aut}(\mathbb{S}^2_K)$ has been described as an amalgamated product in [GD], Sect. 10.1 if $\mathrm{char}(K) \neq 2$ and in [Lam], Thm. 4 in general.

For $(f,t^m) \in \Theta_K$ with $m \in \mathbb{N}$, the surfaces $\mathbb{S}_{f,t^m,K}$ are called (often special or classical) Danielewski surfaces in [FM-J], [Du1], etc. after the author of an unpublished 1989 manuscript which was first quoted in [F] and in which the particular case $f(t)=t-1$ over $\mathbb{C}$ is studied in connection to the cancellation problem (by $\mathbb{A}^1_{\mathbb{C}}$); if $m=0$ (resp. if $m \geq 1$), generators of their groups of automorphisms (resp. generators of their groups of automorphisms and isomorphism classes) are described in [M-L1], Thm. (resp. in [M-L2], Thms. 1 and 2 for $K=\mathbb{C}$ and in [Cr], Thm. 4.2 and Cor. 4.3 applied to $h(x)=-1$). Generalized Danielewski surfaces are zero loci $L(x,y)+y^nz=0$ (hypersurfaces) in $\mathbb{A}^3_K$ with $L(x,y) \in K[x,y]$ such that $\deg\big(L(x,0)\big) \geq 2$; over $\mathbb{C}$ they were introduced and classified in [Po], Def. 2 and Sects. 4 and 5. The terminology 'generalized' is supported by the two facts that (i) for $n \geq 1$ and $g \in K[t]$ such that $\deg(g) < n$ the zero loci $x^2+xg(y)+y^nz=0$ classify surfaces that are non-trivial line bundles over the affine line over $\mathrm{Spec}\,K$ with the origin doubled (see [Wi], Sect. 2), and (ii) over $\mathbb{C}$, for $n \geq 2$ and $(f,g) \in K[t]^2$ with $\deg(f) \geq 2$ and $\deg(g) \leq n-1$, the surfaces that are the zero loci $f(x)+y^nz=0$ and $f(x)g(y)+y^nz=0$ are isomorphic by [FM-J], Thm. 2.

There exist many morphisms between these surfaces over $\mathrm{Spec}\,R$, among which two types are introduced here. We have an open embedding

$$\imath_{f,g,R} : \mathbb{A}^2_R \to \mathbb{S}^2_{f,g,R}$$

defined on valued points by the rule $(x,y) \mapsto \big(xyg(y), y, -xf(xyg(y))\big)$; its complement $\mathbb{D}^1_{f,g,R} := \mathbb{S}_{f,g,R} \setminus \mathrm{Im}(\imath_{f,g,R})$ is the zero locus $f(x)=yg(y)=0$. Over a field $\mathcal{K}$ of characteristic not dividing $n-1$ that contains the $n-1$-th roots of 1 and all zeros of $g(y)$ in an algebraic closure of $\mathcal{K}$, $\mathbb{D}^1_{t^{n-1}-1,g,\mathcal{K}}$ is a disjoint union of $(n-1)\mathrm{z}(tg)$ copies of $\mathbb{A}^1_{\mathcal{K}}$; e.g., $\mathbb{D}^1 := \mathbb{D}^1_{t-1,1,\mathbb{Z}} \cong \mathbb{A}^1$ and $\mathbb{D}^1_{t^{n-1}-1,1,\mathcal{K}}$ is isomorphic to $n-1$ copies of $\mathbb{A}^1_{\mathcal{K}}$. Thus for $n \leq 2$ and $\deg(g) \in \mathbb{N}$ we get the following fact.

Fact 2.8. *For $m \in \mathbb{N}^*$ (resp. $m = 0$), there exists a (monic) connected smooth hypersurface in $\mathbb{A}^3_K$ which is the disjoint union of $\mathbb{A}^2_K$ and of m copies of $\mathbb{A}^1_K$ and which is a finite flat cover of $\mathbb{A}^2_K$ of degree 2 (resp. 1).*

The existence of the open embeddings $\imath_{f,g,K}$ imply that we have an inclusion

$$\Theta_K \subset \mathrm{SmoIntRat}_2(K) \cap \mathrm{AffIntRat}_2(K).$$

If $(f,g) \in \Theta_R$ is such that we have a product decomposition $f = f_1 f_2$ of monic polynomials, then $(f_1,g) \in \Theta_R$ and the rule on valued points $(x,y,z) \mapsto \bigl(x,y,f_2(x)z\bigr)$ defines an open embedding $\imath_{f_1,g;f,R} : \mathbb{S}^2_{f_1,g,R} \to \mathbb{S}^2_{f,g,R}$ whose complement is the zero locus $f_2(x)yg(y) = 0$; note that $\imath_{f_1,g;f,R} \circ \imath_{f_1,g,R} = \imath_{f,g,R}$.

Let

$$c_{f,g,R} : \mathbb{S}^2_{f,g,R} \to \mathbb{A}^2_R$$

be the finite flat morphism defined by the rule on valued points $(x,y,z) \mapsto (y,z)$; the closed subscheme $c_{f,g,R}(\mathbb{D}^1_{f,g,R})$ of $\mathbb{A}^2_R$ is defined by $x_1 g(x_1) = 0$.

If R is an $\mathbb{F}_p$-algebra and $m \in \mathbb{N}^*$, let

$$\Theta^1_{R,m} := \{(f,g) \in \Theta_R | f(t) + tf'(t) \in R^*, \deg(f) = pm - 1\}.$$

If $(f,g) \in \Theta^1_{k,m}$, then $c_{f,g,k} : \mathbb{S}^2_{f,g,k} \to \mathbb{A}^2_k$ is a finite étale cover of degree pm which is Galois if $tf(t)$ is an additive polynomial; its composite with the open embedding $\imath_{f,g,k} : \mathbb{A}^2_k \to \mathbb{S}^2_{f,g,k}$ is a surjective non-finite endomorphism

$$e_{f,g,k} := c_{f,g,k} \circ \imath_{f,g,k} \in \mathrm{EE}_2(k)$$

with $\deg(e_{f,g,k}) = pm$ which is defined over the $\mathbb{F}_p$-subalgebra of k generated by the coefficients of $f(t)$ and $g(t)$ and is finite over the complement of the zero locus $x_1 g(x_1) = 0$ in $\mathbb{A}^2_{k,\mathrm{t}}$.

For each $(l,q) \in \mathbb{N}^2$ we construct (see Corollary 17.7) étale endomorphisms of suitable affine open subvarieties of the surfaces $\mathbb{S}^2_{f,t,k}$ whose images have complements that are disjoint unions of l copies of $\mathbb{A}^1_k$ and of q points (q copies of $\mathbb{A}^0_k$).

E.g., if $(f,g) \in \Theta^1_{k,m}$, then the composite $\imath_{f,g,k} \circ c_{f,g,k} : \mathbb{S}^2_{f,g,k} \to \mathbb{S}^2_{f,g,k}$ is the étale Jacobian surface $\psi_{e_{f,g,k}}$ associated to $e_{f,g,k}$, so $\Gamma_{\psi_{e_{f,g,k}}}$ is isomorphic to $(pm-1)\mathrm{z}(tg)$ copies of $\mathbb{A}^1_k$ and we have $\rho_{\text{ét}}(e_{f,g,k}) = \rho(e_{f,g,k}) = (pm-1)\mathrm{z}(tg)$. This proves Theorem 2.4(2) for $s = 0$; the proof for $s > 0$ is similar but using a different open embedding $\mathbb{A}^2_{k,\mathrm{s}} \to \mathbb{S}^2_{f,g,k}$ (see Proposition 13.1).

If $\mathrm{char}(K) = 0$, then for each $(f,g) \in \Theta_K$ the surface $\mathbb{S}^2_{f,g,K}$ is simply connected as it has an open subvariety isomorphic to $\mathbb{A}^2_K$. In contrast, for each $(f,g) \in \Theta_k$ there exists a finite étale cover $\mathbb{S}^2_{f,g,k} \to \mathbb{A}^2_k$ (see Proposition 20.2(1)) and hence $\mathbb{S}^2_{f,g,K}$ is not simply connected and (see Proposition 20.2(2)) there exists $e \in \mathrm{EE}_2(k)$ such that ψ_e is étale and $X_e \cong \mathbb{S}^2_{f,g,k}$ (recall that $\mathbb{S}^2_{1,g,k} \cong \mathbb{A}^2_k$).

If $(f,g) \in \mathrm{SmoIntRat}_n(K)$, then the tangent bundle $T_{\mathbb{H}_{f,g,K}}$ is trivial, hence each open affine non-empty subvariety W of $T_{\mathbb{H}_{f,g,K}}$ is a complete intersection by Theorem 7.2(2). If $n = 2$, $(f,g) \in \Theta_K$, and W contains an open subvariety isomorphic to $\mathbb{A}^n_K$, then there exist examples in which $W \cong \mathbb{H}_{\tilde{f},\tilde{g},K}$ for a suitable pair $(\tilde{f},\tilde{g}) \in \mathrm{SmoIntRat}_2(K)$ and moreover either $W \cong \mathbb{S}^2_{\tilde{f}_1,\tilde{g}_1,K}$ for a suitable pair $(\tilde{f}_1,\tilde{g}_1) \in \Theta_K$ (for $f = f_1 f_2$ and $(f,g) \in \Theta_K$, see $\mathrm{Im}(\imath_{f_1,g;f,K})$) or $W \ncong \mathbb{S}^2_{\tilde{f}_1,\tilde{g}_1,K}$ for every $(\tilde{f}_1,\tilde{g}_1) \in \Theta_K$ (see Example 20.9).

If $e \in \mathrm{EE}_2(K)$ is non-finite, then X_e (resp. the étale locus $X_e^{\text{ét}}$ of ψ) has a canonical open affine cover associated to $\imath_e$ formed by connected normal surfaces that are disjoint unions $\mathrm{Im}(\imath_e) \cup Y$ with $Y \in \mathrm{Irr}(\Gamma_\psi)$ (resp. $Y \in \mathrm{Irr}(\Gamma_\psi)$ and $Y \subset X_e^{\text{ét}}$) by Definition 11.6(1) (resp. Corollary 11.8(2)). This leads to the following formalism of sphere-like surfaces.

Definition 2.9. *An affine normal connected surface X over K is called a sphere-like surface if it has an open subvariety U with the properties that $U \cong \mathbb{A}^2_K$ and $X \setminus U$ is non-empty irreducible. A sphere-like surface X is called:*

(1) *a pseudo-sphere if it is not regular;*

(2) *a quasi-sphere if it is regular but its tangent bundle is non-trivial;*

(3) *an almost sphere if it is regular with trivial tangent bundle and $X \not\cong \mathbb{S}^2_k$;*

(4) *a sphere if it is isomorphic to $\mathbb{S}^2_k$.*

In this paper we develop the theory of sphere-like surfaces only as far as needed for computing invariants and for relating to prior works. We recall that a morphism $\Sigma : W \to X$ of varieties over K is called an $\mathbb{A}^1$-fibration if for a general point $x \in X$, the fiber $\Sigma^{-1}(x)$ is an affine line over the spectrum of the residue field of x. Each sphere-like surface over K is an $\mathbb{A}^1$-fibration over $\mathbb{P}^1_K$ and a disjoint union of an open subvariety isomorphic to $\mathbb{A}^2_K$ and a closed subvariety isomorphic to $\mathbb{A}^1_K$ by Theorem 11.2(1) and (5); if $\mathrm{char}(K) = 0$ then the first property and, in the case when X is regular, the second property were known and used before (e.g., see [Mi2], Sect. 2 and Lem. 1) but the proofs we include are different. If $e \in \mathrm{EE}_2(K)$, then the canonical open affine cover of X_e associated to $\imath_e$ consists of exactly $|\mathrm{Sing}(X_e)|$ pseudo-spheres, $\rho(e) - \rho_{\text{ét}}(e) - |\mathrm{Sing}(X_e)|$ quasi-spheres, and of $\rho_{\text{ét}}(e)$ almost spheres or spheres (see Corollary 11.10); in particular, ψ_e is étale iff the canonical open affine cover of X_e associated to $\imath_e$ consists only onf almost spheres and spheres (see Proposition 11.8(2)). For pseudo-spheres see Example 13.4 and Proposition 13.6. For quasi-spheres see all $e \in \mathrm{EE}_2(k)$ with X_e regular and non-étale. If $m \geq 1$, then $\mathbb{S}^2_{t-1,t^m,K}$ is an almost sphere (see Example 20.6).

The geometries of $\mathbf{SL}_2$, $\mathbb{S}^2$, and $\mathbb{A}^2$ are interrelated. E.g., $\mathbb{S}^2$ is a quotient of $\mathbf{SL}_2$ and a line bundle over $\mathbb{P}^1$ and we have an isomorphism $\mathbb{A}^2 \to \mathbb{S}^2 \setminus \mathbb{D}^1$ induced by $\imath_{t-1,1,\mathbb{Z}}$. The interrelations are explained and used in Section 23.

Sections 3 to 10 contain different set-theoretical, algebraic, complete intersection and geometric preliminaries on iterates of functions, inner, left and right algebraic degrees and cosets of endomorphisms $e \in \mathrm{End}_n(K)$, the submonoid of $\mathrm{EE}_2(k)$ formed by p-morphisms introduced in [Lang-J], Def. 2.2, normalizations of finite flat extensions of $k[[x_1, x_2]]$, complete intersection embedding dimensions, and rational singularities. Though $\mathrm{EE}_1(k) = \mathrm{BE}_1(k)$, the set $\mathrm{EE}_2(k) \setminus \mathrm{BE}_2(k)$ is very large as, for instance, it contains all endomorphisms constructed in [No], Ex. (6.10) with $(a, b) \in (k^*)^2$ (see also [Lang-J], Ex. 4.1) and generalized in [Lang-J], Props. 4.5 and 4.8 and Rm. 4.9 by Proposition 6.9(3).

Section 11 presents properties of sphere-like surfaces and of the open embedding $X_e^{\text{ét}} \subset X_e$. Section 12 proves Theorem 2.4(1) (see Example 12.3). Section 13 proves Theorem 2.4(2) (see Example 13.2). Section 14 proves Theorem 2.4(3) (see Example 14.1) and Corollary 2.6 (see Subsection 14.1).

Section 15 constructs finite endomorphisms and 'good' extensions of endomorphisms of affine spaces; they are used extensively in the subsequent sections in order to compute geometric degrees, extend properties in dimension n to all dimensions greater than n, and study moduli spaces of endomorphisms.

Properties of iterates are gathered in Section 16. To describe them, let $e : X \to X$ be an endomorphism of a variety over K. Theorem 16.8(1) proves that if e is quasi-finite then $\iota(e) \in \mathbb{N}$, again by *reduction to positive characteristic*. Previously, this was known only when $K = \mathbb{C}$ (hence when $\mathrm{char}(K) = 0$), X is affine, and e is universally open with Γ_e finite (see [PNCG], Thm. 1; recall that if $K = \mathbb{C}$, e is universally open iff the holomorphic endomorphism $e^{\mathrm{an}} : X^{\mathrm{an}} \to X^{\mathrm{an}}$ is open by [MB], Thm. 1.1). If Γ_e is a finite set and $\dim(X) \leq 2$ (resp. and e is defined over a finite field), then $\iota(e) \in \mathbb{N}$ by Proposition 16.15 (resp. Corollary 16.2). If $n \geq 3$ and K is not an algebraic closure of a finite field, then for each $l \in \mathbb{N}^*$ there exists $d \in \mathrm{End}_n(K)$ such that $\varphi(d) = l$ and $\iota(d) = \infty$ by Proposition 16.13; if $n = 3$, then $\deg(d)$ can be an arbitrary integer greater or equal to 3. Corollary 16.2 bounds $\iota(e)$ when Γ_e is a finite set, $\mathrm{char}(K) = p$ and e is defined over a finite field.

Section 17 proves Theorem 17.6; for the passage from Theorem 2.4(3) to Theorem 17.6 we use over k the $n = 2$ case of the m-transitivity of the group of tame automorphisms of affine spaces over fields (see Proposition 17.1).

Section 18 proves Corollary 2.5 based on a general lemma (see Lemma 15.2) that gives large algebraic and geometric degrees and includes Example 18.2 that illustrates how one can work around to control these degrees.

Section 19 proves Theorem 2.7.

Section 20 provides examples of finite étale covers of $\mathbb{A}^2_k$ and applies them to show that for each $r \in \mathbb{N}$ and $m \in 2\mathbb{N}^*$ there exists $e \in \mathrm{EE}_2(k)$ such that $\rho(e) = \rho_{\text{ét}}(e) = r$ and $\deg(e) = pm$ (see Corollary 20.5(1)). Example 20.3 shows the Nollet–Xavier Conjecture over $\mathbb{C}$ (see [Je], Sect. 1 and [NX], Quest. 6) becomes false over k even for $n = 2$: for each $l \in \mathbb{N}^*$, there exists $e \in \mathrm{EE}_2(k)$ such that the non-proper locus of e is isomorphic to l copies of $\mathbb{A}^1_k$ (so for $l = 1$ it is smooth and integral).

Section 21 shows that the differential equation of [Lang-J], Prop. 4.8 generalizes to all $n \geq 2$ producing many examples of étale endomorphisms in $\mathrm{EE}_n(k)$ whose invariants are studied in order to get applications and results that are in contrast to the analogous results in characteristic 0 (see Remarks 21.6(1) and 21.10 that pertain to prior works [Gw], [YD] and [Zh]) or for $n > 2$ are in contrast to results for $n = 2$ (see Example 21.5). In particular we show that for each $n \geq 3$ there exist plenty of $e \in \mathrm{EE}_n(k)$ for which ψ_e is non-regular (see Example 21.8).

Section 22 gathers additional examples that are either required in the subsequent sections or are of interest in their own.

Section 23 proves variants of Theorem 2.4(2) that often compute the non-generic fibers set invariants.

Section 24 studies moduli affine schemes of different types of étale endomorphisms or of automorphisms of $\mathbb{A}^n$ that were introduced in earlier works such as [No], Sect. 7 and [BCW2], Sect. I, Subsect. 1. E.g., we show that for each $q \in \mathbb{N}^*$ the moduli affine scheme $SEE_{n,q,K}$ over $\mathrm{Spec}\, K$ that parametrizes étale endomorphisms defined by n-tuples $(f_1, \ldots, f_n) \in R[x_1, \ldots, x_n]$ with $\pi(f_1, \ldots, f_n) \leq q$ and with Jacobian determinants 1 are connected (see Corollary 24.9) and are smooth iff either $\min\{n, q\} = 1$ or $n = q = \mathrm{char}(K) = 2$ (see Theorem 24.13). Conjecture [MR], Conj. 3.4 (resp. 5.3(2)) is disproved in Example 24.5 (resp. Remark 24.8).

Section 25 studies a conjecture of Adjamagbo (see [Ad], Conj. 3.1 or $\mathrm{CJC}(n, p)$; see also Conjecture 1.3) over subrings of our fields k (i.e., over integral domains of characteristic p) but it suffices to prove it over the fields k. With our notation, this conjecture over k can be reformulated as follows.

Conjecture 2.10 (Adjamagbo 1995). *If* $e \in \mathrm{EE}_n(k)$*, then* $\deg(e) \in \{1\} \cup p\mathbb{N}^*$.

Corollary 25.3 proves the following positive characteristic analogue of classical results of Razar (see [Raz], Thm. 2) and Wright (see [Wr1], Thms. 3.2, 3.3 and 3.7) in characteristic 0: if $e \in \mathrm{EE}_n(k)$ is such that either it induces a Galois field extension $k_{\mathrm{t}}(x_1, \ldots, x_n) \to k_{\mathrm{s}}(x_1, \ldots, x_n)$ of degree prime to p or it is finite with $\deg(e) < p$, then e is an isomorphism. Corollary 25.7 proves that if $\mathrm{char} K = 0$ (resp. $\mathrm{char}(K) = p$) and $e \in \mathrm{EE}_n(K)$ factors as the composite of a quasi-finite morphism $\mathbb{A}^n_K \to \mathbb{A}^{n+1}_K$ which is a locally closed embedding at points of codimension 1 outside a specific finite subset with a projection $\mathbb{A}^{n+1}_K \to \mathbb{A}^n_K$ on the last n coordinates, then the Jacobian Conjecture (resp. Conjecture 2.10) holds for e. This result generalizes [Fo1], Thm. 1 which assumes that $\mathrm{char}(K) = 0$ and that the morphism $\mathbb{A}^n_K \to \mathbb{A}^{n+1}_K$ is actually a closed embedding.

There exists no $e \in \mathrm{EE}_2(K)$ with $\deg(e) \in \{2, 3, 4, 5\}$ and $\mathrm{char}(K) = 0$ by [Żo], Thm. 6.12. We do not know other general results on geometric degrees at least 3. Section 26 studies endomorphisms $e \in \mathrm{EE}_n(K)$ with $\deg(e) = 3$ for all Ks, including estimates of complete intersection dimensions as defined in Section 7 and their usage to get partial results on the non-existence of such es when $\mathrm{char}(K) \neq 3$.

Section 27 is a first appendix that introduces an ad-hoc notion of 'weakly separated scheme' that is used to give alternative proofs to parts of Section 16 that involve iterations of non-separated endomorphisms.

Section 28 is a second appendix that refines and corrects Nori–Kambayashi's work on pullbacks of Lang torsors (see [Kam]) based on the more recent work [CGP] on split unipotent groups.

Section 29 is a third appendix that lists open problems, questions and conjectures with a few examples, remarks and comments.

Theorem 2.4 neither is implied by nor implies Conjecture 2.10 (see also [vdE], Part II, Ch. 10, Sect. 10.3, Subsect. (10.3.16), Conj. JC($\mathbb{F}_p$, n, p)). But Conjecture 2.10 would imply that the geometric degrees parts of many of our results (such as Theorem 2.4(2) and (3) and Example 21.5) are optimal.

In characteristic 0 there exist étale endomorphisms of surfaces that are not automorphisms (e.g., see [Mi3], Sect. 4, Exs. 1 and 3 and [DP], Thm. B) but it seems that all of them are surjective (and in most cases one even expects that they are finite, see [Mi5], Sect. 4). Replacing étaleness by the weaker quasi-finiteness, the following general example improves [vdE], Part III, Ch. D, Sect. D.2, Ex. D.2.3.

Example 2.11. Let $q \in \mathbb{N}^*$. Let $f(t) \in K[t]$ be a polynomial with $f(0) \neq 0$ and $\mathrm{z}(f) = q$. Let $e \in \mathrm{End}_2(K)$ be given on valued points by the rule

$$(x, y) \mapsto \big(xf(xy) + y, xy\big).$$

Then we have $e \in \mathrm{QFE}_2(K)$ with $\varphi(e) = q$, $\deg(e) = 2$, and $\mathcal{F}(e) = \{0, 1\}$. More precisely, the field extension $K_{\mathrm{t}}(x_1, x_2) \to K_{\mathrm{s}}(x_1, x_2)$ induced by e is isomorphic to $K_{\mathrm{t}}(x_1, x_2) \to K_{\mathrm{t}}(x_1, x_2)[z]/\big(z^2 - \frac{x_1}{f(x_2)}z + \frac{x_2}{f(x_2)}\big)$.

For $(\alpha,\beta)\in K^2$, the system of equations $xf(xy)+y-\alpha=0=xy-\beta$ has no solution if $\alpha=0=f(\beta)$, has one solution if $\alpha\neq 0=f(\beta)$ or $f(\beta)\neq 0=\alpha^2-4\beta f(\beta)$, and has two solutions if $f(\beta)[\alpha^2-4\beta f(\beta)]\neq 0$.[5]

For $e\in \mathrm{QFE}_2(K)$ we have $\deg(e)=1$ iff $e\in \mathrm{GA}_2(K)$ (see Lemma 6.1(4)). From this and the non-existence of $e\in \mathrm{EE}_2(K)$ with $\deg(e)=2$ and $\mathrm{char}(K)\neq 2$, we get that for $\varphi(e)>0$, the equalities $\deg(e)=2$ in characteristic different from 2 of Example 2.11 and $\deg(e)=2$ in characteristic 2 of Theorem 2.4(2) or (3) are optimal in the sense that they give the smallest possible geometric degrees. We also note that if $\mathrm{char}(K)\neq 2$, then there exists no $e\in \mathrm{EE}_2(K)\setminus \mathrm{GA}_2(K)$ with $\pi(e)=2$ (see [Wa2], Thm. 62; see also [BCW2], Thm. (2.4) and Cor. (1.4)).

3. Set-theoretical preliminaries

For a set $\mathcal{B}$, let $|\mathcal{B}|\in\mathbb{N}\cup\{\infty\}$ be its cardinality if it is finite and be ∞ if it is infinite. We introduce notations and prove basic properties of iterates of endomorphisms of sets.

Notation 3.1. Let $e:\mathcal{B}\to\mathcal{B}$ be an endomorphism of a set $\mathcal{B}$. Recall that $\Gamma_e:=\mathcal{B}\setminus \mathrm{Im}(e)$ is the complement of e and $\varphi(e)=|\Gamma_e|$ is the gap invariant of e. For $l\in\mathbb{N}$, let e^l be the l-th iterate of e (so $e^0=1_{\mathcal{B}}$ and $e^{l+1}:=e^l\circ e$), let

$$\mathcal{E}_l(e):=\mathrm{Im}(e)\setminus\mathrm{Im}(e^{l+1})=\{P\in\mathrm{Im}(e)|e^{-1}(P)\subset\Gamma_{e^l}\}$$

and let $\mathcal{E}^1_l(e):=\{P\in\mathcal{E}_l(e)||e^{-1}(P)|=1\}$. See Section 2 for $\iota(e)\in\mathbb{N}\cup\{\infty\}$. Let

$$\Gamma_e^+:=\cup_{i\in\mathbb{N}}e^i(\Gamma_e)\ \text{ and }\ \mathcal{E}^1(e):=\cup_{l\geq 0}\mathcal{E}^1_l(e).$$

Let $\varrho_e:=|\mathcal{E}^1(e)|$. If $\varrho_e\in\mathbb{N}$, let $\xi_e\in\mathbb{N}$ be the smallest such that $\mathcal{E}^1(e)=\mathcal{E}^1_{\xi_e}(e)$. Let $\varsigma_e:=\min\{|e^{-1}(P)||P\in\mathcal{B},e^{-1}(P)\neq\emptyset\}\in\mathbb{N}\cup\{\infty\}$.

Lemma 3.2. *Assume $e:\mathcal{B}\to\mathcal{B}$ is such that Γ_e is finite. Then the following properties hold.*

(1) *The sequence $\big(\varphi(e^{l+1})-\varphi(e^l)\big)_{l\geq 0}$ is a non-increasing sequence in $\mathbb{N}$ (thus $\varphi(e^l)\in\mathbb{N}$ for all $l\in\mathbb{N}$).*

(2) *For each $l\in\mathbb{N}$ we have an inequality $\varphi(e^l)\leq l\varphi(e)$.*

(3) *For each $l\in\mathbb{N}$ we have $|\mathcal{E}_l(e)|=\varphi(e^{l+1})-\varphi(e)$.*

(4) *For each $l\in\mathbb{N}$ we have inclusions $\mathrm{Im}(e^l)\setminus\mathrm{Im}(e^{l+1})\subset e^l(\Gamma_e)$ and therefore $\mathcal{E}^1_l(e)\subset\mathcal{E}_l(e)\subset\cup_{i=1}^l e^i(\Gamma_e)\subset\Gamma_e^+$.*

(5) *For all integers $n\geq l\geq 0$ we have inclusions*

$$\Gamma_{e^{n+1}}\subset\Gamma_{e^l}\cup[\cup_{i=l}^n e^i(\Gamma_e)]\subset\cup_{i=0}^n e^i(\Gamma_e).$$

(6) *If Γ_e^+ is a finite set, then $\iota(e)\leq|\Gamma_e^+\setminus\Gamma_e|$.*

(7) *If $\varsigma_e\geq 2$, then for each $l\in\mathbb{N}$ we have $\varphi(e^{l+1})-\varphi(e)\leq\lfloor\frac{\varphi(e^l)}{\varsigma_e}\rfloor\leq\frac{\varphi(e^l)}{\varsigma_e}$; hence $\varphi(e^{l+1})\leq\lfloor\frac{\varsigma_e\varphi(e)}{\varsigma_e-1}\rfloor$ and if $\varphi(e)>0$ we have $\varphi(e^{l+1})<\frac{\varsigma_e\varphi(e)}{\varsigma_e-1}$.*

(8) *If $\varsigma_e=1$, then we have $|\mathcal{E}^1_l(e)|\geq 2\varphi(e^{l+1})-\varphi(e^l)-2\varphi(e)\geq\varphi(e^l)-2\varphi(e)$ for each $l\in\mathbb{N}$.*

[5] We have $\pi_{\mathrm{r}}(e)=\pi_{\mathrm{l}}(e)=2\deg(f)+1$ by Proposition 4.3(1) and (2). Hence for $\deg(f)=q$ we get that $\varphi(e)=q$ and $\pi_{\mathrm{r}}(e)=\pi_{\mathrm{l}}(e)=2q+1$ while Theorem 2.4(3) (resp. Theorem 2.4(2)) provides for $q\in p\mathbb{N}^*-1$ (resp. $q\in\mathbb{N}$) endomorphisms $e\in\mathrm{EE}_2(k)$ with $\pi_{\mathrm{r}}(e)=\pi_{\mathrm{l}}(e)=2q+2$ (resp. $\pi_{\mathrm{r}}(e)=\pi_{\mathrm{l}}(e)=3(p-1)q+2p-3$) and $\varphi(e)=q$.

(9) *Assume $\iota(e)=\infty$. Then $\varsigma_{e^s}=1$ for each $s\in\mathbb{N}^*$.*

(10) *We have $\iota(e)=\infty$ iff the subset $\mathcal{E}^1(e)$ of Γ_e^+ is infinite.*

(11) *We have $\mathcal{E}^1(e)=\{P\in\Gamma_e^+||e^{-1}(P)|=1|\}\setminus\cap_{l\in\mathbb{N}}\mathrm{Im}(e^l)$.*

(12) *Assume e is not surjective, $\iota(e)\in\mathbb{N}$ and $\varsigma_e=1$. Then $\varphi(e^l)\le 2\varphi(e)+\varrho_e-1$ for each $l\in\mathbb{N}$, $\xi_e+1\le\iota(e)\le\varphi(e)+\varrho_e\le\varphi(e)(\xi_e+1)$.*

Proof. As e maps $\mathrm{Im}(e^n)$ onto $\mathrm{Im}(e^{n+1})$ for $n\in\mathbb{N}$, it also maps $\mathrm{Im}(e^l)\setminus\mathrm{Im}(e^{l+1})$ onto $\mathrm{Im}(e^{l+1})\setminus\mathrm{Im}(e^{l+2})$. From this part (1) follows. As $\varphi(e^0)=0$, we have

$$\varphi(e^l)=\sum_{i=1}^{l}\big(\varphi(e^i)-\varphi(e^{i-1})\big)\le\sum_{i=1}^{l}\big(\varphi(e)-\varphi(e^0)\big)=l\varphi(e),$$

where the inequality follows from part (1). Part (3) is clear.

To prove part (4), let $Q\in\mathrm{Im}(e^l)\setminus\mathrm{Im}(e^{l+1})$. Let $P\in\mathcal{B}$ be such that $Q=e^l(P)$. If $P\notin\Gamma_e$, then there exists $P'\in\mathcal{B}$ such that $P=e(P')$, hence $Q=e^{l+1}(P')\in\mathrm{Im}(e^{l+1})$, a contradiction. Thus $P\in\Gamma_e$, hence $Q\in e^l(\Gamma_e)$, and part (4) holds. Part (5) is a standard application of part (4).

To prove part (6), we can assume that $\Gamma_e\neq\emptyset$; hence $\Gamma_e\neq\Gamma_e^+$. For $n\in\mathbb{N}^*$ we have inclusions $e^n(\Gamma_e)\subset e(\Gamma_e^+)\subset\Gamma_e^+\setminus\Gamma_e$. Let $m\in[[1,|\Gamma_e^+\setminus\Gamma_e|]]$ be the smallest integer such that $e^{m-1}(\Gamma_e^+\setminus\Gamma_e)=e^{m-1+l}(\Gamma_e^+\setminus\Gamma_e)$ for all $l\in\mathbb{N}$. As for $s\in\mathbb{N}$ we have $e^{m+s}(\Gamma_e)\subset e^{m-1}(\Gamma_e^+\setminus\Gamma_e)$, we get that $\Gamma_{e^{m+l}}\cap e^{m+s}(\Gamma_e)=\emptyset$. Thus, by taking (l,n) in part (5) to be $(m,m+l-1)$ with $l\in\mathbb{N}^*$, it follows that we have an inclusion $\Gamma_{e^{m+l}}\subset\Gamma_{e^m}$. As $\mathrm{Im}(e^{m+l})\subset\mathrm{Im}(e^m)$ holds in general, we incur that $\mathrm{Im}(e^{m+l})=\mathrm{Im}(e^m)$ for all $l\in\mathbb{N}^*$ and hence for all $l\in\mathbb{N}$, so $\iota(e)\le m\le|\Gamma_e^+\setminus\Gamma_e|$.

For part (7) we note that we have a disjoint union decomposition $\sqcup_{P\in\mathcal{E}_l(e)}e^{-1}(P)$ of a subset of Γ_{e^l} into sets that have at least ς_e elements, so $\varsigma_e|\mathcal{E}_l(e)|\le\varphi(e^l)$. Thus the first part of part (7) follows from part (3). By induction on $l\in\mathbb{N}$ we get that $\varphi(e^{l+1})\le\varphi(e)(\sum_{i=0}^{l}\varsigma_e^{-i})$, from which the second part of part (7) follows.

For part (8), let $\mathcal{E}_l^{\ge 2}(e):=\mathcal{E}_l(e)\setminus\mathcal{E}_l^1(e)$. We have relations

$$\varphi(e^{l+1})-\varphi(e)=|\mathcal{E}_l^1(e)|+|\mathcal{E}_l^{\ge 2}(e)|\le|\mathcal{E}_l^1(e)|+2|\mathcal{E}_l^{\ge 2}(e)|\le\varphi(e^l)$$

by part (3) and the mentioned disjoint union decomposition. Therefore we have $|\mathcal{E}_l^{\ge 2}(e)|\le\varphi(e)+\varphi(e^l)-\varphi(e^{l+1})$ and thus

$$|\mathcal{E}_l^1(e)|=\varphi(e^{l+1})-\varphi(e)-|\mathcal{E}_l^{\ge 2}(e)|\ge 2\varphi(e^{l+1})-\varphi(e^l)-2\varphi(e).$$

If $\iota(e)=\infty$, then $\iota(e^s)=\infty$, and hence part (9) follows from part (7).

As we have inequalities $\varphi(e^l)-2\varphi(e)\le|\mathcal{E}_l^1(e)|\le\varphi(e^{l+1})-\varphi(e)$ by parts (3) and (8) and an inclusion $\mathcal{E}_l^1(e)\subset\mathcal{E}_{l+1}^1(e)$ for each $l\in\mathbb{N}$, $\mathcal{E}^1(e)$ is an infinite set iff the non-decreasing sequence $\varphi(e^l)_{l\in\mathbb{N}}$ diverges to ∞ and hence iff $\iota(e)=\infty$.

The '$\subset$' of part (11) follows from definitions. For $Q\in\{P\in\Gamma_e^+||e^{-1}(P)|=1|\}\setminus\cap_{l\in\mathbb{N}}\mathrm{Im}(e^l)$, let $l\in\mathbb{N}$ be the smallest such that $Q\notin\mathrm{Im}(e^{l+1})$. Thus $Q\in\mathcal{E}_l^1(e)$. As $|e^{-1}(Q)|=1$, we get that $Q\in\mathcal{E}_l^1(e)\subset\mathcal{E}^1(e)$ and the '$\supset$' of part (11) holds.

Taking l in part (8) to be $\iota(e)-1$, we have $\varphi\big(e^{\iota(e)}\big)-\varphi\big(e^{\iota(e)-1}\big)\ge 1$ and hence $\varphi\big(e^{\iota(e)}\big)\le 2\varphi(e)+|\mathcal{E}_{\iota(e)-1}^1(e)|-1\le 2\varphi(e)+\varrho_e-1$. As for each $l\in\mathbb{N}$ we have $\varphi(e^l)\le\varphi\big(e^{\iota(e)}\big)$, the first part of part (12) holds. As $\varphi(e)<\varphi(e^2)<\cdots<\varphi\big(e^{\iota(e)}\big)$, we have $\iota(e)-1\le\varphi\big(e^{\iota(e)}\big)-\varphi(e)\le\varphi(e)+|\mathcal{E}_{\iota(e)-1}^1(e)|-1$ and therefore we have $\iota(e)\le\varphi(e)+|\mathcal{E}_{\iota(e)-1}^1(e)|\le\varphi(e)+\varrho_e$. To prove that $\xi_e+1\le\iota(e)$ we can assume that $\xi_e>0$ and hence the set $\mathcal{E}_{\xi_e}^1(e)\setminus\mathcal{E}_{\xi_e-1}^1(e)$ makes sense and it is non-empty, which implies that the inclusion $\mathrm{Im}(e^{\xi_e+1}) \subset \mathrm{Im}(e^{\xi_e})$ is strict. Thus $\xi_e+1 \le \iota(e)$. By parts (2) and (3), we compute $\varrho_e = |\mathcal{E}^1_{\xi_e}(e)| \le |\mathcal{E}_{\xi_e}(e)| = \varphi(e^{\xi_e+1}) - \varphi(e) \le \xi_e\varphi(e)$. $\square$

4. Polynomial preliminaries

In this section we study affine coordinates in relation to algebraic degrees and (left, right, or double) cosets of e.

Notation 4.1. If $e \in \mathrm{End}_n(K)$ is defined by $(g_1, \ldots, g_n) \in K_{\mathrm{s}}[x_1, \ldots, x_n]^n$, let

$$e^{\#} : K_{\mathrm{t}}[x_1, \ldots, x_n] \to K_{\mathrm{s}}[x_1, \ldots, x_n], \ \ \mathrm{Frac}(e^{\#}) : K_{\mathrm{t}}(x_1, \ldots, x_n) \to K_{\mathrm{s}}(x_1, \ldots, x_n)$$

be the K-algebra endomorphisms such that we have $e^{\#}(x_i) = g_i$ for each $i \in [[1, n]]$; so $e = \mathrm{Spec}\,(e^{\#})$. The elements of the set

$$\mathcal{A}_n(K) := \{\{y_1, \ldots, y_n\} \subset K[x_1, \ldots, x_n] | K[y_1, \ldots, y_n] = K[x_1, \ldots, x_n]\}$$

will be called affine coordinates of $K[x_1, \ldots, x_n]$.

The following lemma computes (left or right) algebraic degrees.

Lemma 4.2. *Let $e \in \mathrm{End}_n(K)$ be defined by $(g_1, \ldots, g_n) \in K_{\mathrm{s}}[x_1, \ldots, x_n]^n$. Then $\pi(e)$ is the smallest possible value one gets for*

$$\max\{\deg\big(h_i(g_1(y_1, \ldots, y_n), \ldots, g_n(y_1, \ldots, y_n))\big) | i \in [[1, n]]\}$$

when $(\{h_1, \ldots, h_n\}, \{y_1, \ldots, y_n\})$ varies through all elements of $\mathcal{A}_n(K)^2$. Similarly,

$$\pi_{\mathrm{l}}(e) = \min\big\{\max\{\deg\big(h_i(g_1(\underline{x}), \ldots, g_n(\underline{x}))\big) | i \in [[1, n]]\} | (h_1, \ldots, h_n) \in \mathcal{A}_n(K)\big\},$$

where $\underline{x} := x_1, \ldots, x_n$, and

$$\pi_{\mathrm{r}}(e) = \min\big\{\max\{\deg\big(g_1(y_1, \ldots, y_n), \ldots, g_n(y_1, \ldots, y_n)\big)\} | (y_1, \ldots, y_n) \in \mathcal{A}_n(K)\big\}.$$

Proof. Let $a, b \in \mathrm{GA}_n(K)$. For $i \in [[1, n]]$ let $h_i := a^{\#}(x_i)$ and $y_i := b^{\#}(x_i)$. The rule $(a, b) \mapsto (\{h_1, \ldots, h_n\}, \{y_1, \ldots, y_n\})$ defines a bijection $\mathrm{GA}_n(K)^2 \to \mathcal{A}_n(K)^2$ and we compute

$$(aeb)^{\#}(x_i) = (b^{\#} \circ e^{\#} \circ a^{\#})(x_i) = h_i\big(g_1(y_1, \cdots, y_n), \ldots, g_n(y_1, \cdots, y_n)\big).$$

Thus the first part of the lemma follows from the definition of $\pi(e)$. The second part of the lemma for $\pi_{\mathrm{l}}(e)$ (resp. $\pi_{\mathrm{r}}(e)$) follows from its definition and the first part of the lemma applied only with b (resp. a) being $1_{\mathbb{A}^2_K}$, i.e., with $y_i = x_i$ (resp. $h_i = x_i$) for each $i \in [[1, n]]$. $\square$

Proposition 4.3. *Let $e \in \mathrm{End}_n(K)$ be defined by an n-tuple $(g_1, \ldots, g_n)$ in $K_{\mathrm{s}}[x_1, \ldots, x_n]^n$ such that for each $i \in [[1, n]]$ we can decompose $g_i = \sum_{j=1}^{d_i} \mu_{i,j}$ as a sum of non-zero monomials with $d_i \in \mathbb{N}^*$. We consider two conditions.*

(i) *If $i \in [[1, n]]$ and $j \in [[1, d_i]]$, then $\mu_{i,j}$ divides μ_{i,d_i} and $\deg(\mu_{i,j}) < \deg(\mu_{i,d_i})$.*

(ii) *The primitive monomials proportional to the μ_{i,d_i}s ($i \in [[1, n]]$) are multiplicatively independent.*

Let $\Pi := \max\{\deg(\mu_{i,d_i}) | i \in [[1, n]]\}$. Then the following properties hold.

(1) *If condition (i) holds, then $\pi_{\mathrm{r}}(e) = \Pi$.*

(2) *If conditions (i) and (ii) hold, then $\pi_{\mathrm{l}}(e) = \Pi$.*

(3) *For $i \in [[1, n]]$ let $s_i \in \mathbb{N}$ be the greatest such that $x_i^{s_i} | \mu_{i,d_i}$ and let $\epsilon_i \in \{0, 1\}$ be such that $\epsilon_i = 0$ iff $\mu_{i,d_i} \in K^* x_i^{s_i}$. If condition (i) holds, then we have inequalities $\min\{\deg(\mu_{i,d_i}) | i \in [[1, n]]\} \le \pi_{\mathrm{i}}(e) \le \Pi$ and $\max\{s_1 + \epsilon_1, \ldots, s_n + \epsilon_n\} \le \pi_{\mathrm{i}}(e)$.*

(4) *Assume $n = 2$, condition (i) holds, and (to fix the ideas) $\Pi = \deg(\mu_{2,d_2})$. If either $\mu_{2,d_2} \in K^* x_2^{\deg(\mu_{2,d_2})}$ or $x_1 x_2 | \mu_{1,d_1}$ and $\deg(\mu_{1,d_1}) = \Pi - 1$, then $\pi_{\mathrm{i}}(e) = \Pi$.*

Proof. The '$\leq \Pi$' is clear for parts (1) to (3). To prove the '$\geq$' for part (1), based on Lemma 4.2 it suffices to show that for each $\{y_1, \ldots, y_n\} \in \mathcal{A}_n(K)$ and every $i \in [[1, n]]$ we have $\deg\big(g_i(y_1, \ldots, y_n)\big) \geq \deg(\mu_{i,d_i})$. As condition (i) holds, we have an identity $\deg\big(g_i(y_1, \ldots, y_n)\big) = \deg\big(\mu_{i,d_i}(y_1, \ldots, y_n)\big)$. As $\deg(y_j) \geq 1$ for each $j \in [[1, n]]$ and μ_{i,d_i} is a monomial, $\deg\big(\mu_{i,d_i}(y_1, \ldots, y_n)\big) \geq \deg(\mu_{i,d_i})$, and hence $\deg\big(g_i(y_1, \ldots, y_n)\big) \geq \deg(\mu_{i,d_i})$.

To prove the '$\geq$' for part (2), based on Lemma 4.2 it suffices to show that for each $\{h_1, \ldots, h_n\} \in \mathcal{A}_n(K)$ and every $i \in [[1, n]]$, there exists $j \in [[1, n]]$ such that for $L_j = L_j(x_1, \ldots, x_n) := h_j\big(g_1(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots, x_n)\big)$, we have an inequality $\deg(L_j) \geq \deg(\mu_{i,d_i})$. We choose j such that $\frac{\partial h_j}{\partial x_i}(0) \neq 0$. Let

$$L_j^{\mathrm{top}} = L_j^{\mathrm{top}}(x_1, \ldots, x_n) := h_j\big(\mu_{1,d_1}(x_1, \ldots, x_n), \ldots, \mu_{n,d_n}(x_1, \ldots, x_n)\big).$$

Condition (i) implies that each monomial of $L_j - L_j^{\mathrm{top}}$ divides a suitable monomial of L_j^{top} in such a way that the quotient has positive degree, and condition (ii) implies that $\deg(L_j^{\mathrm{top}})$ is the maximum of all degrees

$$\deg\big(h_{j,s}(\mu_{1,d_1}(x_1, \ldots, x_n), \ldots, \mu_{n,d_n}(x_1, \ldots, x_n))\big),$$

where $h_{j,s}$s are the non-zero monomials of h_j. Thus $\deg(L_j - L_j^{\mathrm{top}}) < \deg(L_j^{\mathrm{top}})$, hence $\deg(L_j) = \deg(L_j^{\mathrm{top}})$. As $\frac{\partial h_j}{\partial x_i}(0) \neq 0$, one of the $h_{j,s}$s is in $K^* x_i$ and hence $\deg(L_j^{\mathrm{top}}) \geq \deg(\mu_{i,d_i})$. Thus $\deg\big(L_j(x_1, \ldots, x_n)\big) \geq \deg(\mu_{i,d_i})$.

For parts (3) and (4), let $a \in \mathrm{GA}_2(K)$ and $d := aea^{-1} \in \mathrm{End}_2(K)$. For $i \in [[1, n]]$ let $y_i := (a^{-1})^{\#}(x_i)$. As $d^{\#}(y_i) = [(a^{-1})^{\#} \circ e^{\#} \circ a^{\#}](y_i) = g_i(y_1, \ldots, y_n)$, from condition (i) it follows that

$$\deg\big(d^{\#}(y_i)\big) = \deg\big(\mu_{i,d_i}(y_i, \ldots, y_n)\big). \tag{3}$$

Let $q \in [[1, n]]$ be such that $\deg(y_q) = \min\{\deg(y_i) | i \in [[1, n]]\}$. For $i \in [[1, n]]$ let $f_i := d^{\#}(x_i)$. Let $l := \pi(f_1, \ldots, f_n)$. As $\deg\big(\mu_{q,d_q}(y_1, \ldots, y_n)\big) \geq \deg(\mu_{q,d_q}) \deg(y_q)$ and $\deg\big(d^{\#}(y_q)\big) \leq l \deg(y_q)$, from Equation (3) applied to $i = q$ it follows that $l \deg(y_q) \geq \deg(\mu_{q,d_q}) \deg(y_q)$, hence $l \geq \deg(\mu_{q,d_q}) \geq \min\{\deg(\mu_{i,d_i}) | i \in [[1, n]]\}$. Thus $\min\{\deg(\mu_{i,d_i}) | i \in [[1, n]]\} \leq \pi_{\mathrm{i}}(e)$. For $i \in [[1, n]]$, as $\deg\big(d^{\#}(y_i)\big) \leq l \deg(y_i)$ and $s_i \deg(y_i) + \epsilon_i \leq \deg\big(\mu_{i,d_i}(y_i, \ldots, y_n)\big)$, we have $s_i + \epsilon_i \leq l$ by Equation (3). So part (3) holds. For part (4) it suffices to show that $l \geq \deg(\mu_{2,d_2})$. We can assume that $q = 1$ and $\deg(y_1) < \deg(y_2)$. If $\mu_{2,d_2} \in K^* x_2^{\deg(\mu_{2,d_2})}$, then $l \geq s_2 = \deg(\mu_{2,d_2})$. If $x_1 x_2$ divides μ_{1,d_1} and $\deg(\mu_{1,d_1}) = \Pi - 1$, then $\deg\big(\mu_{1,d_1}(y_1, y_2)\big) > \deg(\mu_{1,d_1}) \deg(y_1)$ and thus $l > \deg(\mu_{1,d_1})$. Hence $l \geq \deg(\mu_{1,d_1}) + 1 = \Pi$. □

Example 4.4. Let $(i, j, l) \in (\mathbb{N}^*)^3$ with $l \geq 2$. Let $e \in \mathrm{End}_2(K)$ be defined by the pair $(x_1 + \mu^l, x_2 + \mu)$; if $\mathrm{char}(K) = p$ and $(li, j) \in p\mathbb{N}^*$ (resp. $(i, j) \in p\mathbb{N}^*$), then $e \in \mathrm{EE}_2(K)$ (resp. $e \in \mathrm{BE}_2(K)$). We have $\pi_{\mathrm{r}}(e) = l \deg(\mu) = l(i+j)$ by Proposition 4.3(1). We have $\pi_{\mathrm{l}}(e) \leq (l-1)\deg(\mu) + 1 < \pi_{\mathrm{r}}(e)$ as for $a \in \mathrm{GA}_2(K)$ defined by the pair $(x_1 - x_2^l, x_2)$, ae is defined by the pair $(x_1 - \sum_{i=1}^{l} \binom{l}{i} x_2^i \mu^{l-i}, x_2 + \mu)$. If $\mathrm{char}(K) = p$ and $l = p$, then we have $\pi_{\mathrm{l}}(e) \leq \max\{i+j, p\}$. If $p \nmid l$, then Proposition 4.3(2) applied to ae gives that $\pi_{\mathrm{l}}(e) = \pi_{\mathrm{l}}(ae) = (l-1)\deg(\mu) + 1 < \pi_{\mathrm{r}}(e)$. The last two sentences imply that Proposition 4.3(2) does not hold without its condition (ii).

We have $\max\{i+j, il+1\} \leq \pi_{\mathrm{i}}(e)$ by the first and third inequalities of Proposition 4.3(3). If $\mathrm{char}(K)=p$, $(i,j)\in(p\mathbb{N}^*)^2$ and $p\nmid l$, then $e\in \mathrm{BE}_2(K)$ and $p\nmid \pi_{\mathrm{l}}(e)$.

Assume $l=2$. The endomorphisms ae, a^{-1}, and aea^{-1} are defined by pairs $(x_1-x_2^2-2x_2\mu, x_2+\mu)$, $(x_1+x_2^2, x_2)$, and $\big(x_1-x_2^2-2x_2(x_1+x_2^2)^i x_2^j, x_2+(x_1+x_2^2)^i x_2^j)\big)$ (respectively). Therefore $\pi_{\mathrm{i}}(e)\leq 2i+j$ if $\mathrm{char}(K)=2$ and $\pi_{\mathrm{i}}(e)\leq 2i+j+1$ if $\mathrm{char}(K)\neq 2$. If $j=1$, then for $\mathrm{char}(K)\neq 2$ we get the following relations

$$\pi_{\mathrm{l}}(e)=\deg(\mu)+1=i+2\leq 2i+1\leq \pi_{\mathrm{i}}(e)\leq 2i+2=\pi_{\mathrm{r}}(e)$$

and for $\mathrm{char}(K)=2$ we get the following relations

$$\pi_{\mathrm{l}}(e)\leq \deg(\mu)=i+1<2i+1=\pi_{\mathrm{i}}(e)<2i+2=\pi_{\mathrm{r}}(e).$$

Similarly, if $2\leq j<i$, then for $\mathrm{char}(K)\neq 2$ we get the following relations

$$\pi_{\mathrm{l}}(e)=i+j+1<2i+1\leq \pi_{\mathrm{i}}(e)\leq 2i+j+1<2i+2j=\pi_{\mathrm{r}}(e)$$

and for $\mathrm{char}(K)=2$ we get the following relations

$$\pi_{\mathrm{l}}(e)\leq i+j<2i+1\leq \pi_{\mathrm{i}}(e)\leq 2i+j<2i+2j=\pi_{\mathrm{r}}(e).$$

5. Formal power series preliminaries

We present regularity criteria for local complete normal spectra that are finite flat over $\mathrm{Spec}\,(k[[x,y]])$ and that are obtained by taken normalizations.

Lemma 5.1. *Let $q\in\mathbb{N}^*$, $R:=k[[x,y]]$ and $\alpha\in R^*$. For $m\in\mathbb{N}^*$ we consider the R-algebra $R_m:=k[[x^{\frac{1}{m}},y]]$ (so $R_1=R$). For $i\in [[0,q-1]]$ let $(l_i,m_i,u_i)\in \mathbb{N}^2\times(\{0\}\cup R^*)$ with $u_0\in R^*$. Let*

$$h(z):=z^{pq}+\alpha z^{pq-1}+\sum_{i=0}^{q-1}u_ix^{l_i}y^{m_i}z^{pi}\in R[z]\quad\text{and}\quad A:=R[z]/\big(h(z)\big).$$

Then the following properties hold for A and its normalization A^{n}.

(1) *The R-algebra A is generically étale.*

(2) *If $p=2$ and $q=1$, then the R-algebra A is étale (so $A=A^{\mathrm{n}}\cong R^2$).*

(3) *If $pq>2$, then the non-étale locus of* $\mathrm{Spec}\,A\to\mathrm{Spec}\,R$ *is given by $x^{l_0}y^{m_0}=0$.*

(4) *Assume that $pq-1$ divides m_0 and that for each $i\in [[0,q-1]]$ with $u_i\neq 0$ the inequalities $m_i\geq \frac{pq-1-pi}{pq-1}m_0$ and $l_i\geq\frac{pq-1-pi}{pq-1}l_0$ hold. Then the R-algebra A^{n} is regular. If moreover $pq-1$ and l_0 are relatively prime and $l_i>\frac{pq-1-pi}{pq-1}l_0$ for each $i\in [[0,q-1]]$ with $u_i\neq 0$, then the R-algebra A^{n} is isomorphic to $R\times R_{pq-1}$.*

(5) *If $(l_0,m_0)\in\{(0,1),(1,0)\}$, then A is regular.*

(6) *If $(l_0,m_0)=(1,1)$, then $A=A^{\mathrm{n}}$ and the singular locus of* $\mathrm{Spec}\,A$ *is only one point defined by $x=y=z=0$.*

Proof. Parts (1) to (3) follow directly from the fact that $\frac{\partial h}{\partial z}=-\alpha z^{pq-2}$.

By substituting $z:=\alpha z'$ in $\alpha^{-pq}h(z)$ we can assume that $\alpha=1$.

For part (4), let d_0 be the greatest common divisor of $pq-1$ and l_0. Writing $m_0=s_0(pq-1)$ and $l_0=d_0r_0$ with $s_0,r_0\in\mathbb{N}$ and r_0 relatively prime to $n_0:=\frac{pq-1}{d_0}$, for $t:=u_0^{\frac{1}{pq-1}}x^{\frac{r_0}{n_0}}y^{s_0}\in R_{n_0}$ we have $t^{pq-1}=u_0x^{l_0}y^{m_0}$. Substituting $w:=tz^{-1}$ in

$tz^{-pq}h(z)$ we get that the normalization $(A\otimes_R R_{n_0})^{\mathrm{n}}$ of the R_{n_0}-algebra $A\otimes_R R_{n_0}$ is isomorphic as an R_{n_0}-algebra to the finite étale R_{n_0}-algebra $R_{n_0}[t]/\big(g(w)\big)$ with

$$g(w):=w^{pq}+\Big(\sum_{i=1}^{q-1}u_ix^{l_i}y^{m_i}t^{pi-pq+1}w^{p(q-i)}\Big)+w+t\in R_{n_0}[w].$$

This makes sense as $u_ix^{l_i}y^{m_i}t^{pi-pq+1}\in R_{n_0}$ for $i\in[[0,q-i]]$; more precisely, we have $u_ix^{l_i}y^{m_i}t^{pi-pq+1}\in\{0\}\cup R^*x^{r_i}y^{s_i}$, where $r_i:=l_i+\frac{pil_0}{pq-1}-l_0\in\frac{1}{pq-1}\mathbb{N}$ and $s_i:=m_i+pis_0-m_0\in\mathbb{N}$ if $u_i\neq 0$ by hypotheses. Therefore $(A\otimes_R R_{n_0})^{\mathrm{n}}\cong R_{n_0}^{pq}$. From Galois theory we get that the only intermediate fields between $\mathrm{Frac}(R)$ and $\mathrm{Frac}(R_{n_0})$ are $\mathrm{Frac}(R_n)$ with n a divisor of n_0. Hence each local direct factor of A^{n} is isomorphic to such an R_n and therefore A^{n} is regular. We are left to prove that if $d_0=1$, so $n_0=pq-1$, and $r_i>0$ if $u_i\neq 0$, then $A^{\mathrm{n}}\cong R\times R_{pq-1}$. Based on part (2) we can assume $pq>2$ and it suffices to show that there exist R-algebra homomorphisms $\sigma_1:A\to R$ and $\sigma_2:A\to R_{pq-1}$ which extend to R-algebra homomorphisms $\sigma_1^{\mathrm{n}}:A^{\mathrm{n}}\to R$ and $\sigma_2^{\mathrm{n}}:A^{\mathrm{n}}\to R_{pq-1}$ that are surjective.

There exists a unique R-algebra retraction $\sigma_1:A\to R$ that maps $z+\big(h(z)\big)$ to $-1+(x,y)$ by Hensel's Lemma; as σ_1 is surjective, so is σ_1^{n}. Hensel's Lemma also implies that there exists $v\in R_{pq-1}^*$ such that $tv^{pq}+v^{pq-1}+[\sum_{i=1}^{q-1}u_ix^{r_i}y^{s_i}v^{pi}]+1=0$ (recall $r_i>0$ if $u_i\neq 0$). For $z_0:=tv\in R_{pq-1}$ we have $z_0^{pq}+z_0^{pq-1}+\sum_{i=0}^{q-1}u_ix^{l_i}y^{m_i}z_0^{pi}=0$, hence there exists an R-algebra homomorphism $\sigma_2:A\to R_{pq-1}$ that maps $z+\big(h(z)\big)$ to z_0. As l_0 is relatively prime to $pq-1$, for a divisor n of $n_0=pq-1$ we have $z_0\in\mathrm{Frac}(R_n)$ iff $n=pq-1$. Thus, as $z_0\in\mathrm{Im}(\sigma_2)$, $\sigma_2^{\mathrm{n}}\otimes 1_{\mathrm{Frac}(R)}$ is surjective and this implies that σ_2^{n} is surjective.

Part (5) follows from Nagata Jacobian Criterion (see [Gro2], Prop. 22.7.2) as h and its partial derivatives generate $R[z]$. For part (6), h and its partial derivatives generate the ideal (x,y,z) of $R[z]$, so $\{(0,0,0)\}$ is the singular locus of $\mathrm{Spec}\,A$ by loc. cit. Due to this and the fact that A is a free module of finite rank over the regular local ring R of dimension 2, properties (R_1) and (S_2) hold for A; so A is normal by Serre's Criterion (see [Gro3], Thm. (5.8.6) or [Ma1], Ch. 8, Thm. 23.8). □

Lemma 5.2. *Let $S:=k[[x]]$ and $R:=S[[y]]=k[[x,y]]$. For $q\in\mathbb{N}^*$ and $f_1,\ldots,f_q\in S$ with $f_q\neq 0$, let $h(z):=z+\sum_{i=1}^q f_i(x)z^{pi}\in S[z]$. Assume that either q is a power of p and $h(z)$ is additive (i.e., and $f_i=0$ if $i\in[[1,q-1]]$ is not a power of p) or $q=2$. Then the normalization A of R in the étale $k((x,y))$-algebra $k((x,y))[z]/\big(y+h(z)\big)$ is regular.*

Proof. If $f_q(x)\in S^*$, then $A\cong R[z]/\big(y+h(z)\big)$ is a finite étale R-algebra isomorphic to R^{pq}. So we can assume that $f_q(0)=0$.

Assume h is additive. Let $\prod_{i=1}^s S_i$ be the product decomposition into complete discrete valuation rings of the normalization of S in $\mathrm{Frac}(S)[z]/\big(h(z)\big)$. For each $i\in[[1,s]]$ let $z_i\in\mathrm{Frac}(S_i)$ be the image of $z+\big(h(z)\big)$ in $\mathrm{Frac}(S_i)$; it generates the separable field extension $\mathrm{Frac}(S)\to\mathrm{Frac}(S_i)$ and we have $h(z_i)=0$. Let $z_0\in R$ be the only solution of the equation $y+h(z_0)=0$ with the property that $z_0\in -y+Ry^p$; it exists by Hensel's Lemma. As h is additive, $y+h(z_0+z_i)=y+h(z_0)+h(z_i)=0$ for each $i\in[[1,s]]$. Hence for each $i\in[[1,s]]$ there exists an R-algebra epimorphism $\sigma_i:A\to S_i[[y]]$ under which at the level of rings of fractions $z+\big(y+h(z)\big)$ is mapped to z_0+z_i; it is surjective as z_i generates $\mathrm{Frac}(S_i)$. The resulting R-algebra homomorphism $(\sigma_1,\ldots,\sigma_s) : A \to \prod_{i=1}^s S_i[[y]]$ is an isomorphism as it is so after tensoring with $\mathrm{Frac}(S)$ over S (one can see this by working modulo y).

Assume $q=2$. As $h(z)$ is additive if $p=2$, we can assume that $p\ge 3$. This case is similar to the previous one except that under σ_i, $z+\bigl(y+h(z)\bigr)$ maps to $z_0+z_i+t_i$ for some $t_i\in S_i[[y]]y^p$. We only show the existence of t_i. As we have $y+z_0+f_1(x)z_0^p+f_2(x)z_0^{2p}=0=z_i+f_1(x)z_i^p+f_2(x)z_i^{2p}$, we get

$$y+z_0+z_i+t_i+f_1(x)(z_0+z_i+t_i)^p+f_2(x)(z_0+z_i+t_i)^{2p}$$
$$=2f_2(x)z_0^pz_i^p+t_i+[f_1(x)+2f_2(x)z_0^p+2f_2(x)z_i^p]t_i^p+f_2(x)t_i^{2p}.$$

As $z_0^p\in Ry^p$, the existence of t_i follows from Hensel's Lemma once we check that $f_2(x)z_i^p\in S_i$. This is clear if $z_i=0$, hence we can assume that the index $i\in[[1,s]]$ is such that $1+f_1(x)z_i^{p-1}+f_2(x)z_i^{2p-1}=0$. Thus $-1=z_i^{p-1}[f_1(x)+f_2(x)z_i^p]$, hence $z_i^{-p+1}\in S_i$ and $f_2(x)z_i^p=-z_i^{-p+1}-f_1(x)\in S_i$. □

Lemma 5.3. *Let $R:=k[[x,y]]$. For $q\in\mathbb{N}^*$, $f_0,\ldots,f_{q-1}\in R$, and $u\in R^*$, let $h(z):=z+(\sum_{i=0}^{q-1}f_i(x,y)z^{pi})+uxz^{pq}\in R[z]$. Then the normalization A of R in the étale $k((x,y))$-algebra $k((x,y))[z]/\bigl(h(z)\bigr)$ is regular.*

Proof. If $f_0(0,0)\in k^*$, then using the substitution $w:=z^{-1}$ in $f_0^{-1}z^{-pq}h(z)$, it follows from Lemma 5.1(5) that A is regular. So we can assume that $f_0(0,0)=0$. Hensel's Lemma implies that there exists a unique $z_0\in -f_0+Rx^p+Ry^p$ such that $h(z_0)=0$. Using the substitution $u:=z_0+z$, we can assume that $f_0=0$. It suffices to show that the normalization of R in the étale $k((x,y))$-algebra $k((x,y))[z]/\bigl(g(z)\bigr)$ is regular, where $g(z):=1+(\sum_{i=1}^{q-1}f_i(x,y)z^{pi-1})+uxz^{pq-1}$. Using the substitution $w:=z^{-1}$ in $z^{-pq+1}g(z)$, it suffices to show that for $f(w):=w^{pq-1}+\bigl(\sum_{i=1}^{q-1}f_i(x,y)w^{p(q-i)}\bigr)+ux\in R[w]$, the R-algebra $R[w]/\bigl(f(w)\bigr)$ is regular, which is so as f and its partial derivatives f_x and f_w generate $R[w]$. □

6. Preliminaries on endomorphisms of affine spaces

In this section we include properties of several classes of endomorphisms of affine spaces over K which are either of theoretical nature or pertain to new invariants. In particular, for $n\ge 2$ the study of basic endomorphisms $e\in\mathrm{BE}_n(k)$ relates closely to the study of endomorphism of the submonoid $\text{p-Mor}_n(k)\subset\mathrm{EE}_n(k)$ of p-morphisms which for $n=2$ were introduced in [Lang-J].

We recall the argument for the fact that $\mathrm{Im}(\varphi_{2,K})\subset\mathbb{N}$ in the following general form, some part of which is only a variation of [Raz], Lem. 1.

Lemma 6.1. *Let $d\in\mathrm{End}_n(K)$. Then $d\in\mathrm{QF}_n(K)$ iff d is flat. Moreover, if $d\in\mathrm{QF}_n(K)$, then the following properties hold.*

(1) *The endomorphism d is universally open.*

(2) *The complement Γ_d is closed and either empty or of codimension at least 2 (hence it is empty if $n=1$ and it is finite if $n=2$).*

(3) *Viewing $d^\#$ as an inclusion, we have*

$$K_{\mathrm{t}}[x_1,\ldots,x_n]=K_{\mathrm{s}}[x_1,\ldots,x_n]\cap K_{\mathrm{t}}(x_1,\ldots,x_n).$$

(4) *We have $d\in\mathrm{GA}_n(K)$ iff d is birational (i.e., $\deg(d)=1$ and, if $\mathrm{char}(K)=p$, d is generically étale).*

Proof. We have $d \in \mathrm{QF}_n(K)$ iff d is flat by [Ma1], Ch. 5, Thms. 15.1 and 23.1. Part (1) is a particular case of [Gro4], Cor. (14.4.3); so $\mathrm{Im}(d)$ is a non-empty open in $\mathbb{A}^n_{K,\mathrm{t}}$.

We show that the assumption that the reduced closed subvariety Γ_d has an irreducible component of dimension $n-1$ leads to a contradiction. We incur the existence of a prime element $h \in K_{\mathrm{t}}[x_1,\ldots,x_n]$ such that the zero locus $h=0$ does not intersect $\mathrm{Im}(d)$. This implies that $d^{\#}(h) \in K_{\mathrm{s}}[x_1,\ldots,x_n]^* = K^*$, a contradiction with h being a prime element. Thus part (2) holds.

For part (3), the '$\subset$' part is clear and to show the '$\supset$' part let O_{t} be a local ring of $\mathbb{A}^n_{k,\mathrm{t}}$ which is a discrete valuation ring. From part (2) we get that there exists a local ring O_{s} of $\mathbb{A}^n_{k,\mathrm{s}}$ which dominates O_{t}; so O_{s} is a discrete valuation ring. We get that $K_{\mathrm{s}}[x_1,\ldots,x_n] \cap K_{\mathrm{t}}(x_1,\ldots,x_n)$ is contained in $O_{\mathrm{s}} \cap K_{\mathrm{t}}(x_1,\ldots,x_n) = O_{\mathrm{t}}$. Thus, as $K_{\mathrm{t}}[x_1,\ldots,x_n]$ is the intersection of all such local rings O_{t} of $\mathbb{A}^n_{k,\mathrm{t}}$ (see [Ma1], Ch. 4, Thm. 11.5), we get that the '$\supset$' part also holds. Thus part (3) holds.

For part (4), the 'only if' part is clear and the 'if' part follows from part (3) as d birational implies $K_{\mathrm{s}}[x_1,\ldots,x_n] \cap K_{\mathrm{t}}(x_1,\ldots,x_n) = K_{\mathrm{s}}[x_1,\ldots,x_n]$. $\square$

Remark 6.2. Lemma 6.1(2) implies that the rule $(d,e) \mapsto de$ defines a composite map $[\mathrm{D}_2(K) \setminus \mathrm{QF}_2] \times \mathrm{QF}_2(K) \to \mathrm{D}_2(K) \setminus \mathrm{QF}_2$. However, if $n \geq 3$, then there exists $(d,e) \in [\mathrm{D}_n(K) \setminus \mathrm{QF}_n(K)] \times \mathrm{QF}_n(K)$ such that $de \in \mathrm{QF}_n(K)$. To exemplify this we can assume that $n=3$. Let d be defined by the rule $(x,y,z) \mapsto (xy, xz+y, z)$, so $d^{-1}(0,0,0) = \mathbb{A}^1_K \times_{\mathrm{Spec}\, K} \{(0,0)\}$ is its only infinite fiber. Let e be such that its image is contained in $\mathbb{A}^3_{K,\mathrm{t}} \setminus [\mathbb{A}^1_K \times_{\mathrm{Spec}\, K} \{(0,0)\}]$, say e is $1_{\mathbb{A}^1_K}$ times the translation by $(0,1)$ of an endomorphism in $QF_2(K)$ as in Example 2.11 for $f(x) = x+1$ and $q=1$; concretely, e is defined by the rule $(x,y,z) \mapsto (x, y^2z+y+z, yz+1)$.

Let $\mathrm{Der}_n(K)$ denote the free $K_{\mathrm{s}}[x_1,\ldots,x_n]$-module of K-linear derivations of $K_{\mathrm{s}}[x_1,\ldots,x_n]$. Let $\{y_1,\ldots,y_n\} \in \mathcal{A}_n(K)$. For each $i \in [[1,n]]$ let $D_{y_i} \in \mathrm{Der}_n(K)$ be such that for every $j \in [[1,n]]$, $D_{y_i}(y_j)$ is 1 if $j=i$ and is 0 otherwise. Then $\{D_{y_1},\ldots,D_{y_n}\}$ is a basis of the $K_{\mathrm{s}}[x_1,\ldots,x_n]$-module $\mathrm{Der}_n(K)$. Moreover, the commutator bracket $[,]$ makes $\mathrm{Der}_n(K)$ a Lie algebra over $K_{\mathrm{s}}[x_1^p,\ldots,x_n^p]$ and we have $[D_{y_i}, D_{y_j}] = 0$ for all $i,j \in [[1,n]]$.

Lemma 6.3. *Let $e,d \in \mathrm{D}_n(K)$ be defined by n-tuples $(g_1,\ldots,g_n)$ and $(h_1,\ldots,h_n)$ in $K_{\mathrm{s}}[x_1,\ldots,x_n]^n$ (respectively). Then the following properties hold.*

(1) *We have $\mathrm{GA}_n(K)e = \mathrm{GA}_n(K)d$ iff $K_{\mathrm{s}}[g_1,\ldots,g_n] = K_{\mathrm{s}}[h_1,\ldots,h_n]$.*

(2) *We have $\mathrm{GA}_n(K)e\mathrm{GA}_n(K) = \mathrm{GA}_n(K)d\mathrm{GA}_n(K)$ iff there exits an identification $K_{\mathrm{s}}[x_1,\ldots,x_n] = K_{\mathrm{s}}[y_1,\ldots,y_n]$ that restricts to (i.e., there exist affine coordinates $y_1,\ldots,y_n$ of $K[x_1,\ldots,x_n]$ that induce) an identification*

$$K_{\mathrm{s}}[g_i(x_1,\ldots,x_n) | i \in [[1,n]]] = K_{\mathrm{s}}[h_i(y_1,\ldots,y_n) | i \in [[1,n]]]. \tag{4}$$

(3) *Assume that $e,d \in \mathrm{EE}_n(K)$ and $\deg(e) = \deg(d)$. Then part (2) holds with the equality in Equation (4) replaced by an inclusion.*

Proof. For $a \in \mathrm{GA}_n(K)$ we have $ae = d$ iff $d^{\#} = e^{\#} \circ a^{\#}$. Thus, such an a exists iff $\mathrm{Im}(e^{\#}) = \mathrm{Im}(d^{\#})$ and part (1) follows.

For $a,b \in \mathrm{GA}_n(K)$ we have $aeb = d$ iff $bae = bdb^{-1}$. As $(bdb^{-1})^{\#} = (b^{-1})^{\#} \circ d^{\#} \circ b^{\#}$, for $i \in [[1,n]]$ we define $y_i := (b^{\#})^{-1}(x_i) = (b^{-1})^{\#}(x_i)$. Thus $(y_1,\ldots,y_n) \in \mathcal{A}_n(K)$ is defined by b^{-1} and determines b^{-1} (hence also b) uniquely. So, as

$$\mathrm{Im}\big((bdb^{-1})^{\#}\big) = K_{\mathrm{s}}[h_1(y_1,\ldots,y_n),\ldots,h_n(y_1,\ldots,y_n)] \subset K_{\mathrm{s}}[y_1,\ldots,y_n],$$

part (2) follows from the last paragraph applied to $(a, e, d) = (ba, e, bdb^{-1})$.

For part (3) it suffices to show that if $K_{\mathrm{s}}[g_1(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots, x_n)]$ is included in $K_{\mathrm{s}}[h_1(y_1, \ldots, y_n), \ldots, h_n(y_1, \ldots, y_n)]$, then Equation (4) holds. Due to this inclusion we have a factorization $e = cd$, where the endomorphism $c : \mathbb{A}^n_{K,\mathrm{t}} \to \mathbb{A}^n_{K,\mathrm{t}}$ is birational (as $\deg(e) = \deg(d)$). As d and e are étale, the étale locus of c contains $\mathrm{Im}(d)$ and thus (see Lemma 6.1(2)) the non-étale locus of c is either empty or its codimension in $\mathbb{A}^n_{K,\mathrm{t}}$ is at least 2. Thus, being the zero locus of the Jacobian determinant, the non-étale locus of c is empty, i.e., c is étale; from Lemma 6.1(4) we get that c is an isomorphism. Thus Equation (4) holds. □

Lemma 6.4. *The following properties hold for $e \in \mathrm{D}_n(k)$.*

(1) *The k-vector subspace $\Re_e := \mathrm{Im}(e^{\#}) + k_{\mathrm{s}}[x_1^p, \ldots, x_n^p]$ of $k_{\mathrm{s}}[x_1, \ldots, x_n]$ is a left invariant and a $k_{\mathrm{t}}[x_1^p, \ldots, x_n^p]$-module.*

(2) *If $e \in \mathrm{BE}_n(k)$, then there exists $\{y_1, \ldots, y_n\} \in \mathcal{A}_n(k)$ such that $\{y_1, \ldots, y_n\} \subset \Re_e$.*

(3) *If there exists $\{y_1, \ldots, y_n\} \in \mathcal{A}_n(k)$ such that $\{y_1, \ldots, y_n\} \subset \Re_e$, then $e \in \mathrm{EE}_n(k)$ and there exists $d \in \mathrm{End}_n(k)$ such that $de \in \mathrm{BE}_n(k)$.*

Proof. Part (1) follows from definitions. To prove part (2), we can replace e by aeb with $(a, b) \in \mathrm{GA}_n(k)$, and hence we can assume that $e = \mathrm{e}(x_1 + f_1^p, \ldots, x_n + f_n^p)$ with $(f_1, \ldots, f_n) \in (k_{\mathrm{s}}[x_1, \ldots, x_n])^n$. As $\{x_1, \ldots, x_n\} \subset \Re_e$, part (2) holds.

To prove part (3), for $i \in [[1, n]]$ let $v_i \in k_{\mathrm{s}}[x_1, \ldots, x_n]$ be such that we write $y_i - v_i^p \in \mathrm{Im}(e^{\#})$. If $e_1 \in \mathrm{BE}_n(k)$ is defined by the n-tuple $(y_1 - v_1^p, \ldots, y_n - v_n^p)$, then we have a factorization $e_1 = de$ with $d^{\#}$ defined by the inclusion $\mathrm{Im}(e_1^{\#}) \subset \mathrm{Im}(e^{\#})$. As $de \in \mathrm{EE}_n(k)$ implies $e \in \mathrm{EE}_n(k)$, part (3) holds. □

Remark 6.5. If $f(x_1, \ldots, x_{n-1}) \in k_{\mathrm{s}}[x_1, \ldots, x_{n-1}] \setminus k_{\mathrm{s}}[x_1^p, \ldots, x_{n-1}^p]$, then the n-tuple $\big(x_1, \ldots, x_{n-1}, x_n + f(x_1, \ldots, x_{n-1})x_n^{pm}\big) \in k_{\mathrm{s}}[x_1, \ldots, x_n]^n$ defines $e \in \mathrm{EE}_n(k)$ with $x_n \notin \Re_e$. To show this, by substituting $x_i = x^{q_i}$ for $i \in [[1, n-1]]$ with $q_i \in \{1\} \cup p\mathbb{N}^*$ such that $f(x^{q_1}, \ldots, x^{q_{n-1}}) \in k_{\mathrm{s}}[x] \setminus k_{\mathrm{s}}[x^p]$, we can assume that $n = 2$. For $n = 2$ we show that $(k_{\mathrm{s}}[x_2] \setminus k_{\mathrm{s}}[x_2^p]) \cap \Re_e = \emptyset$. It suffices to show that the assumption that there exists $(g, h, F) \in (k_{\mathrm{s}}[x, y])^2 \times (k_{\mathrm{s}}[x_2] \setminus k_{\mathrm{s}}[x_2^p])$ such that

$$g\big(x_1, x_2 + f(x_1)x_2^{pm}\big) = F(x_2) + h(x_1^p, x_2^p) \tag{5}$$

leads to a contradiction. Applying D_{x_1} to this identity, we get that

$$g_x\big(x_1, x_2 + f(x_1)x_2^{pm}\big) + g_y\big(x_1, x_2 + f(x_1)x_2^{pm}\big)f'(x_1)x_2^{pm} = 0.$$

If $g_y = 0$, then also $g_x = 0$. So $g \in k_{\mathrm{s}}[x^p, y^p]$ and the left-hand side of Equation (5) is in $k_{\mathrm{s}}[x_1^p, x_2^p]$ and cannot be equal to $F(x_2) + h(x_1^p, x_2^p)$. So $g_y \neq 0$ and

$$f'(x_1)x_2^{pm} = \frac{-g_x\big(x_1, x_2 + f(x_1)x_2^{pm}\big)}{g_y\big(x_1, x_2 + f(x_1)x_2^{pm}\big)}$$

belongs to the intersection of the field of fractions of $k_{\mathrm{s}}[x_1, x_2 + f(x_1)x_2^{pm}]$ with $k_{\mathrm{s}}[x_1, x_2]$ and hence (see Lemma 6.1 (3)) it belongs to $k_{\mathrm{s}}[x_1, x_2 + f(x_1)x_2^{pm}]$. The relation $f(x_1) \in k_{\mathrm{s}}[x_1] \setminus k_{\mathrm{s}}[x_1^p]$ implies that $0 \leq \deg\big(f'(x_1)\big) < \deg\big(f(x_1)\big)$, and it is easy to see that $f'(x_1)x_2^{pm} \notin k_{\mathrm{s}}[x_1, x_2 + f(x_1)x_2^{pm}]$, contradiction.

The following notions were either explicitly or in essence introduced in [Lang-J], Defs. 2.2 and 2.15 for $n = 2$.

Definition 6.6. *Assume $n \geq 2$. For $f \in k[x_1, \ldots, x_n]$, let $e_f \in \mathrm{EE}_n(k)$ be defined by the rule $(x_1, \ldots, x_n) \mapsto \big(x_1, \ldots, x_n + f(x_1, \ldots, x_{n-1}, x_n^p)\big)$. Let $\text{p-Mor}_n(k)$ be the submonoid of $\mathrm{EE}_n(k)$ generated by $\mathrm{GA}_n(k) \cup \{e_f \in \mathrm{EE}_n(k) | f \in k[x_1, \ldots, x_n]\}$.*

(1) *The elements of $\text{p-Mor}_n(k)$ are called p-morphisms and the elements in $\mathrm{GA}_n(k)\{e_f \in \mathrm{EE}_n(k)|f \in k[x_1, \ldots, x_n]\}\mathrm{GA}_n(k)$ are called elementary p-morphisms.*

(2) *Assume $f \notin k[x_1^p, \ldots, x_n^p]$. We decompose $f = \sum_{i=0}^{\deg(f)} f_i$, where each f_i is either 0 or homogeneous of degree i. By the modulo p-th powers degree of f, to be denoted as $\deg_p(f)$, we mean $m \in [[1, \deg(f)]]$ such that $f_m \notin k[x_1^p, \ldots, x_n^p]$ but $f_i \in k[x_1^p, \ldots, x_n^p]$ for each $i \in [[m+1, \deg(e)]]$.*

(3) *Assume $f \notin k[x_1^p, \ldots, x_n^p]$. By a standard writing modulo p-th powers of F, shortly a standard writing of f, we mean a decomposition $f = g^p + \sum_{i=1}^{\deg_p(f)} h_i$ with $g \in k[x_1, \ldots, x_n]$ and each $h_i \in k[x_1, \ldots, x_n]$ either 0 or homogeneous of degree i. A standard writing $f = g^p + \sum_{i=1}^{\deg_p(f)} h_i$ of f is called reduced if either $p \nmid \deg_p(p)$ or $p \mid \deg_p(f)$ and each primitive monomial in $[x_1^p, \ldots, x_n^p]$ of degree $\deg_p(f)$ has coefficient 0 in $h_{\deg_p(f)}$.*

(4) *Let $r \in \mathbb{N}^*$. Assume $f \notin k[x_1^p, \ldots, x_n^p]$. We say that f has r points at infinity modulo p, and we write $\infty_p(f) = r$, if there exists one (equivalently, if in each) reduced standard writing $f = g^p + \sum_{i=1}^{\deg_p(f)} h_i$ of f such that $h_{\deg_p(f)}$ is a product $h^p \prod_{i=1}^r \ell_i^{s_i}$ with $h \in k[x_1, \ldots, x_n] \setminus \{0\}$, $(s_1, \ldots, s_r) \in [[1, p-1]]^r$, and $\ell_1, \ldots, \ell_s$ non-associated forms (if $n = 2$, they are linear forms).*

(5) *Let $e \in \mathrm{EE}_n(k)$. We say that e has 1 point at infinity modulo p if it is defined by an n-tuple $(g_1, \ldots, g_n) \in k_{\mathrm{s}}[x_1, \ldots, x_n]^n$ with $\infty_p(g_1) = \cdots = \infty_p(g_n) = 1$.*

Remark 6.7. Let $f \in k[x_1, \ldots, x_n] \setminus k[x_1^p, \ldots, x_n^p]$.

(1) If $\mathrm{e}(g_1, g_2) \in \mathrm{EE}_2(k)$, then it has 1 point at infinity modulo p iff $1 \in \{\infty_p(g_1), \infty_p(g_2)\}$ by [Lang-J], Lem. 2.17.

(2) If $\infty_p(f) = 1$, then $p \nmid \deg_p(f)$.

(3) If $p \nmid \deg_p(f)$ (resp. $p \mid \deg_p(f)$), then in each standard (resp. reduced standard) writing $f = g^p + \sum_{i=1}^{\deg_p(f)} h_i$ of f, the form $h_{\deg_p(f)}$ is uniquely determined.

(4) For each $g \in k[x_1, \ldots, x_n]$, we have $\infty_p(f) = \infty_p(f + g^p)$.

(5) If $n = 2$ and $p \nmid \deg_p(f)$, then $\infty_p(f)$ is as in [Lang-J], Def. 2.15. But if $p \mid \deg_p(f)$, then $\infty_p(f)$ is in general different from the one defined in loc. cit.: the change is made so that part (4) holds. If $p = 2$ and $f := x_1x_2(x_1 + x_2)(x_1 + \alpha x_2)$ with $\alpha \in k \setminus \{0, 1\}$, then $f = x_1^3x_2 + (\alpha+1)x_1^2x_2^2 + \alpha x_1x_2^3$, $\deg(f) = \deg_p(f) = 4$, in the reduced standard writings of f we have $h_4 = x_1^3x_2 + \alpha x_1x_2^3 = x_1x_2(x_1 + \sqrt{\alpha}x_2)^2$, hence $\infty_2(f) = 2$ while loc. cit. gives the value 4.

(6) If $e \in \mathrm{EE}_n(k)$ is a p-morphism which is not of the form adb with $(a, b) \in \mathrm{GA}_n(k)$ and $d \in \mathrm{EE}_n(k)$ an elementary p-morphism, then $p^2 \mid \deg(e)$. For such a composite adb we have $\mathcal{F}(adb) \subset \{1\} \cup p\mathbb{N}^*$.

Lemma 6.8. *Let $f \in k[x, y] \setminus k[x^p, y^p]$. Let $l := \max\{\deg(f_x), \deg(f_y)\} \in \mathbb{N}$ and $m := \infty_p(f)$. For $z \in \{x, y\}$ let $h_z \in k[x, y]$ be the homogeneous component of f_z of degree l. Then $\infty_p(f) = 1$ iff $l = m - 1$ and one of the following two disjoint conditions holds:*

(i) *we have* $h_x h_y \neq 0$ *and* $kh_x = kh_y = k\ell^s h_0^p$ *with* $h_0 \in k[x,y]$, $s \in [[0, p-2]]$, *and* $\ell \in k[x,y]$ *a linear fom;*

(ii) *we have* $h_x h_y = 0$ *and the non-zero factor* h_z *with* $z \in \{x, y\}$ *is of the form* $z^s h_0^p$ *with* $h_0 \in k[x,y]$ *and* $s \in [[0, p-2]]$.

Proof. Assume $\infty_p(f) = 1$. We consider a standard writing $f = g^p + \sum_{i=1}^m h_i$ of f; so each $h_i \in k[x,y]$ is either 0 or homogeneous of degree i and $h_m = \ell^s h^p$ with $h \in k[x,y] \setminus \{0\}$, $s \in [[1, p-1]]$ and $\ell = \alpha x + \beta y$ with $(\alpha, \beta) \in k^2 \setminus \{(0,0)\}$. We have $m = s + p\deg(h)$. If $\alpha \neq 0$ (resp. $\alpha = 0$), then $h_x = s\alpha \ell^{s-1} h^p$ (resp. $\deg(h_x) < l$). Similarly, if $\beta \neq 0$ (resp. $\beta = 0$), then $h_y = s\beta \ell^{s-1} h^p$ (resp. $\deg(h_y) < l$). Thus for $\alpha\beta \neq 0$ (resp. $\alpha\beta = 0$), condition (i) (resp. (ii)) holds and $l = s - 1 + p\deg(h) = m - 1$.

Assume now that $l = m - 1$ and one of the two conditions (i) and (ii) holds. Let $r := \infty_p(f) \in \mathbb{N}^*$. We consider a reduced standard writing $f = g^p + \sum_{i=1}^m h_i$ of f; we have a product decomposition $h_m = h^p \prod_{i=1}^r \ell_i^{s_i}$ with $h \in k[x_1, \ldots, x_n] \setminus \{0\}$, $(s_1, \ldots, s_r) \in [[1, p-1]]^r$, and each ℓ_i a homogeneous polynomial of degree 1. Thus $l = m - 1 = p\deg(h) - 1 + \sum_{i=1}^r s_i$. Let $F := \prod_{i=1}^r \ell_i^{s_i}$. We have $f_x = F_x h^p + L_x$ and $f_y = F_y h^p + L_y$, with $L_z \in k[x,y]$ such that $\deg(L_z) \leq m - 2$.

Assume condition (ii) holds. To fix the ideas we can assume that $z = x$, so $h_x = x^s h_0^p$ and $h_y = 0$. Thus $h_x = x^s h_0^p = F_x h^p$, hence $h | h_0$. Let $g_0 \in k[x,y]$ be such that $h = h_0 g_0$; the polynomials h, h_0 and g_0 are homogeneous polynomials. Thus $F_x = x^s g_0^p$. Hence $F = \frac{x^{s+1} g_0^p}{s+1} + g_1(x^p, y)$ with $g_1 \in k[x,y]$ such that $g_1(x^p, y)$ is a homogeneous polynomial of degree $s + 1 + p\deg(g_0) \leq p - 1 + p\deg(g_0)$. By reasons of degree we get that $g_1 = y^{s+1} g_2$ with $g_2 \in k[x,y]$ a homegeneous polynomial $\deg\bigl(g_2(x^p, y)\bigr) = p\deg(g_0)$. This implies that there exists a homogeneous polynomial $g_3 \in k[x,y]$ such that $g_2(x^p, y) = g_3(x,y)^p$ and $\deg(g_3) = \deg(g_0)$; hence $F = \frac{x^{s+1} g_0^p}{s+1} + y^{s+1} g_3(x,y)^p$. As $h_y = 0$, we get that $\deg(F_y) < s + p\deg(h_0) = l = m - 1$. This implies that $g_3 = 0$. Hence $g_1 = 0$ and $F = \frac{x^{s+1}}{s+1}$. Thus $r = 1$.

Assume condition (i) holds. We write $\ell = \alpha x + \beta y$ with $(\alpha, \beta) \in k^2 \setminus \{(0,0)\}$. To fix the ideas we can assume that $\alpha \neq 0$. By replacing h_0 with $\sqrt[p]{\alpha} h_0$ we can assume that $\alpha = 1$. As in the previous paragraph we argue that $F = \frac{\ell^{s+1} g_0^p}{s+1} + y^{s+1} g_3(x,y)^p$ with $g_3 \in k[x,y]$ homogeneous. As $F_x h^p = kf_x = kf_y = F_y h^p$ we get that $ky^s g_3(x,y)^p \subset k\ell^s h_0^p$, thus $g_3 = 0$, hence $r = 1$. □

Proposition 6.9. *Let* $e \in \mathrm{EE}_2(k)$. *Then the following properties hold.*

(1) *If* e *has* 1 *point at infinity modulo* p, *then for each* $a \in \mathrm{GA}_2(k)$ *the endomorphism* ae *has* 1 *point at infinity modulo* p.

(2) **(Lang)** *If* $e \in$ p-$\mathrm{Mor}_2(k)$, *then* e *has* 1 *point at infinity modulo* p.

(3) *Each* $e \in \mathrm{BE}_2(k)$ *has* 1 *point at infinity modulo* p.

Proof. Let $(f_1, f_2) \in k_{\mathrm{s}}[x_1, x_2]$ be such that $e = \mathrm{e}(f_1, f_2)$; by definition we have $\infty_p(f_1) = \infty_p(f_2) = 1$. Based on the amalgamated structure of $\mathrm{GA}_2(k)$ (see [vdK], Thms. 1 and 2; cf. also [Kr], Sect. 5, Thm. 7), to prove part (1) we can assume that a is defined by a pair $(F_1, F_2) \in k[x_1, x_2]$ such that either $\pi(F_1) = \pi(F_2) = 1$ or $F_1 = x_1$. If $F_1 = x_1$, then ae, being defined by a pair (f_1, f_3) with $f_3 \in k_{\mathrm{s}}[x_1, x_2]$, has 1 point at infinity modulo p. We assume now that $\pi(F_1) = \pi(F_2) = 1$; so $J := J(F_1, F_2)(x_1, x_2) = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{13} & \gamma_{14} \end{bmatrix} \in \mathbf{GL}_2(k)$. So $ae = \mathrm{e}(\gamma_{11} f_1 + \gamma_{12} f_2, \gamma_{21} f_1 + \gamma_{22} f_2)$.

For $i \in \{1,2\}$, let $m_i := \deg_p(f_i)$ and a standard writing $f_i = g_i^p + \sum_{j=1}^{m_i} h_{i,j}$ of f_i with $g_i \in k_{\mathrm{s}}[x_1,x_2]$ and each $h_{ij} \in k_{\mathrm{s}}[x_1,x_2]$ either 0 or homogeneous of degree j. We write $h_{im_i} = H_i^p \ell_i^{s_i}$ with $H_i \in k_{\mathrm{s}}[x_1,x_2] \setminus \{0\}$, $s_i \in [[0,p-2]]$ and $\ell_i = \alpha_i x_1 + \beta_i x_2$ with $(\alpha_i,\beta_i) \in k^2 \setminus \{(0,0)\}$. To show that we have the relation $1 \in \{\infty_p(\gamma_{11}f_1+\gamma_{12}f_2), \infty_p(\gamma_{21}f_1+\gamma_{22}f_2)\}$ we consider two cases as follows.

Case 1: $m_1 \neq m_2$**.** To fix the ideas we can assume that $m_1 > m_2$. This implies that $\gamma_{11}f_1+\gamma_{12}f_2 = \gamma_{11}g_1^p + \gamma_{12}g_2^p + \sum_{j=1}^{m_1-1} h_{a,j} + \gamma_{11}H_1^p\ell_1^{s_1}$ with each $h_{a,j} \in k_{\mathrm{s}}[x_1,x_2]$ homogeneous of degree j. Thus $\infty_p(\gamma_{11}f_1+\gamma_{12}f_2) = 1$.

Case 2: $m_1 = m_2$**.** The case $m_1 = 1$ is trivial, thus we can assume that $m_1 \geq 2$. As $m_i \equiv s_i \pmod p$, we have $s_1 = s_2$ and hence $\deg(H_1) = \deg(H_2)$. As $p \nmid m_1m_2$, from [Lang-J], Prop. 2.16 we get that we can assume that $\ell_1 = \ell_2$. As $J \in \mathbf{GL}_2(k)$, either $\gamma_{11}H_1^p + \gamma_{12}H_p^2$ or $\gamma_{21}H_1^p + \gamma_{22}H_p^2$ is H_a^p with H_a a non-zero homogeneous polynomial of degree $\deg(H_1)$. Thus $1 \in \{\infty_p(\gamma_{11}f_1+\gamma_{12}f_2), \infty_p(\gamma_{21}f_1+\gamma_{22}f_2)\}$ as one can easily check based on Lemma 6.8.

As $1 \in \{\infty_p(\gamma_{11}f_1+\gamma_{12}f_2), \infty_p(\gamma_{21}f_1+\gamma_{22}f_2)\}$ in both cases, part (1) holds by Remark 6.7(1). See [Lang-J], Thm. 2.18 for part (2).

To prove part (3) we write $e = bdc$ with $(b,c) \in \mathrm{GA}_2(k)^2$ and $d \in \mathrm{EE}_2(k)$ defined by a pair $\big(x_1 + g_1(x_1^p,x_2^p), x_2 + g_2(x_1^p,x_2^p)\big)$ with $(g_1,g_2) \in k_{\mathrm{s}}[x_1,x_2]^2$. Based on part (1) we can assume that $b = 1_{\mathbb{A}^2_k}$, thus $e = dc$. Let $(y_1,y_2) \in \mathcal{A}_2(k)$ be such that $c = \mathrm{e}(y_1,y_2)$, then $dc = \mathrm{e}\big(y_1 + f_1(y_1^p,y_2^p), y_2 + f_2(y_1^p,y_2^p)\big)$. As c has 1 point at infinity modulo p by part (2), we have $\infty_p(y_1) = \infty_p(y_2) = 1$ and therefore dc has 1 point at infinity modulo p. Thus part (c) holds. □

7. Preliminaries on complete intersections

For a scheme W, let $\mathcal{O}_W$ be its structure ring sheaf. Let $\mathcal{O}(W) := \mathcal{O}_W(W)$.

For a variety W over K, the partial order on the set

$$\mathrm{Triv}(W) := \{Z \subset W | Z \text{ is affine open regular and } T_Z \text{ is trivial}\}$$

is the inclusion. After recalling two definitions, we introduce four more.

Definition 7.1. *Let $X = \operatorname{Spec} R$ be a non-empty equidimensional affine variety over K.*

(1) *Assume* $\mathrm{char}(K) = p$. *A differential basis of R is a subset $\{f_1,\ldots,f_{\dim(X)}\}$ of R such that $\{df_1,\ldots,df_{\dim(X)}\}$ is an R-basis of the R-module $\Omega_R = \Omega_{R/K}$ of (relative) differentials.*

(2) *A closed embedding $X \to \mathbb{A}^n_K$, i.e., an identification $R = K[x_1,\ldots,x_n]/I$ of K–algebras, is called a complete intersection if the ideal I is generated by $n-\dim(X)$ elements.*

(3) *The complete intersection embedding dimension of X is* $\mathrm{ciedim}(X) \in \mathbb{N}\cup\{\infty\}$ *defined as follows. If there exists a complete intersection closed embedding $X \to \mathbb{A}^N_K$, then* $\mathrm{ciedim}(X)$ *is the smallest possible value for N. If no such N exists, then* $\mathrm{ciedim}(X) := \infty$. *We say that X is a complete intersection iff* $\mathrm{ciedim}(X) \in \mathbb{N}$.

(4) *Assume X is regular. By the weak complete intersection embedding dimension of X we mean* $\mathrm{wciedim}(X) \in \mathbb{N}\cup\{\infty\}$ *defined as follows. If there exists a closed embedding $\imath : X \to \mathbb{A}^N_K$ whose normal bundle $N_\imath$ is trivial, then* $\mathrm{wciedim}(X)$ *is the smallest possible value for N. If no such N exists, then* $\mathrm{wciedim}(X) := \infty$.

(5) *Assume* $\mathrm{ciedim}(X) \in \mathbb{N}$. *By the strong complete intersection embedding dimension of X we mean the smallest* $\mathrm{sciedim}(X) \in \mathbb{N}$ *such that the inequality*

$\mathrm{ciedim}(X) \leq \mathrm{sciedim}(X)$ *holds and each closed embedding* $X \to \mathbb{A}_K^{\mathrm{sciedim}(X)+m}$ *with* $m \in \mathbb{N}$ *is a complete intersection.*

(6) *Assume* X *is normal connected and contains an open subvariety isomorphic to* $\mathbb{A}_K^{\dim(X)}$. *Let* $\rho_{\text{ét}}(X) \in \mathbb{N}$ *be the largest* n *such that there exists* $U \in \mathrm{Triv}(X)$ *and an open subvariety* V *of* U *such that* $V \cong \mathbb{A}_K^{\dim(X)}$ *and* $|\mathrm{Irr}(U \setminus V)| = n$.

If $\mathrm{ciedim}(X) \in \mathbb{N}$, then for each $m \in \mathbb{N}$ there exists a complete intersection closed embedding $X \to \mathbb{A}_K^{\mathrm{ciedim}(X)+m}$, hence $\mathrm{sciedim}(X)$ exists by [Sr], Thm. 2.

We first group together several results in the literature and then apply them in the context of X_e with $e \in \mathrm{EE}_n(K)$.

Theorem 7.2. *Let* X *be a non-empty equidimensional affine regular variety over* K. *Then the following properties hold.*

(1) *We have* $\mathrm{edim}(X) \leq 2\dim(X) + 1$.

(2) *We have* $\mathrm{ciedim}(X) \in \mathbb{N}$ *iff* T_X *is trivial.*

(3) *Assume* T_X *is trivial. Then we have inequalities*

$$\mathrm{edim}(X) \leq \mathrm{wciedim}(X) \leq \mathrm{ciedim}(X) \leq \mathrm{sciedim}(X) \leq 2\dim(X) + 2$$

with $\mathrm{ciedim}(X) - \mathrm{wciedim}(X) \in \{0, 1\}$.

(4) *Assume* T_X *is trivial. If* $\mathrm{edim}(X) = 2\dim(X)$ *with* $\dim(X) \geq 1$, *then* $\mathrm{ciedim}(X) \leq 2\dim(X) + 1$ *if* $\dim(X) \geq 3$ *and* $\mathrm{ciedim}(X) = 2\dim(X)$ *if* $\dim(X) \in \{1, 2\}$. *If* $\mathrm{edim}(X) = 2\dim(X) - l$ *with* $l \in [[1, \dim(X)]]$ *and* $\dim(X) \geq 1 + l$ *and if for* $\dim(X) \geq 3 + l$ *the factorial number* $\big(\dim(X) - 1\big)!$ *is invertible in* K, *then we have inequalities* $\mathrm{ciedim}(X) \leq 2\dim(X) - \min\{1, l - 1\}$ *if* $\dim(X) \geq 3 + l$ *and* $\mathrm{ciedim}(X) = 2\dim(X) - l$ *if* $\dim(X) \in \{1 + l, 2 + l\}$.

(5) *Let* R *be a finitely generated* K*-algebra with equidimensional spectrum. If* $\mathrm{char}(K) = p$, *then* R *has a* p*-basis iff* R *is regular and has a differential basis.*

Proof. Part (1) is well-known (e.g., see [Sr], Cor. 1). Clearly, $\mathrm{edim}(X) \leq \mathrm{ciedim}(X)$. If $\mathrm{ciedim}(X) \in \mathbb{N}$, then $\mathrm{sciedim}(X) \leq 2\dim(X)+2$ by [Kum], Thm.2) (this inequality also follows from [Sr], Thm. 2).

For a closed embedding $\jmath : X \to \mathbb{A}_K^{\mathrm{edim}(X)+\epsilon}$ with $\epsilon \in \mathbb{N}$ we have a normal bundle $N_\jmath$ over X defined by a short exact sequence $0 \to T_X \to \jmath^*(T_{\mathbb{A}_K^{\mathrm{edim}(X)+\epsilon}}) \to N_\jmath \to 0$ of vector bundles over X. If $\jmath$ is a complete intersection, then the dual of $N_\jmath$ (the conormal bundle) is trivial and thus $N_\jmath$ is trivial; thus T_X is stable free and hence trivial by [Su], Thm. 1. Thus the 'only if' of part (2) holds.

To prove the 'if' of part (2) and parts (3) and (4), we can assume that $\dim(X) > 0$, $X \not\cong \mathbb{A}_K^{\dim(X)}$ and T_X is trivial. We write $\mathrm{edim}(X) = 2\dim(X) + 1 - i$ with $i \in [[0, \dim(X)]]$ by part (1). As $N_\jmath$ has rank $\dim(X) + 1 - i + \epsilon \geq 1$ and is stable free by the above short exact sequence, it is trivial if $\epsilon \geq i$ by [Ba1], Thm. 9.3, if $\epsilon = i - 1$ by [Su], Thm. 1, and if $\epsilon = i - 2$ and for $\dim(X) \geq 4$ the factorial $\big(\dim(X) - 1\big)!$ is invertible in K by [FRS], Thm. Thus $\mathrm{wciedim}(X) \leq \mathrm{edim}(X) + i$; to refine this inequality let $\mathrm{n}(X) := \mathrm{wciedim}(X) - \mathrm{edim}(X)$. If $i \in \{0, 1\}$, then $\mathrm{n}(X) = 0$. If $i \geq 2$, then $\mathrm{n}(X) \leq i - 1$; if moreover either $\dim(X) \in \{2, 3\}$ or $\dim(X) \geq 4$ and $\big(\dim(X) - 1\big)!$ is invertible in K, then $\mathrm{n}(X) \leq i - 2$. If $i \geq 2$, then in the second part (4) we have $l = i - 1$.

Let $\imath : X \to \mathbb{A}_K^{\mathrm{edim}(X)+\mathrm{n}(X)}$ be a closed embedding with trivial normal bundle $N_\imath$. Clearly, $\mathrm{ciedim}(X) \geq \mathrm{edim}(X) + \mathrm{n}(X)$. As the dual of $N_\imath$ is also trivial, X is a connected component of a complete intersection $X_+ = X_+(\imath)$ in $\mathbb{A}_K^{\mathrm{edim}(X)+\mathrm{n}(X)}$.

If $\dim(X) - i + \mathrm{n}(X) = 0$, then X is a hypersurface in $\mathbb{A}_K^{\dim(X)+1} = \mathbb{A}_K^{\mathrm{edim}(X)+\mathrm{n}(X)}$ and thus we can assume that $X_+ = X$. If $\dim(X) - i + \mathrm{n}(X) = 1$, then we can assume that $X_+ = X$ by [Kum], Thm.1). Thus for $\dim(X) - i + \mathrm{n}(X) \in \{0, 1\}$ we have $X_+ = X$ and $\mathrm{ciedim}(X) \leq \mathrm{edim}(X) + \mathrm{n}(X)$, hence $\mathrm{ciedim}(X) = \mathrm{edim}(X) + \mathrm{n}(X)$. Similarly, if $\dim(X) - i + \mathrm{n}(X) \geq 2$ and $X_+ = X$, then $\mathrm{ciedim}(X) = \mathrm{edim}(X) + \mathrm{n}(X)$.

Assume now that $\dim(X) - i + \mathrm{n}(X) \geq 2$ and $W = W(\imath) := X_+ \setminus X$ is non-empty for each embedding $\imath$ (i.e., and $\mathrm{ciedim}(X) \geq \mathrm{edim}(X) + \mathrm{n}(X) + 1$). For $(f_1, \ldots, f_{\dim(X)-i+\mathrm{n}(X)+1}, g) \in K[x_1, \ldots, x_{\mathrm{edim}(X)+\mathrm{n}(X)+1}]^{\dim(X)-i+\mathrm{n}(X)+2}$ with $g(W) = \{0\}$, $g(X) = \{1\}$ and the ideal $(f_1, \ldots, f_{\dim(X)-i+\mathrm{n}(X)+1})$ defines X_+,

$$\mathrm{Spec}\,\big(K[x_1, \ldots, x_{\mathrm{edim}(X)+\mathrm{n}(X)+1}]/(f_1, \ldots, f_{\dim(X)-i+\mathrm{n}(X)+1}, 1 - g x_{\mathrm{edim}(X)+\mathrm{n}(X)+1})\big)$$

is isomorphic to X_+, hence $\mathrm{ciedim}(X) = \mathrm{edim}(X) + \mathrm{n}(X) + 1$.

The last two paragraphs and the above relations on $\mathrm{n}(X)$ imply that 'if' of part (2) and parts (3) and (4) hold.

For part (5), if R has a p-basis then R is regular by [Kun], Cor. 2.7 applied to local rings of R. Thus to prove part (5) we can assume that R is regular and, due to the equidimensional hypothesis, an integral domain. Thus part (5) holds by either [KN], Thm. or [Aba], Prop. 3.2 and Cor. 4.2. □

Theorem 7.3. *Let $e \in \mathrm{EE}_n(K)$ with $n \geq 2$. Let $Y = \mathrm{Spec}\, R$ be an open affine subvariety of X_e.*

(1) *Assume $\mathrm{Im}(\imath_e) \subset Y$. Then the following statements are equivalent.*

(i) *We have $Y \subset X_e^{\text{ét}}$.*

(ii) *The generic points of the irreducible components of $Y \setminus \mathrm{Im}(\imath_e)$ belong to $X_e^{\text{ét}}$.*

(iii) *We have $Y = \mathrm{Reg}(Y)$ and T_Y is trivial.*

(iv) *The tangent bundle $T_{\mathrm{Reg}(Y)}$ over $\mathrm{Reg}(Y)$ is trivial.*

(v) *We have $Y = \mathrm{Reg}(Y)$ and $\mathrm{ciedim}(Y) \in \mathbb{N}$.*

Moreover, if $\mathrm{char}(K) = p$, then statements (i) to (v) are also equivalent to either one of the following extra statements.

(vi) *The ring R has a p-basis.*

(vii) *The ring (or the K-algebra) R has a differential basis.*

(2) *The open regular subset $X_e^{\text{ét}}$ of X_e is a maximal element of $\mathrm{Triv}(X_e)$.*

(3) *Assume $n = 2$, ψ_e is non-finite étale, and $\mathrm{Im}(\imath_e) \subsetneq Y$. If $\mathrm{edim}(Y) \in \{3, 4\}$, then $\mathrm{ciedim}(Y) = \mathrm{edim}(Y)$. If $\mathrm{edim}(Y) = 5$, then $\mathrm{ciedim}(Y) \in \{5, 6\}$.*

(4) *We have relations $\rho_{\text{ét}}(e) = \rho_{\text{ét}}(X_e^{\text{ét}}) \leq \rho_{\text{ét}}(X_e)$.*

(5) *If ψ_e is regular but non-étale, then $\mathrm{ciedim}(X_e) = \infty$.*

Proof. To prove part (1), let $Z := \mathrm{Reg}(Y)$. As Y is normal, we have $\dim(Y \setminus Z) \leq \dim(Y) - 2$. Let $\psi_Y : Y \to \mathbb{A}^n_{K,\mathrm{s}}$ and $\psi_Z : Z \to \mathbb{A}^n_{K,\mathrm{s}}$ be defined by ψ_e by restricting the source and target; we have a functorial morphism $T(\psi_Z) : T_Z \to \psi_Z^*(\mathbb{A}^n_{K,\mathrm{s}})$ between bundles over Z whose target is trivial. Let $Y^{\text{ét}} := Y \cap X_e^{\text{ét}}$; it is the étale locus of ψ_Y. See [Stacks1] or [GZ], Thm. 1 for the affineness of $X_e^{\text{ét}}$ and hence also of $Y^{\text{ét}}$. Thus the complement $Y \setminus Y^{\text{ét}}$ is either empty or of pure codimension 1 (see [SGA2], Exp. V, Ex. 3.4). Hence $Y = Y^{\text{ét}}$ iff $\dim(Y \setminus Y^{\text{ét}}) \le \dim(Y) - 2$.

Clearly, $(i) \Rightarrow (ii)$ and $(iii) \Rightarrow (iv)$. Moreover, $(iii) \Leftrightarrow (v)$ by Theorem 7.2(2). If (ii), holds, then $\dim(Y \setminus Y^{\text{ét}}) \le \dim(Y) - 2$ and hence $Y = Y^{\text{ét}}$; thus $Z = Y$, ψ_Y is étale and therefore $T(\psi_Y)$ is an isomorphism. Hence $(ii) \Rightarrow (iii)$. If (iv) holds, then the morphism $T(\psi_Z)$ is between trivial bundles of dimension n and an isomorphism over $\mathrm{Im}(\imath_e)$. Thus its determinant is a global function on Z that maps to $\mathcal{O}\big(\mathrm{Im}(\imath_e)\big) = K^*$ and hence $T(\psi_Z)$ is an isomorphism, which implies that ψ_Z is étale. Hence $Z \subset Y^{\text{ét}}$. Thus $\dim(Y \setminus Y^{\text{ét}}) \le \dim(Y) - 2$ and hence $Y = Y^{\text{ét}}$. Thus $(iv) \Rightarrow (i)$. We conclude that statements (i) to (v) are equivalent.

In this paragraph we assume that $\mathrm{char}(K) = p$. Clearly, $(i) \Rightarrow (vi)$. We have $(vi) \Leftrightarrow (vii)$ by Theorem 7.2(5). To show that $(vi) \Rightarrow (i)$ we consider the K-algebra homomorphism $K_{\mathrm{t}}[x_1, \ldots, x_n] \to R$ that defines ψ_Y and we view it as an inclusion. We also consider the R^p-subalgebra $S := \sum_{(i_1,\ldots,i_n)\in[[0,p-1]]^n} R^p \prod_{j=1}^n x_j^{i_j}$ of R. From the equivalence $(N1) \Leftrightarrow (N3)$ of Section 2 it follows that in fact we have a direct sum decomposition $S = \oplus_{(i_1,\ldots,i_n)\in[[0,p-1]]^n} R^p \prod_{j=1}^n x_j^{i_j}$ of R^p-modules. The assumption that R has a p-basis implies that the R^p-module R is free of rank p^n. The inclusion $S \to R$ is an R^p-linear transformation between free modules of rank p^n; its determinant $u \in R^p$ is uniquely determined up to multiplication by an element in $R^* = K^*$. The mentioned equivalence also gives that $u \in (R^p)^* = K^*$. Thus $S = R$, hence R has a p-basis formed by elements of $K_{\mathrm{t}}[x_1, \ldots, x_n]$. This implies that ψ_Y is unramified and thus, being also flat in codimension at most 2 (as so is ψ_e), it is étale in codimension at most 2, i.e., $\dim(Y \setminus Y^{\text{ét}}) \le \dim(Y) - 2$. Thus $Y = Y^{\text{ét}}$ and hence $(vi) \Rightarrow (i)$. We conclude that part (1) holds.

As $X_e^{\text{ét}}$ is an affine variety, we have $X_e^{\text{ét}} \in \mathrm{Triv}(X_e)$ by part (1) applied to $Y := X_e^{\text{ét}}$. If $W \in \mathrm{Triv}(X_e)$ contains $X_e^{\text{ét}}$, then $W \subset X_e^{\text{ét}}$ by part (1) applied to $Y := W$, hence $W = X_e^{\text{ét}}$. Thus part (2) holds.

Part (3) follows from Theorem 7.2(4) and part (1). Based on part (2), the inequality $\rho_{\text{ét}}(X_e^{\text{ét}}) \le \rho_{\text{ét}}(X_e)$ follows from definitions. For an open subvariety U of $X_e^{\text{ét}}$ with $U \cong \mathbb{A}_K^n$, $Cl(X_e^{\text{ét}}) \cong \mathbb{Z}^{|\mathrm{Irr}(X_e^{\text{ét}} \setminus U)|}$. As $|\mathrm{Irr}(X_e^{\text{ét}} \setminus U)|$ does not depend on U, it is $\rho_{\text{ét}}(X_e^{\text{ét}})$. When $U = \mathrm{Im}(\imath_e)$ we get that $Cl(X_e^{\text{ét}}) \cong \mathbb{Z}^{\rho_{\text{ét}}(e)}$ and $\rho_{\text{ét}}(X_e) = \rho_{\text{ét}}(e)$. So part (4) holds. Part (5) follows from Theorem 7.2(2) and part (2). $\square$

8. Preliminaries on models of $\mathbb{P}^1_{\mathcal{K}}$

Let D be a discrete valuation ring. Let D^{h} be its henselization. Let $\mathcal{K} := \mathrm{Frac}(D)$ and $\mathcal{K}^{\mathrm{h}} := \mathrm{Frac}(D^{\mathrm{h}})$. Let κ be the residue field of D. In this section we study normal projective flat models X of $\mathbb{P}^1_{\mathcal{K}}$ over $\mathrm{Spec}\, D$; so $X_{\mathcal{K}} = \mathbb{P}^1_{\mathcal{K}}$ and X is integral. In practice, D is (the completion) of a local ring $K[x]_{(x-\alpha_0)}$ with $\alpha_0 \in K$ and X is a subscheme of some X_e with $e \in \mathrm{EE}_2(K)$, but for the sake of generality, we work over more general settings that involve contractions of curves and Stein factors.

Such a model X corresponds to finite non-empty sets of divisorial valuations of the field of fractions $\mathcal{K}(t)$ of $\mathbb{P}^1_{\mathcal{K}}$, which are (equivalent classes of) discrete valuations coming from irreducible curves in the special fiber X_κ of X. The terminology 'divisorial valuations' is more used and established and pertains to models X which are known to be of finite type over $\mathrm{Spec}\, D$. Without this restriction, i.e., in the most general case, "Zariski prime divisor" terminology is used in [Ga1], p. 108.

So all divisorial valuation here are Zariski prime divisors of $\mathbb{P}^1_D$, i.e., are discrete valuation rings of the field of fractions $\mathcal{K}(t)$ that dominate D.

Remark 8.1. **(1)** Let A be a henselian local ring of residue field κ. Let Y be a proper algebraic space over $\operatorname{Spec} A$ of relative dimension ≤ 1. The special fiber Y_κ of Y is a proper algebraic space of dimension ≤ 1 over $\operatorname{Spec}\kappa$ and hence it is a proper scheme over $\operatorname{Spec}(\kappa)$ by either [Stacks6] or [Kn], Ch. 5, Thm. 4.9 and Ch. 2, Cor. 6.16. Being proper and of dimension ≤ 1, Y_κ is projective over $\operatorname{Spec}(\kappa)$. If Y is a scheme, then Y is projective by [Gro5], Cor. (21.9.12) and Rm. 21.9.13.[6]

(2) If Z is an algebraic space over $\operatorname{Spec} D$ with $Z_{\mathcal{K}} = \mathbb{P}^1_{\mathcal{K}}$ and special fiber of dimension at most 1 (e.g., if $Z_{\mathcal{K}}$ is Zariski dense in Z) and $Z_{D^{\mathrm{h}}} := Z \times_{\operatorname{Spec} D} \operatorname{Spec} D^{\mathrm{h}}$ is a scheme, then we have an ample line bundle over $Z_{D^{\mathrm{h}}}$ by part (1); as the pullback homomorphism $\operatorname{Pic}(\mathbb{P}^1_{\mathcal{K}}) \to \operatorname{Pic}(\mathbb{P}^1_{\mathcal{K}^{\mathrm{h}}})$ is an isomorphism, it descends to Z. The descended line bundle is still ample and thus Z is projective.[7]

(3) Following [BLR], Ch. 6, Sect. 6.7, assume κ is algebraically closed but not an algebraic closure of a finite field. Let $D := \kappa[t]_{(t)}$. Let $(f,g) \in \kappa[x,y,z]^2$ be a pair of non-zero non-proportional homogeneous polynomials of degree 3. We assume that the zero locus $f = 0$ is smooth and has a κ-valued inflection point P. The zero locus $f + tg = 0$ is a cubic X in $\mathbb{P}^2_D$ and we endow the special fiber X_κ of X with the group law whose identity element is P. Let W be the blow up of X at a κ-valued point of infinite order modulo the subgroup generated by the zero locus $f = g = 0$. The blow down $f : W \to Z$ of the proper transform of X_κ in the category of algebraic spaces (see [Em], Cor. or [Pi], Cor. 4 for the pullback of f to $\operatorname{Spec} D^{\mathrm{h}}$ which is a morphism of projective schemes), gives a normal proper flat algebraic space model Z of the generic fiber of W which is not a scheme.

Recall the following definition introduced in [Kl], Sect. 5, Def. 2.

Definition 8.2. *Let R be a local ring. Let W be a proper scheme over* $\operatorname{Spec} R$ *of relative dimension ≤ 1. By a Stein factor of W we mean a proper morphism $f : W \to V$ of schemes over* $\operatorname{Spec} R$ *such that the homomorphism $\mathcal{O}_V \to f_*(\mathcal{O}_W)$ is an isomorphism. An isomorphism between two Stein factors $f_1 : W \to V_1$ and $f_2 : W \to V_2$ of W is an isomorphism $g : V_1 \to V_2$ over* $\operatorname{Spec} R$ *such that $g \circ f_1 = f_2$.*

Proposition 8.3. *Let R be a local ring of residue field κ. Let W be a proper scheme over* $\operatorname{Spec} R$ *of relative dimension* 1. *Let $\mathcal{B}$ be the set of* 1*-dimensional irreducible components of the special fiber W_κ of W. Then the following properties hold.*

(1) *Each Stein factor $f : W \to V$ of W is uniquely determined up to isomorphism by the subset $\mathcal{C}_f := \{C \in \mathcal{B} | f(C) \text{ is one point}\}$ of $\mathcal{B}$.*

(2) *Assume $R = D$ is a discrete valuation ring and W is flat over* $\operatorname{Spec} D$ *with integral generic fiber.*

(2.a) *For each subset $\mathcal{C}$ of $\mathcal{B}$ distinct from $\mathcal{B}$ there exists a Stein factor $f^{\mathrm{h}}_{\mathcal{C}} : W_{D^{\mathrm{h}}} \to V_{D^{\mathrm{h}}}$ of $W_{D^{\mathrm{h}}}$ such that $\mathcal{C}_f = \mathcal{C}$ and its generic fiber is an isomorphism and thus $f^{\mathrm{h}}_{\mathcal{C}}$ descends to a morphism $f_{\mathcal{C}} : W \to V$ of algebraic spaces over* $\operatorname{Spec} D$.[8]

[6] In general, one can lift an ample line bundle on Y_κ to an ample line bundle on Y as in the proof of the proper base change theorem of étale cohomology (see [SGA4-3], Exp. XVIII, Cor. 3.2), hence Y is projective over $\operatorname{Spec} A$ and in particular it is a scheme.

[7] Based on Footnote 6, the same holds without assuming that $Z_{D^{\mathrm{h}}}$ is a scheme.

[8] The Stein factor $f_{\mathcal{B}} : W \to V$ that contracts W_κ exists but $V = \mathcal{O}(W)$ is finite flat over $\operatorname{Spec} D$, so the generic fiber of $f_{\mathcal{B}}$ is not an isomorphism.

(2.b) *If W is normal, then $V_{D^{\mathrm{h}}}$ and V are normal.*

(2.c) *If $W_{\mathcal{K}} = \mathbb{P}^1_{\mathcal{K}}$, then $f_{\mathcal{C}} : W \to V$ is projective.*

(2.d) *Let $Q \subset Z$ be of pure codimension 1 with non-empty general fiber. The morphism $Z \setminus Q \to \operatorname{Spec} \mathcal{O}(Z \setminus Q)$ is proper and is an isomorphism iff Q meets all irreducible components of the special fiber (hence if Z_κ is irreducible, then $Z \setminus Q \to \operatorname{Spec} \mathcal{O}(Z \setminus Q)$ is an isomorphism).*

Proof. Part (1) is stated in [Sc], Sect. 2 when R is noetherian by just quoting [Gro1], Thm. (5.4.1) but the last reference does not suffice. If R is a discrete valuation ring and W is flat over $\operatorname{Spec} R$, then part (1) holds by [Pi], Prop. 4. Let $f_1 : W \to V_1$ and $f_2 : W \to V_2$ be two Stein factors of W such that $\mathcal{C}_{f_1} = \mathcal{C}_{f_2}$. Let V_3 be the schematic image of the morphism $f_1 \times f_2 : X \to V_1 \times_{\operatorname{Spec} R} V_2$. For $i \in \{1, 2\}$, the i-th projection $\pi_i : V_3 \to V_i$ is finite on special fibers over $\operatorname{Spec} \kappa$, hence it has finite fibers by Chevalley's Semi-continuity Theorem (see [Gro4], Thm. (13.1.3)). Thus π_i is finite; by the Stein property it is an isomorphism. So $g := \pi_2 \circ \pi_1^{-1} : V_1 \to V_2$ is an isomorphism between $f_1 : W \to V_1$ and $f_2 : W \to V_2$. Thus part (1) holds.

Part (2.a) holds by [Em], Cor. or [Pi], Cor. 4. Part (2.b) holds as $V_{D^{\mathrm{h}}}$ is normal by [Pi], Prop. 3. Part (2.c) follows from part (2) and Remark 8.1(2) applied to V over $\operatorname{Spec} D$ and the scheme $V_{D^{\mathrm{h}}}$ over $\operatorname{Spec} D^{\mathrm{h}}$. Part (2.d) follows from the arguments i the proof of citeSc, Thm. 2.2. □

Corollary 8.4. *Let $\operatorname{Model}(\mathbb{P}^1_{\mathcal{K}})$ be the set of isomorphism classes of normal proper (equivalently projective) flat models of $\mathbb{P}^1_{\mathcal{K}}$. Let $\mathcal{P}^0_f\big(\operatorname{Div}(k(t)\big)$ be the set of non-empty finite sets of divisorial valuations of $\mathbb{P}^1_D$ (i.e., of Zariski prime divisors of $\mathbb{P}^1_D$) that dominate D. Then there exists a bijection*

$$\Psi_{\mathcal{K}} : \operatorname{Model}(\mathbb{P}^1_{\mathcal{K}}) \to \mathcal{P}^0_f\big(\operatorname{Div}(k(t)\big)$$

defined by the rule $\Psi_{\mathcal{K}}([X]) = Div_X$, where to a normal proper flat model X of $X_{\mathcal{K}} = \mathbb{P}^1_{\mathcal{K}}$ we associate the set $\mathcal{B}_X$ of 1-dimensional irreducible components of X_κ and the set of divisorial valuations

$$\operatorname{Div}_X := \{\mathcal{O}_{X,\eta_C} | C \in \mathcal{B}_X, \eta_C \text{ is the generic point of C}\}.$$

Proof. Let X' be a normal proper flat model of $X_{\mathcal{K}} = \mathbb{P}^1_{\mathcal{K}}$ with $\operatorname{Div}_{X'} = \operatorname{Div}_X$. Let X^+ be the normalization of the Zariski closure in $X \times_{\operatorname{Spec} D} X'$ of the diagonal embedding of $\mathbb{P}^1_{\mathcal{K}} \to (X \times_{\operatorname{Spec} D} X') \times_{\operatorname{Spec} D} \operatorname{Spec} \mathcal{K} = \mathbb{P}^1_{\mathcal{K}} \times_{\operatorname{Spec} \mathcal{K}} \mathbb{P}^1_{\mathcal{K}}$. The two projections $X^+ \to X$ and $X^+ \to X'$ are Stein factors of X^+. Thus, as $\operatorname{Div}_X = \operatorname{Div}_{X'}$, these two Stein factors of X^+ are isomorphic by Proposition 8.3(1), hence there exists an isomorphism $X \to X'$ that extends $1_{\mathbb{P}^1_{\mathcal{K}}}$. Thus $\Psi_{\mathcal{K}}$ is injective.

Given $\operatorname{Div} \in \mathcal{P}^0_f\big(\operatorname{Div}(k(t)\big)$, let Y be an affine normal flat scheme of finite type over $\operatorname{Spec} D$ such that $Y \times_{\operatorname{Spec} D} \operatorname{Spec} \mathcal{K}$ is an open subvariety of $\mathbb{P}^1_{\mathcal{K}}$ and the set of local rings of Y at generic points of irreducible components of its special fibre Y_κ is Div. Let Y^+ be the quasi-projective variety obtained by gluing Y and $\mathbb{P}^1_{\mathcal{Y}}$ and let X be a projective normal scheme over $\operatorname{Spec} D$ that contains Y^+ as an open Zariski dense subscheme; so X is a normal proper flat model of $\mathbb{P}^1_{\mathcal{K}}$ with $\operatorname{Div} \subset \operatorname{Div}_X$. Let $f : X \to V$ be a Stein factor such that V is a normal projective flat model of $\mathbb{P}^1_K$ with $\operatorname{Div}_V = \operatorname{Div}$ by Proposition 8.3(2.a) to (2.c). Thus $\Psi_{\mathcal{K}}$ is surjective. □

Example 8.5. We take K to be an algebraic closure of $\mathcal{K}$. Let $\widehat{D}$ be the completion of D. Let $\widehat{\mathcal{K}} := \operatorname{Frac}(\widehat{D})$. Let $\overline{\widehat{\mathcal{K}}}$ be an algebraic closure of $\widehat{\mathcal{K}}$. We fix a divisorial valuation O of $\mathcal{K}(t)$. Let $\mathfrak{w} : \mathcal{K}(t) \to \Gamma_{\mathcal{K}(t)} \cup \{\infty\}$ be the discrete valuation that defines it; so $\mathfrak{w}$ is surjective, its valuation ring is O and we have an isomorphism $\Gamma_{\mathcal{K}(t)} \cong \mathbb{Z}$ of totally ordered groups. Let $\overline{\mathfrak{w}} : K(t) \to \Gamma_{K(t)} \cup \{\infty\}$ be a valuation that extends $\mathfrak{w}$; let $O_{\overline{\mathfrak{w}}}$ be its valuation ring. Let $\bar{D} := O_{\overline{\mathfrak{w}}} \cap K$ and let $\overline{\mathfrak{v}} : K \to \Gamma_K \cup \{\infty\}$ and $\mathfrak{v} : \mathcal{K} \to \Gamma_{\mathcal{K}} \cup \{\infty\}$ be induced by $\overline{\mathfrak{w}}$; so D is the valuation ring of $\mathfrak{v}$. We denote also by $\overline{\mathfrak{v}}$ the valuation of the completion $\widehat{K}$ of K with respect to $\overline{\mathfrak{v}}$. We naturally identify $\widehat{\mathcal{K}}$ with a subfield of $\widehat{K}$. We have

$$\Gamma_{\mathcal{K}} \leqslant \Gamma_{\mathcal{K}(t)} \leqslant \Gamma_{K(t)} = \Gamma_K = \Gamma_{\mathcal{K}} \otimes_{\mathbb{Z}} \mathbb{Q} = \Gamma_{\mathcal{K}(t)} \otimes_{\mathbb{Z}} \mathbb{Q}$$

(see [Epp], Thm. (2.0) and [Kuh] for the first equality).

Let k be the residue field of $\bar{D}$; it is an algebraic closure of κ. The residue field $k(\overline{\mathfrak{w}})$ of $O_{\overline{\mathfrak{w}}}$ has transcendental degree 1 over k, i.e., $\overline{\mathfrak{w}}$ is a residual transcendental extension of $\overline{\mathfrak{v}}$ in the sense of [APZ1], Sect. 1; such extensions were classified in [AP], [APZ1] and [APZ2]. In particular, there exists $(\alpha_0, r) \in K \times \overline{\mathfrak{v}}(K^*)$ such that

$$\mathfrak{w}\Big(\sum_{i=0}^{s} \delta_i (x - \alpha_0)^i\Big) = \inf\{\overline{\mathfrak{v}}(\delta_i) + ir | i \in [[0, s]]\} \in \Gamma_{K(t)} \cup \{\infty\}$$

for all $(s, \delta_0, \ldots, \delta_s) \in \mathbb{N} \times K^{s+1}$ (see [AP], Prop. 2), equivalently the norm on $K(t)$ is the sup norm on $\mathcal{B}_K(\alpha_0, r) := \{x \in K | \overline{\mathfrak{v}}(x - \alpha_0) \geq r\}$ or on the (rigid analytic) disk $\mathcal{B}_{\widehat{K}}(\alpha_0, r) := \{x \in \widehat{K} | \overline{\mathfrak{v}}(x - \alpha_0) \geq r\}$.

The divisorial valuations of $\mathcal{K}(t)$ are in bijection to those of $\mathcal{K}^{\mathrm{h}}(t)$, which correspond to $\mathrm{Gal}(K/\mathcal{K}^{\mathrm{h}})$-orbits of the action of $\mathrm{Gal}(K/\mathcal{K}^{\mathrm{h}})$ on the set of such disks (see [APZ2], Thm. 2.2; cf. [AP], Prop. 3).

Let $\beta_0 \in \widehat{\mathcal{K}}$ be of minimal degree l over $\widehat{\mathcal{K}}$ such that $\mathcal{B}_{\widehat{K}}(\beta_0, r) = \mathcal{B}_{\widehat{K}}(\alpha_0, r)$. We approximate β_0 by separable elements (if $\mathrm{char}(\mathcal{K}) = p$, approximate a solution of $X^{p^q} = \gamma$ by a solutions of $X^{p^q} + \epsilon X = \gamma$ with ϵ near 0). Then by Krasner's Lemma we can assume that $\beta_0 \in K$ is separable over $\mathcal{K}$ of degree l; this was first proved in [APZ2], Thm. 3.1. If $\beta \in K \cap \mathcal{B}_{\widehat{K}}(\alpha_0, r)$ is algebraic over $\mathcal{K}$ of degree l, then there exists a unique extension of $\mathfrak{v}$ to $\mathcal{K}(\beta)$, the local degree (i.e., the degree after the completion) is still l, hence the valuation ring of $\mathcal{K}(\beta)$ is a finite D-module.

Theorem 8.6. *Given a divisorial valuation O of $\mathcal{K}(t)$, let $X = X_O$ be the flat normal model of $X_{\mathcal{K}} = \mathbb{P}^1_{\mathcal{K}}$ with the property that $\Psi_{\mathcal{K}}([X]) = \{O\}$ (see Corollary 8.4). Then the following properties hold.*

(1) *If Q is the closure of $(1 : 0) \in \mathbb{P}^1_{\mathcal{K}}(\mathcal{K})$, then $X \setminus Q$ is affine and it is* $\mathrm{Spec}\, \mathcal{K}[t] \cap O$.

(2) *The flat model* $\mathrm{Spec}\, \mathcal{K}[t] \cap O$ *of $\mathbb{A}^1_{\mathcal{K}}$ has at most one singular point.*

(3) *The flat normal model X of $\mathbb{P}^1_{\mathcal{K}}$ has at most two singular points.*

Proof. Part (1) follows from Proposition 8.3(2.d). For parts (2) and (3), we use the notation of Example 8.5, with the finite field extension $\mathcal{K} \to \mathcal{K}(\alpha_0)$ separable of degree l. Let $L \in \mathcal{K}[t]$ be the monic minimal polynomial of α_0 over $\mathcal{K}$. Let $\mathcal{K}' := \mathcal{K}(\alpha_0)$ with valuation ring D' a free D-module of rank l, residue field κ', ramification index $i_{D'}$. We have $\Gamma_{\mathcal{K}} \subset \Gamma_{\mathcal{K}'} \subset \Gamma_{\mathcal{K}'} + \mathbb{Z}r \subset \Gamma_{\mathcal{K}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and inclusions $\mathcal{K}^{\mathrm{h}} \subset \mathcal{K}'' \subset (\mathcal{K}')^{\mathrm{h}}$, where $\mathrm{Gal}\big((\mathcal{K}')^{\mathrm{h}}/\mathcal{K}''\big)$ is the subgroup that fixes $\mathcal{B}_K(\alpha_0, r)$.[9]

[9] The field extension $\mathcal{K}'' \to (\mathcal{K}')^{\mathrm{h}}$ is an extension with trivial tame part, in particular if it is non-trivial then $\mathrm{char}(\kappa) = p$ and it has degree a power of p.

For $\mathcal{B}_{\mathrm{Frac}(\widehat{D'})}(\alpha_0, r)$, where $\widehat{D'}$ is the completion of D', the valuation group is $\Gamma_{\mathcal{K}'} + \mathbb{Z}r$. We have $\overline{\mathfrak{w}}(L) \in \Gamma_{\mathcal{K}'} + \mathbb{Z}r$; let m be the order of $\mathfrak{w}(L)$ mod $\Gamma_{\mathcal{K}'}$. Thus $\overline{\mathfrak{w}}(L^m) = \bar{\mathfrak{v}}(\delta)$ for some $\delta \in (\mathcal{K}')^*$. We have a $\mathcal{K}$-linear natural isomorphism $\oplus_{j=0}^{l-1}\mathcal{K}t^j \cong \mathcal{K}'$. If $0 \neq \gamma \in \mathcal{K}'$ is represented by $g_\gamma(t) \in \oplus_{j=0}^{l-1}\mathcal{K}t^j$, then g_γ has no zeros in the disk so it must be a constant $\gamma_0 \in \mathcal{K}$ plus something with greater value of $\bar{\mathfrak{v}}$. Thus $\overline{\mathfrak{w}}(L^m/g_\delta) = 0$. We have to compare the disk $\mathcal{B}_{\mathcal{K}(t)}(\alpha_0, r)$ and its ring of polynomial functions $\mathcal{K}[t] \cap O$ to $\mathcal{B}_{\widehat{\mathcal{K}'}}(0, \overline{\mathfrak{w}}(L))$. Note that $L^m/g_\delta \in O_{\overline{\mathfrak{w}}}$ defines an element $\zeta \in k(\overline{\mathfrak{w}})$ which is transcendental over κ' (see [APZ1], Thm, 2.1b)).

Let O' be the discrete valuation ring of the 'standard' valuation $\mathfrak{w}'$ of $\mathcal{K}'(t)$ with the properties: (i) the restrictions of $\mathfrak{w}'$ and $\overline{\mathfrak{w}}$ to $\mathcal{K}'$ coincide, and (ii) $\mathfrak{w}'(T) = \overline{\mathfrak{w}}(L)$.

For $h \in \mathcal{K}[t]$ we consider its L-expansion $h = \sum_{j=0}^{n_h} h_j L^j$; so $n_h \in \mathbb{N}$ is the smallest such that $\deg(h_j) < l$ for each $j \in [[0, n_h]]$. By sending each h_j to $h_j(\alpha_0) \in \mathcal{K}'$ and L to T we get a $\mathcal{K}$-linear isomorphism $\mathcal{K}[t] \to \mathcal{K}'[T]$, which preserves the sup norm on the relevant disks. It is not an algebra isomorphism in general but we have equality of products modulo elements with higher sup norm. This gives an isomorphism of associated graded rings

$$\mathrm{gr}(\mathcal{K}[t] \cap O) \to \mathrm{gr}(\mathcal{K}'[T] \cap O'),$$

the filtrations being $\big((K[t]L^j) \cap O\big)_{j\in\mathbb{N}}$ and $\big((K'[T]T^j) \cap O'\big)_{j\in\mathbb{N}}$. This implies that the residue field of O' is $\kappa'(\zeta)$ and that the value group $\Gamma_{K'(t)}$ is $\Gamma_{\mathcal{K}'} + \mathbb{Z}\overline{\mathfrak{w}}(L)$.

We consider two cases as follows.

Case 1: $\overline{\mathfrak{w}} \in \Gamma_{\mathcal{K}'}$ (i.e., $m = 1$). Then $O' \cap \mathcal{K}'[T] \cong D'[T/\delta]$ is regular. From this and the information on graded rings we get that $\mathcal{K}[t] \cap O$ is regular by [Ma1], Ch. 6, Thm. 17.10.

Case 2: $\overline{\mathfrak{w}} \notin \Gamma_{\mathcal{K}'}$ (i.e., $m > 1$). Then we can still verify the regularity after inverting L^m/g_δ. This leaves one possible singular point in the special fiber $\operatorname{Spec} \kappa'[\zeta]$ of $\operatorname{Spec} \mathcal{K}[t] \cap O$ which is the zero locus $\zeta = 0$.

Part (1) follows from the two cases above. Part (2) follows from part (1) and the union $X = Q \cup [\operatorname{Spec} \mathcal{K}[t] \cap O]$. □

Remark 8.7. We refer to the proof of Theorem 8.6. Let $C := (X_\kappa)_{\mathrm{red}}$. Let $(C \cap Q)_{\mathrm{red}}$ define the closed point P of C at infinity.

(1) The special fiber $C \setminus \{P\}$ of $\operatorname{Spec} \mathcal{K}[t] \cap O$ is $\operatorname{Spec} \kappa'[\zeta] \cong \mathbb{A}^1_{\kappa'}$.

(2) In Case 2, the point $\zeta = 0$ of $\operatorname{Spec} \kappa'[\zeta]$ lifts to a D'-valued point of $\operatorname{Spec} \mathcal{K}[t] \cap O$ given by evaluation at α_0, in fact we have a D-algebra surjection $\mathcal{K}[t] \cap O \to D'$ as a uniformizer v' of D' lifts to an element given by a polynomial $h_{v'} \in \mathcal{K}[t]$ of degree $< l$. It follows that the ideal of the reduced special fiber is not principal at this point (as otherwise it would be generated by $h_{v'}$ which does not have the right valuation at the generic point of the special fiber), so the point is singular.

(3) From (1) and Case 1 we get that $\operatorname{Spec} \mathcal{K}[t] \cap O$ is singular iff $m > 1$.

(4) The reduced fiber C of $X = X_O$ is the pinching pushout

$$\begin{array}{ccc} \operatorname{Spec} \kappa' & \longrightarrow & \operatorname{Spec} \kappa \\ \downarrow & & \downarrow \\ \mathbb{P}^1_\kappa & \longrightarrow & C. \end{array}$$

(5) The ideal of C at p is not principal if $i_{D'}m > 1$ and so we have a singularity. It is principal generated by a uniformizer v of D if $i_{D'} = m = 1$, but if $\kappa' \neq \kappa$ we still have a singularity as the zero locus $v = 0$ is singular and v is not in the square of the maximal ideal.

(6) If $\mathcal{K}' = \mathcal{K}$ and $m = 1$ in the proof of Theorem 8.6, then $X = \mathbb{P}^1_D$.

Corollary 8.8. *Let $R := k[[x]]$. Let $R \to S$ be an extension of discrete valuation rings which is generically finite Galois of Galois group G. If the action of G on S extends to an action of G on $S[y]$, then $S[y]^G$ has at most one singular point.*

Proof. We take $D := R$, $\mathcal{K} = k((x))$ and $O := \mathcal{O}_{\operatorname{Spec} S[t]^G,\eta}$, where η is the generic point of $\operatorname{Spec}(S[t]^G/S[t]^G x)_{\mathrm{red}}$. This makes sense as $\operatorname{Spec} S[t]^G[\frac{1}{x}]$, being a form of $\mathbb{A}^1_{k((x))}$, is isomorphic to $\mathbb{A}^1_{k((x))}$ and thus we identify it with $\operatorname{Spec} k((x))[t]$. We have $\kappa = \kappa' = k$ and $\mathcal{K}[t] \cap O = S[t]^G$. So the corollary holds by Theorem 8.6(2). $\square$

9. Preliminaries on rational singularities

For (quasi-)excellent rings and schemes we refer to [Gro3], Subsect. 7.8 and [Travaux], Exp. 1. We recall the following general result

Lemma 9.1. *Let W be a noetherian scheme. The following two properties hold.*

(1) *If W is normal, then it is quasi-excellent iff it is excellent.*

(2) *Let $f : V \to W$ be a finite surjective morphism. Then V is quasi-excellent iff W is quasi-excellent.*

Proof. The 'if' of part (1) is clear. For the 'only if' of part (1) it suffices show that a quasi-excellent normal noetherian ring R is universally catenary. To check this we can assume that R is local; its formal fibers are geometrically regular and thus also geometrically normal and from [Stacks5] we get that R is universally catenary. To prove part (2) we can assume that W is affine, hence V is also affine, and this case was proved in [Gre], Thm. 3.1i). $\square$

For rational singularities see [Lip1], Def. 1.1. Part (2) of the following result in characteristic 0 is a particular case of either [Bout], Thm. or [Kem], Thm. 7.

Proposition 9.2. *Let a finite group G act on a noetherian scheme W over $\operatorname{Spec}\mathbb{Z}[\frac{1}{|G|}]$ in such a way that for each $x \in W$ the orbit $G \cdot x$ is contained in an affine open subscheme of W. Then the following properties hold.*

(1) *The universal geometric quotient $W \to W/G$ in the sense of* [MFK]*, Ch. 0, Defs. 0.6 and 0.7 exists and it is an affine morphism.*

(2) *If $\dim(W) = 2$ and W is normal, excellent and has at most rational singularities, then X/G has at most rational singularities.*

Proof. We can assume that $W = \operatorname{Spec} R$ is affine. Taking $W/G := \operatorname{Spec} R^G$, clearly $W \to W/G$ is surjective and G-invariant. If W is normal, so is W/G. The surjective morphism $W \to W/G$ is finite by [Travaux], Exp. IV, Prop. 2.2.3(ii)(a). The commutative $\mathbb{Z}$-algebra R^G is noetherian due to the existence of the Raynolds operator $R \to R^G$, i.e., of the R^G-linear retraction that maps $x \in R$ to $\frac{1}{|G|}\sum_{g\in G} gx \in R^G$ (e.g., see [Travaux], Exp. IV, Prop. 2.2.3(i)(a)). The Reynolds operator implies that for each R^G-algebra S we have $(R \otimes_{R^G} S)^G = S$. If S is an algebraically closed field, then G acts transitively on $\operatorname{Spec} R \otimes_{R^G} S$ as the restriction function

$\mathrm{Hom}(R,S)/G \to \mathrm{Hom}(R^G,S)$ is a bijection by [Bour], Ch. V, Subsect. 2.1, Cor. 4 after Thm. 1 and Subsect. 2.2, Cor. after Thm. 2. So $W \to W/G$ is a universal geometric quotient and part (1) holds.

For part (2), R^G is excellent by Lemma 9.1(1) and (2). Being also normal, we can strictly henselize by [Lip1], Prop. 16.5 and [Gre], Cor. 5.6, i.e., we can assume that $W = \mathrm{Spec}\, R$ and $W/G = \mathrm{Spec}\, R^G$ are affine normal strictly henselian local of dimension 2 and that G acts trivially on the residue field of R. Let $\Sigma : V \to W/G$ be a resolution of singularities by [Lip2], Thm. We can desingularize $Z := (V \times_{W/G} W)'$ (the $'$ means that we take the schematic closure of the punctured spectrum of R, equivalently the reduced scheme) by normalizations and blowing up of closed points. This is G-equivariant. We consider a G-equivariant desingularization $Z_1 \to W$ that maps to V. For the resulting morphism $\phi : Z_1/G \to V$, the homomorphism of ringed sheaves $\mathcal{O}_V \to \phi_*(\mathcal{O}_{Z_1/G})$ is an isomorphism. Then a Čech computation gives $H^1(Z_1/G, \mathcal{O}_{Z_1/G}) = H^1(Z_1, \mathcal{O}_{Z_1})^G = 0$ and $H^1(V, \mathcal{O}_V) \to H^1(Z_1/G, \mathcal{O}_{Z_1/G})$ is injective by the Leray spectral sequence. Thus $H^1(V, \mathcal{O}_V) = 0$, i.e., R/G has a rational singularity. $\square$

Proposition 9.3. *Let D be a discrete valuation ring. Let R be a D-subalgebra of* $\mathrm{Frac}(D)[t]$ *which is of finite type over D. Then the normalization R^{n} of R has only rational singularities.*

Proof. We can assume that $D \neq R \neq \mathrm{Frac}(D)$; so $\mathrm{Spec}\, R^{\mathrm{n}} \otimes_D \mathrm{Frac}(D) \cong \mathbb{A}^1_{\mathrm{Frac}(D)}$. Hence the R-module R^{n} is finite by [GLL], Prop. 7.6(a). We compactify $\mathrm{Spec}\, R^{\mathrm{n}}$ to a normal integral projective scheme X over $\mathrm{Spec}\, D$ with general fiber $\mathbb{P}^1_{\mathrm{Frac}(D)}$. The rational map $\mathbb{P}^1_D \dashrightarrow X$ becomes regular
$$\begin{tikzcd} W \arrow[rr, "\phi"] \arrow[dr] & & \mathbb{P}^1_D \arrow[dl, dashed] \\ & X & \end{tikzcd}$$
after blowing up finitely many points. As $X \to \mathrm{Spec}\, D$ is cohomologically flat by (the implication $(ii) \Rightarrow (iv)$ in) [Ray], Thm. 8.2.1, we have $H^1(X, \mathcal{O}_X) = 0$. As the blow-ups do not change the cohomology of the structure sheaf, $H^1(W, \mathcal{O}_W) = 0$ and $R^1\phi_*(\mathcal{O}_W)$ is a skyscrapped sheaf supported at finitely many closed points. To show a singular point $P \in X$ is rational we have to show that $\big(R^1\phi_*(\mathcal{O}_W)\big)_P = 0$. As $H^0\big(X, R^1\phi_*(\mathcal{O}_W)\big)$ surjects to this, it suffices to show that $H^0\big(X, R^1\phi_*(\mathcal{O}_W)\big) = 0$. The differential $H^0\big(X, R^1\phi_*(\mathcal{O}_W)\big) \to H^2\big(X, \phi_*(\mathcal{O}_W)\big)$ vanishes by a general cohomological dimension fact on proper schemes in terms of the dimension of fibers. Thus $H^0\big(X, R^1\phi_*(\mathcal{O}_W)\big) = 0$ as $H^1(W, \mathcal{O}_W) = 0$, hence $\big(R^1\phi_*(\mathcal{O}_W)\big)_P = 0$. $\square$

10. Geometric preliminaries

For $n, m \in \mathbb{N}^*$ with $n \geq m$, a smooth morphism $\mathbb{A}^n_R \to \mathbb{A}^m_R$ is called a projection if it is isomorphic to the one given on valued points by the rule $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_m)$. We list the geometric notations for $e \in \mathrm{EE}_n(K)$.

Notation 10.1. For $e \in \mathrm{EE}_n(K)$, let the Jacobian variety $\psi_e : X_e \to X_e$, the open embedding $\imath_e : \mathbb{A}^n_{K,\mathrm{s}} \to X_e$, the finite subset $\mathcal{F}(e)$ of $\mathbb{N}$, and the numerical invariants $\rho(e)$, $\rho_{\text{ét}}(e)$ and $\deg(e)$ be as in Section 2. We write $X_e = \mathrm{Spec}\, A_e$. Let $N_e := \psi_e(\Gamma_{\psi_e})$ be the non-proper locus of e. Let $\nu(e) := |\mathrm{Irr}(N_e)|$. Let $X_e^{\text{ét}}$ be the open subvariety of X_e which is the étale locus of ψ_e. Let $N_e^{\mathrm{n-ét}} := \psi_e(X_e \setminus X_e^{\text{ét}})$. Let $\nu_{\mathrm{n-ét}}(e) := |\mathrm{Irr}(N_e^{\mathrm{n-ét}})| \in \mathbb{N}$.

We have $N_e = \emptyset$ iff $\Gamma_{\psi_e} = \emptyset$ iff $\imath_e$ is an isomorphism (i.e., $X_e = \mathbb{A}^n_{K,s}$) and iff e is finite. If $N_e \neq \emptyset$, then as ψ_e is quasi-finite and $\Gamma_{\psi_e} = X_e \setminus \mathrm{Im}(\imath_e)$ is of pure codimension 1 (see [SGA2], Exp. V, Ex. 3.4), $N_e = \psi_e(\Gamma_{\psi_e})$ is equidimensional of dimension $n-1$. In general, the inequalities $\nu(e) \leq \rho(e) \leq \nu(e)[\deg(e)-1]$ hold and N_e is the smallest (hence reduced) closed subvariety of $\mathbb{A}^n_{K,\mathrm{t}}$ such that e is finite étale over $\mathbb{A}^n_{K,\mathrm{t}} \setminus N_e$. Similarly, $N_e^{\mathrm{n-\acute{e}t}}$ is equidimensional of dimension $n-1$, the inequalities $\nu(e)_{\mathrm{n-\acute{e}t}} \leq \rho(e) - \rho_{\mathrm{\acute{e}t}}(e) \leq \nu_{\mathrm{n-\acute{e}t}}(e)[\deg(e)-1]$ hold and $N_e^{\mathrm{n-\acute{e}t}}$ is the smallest closed subvariety of $\mathbb{A}^n_{K,\mathrm{t}}$ such that ψ_e is finite étale over $\mathbb{A}^n_{K,\mathrm{t}} \setminus N_e^{\mathrm{n-\acute{e}t}}$.

Remark 10.2. Let $e \in \mathrm{QFE}_n(K)$. For $l \in \mathcal{F}(e) \cup \{\deg(e)\}$ we consider the reduced locally closed subvariety $N_l(e)$ of $\mathbb{A}^n_{K,\mathrm{t}}$ defined by

$$N_l(e)(K) := \{P \in \mathbb{A}^n_{K,\mathrm{t}}(K) | |e^{-1}(P)| = l\}.$$

If N_e is the smallest closed subvariety of $\mathbb{A}^n_{K,\mathrm{t}}$ such that e is finite étale over $\mathbb{A}^n_{K,\mathrm{t}} \setminus N_e$, then $N_e = \cup_{l \in \mathcal{F}(e)} N_l(e)$. As $N_0(e)$ is either empty or of codimension at least 2 (see Lemma 6.1(2)) it follows that if e is not finite étale (i.e., if $\mathcal{F}(e) \neq \emptyset$), then $\mathcal{F}(e) \cap \mathbb{N}^*$ is non-empty and hence $\varsigma_e = \min(\mathcal{F}(e) \cap \mathbb{N}^*) \in \mathbb{N}^*$ (see Notation 3.1 for ς_e). The invariant $\upsilon(e) := \max(\mathcal{F}(e) \cap \mathbb{N}^*) \in \mathbb{N}^*$ helps in the study of N_e.

Example 10.3. Let $e \in \mathrm{EE}_n(K)$ be non-finite. For $l \in \mathcal{F}(e)$, as the Zariski closure of $N_l(e)$ in $\mathbb{A}^n_{K,\mathrm{t}}$ (equivalently, in N_e) is contained in $\cup_{m \in \mathcal{F}(e), m \leq l} N_m(e)$ by [Gro5], Prop. (18.2.8), we have $\mathrm{Irr}\big(N_{\upsilon(e)}(e)\big) \subset \mathrm{Irr}(N_e)$, hence $N_{\upsilon(e)}(e)$ is equidimensional of dimension $n-1$. If $\upsilon(e) = \deg(e) - 1$, then ψ_e is étale above the generic points of irreducible components of $N_{\upsilon(e)}(e)$ and thus $\rho_{\mathrm{\acute{e}t}}(e) \geq 1$; if moreover, $\mathcal{F}(e) = \{\deg(e)-1\}$, then X_e is a finite étale cover of $\mathbb{A}^n_{K,\mathrm{t}}$ by the purity of the branch locus (see [SGA2], Exp. X, Thm. 3.4 (i)) and we have $\nu(e) = \rho_{\mathrm{\acute{e}t}}(e) = \rho(e) > 0$.

Example 10.4. For $e \in \mathrm{EE}_n(K)$, let $W_e^+ := \mathbb{A}^n_{K,\mathrm{s}} \times_{e, \mathbb{A}^n_{k,\mathrm{t}}, \psi_e} X_e$. The first projection $f_e^+ : W_e^+ \to \mathbb{A}^n_{K,\mathrm{s}}$ is a finite surjective generically étale morphism between normal schemes which is flat in codimension at most 2 and has a section $1_{\mathbb{A}^n_{K,\mathrm{s}}} \times \imath_e : \mathbb{A}^n_{K,\mathrm{s}} \to W_e^+$. Let $W_e := W_e^+ \setminus \mathrm{Im}(1_{\mathbb{A}^n_{K,\mathrm{s}}} \times \imath_e)$; it is a clopen subscheme of W_e^+ equipped with a finite surjective morphism $f_e : W_e \to \mathbb{A}^n_{K,\mathrm{s}}$ between normal schemes which generically is étale of degree $\deg(e) - 1$ and flat in codimension at most 2. Note that ψ_e is regular (resp. étale) above $\mathrm{Im}(e)$ iff W_e is regular (resp. iff f_e is étale).

Recall the following well-known lemma.

Lemma 10.5. *Let $f : X \to Y$ be a dominant morphism of irreducible separated curves over K. For $W \in \{X, Y\}$, let W^{n} be the normalization of W and let W^{c} be the complete smooth connected curve having W^{n} as an open subvariety. Then the following properties hold.*

- **(1)** *If f is surjective, then $|X^{\mathrm{c}} \setminus X^{\mathrm{n}}| \geq |Y^{\mathrm{c}} \setminus Y^{\mathrm{n}}|$.*
- **(2)** *If X is rational, then Y is rational.*
- **(3)** *If $|X^{\mathrm{c}} \setminus X^{\mathrm{n}}| = |Y^{\mathrm{c}} \setminus Y^{\mathrm{n}}|$, then f is finite surjective.*

Proof. We can assume that X and Y are normal. The morphism f extends to a finite surjective morphism $f^{\mathrm{c}} : X^{\mathrm{c}} \to Y^{\mathrm{c}}$. Hence $f^{\mathrm{c}}(X^{\mathrm{c}} \setminus X)$ has at least $|Y^{\mathrm{c}} \setminus Y|$ elements and part (1) holds. Part (2) follows from Lüroth's Theorem. For part (3), the inequalities $|Y^{\mathrm{c}} \setminus Y| \leq |(f^{\mathrm{c}})^{-1}(Y^{\mathrm{c}} \setminus Y)| \leq |X^{\mathrm{c}} \setminus X|$ must be equalities, so the inclusion $(f^{\mathrm{c}})^{-1}(Y^{\mathrm{c}} \setminus Y) \subset X^{\mathrm{c}} \setminus X$ is a bijection. Thus $(f^{\mathrm{c}})^{-1}(Y) = X$ and the finite surjective morphism $(f^{\mathrm{c}})^{-1}(Y) \to Y$ is f. □

Example 10.6. The hypersurface $X_1 := \mathrm{Spec}\big(k[x,y,z]/(x^{p+1}+xy+z^n)\big)$ in $\mathbb{A}_k^3$ is regular for $n=1$ and normal with $\mathrm{Sing}(X_1)=\{(0,0,0)\}$ if $n\geq 2$. The non-étale locus Y of the finite flat morphism $\phi_1 : X_1 \to \mathbb{A}_k^2$ defined by the projection on the last two coordinates is the zero locus $x^p+y=0$. The induced morphism $Y \to \psi_1(Y)$ is radicial and $\psi_1^{-1}\big(\psi_1(Y)\big)$ has two irreducible components Y and W with $(Y\cap W)_{\mathrm{red}}=\{(0,0,0)\}$. If $n=1$, then $X_1\cong \mathbb{A}_k^2$. If $n\geq 2$, then the rule $(x,y)\mapsto (x,-x^{n-1}y^n-x^p,xy)$ defines a birational morphism $\mathbb{A}_k^2\to X_1$ which is not an open embedding.

If $p=2$ let $l=3$ and if $p\geq 3$ let $l=2$. For the normal hypersurface $X_2 := \mathrm{Spec}\big(k[x,y,z]/(x^{pl}+xy-z^l)\big)$ in $\mathbb{A}_k^3$ we have $\mathrm{Sing}(X)=\{(0,0,0)\}$. The non-étale locus of the finite flat morphism $\phi_2 : X_2\to \mathbb{A}_k^2$ defined by the projection on the last two coordinates is the zero locus $y=0$ and has two irreducible components Y_1 and Y_2 with $(Y_1\cap Y_2)_{\mathrm{red}}=\mathrm{Sing}(X)$ and with Y_1 as the zero locus $x^p-z=0$. The induced morphism $Y_1\to\phi_2(Y_1)$ is radicial and the morphism $\mathbb{A}_k^2\to X_2$ defined by the rule $(x,y)\mapsto(x,x^{l-1}y^l-x^{pl-1},xy)$ is birational but not an open embedding.

Either one of the last two examples implies that the result [Je], Thm. 2.1 over $\mathbb{C}$ does not hold in positive characteristic even when $n=2$.

The hypersurface $X_3 := \mathrm{Spec}\big(k[x,y,z]/(x^p+xy+yz)\big)$ in $\mathbb{A}_k^3$ is normal with $\mathrm{Sing}(X_3)=\{(0,0,0)\}$. The non-étale locus W of the finite flat morphism $\phi_3 : X_3\to\mathbb{A}_k^2$ defined by the projection on the last two coordinates is the zero locus $y=0$, thus the induced morphism $W\to\phi_3(W)$ is an isomorphism.

Example 10.6 implies that the following regularity criterion, which is only a variation of Abhyankar's Lemma, is the maximum one can get using local arguments.

Criterion 10.7. *Assume $n\geq 2$. Let $e\in \mathrm{EE}_n(K)$. For $Y\in\mathrm{Irr}(X_e\setminus X_e^{\text{ét}})$ let m_Y be the multiplicity of Y in $\psi_e^{-1}\big(\psi_e(Y)\big)$. Let*

$$\mathrm{Irr}_{\mathrm{t}}(X_e\backslash X_e^{\text{ét}}) := \{Y\in\mathrm{Irr}(X_e\backslash X_e^{\text{ét}})|\mathrm{char}(K)\nmid m_Y,\ Y\to\psi_e(Y) \text{ is generically étale}\}.$$

Then the following properties hold.

(1) *Assume that $Y\in\mathrm{Irr}_{\mathrm{t}}(X_e\setminus X_e^{\text{ét}})$. Let $Q\in Y(K)$ be such that $\psi_e(Q)\in\mathrm{Reg}\big(\psi_e(Y)\big)$ and $Q\notin\cup_{Z\in\mathrm{Irr}(X_e\backslash X_e^{\text{ét}})\backslash\mathrm{Irr}_{\mathrm{t}}(X_e\backslash X_e^{\text{ét}})}Z(K)$.*[10] *Then $Q\in\mathrm{Reg}(X_e)$.*

(2) *If $\mathrm{Irr}(X_e\setminus X_e^{\text{ét}})=\mathrm{Irr}_{\mathrm{t}}(X_e\setminus X_e^{\text{ét}})$, then*

$$\mathrm{Sing}(X_e)\subset\big(\psi_e^{-1}(\cup_{Y\in\mathrm{Irr}(N_e^{\mathrm{n-\acute{e}t}})}\mathrm{Sing}(Y))\big)\cap(X_e\setminus X_e^{\text{ét}}).$$

(3) *If $\psi_e(Y)\cong\mathbb{A}_K^1$ (so $n=2$) and $Y\in\mathrm{Irr}_{\mathrm{t}}(X_e\setminus X_e^{\text{ét}})$, then $Y\subset\mathrm{Reg}(X_e)$.*

Proof. Let S and R be the completions of the local rings $\mathcal{O}_{X_e,Q}$ and $\mathcal{O}_{\mathbb{A}^n_{K,\mathrm{t}},\psi_e(Q)}$ (respectively); the R-algebra S is integral non-étale and normal. As $\psi_e(Q)\in\mathrm{Reg}\big(\psi_e(Y)\big)$, we can identify $R=K[[x_1,\ldots,x_n]]$ in such a way that $x_1=0$ defines the inverse image of $\psi_e(Y)$ to $\mathrm{Spec}\,R$. The non-étale locus of $\mathrm{Spec}\,(S)\to\mathrm{Spec}\,R$ is $\mathrm{Spec}\,(S/I)$ for some radical ideal I of S that contains x and that (by hypotheses) defines the inverse image of Y to $\mathrm{Spec}\,(S)$. From this and the hypotheses of part (1) we get that $\mathrm{Spec}\,(S/I)$ defines the inverse image of Y to $\mathrm{Spec}\,(S)$ for all $n\geq 2$. Hence $R_x\to A_x$ is étale and the morphism $\mathrm{Spec}\,A\to\mathrm{Spec}\,R$ is tamely ramified in codimension 1. From Abhyankar's Lemma for regular divisors (see [Stacks2]) we get that $S\cong R[t]/(t^{m_Y}-x_1)$, hence S is regular. So $Q\in\mathrm{Reg}(X_e)$ and part (1) holds.

[10] If $n=2$, the second condition always holds by Corollary 11.4(1) below.

Part (2) follows from part (1). If $n = 2$ and $\psi_e(Y) \cong \mathbb{A}^1_K$, then $Y(K) \subset \mathrm{Reg}(X_e)$ by part (1), hence part (3) holds. $\square$

If $\mathrm{char}(K) = 0$ and $e \in \mathrm{EE}_n(K)$, then $\Gamma_e \subset \mathrm{Sing}(N_e)$ by [Je], Cor. 1.2. We have the following version of loc. cit. which holds over all K and which is weaker (cf. Example 10.6) as it uses in essence local arguments.

Proposition 10.8. *Let $e \in \mathrm{EE}_n(K)$. Then the following properties hold.*

- **(1)** *We have $\Gamma_e \subset \mathrm{Sing}(N_e) \cup [\mathrm{Reg}(N_e) \cap \psi_e(X_e \setminus X_e^{\text{ét}})]$.*
- **(2)** *If ψ_e is étale, then $\Gamma_e \subset \mathrm{Sing}(N_e)$.*

Proof. As $(\mathbb{A}^n_{K,\mathrm{t}} \setminus N_e) \subset \mathrm{Im}(e)$, we have $\Gamma_e \subset N_e$. We show that the assumption that there exists $P \in \Gamma_e \cap \mathrm{Reg}(N_e)$ such that $\psi_e^{-1}(P) \subset X_e^{\text{ét}}$ leads to a contradiction. Let $Y \in \mathrm{Irr}(N_e)$ be the unique irreducible component such that $P \in Y$. Let $(l, s) \in (\mathbb{N}^*)^2$ be such that $\mathrm{Irr}\big(\psi_e^{-1}(Y)\big) = \{Y_i | i \in [[1, l]]\} \cup \{Z_j | j \in [[1, s]]\}$ with each Y_i a closed irreducible curve of $\mathbb{A}^n_{K,\mathrm{s}}$ and every $Z_j \in \mathrm{Irr}(\Gamma_{\psi_e})$. For each $i \in [[1, l]]$, the Zariski closure $\overline{Y_i}$ of Y_i in X_e maps onto Y, hence there exist $j_i \in [[1, s]]$ and $P_{j_i} \in [\overline{Y_i} \setminus Y_i] \cap Z_{j_i}$ such $\psi_e(P_{j_i}) = P$. As $P_{j_i} \in \mathrm{Sing}\big(\psi_e^{-1}(Y)\big) \cap X_e^{\text{ét}}$ maps to $P \in \mathrm{Reg}(Y)$, we reached a contradiction; so part (1) holds.[11] Clearly, (1) $\Rightarrow$ (2). $\square$

For $e \in \mathrm{EE}_2(k)$ with $\Gamma_e \subsetneq \mathrm{Sing}(N_e)$ (resp. $\Gamma_e = \mathrm{Sing}(N_e)$) see Example 13.2 with $\deg(f) = 1$ (resp. Example 13.2 with $\deg(f) \geq 2$ and Example 14.1).

11. On sphere-like surfaces

The next lemma is also well-known (e.g., see [Mi2], Sect. 2 for the case $\mathrm{char}(K) = 0$), but due to its importance in the paper we include a self-contained proof of it.

Lemma 11.1. *Let X be a connected affine normal surface over K with an open embedding $\imath : \mathbb{A}^2_K \to X$. For a projection $\pi : \mathbb{A}^2_K \to \mathbb{A}^1_K$, let $\chi(\pi) : \mathbb{A}^2_K \to \mathbb{P}^1_K$ be its composite with the open embedding $\mathbb{A}^1_K \subset \mathbb{P}^1_K$. Then the following properties hold.*

- **(1)** *There exists a unique morphism $\chi(\pi)_X : X \to \mathbb{P}^1_K$ such that $\chi(\pi)_X \circ \imath = \chi(\pi)$.*
- **(2)** *For each $Y \in \mathrm{Irr}\big(X \setminus \mathrm{Im}(\imath)\big)$, the restriction of $\chi(\pi)_X$ to Y is constant.*

Proof. We consider a second projection $\pi^\perp : \mathbb{A}^2_K \to \mathbb{A}^1_K$ such that the morphism $\pi \times \pi^\perp : \mathbb{A}^2_K \to \mathbb{A}^1_K \times_{\mathrm{Spec}\, K} \mathbb{A}^1_K$ is an isomorphism. Let $\phi : \mathbb{A}^2_K \to \mathbb{P}^1_K \times_{\mathrm{Spec}\, K} \mathbb{P}^1_K$ be the composite of $\pi \times \pi^\perp$ with the product open embedding $\mathbb{A}^1_K \times_{\mathrm{Spec}\, K} \mathbb{A}^1_K \subset \mathbb{P}^1_K \times_{\mathrm{Spec}\, K} \mathbb{P}^1_K$. It suffices to show that ϕ extends uniquely to a morphism $\phi_X : X \to \mathbb{P}^1_K \times_{\mathrm{Spec}\, K} \mathbb{P}^1_K$ and that for each $Y \in \mathrm{Irr}\big(X \setminus \mathrm{Im}(\imath)\big)$, the restriction $\phi_X | Y : Y \to \mathbb{P}^1_K \times_{\mathrm{Spec}\, K} \mathbb{P}^1_K$ of ϕ to it is constant. As $\mathbb{P}^1_K \times_{\mathrm{Spec}\, K} \mathbb{P}^1_K$ is projective, there exists an open subvariety X_0 of X such that $X \setminus X_0$ is finite and we have a morphism $\phi_{X_0} : X_0 \to \mathbb{P}^1_K \times_{\mathrm{Spec}\, K} \mathbb{P}^1_K$ that extends ϕ.

Let W be the normalization of the Zariski closure of the diagonal embedding $X_0 \to X \times_{\mathrm{Spec}\, K} \mathbb{P}^1_K \times_{\mathrm{Spec}\, K} \mathbb{P}^1_K$ in $X \times_{\mathrm{Spec}\, K} \mathbb{P}^1_K \times_{\mathrm{Spec}\, K} \mathbb{P}^1_K$. We have two projections $\Sigma_1 : W \to X$ and $\Sigma_2 : W \to \mathbb{P}^1_K \times_{\mathrm{Spec}\, K} \mathbb{P}^1_K$, with Σ_1 projective and birational as $\Sigma_{1,X_0} : W \times_X X_0 \to X_0$ is an isomorphism and with Σ_2 affine. Thus $\mathcal{O}(W) = \mathcal{O}(X) = \mathcal{O}(X_0)$. Hence the fibers of Σ_1 are connected by properties of Stein Factorization (see [Stacks3] or [H2], Ch. III, Cor. 11.5 and its proof).

[11] If $P \in \Gamma_e \cap \mathrm{Reg}(N_2) \cap \psi_e(X_e \setminus X_e^{\text{ét}})$, then by reasons of multiplicities each $W \in \mathrm{Irr}\big(\psi_e^{-1}(Y)\big)$ passes through a unique point in $\psi_e^{-1}(P)$.

We show that the assumption that Σ_1 is not an isomorphism leads to a contradiction. This assumption implies that, by enlarging X_0, we can assume that $X_0 \neq X$ and that for each $x \in X \setminus X_0$, the fiber $\Sigma_1^{-1}(x)$ is a connected curve. As $W \to X \times_{\operatorname{Spec} K} \mathbb{P}^1_K \times_{\operatorname{Spec} K} \mathbb{P}^1_K$ is quasi-finite, the restriction of the second projection $W \to \mathbb{P}^1_K \times_{\operatorname{Spec} K} \mathbb{P}^1_K$ to each irreducible component of $\Sigma_1^{-1}(x)$ is non-constant. Let C be the Zariski closure of $X_0 \setminus \operatorname{Im}(\imath)$ in W. The open embedding $W \setminus C \to W$ is affine by Nagata's Theorem (see [H1], Cor. 3.3). Thus the resulting morphism $W \setminus C \to \mathbb{P}^1_K \times_{\operatorname{Spec} K} \mathbb{P}^1_K$ is affine and quasi-finite, hence an open embedding (by Zariski's Main Theorem) which contains $\mathbb{A}^1_K \times_{\operatorname{Spec} K} \mathbb{A}^1_K$ and whose complement is a non-empty divisor. Therefore $W \setminus C$ is either $\mathbb{P}^1_K \times_{\operatorname{Spec} K} \mathbb{P}^1_K$ or isomorphic to $\mathbb{P}^1_K \times_{\operatorname{Spec} K} \mathbb{A}^1_K$. Thus the K-algebra $\mathcal{O}(W \setminus C)$ is either K or isomorphic to $K[t]$ which contradicts the fact that it has a K-subalgebra isomorphic to $\mathcal{O}(X)$. Thus Σ_1 is an isomorphism, hence we can take $\phi_X := \Sigma_2 \circ \Sigma_1^{-1} : X \to \mathbb{P}^1_K \times_{\operatorname{Spec} K} \mathbb{P}^1_K$. The uniqueness of ϕ_X and $\chi(\pi)_X$ holds as their targets are separated.

We show that the assumption that there exists $Y \in \operatorname{Irr}\big(X \setminus \operatorname{Im}(\imath)\big)$ such that $\phi_X|Y$ is non-constant leads to a contradiction. By replacing X with its open subvariety $X_Y := X \setminus (\cup_{Z \in \operatorname{Irr}(X \setminus \operatorname{Im}(\imath)), Z \neq Y} Z)$ which is affine by Nagata's Theorem, we can assume that $\operatorname{Irr}\big(X \setminus \operatorname{Im}(\imath)\big) = \{Y\}$. As $\phi_X|Y$ is non-constant, ϕ_X is quasi-finite, hence an open embedding by Zariski's Main Theorem. This implies that X is isomorphic to $\mathbb{A}^1_K \times_{\operatorname{Spec} K} \mathbb{P}^1_K$ minus a finite number of points, hence the K-algebra of global functions on X is isomorphic to $K[t]$, a contradiction. $\square$

Theorem 11.2. *Let X be a connected affine normal surface over K with an open embedding $\imath : \mathbb{A}^2_K \to X$. Assume that $Y := X \setminus \operatorname{Im}(\imath)$ is irreducible (i.e., X is a sphere-like surface). Then the following properties hold.*

(1) *There exists an $\mathbb{A}^1$-fibration $\Sigma : X \to \mathbb{P}^1_K$ such that $\operatorname{Im}(\imath) = \Sigma^{-1}(\mathbb{A}^1_K)$ (hence $Y = \big(\Sigma^{-1}(1:0)\big)_{\mathrm{red}}$) and the induced morphism $\Sigma^{-1}(\mathbb{A}^1_K) \to \mathbb{A}^1_K$ is a line bundle.*

(2) *Let $q(\Sigma) \in \mathbb{N}^*$ be the multiplicity of Y in $\Sigma^{-1}\big((1:0)\big)$. We have $q(\Sigma) = 1$ iff Σ is a line bundle.*

(3) *Let $\Sigma : X \to \mathbb{P}^1_K$ be an $\mathbb{A}^1$-fibration as in part (1) and let $q := q(\Sigma)$. Then there exists a finite flat morphism $c : C \to \mathbb{P}^1_K$ with C a smooth connected complete curve over K such that the following properties hold.*

(3.a) *The normalization of $X \times_{\Sigma, \mathbb{P}^1_K, c} C$ has an affine open cover formed by line bundles over C.*

(3.b) *We can assume (as needed) that either $|c^{-1}(1:0)| = 1$ or the field extension $\operatorname{Frac}(c)$ induced by c at the level of fields of fractions is separable (or Galois).*

(3.c) *If $\operatorname{char}(K) \nmid q$, then we can take $c : \mathbb{P}^1_{K,\mathrm{s}} \to \mathbb{P}^1_{K,\mathrm{t}}$ to be defined by the rule $(x : y) \mapsto (x^q : y^q)$ (hence $C = \mathbb{P}^1_K$, $|c^{-1}(1:0)| = 1$, and $\operatorname{Frac}(c)$ is cyclic).*

(4) *The curve Y is rational with one point at infinity.*

(5) *We have $Y \cong \mathbb{A}^1_K$.*

(6) *We have $|\operatorname{Sing}(X)| \leq 1$.*

(7) *If $\operatorname{char}(K) \nmid q$ and $|\operatorname{Sing}(X)| = 1$, then the singularity of X is a quotient singularity by the cyclic group $\mathbb{Z}/q\mathbb{Z}$. If $\operatorname{char}(K)|q$ and $|Sing(X)| = 1$, then the singularity of X is a quotient singularity by a finite group.*

(8) *Each singularity of X is rational. If $\operatorname{char}(K) = 0$ and $|\operatorname{Sing}(X)| = 1$, then the singularity of X is log-terminal.*

Proof. We identify $\mathbb{A}^1_K = \operatorname{Spec} K[t]$ and $K(t)$ with the field of fractions of $\mathbb{P}^1_K$.

For $i \in \{1,2\}$ let $\pi_i : \mathbb{A}^2_K \to \mathbb{A}^1_K$ be a projection such that the morphism $\pi_1 \times \pi_2 : \mathbb{A}^2_K \to \mathbb{A}^1_K \times_{\operatorname{Spec} K} \mathbb{A}^1_K$ is an isomorphism. Let $\phi : \mathbb{A}^2_K \to \mathbb{P}^1_K \times_{\operatorname{Spec} K} \mathbb{P}^1_K$ be the composite of $\pi_1 \times \pi_2$ with the product open embedding $\mathbb{A}^1_K \times_{\operatorname{Spec} K} \mathbb{A}^1_K \subset \mathbb{P}^1_K \times_{\operatorname{Spec} K} \mathbb{P}^1_K$ and let $\phi_X : X \to \mathbb{P}^1_K \times_{\operatorname{Spec} K} \mathbb{P}^1_K$ be its unique extension to X (see Lemma 11.1(1)). The image $\phi_X(Y)$ is one point $(P_1, P_2) \in (\mathbb{P}^1_K \times_{\operatorname{Spec} K} \mathbb{P}^1_K)(K)$ by Lemma 11.1(2). We write $P_i = (\alpha_i : \beta_i)$ with $(\alpha_i, \beta_i) \in K^2 \setminus \{(0,0)\}$. As X is separated, its local ring $\mathcal{O}_{X,Q}$ at a point $Q \in Y(K)$ cannot dominate its local ring $\mathcal{O}_{X,\iota(\gamma_1,\gamma_2)}$ with $(\gamma_1, \gamma_2) \in K^2$, hence $\beta_1\beta_2 = 0$. If $\beta_i = 0$, then the composite of ϕ_X with the i-th projection $\mathbb{P}^1_K \times_{\operatorname{Spec} K} \mathbb{P}^1_K \to \mathbb{P}^1_K$ is a flat surjective morphism $\Sigma : X \to \mathbb{P}^1_K$ whose generic fiber is isomorphic to $\mathbb{A}^1_{K(t)}$. For $(\alpha, \beta) \in K^2 \setminus \{(0,0)\}$, the fiber $\Sigma^{-1}\big((\alpha : \beta)\big)$ is isomorphic to $\mathbb{A}^1_K$ if $\beta \neq 0$ and its reduced variety is Y if $\beta = 0$. So Σ is an $\mathbb{A}^1$-fibration on X. As the induced morphism $\Sigma^{-1}(\mathbb{A}^1_K) \to \mathbb{A}^1_K$ is isomorphic to a projection $\mathbb{A}^2_K \to \mathbb{A}^1_K$, it is a line bundle. So part (1) holds.

The ‘if’ of part (2) is clear. If $q = 1$, i.e., if $\Sigma^{-1}\big((1:0)\big)$ is reduced (equivalently integral), then Σ is an $\mathbb{A}^1$-line bundle over $\mathbb{A}^1_K$ by [KM], Thm. 1 and thus a line bundle by [BCW1], Thm. 4.4. So the ‘only if’ of part (2) holds, X is regular and $Y \cong \mathbb{A}^1_K$ which also follow directly from [KM], Lem. 1.3. So part (2) holds and parts (3) to (8) hold for $q = 1$; to prove (3) to (8) in general we assume $q > 1$.

For a finite field extension $\mathcal{K}$ of $K(t)$, let $C := C(\mathcal{K})$ be the normalization of $\mathbb{P}^1_K$ in $\mathcal{K}$; we have a finite flat morphism $c : C \to \mathbb{P}^1_K$. Let X^{n}_C be the normalization of $X \times_{\Sigma, \mathbb{P}^1_K, c} C$. We have two projections: a finite surjective morphism $\Sigma_1 : X^{\mathrm{n}}_C \to X$ and an $\mathbb{A}^1$-fibration $\Sigma_2 : X^{\mathrm{n}}_C \to C$.

From [Epp], Thm. (2.0) and [Kuh] we get that we can choose $\mathcal{K}$ such that the fiber $c^{-1}\big((1:0)\big)$ is one point P, i.e., $c^{-1}\big((1:0)\big)_{\mathrm{red}} \cong \operatorname{Spec} K$, with $\Sigma_2^{-1}(P)$ reduced (hence a curve over K). Let $s := |\operatorname{Irr}\big(\Sigma_2^{-1}(P)\big)| \in [[1, [\mathcal{K} : K(t)]]]$; we write $\operatorname{Irr}\big(\Sigma_2^{-1}(P)\big) = \{Y_1, \ldots, Y_s\}$. For $i \in [[1,s]]$, let $W_i := \cup_{Z \in \operatorname{Irr}\left(\Sigma_2^{-1}(P)\right) \setminus \{Y_i\}} Z$, and let $X_i := X^{\mathrm{n}}_C \setminus W_i$. So X_i is affine by Nagata’s Theorem and the fibers of the restriction $\Sigma_2|X_i : X_i \to C$ are integral. Thus as above we argue that $\Sigma_2|X_i$ is an $\mathbb{A}^1$-line bundle over C. As $(\Sigma_2|X_i)^{-1}(P) = Y_i \setminus (Y_i \cap W_i) \cong \mathbb{A}^1_K$ and Y_i is affine, we have $Y_i \cap W_i = \emptyset$ and $Y_i \cong \mathbb{A}^1_K$. We have $X^{\mathrm{n}}_C = \cup_{i=1}^s X_i$.

From [Stacks4] we get that we can choose $\mathcal{K}$ to be a separable extension of $K(t)$ such that for each $P \in c^{-1}\big((1:0)\big)(K)$ the fiber $\Sigma_2^{-1}(P)$ is reduced. As above we argue that X^{n}_C has an affine open cover formed by line bundles over C. Concretely, if for $l := |c^{-1}\big((1:0)\big)(K)| \in \mathbb{N}^*$ we write $c^{-1}\big((1:0)\big)(K) = \{P_1, \ldots, P_l\}$ and if for each $j \in [[1,l]]$ we write $\operatorname{Irr}\big(\Sigma_2^{-1}(P_j)\big) = \{Y_{1,j}, \ldots, Y_{s_j,j}\}$ with $s_j \in [[1, [\mathcal{K} : K(t)]]]$, then for $\lambda := (i_1, \ldots, i_l) \in \prod_{j=1}^l [[1, s_j]]$, let $W_{i_j,j} := \cup_{Z \in \operatorname{Irr}\left(\Sigma_2^{-1}(P_j)\right) \setminus \{Y_{i_j,j}\}} Z$, and it follows that $X_\lambda := X^{\mathrm{n}}_C \setminus \cup_{j=1}^l W_{i_j,j}$ is a line bundle over C, $Y_{i_j,j} \cap W_{i_j,j} = \emptyset$, and $Y_{i_j,j} \cong \mathbb{A}^1_K$. We have $X = \cup_{\lambda \in \prod_{j=1}^l [[1,s_j]]} X_\lambda$. By enlarging $\mathcal{K}$, we can assume it is a Galois extension of $K(t)$ of finite Galois group G; clearly, $X = (X^{\mathrm{n}}_C)/G$.

From the last two paragraphs we get that parts (3.a) and (3.b) hold. Note that part (3.a) implies that Σ_2 is smooth.

In this paragraph we assume that $\operatorname{char}(K) \nmid q$. We take $\mathcal{K} = K(t^{\frac{1}{q}})$; the field extension $K(T) \to \mathcal{K}$ is cyclic (thus Galois) and the fiber $c^{-1}\big((1:0)\big)$ is one point P with $\Sigma_2^{-1}(P)$ reduced. Clearly, $c : C = \mathbb{P}^1_{K,\mathrm{s}} \to \mathbb{P}^1_{K,\mathrm{t}}$ is defined by the rule $(x:y) \mapsto (x^q : y^q)$ and $s \mid q$. Thus part (3c) holds. So part (3) holds.

If $c^{-1}\big((1:0)\big)$ is one point (resp. if $\mathcal{K}$ is a separable extension of $K(t)$), then as the morphism $\mathbb{A}^1_K \cong Y_i \to Y$ (resp. $\mathbb{A}^1_K \cong Y_{i_j,j} \to Y$) induced by Σ_1 is finite surjective, part (4) holds by Lemma 10.5(1) and (2).

To prove that parts (5) to (7) hold when $\mathrm{char}(K) \nmid q$, we choose c as in part (3.c), so c is a finite Galois cover over $\mathbb{P}^1_{K,\mathrm{t}} \setminus \{(1:0),(0:1)\}$ of Galois group $G \cong \mathbb{Z}/q\mathbb{Z}$. Hence $\Sigma_1 : X^{\mathrm{n}}_C \to X$ is a finite Galois cover over the complement of $\Sigma^{-1}(0:1) \cup \mathcal{B}$ in X, where $\mathcal{B}$ is a finite subset of $Y(K)$. The group G acts on Σ_1 and we have $X = (X^{\mathrm{n}}_C)^G$. As $\mathrm{char}(K) \nmid q$, all G-modules over K are semisimple and the functor $\star^G$ on G-modules $\star$ over K is exact. Thus Y is the quotient of the action of G on either $\Sigma_1^{-1}(Y)$ or (as Y is reduced) $\Sigma_1^{-1}(Y)_{\mathrm{red}} = \sqcup_{i=1}^s Y_i$. Hence $Y = (Y_i)^{G_i}$ with G_i a cyclic group of order $\frac{q}{s}$ which is a subgroup of $\mathrm{Aut}(Y_i)$. Thus Y is normal and hence regular; from this and part (4) we get that $Y \cong \mathbb{A}^1_K$. Moreover, there exists a unique point $P_i \in Y_i(K)$ such that the stabilizer of P_i in G_i is non-trivial, and in fact this stabilizer is G_i itself. So for $Q_i \in Y_i(K)\setminus\{P_i\}$, the completions of the local rings $\mathcal{O}_{X,\Sigma_1(Q_i)}$ and $\mathcal{O}_{X^{\mathrm{n}}_C,Q_i}$ are naturally identified; hence $\Sigma_1(Q_i) \notin \mathrm{Sing}(X)$. As $Y_i \to Y$ is surjective, it follows that $\mathrm{Sing}(X) \subset \{\Sigma_1(P_i)\}$ and hence $|\mathrm{Sing}(X)| \leq 1$. So parts (5) to (7) hold if $\mathrm{char}(K) \nmid q$. Note that we can take $\mathcal{B} = \mathrm{Sing}(X) \subset Y(K)$ and, if X is singular, that $\Sigma_1(P_i)$ does not depend on $i \in [[1,s]]$.

To prove that parts (5) to (7) hold we could assume that $\mathrm{char}(K)|q$ but we include arguments which actually work in general. For part (5), let $R \cong K[[x]]$ be the completion of $\mathcal{O}_{\mathbb{P}^1_K,(1:0)}$ and let $\Sigma_R : X \times_{\Sigma,\mathbb{P}^1_K} \mathrm{Spec}\, R \to \mathrm{Spec}\, R$ be the pullback of Σ via the natural morphism $\mathrm{Spec}\, R \to \mathbb{P}^1_K$. Let $\Sigma^+_R : X^+_R \to \mathrm{Spec}\, R$ be a projective morphism that extends Σ_R with X^+_R an integral normal scheme that contains X_R as an open subscheme. The K-algebra of global functions on X_R and hence also on X^+_R is R. From this and the fact that Σ^+_R is flat, we get that the K-algebra of global functions on the special fiber X^+_K of Σ^+_R (over $\mathrm{Spec}\, K$) is K. As the generic fiber of Σ^+_R (over $\mathrm{Spec}\,\mathrm{Frac}R$) is isomorphic to $\mathbb{A}^1_{\mathrm{Frac}(R)}$, it has a divisor of degree 1. So Σ^+_R is cohomologically flat by (the implication $(ii) \Rightarrow (iv)$ in) [Ray], Thm. 8.2.1. Thus $H^1(X^+_R, \mathcal{O}_{X^+_R}) = 0$, hence for the curve $Y^+ \in \mathrm{Irr}(X^+_K)$ that contains Y we have $H^1(Y^+, \mathcal{O}_{Y^+}) = 0$. Thus $Y^+ \cong \mathbb{P}^1_K$. From this and part (4) we get that part (5) holds.[12]

For parts (6) and (7), we chose c such that $\mathcal{K}$ is a Galois extension of $k(t)$ and we use the above notation for the curves $Y_{i_j,j}$ and the Galois group G. Fixing a $j \in [[1,l]]$ and an $i_j \in [[1,s_j]]$, let $H_{i_j,j}$ be the subgroup of G that stabilizes $Y_{i_j,j}$. Let S be the completion of the local ring $\mathcal{O}_{C,P_j}$. If i_j is the j-th component of $\lambda = (i_1,\ldots,i_l)$, then $X_\lambda \times_C \mathrm{Spec}\, S \cong \mathbb{A}^1_S$, $H_{i_j,j}$ is a subgroup of $\mathrm{Gal}\big(\mathrm{Frac}(S)/\mathrm{Frac}(R)\big)$ and the morphism $(X_\lambda \times_C \mathrm{Spec}\, S)/H_{i_j,j} \to \widehat{X} := X \times_{\Sigma,\mathbb{P}^1_K} \mathrm{Spec}\, R$ induces isomorphisms at the level of completions of local rings of residue fields k. Thus $|\mathrm{Sing}(\widehat{X})|$ is the number of singular points of $(X_\lambda \times_C \mathrm{Spec}\, S)/H_{i_j,j}$. From this and Corollary 8.8 applied to R equal to $S^{H_{i_j,j}}$ we get that $|\mathrm{Sing}(\widehat{X})| \leq 1$. Thus part (6) holds. As $X = (X^{\mathrm{n}}_C)/G$, part (7) holds.

The first part of (8) follows from Proposition 9.2(3) (resp. 9.3) and part (7) if $\mathrm{char}(K) \nmid q$ (resp. in general). The second part of (8) follows from the first part of (7) and [Kaw], Prop. 1.7 and the paragraph after it. $\square$

[12] Part (5) is also a particular case of Remark 8.7(1).

Remark 11.3. **(1)** For $(l_1, l_2, l_3) \in (\mathbb{N}^*)^3$ with $g.c.d.(l_1, l_2) = 1$ and $\min\{l_1, l_2\} \geq 2$ the flat integral homomorphism $k[t^{l_1}, t^{l_2}] \to k[x, t^{l_1}, t^{l_2}]/(x-t)^{p^{l_3}}$ has a non-regular integral domain source and a non-reduced target whose reduction is regular. Similarly, if $p \nmid n$, for the non-regular integral domain $R := k[x, y]/(x^n - y^p)$, and the flat R-algebra $S := A[z]/(z^p - x)$ whose reduction is isomorphic to $k[x]$, the rule $t \mapsto z^n - y$ defines an R-algebra homomorphism $A \otimes_k k[t]/(t^p) \to S$ that makes S a flat $k[t]/(t^p)$-algebra. These examples explain why the proof of Theorem 11.2(5) is not worked out using only schemes over Y.

(2) Theorem 11.2(1), (4) and (6) in general and (5) in the case when X is regular are proved in [Mi2], Sect. 2 and Lem. 1 provided $\mathrm{char}(K) = 0$.

(3) If $\mathrm{char}(K) = 0$ and $|\mathrm{Sing}(X)| = 1$, then the fact that the singularity of X is rational was first proved in [Mi2], Thm. 1(1) and it also follows from the more general result [FZ], Thm. 0.3 or 1.12 which applies as $X \setminus Y = \mathrm{Im}(\imath)$ is a union of closed $\mathbb{A}^1_K$ curves in X that do not pass through the singular point of X in Y.

(4) If in the proof of Theorem 11.2(6) we have $|H_{i_j,j}| = \mathrm{char}(K) \mid q$, then either $(X_\lambda \times_C \mathrm{Spec}\, S)/H_{i_j,j} \to \mathrm{Spec}\, S^{H_{i_j,j}}$ or $X_\lambda \times_C \mathrm{Spec}\, S \to (X_\lambda \times_C \mathrm{Spec}\, S)/H_{i_j,j}$ is a smooth morphism, hence $\widehat{X}$ is regular; so X is also regular.

Corollary 11.4. *Let X be a connected affine normal surface over K with an open embedding $\imath : \mathbb{A}^2_K \to X$ which is not surjective. The following properties hold.*

(1) *Each connected component of $X \setminus \mathrm{Im}(\imath)$ is irreducible and isomorphic to $\mathbb{A}^1_K$.*

(2) *Let $Y \in \mathrm{Irr}\big(X \setminus \mathrm{Im}(\imath)\big)$. The union $\mathbb{S}^2_Y := \mathrm{Im}(\imath) \cup Y$ is an open affine subvariety of X and we have $|Y \cap \mathrm{Sing}(X)| \leq 1$.*

Proof. Let $Y \in \mathrm{Irr}\big(X \setminus \mathrm{Im}(\imath)\big)$. Let $W := \cup_{Z \in \mathrm{Irr}(X \setminus \mathrm{Im}(\imath)) \setminus \{Y\}} Z$. Then $X \setminus W$ is affine by Nagata's Theorem and the union of $\mathrm{Im}(\imath)$ and $Y \setminus (Y \cap W)$. If $Y \cap W \neq \emptyset$, then $Y \setminus (Y \cap W)$ is an irreducible curve with at least 2 points at infinity, a contradiction to Theorem 11.2(4) applied to $X \setminus W$. Thus $Y \cap W = \emptyset$. So the connected components of $X \setminus \mathrm{Im}(\imath)$ are irreducible. As $Y = Y \setminus (Y \cap W)$, $\mathbb{S}^2_Y = X \setminus W$ is affine. From these and Theorem 11.2(5) and (6) we get that corollary holds. □

Corollary 11.5. *Let X be a connected affine normal surface over K. Then X is a sphere-like surface iff there exists a surjective $\mathbb{A}^1$-fibration $\Sigma : X \to \mathbb{P}^1_K$ with all fibers irreducible and with at most one non-reduced fiber.*

Proof. The 'only if' holds by Theorem 11.2(1). For the 'if' part assume that all fibers of $\Sigma : X \to \mathbb{P}^1_K$ except $\Sigma^{-1}(1:0)$ are integral. Thus $\Sigma^{-1}(\mathbb{A}^1_K) \to \mathbb{A}^1_K$ is an $\mathbb{A}^1_K$-bundle by [KM], Thm. 1 and hence a line bundle by [BCW1], Thm. 4.4. So $\Sigma^{-1}(\mathbb{A}^1_K) \cong \mathbb{A}^2_K$ and the 'if' part holds as $\mathrm{Irr}(X \setminus \Sigma^{-1}(\mathbb{A}^1_K)) = \{\big(\Sigma^{-1}(1:0)\big)_{\mathrm{red}}\}$. □

Definition 11.6. **(1)** *Let X be a connected affine normal surface over K with an open embedding $\imath : \mathbb{A}^2_K \to X$ which is not an isomorphism. We call the open affine cover $X = \cup_{Y \in \mathrm{Irr}(X \setminus \mathrm{Im}(\imath))} \mathbb{S}^2_Y$ as the canonical open cover of X by sphere-like surfaces associated to $\imath$, where $\mathbb{S}^2_Y := \mathrm{Im}(\imath) \cup Y$.*

(2) *Given $e \in \mathrm{EE}_n(K)$, by an approximation of $X_e = \mathrm{Spec}\, A_e$ we mean an affine variety $Z_e = \mathrm{Spec}\, B_e$ with B_e a $K_{\mathrm{t}}[x_1, \ldots, x_n]$-subalgebra of A_e. The approximation Z_e of X_e is called flat if B_e is a free $K_{\mathrm{t}}[x_1, \ldots, x_n]$-module and is called a model of X_e if $\mathrm{Frac}(B_e) = \mathrm{Frac}(A_e)$.*

(3) *Let X be a sphere-like surface over K for which there exists an $\mathbb{A}^1$-fibration $\Sigma : X \to \mathbb{P}^1_K$ such that $\Sigma^{-1}(\mathbb{A}^1_K) \to \mathbb{A}^1_K$ is a line bundle and the fiber $\Sigma^{-1}(1:0)$ is irreducible but non-reduced (so Σ is as in Theorem 11.2(1) and (2) with $q = q(\Sigma) \geq 2$). For each such Σ and every $P \in \mathbb{P}^1_K(K) \setminus \{(1:0)\}$, $X_{\Sigma,P} := X \setminus \Sigma^{-1}(P)$ is an affine pseudo-plane over K in the sense of* [MM1]*, Def. 2.1*[13] *and hence we call it a standard affine pseudo-plane of X (of type q).*

Remark 11.7. **(1)** Let $e \in \mathrm{EE}_n(K)$. For each approximation $Z_e = \operatorname{Spec} B_e$ of X_e, the inclusion $B_e \to A_e$ defines a finite morphism $X_e \to Z_e$ and we have an inequality $\rho(e) \geq |\mathrm{Irr}(Z_e \setminus V_e)|$, where V_e is the image of $\mathrm{Im}(\imath_e)$ in Z_e. Note that X_e is the normalization of each model of it.

(2) Let $e \in \mathrm{EE}_2(K)$. For two projections $\pi_1, \pi_2 : \mathbb{A}^2_{K,\mathrm{s}} \to \mathbb{A}^1_K$, let $Z_e(\pi_1, \pi_2)$ be the Zariski closure in $\mathbb{P}^1_K \times_{\operatorname{Spec} K} \mathbb{P}^1_K \times_{\operatorname{Spec} K} \mathbb{A}^2_{K,\mathrm{t}}$ of the image of the locally closed embedding $\chi(\pi_1) \times \chi(\pi_2) \times e : \mathbb{A}^2_{K,\mathrm{s}} \to \mathbb{P}^1_K \times_{\operatorname{Spec} K} \mathbb{P}^1_K \times_{\operatorname{Spec} K} \mathbb{A}^2_{K,\mathrm{t}}$. Then $\chi(\pi_1) \times \chi(\pi_2) \times e$ extends to a finite surjective morphism $X_e \to Z_e(\pi_1, \pi_2)$ defined by $\chi_e(\pi_1) \times \chi_e(\pi_2) \times \psi_e$. Note that $Z_e(\pi_1, \pi_2)$ is an approximation of X_e; if $\pi_1 \times \pi_2 : \mathbb{A}^2_{K,\mathrm{s}} \to \mathbb{A}^1_K \times_{\operatorname{Spec} K} \mathbb{A}^1_K$ is an isomorphism, then $Z_e(\pi_1, \pi_2)$ is a model of X_e.

(3) Referring to Definition 11.6(3), the $\mathbb{A}^1$-fibration $\Sigma_P : X_{\Sigma,P} \to \mathbb{P}^1_K \setminus \{P\} \cong \mathbb{A}^1_K$ induced by Σ has only one non-reduced fiber $\Sigma^{-1}(1:0) = \Sigma_P^{-1}(1:0)$ of multiplicity q and has all fibers over K-valued points such that their reduced varieties are isomorphic to $\mathbb{A}^1_K$ (by Theorem 11.2(5)).

For $e \in \mathrm{EE}_2(K)$, the étale loci $X_e^{\text{ét}}$ can be described as follows.

Corollary 11.8. *For a non-finite $e \in \mathrm{EE}_2(K)$ we consider the canonical open affine cover $X_e = \cup_{Y \in \mathrm{Irr}(\Gamma_{\psi_e})} \mathbb{S}^2_Y$ associated to $\imath_e$. The following properties hold.*

(1) *We write $\mathbb{S}^2_Y = \operatorname{Spec} A_Y$. Then the following three statements are equivalent.*

(i) *The open subvariety $\mathbb{S}^2_Y$ is contained in the étale locus $X_e^{\text{ét}}$ of ψ_e.*

(ii) *The generic point η_Y of Y belongs to $X_e^{\text{ét}}$.*

(iii) *The tangent bundle $T_{\mathbb{S}^2_Y}$ over $\mathbb{S}^2_Y$ is trivial.*

Moreover, if $\mathrm{char}(K) = p$, then statements (i) to (iii) are also equivalent to either one of the following extra statements.

(iv) *The ring A_Y has a p-basis.*

(v) *The ring (or the K-algebra A_Y has a differential basis.*

(2) *We have an identity $X_e^{\text{ét}} = \mathrm{Im}(\imath_e) \bigcup_{Y \in \mathrm{Irr}(\Gamma_{\psi_e}), T_{\mathbb{S}^2_Y} \text{ is a trivial bundle}} \mathbb{S}^2_Y$. If $\mathrm{char}(K) = p$ then the identity $X_e^{\text{ét}} = \mathrm{Im}(\imath_e) \bigcup_{Y \in \mathrm{Irr}(\Gamma_{\psi_e}), A_Y \text{ has a } p\text{-basis}} \mathbb{S}^2_Y$ holds.*

(3) *The open subvariety $X_e^{\text{ét}}$ of X_e is affine regular, contains $\mathrm{Im}(\imath_e) = \mathrm{Im}(\psi_e)$, and the complement $X_e^{\text{ét}} \setminus \mathrm{Im}(\imath_e)$ is a disjoint union of $\rho_{\text{ét}}(e)$ irreducible components of Γ_{ψ_e} isomorphic to $\mathbb{A}^1_K$.*

(4) *The complement $X_e \setminus \mathrm{Im}(\imath_e)$ is a disjoint union of $\rho(e)$ irreducible components of Γ_{ψ_e} isomorphic to $\mathbb{A}^1_K$.*

(5) *If $Y \subset X_e^{\text{ét}}$ then $\mathbb{S}^2_Y$ is regular, and if $Y \subset X_e \setminus X_e^{\text{ét}}$ then $|\mathrm{Sing}(\mathbb{S}^2_Y)| \leq 1$.*

(6) *We have $|\mathrm{Sing}(X_e)| \leq \rho(e) - \rho_{\text{ét}}(e)$.*

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[13] Based on Theorem 11.2(5), the hypotheses $F \cong \mathbb{A}^1$ in [MM1], Def. 2.1(2) is superfluous.

Proof. Part (1) is a particular case of Theorem 7.3(1). Part (2) follows from part (1). For part (3), $X_e^{\text{ét}}$ is regular as $\mathrm{Im}(\psi_e)=\mathbb{A}^2_{K,\mathrm{s}}$ and is affine by Nagata's Theorem as $X_e\setminus X_e^{\text{ét}}$ is a union of irreducible divisors of X_e by part (2) and Corollary 11.4(1). The last statement of part (3) follows from part (2), Corollary 11.4(1) and the definition of $\rho_{\text{ét}}(e)$. So part (3) holds. Similarly, part (4) follows from Corollary 11.4(1) and the definition of $\rho(e)$. The first part of (5) is clear and the second part of (5) follows from Corollary 11.4(2). Part (6) follows from part (5) as the number of $Y\in\mathrm{Irr}(\Gamma_{\psi_e})$ with A_Y not having a p-basis is $\rho(e)-\rho_{\text{ét}}(e)$. □

Corollary 11.8(2) justifies the following definition.

Definition 11.9. *Let $e\in\mathrm{EE}_2(K)$ be non-finite. We refer to the open affine cover $X_e^{\text{ét}}=\mathrm{Im}(\imath_e)\bigcup_{Y\in\mathrm{Irr}(\Gamma_{\psi_e}),T_{\mathbb{S}^2_Y}\text{ is a trivial bundle}}\mathbb{S}^2_Y$ as the canonical open cover of $X_e^{\text{ét}}$ by sphere-like surfaces associated to $\imath_e$.*

The *sphere class rank invariant* (resp. *almost sphere class rank invariant*)

$$\rho_1=\rho_{1,K,2}:\mathrm{EE}_2(K)\to\mathbb{N}\quad (\textit{resp. } \rho_2=\rho_{2,K,2}:\mathrm{EE}_2(K)\to\mathbb{N})$$

is defined for $e\in\mathrm{EE}_2(K)$ by: if e is finite then $\rho_1(e):=0$ and if e is non-finite then

$$\rho_1(e):=|\{Y\in\mathrm{Irr}(\Gamma_{\psi_e})|\mathbb{S}^2_Y\cong\mathbb{S}^2_k\}|.$$

If $e\in\mathrm{EE}_2(K)$, then $\rho_2(e):=\rho_{\text{ét}}(e)\setminus\rho_1(e)$.

The *quasi-sphere class rank invariant* (resp. *pseudo-sphere class rank invariant*)

$$\rho_3=\rho_{3,K,2}:\mathrm{EE}_2(K)\to\mathbb{N}\quad (\textit{resp. } \rho_4=\rho_{4,K,2}:\mathrm{EE}_2(K)\to\mathbb{N})$$

is defined by the rule $\rho_3(e):=\rho(e)-\rho_{\text{ét}}(e)-|\mathrm{Sing}(X_e)|$ (resp. $\rho_4(e):=|\mathrm{Sing}(X_e)|$).

From definitions, Corollaries 11.8(2) and (4) and 11.4 we get directly the following result.

Corollary 11.10. *Let $e\in\mathrm{EE}_2(K)$. Then the canonical open cover of X_e by sphere-like surfaces associated to $\imath_e$ consists of exactly $\rho_1(e)$ spheres, $\rho_2(e)$ almost spheres, $\rho_3(e)$ quasi-spheres, and $\rho_4(e)$ pseudo-spheres.*

For $e\in\mathrm{EE}_2(k)$, all cases of 'strictness-equality' in the general inequalities $0\le\rho_{\text{ét}}(e)\le\rho(e)$ occur with the Jacobian surface ψ_e regular: (i) for $0<\rho_{\text{ét}}(e)=\rho(e)$ see $e_{f,g,k}$ of Section 2 with $(f,g)\in\Theta^1_{k,m}$, Example 20.3, Example 21.5 for $n=2$, and the case $p=3$ of Examples 22.6 and 22.7; (ii) for $0=\rho_{\text{ét}}(e)<\rho(e)$ see Remark 23.4 and the case $p\ge5$ of Examples 22.6 and 22.7; (iii) for $0<\rho_{\text{ét}}(e)<\rho(e)$ see end of Example 14.1 for $mq>2$; (iv) for $0=\rho_{\text{ét}}(e)=\rho(e)$ see Remark 21.9.

12. Proof of Theorem 2.4(1)

We construct two families of examples. The easier one, Example 12.1, works for almost all l, and the harder one, Example 12.3, works for all l.

Example 12.1. Let $m\in\mathbb{N}^*$ and $s\in[[0,pm]]$. Let $\wp(m,s):=p^2m-s(p-1)$. Let $g(x)\in k[x]$ be a polynomial of degree m and let $\gamma_1,\ldots,\gamma_{pm}\in k$ be the pm distinct zeros of the equation $x+g(x)^p=0$. Let $f(x)\in k[x]$ be such that for $i\in[[1,pm]]$ we have $f(\gamma_i)=0$ iff $i\in[[1,s]]$; thus $\deg(f)\ge s$. Then the F_2-system $\mathcal{W}$

$$x+g(x)^p=0=y+y^pf(x)^p$$

over k has $\mathrm{s}(\mathcal{W})=s+p(pm-s)=\wp(m,s)$ solutions computed as follows: for $i\in[[1,s]]$ we have the solution $(\gamma_i,0)$ and for $i\in[[1,pm]]\setminus[[1,s]]$ there exist precisely p solutions with the first coordinate equal to γ_i. From Proposition 4.3(1) to (3) we get the second inequality and the first two equalities in

(6) $\quad pm \le \pi_{\mathrm{i}}(e) \le \pi_{\mathrm{l}} = \pi_{\mathrm{r}} = \pi(\mathcal{W}) = p\max\{m, \deg(f)+1\} \ge p\max\{m, s+1\}.$

In particular, for $s \le \deg(f) \le m-1$ we have $\pi_{\mathrm{i}}(e) = \pi_{\mathrm{l}} = \pi_{\mathrm{r}} = \pi(\mathcal{W}) = pm$.

Lemma 12.2. *For* $\mathcal{D} := \{\wp(m,s)|m \in \mathbb{N}^*, s \in [[0,pm]]\}$ *we have inclusions*

$$[[0,p-1]] \subset \mathbb{N}\setminus\mathcal{D} = \{0\} \sqcup \{pq-r|q,r \in [[1,p-1]], q \le r\} \subset [[0,(p-1)^2]].$$

Proof. Let $r := pm - s \in [[0,pm]]$. We have $\wp(m,s) = pm + (p-1)r$ with $m \in \mathbb{N}^*$ and $r \in [[0,pm]]$. Thus the set $\mathcal{D}$ is equal to

$$\{pm+(p-1)r|m \in \mathbb{N}^*, r \in [[0,pm]]\} = \{pm+(p-1)r|(m,r) \in \mathbb{N}^* \times [[0,p-1]]\}.$$

Denoting $q := m+r$, we have $\wp(m,s) = pq - r$ with $q \ge r+1$. Therefore $\mathcal{D} = \{pq-r|(q,r) \in \mathbb{N}^* \times [[0,p-1]], q \ge r+1\}$. For $r=0$ we get $0 \notin \mathcal{D} \supset p\mathbb{N}^*$ and for $r \in [[1,p-1]]$ the smallest and the largest numbers in $\mathbb{N}\setminus\mathcal{D}$ congruent to $-r$ modulo p are $p-r$ and $pr-r$ (respectively), which for $r=p-1$ are 1 and $(p-1)^2$ (respectively). The lemma follows. $\square$

Example 12.3. Let $m, q \in \mathbb{N}^*$. Let $f(y) \in k[y]$ be such that $f(0) \neq 0$ and the equation $f(y) = 0$ has q distinct solutions in k, to be denoted as $\gamma_1, \ldots, \gamma_q$; so $\deg(f) \ge q$. For $i \in [[1,q]]$, let $\varepsilon_i := \sqrt[p]{\gamma_i} \in k$. We consider the k-algebra

$$A := k[x,y]/\big(y - x^p f(y)^p\big).$$

As $f(0) \neq 0$, we have $u := f(y) + \big(y - x^p f(y)^p\big) \in A^*$. Moreover, u is a p-th power of an element of A as so is $y + \big(y - x^p f(y)^p\big) \in A$. So the inverse of u is a p-th power. Thus there exists a polynomial $g \in k[x,y]$ such that $f(y)\big(g(x,y)\big)^p - 1$ is divisible by $y - x^p f(y)^p$. If $z := \sqrt[p]{y}$, so $y = z^p$, we get that we have an isomorphism

$$A \cong k[z]\Big[\frac{1}{\prod_{i=1}^q (z-\varepsilon_i)}\Big]$$

induced by the k-algebra epimorphism

$$\theta : k[x,y] \to k[z, f(z^p)^{-1}]$$

that maps y to z^p and x to $zf(z^p)^{-1}$; note that $\theta\big(f(y)x\big) = z$, $\theta\big(y - x^p f(y)^p\big) = 0$, and $\theta\big(g(x,y)^p\big) = f(z^p)^{-1}$.

We now study the solution set $\mathcal{J}_0 \subset k^2$ of the F_2-system $\mathcal{U}_0$

$$x - \big(g(x,y)\big)^p - x^{pm} f(y)^{pm}\big(g(x,y)\big)^p = y - x^p f(y)^p = 0$$

over k which satisfies

(7) $$\pi(e_{\mathcal{U}_0}) \le \pi(\mathcal{U}_0) = p[m + m\deg(f) + \deg(g)].$$

As $\theta\big(x - (g(x,y))^p - x^{pm} f(y)^{pm}(g(x,y))^p\big) = f(z^p)^{-1}(-z^{pm} + z - 1)$, we have

$$\mathcal{J}_0 = \Big\{\Big(\frac{z}{f(z^p)}, z^p\Big)|z \in k \setminus \{\varepsilon_i|i \in [[1,q]]\},\ z^{pm} - z + 1 = 0\Big\}.$$

As the equation $z^{pm} - z + 1 = 0$ has exactly pm solutions $\delta_1, \ldots, \delta_{pm}$ in k, we have $\mathrm{s}(\mathcal{U}_0) = |\mathcal{J}_0| = pm - r$ with

$$r := |\{\varepsilon_i|i \in [[1,q]]\} \cap \{\delta_i|i \in [[1,pm]]\}| \in [[0,pm]].$$

Let $\alpha \in k^*$; if $\deg(f) = m$, we assume that $-\alpha f(y)$ is not monic. A similar argument shows that the solution set of the F_2-system $\mathcal{U}_\alpha$

$$x - \big(g(x,y)\big)^p - x^{pm} f(y)^{pm} \big(g(x,y)\big)^p - \alpha = y - x^p f(y)^p = 0$$

is $\mathcal{J}_\alpha := \big\{\big(\frac{z}{f(z^p)}, z^p\big) | z \in k \setminus \{\varepsilon_i | i \in [[1,q]]\},\ z^{pm} - z + 1 + \alpha f(z^p) = 0\big\}$. We have $\deg\big(z^{pm} - z + 1 + \alpha f(z^p)\big) = p \max\{m, \deg(f)\}$. Hence, as ε_i with $i \in [[1,q]]$ is a zero of $z^{pm} - z + 1$ iff it is a zero of $z^{pm} - z + 1 + \alpha f(z^p)$, we get the identity

$$\mathrm{s}(\mathcal{U}_\alpha) = |\mathcal{J}_\alpha| = p \max\{m, \deg(f)\} - r.$$

Let now $l \in \mathbb{N}$. We take $m \in \mathbb{N}^*$, $m \geq \lceil \frac{l}{p} \rceil$, $r = pm - l$, and we choose $f(y)$ so that $\{\varepsilon_i | i \in [[1,q]]\} \cap \{\delta_i | i \in [[1,pm]]\}$ has precisely r elements; so $\deg(f) \geq q \geq r$. We get that $\mathrm{s}(\mathcal{U}_0) = l$. If $\deg(f) > m$, then we have relations

$$\mathrm{s}(\mathcal{U}_0) = l = pm - r < p \deg(f) - r = \mathrm{s}(\mathcal{U}_\alpha) \leq \deg(e_{\mathcal{U}_0}) = \deg(e_{\mathcal{U}_\alpha});$$

see [Gro5], Prop. (18.2.8) for the last (i.e., non-strict) inequality.

Assume $l = 0$. We take $m = 1$, $q = r = p$, and $f(y) = y^p - y + 1 \in \mathbb{F}_p[y]$; note that the set of zeros of f is $\{\delta_i | i \in [[1,p]]\} = \{\delta_i^{\frac{1}{p}} | i \in [[1,p]]\}$.

Assume $l = 1$ and $p > 2$. We choose m such that $p | 2^m - 1$, i.e., 2 is a zero of $z^{pm} - z + 1$, say $2 = \delta_1$. We take q to be $r = pm - 1$ and $f(y) = \prod_{i=2}^{pm}(y - \delta_i^{\frac{1}{p}})$; as $\{\delta_i | i \in [[1,pm]]\} = \{\delta_i^{\frac{1}{p}} | i \in [[1,pm]]\}$ we have $f(y) = \frac{y^{pm} - y + 1}{y - 2} \in \mathbb{F}_p[y]$.

If $f(x) \in \mathbb{F}_p[x]$, then we can choose $g(x,y) \in \mathbb{F}_p[x,y]$, so $\mathcal{U}_0$ is defined over $\mathbb{F}_p$.

Theorem 2.4(1) follows from Example 12.3.

For $l \in \mathcal{D}$ (see Lemma 12.2), Equation (6) gives simpler and smaller upper bounds for algebraic degrees than Relations (7) applied to an f with $\mathrm{s}(\mathcal{U}_0) = l$.

Remark 12.4. With m, q and $f(y)$ as in Example 12.3, we can write everything explicitly, which is how Example 1.2 was obtained. Let $f_1(y) \in k[y]$ be the unique polynomial such that $f(y) = f(0) + y f_1(y)$. Working modulo $\big(y - x^p f(y)^p\big)$ we get

$$1 = f(0)^{-1}\big(f(y) - y f_1(y)\big) \equiv f(0)^{-1}\big(f(y) - x^p f(y)^p f_1(y)\big)$$

$$= f(0)^{-1} f(y)\big(1 - x^p f(y)^{p-1} f_1(y)\big) \equiv f(0)^{-1} f(y)\big(1 - x^p f(x^p f(y)^p)^{p-1} f_1(x^p f(y)^p)\big),$$

i.e., we can take $g(x,y)^p$ to be $f(0)^{-1}\big(1 - x^p f(x^p f(y)^p)^{p-1} f_1(x^p f(y)^p)\big)$. Thus

$$(x,y) \mapsto \big(x - f(0)^{-1}(1 + x^{pm} f(y)^{pm})(1 - f_1(x^p f(y)^p) x^p f(x^p f(y)^p)^{p-1}), y - x^p f(y)^p\big)$$

is the rule that defines $e_{\mathcal{U}_0}$.

If $f(y) = 1 - y + y^p$, then $q = p$, $f(0) = 1$ and $f_1(y) = -1 + y^{p-1}$; if $m = 1$, then $\mathcal{U}_0$ is the F_2-system of Example 1.2. With $m \in \mathbb{N}^*$ arbitrary, let $e =: e_{\mathcal{U}_0}$ (it depends on m and f). To compute $\deg(e)$ we substitute $z := x f(y)$. The system

$$x - \big(1 + x^{pm} f(y)^{pm}\big)\big(1 - x^p f(x^p f(y)^p)^{p-1} f_1(x^p f(y)^p)\big) - x_1 = 0 = y - x^p f(y)^p - x_2,$$

gives

$$y = z^p + x_2 \ \text{ and } \ x = \frac{z}{z^{p^2} - z^p + 1 + h_2}$$

with $h_2 := x_2^p - x_2 \in k[x_1, x_2]$. Substituting these expressions in $f(y)^p$ times the first equation of the system and $z f_1(z) = f(z) - 1$ we get that

$$z f(z^p + x_2)^{p-1} - x_1 f(z^p + x_2)^p = (z^{pm} + 1)[f(z^p + x_2)^p - z^p f(z^p)^{p-1} f_1(z^p)]$$

$$= (z^{pm} + 1)[f(z^p + x_2)^p - f(z^p)^p + f(z^p)^{p-1}] = (z^{pm} + 1)[f(z^p)^{p-1} + h_2^p].$$

The left (resp. right) hand side is a polynomial in z of degree p^3 (resp. p^3-p^2+pm) with leading coefficient $-x_1$ (resp. 1). Thus $\deg(e)=\max\{p^3-p^2+pm,p^3\}$ and the field extension $\mathrm{Frac}(e^{\#})$ is isomorphic to $k_{\mathrm{t}}(x_1,x_2)\to k_{\mathrm{t}}(x_1,x_2)[z]/\big(h(z)\big)$, where

$$h(z):=(z^{pm}+1)[(z^{p^2}-z^p+1)^{p-1}+h_2^p]+x_1(z^{p^2}-z^p+1+h_2)^p-z(z^{p^2}-z^p+1+h_2)^{p-1}$$

is in $k[x_1,x_2][z]$. If $m>p$, then $\mathrm{Spec}\,\big(k_{\mathrm{t}}(x_1,x_2)[z]/(h(z))\big)$ is a flat model of X_e.

13. Proof of Theorem 2.4(2)

For the proof we need a new type of open embeddings of $\mathbb{A}^2_K$ into the surfaces $\mathbb{S}^2_{f,g,K}=\mathrm{Spec}\,\Big(K[x,y,z]/\big(xf(x)+yg(y)z\big)\Big)$ of Section 2.

Proposition 13.1. *Let* $(f,g)\in\Theta_K$ *with* $\deg(f)\geq 1$ *and* $g(0)\neq 0$. *Let* $\delta\in K^*$ *be such that* $f\big(\delta g(0)\big)=0$. *Let* $h=h(x,y):=\delta+xy\in K[x,y]$. *Then the rule on valued points*

$$(x,y)\mapsto\Big(hg(yh),yh,\frac{-f\big(hg(yh)\big)}{y}\Big)$$

defines an open embedding $\jmath_{f,g,K;\delta}:\mathbb{A}^2_K\to\mathbb{S}_{f,g,K}$ *with the property that* $\mathbb{S}_{f,g,K}\setminus\mathrm{Im}(\jmath_{f,g,K;\delta})$ *is the disjoint union of* $\deg(f)\mathrm{z}(tg)$ *curves isomorphic to* $\mathbb{A}^1_K$.

Proof. As $hg(yh)\in\delta g(0)+yK[x,y]$ and $f\big(\delta g(0)\big)=0$, y divides $-f\big(hg(yh)\big)$ and hence the rule is well defined. By computing $-f\big(hg(yh)\big)$ modulo y^2 we get

$$\frac{-f\big(hg(yh)\big)}{y}\in -f'\big(\delta g(0)\big)g(0)x-\delta^2f'\big(\delta g(0)\big)g'(0)+yK[x,y]. \tag{8}$$

As y (resp. h) are rational functions in the first and third (resp. second) coordinates of the rule and x is a rational function in y and yh, $\jmath_{f,g,K;\delta}$ is birational. To show that it is an open embedding and to identify its complement, for $(\alpha,\beta,\gamma)\in\mathbb{S}^2_{f,g,K}(K)$, i.e., for $(\alpha,\beta,\gamma)\in K^3$ with $\alpha f(\alpha)+\beta g(\beta)\gamma=0$, we solve in K^2 the system of two equations

$$hg(yh)-\alpha=yh-\beta=\frac{-f\big(hg(yh)\big)}{y}-\gamma=0$$

in the indeterminates x and y by considering three cases as follows.

Case 1: $\alpha=0$ **(so** $\beta g(\beta)\gamma=0$**).** This implies that $y\neq 0$ and therefore $\frac{-f\big(hg(yh)\big)}{y}=\frac{-f(0)}{y}\in K^*y^{-1}$. So the system is inconsistent if $\gamma=0$. We now assume that $\gamma\neq 0$. Thus $y=\frac{-f(0)}{\gamma}$ and, as $y(\delta+xy)=\beta$, we get that $x=\frac{\gamma[\beta\gamma+\delta f(0)]}{f(0)^2}$. As $hg(yh)=\frac{-\beta g(\beta)\gamma}{f(0)}=0=\alpha$, $\big(\frac{\gamma[\beta\gamma+\delta f(0)]}{f(0)^2},\frac{-f(0)}{\gamma}\big)$ is the only solution when $\gamma\neq 0$.

Case 2: $\alpha\neq 0$ **and** $g(\beta)=0$ **(hence** $f(\alpha)=0$**).** As $\alpha\neq hg(\beta)=0$, the system is inconsistent.

Case 3: $\alpha g(\beta)\neq 0$. By replacing $yh=\beta$ in $hg(yh)=\alpha$ we get that $h=\frac{\alpha}{g(\beta)}\neq 0$, so $y=\frac{\beta g(\beta)}{\alpha}$ and $xy=h-\delta=\frac{\alpha-\delta g(\beta)}{g(\beta)}$ and we consider two subcases as follows.

Subcase 3.1: $\beta=0$ **(hence** $f(\alpha)=0$**).** The second equation gives $y=0$. Thus, if α is a zero of f different from $h(x,0)g(0)=\delta g(0)$, then the system is inconsistent. If $\alpha=\delta g(0)$, then the first equation and the equation $xy=\frac{\alpha-\delta g(\beta)}{g(\beta)}=0$ hold. As

$y = 0$, the third equation becomes $-f'\big(\delta g(0)\big)g(0)x - \delta^2 f'\big(\delta g(0)\big)g'(0) = \gamma$ by Relation (8), so the only solution is $\left(\frac{-\gamma-\delta^2 f'\left(\delta g(0)\right)g'(0)}{f'\left(\delta g(0)\right)g(0)}, 0\right)$.

Subcase 3.2: $\beta \neq 0$. Thus $y \neq 0$ and therefore $x = \frac{\alpha[\alpha-\delta g(\beta)]}{\beta[g(\beta)]^2}$. The pair $\big(\frac{\alpha[\alpha-\delta g(\beta)]}{\beta[g(\beta)]^2}, \frac{\beta g(\beta)}{\alpha}\big)$ is indeed the only solution as $\alpha f(\alpha) + \beta g(\beta)\gamma = 0$ implies that the third equation $\frac{-f\left(hg(yh)\right)}{y} - \gamma = 0$ also holds.

These cases imply that the birational morphism $\jmath_{f,g,K;\delta}$ is also radicial, hence it is an open embedding. The complement $\mathbb{S}_{f,g,K} \setminus \mathrm{Im}(\jmath_{f,g,K;\delta})$ is the disjoint union of curves isomorphic to $\mathbb{A}^1_K$ and given by equations $x = z = 0$ (see Case 1) and equations $x - \alpha = y - \beta$ with $(\alpha, \beta) \in K^2$ satisfying $f(\alpha) = g(\beta) = 0$ (see Case 2) and equations $f(\alpha) = \beta = 0$ with $\alpha \neq \delta g(0)$ (see Subcase 3.1), their number being $1 + \deg(f)\mathrm{z}(g) + [\deg(f) - 1] = \deg(f)[\mathrm{z}(g) + 1] = \deg(f)\mathrm{z}(tg)$. $\square$

The following concrete application of Proposition 13.1 proves Theorem 2.4(2).

Example 13.2. Let $(m, s) \in \mathbb{N}^* \times \mathbb{N}$. Let $\alpha_0 := 0 \in k$ and $\alpha_1 := 1 \in k$. If $s \geq 2$, let $a_2, \ldots, a_s$ be distinct elements of $k^* \setminus \{1\}$. If $s = 0$ let $g(t) := 1$ and if $s \geq 1$ let $(j_1, \ldots, j_s) \in (\mathbb{N}^*)^s$ and $g(t) := \prod_{i=1}^s (t - \alpha_i)^{j_i} \in k[t]$. So g is monic with $g(0) \neq 0$, $\mathrm{z}(g) = s$ and $\deg(g) = \sum_{i=1}^s j_i \geq s$. Let $(f, g) \in \Theta^1_{k,m}$ with $\deg(f) \geq 1$; e.g., we can take $f(t) = t^{pm-1} - 1$. The smooth surface $\mathbb{S}^2_{f,g,k}$ is equipped with the finite étale projection $c_{f,g,k} : \mathbb{S}^2_{f,g,k} \to \mathbb{A}^2_{k,\mathrm{t}}$ on the last two coordinates (see Section 2). Let $\delta \in k^*$ be such that $f\big(\delta g(0)\big) = 0$. For $h = h(x, y) := \delta + xy \in k_{\mathrm{s}}[x, y]$, the degree of $\frac{-f\left(hg(yh)\right)}{y} \in k_{\mathrm{s}}[x, y]$ is $\big(3\deg(g) + 2\big)(pm - 1) - 1 \geq (3s + 2)(pm - 1) - 1$ with equality iff for $s \geq 1$ we have $j_1 = \cdots = j_s = 1$. Let (see Proposition 13.1)

$$e := c_{f,g,k} \circ \jmath_{f,g,k;\delta} \in \mathrm{End}_2(k).$$

As e is the composite of an open embedding with a finite étale morphism of degree pm, we have $e \in \mathrm{EE}_2(k)$ and $\deg(e) = pm$. Clearly, $X_e = \mathbb{S}^2_{f,g,k}$ is a finite étale cover of $\mathbb{A}^2_{k,\mathrm{t}}$, so ψ_e is étale. The last part of the proof of Proposition 13.1 gives

$$\Gamma_{\psi_e}(k) = \{(0, \beta, 0) | \beta \in k\} \cup \{(\alpha, \beta, \gamma) \in k^3 | f(\alpha) = \beta g(\beta) = 0, (\alpha, \beta) \neq \big(\delta g(0), 0\big)\},$$

$$\Gamma_e = \{(\beta, \gamma) \in k^2 | \nexists\, \alpha \in k \text{ with } (\alpha, \beta, \gamma) \in [\mathbb{S}^2_{f,g,k} \cap \mathrm{Im}(\jmath_{f,g,k;\delta})](k)\}$$
$$= \{(\beta, 0) \in k^2 | g(\beta) = 0\}.$$

Thus $\rho_{\text{ét}}(e) = \rho(e) = \deg(f)\mathrm{z}(tg) = (pm - 1)(s + 1)$ and $\varphi(e) = s$.

For $(\beta, \gamma) \in k^2$, from the three cases of the proof of Proposition 13.1 we get that $|e^{-1}(\beta, \gamma)|$ is 1 if either $g(\beta) = 0 \neq \gamma$ or $\beta = \gamma = 0$, is 2 if $\beta = 0 \neq \gamma$, is $pm - 1$ if $\beta g(\beta) \neq 0 = \gamma$, and is pm otherwise; so for $pm > 2$ we have $\mathcal{F}(e) = \{1, 2, pm - 1\}$ if $s = 0$ and $\mathcal{F}(e) = \{0, 1, 2, pm - 1\}$ if $s \geq 1$ and for $pm = 2$ we have $\mathcal{F}(e) = \{1\}$ if $s = 0$ and $\mathcal{F}(e) = \{0, 1\}$ if $s \geq 1$. If $\deg(f) = 1$, then $N_e = c_{f,g,k}(\Gamma_{\psi_e})$ is the zero locus $g(x_1)x_2 = 0$, hence $\nu(e) = s + 1$ and $\Gamma_e = \mathrm{Sing}(N_e)$. If $\deg(f) \geq 2$, then $N_e = c_{f,g,k}(\Gamma_{\psi_e})$ is the zero locus $x_1 g(x_1)x_2 = 0$, hence $\nu(e) = s + 2$ and $\Gamma_e \subsetneq \mathrm{Sing}(N_e)$. As the conditions (i) and (ii) of Proposition 4.3 hold for the pair $\left(yh, \frac{-f\left(hg(yh)\right)}{y}\right)$ that defines e, Proposition 4.3(1) and (2) gives

$$\pi(e) \leq \pi_{\mathrm{l}}(e) = \pi_{\mathrm{r}}(e) = (3\deg(g) + 2)(pm - 1) - 1 \geq (3s + 2)(pm - 1) - 1,$$

and the last inequality is an equality iff for $s \geq 1$ we have $j_1 = \cdots = j_s = 1$.

Let $l \in \mathbb{N}^*$ with $p^l \geq s+1$. If we take $f(t) = t^{pm-1} - 1$, $\alpha_2, \ldots, \alpha_s \in \mathbb{F}_{p^l} \setminus \{0, 1\}$, and $\delta = g(0)^{-1}$, then e is defined over $\mathbb{F}_{p^l}$. Thus Theorem 2.4(2) holds.

Fact 13.3. *Let $(f, g) \in \Theta_K$ with $\deg(f) \geq 1$, $g(0) \neq 0$ and $\deg(f)+\deg(g) \geq 2$. Let $\delta \in K^*$ be such that $f\big(\delta g(0)\big) = 0$. Then the two open embeddings $\imath_{f,g,K}$ and $\jmath_{f,g,K;\delta}$ are not isomorphic and we have an identity $\mathrm{Im}(\imath_{f,g,K}) \setminus Y_1 = \mathrm{Im}(\jmath_{f,g,K;\delta}) \setminus Y_2$, where Y_1 and Y_2 are the zero loci $x = z = 0$ and $x - \delta g(0) = y = 0$ (respectively) in $\mathbb{S}^2_{f,g,K}$.*

Proof. The identity follows from the descriptions of the complements of the images of the two open embeddings (see Section 2 and proof of Proposition 13.1). We have $\deg(f) \geq 1$ and there exists a unique monic polynomial $f_1(t) \in K[t]$ such that $f(t) = \big(t - \delta g(0)\big) f_1(t)$. We write $g(t) = g(0) + \sum_{i=1}^{\deg(g)} \alpha_i t^i$ with $\alpha_{\deg(g)} = 1$ and, if $\deg(g) \geq 2$, $(\alpha_1, \ldots, \alpha_{\deg(g)-1}) \in k^{\deg(g)-1}$. Thus

$$h_1(x,y) := \frac{h(x,y) g\big(yh(x,y)\big) - \delta g(0)}{y} = \tag{9}$$

$$x\big[g\big(yh(x,y)\big) + \delta \sum_{i=1}^{\deg(g)} \alpha_i [h(x,y) - \delta]^{i-1} h(x,y)^i\big] \in K[x,y].$$

The open embeddings $\imath_{f,g,K} \to \mathbb{S}^2_{f,g,K}$ and $\jmath_{f,g,K;\delta} \to \mathbb{S}^2_{f,g,K}$ are given by the rules $(x,y) \mapsto \big(xyg(y), y, -xf(xyg(y))\big)$ and $(x,y) \mapsto \big(hg(yh), yh, -h_1(x,y) f_1(hg(yh))\big)$ with $h = \delta + xy$ (respectively). To show that these two open embeddings are not isomorphic it suffices to show that exists no $a \in \mathrm{GA}_2(K)$ such that $a^{\#}(B_1) = B_2$, where B_1 and B_2 are the K-subalgebras $K_{\mathrm{s}}[x_1 x_2 g(x_2), x_2, x_1 f\big(x_1 x_2 g(x_2)\big)]$ and

$$K_{\mathrm{s}}\big[h(x_1,x_2) g\big(x_2 h(x_1,x_2)\big), x_2 h(x_1,x_2), h_1(x_1,x_2) f_1\big(h(x_1,x_2) g(x_2 h(x_1,x_2))\big)\big]$$

(respectively) of $K_{\mathrm{s}}[x_1, x_2]$ that define $\imath_{f,1,K}$ and $\jmath_{f,1,K;1}$ (respectively).

That no such a exists follows from the fact that there exists an element $y_1 \in B_1$ that can be extended to $\{y_1, y_2\} \in \mathcal{A}_2(K)$ (e.g., $y_1 = x_2$) but no such element exists in B_2. To check this we can enlarge the K-subalgebra B_2 of $K_{\mathrm{s}}[x_1, x_2]$ and hence, based on the definition of B_2 and Equation (9) we can assume that either $g = 1$ or $f_1 = 1$. As the case $f_1 = 1$ is similar to the case $g = 1$, in what follows we assume that $g = 1$. Thus $B_2 = K_{\mathrm{s}}[\delta + x_1 x_2, -x_1 f_1(\delta + x_1 x_2), x_2(\delta + x_1 x_2)]$ and $\deg(f_1) \geq 1$. Note that $\{1\} \cup \{\big(-x_1 f_1(\delta + x_1 x_2)\big)^i, \big(-x_2(1 + x_1 x_2)\big)^i | i \in \mathbb{N}^*\}$ is a basis of the $K[\delta + x_1 x_2]$-module B_2. Thus for each irreducible polynomial $z_1 \in B_2 \setminus K$, the zero locus $z_1 = 0$ is an irreducible closed curve C_1 of $\mathbb{A}^2_{K,\mathrm{s}}$ which is not isomorphic to $\mathbb{A}^1_K$ as the first or second projection $C_1 \to \mathbb{A}^1_K$ is a quasi-finite non-finite morphism as z_1 is a non-monic polynomial in either x_1 or x_2. Thus there exists no $z_2 \in K_{\mathrm{s}}[x_1, x_2]$ such that $\{z_1, z_2\} \in \mathcal{A}_2(K)$, hence indeed $B_1 \not\cong B_2$. □

Example 13.4. The normal and general analogues of Θ_R over a field $\mathcal{K}$ are

$\Theta^{\mathrm{n}}_{\mathcal{K}} := (\mathbb{N} \setminus \{0,1\}) \times \{1\} \times \{(f,g) \in (\mathcal{K}[t])^2 | f$ are g are monic separable, $f(0)g(0) \neq 0\}$ and $\Theta^{\mathrm{gen}}_{\mathcal{K}} := (\mathbb{N} \setminus \{0,1\}) \times \mathbb{N}^* \times \{(f,g) \in \Theta_{\mathcal{K}} | g(0) \neq 0\} \supset \Theta^{\mathrm{n}}_{\mathcal{K}}$. For $(q,l,f,g) \in \Theta^{\mathrm{gen}}_K$ the affine surface

$$\mathbb{S}^2_{q,l,f,g,K} := \mathrm{Spec}\,\big(K[x,y,z]/(x^q f(x) + y^l g(y) z)\big)$$

is normal iff $(q,l,f,g) \in \Theta^{\mathrm{n}}_{\mathcal{K}}$; similarly, $(0,0,0)$ is a normal point of it iff $l = 1$ and iff $(0,0,0)$ is a normal singular point of it. If $(q,1,f,g) \in \Theta^{\mathrm{n}}_K$, then $\mathrm{Sing}(\mathbb{S}^2_{q,1,f,g,K}) = \{(0,\beta,0) | \beta \in K, g(\beta) = 0\}$. The open embeddings $\imath_{f,g,K}$ or $\jmath_{f,g,K;\delta}$ have analogues for the general context in many cases. To exemplify this, let $(q,1,f,g) \in \Theta_K^{\mathrm{gen}}$ with $\deg(f) \geq 1$ and $g = g_1^q$ with $g_1(t) \in K[t]$ monic. Let $\delta \in K^*$ and $f_1(t) \in K[t]$ be such that $f(t) = \big(t - \delta g_1(0)\big) f_1(t)$. Let $\jmath_{q,1,f,g,K;\delta} : \mathbb{A}_K^2 \to \mathbb{S}_{q,1,f,g,K}^2$ be given by the rule $(x,y) \mapsto \big(hg_1(yh^q), yh^q, \frac{-f(hg_1(yh^q))}{y}\big)$, where $h := \delta + xy$. The proof of Proposition 13.1 applies to give that $\jmath_{q,1,f,g,K;\delta}$ is a birational quasi-finite morphism and $\mathbb{S}_{q,1,f,g,K}^2 \setminus \mathrm{Im}(\jmath_{q,f,g,1,K;\delta})$ is the disjoint union of $\deg(f)\mathrm{z}(tg_1)$ closed subvarieties isomorphic to $\mathbb{A}_K^1$, the zero locus $x = z = 0$ being the one that contains the normal singular point $(0,0,0)$. The birational morphism $\jmath_{q,1,f,g,K;\delta}$ is an open embedding (i.e., is injective) iff either $(q,1,f,g) \in \Theta_K^{\mathrm{n}}$ (i.e., $g = g_1 = 1$) or $q = \mathrm{char}(K)^m$ with $m \in \mathbb{N}^*$. If $f_1(t) = g(t) = 1$, so $(q,1,f,g) \in \Theta_K^{\mathrm{n}}$, then the rule $(x,y) \mapsto (h, y, -xh^q)$ defines an open embedding $\imath_{q,1,t-\delta,1,K} : \mathbb{A}_K^2 \to \mathbb{S}_{q,1,t-\delta,1,K}^2$ and we have $\imath_{q,1,t-\delta,1,K} \circ \mathrm{Inv}_{\mathbb{A}_K^2} = \mathrm{Inv}_{\mathbb{S}_{q,1,t-\delta,1,K}^2} \circ \jmath_{q,1,t-\delta,1,K;\delta}$, where $\mathrm{Inv}_{\mathbb{A}_K^2}$ and $\mathrm{Inv}_{\mathbb{S}_{q,1,t-\delta,1,K}^2}$ are the involutions of $\mathbb{A}_K^2$ and $\mathbb{S}_{q,1,t-\delta,1,K}^2$ (respectively) defined by the rules $(x,y) \mapsto (y,x)$ and $(x,y,z) \mapsto (x,-z,-y)$ (respectively).

Example 13.5. Let $\Sigma : \mathbb{S}_{q,1,t-1,1;K} \to \mathbb{P}_K^1$ be the morphism that maps $(\alpha,\beta,\gamma) \in \mathbb{S}_{q,1,t-1,1;K}(K)$ to either $\big(1-\alpha : \beta\big)$ or $(\gamma : \alpha^q)$ (whichever is defined; if both are defined, they are equal). The multiplicity of $\big(\Sigma^{-1}(1:0)\big)_{\mathrm{red}}$ in $\Sigma^{-1}(1:0)$ is q and Σ is a surjective $\mathbb{A}^1$-fibration as in Corollary 11.5. So $\mathbb{S}_{q,1,t-1,1;K}$ is a sphere-like surface with $|\mathrm{Sing}(\mathbb{S}_{q,1,t-1,1;K})| = 1$ by Example 13.4; note that $(q,1,t-1,1) \in \Theta_K^{\mathrm{n}}$.

Proposition 13.6. *Let $m \in \mathbb{N}^*$. Let B_m be the k-subalgebra of $k_{\mathrm{s}}[x_1,x_2]$ generated by x_1 and $x_2^i(1+x_1x_2)^{i+1}$ with $i \in [[0,m]]$. Then the following properties hold.*

(1) *There exists no pair $(g_1,g_2) \in B_m^2$ such that $\mathrm{e}(g_1,g_2) \in \mathrm{EE}_2(k)$.*

(2) *Let $(q,1,f,g_1^q) \in \Theta_k^{\mathrm{gen}}$ with $\deg(f) \geq 1$ and $g_1(t) \in k[t]$ monic. Then there exists no $e \in \mathrm{EE}_2(k)$ such that ι_e factors through the normalization of $\jmath_{q,1,f,g_1^q,k;\delta}$.*

Proof. We have a chain of inclusions $B_1 \subset \cdots \subset B_m \subset \cdots$. Let $h := 1 + x_1x_2$ and we consider the k-linear derivations $h^2D_{x_1}$ and D_{x_2} of $k_{\mathrm{s}}[x_1,x_2]$ and its k-subalgebra $B_\infty := \cup_{i\geq 1} B_i$. Let $D := h^2D_{x_1} + D_{x_2}$. One computes that $D(x_1) = h^2$ and $D(h) = x_1 + x_2h^2$. For $i \in \mathbb{N}^* \setminus \{1\}$ we have

$$D\big(x_2^{i-1}h^i\big) = ix_2^ih^{i+1} + (2i-1)x_2^{i-2}h^i - ix_2^{i-2}h^{i-1}. \tag{10}$$

It follows that D is a k-linear derivation of B_m iff $p|m+1$.

We consider the normal affine surface

$$\mathbb{T}_m^2 := \mathrm{Spec}\,\big(K[y_0,\ldots,y_{m+1}]/(y_1^3 - y_1^2 + y_0y_2, y_i^2 - y_{i-1}y_{i+1}|i \in [[2,m]])\big);$$

it is a complete intersection in $\mathbb{A}_K^{m+2}$ with $\mathrm{Sing}(\mathbb{T}_m^2) = \{(0,\ldots,0)\}$. The rule $(x,y) \mapsto (-x, h, yh^2, y^2h^3, \ldots, y^mh^{m+1})$ defines a morphism $\imath_{m,K} : \mathbb{A}_K^2 \to \mathbb{T}_m^2$ which is an open embedding as already the rule $(-x,h,yh^2)$ defines an open embedding by Example 13.4 applied to (z,x,y) instead of (x,y,z). For a point $P = (\alpha_0,\ldots,\alpha_{m+1}) \in \mathbb{T}_m^2(K)$, if $\alpha_0 = \alpha_1 = 0$, then we have $\alpha_i = 0$ for all $i \in [[0,m]]$; hence P is the singular point iff $\alpha_0 = \alpha_1 = \alpha_{m+1} = 0$.

To prove part (1) we can assume that $p|m+1$; hence D is a k-linear derivation of B_m. Let T_D be the global section of the tangent bundle $T_{\mathrm{Reg}(\mathbb{T}_m^2)}$ defined by D. We show that the assumption that T_D is zero at some point $Q = (\beta_0,\ldots,\beta_{m+1}) \neq (0,\ldots,0)$ leads to a contradiction. Based on the formulas for $D(x_1)$ and $D(h)$, this assumption implies that $\beta_1^2 = \beta_0 + \beta_2 = 0$ and thus, as we also have $\beta_1^3 - \beta_1^2 = \beta_0\beta_2$, we get that $\beta_0 = \beta_1 = \beta_2 = 0$ and therefore $m > 1$ and we have $\beta_i = 0$ for all

$i \in [[0,m]]$. From Equation (10) applied with $i = m-1 \geq 1$, as $\beta_{m-2} = \beta_{m-1} = 0$, we get that $m\beta_{m+1} = 0$. As $p|m+1$, it follows that $\beta_{m+1} = 0$, so Q is the singular point, a contradiction. Thus T_D is nowhere zero and it generates a direct summand of $T_{\mathrm{Reg}(\mathbb{T}^2_m)}$ which is free of rank 1. Based on this, as moreover the conormal bundle of the complete intersection closed embedding $\mathbb{T}^2_m \to \mathbb{A}^{m+2}_K$ is trivial and the tangent bundle $T_{\mathbb{A}^{m+2}_K}$ is trivial, it follows that $T_{\mathrm{Reg}(\mathbb{T}^2_m)}$ itself is trivial.

We show that the assumption that there exists a pair $(g_1, g_2) \in B^2_m$ such that $\mathrm{e}(g_1, g_2) \in \mathrm{EE}_2(K)$ leads to a contradiction. This assumption implies that there exists a morphism $\psi : \mathbb{T}^2_m \to \mathbb{A}^2_K$ whose restriction to $\mathrm{Im}(\imath_{m,K})$ is étale. Let $\psi_0 := \psi|\mathrm{Reg}(\mathbb{T}^2_m)/$ As in the proof of the implication $(iv) \Rightarrow (i)$ of Theorem 7.3(1) we argue that the functorial morphism $T_{\mathbb{T}^2_m} \to \psi_0^*(T_{\mathbb{A}^2_K})$ is an isomorphism, hence ψ_0 is étale. By the purity of the branch locus (see [SGA2], Exp. X, Thm. 3.4 (i)) we get that ψ itself is étale, which contradicts the identity $|\mathrm{Sing}(\mathbb{T}^2_m)| = 1$.

Part (2) follows from part (1) applied to $m = 1$ once we remark that up to isomorphism we can assume that $\delta = 1$ and that the k-subalgebra $B_{q,1,f,g_1^q,K}$ of $k[x,y]$ generated by $hg_1(yh^q)$, yh^q and $\frac{-f(hg_1(yh^q))}{y}$ is contained in $B_{2,1,t-1,1,k}$ which, under the identification $(x,y) = (x_1,x_2)$, is B_2. □

14. Proof of Theorem 2.4(3)

Theorem 2.4(3) follows from the following example.

Example 14.1. Let $m, q \in \mathbb{N}^*$ be such that p divides mq and let $e \in \mathrm{EE}_2(k)$ be defined over $\mathbb{F}_p$ by the rule on valued points

$$(x,y) \mapsto \big(x + (1 - x^m y^{mq-1})^q, y - x^m y^{mq}\big).$$

For $(\alpha,\beta) \in k^2$ let $\mathrm{s}(\alpha,\beta) \in \mathbb{N}$ be such that $e^{-1}(\alpha,\beta) \cong \mathrm{Spec}\,\big(k^{\mathrm{s}(\alpha,\beta)}\big)$. So $\mathrm{s}(\alpha,\beta)$ is the number of solutions in k^2 of the system $\mathcal{J}_{\alpha,\beta}$ of equations

$$x + (1 - x^m y^{mq-1})^q - \alpha = 0 = y - x^m y^{mq} - \beta. \tag{11}$$

Let $z := 1 - x^m y^{mq-1}$. Then $\mathcal{J}_{\alpha,\beta}$ can be rewritten as

$$x + z^q = \alpha \ \text{ and } \ yz = \beta. \tag{12}$$

To compute every $\mathrm{s}(\alpha,\beta)$, we first consider the case when $\beta \neq 0$. Thus $y \neq 0$ and $z = \beta y^{-1}$. From this and the equation $x + z^q = \alpha$ we get that $x = \alpha - \beta^q y^{-q}$. Thus $xy^q = \alpha y^q - \beta^q$. Hence $\mathcal{J}_{\alpha,\beta}$ is equivalent to the system of equations

$$x = \alpha - \beta^q y^{-q} \ \text{ and } \ y - \beta = (xy^q)^m = (\alpha y^q - \beta^q)^m.$$

If $\alpha \neq 0$, then $p \mid mq$ implies that the equation $y - \beta = (\alpha y^q - \beta^q)^m$ in y has mq distinct solutions in k, 0 being a solution of it iff β belongs to the set $\mathbb{I}^*_{p,m,q}$, where

$$\mathbb{I}_{p,m,q} := \{x \in k | x = (-1)^{m-1} x^{mq}\} \ \text{ and } \ \mathbb{I}^*_{p,m,q} := \mathbb{I}_{p,m,q} \cap k^* = \mathbb{I}_{p,m,q} \setminus \{0\}.$$

Thus for $\beta \notin \mathbb{I}_{p,m,q}$ and $\alpha \neq 0$, we have $\mathrm{s}(\alpha,\beta) = mq$, which implies that $\deg(e) = mq$, and for $\beta \in \mathbb{I}^*_{p,m,q}$ and $\alpha \neq 0$ we have $\mathrm{s}(\alpha,\beta) = mq - 1$. Similarly, if $\beta \notin \mathbb{I}_{p,m,q}$, we have $\mathrm{s}(0,\beta) = 1$, the only solution being

$$\big(-[1 + (-1)^m \beta^{mq-1}]^{-q}, \beta + (-1)^m \beta^{mq}\big),$$

and if $\beta \in \mathbb{I}^*_{p,m,q}$ we have $\mathrm{s}(0,\beta) = 0$.

We now consider the case $\beta = 0$. Thus $yz = 0$. If $y = 0$, then $z = 1$ and we get one solution $(\alpha - 1, 0)$ of $\mathcal{J}_{\alpha,0}$ (see System (11)). If $z = 0$, then $x = \alpha$ (see System

(12)) and hence $z = 0 = 1 - \alpha^m y^{mq-1}$ and we get no solutions if $\alpha = 0$ and $mq - 1$ solutions of the form $(\alpha, \sqrt[mq-1]{\alpha^{-m}})$ if $\alpha \neq 0$, all of them being different from the prior solution $(\alpha - 1, 0)$. We conclude that $\mathrm{s}(0, 0) = 1$ and $\mathrm{s}(\alpha, 0) = mq$ if $\alpha \neq 0$.

From the last two paragraphs we get first that $\mathcal{F}(e) = \{0, 1, mq - 1\}$ and second that $\Gamma_e = \{(0, \beta) | \beta \in \mathbb{I}^*_{p,m,q}\}$, hence $\varphi(e) = mq - 1$. Moreover, N_e is the zero locus $x_1 \prod_{\beta \in \mathbb{I}^*_{p,m,q}} (x_2 - \beta) = 0$, hence $\nu(e) = 1 + |\mathbb{I}^*_{p,m,q}| = mq$ and $\Gamma_e = \mathrm{Sing}(N_e)$.

For $\beta \in \mathbb{I}^*_{p,m,q}$ we have $e(0, \beta) = (1, \beta)$ and $\mathrm{s}(1, \beta) = mq - 1$, and therefore $|e^{-1}(1, \beta) \setminus \Gamma_e| = mq - 2$. Thus, for $mq > 2$ we have $|\mathcal{E}_2(e)| = 0$ and from Lemma 3.2(3) we get that $\mathrm{Im}(e) = \mathrm{Im}(e^2)$; so $\iota(e) = 1$. Similarly, if $mq = 2$ (so $p = 2$), then $\mathbb{I}_{2,m,q} = \mathbb{F}_2$, $\mathrm{Im}(e) \setminus \mathrm{Im}(e^2) = \{(1, 1)\}$ and thus $\varphi(e^2) = 2$; as $e(1, 1) = (1, 0)$ and $\mathrm{s}(1, 0) = 2$, we get $\mathcal{E}_2(e) = \mathcal{E}_1(e)$, so $\varphi(e^3) = \varphi(e^2)$ by Lemma 3.2(3) and $\iota(e) = 2$.

Clearly, if p divides both m and q, then $e \in \mathrm{BE}_2(k)$. Moreover, if $q = 1$ we have $\pi_{\mathrm{i}}(e) = \pi_{\mathrm{l}}(e) = \pi_{\mathrm{r}}(e) = 2m$ and if $q \geq 2$ we have $mq + m \leq \pi_{\mathrm{i}}(e) \leq \pi_{\mathrm{l}}(e) = \pi_r(e) = mq + (mq - 1)q$ by Proposition 4.3(1) to (4). For $r := mq \in p\mathbb{N}^*$ fixed, the smallest possible value of $\pi_{\mathrm{l}}(e) = \pi_{\mathrm{r}}(e)$ is $2r = 2mq$ obtained only for $(m, q) = (r, 1)$.

Recall $X_e = \mathrm{Spec}\, A_e$; we have inclusions $k_{\mathrm{t}}[x_1, x_2] \subset A_e \subset k_{\mathrm{s}}[x_1, x_2] = k[x, y]$ defined by e, with $x_1 = x + z^q$, $x_2 = yz$, and $z = 1 - x^m y^{mq-1}$. It follows that e is the composite of birational quasi-finite morphisms $\mathbb{A}^2_{k,\mathrm{s}} \to X \to Y$ with a finite flat morphism $Y \to \mathbb{A}^2_{k,\mathrm{t}}$ of degree mq, where $X := \mathrm{Spec}\,\big(k_{\mathrm{t}}[x_1, x_2][z]/(f(z))\big)$ with

$$f(z) := z^{mq} - z^{mq-1} + (x_1 - z^q)^m x_2^{mq-1}$$

and where $Y := \mathrm{Spec}\,\big(k_{\mathrm{t}}[x_1, x_2][w]/(g(w)\big)$ is a flat model of X_e with

$$g(w) := w^{mq} - w^{mq-1} + \sum_{i=1}^{m} (-1)^{m-i} \binom{m}{i} x_1^i x_2^{mq-1} \big(1 + (-1)^m x_2^{mq-1}\big)^{qi-1} w^{q(m-i)}$$

in $k_{\mathrm{t}}[x_1, x_2][w]$ obtained from $\big(1 + (-1)^m x_2^{mq-1}\big)^{mq-1} f(z)$ through the substitution $w := z\big(1 + (-1)^m x_2^{mq-1}\big)$. Thus $w \in A_e$. If $mq > 2$, then for $\alpha \in k^*$ the $mq - 1$ points of $\mathbb{A}^2_{k,\mathrm{s}}$ of the form $(\alpha, \sqrt[mq-1]{\alpha^{-m}})$ map to the same point of X and therefore Y is not normal by Zariski's Main Theorem, i.e., $Y \neq X_e$. If $mq = 2$ (so $p = 2$) then $\mathbb{I}_{p,m,q}$ is $\mathbb{F}_2$ and $g(w)$ is $w^2 + w + x_1^2 x_2(x_2 + 1)$ if $(m, q) = (2, 1)$ and is $w^2 + w + x_1 x_2(x_2 + 1)$ if $(m, q) = (1, 2)$; hence $Y = X_e \to \mathbb{A}^2_{k,\mathrm{t}}$ is a finite Galois cover, $A_e \cong k_{\mathrm{t}}[x_1, x_2][w]/\big(g(w)\big)$, and the complement of the open embedding $\mathbb{A}^2_{k,\mathrm{s}} \to Y$ consists of 2 disjoint $\mathbb{A}^1_k$ curves given by the equations $w = x_2 = 1$ and $w = x_1 = 0$ implying that $\rho_{\text{ét}}(e) = \rho(e) = 2$. For $(m, q) = (1, 2)$, e is $e_{t+1,t+1,k}$ of Section 2.

Let $\epsilon \in \{1, 2\}$ be 1 iff $q = 1$. Similarly, e factors through a finite flat morphism $e_Z : Z \to \mathbb{A}^2_{k,\mathrm{t}}$, where $Z := \mathrm{Spec}\,\big(k_{\mathrm{t}}[x_1, x_2][v]/(h(v))\big)$ is a flat model of X_e with

$$h(v) = v^{mq} + \sum_{i=1}^{m-\epsilon} h_{mq-iq} v^{mq-iq} + h_1 v + h_0 \in k_{\mathrm{t}}[x_1, x_2][v]$$

obtained from $x_1^{m(q+\epsilon-2)} x_2 z^{-mq} f(z)$ via the substitution $v := x_1^{\epsilon} x_2 z^{-1}$; so each $h_j \in k_{\mathrm{t}}[x_1, x_2]$ is uniquely determined by m, q, the index j and $v \in A_e$. E.g., $h_0(x_1, x_2) = x_1^{m(q+\epsilon-2)} x_2$ and $h_1(x_1, x_2) = -x_1^{m(q+\epsilon-2)-\epsilon}$; also, if $q \geq 2$, then

$$h_{mq-iq}(x_1, x_2) = x_1^{mq-m} x_2 (-1)^i \binom{m}{i} x_1^{m-i} x_2^{mq-1} (vz)^{qi-mq} = (-1)^i \binom{m}{i} x_1^{i(q-1)} x_2^{qi}.$$

As $h_1(x_1,x_2)=-x_1^{m(q+\epsilon-2)-\epsilon}$ becomes -1 when $mq=2$ (so $p=2$), it follows that for $mq=2$ the morphism e_Z is a finite Galois cover and thus $X_e=Z$, and for $mq>2$ the étale locus of e_Z is where x_1 is invertible and therefore the $k_{\mathrm{t}}[x_1,x_2]_{x_1}$-algebra $A_{e,x_1}\cong k_{\mathrm{t}}[x_1,x_2]_{x_1}[v]/(h(v))$ is étale.

We show that A_e is always regular and that ψ_e is non-étale iff $mq>2$. It suffices to show that for each $\beta\in k$, by denoting $R:=k_{\mathrm{t}}[[x_1,x_2-\beta]]$ the completion of $k_{\mathrm{t}}[x_1,x_2]$ at the point $(0,\beta)$, the normalization A_β^{n} of the R-algebra $A_\beta:=R[w]/\big(g(w)\big)$ is isomorphic to the product $R\times R_{mq-1}$, where $R_{mq-1}:=k[[x_1^{\frac{1}{mq-1}},x_2]]$ is regular. We write $g(w)=w^{mq}-w^{mq-1}+\sum_{i=0}^{\frac{mq}{p}-1}u_ix_1^{l_i}(x_2-\beta)^{m_i}w^{pi}$ with each $u_i\in\{0\}\cup R^*$ and with $(l_i,m_i)\in\mathbb{N}^2$; this is possible as $\binom{m}{i}w^{q(m-i)}$ is 0 when p does not divide qi as in this case p divides both m and $\binom{m}{i}$. One computes $l_0=m$ and if $u_i\neq 0$ then q divides pi and l_i is the solution of the equation $q(m-x)=pi$ in x, hence $l_i=m-\frac{pi}{q}>l_0\frac{mq-1-pi}{mq-1}$. If $\beta\big(1+(-1)^m\beta^{mq-1}\big)\neq 0$ then $m_0=0$ and if $u_i\neq 0$ the inequality $m_i\geq\frac{mq-1-pi}{mq-1}m_0$ holds. If $\big(1+(-1)^m\beta^{mq-1}\big)=0$ (resp. $\beta=0$) then $m_0=mq-1$ and if $u_i\neq 0$ then m_i is the value of $qx-1$ computed at the solution of the equation $q(m-x)=pi$ in x and hence $m_i=mq-1-pi=\frac{mq-1-pi}{mq-1}m_0$ (resp. $m_i=mq-1\geq\frac{mq-1-pi}{mq-1}m_0$). Thus $mq-1$ divides m_0 and the statement on A_β^{n} follows from Lemma 5.1(4) if $mq>2$ and from Lemma 5.1(2) if $mq=2$.

As for $(\alpha,\beta)\in k^*\times k$, $\mathrm{s}(\alpha,\beta)\in\{mq-1,mq\}$ is $mq-1$ iff $\beta\in\mathbb{I}_{p,m,q}^*$, $\operatorname{Spec}A_{e,x_1}\setminus\operatorname{Im}(\imath_e)$ is $mq-1=|\mathbb{I}_{p,m,q}^*|$ copies of $\mathbb{G}_{m,k}$; thus $\rho_{\text{ét}}(e)\geq mq-1$. From this, the fact that $\mathrm{s}(0,\beta)\in\{0,1\}$ is 0 iff $\beta\in\mathbb{I}_{p,m,q}^*$, and the above part on complete local rings it follows that $\rho(e)-\rho_{\text{ét}}(e)$ is 1 if $mq>2$ and is 0 if $mq=2$. Thus $0<mq-1=\rho_{\text{ét}}(e)<mq=\rho(e)$ if $mq>2$ and $\rho_{\text{ét}}(e)=\rho(e)=2$ if $mq=2$.

We say more in the following cases.

Case 1: $m=p$**.** As $\binom{p}{i}=0$ if $i\in[[1,p-1]]$, we compute $g(w)=w^{pq}-w^{pq-1}+x_1^px_2^{pq-1}(1-x_2^{pq-1})^{pq-1}$. If $q=1$ then $h(v)=v^p-x_1^{p-2}v+x_1^px_2(1-x_2^{p-1})$ and if $q\geq 2$ then $h(v)=v^{pq}-x_1^{pq-p-1}v+x_1^{pq-p}x_2(1-x_2^{pq-1})$.

Case 2: $m=1$**.** So $p|q$ and $\epsilon=1$. As in the prior paragraph we compute $g(w)=w^q-w^{q-1}+x_1x_2^{q-1}(1-x_2^{q-1})^{q-1}$ and $h(v):=v^q-x_1^{q-2}v+x_1^{q-1}x_2(1-x_2^{q-1})$.

14.1. **Proof of Corollary 2.6.** To prove Corollary 2.6 we can assume that $n=2$ as the passage from $n=2$ to an arbitrary $n>2$ is achieved by adding $n-2$ indeterminates $x_3,\ldots,x_n$ and equations $x_i=0$ for $i\in\{3,\ldots,n\}$. We apply Example 14.1 with $mq=p$; so $\deg(e)=p$ and $\mathbb{I}_{p,m,q}=\mathbb{F}_p$. For $n=2$, Corollary 2.6 follows from the identity $\mathrm{s}(0,0)=1$ (resp. $\mathrm{s}(0,1)=0$). Concretely, say $m=p$ and $q=1$, then we can take (g_1,g_2) to be $\big(x-(x-1)^py^{p-1},y-(x-1)^py^p\big)$ obtained by replacing x with $x-1$ (resp. $(1+x-x^py^{p-1},y-x^py^p)$). Similar examples can be obtained based on Example 13.2 by taking $(m,q,s)=(p,1,1)$.

15. Two general properties of endomorphisms of affine spaces

We include practical results on $e\in\operatorname{End}_n(K)$ that are often used in what follows and that either estimate $\deg(e)$ if $e\in\mathrm{QF}_n(K)$ or provide a criteria on when e is finite or, for inductive purposes, construct $d\in\operatorname{End}_{n+1}(k)$ from e that has a few iteration properties that are identical to those of e.

Fact 15.1. *Let $e := \mathrm{e}(g_1, \ldots, e_n) \in \mathrm{End}_n(K)$ with $(g_1, \ldots, g_n) \in K_{\mathrm{s}}[x_1, \ldots, x_n]^n$ such that $q := \prod_{i=1}^n \deg(g_i) > 0$. For $i \in [[1, n]]$, let $h_i \in K_{\mathrm{s}}[x_1, \ldots, x_n]$ be the homogeneous component of g_i of degree $\deg(g_i)$. Then the following properties hold.*

(1) *If $e \in \mathrm{QF}_n(K)$, then $\deg(e) \leq q$.*

(2) *If the homogeneous system of n equations*

$$h_1 = \cdots = h_n = 0$$

in the indeterminates $y_0, \ldots, y_n$ has no solution in $\mathbb{P}^{n-1}(K)$, then e is finite. Moreover, if $e \in \mathrm{QFE}_n(K)$, then $\deg(e) = q$.

Proof. For $i \in [[1, n]]$ we write $g_i = \sum_{j=0}^{\deg(g_i)} h_{i,j}$, with each $h_{i,j} \in K_{\mathrm{s}}[x_1, \ldots, x_n]$ homogeneous of degree j; so $h_i = h_{i,\deg(g_i)}$. We consider the closed subscheme W of $\mathbb{P}^n_{K_{\mathrm{t}}[x_1,\ldots,x_n]}$ defined by the system of n homogeneous equations

$$\sum_{j=0}^{\deg(g_i)} h_{i,j}(y_1, \ldots, y_n) y_0^{\deg(g_i)-j} - x_i y_0^{\deg(g_i)} = 0, \quad i \in [[1, n]].$$

Let $\phi : W \to \mathbb{A}^n_K$ be the natural morphism. For $P \in K^n$, $\phi^{-1}(P)_{\mathrm{red}}$ has $|e^{-1}(P)|$ irreducible components of dimension 0 in $\mathbb{A}^n_{K_{\mathrm{t}}[x_1,\ldots,x_n]}$ and possibly other irreducible components in $\mathbb{P}^{n-1}_{K_{\mathrm{t}}[x_1,\ldots,x_n]} = \mathbb{P}^n_{K_{\mathrm{t}}[x_1,\ldots,x_n]} \setminus \mathbb{A}^n_{K_{\mathrm{t}}[x_1,\ldots,x_n]}$. Thus part (1) follows from Bézout's inequality (see [Fu], Ch. Ex. 8.4.6).

For part (2), due to the hypotheses, $W \cap \mathbb{P}^{n-1}_{K_{\mathrm{t}}[x_1,\ldots,x_n]} = \emptyset$. So W is $\mathbb{A}^n_{K,\mathrm{s}}$, i.e., is the spectrum of $K_{\mathrm{t}}[x_1, \ldots, x_s, y_1, \ldots, y_s]\big(g_1(y_1, \ldots, y_n) - x_1, \ldots, g_n(y_1, \ldots, y_n) - x_n\big)$. Hence e is projective; being also affine, it is finite. If $e \in \mathrm{QFE}_n(K)$, then for a generic P, $\phi^{-1}(P) = e^{-1}(P) \cong \mathrm{Spec}\, k^{\deg(e)}$, so $\deg(e) = q$ by Bézout's Theorem (see [Fu], Ch. 8, Prop. 8.4; see also [H2], Ch. I, Thm. 7.7 applied $n-1$ times). □

Lemma 15.2. *Let $e \in \mathrm{End}_n(K)$ be defined over a subfield $\mathcal{K}$ of K. Then the following properties hold.*

(1) *There exists $d \in \mathrm{End}_{n+1}(K)$ defined over $\mathcal{K}$ and such that for each $m \in \mathbb{N}^*$ the sets Γ_{e^m} and Γ_{d^m} are compatibly isomorphic under isomorphisms induced by a projection $\mathbb{A}^{n+1}_{k,\mathrm{t}} \to \mathbb{A}^n_{k,\mathrm{t}}$ defined over $\mathcal{K}$ and hence we have $\iota(d) = \iota(e)$.*

(2) *If $\mathrm{char}(K) = p$ and $e \in \mathrm{EE}_n(K)$ (resp. $e \in \mathrm{BE}_n(K)$), then we can choose d such that also $d \in \mathrm{EE}_{n+1}(K)$ (resp. $d \in \mathrm{BE}_{n+1}(K)$) and $\mathcal{F}(d) = \mathcal{F}(e) \cup \{\deg(e)\}$.*

Proof. Let $(g_1, \ldots, g_n) \in \mathcal{K}[x_1, \ldots, x_n]^n$ define e. For $i \in [[1, n]]$ let $q_i \in \mathbb{N}^*$ be such that $q_i > \deg(g_i)$. Let $j \in \mathbb{N}^*$. We take d to be defined by the $n+1$-tuple

$$\big(g_1(x_1, \ldots, x_n) + x_1^{q_1} x_{n+1}^j, \ldots, g_n(x_1, \ldots, x_n) + x_n^{q_n} x_{n+1}^j, x_{n+1}\big).$$

For $P \in K^n$ we have $d(P, 0) = \big(e(P), 0\big)$ and $|e^{-1}(P)| = |d^{-1}(P, 0)|$. For $\alpha \in K^*$, the endomorphism $d_\alpha \in \mathrm{End}_n(K)$ defined by the n-tuple

$$\big(g_1(x_1, \ldots, x_n) + \alpha^j x_1^{q_1}, \ldots, g_n(x_1, \ldots, x_n) + \alpha^j x_n^{q_n}\big)$$

is finite (see Fact 15.1(2)), so $d(K^n \times \{\alpha\}) = K^n \times \{\alpha\}$. From the last two sentences we get that for each $m \in \mathbb{N}$ the projection $\mathbb{A}^{n+1}_{k,\mathrm{t}} \to \mathbb{A}^n_{k,\mathrm{t}}$ on the first n coordinates is defined over $\mathcal{K}$ and induces bijections $\Gamma_{d^m} \to \Gamma_{e^m}$ compatibly. Thus $\mathrm{Im}(d^m) = \mathrm{Im}(d^{m+1}) \Leftrightarrow \mathrm{Im}(e^m) = \mathrm{Im}(e^{m+1})$, hence $\iota(d) = \iota(e)$ and part (1) holds.

To check part (2), we take $q_1, \ldots, q_n$ (resp. $q_1, \ldots, q_n, j$) to be in $p\mathbb{N}^*$; we have $d \in \mathrm{EE}_{n+1}(K)$ and $d_\alpha \in \mathrm{EE}_n(K)$ (resp. $d \in \mathrm{BE}_{n+1}(K)$ and $d_\alpha \in \mathrm{BE}_n(K)$) for all $\alpha \in K^*$. For each $\alpha \in K^*$ we have $\deg(d_\alpha) = \prod_{i=1}^n q_i$ by Fact 15.1(2). As $\prod_{i=1}^n q_i$ is greater that $\prod_{i=1}^n \deg(g_i)$ and hence than $\deg(e)$ by Fact 15.1(1), it follows that $\deg(d) = \prod_{i=1}^n q_i$ and that $\mathcal{F}(d)$ is the set of cardinals of fibers of e. Thus $\mathcal{F}(d) = F(e) \cup \{\deg(e)\}$ and part (2) holds. □

Example 15.3. Let $e := e\big(x_1 + x_1^{p^m} - x_2^{p^m(p^m+1)}, x_2 + x_1^{p^m} - x_2^{p^m(p^m+1)}\big) \in \mathrm{BE}_2(k)$ with $m \in \mathbb{N}$. Let $a \in \mathrm{GA}_2(k)$ be defined by the pair $(x_1 + x_2^{p^m+1}, x_2)$. Then $d := ea$ is defined by the pair $(x_1 + x_1^{p^m} + x_2^{p^m+1}, x_2 + x_1^{p^m})$. Thus we have inequalities $\pi(d) \le \min\{\pi_{\mathrm{i}}(d), \pi_{\mathrm{l}}(d), \pi_{\mathrm{r}}(d)\} \le \pi_{\mathrm{r}}(d) \le p^m + 1$ and d is finite with $\deg(d) = p^m(p^m+1)$ by Fact 15.1(2). If $\pi(d) \le p^m$, then there exists $(b, c) \in \mathrm{GA}_2(k)^2$ such that $bdc = \mathrm{e}(g_1, g_2)$ with $(g_1, g_2) \in k_{\mathrm{s}}[x_1, x_2]^2$ satisfying $\pi(g_1, g_2) \le p^m$, hence we have $p^m(p^m+1) = \deg(d) = \deg(bdc) \le \deg(g_1)\deg(g_2) \le (p^m)^2$ by Fact 15.1(1), a contradiction. Thus $p^m + 1 = \pi(d) = \pi_{\mathrm{i}}(d) = \pi_{\mathrm{l}}(d) = \pi_{\mathrm{r}}(d)$. Note that $d \in \mathrm{BE}_2(k)$ and $p^m + 1 = \pi_{\mathrm{r}}(e) = \pi(e) \le \min\{\pi_{\mathrm{l}}(e), \pi_{\mathrm{i}}(e)\}$.

16. On iterate inner invariants

We first include examples and applications of Section 3 using Notation 3.1.

Example 16.1. Let $e : X \to X$ be an endomorphism of a variety over K such that $\varphi(e) \in \mathbb{N}^*$. If $\varsigma_e \ge 2$, then $\varphi(e^{l+1}) < \frac{\varsigma_e \varphi(e)}{\varsigma_e - 1} \le 2\varphi(e)$ by Lemma 3.2(7) and therefore $\iota(e) \le \max\{1, \frac{\varphi(e)}{\varsigma_e - 1}\} \le \varphi(e)$ and $\varphi\big(e^{\iota(e)}\big) < \frac{\varsigma_e \varphi(e)}{\varsigma_e - 1} \le 2\varphi(e)$. Similarly, if there exists $s \in \mathbb{N}^*$ such that $\varsigma_{e^s} \ge 2$, then the inequalities $\iota(e^s) \le \frac{\varphi(e^s)}{\varsigma_{e^s} - 1} \le \frac{s\varphi(e)}{\varsigma_{e^s} - 1}$ (the second one by Lemma 3.2(2)) imply that $\iota(e) \le s\iota(e^s) \le \max\{s, \frac{s^2\varphi(e)}{\varsigma_{e^s} - 1}\}$, thus $\iota(e) \in \mathbb{N}$.

Corollary 16.2. *Let $e : X \to X$ be an endomorphism of a variety over k such that Γ_e is finite. We assume that there exists $q \in \mathbb{N}^*$ such that e is the pullback of an endomorphism $d : Y \to Y$ over $\mathbb{F}_{p^q}$. Then $\iota(e) \in \mathbb{N}$. More precisely, if we choose q and d such that $\Gamma_e \subset Y(\mathbb{F}_{p^q})$, then $\iota(e) \le |Y(\mathbb{F}_{p^q}) \setminus \Gamma_e|$.*

Proof. As $\Gamma^+_{e(k)} \subset Y(\mathbb{F}_{p^q})$ is finite, this follows from Lemma 3.2(6). □

Example 16.3. Let $e \in \mathrm{QF}_2(k)$ be non-surjective (i.e., $0 \in \mathcal{F}(e)$ or $\varphi(e) > 0$). If e and Γ_e are defined over $\mathbb{F}_{p^q}$, then $\iota(e) \le p^{2q} - \varphi(e)$ by Corollary 16.2 and hence $\varphi\big(e^{\iota(e)}\big) \le \varphi(e)\iota(e) \le \varphi(e)[p^{2q} - \varphi(e)]$ by Lemma 3.2(2).

For an endomorphism $e : \mathcal{B} \to \mathcal{B}$ of a set $\mathcal{B}$ and $P \in \mathcal{B}$ we consider two conditions.

($\natural_{e,P}$) *The sequence $\big(e^m(P)\big)_{m \ge 0}$ has distinct elements and for $i, j \in \mathbb{N}$, we have $|e^{-i}\big(e^j(P)\big)| > 0$ iff $i \le j$ and iff $e^{-i}\big(e^j(P)\big) = \{e^{j-i}(P)\}$.*

($\flat_{e,P}$) *For each $m \in \mathbb{N}$, $e^m(P) \notin \mathrm{Im}(e^{m+1})$.*

Notation 16.4. For a non-empty constructible subset V of a variety W over K, let $\bar{V}$ be its Zariski closure in W, $\digamma(V) = \digamma(\bar{V}) := |\{Y \in \mathrm{Irr}(\bar{V}) | \dim(Y) = \dim(\bar{V})\}|$ and $\hbar(V) = \hbar\bar{V} := \big(\dim(\bar{V}), \digamma(\bar{V})\big) \in \mathbb{N}^2$. If W is irreducible, let $\eta_W \in W$ be its generic point. Endowing $\mathbb{N}^2$ with the lexicographic order (so $(0,1) < (1,0)$), for a dominant morphism $W_1 \to W_2$ of varieties over K we have $\hbar(W_1) \ge \hbar(W_2)$.

Lemma 16.5. *Let $e : X \to X$ be an endomorphism of a variety over K. For $m \in \mathbb{N}$, let $e_m : \mathrm{Im}(e^m) \to \mathrm{Im}(e^m)$ be the endomorphism of constructible sets induced by e. Then the following statements are equivalent.*

(1) *We have* $\iota(e) = \infty$.

(2) *There exists* $m \in \mathbb{N}$ *and a point* $P \in \mathrm{Im}(e^m)$ *such that* $(\natural_{e_m,P})$ *holds.*

(3) *There exists* $m \in \mathbb{N}$ *and a point* $P \in \mathrm{Im}(e^m)$ *such that* $(\flat_{e_m,P})$ *holds.*

(4) *There exists an algebraically closed field* K_1 *that contains* K*, an* $m \in \mathbb{N}$*, and a point* $P \in \mathrm{Im}(e^m)(K_1)$ *such that* $(\flat_{(e_m)_{K_1},P})$ *holds.*

If Γ_e *is finite, then the statements are also equivalent to the following two.*

(5) *There exists* $m \in \mathbb{N}$ *and* $P \in \mathrm{Im}(e^m)(K)$ *such that* $(\natural_{e_m,P})$ *holds.*

(6) *There exists* $m \in \mathbb{N}$ *and* $P \in \mathrm{Im}(e^m)(K)$ *such that* $(\flat_{e_m,P})$ *holds.*

Proof. The implications (2) $\Rightarrow$ (3) and (5) $\Rightarrow$ (6) always hold. Clearly, (6) $\Rightarrow$ (3).

We show that (1) $\Rightarrow$ (2). For $i \in \mathbb{N}$ the constructible set $Z_i := \mathrm{Im}(e^i) \setminus \mathrm{Im}(e^{i+1})$ (by Chevalley's Theorem, see [Gro2], Thm. (1.8.4)) is non-empty. Let Y_i be the Zariski closure of Z_i in X. As e induces surjective morphisms $Z_i \to Z_{i+1}$, it also induces dominant morphisms $Y_i \to Y_{i+1}$. As $\big(\hbar(Y_n)\big)_{n\geq 0}$ is a non-increasing sequence of the well ordered $\mathbb{N}^2$, there exists $(m,q,l) \in \mathbb{N}^3$ such that $\hbar(Y_i) = \hbar(Y_m) = (q;l)$ for each integer $i \geq m$. For $i \geq m$ let $\eta_i := \{\eta_Y | Y \in \mathrm{Irr}(Y_i), \dim(Y) = q\}$; so $|\eta_i| = l > 0$ and $\eta_i \subset Z_i$. An induction on $j \in \mathbb{N}$ gives that $e^j(\eta_m) = \eta_{m+j}$; so $(\natural_{e_m,P})$ holds for each $P \in \eta_m$.

If Γ_e is finite, then $q = 0$, $\mathrm{Irr}_m \subset X(K)$, thus (1) $\Rightarrow$ (5) by previous paragraph.

To show that (3) $\Rightarrow$ (4), let K_1 be an uncountable algebraically closed field that contains the residue field of P. If $\overline{P}$ is the Zariski closure in X of P, then $\overline{P}_{K_1}$ is not the union of its proper subvarieties $\overline{P}_{K_1} \cap e_{m,K_1}^{-l}\big(\mathrm{Im}(e_{m,K_1}^{l+1})\big)$ indexed by $l \in \mathbb{N}$ and hence there exists a point $Q \in \overline{P}(K_1)$ such that $(\flat_{e_{m,K_1},Q})$ holds. Thus (3) $\Rightarrow$ (4).

We show that (4) $\Rightarrow$ (1). As $\iota(e) = \iota(e_{K_1})$ and $\iota(e) = \infty \Leftrightarrow \iota(e_m) = \infty$, we can assume that $K = K_1$ and $m = 0$. As $(\flat_{e,P})$ holds, we have $\mathrm{Im}(e^l) \neq \mathrm{Im}(e^{l+1})$ for all $l \in \mathbb{N}$, hence $\iota(e) = \infty$. $\square$

Remark 16.6. Assume Γ_e is finite and $\iota(e) = \infty$. We use Lemma 3.2 to show that statement (5) of Lemma 16.5 holds. Let $(l,N) \in \mathbb{N}^2$ be such that $\varphi(e^{n+1}) - \varphi(e^n) = l$ for all integers $n \geq N$ by Lemma 3.2(1). By replacing e with e_N we can assume that $N = 0$; so $\varphi(e^n) = nl$ for all $n \in \mathbb{N}$, with $l \in \mathbb{N}^*$ as $\iota(e) = \infty$. For $n \in \mathbb{N}$, as $|\cup_{i=0}^n e^i(\Gamma_e)| \leq l(n+1) = \varphi(e^{n+1}) = |\Gamma_{e^{n+1}}|$, we get that $\Gamma_{e^{n+1}} = \sqcup_{i=0}^n e^i(\Gamma_e)$ by Lemma 3.2(5); so $(\natural_{e,P})$ holds for each $P \in \Gamma_e \subset X(K)$.

Remark 16.7. Let $e : X \to X$ be an endomorphism of a non-empty variety over K which is dominant (resp. which is dominant outside a closed subvariety of the target of dimension $m < \dim(X)$). A decreasing induction on $i \in [[0, \dim(X)]]$ (resp. $i \in [[m+1, \dim(X)]]$) gives that e permutes the generic points of irreducible components of X of dimension i. So there exists $N \in \mathbb{N}^*$ such that e^N fixes the generic points of all irreducible components (resp. irreducible components of dimension $> m$) of X. If $d : X \to X$ is quasi-finite, then $1 \leq \digamma\big(\mathrm{Im}(d)\big) \leq \digamma(X)$; so there exists $i \in [[0, \digamma(X) - 1]]$ such that $\digamma\big(\mathrm{Im}(d^{i+j})\big) = \digamma\big(\mathrm{Im}(d^i)\big) \in \mathbb{N}$ for $j \in \mathbb{N}$.

Next theorem was first obtained in [PNCG], Thm. 1 under the extra hypotheses that $K = \mathbb{C}$, X is affine, e is universally open, and Γ_e is finite.

Theorem 16.8. *Let* $e : X \to X$ *be a quasi-finite endomorphism of a variety over* K*. Then the following properties hold.*

(1) *We have* $\iota(e) \in \mathbb{N}$.

(2) *There exists no $P \in X$ for which condition $(\flat_{e,P})$ (or $(\natural_{e,P})$) holds.*

Proof. Part (2) follows from part (1) and Lemma 16.5.

To prove part (1), let $\overline{Z}$ be the Zariski closure in X of a subset Z of X. The decreasing sequence $\big(\overline{\mathrm{Im}(e^m)}\big)_{m\in\mathbb{N}}$ is stationary. Thus, by replacing X with $Y := \cap_{m\geq 0}\overline{\mathrm{Im}(e^m)}$, we can assume that $X = \overline{\mathrm{Im}(e)}$ (i.e., e is dominant). Note that if $l \in \mathbb{N}$ is such that $Y = \overline{\mathrm{Im}(e^l)}$, then $\mathrm{Im}(e^{m+l}) = e^m(Y) \subset \mathrm{Im}(e^m)$, thus the decreasing sequence $\big(\mathrm{Im}(e^m)\big)_{m\in\mathbb{N}}$ is stationary iff the decreasing sequence $\big(e^m(Y)\big)_{m\in\mathbb{N}}$ is so.

By replacing e with e^N for some $N \in \mathbb{N}^*$, we can assume that e fixes each generic point of an $Y \in \mathrm{Irr}(X)$ (see Remark 16.7); so e induces an endomorphism $e_Y : Y \to Y$. If $\iota(e_Y) \in \mathbb{N}$ for each $Y \in \mathrm{Irr}(X)$, then $\iota(e) \in \mathbb{N}$ as in fact we have $\iota(e) \leq \max\{e_Y | Y \in \mathrm{Irr}(X)\}$. Thus, by replacing e with the normalizations of e_Ys we can assume that X is normal irreducible.

To end the proof it suffices to show that the assumption that $\iota(e) = \infty$ leads to a contradiction. From Lemma 16.5 we get that, by replacing e with the endomorphism it induces on the open subvariety $\mathrm{Im}(e^m)$ for some $m \in \mathbb{N}$, we can assume that there exists $P \in X$ such that condition $(\flat_{e,P})$ holds.

Let R be a finitely generated $\mathbb{Z}$-subalgebra of K such that e is defined over R (see [Gro4], Prop. (8.9.1)(iii)) and the Zariski closure Z_P of P in X is defined over $\mathcal{K} := \mathrm{Frac}(R)$; let $e_R : X_R \to X_R$ be an endomorphism of a $\operatorname{Spec} R$-scheme X_R of finite type such that its pullback to $\operatorname{Spec} K$ is e. The fiber $e_0 : X_0 \to X_0$ of e_R over $\mathcal{K}$ is quasi-finite and there exists $P_0 \in X_0$ with the property that the pullback of its Zariski closure in X_0 to X is Z_P. As $(\flat_{e,P})$ holds, $(\flat_{e_0,P_0})$ also holds. By localizing R we can assume that X_R is normal irreducible and (see [Gro4], Thm. (8.10.5)(xi)) that e_R is quasi-finite; so e_R is universally open by [Gro4], Cor. (14.4.3).

Let Q be a closed point of the Zariski closure of P_0 in X_0; its residue field κ is finite and we have an inclusion $\{e^m(Q)|m \in \mathbb{N}\} \subset X_0(\kappa)$ of finite sets. Let $(q,s) \in \mathbb{N} \times \mathbb{N}^*$ be such that $e_0^q(Q) = e_0^{q+s}(Q)$. As $e_0^q(Q)$ belongs to the open subscheme $\mathrm{Im}(e_0^{q+s})$ of X_0, $e_0^q(P_0) \in \mathrm{Im}(e_0^{q+s}) \subset \mathrm{Im}(e_0^{q+1})$, which contradicts $(\flat_{e_0,P_0})$. $\square$

Corollary 16.9. *Let $f : W \to Y$ be a morphism of finite presentation between quasi-compact schemes. Let $e : W \to W$ be an endomorphism of schemes over Y (i.e., $f = f \circ e$) with the property that all its fibers $e_y : W_y \to W_y$ over geometric points y of Y are quasi-finite. Then there exists $n \in \mathbb{N}$ such that $\mathrm{Im}(e^n) = \mathrm{Im}(e^{n+1})$.*

Proof. For $m \in \mathbb{N}$, as $\mathrm{Im}(e^m)$ is a constructible subset of W, from [Gro4], Cor. (9.5.2) we get that $Y_m := \{y \in Y | \mathrm{Im}(e^m)_y = \mathrm{Im}(e^{m+1})_y\}$ is a constructible subset of Y. As $\mathrm{Im}(e^m)_y = \mathrm{Im}(e_y^m)$, it follows first that $Y_m \subset Y_{m+1}$ for all $m \in \mathbb{N}$ and second that $\cup_{m\geq 0} Y_m = Y$ by Theorem 16.8. So there exists $n \in \mathbb{N}$ such that $Y_n = Y$ by [Gro2], Cor. (1.9.9); hence $\mathrm{Im}(e^n) = \mathrm{Im}(e^{n+1})$. $\square$

Our examples of endomorphisms $e : X \to X$ with $\iota(e) = \infty$ are in essence based on the notion 'trap' used in [Bori] and formalized here as follows.

Definition 16.10. *Let $e : X \to X$ be an endomorphism of a non-empty variety over K. By a trap of e we mean a non-empty closed subvariety Y of X such that $\dim(Y) < \dim(X)$, $e(Y) \subset Y$, the endomorphism $e_Y : Y \to Y$ induced by e is dominant, and we have a strictly increasing sequence*

$$\big(|\{Z \in \mathrm{Irr}(e^{-n}(Y)) |\, \dim(Z) > \dim(Y) \text{ or } \dim(e^n(Z)) < \dim(Z) = \dim(Y)\}|\big)_{n\in\mathbb{N}^*}.$$

Example 16.11. We list several examples of endomorphisms in $\mathrm{D}_n(K) \setminus \mathrm{QF}_n(K)$ with $n \in \{2,3\}$ whose iterate inner invariants are ∞ and often mention zero loci that are traps of them. Let $(l,m) \in (\mathbb{N}^*)^2$. Let $C_m \subset K^2$ be the empty set if $m=1$ and be $\{0\} \times K^*$ if $m \geq 2$.

(1) The rule $(x,y) \mapsto (xy, xy+y)$ defines $e \in \mathrm{D}_2(K) \setminus \mathrm{QF}_2(K)$ which is birational (i.e., generically étale with $\deg(e)=1$); the zero locus $x_1 = x_2 = 0$ is a trap of it of dimension 0. If K_1 is an algebraically closed field that contains K and a transcendental element α over the prime field, then $(\natural_{e_{K_1},(\alpha,\alpha)})$ holds, therefore $\iota(e) = \iota(e_{K_1}) = \infty$ by Lemma 16.5. Similarly, if $\mathrm{char}(K) = 0$ and $\beta = 1$ (resp. $\mathrm{char}(K) = p$ and $\beta \in K$ is a transcendental element over $\mathbb{F}_p$), then the rule $(x,y) \mapsto (xy, \beta y + 1)$ defines $d \in \mathrm{D}_2(K) \setminus \mathrm{QF}_2(K)$ which is birational with $\iota(d) = \infty$; the zero locus $x_1 = 0$ is a trap of it of dimension 1.

(2) Similarly to example (1), either one of the rules $(x,y) \mapsto (x^p y^p, y^p + x^p y^p)$ and $(x,y) \mapsto \big(x(x-y)^{p-1}, y^p(x-y)^{p^2-p}\big)$ defines an $e \in \mathrm{D}_2(K) \setminus \mathrm{QF}_2(K)$ which is generically radicial (i.e., $\deg(e) = 1$) and defined over the prime field with $\iota(e) = \infty$.

(3) The rule $(x,y) \mapsto \big(x^m y, x^{lm} y^l + x^{l(m-1)} y^{l+1}\big)$ defines an endomorphism $e_{\mathrm{a},l,m} \in \mathrm{D}_2(K) \setminus \mathrm{QF}_2(K)$. If either $y=0$ or $x=0$ and $m \geq 2$, then $(x,y) \mapsto (0,0)$. If $m=1$, then $(0,y) \mapsto (0, y^{l+1})$. If $xy \neq 0$, then we have the additive property

$$\frac{x^{lm} y^l + x^{l(m-1)} y^{l+1}}{(x^m y)^l} = 1 + \frac{y}{x^l}. \tag{13}$$

If $\mathrm{char}(K) \nmid l+m$ and $(\alpha,\beta) \in (K^*)^2$ with $\beta \neq \alpha^l$, then the system of equations $x^m y - \alpha = 0 = x^{lm} y^l + x^{l(m-1)} y^{l+1} - \beta$ gives based on Equation (13) that $\frac{\beta}{\alpha^l} = \frac{y}{x^l} + 1$, hence $y = \frac{(\beta-\alpha^l)x^l}{\alpha^l}$; substituting this in the first equation we get that $x^{l+m} = \frac{\alpha^{l+1}}{\beta - \alpha^l}$, hence the system of equations has $l+m$ solutions, therefore $\deg(e_{\mathrm{a},l,m}) = l+m$ and $e_{\mathrm{a},l,m}$ is generically étale. If $\mathrm{char}(K) = 0$, then $\deg(e_{\mathrm{a},l,m}) = l+m \geq 2$, $\iota(e_{\mathrm{a},l,m}) = \infty$ and the zero locus $x_1 = x_2 = 0$ is a trap of e_{a} of dimension 0; more precisely, we have $\iota(e_{\mathrm{a},l,m}) = \infty$ as for $n \in \mathbb{N}^*$, one computes

$$\Gamma_{e^n_{\mathrm{a},l,m}} = \{(x,y) \in (K^*)^2 | x^{-l} y \in [[1,n]]\} \cup C_m.$$

The case $(l,m) = (1,2)$ is the "Additive Trap" of [Bori], Ex. 1.

(4) The multiplicative analogue of part (3) works in all characteristics. Let $(q,s) \in \mathbb{N}^2$. Let $h(t) \in K[t]$ be such that $\deg(h) = s$ and the derivative of $t^{l(q+1)+m} h(t)$ is non-zero. Assume there exists $\delta \in K^*$ of infinite multiplicative order. Let $e := e_{\mathrm{m},l,m,q,s,h,\delta} \in \mathrm{D}_2(K) \setminus \mathrm{QF}_2(K)$ be defined by the rule

$$(x,y) \mapsto \big(x^m y^q (x^l - y) h(x), \delta x^{l(m-1)} y^{ql+1} (x^l - y)^l h(x)^l\big).$$

If either $(x^l - y)h(x) = 0$ or $x = 0$ and $m \geq 2$ or $y = 0$ and $q \geq 1$, then $(x,y) \mapsto (0,0)$. If $m = 1$, then $(0,y) \mapsto \big(0, (-1)^l \delta h(0)^l y^{(q+1)l+1}\big)$. If $q = 0$, then $(x,0) \mapsto (x^m h(x), 0)$. If $xy(x^l - y)h(x) \neq 0$, then we have the multiplicative property

$$\frac{\delta x^{l(m-1)} y^{lq+1} (x^l - y)^l h(x)^l}{[x^m y^q (x^l - y) h(x)]^l} = \delta \frac{y}{x^l}. \tag{14}$$

For a generic $(\alpha,\beta) \in (K^*)^2$ with $\beta \neq \alpha^l$ we consider the system of equations $x^m y^q (x^l - y) h(x) - \alpha = 0 = \delta x^{l(m-1)} y^{ql+1} (x^l - y)^l h(x)^l - \beta$. From Equation (14) we get that $\delta \frac{y}{x^l} = \frac{\beta}{\alpha^l}$, hence $y = \frac{\beta x^l}{\alpha^l}$. Substituting this in the first equation we get that $x^{l(q+1)+m} h(x) = \frac{\alpha^{(q+1)l+1}}{(\alpha^l - \beta)\beta^q}$. From this and the assumptions $[t^{l(q+1)+m} h(t)]' \neq 0$

and (α,β) generic, we get that the system has $l(q+1)+m+s$ solutions. Thus $\deg(e)=l(q+1)+m+s$ and e is generically étale. The zero locus $x_1=x_2=0$ is a trap of e of dimension 0. We have $\iota(e_{\mathrm{m},l,m,\delta})=\infty$ as for $n\in\mathbb{N}^*$ we have an inclusion

$$\Gamma_{e^n_{\mathrm{m},l,m,\delta}}(K)\supset\big\{(x,y)\in(K^*)^2|x^{-l}y\in\{\delta^l|l\in[[1,n]]\}\big\}\cup C_m\cup(K^*\times\{0\})$$

which in fact is an equality if $h(t)=1$ and $q\geq 1$. The case $(l,m,q,h)=(1,2,1,1)$ is the "Multiplicative Trap" of [Bori], Ex. 3.

If k is an algebraic closure of $\mathbb{F}_p$ (i.e., if k^* has no element of infinite multiplicative order), the rule $(x,y,z)\mapsto\big(x^my^q(x^l-y)h(x),x^{l(m-1)}y^{l+1}(x^l-y)^lh(x)^lz,z\big)$ defines an endomorphism $d:=d_{\mathrm{m},m,l,q,s,h}\in\mathrm{D}_3(k)\setminus\mathrm{QF}_3(k)$. For $n\in\mathbb{N}^*$, Γ_{d^n} contains $\big\{(x,x^lz^q,z)|q\in[[1,n]],(x,z)\in(k^*)^2\big\}$ and does not contain the generic point of the zero locus $y-x^lz^{n+1}=0$, so $\iota(d)=\infty$. We have $\deg(d)=l(q+1)+m+s$.

Corollary 16.12. *If* $\mathrm{char}(K)\neq 2$ *(resp.* $\mathrm{char}(K)=2$*) let* $N\geq 2$ *(resp.* $N\geq 3$*) be an integer. Let* $n\in\{2,3\}$ *be such that we have* $n=2$ *iff* K^* *has an element of infinite multiplicative order. Then there exists* $e\in\mathrm{D}_n(K)\setminus\mathrm{QF}_n(K)$ *such that* $\iota(e)=\infty$*,* $\deg(e)=N$ *and* e *is generically étale.*

Proof. This follows from Example 16.11(4) as we can choose (l,q,m,s,h) such that $N=l(q+1)+m+s$. E.g., for $l=m=1$, if $\mathrm{char}(K)\nmid N$ we can take $(q,s,h)=(N-2,0,1)$ and if $\mathrm{char}(K)\mid N$ we can take $(q,s,h)=(N-3,1,t+1)$. $\square$

Proposition 16.13. *If* $\mathrm{char}(K)=p$*, we assume that* K *is not an algebraic closure of* $\mathbb{F}_p$*. Let* $n\geq 3$ *be an integer. Then there exists an endomorphism* $e\in\mathrm{D}_n(K)$ *such that* $\varphi(e^l)=l$ *for each* $l\in\mathbb{N}$ *(thus* $\iota(e)=\infty$*). If* $n=3$*, then for each integer* $s\geq 3$ *we can choose* e *such that moreover we have* $\deg(e)=s$*.*

Proof. We can assume that $n=3$ by Lemma 15.2(1). Let $\alpha\in K$ be 1 if $\mathrm{char}(K)=0$ and be transcendental over $\mathbb{F}_p$ if $\mathrm{char}(K)=p$. The rule $(x,y)\mapsto\big(xy,\alpha(y+1)\big)$ defines an element $d\in\mathrm{D}_2(K)\setminus\mathrm{QF}_2(K)$ which is birational (cf. Example 16.11(1)). As $(\natural_{d,(\beta,\alpha)})$ holds for each $\beta\in K^*$, we have $\iota(d)=\infty$ by Lemma 16.5. Let Y be the plane in $\mathbb{A}^3_K$ defined by the equation $x_3=0$ and let C be its curve defined by the equations $x_2-\alpha=x_3=0$. We have $\Gamma_d\times\{0\}=C\setminus\{(0,\alpha,0)\}\cong\mathbb{A}^1_K\setminus\{0\}$.

For $(m,f,g)\in\mathbb{N}^*\times(K[x_1,x_2,x_3]\setminus\{0\})^2$ let $e_{m,f,g}\in\mathrm{End}_3(K)$ be defined by the triple

$$\big(x_1x_2+gx_3(g-x_1x_2+1),\alpha(gx_3-1)(fx_3-x_2)+\alpha,x_3^m(gx_3-1)\big).$$

It extends d as $e_{m,f,g}(x,y,0)=\big(xy,\alpha(y+1),0\big)$. We have $e^{-1}_{m,f,g}(Y)=Y\sqcup Z$, where Z is the surface defined by the equation $gx_3=1$. As $e_{m,f,g}(Z)\subset C$, we get that if there exists $Q\in C(K)\setminus[\{(0,\alpha,0)\}\cup e_{m,f,g}(Z)]$, then condition $(\natural_{e_{m,f,g},Q})$ holds and thus $\iota(e_{m,f,g})=\infty$ by Lemma 16.5.

Let $P:=(1,\alpha,0)$ and $g:=x_2x_3^q$. For each $(m,f)\in\mathbb{N}^*\times K[x_1,x_2,x_3]$ we have

$$\begin{aligned}e_{m,f,g}\big(Z(K)\big)&=\{e_{m,f,g}(\gamma,\delta^{-q-1},\delta)|(\gamma,\delta)\in K\times K^*\}\\&=\{(\delta^{-1}+1,\alpha,0)|\delta\in K^*\}=C(K)\setminus\{P\}.\end{aligned}$$

We take either $f=(\alpha+1)x_2-\alpha x_2x_3$ with $m\geq 1$ or $f=1$ with $m\geq 2$; for $e:=e_{m,f,x_2x_3^q}$ we show that we have

$$C\setminus[\{(0,\alpha,0)\}\cup e(Z)]=\{P\}=\Gamma_e. \tag{15}$$

We already know that the first identity of Equation (15) holds. To show that the second identity of Equation (15) holds, let $(\beta, \alpha\gamma, \delta) \in K^2 \times K^*$ and we consider the system of equations (without the non-zero factor α of the second equation)

$$xy+yz^{q+1}(yz^q-xy+1)-\beta = (yz^{q+1}-1)\big(f(y,z)z-y\big)+1-\gamma = z^m(yz^{q+1}-1)-\delta = 0.$$

The third equation gives $y = \frac{\delta+z^m}{z^{m+q+1}}$. Based on this, the first equation gives $xy = \frac{\beta-yz^{q+1}(yz^q+1)}{1-yz^{q+1}}$. Moreover, z must be a root of the polynomial $h(z) = h_{m,f,\gamma,\delta}(z) \in K[z]$ we get by substituting $y = \frac{\delta+z^m}{z^{m+q+1}}$ in $(yz^{q+1}-1)\big(f(y,z)z-y\big)+1-\gamma$ and clearing the denominators. If $f = (\alpha+1)x_2 - \alpha x_2 x_3$ with $m \geq 1$, then $f(y,z)z-y = (\alpha+1)yz - \alpha yz^2 - y = -y(z-1)(\alpha z-1)$ and

$$h(z) = (1-\gamma)z^{2m+q+1} - \delta(z^m+\delta)(z-1)(\alpha z-1).$$

If $f = 1$ with $m \geq 2$ then $h(z) = \frac{(\gamma-1)}{\delta}z^{2m+q+1} - z^{m+q+2} + z^m + \delta$.

We show that the assumption that all roots of $h(z) = 0$ are m-th root of $-\delta$ leads to a contradiction. We can assume that there exists $z_0 \in K$ such that $h(z_0) = 0 = z_0^m + \delta$. Clearly, $z_0 \neq 0$.

If $f = (\alpha+1)x_2 - \alpha x_2 x_3$ with $m \geq 1$, then $h(z_0) = (\gamma-1)z_0^{2m+q+1} = 0$ implies $1-\gamma = 0$ and hence $h(z) = \delta(z^m+\delta)(z-1)(\alpha z-1)$. It follows that $\delta + 1 = \delta + \alpha^{-m} = 0$, hence $\alpha^m = -1$, a contradiction.

If $f = 1$ with $m \geq 2$, then we can assume that $\gamma_1 := \frac{(\gamma-1)}{\delta} \neq 0$. It follows that $\gamma_1\delta^2 z_0^{q+1} + \delta z_0^{q+2} = 0$ and hence, as $z_0\delta \neq 0$, we have $z_0 = -\gamma_1\delta$. As $m \geq 2$, we have $2m+q+1 > m+q+2$, hence $h(z)$ has degree $2m+q+1$. Thus $h(z) = -\gamma_1(z+\gamma_1\delta)^{2m+q+1}$ as z_0 is the only possible root. By identifying the coefficient of z, as $m \geq 2$ we get that $0 = (2m+q+1)\gamma_1^{2m+q+1}\delta^{2m+q}$, hence $\mathrm{char}(K) = p$ with p dividing $2m+q+1$. So all positive exponents of powers of z that show up in $h(z)$ with non-zero coefficients are divisible by p, i.e., p divides also m and $m+q+2$, a contradiction.

For a root of $h(z)$ which is not an m-th root of $-\delta$, we have $y \neq 0$ and thus we can solve for x and hence the system is consistent. From this and the descriptions of Γ_d and $e_{m,f,g}\big(Z(K)\big) = e\big(Z(K)\big)$ we get that the second identity of Equation (15) also holds. Note that it follows that Z is a trap of e of dimension 2.

If $f = (\alpha+1)x_2 - \alpha x_2 x_3$ with $m \geq 1$, then for $\gamma = 1-\delta$, the resulting polynomial $h(z) = \delta[z^{2m+q+1} - (z^m+\delta)(z-1)(\alpha z-1)]$ is separable of degree $2m+q+1$ for all δs outside a finite set, hence $\deg(e) = 2m+q+1$; for each integer $s \geq 3$, there exist integers $m \geq 1$ and $q \geq 0$ such that $s = 2m+q+1$.

Similarly, if $f = 1$ with $m \geq 2$, then for $\gamma = \delta + 1$ the resulting polynomial $h(z) = z^{2m+q+1} - z^{m+q+2} + z^m + \delta$ is separable of degree $2m+q+1$ for all δs outside a finite set, hence $\deg(e) = 2m+q+1$; for each integer $s \geq 5$, there exist integers $m \geq 2$ and $q \geq 0$ such that $s = 2m+q+1$.

As $\iota(e) = \infty$, the sequence $\big(\varphi(e^l)\big)_{l\in\mathbb{N}}$ is strictly increasing. As $\varphi(e) = 1$ by Equation (15), from Lemma 3.2(2) we get that $\varphi(e^l) \leq l$. From the last two sentences we get that $\varphi(e^l) = l$ for each $l \in \mathbb{N}$. □

Proposition 16.13 shows that Theorem 16.8 (resp. Corollary 16.2) is optimal, i.e., it does not hold in general for endomorphisms that are not quasi-finite (resp. not defined over finite fields) in dimensions at least 3. In order to supplement Corollary 16.2 for dimensions at most 2, we first prove the following general lemma.

Lemma 16.14. *Let $e : X \to X$ be an endomorphism of a variety over K with the property that there exists $P \in X(K)$ such that ($\natural_{e,P}$) holds. For $(q, r) \in \mathbb{N} \times \mathbb{N}^*$ with $q < r$, let $Y_{q,r}$ be the Zariski closure in X of the sequence $\big(e^{q+ri}(P)\big)_{i\geq 0}$ and let $Y_{q,r}^- := \cup_{\{Y\in \mathrm{Irr}(Y_{q,r})|\dim(Y)>0\}} Y$. Then the following properties hold.*

(1) *The generic points of the irreducible components of $Y_{q,r}^-$ are permuted transitively by e^r (hence $Y_{q,r}^-$ is equidimensional).*

(2) *There exists no $n \in \mathbb{N}$ such that $e^n(P)$ belongs to two or more irreducible components of $Y_{q,r}^-$.*

(3) *The number $|\mathrm{Irr}(Y_{q,r}^-)|$ does not depend on $q \in [[0, r-1]]$.*

(4) *The number $\dim(Y_{q,r}^-)$ does not depend on $(q, r) \in \mathbb{N} \times \mathbb{N}^*$ with $q < r$.*

(5) *Each $Y_{q,r}^-$ is a union of irreducible components of $Y_{0,1}^-$.*

(6) *If $Z \in \mathrm{Irr}(Y_{0,1}^-)$ contains the sequence $\big(e^{q+ri}(P)\big)_{i\geq 0}$, then Z is the Zariski closure of this sequence.*

(7) *Let $Z \in \mathrm{Irr}(Y_{0,1}^-)$ and $s \in \mathbb{N}^*$ be such that $e^s(Z) \subset Z$. Then the endomorphism $d : Z \to Z$ induced by e^s is dominant and generically radicial. Moreover, there exists $m \in \mathbb{N}$ such that $e^m(P) \in Z$, condition ($\natural_{d,e^m(P)}$) holds and the orbit $\{d^i\big(e^m(P)\big)|i \in \mathbb{N}\}$ is Zariski dense in Z.*

Proof. As $e^r(Y_{q,r}) \cup \{e^q(P)\}$ is Zariski dense in $Y_{q,r}$, e^r induces a dominant endomorphism of $Y_{q,r}^-$ and thus (see Remark 16.7) it permutes the generic points of varieeties in $\mathrm{Irr}(Y_{q,r}^-)$. Let $Z_1, Z_2 \in \mathrm{Irr}(I_{q,r}^-)$ with $Z_1 \neq Z_2$. For $i \in \{1, 2\}$ let $m_i \in q + r\mathbb{N}$ be such that $e^{m_i}(P) \in Z_i$. We can assume that $m_1 < m_2$ and that $e^{m_2}(P)$ does not lie in any other irreducible component of $Y_{q,r}^-$ different from Z_2. So, as r divides $m_2 - m_1$, $e^{m_2-m_1}(Z_1)$ must be Z_2 and part (1) holds. Due to the transitivity property of part (1), Z_1 is uniquely determined by m_1 and thus part (2) follows by taking $m_1 = n$.

As $Y_{0,r}^-$ is equidimensional of some dimension $l \in \mathbb{N}^*$, from part (1) it follows that for each $Z \in \mathrm{Irr}(Y_{0,r}^-)$, the Zariski closure of every $e^i(Z)$ with $i \in \mathbb{N}$ is of dimension l. Thus, as $Y_{q,r}$ is the Zariski closure of $e^q(Y_{0,r})$ and $Y_{0,1} = \cup_{j=0}^{r-1} Y_{j,r}$, parts (3) to (5) hold. Part (6) follows from part (4).

If $m \in \mathbb{N}$ is the smallest such that $e^m(P) \in Z$, then ($\natural_{d,e^m(P)}$) holds as ($\natural_{e,P}$) holds. For $j \in \mathbb{N}^*$ such that $js > m$, the sequence $\big(e^{m+jsi}(P)\big)_{i\geq 0}$ is a subsequence of the orbit $\{d^i\big(e^m(P)\big)|i \in \mathbb{N}\}$; thus from part (6) it follows that this orbit is Zariski dense in Z. Let V be an open dense subvariety of Z such that $d_V := d \times_Z 1_V : e^{-1}(V) \to V$ is finite. By shrinking V we can assume that d_V is finite flat of degree $o \in \mathbb{N}$ with (see [Gro4], Cor. (9.7.9)) all geometric fibers of the same cardinality $o_{\mathrm{sep}} \in [[1, o]]$. As $V \cap \{d^i\big(e^m(P)\big)|i \in \mathbb{N}^*\} \neq \emptyset$ by the Zariski density of the mentioned orbit, we have $o_{\mathrm{sep}} = 1$ by ($\natural_{d,e^m(P)}$). Thus d_V is finite radicial; so d is dominant and generically radicial. □

Proposition 16.15. *Let $e : X \to X$ be an endomorphism of a variety over K of dimension at most 2. If Γ_e is finite, then $\iota(e) \in \mathbb{N}$.*

Proof. We show that the assumption that $\iota(e) = \infty$ leads to a contradiction. We can assume that there exists $P \in X(K)$ such that ($\natural_{e,P}$) holds by Lemma 16.5. Let $Y_{0,1}^-$ be the closed subvariety of X obtained from P as in Lemma 16.14; it is equidimensional of dimension 1 or 2 and its irreducible components are permuted transitively by e by Lemma 16.14(1). Thus, if $\dim(Y_{0,1}^-) = 1$ then the endomorphism of $Y_{0,1}^-$ induced by e is dominant hence quasi-finite, so $(\natural_{e,P})$ does not hold by Theorem 16.8 and Lemma 16.5, a contradiction.[14] Thus $\dim(Y_{0,1}^-) = 2$.

By replacing e with a power of it, we can assume that there exists $Z \in \mathrm{Irr}(Y_{0,1}^-) \subset \mathrm{Irr}(X)$ (so $\dim(Z) = 2$) such that the endomorphism $e_Z : Z \to Z$ induced by e is dominant and generically radicial, $(\natural_{e_Z, e^m(P)})$ holds for some $m \in \mathbb{N}$, and the orbit $\{e_Z^i(e^m(P)) | i \in \mathbb{N}\}$ is Zariski dense in Z (see Lemma 16.14(7)).

Let $W := \cup_{Z' \in \mathrm{Irr}(X) \setminus \{Z\}} Z \cap Z'$; we have $e(W) \subset W$ by Remark 16.7 and hence the mentioned Zariski density gives that $e_Z^i(e^m(P)) \notin W$ for all $i \in \mathbb{N}$. If $Q_1 \in (Z \setminus W)(K)$ is $e(Q_0)$ for some $Q_0 \in X(K)$, then either $Q_0 \in (Z \setminus W)(K)$ or Q_0 is one of the finitely many isolated points of X. Thus there exists a finite set W_0 of K-valued points of $Z \setminus W$ such that $[Z \setminus (W \cup W_0)] \subset \mathrm{Im}(e_Z)$. Let $Y \subset [Z \setminus (W \cup W_0)]$ be a dense open subset such that the morphism $e_Z^{-1}(Y) \to Y$ induced by e_Z (or e) is finite radicial. Clearly, $e_Z^{-1}(Y) \subset Z \setminus W$.

By counting points over finite fields (i.e., by considering a model e_R of e over a finitely generated $\mathbb{Z}$-subalgebra of K and by reducing it modulo a maximal ideal that has finite residue field) we see that the number of 1-dimensional irreducible components of $(Z \setminus W) \setminus e_Z^{-1}(Y)$ is equal to the number of 1-dimensional irreducible components of $(Z \setminus W) \setminus Y$. But the generic points of irreducible components of $(Z \setminus W) \setminus Y$ lie in the image of e_Z. As $e_Z^{-1}(Y) \subset e_Z^{-1}(Z \setminus W) \subset Z \setminus W$, by counting curves we see that $(Z \setminus W) \setminus e_Z^{-1}(Z \setminus W)$ has no 1-dimensional irreducible component and no 1-dimensional irreducible component of $e_Z^{-1}(Z \setminus W) \setminus e_Z^{-1}(Y)$ is contracted. In particular, the morphism $d : e_Z^{-1}(Z \setminus W) \to Z \setminus W$ induced by e is quasi-finite.

Let $\mathcal{Y} : Z^{\mathrm{n}} \to Z$ be the normalization of Z and let $e_{Z^{\mathrm{n}}} : Z^{\mathrm{n}} \to Z^{\mathrm{n}}$ be the endomorphism induced by e. Let $W_1 := \mathcal{Y}^{-1}(W)$. Let W_2 be the union of 1-dimensional irreducible components of W_1. Let $\Omega := Z^{\mathrm{n}} \setminus W_2$; the open embedding $\Omega \to Z^{\mathrm{n}}$ is affine by Nagata Theorem (see [H1], Cor. 3.3). Thus $e_{Z^{\mathrm{n}}}^{-1}(\Omega) \to Z^{\mathrm{n}}$ is affine, so the complement $Z^{\mathrm{n}} \setminus e_{Z^{\mathrm{n}}}^{-1}(\Omega)$ is purely of codimension 1 (see [SGA2], Exp. V, Ex. 3.4). But by the above part we get that $e_{Z^{\mathrm{n}}}$ sends $\mathcal{Y}^{-1}(Z \setminus W)$ to itself except for finitely many K-valued points. So $Z^{\mathrm{n}} \setminus e_{Z^{\mathrm{n}}}^{-1}(\Omega)$ does not meet $\mathcal{Y}^{-1}(Z \setminus W)$, hence does not meet Ω which is obtained by adding the isolated points of W_1. Thus $e_{Z^{\mathrm{n}}}$ restricts to an endomorphism $e_\Omega : \Omega \to \Omega$ which is quasi-finite as, up to finitely many points, it is the normalization of d.

Let $Q \in \Omega(K)$ that lifts $e^m(P)$. As $(\natural_{e_Z, e^m(P)})$ holds, $(\flat_{e_\Omega, Q})$ holds and hence $\iota(e_\Omega) = \infty$ by Lemma 16.5, which contradicts Theorem 16.8. $\square$

For a normal (possibly not connected) variety X over K, let $\rho_X \in \mathbb{N}$ be the rank of the Néron–Severi group $NS(X)$ of X (e.g., see [Kah], Thm. 3 for the classical result that $NS(X)$ is finitely generated). If l is a prime different from $\mathrm{char}(K)$, then the Kummer sequence in the étale topos of X gives an injective homomorphism $NS(X)/l^m NS(X) \to H^2(\mathrm{Reg}(X), \mu_{l^m})$ for all $m \in \mathbb{N}$, hence the homomorphism $\imath_Z : NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}_l \to H^2(\mathrm{Reg}(X), \mathbb{Q}_l(1)) = H^2(\mathrm{Reg}(X), \mathbb{Q}_l)$ is injective.[15]

[14] For an alternative proof see Example 27.5 of Section 27.

[15] The Kummer sequence applies as the group $\mathrm{Pic}(\mathrm{Reg}(X))_{\mathrm{a}}$ of divisor classes algebraically equivalent to zero is divisible. If X is projective, this group is in bijection to the group of K-valued points of the dual of the Albanese variety $\mathrm{Alb}(X)$ of X (see [Lang-S1], Ch. VI, Sect. 1, Thm. 1; one verifies that the proofs in the projective case give in general a surjection $\mathrm{Pic}(\mathrm{Alb}(X))_{\mathrm{a}} \to \mathrm{Pic}(\mathrm{Reg}(X))_{\mathrm{a}}$).

The following supplement to Theorem 16.8 also generalizes Lemma 6.1(2).

Theorem 16.16. *Let X be a non-empty variety X over K. For $i \in \{0,1\}$ let $r_i(X)$ be the number of irreducible components of X of dimension* $\dim(X)-i$. *Let Y be the normalization of the union of irreducible components of X of dimension* $\dim(X)$. *Let $N(X) := r_0(X) + r_1(X) + \rho(Y) - 1$. Then for each quasi-finite endomorphism $e : X \to X$ and integer $i \geq N(X)$ we have* $\dim\bigl(\mathrm{Im}(e^i) \setminus \mathrm{Im}(e^{i+1})\bigr) \leq \dim(X) - 2$.

Proof. There exists $i_0 \in [[0, r_0(X)-1]]$ such that the number of irreducible components of dimension $\dim(X)$ of the Zariski closure X_0 of $e^{i_0+j}(X)$ in X is independent on $j \in \mathbb{N}$ by Remark 16.7. Similarly, there exists $i_1 \in [[0, r_1(X)]]$ such that the number of irreducible components of dimension $\dim(X)-1$ of the Zariski closure X_1 of $e^{i_0+i_1+j}(X)$ in Z is independent of $j \in \mathbb{N}$. For $i \in \{0,1\}$ let Y_i be the normalization of the union of irreducible components of X_i of dimension $\dim(X)-i$. It follows that e induces endomorphisms $e_0 : Y_0 \to Y_0$ and $e_1 : Y_1 \to Y_1$. As Y_0 is a clopen of Y, we have $\rho_{Y_0} \leq \rho_Y$. To end the proof, by replacing e by e_0, it suffices to show that if X is normal and equidimensional, then for each quasi-finite endomorphism $e : X \to X$, the assumption that there exists $r \in \mathbb{N}$ such that $r > \rho_X$ and $\dim\bigl(\mathrm{Im}(e^r) \setminus \mathrm{Im}(e^{r+1})\bigr) \geq \dim(X) - 1$ leads to a contradiction. This assumption implies that there exists a quasi-finite morphism $\phi : X \to X \setminus (\cup_{i=1}^r Y_i)$, where $Y_1, \ldots, Y_r$ are distinct irreducible closed subvarieties of X of dimension $\dim(X)-1$ (say the one induced by e^r, with $Y_i \subset \mathrm{Im}(e^{i-1}) \setminus \mathrm{Im}(e^i)$). Let

$$\Omega := X \setminus \bigl[\bigl(\mathrm{Sing}(X) \cup_{i=1}^r \mathrm{Sing}(Y_i)\bigr) \bigcup (\cup_{1 \leq i < j \leq r} Y_i \cap Y_j)\bigr];$$

it is an open subvariety of $\mathrm{Reg}(X)$ whose codimension in X is at least 2.

Let $V := \Omega \setminus (\cup_{i=1}^r Y_i)$. We have an exact complex

$$0 = \oplus_{i=1}^r H^1_{Y_i \cap \Omega}(\Omega, \mathbb{Q}_l) \to H^1(\Omega, \mathbb{Q}_l) \to H^1(V, \mathbb{Q}_l)$$

$$\to \oplus_{i=1}^r H^2_{Y_i \cap \Omega}(\Omega, \mathbb{Q}_l) \xrightarrow{\phi^*} H^2\bigl(\mathrm{Reg}(X), \mathbb{Q}_l\bigr)$$

in the étale cohomology with $\mathbb{Q}_l$-coefficients; the last exactness follows from the injectivity of $\imath_{\mathrm{Reg}(X)}$, $\imath_V$ and of the pullback homomorphism $\phi^* : NS(V) \to NS\bigl(\phi^{-1}(V) \cap \mathrm{Reg}(X)\bigr) \cong NS\bigl(\mathrm{Reg}(X)\bigr)$.

As $r > \rho_X$, the divisor classes of the Y_is with $i \in [[i, r]]$ are dependent over $\mathbb{Q}_l$, hence from the compatibility of cycle classes with algebraic equivalence (see [SGA4.5], Cycle, Rm. 2.3.10) it follows that $\mathrm{Ker}(\phi^*) \neq 0$. Thus

$$\dim_{\mathbb{Q}_l}\bigl(H^1(V, \mathbb{Q}_l)\bigr) > \dim_{\mathbb{Q}_l}\bigl(H^1(\Omega, \mathbb{Q}_l)\bigr) = \dim_{\mathbb{Q}_l}\bigl(H^1(\mathrm{Reg}(X), \mathbb{Q}_l)\bigr) \tag{16}$$

with the equality by the purity of branch loci (see [SGA2], Exp. X, Thm. 3.4 (i)).

Let $j : W \to V$ be an open embedding such that ϕ is finite over W. The functorial homomorphism $\phi^* : H^1(W, \mathbb{Q}_l) \to H^1\bigl(\phi^{-1}(W), \mathbb{Q}_l\bigr)$ is injective as it has a left inverse which is $\deg(\phi)^{-1}$ times the trace map $H^1\bigl(\phi^{-1}(W), \mathbb{Q}_l\bigr) \to H^1(W, \mathbb{Q}_l)$ (see [SGA4-3], Exp. XVII, Thm. 6.2.3 and Ex. 6.2.6). The restriction homomorphism $H^1(V, \mathbb{Q}_l) \to H^1(W, \mathbb{Q}_l)$ is injective as for finite coefficients Λ we have $\Lambda = j_*(\Lambda)$.

From the last two sentences we get that the composite of $\phi^* : H^1(V, \mathbb{Q}_l) \to H^1\bigl(\phi^{-1}(V), \mathbb{Q}_l\bigr)$ with the restriction $H^1\bigl(\phi^{-1}(V), \mathbb{Q}_l\bigr) \to H^1\bigl(\phi^{-1}(W), \mathbb{Q}_l\bigr)$ is injective, hence $\phi^* : H^1(V, \mathbb{Q}_l) \to H^1\bigl(\phi^{-1}(V), \mathbb{Q}_l\bigr)$ itself is injective. Also, the restriction $H^1\bigl(\phi^{-1}(V), \mathbb{Q}_l\bigr) \to H^1\bigl(\phi^{-1}(V) \cap \mathrm{Reg}(X), \mathbb{Q}_l\bigr) \cong H^1\bigl(\mathrm{Reg}(X), \mathbb{Q}_l\bigr)$ is injective. From the last two sentences we get that Inequality (16) is false, a contradiction. $\square$

Remark 16.17. Let X_{perf} be the perfection of a variety X over k and we consider an endomorphism $e : X_{\mathrm{perf}} \to X_{\mathrm{perf}}$ with finite fibers. Then the proof of Theorem 16.8 applies to give that $\iota(e) \in \mathbb{N}$ and the proof of Theorem 16.16 applies to give that for every integer $i \geq N(X)$ we have $\dim\bigl(\mathrm{Im}(e^i) \setminus \mathrm{Im}(e^{i+1})\bigr) \leq \dim(X) - 2$. If $\mathrm{Frob}_k : \mathrm{Spec}\, k \to \mathrm{Spec}\, k$ is the Frobenius automorphism, then there exists $m \in \mathbb{Z}$ such that e is defined by a quasi-finite morphism $X \to (\mathrm{Frob}_k^m)^*(X)$. If $\dim(X) \leq 2$ and $d : X_{\mathrm{perf}} \to X_{\mathrm{perf}}$ is an endomorphism such that Γ_d is finite, then the proof of Proposition 16.15 applies to give that $\iota(d) \in \mathbb{N}$; there exists $l \in \mathbb{Z}$ such that d is defined by a morphism $d_l : X \to (\mathrm{Frob}_k^l)^*(X)$ with $(\mathrm{Frob}_k^l)^*(X) \setminus \mathrm{Im}(d_l)$ finite.

17. On the gap invariants for $n = 2$

To compute the set $\{\varphi(e) | e \in \mathrm{EE}_2(k), \psi_e \text{ is non-regular}\}$ we apply in the context of Example 14.1 the following result which is in contrast to the $n = 1$ case where only the strong 2-transitivity holds.

Proposition 17.1. *Assume that $n \geq 2$. Let $m \in \mathbb{N}^*$ and let $\mathcal{K}$ be a field with $|\mathcal{K}| \geq \frac{m(m-1)}{2} + 1$. If $P_1, \ldots, P_m$ and $Q_1, \ldots, Q_m$ are two sequences of m distinct points of $\mathbb{A}^n_{K,\mathrm{s}}(\mathcal{K})$ and $\mathbb{A}^n_{K,\mathrm{t}}(\mathcal{K})$ (respectively), then there exists a tame automorphism $a = \mathrm{e}(f_1, \ldots, f_n) \in \mathrm{GA}_n(\mathcal{K})$ such that for all $i \in [[1, m]]$ we have $a(P_i) = Q_i$, $\pi(f_1, \ldots, f_n) \leq \max\{1, (m-1)^2\}$ and $\pi(F_1, \ldots, F_n) \leq \max\{1, (m-1)^2\}$, where $a^{-1} = \mathrm{e}(F_1, \ldots, F_n)$.*

Proof. As $|\mathcal{K}| \geq \frac{m(m-1)}{2} + 1$, a vector space over $\mathcal{K}$ of dimension $n-1$ is not the union of at most $\frac{m(m-1)}{2}$ affine subsets of dimension $n-2$. Hence there exist $(\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_n) \in \mathcal{K}^n$ such that $\alpha_1 = \beta_2 = 1$ and for the tame automorphisms $a_1 := \mathrm{e}(\sum_{i=1}^n \alpha_i x_i, x_2, \ldots, x_n)$ and $a_2 := \mathrm{e}(x_1, \sum_{i=1}^n \beta_i x_i, x_3, \ldots, x_n)$, by defining $(\alpha_{i1}, \ldots, \alpha_{in}) := a_1(P_i)$ and $(\beta_{i1}, \ldots, \beta_{in}) := a_2(Q_i)$ for $i \in [[1, m]]$, $\alpha_{11}, \ldots, \alpha_{m1}$ and $\beta_{12}, \ldots, \beta_{m2}$ are two sequences of m distinct elements of $\mathcal{K}$. As $|\mathcal{K}| \geq m$, there exist tame automorphisms $b_1 := e\bigl(x_1, x_2 - g_2(x_1), \ldots, x_n - g_n(x_1)\bigr)$ and $b_2 := e\bigl(x_1 - h_1(x_2), x_2, x_3 - h_3(x_2), \ldots, x_n - h_n(x_2)\bigr)$ defined by Lagrange interpolating polynomials $g_2, \ldots, g_n, h_1, h_3, \ldots, h_n \in \mathcal{K}[x]$ of degree at most $m-1$ such that for all $i \in [[1, m]]$ we have $b_1(\alpha_{i1}, \beta_{i2}, 0, \ldots, 0) = a_1(P_i)$ and $b_2(\alpha_{i1}, \beta_{i2}, 0, \ldots, 0) = a_2(Q_i)$. Taking $a := a_2^{-1} b_2 b_1^{-1} a_1$, so $a^{-1} = a_1^{-1} b_1 b_2^{-1} a_2$, the proposition holds. □

In Proposition 17.1, the existence of a over infinite fields is well-known (e.g., [Sr], Thm. 2), so the only new things it brings are the estimates given by inequalities.

Definition 17.2. *Let $n, q \in \mathbb{N}$ with $n \geq 2$. By the affine space of dimension n punctured in q points over K we mean the quasi-affine smooth variety $\mathbb{B}^n_{q,K}$ over K obtained from $\mathbb{A}^n_K$ by removing q distinct K-valued points.*

Proposition 17.1 implies that, as the notation suggests, the isomorphism class of $\mathbb{B}^n_{q,K}$ over K depends only on $n \geq 2$ and q. We have $\mathbb{B}^n_{0,K} = \mathbb{A}^n_K$.

Example 17.3. Let $e \in \mathrm{QF}_2(K)$ be such that $\varphi(e) > 0$ and $\varsigma_e = 1$. So the $\mathcal{E} := \{P \in k^2 | |e^{-1}(P)| = 1\}$ is non-empty. We assume that $\mathcal{E}$ is infinite. E.g., if e is as in Example 2.11 we have $\mathcal{E} \supset K^* \times \{\alpha \in K | f(\alpha) = 0\}$, if $K = k$ and e is as in Example 13.2 with $s \geq 1$ we have $\mathcal{E} = \{\alpha_1, \ldots, \alpha_s\} \times k^*$, and if $K = k$ and e is as in Example 14.1 we have $\mathcal{E} \supset \{0\} \times (k \setminus \mathbb{I}^*_{p,m,q})$.

Let $\Gamma := \Gamma_e$. Let $(l,s) \in [[0,|\Gamma|]] \times \mathbb{N}^*$. Let Γ_0 be a subset of Γ with $|\Gamma_0| = l$. Let $\Gamma_0^\perp := \Gamma \setminus \Gamma_0$; we have $|\Gamma_0^\perp| = \varphi(e) - l$. Let $a_{l,s} \in \mathrm{GA}_2(K)$ be such that by defining $d_{l,s} := ea_{l,s} \in \mathrm{QF}_2(K)$ and $\Gamma_i := d_{l,s}(\Gamma_{i-1})$ and $\Gamma_i^\perp := d_{l,s}(\Gamma_{i-1}^\perp)$ for $i \in [[1,s]]$, for each $i \in [[0,s-1]]$ we have $a_{l,s}(\Gamma_i) \subset [e^{-1}(\mathcal{E})] \setminus \Gamma$ and $a_{l,s}(\Gamma_i^\perp) \cap e^{-1}(\mathcal{E}) = \emptyset$; the existence of $a_{l,s}$ follows from Proposition 17.1 applied to $s\varphi(e)$ points. We have disjoint unions $\sqcup_{i=0}^s \Gamma_i$ and $\sqcup_{i=0}^s \Gamma_i^\perp$ with $(s+1)l$ and $(s+1)[\varphi(e)-l]$ (respectively) elements. We have $\Gamma_i \subset \mathrm{Im}(d_{l,s}^i) \setminus \mathrm{Im}(d_{l,s}^{i+1})$ for each $i \in [[1,s]]$ as $d_{l,m}^{-1}(\Gamma_j) = \Gamma_{j-1}$ for $j \in [[0,i]]$ implies that $d_{l,s}^{-i-1}(\Gamma_i) = d_{l,s}^{-1}(\Gamma_0) = \emptyset$. Thus, with Notation 3.1 applied to $d_{l,s}(K) : K^2 \to K^2$ but using only $d_{l,s}$ instead of $d_{l,s}(K)$, we have $\mathcal{E}_i(d_{l,s}) \subset \cup_{j=1}^i d_{l,s}^j(\Gamma)$ by Lemma 3.2(4), so $\mathcal{E}_i(d_{l,s}) \subset \cup_{j=1}^i (\Gamma_j \cup \Gamma_j^\perp)$. It follows that $\mathcal{E}_i^1(d_{l,s}) = \sqcup_{j=1}^i \Gamma_j$ for each $i \in [[1,s]]$. Thus $\varrho_{d_{l,s}} \geq ls$, $\xi_{d_{l,s}} \geq s$, and $\iota(d_{l,s}) \geq s+1$ but $\varphi(d_{l,s}) = \varphi(e)$ and $\deg(d_{l,s}) = \deg(e)$ do not depend on either l or s.

Directly from Example 17.3 we get the following result.

Corollary 17.4. *For each $r \in \mathbb{N}^*$ the sets* $\{\iota(e)|e \in \mathrm{EE}_2(k), \varphi(e) = r, \psi_e \text{ is étale}\}$, $\{\iota(e)|e \in \mathrm{EE}_2(k), \varphi(e) = r, \psi_e \text{ is non-étale}\}$ *and* $\{\iota(e)|e \in \mathrm{QF}_2(K), \varphi(e) = r\}$ *are infinite.*

Example 17.5. We refer to $e \in \mathrm{EE}_2(k)$ of Example 14.1; hence $\varphi(e) = mq-1$, $\Gamma_e = \{(0,\beta)|\beta \in \mathbb{I}_{p,m,q}^*\}$ and the set $\mathcal{B} := \{P \in k^2 | e^{-1}\big((e(P)\big) = \{P\}\}$ is equal to $\{\big(-(1+(-1)^m\beta^{mq-1})^{-q}, \beta + (-1)^m\beta^{mq}\big)|\beta \in k \setminus \mathbb{I}_{p,m,q}^*\}$. We apply Example 17.3 with $l \in [[0,mq-1]]$ and $s \in \mathbb{N}^*$; e.g., as $\mathcal{F}(e) = \{0,1,mq-1\}$, if $mq > 1$ and $s = 1$, $a_{l,1} \in \mathrm{GA}_2(k)$ is subject to the only constraint that for $\Gamma_0 := \{P \in \Gamma_e | a_{l,1}(P) \in \mathcal{B}\}$ we have $|\Gamma_0| = l$. If $P \in \mathcal{E}_1(d_{l,s}) \setminus \mathcal{E}_1^1(d_{l,s}) \subset d_{l,s}(\Gamma_0) \setminus \Gamma_1$, then $P \in \Gamma_1^\perp$ is such that $e^{-1}(P)$ is contained in $\Gamma \setminus \Gamma_0 = \Gamma_0^\perp$; as $|\Gamma_0^\perp| = mq-1-l$ and $\mathcal{F}(d_{l,s}) = \mathcal{F}(e) = \{0,1,mq-1\}$, such a point P can exist only when $l = 0$ and if it exists it is unique. Thus for $l \in [[1,mq-1]]$, the set $\mathcal{E}_1(d_{l,s}) = \mathcal{E}_1^1(d_{l,s}) = \Gamma_1$ has l elements. Moreover,

$$\varphi(d_{l,s}^2) = \varphi(d_{l,s}) + l \ \text{ and } \ \iota(d_l) \geq 2 \tag{17}$$

by Lemma 3.2(3). Based on Proposition 17.1 applied to $mq-1$ points and Example 14.1, we can choose $a_{l,1}$ such that we have

$$\pi(g_1,g_2) \leq \max\{mq + \max\{(mq-1)q, m\}, (mq-2)^2[mq + \max\{(mq-1)q, m\}]\}.$$

We have $\varphi(d_{mq-1,s}^2) = 2\varphi(d_{mq-1,s})$. If $mq = p^r$ with $r \in \mathbb{N}^*$, then $\mathbb{I}_{p,m,q} = \mathbb{F}_{p^r}$; thus for $l \in [[1,mq-1]]$, $a_{l,s}$ is not defined over $\mathbb{F}_{p^r}$.

Theorem 17.6. *We have identities* $\{\varphi(e)|e \in \mathrm{EE}_2(k), \psi_e \text{ is non-étale}\} = \mathbb{N}$ *and*

$\{\varphi(e)|e \in \mathrm{D}_2(K) \setminus \mathrm{QF}_2(K) \text{ defined over the prime field of } K, \iota(e) \in \mathbb{N}\} = \mathbb{N} \cup \{\infty\}$.

Proof. Let $r \in \mathbb{N}$. To show that there exists $e_r \in \mathrm{EE}_2(k)$ such that $\varphi(e_r) = r$, in order to compute the geometric degrees and estimate the algebraic degrees, we consider four disjoint cases as follows. In all cases e_r are composites of étale endomorphisms among which at least one has a Jacobian surface which is non-étale based on Theorem 2.4(3) and hence ψ_{e_r} is automatically non-étale.

Case 1: $r = 0$. We take e_0 to be a composite cd with d non-étale with $\deg(d) = p$, $\varphi(d) = p-1$, and $\pi(d) \leq 2p$ if $p > 2$ and with $\deg(d) = 4$, $\varphi(d) = 3$, and $\pi(d) \leq 8$ if $p = 2$ (see Theorem 2.4(3)) and with c a finite étale endomorphism with $\deg(c) = p$ and $\pi(c) \leq p$ if $p > 2$ and with $\deg(c) = 4$ and $\pi(c) \leq 4$ if $p = 2$. As $\varphi(d) < \deg(c)$, $e_0 = cd$ is surjective, i.e., $\varphi(e_0) = 0$.

Case 2: $r = po - 1$ with $o \in \mathbb{N}^*$ and $(p, o) \neq (2, 1)$. We take e_r to be the e of Example 14.1 with $m = po$ and $q = 1$, thus $\deg(e_r) = mq = po$, $\varphi(e_r) = r$, and $\pi(e_r) \leq 2mq = 2po$.

Case 3: $po - 1 < r < p(o + 1) - 1$ with $o \in \mathbb{N}^*$. We take e_r to be $d^2_{r-po+1,s}$ of Example 17.5 applied to $m = po$ and $q = 1$; as $r - po + 1 \in [[1, mq - 1]]$ we have $\varphi(e_r) = \varphi(d_{r-po+1,s}) + r - po + 1 = r$, $\deg(e_r) = \deg(d_{r-po+1,s})^2 = (mq)^2 = p^2o^2$, and when $s = 1$ we have $\pi(e_r) \leq \max\{4p^2o^2, 4p^2o^2(po - 2)^4\}$.

Case 4: $r \in [[1, p - 2]]$ or $(p, r) = (2, 1)$. Let $e_{pr} \in \mathrm{EE}_2(k)$ be such that $\varphi(e_{pr}) = pr$ (see Case 3 with $o = r$); we have relations $\deg(e_{pr}) = (pr)^2$ and $\pi(e_{pr}) \leq \max\{4p^2r^2, 4p^2r^2(pr - 2)^4\}$. Based on Proposition 17.1, let $a \in \mathrm{GA}_2(k)$ be such that the set $\Gamma_{ae_{pr}}$ is $\mathbb{F}_p \times \mathbb{I}_r$ with $\mathbb{I}_r$ a subset of k that has r elements. If e_r is the composite of $ae_{pr} : \mathbb{A}^2_{k,\mathrm{s}} \to \mathbb{A}^2_{k,\mathrm{t}}$ with the finite étale endomorphism $\mathbb{A}^2_{k,\mathrm{t}} \to \mathbb{A}^2_{k,\mathrm{t}}$ which is identified naturally with the quotient epimorphism $\mathbb{G}^2_{a,k} \to \mathbb{G}^2_{a,k}/(\mathbb{F}_p \times 0) \cong \mathbb{G}^2_{a,k}$, then $\Gamma_{e_r} = \{0\} \times \mathbb{I}_r$, hence $\varphi(e_r) = r$; we have relations $\deg(e_r) = p\deg(e_{pr}) = p^3r^2$ and $\pi(e_r) \leq \max\{4p^3r^2, 4p^3r^2(pr - 2)^4\}$. As $\varsigma_{e_r} - 1 \geq p - 1 \geq r$, we have $\iota(e_r) = 1$ by Example 16.1.

All endomorphisms of this paragraph are defined over the prime field of K. The rule $(x, y) \mapsto (xy, y)$ (resp. $(x, y) \mapsto (xy, y + 1)$) defines $e_K \in \mathrm{D}_2(K) \setminus \mathrm{QF}_2(K)$ with $\pi(e_K) = 2$, $\varphi(e_K) = \infty$ and $\iota(e_K) = 1$ (resp. and $\iota(e_K)$ equal to ∞ if $\mathrm{char}(K) = 0$ by Example 16.11(1) and equal to p if $\mathrm{char}(K) = p$). The rule $(x, y) \mapsto (xy, y^2 + y)$ defines $d_K \in \mathrm{D}_2(K) \setminus \mathrm{QF}_2(K)$ with $\varphi(d_K) = \iota(d_K) = 0$ and $\pi(d_K) = 2$. If $r \in \mathbb{N}^*$, let $e_{r,K} \in \mathrm{QF}_2(K)$ be such that $\pi(e_{r,K}) \leq 2q + 1$ and $\varphi(e_{r,K}) = r$ (see Example 2.11); we have $\varphi(e_{r,K}d_K) = r$, $\pi(e_{r,K}d_K) \leq 4q + 2$, and Corollary 16.2 gives $\iota(e_{r,K}d_K) \in \mathbb{N}^*$ provided $\mathrm{char}(K) = p$. If $\mathrm{char}(K) = 0$, then for a finite surjective endomorphism $c \in \mathrm{End}_2(K)$ with $\varsigma_c \geq 2$, we have $e_{r,K}d_Kc \in \mathrm{D}_2(K) \setminus \mathrm{QF}_2(K)$ by Remark 6.2, $\varphi(e_{r,K}d_Kc) = r$ and (see Example 16.1) $\iota(e_{r,K}d_Kc) \in \mathbb{N}$. □

We have the following consequence of either Theorem 2.4(2) or Theorem 17.6.

Corollary 17.7. *Let $l, q \in \mathbb{N}$. Then there exist an affine connected smooth surface Y over $\mathrm{Spec}\, k$ that has an open subvariety isomorphic to $\mathbb{A}^2_k$ and that is equipped with an étale endomorphism $d : Y \to Y$ whose complement $Y \setminus d(Y)$ is a disjoint union of l copies of $\mathbb{A}^1_k$ and of q points.*

Proof. We can assume $l \geq 1$ by either Theorem 2.4(2) or Theorem 17.6. Let $e \in \mathrm{EE}_2(k)$ be such that $\varphi(e) = q$ and $\deg(e) \geq p$. Let $m \in \mathbb{N}^*$ be such that $pm - 1 \geq l$. Let $(f, g) := (t^{pm-1} - 1, 1) \in \Theta^1_{k,m}$. We take Y to be an open subvariety of $\mathbb{S}^2_{f,g,k}$ which is the union of $\mathrm{Im}(\imath_{f,g,k})$ and of l connected components of $\mathbb{D}^1_{f,g,k}$ (see Section 2; $|\mathrm{Irr}(\mathbb{D}^1_{f,g,k})| = pm - 1$). Let $\imath_Y : Y \to \mathbb{S}^2_{f,g,k}$ be the open embedding and let $\imath_{f,g,k,Y} : \mathbb{A}^2_k \to Y$ be the factorization of $\imath_{f,g,k}$; as $c_{f,g,k} \circ \imath_{f,g,k}$ is surjective, so is $c_{f,g,k} \circ \imath_Y$. Then the endomorphism

$$d := \imath_{f,g,k,Y} \circ e \circ c_{f,g,k} \circ \imath_Y : Y \to Y$$

is étale and $Y \setminus d(Y) = Y \setminus \mathrm{Im}(\imath_{f,g,k}) \sqcup \Gamma_e$ is a disjoint union of l copies of $\mathbb{A}^1_k$ and of q points; we have $\deg(d) = pm\deg(e) \geq p(l + 1)$. For $\deg(e) = p$ (see Theorem 2.4(2) for $q \geq 1$) and $m := \lceil \frac{l+1}{p} \rceil$ we get the smallest value $\deg(e) = p^2\lceil \frac{l+1}{p} \rceil$. □

Corollary 17.8. *For each $(m, q) \in \mathbb{N}^2$, with the notation of Definition 17.2 the following properties hold.*

(1) *There exists a surjective étale morphism* $d_{m,q}:\mathbb{B}^2_{m,k}\to\mathbb{B}^2_{q,k}$ *which extends (or which does not extend) to a finite étale cover of* $\mathbb{B}^2_{q,k}$.

(2) *There exists a surjective quasi-finite morphism* $d_{m,q}:\mathbb{B}^2_{m,K}\to\mathbb{B}^2_{q,K}$.

Proof. The existence of $d_{0,q}$ in part (1) follows from Theorem 2.4(2) in case we want it to extend to a finite étale cover of $\mathbb{B}^2_{q,k}$ and from Theorem 17.6 in case we do not want it to extend to a finite étale cover of $\mathbb{B}^2_{q,k}$. The existence of $d_{0,q}$ in part (2) follows from Example 2.11. If $q>0$, then $d_{0,q}$ is generically étale and non-birational. For part (1) (resp. (2)), we choose $d_{0,0}$ to be étale (resp. generically étale) and finite non-birational. Hence there exists an infinite number of points $P\in\mathbb{B}^2_{q,k}(k)$ (resp. $P\in\mathbb{B}^2_{q,k}(K)$) such that $|d_{0,q}^{-1}(P)|\geq 2$; by removing from each one of m such distinct fibers one point, $d_{0,q}$ restricts to a surjective morphism $d_{m,q}:\mathbb{B}^2_{m,k}\to\mathbb{B}^2_{q,k}$ (resp. $d_{m,q}:\mathbb{B}^2_{m,K}\to\mathbb{B}^2_{q,K}$) which is étale (resp. quasi-finite). □

18. Proof of Corollary 2.5

Proof of Corollary 2.5. Based on Lemma 15.2(2), to prove Corollary 2.5 we can assume that $n=l+2$. Let $d\in\mathrm{EE}_2(k)$ be such that $\varphi(d)=q$ by Theorem 2.4(2) or Theorem 17.6 and let $(g_1,g_2)\in k[x_1,x_2]^2$ define d. Let $(s_1,s_2)\in(\mathbb{N}^*)^2$ be such that $ps_1>\deg(g_1)$ and $ps_2>\deg(g_2)$. We write $X=\mathrm{Spec}\,\big(k[x_3,\ldots,x_n]/(f_1,\ldots,f_m)\big)$, with $m\in\mathbb{N}$ and $(f_1,\ldots,f_m)$ a radical ideal of $k[x_3,\ldots,x_n]$. Then the n-tuple

$$\big(g_1+\sum_{i=1}^{m}x_1^{p(s_1+i-1)}f_i,\,g_2+\sum_{i=1}^{m}x_2^{p(s_2+i-1)}f_i,\,x_3,\ldots,x_n\big)\in k[x_1,\ldots,x_n]^n$$

defines an endomorphism $e\in\mathrm{EE}_n(k)$ that induces the endomorphism $d\times 1_X$ on $\mathbb{A}^2_k\times_{\mathrm{Spec}\,k}X$ and an étale endomorphism $d_P\times 1_P$ on $\mathbb{A}^2_k\times_{\mathrm{Spec}\,k}P$ for each $P\in[\mathrm{Spec}\,(k[x_3,\ldots,x_n])\setminus X](k)$ which is finite (hence surjective) by Fact 15.1(2). Thus we have $\Gamma_{e^i}=\Gamma_{d^i}\times X$ for every $i\in\mathbb{N}^*$. Hence $\iota(e)=\iota(d)$ and thus $\iota(e)\in\mathbb{N}$ by Theorem 16.8. Corollary 2.5 holds by taking $q_i:=\varphi(d^i)$ for each $i\in\mathbb{N}^*$ (so $q_1=q$).

Newxt we use the proof of Corollary 2.5 in a way that relates to the traps of Example 16.11 in order to show that there exists $e\in\mathrm{QF}_3(K)$ such that $\iota(e)\in\mathbb{N}$ and Γ_e^+ is infinite (cf. Lemma L1(6)).

Example 18.1. Assume K is not an algebraic closure of a finite field. Let $\alpha\in K^*$ of infinite multiplicative order. If $\mathrm{char}(K)=0$ (resp. $\mathrm{char}(K)=p$) let $e\in\mathrm{QF}_2(K)$ (resp. $e\in\mathrm{EE}_2(K)$) be such that $\varphi(e)>0$. Let $(g_1,g_2)\in k[x_1,x_2]^2$ define e and let $(s_1,s_2)\in(\mathbb{N}^*)^2$ (resp. $(s_1,s_2)\in p(\mathbb{N}^*)^2$) be such that $s_1>\deg(g_1)$ and $s_2>\deg(g_2)$. Then the triple $\big(g_1+(x_3-1)x_1^{s_1},g_2+(x_3-1)x_2^{s_2},\alpha x_3\big)$ defines a $d\in\mathrm{QF}_3(K)$ (resp. $d\in\mathrm{EE}_3(k)$) such that $\varphi(d)=\varphi(e)$, Γ_d is the finite set $\Gamma_e\times\{1\}$ with $\Gamma_d^+=\cup_{i\in\mathbb{N}}d^i(\Gamma_d)$ infinite, and $\iota(d)\in\mathbb{N}$ by either Theorem 16.8 or Lemma 3.2(10) as it is easy to see that the set $\{P\in\Gamma_d^+||d^{-1}(P)|=1\}$ is finite. If $\varphi(e)=1$, then d induces a bijection $\Gamma_d^+\to\Gamma_d^+\setminus\Gamma_d$.[16]

Lemma 15.2(2) increases geometric and algebraic degrees considerably as n grows: e.g., if $p>3$ and $e\in\mathrm{EE}_2(k)$ is as in Example 13.2 with $\deg(e)=pm$ and $\deg(g)=1$ (hence $s=1$), then for $d\in\mathrm{EE}_3(k)$ as in the proof of Lemma 15.2(2) we have $\deg(d)\geq 5p^2m$ and for a $d_2\in\mathrm{EE}_4(k)$ obtained from d as in the proof of Lemma 15.2(2) we have $\deg(d_2)\geq 2p^3(5m+1)$. We include an example

[16] If $\mathrm{char}(K)=0$, then we can also take the triple $\big(g_1+x_3x_1^{s_1},g_2+x_3x_2^{s_2},x_3+1\big)$ to define d.

that starts from two cases of Example 14.1, that pertain to Corollary 2.5 with $l = 1$ and with either $q = p^2 - p$ and $e \in \mathrm{BE}_2(k)$ or $q = p - 1$ and $e \in \mathrm{EE}_2(k)$, and that for $n \geq 3$ involve smaller geometric and algebraic degrees than what Lemma 15.2(2) would give when n increases.

Example 18.2. Assume $n \geq 3$. Let $e \in \mathrm{BE}_n(k)$ be defined by the n-tuple

$$\big(x_1 - x_1^{p^2} x_2^{p(p^2-1)}, x_2 - x_1^p x_2^{p^2} - x_3^p, x_3 - x_3^p - \sum_{i=4}^{n} x_i^p x_3^{p^2+p(i-4)}, x_4, x_5, \ldots, x_n\big)$$

in $k_{\mathrm{s}}[x_1, \ldots, x_n]^n$ (cf. Example 14.1 applied to $m = q = p$ and $\mathbb{I}_{p,p,p} = \mathbb{F}_{p^2}$). For $\underline{\alpha} := (\alpha_1, \ldots, \alpha_n) \in k^n$, the number $\mathrm{s}(\underline{\alpha})$ of solutions of the F_3-system

$$x_1 - x_1^{p^2} x_2^{p(p^2-1)} - \alpha_1 = x_2 - x_1^p x_2^{p^2} - x_3^p - \alpha_2 = x_3 - x_3^p - \sum_{i=4}^{n} x_i^p x_3^{p^2+p(i-4)} - \alpha_3 = 0$$

equals the number of solutions of the F_n-system obtained by adding to the F_3-system the $n-3$ equations $x_i - \alpha_i = 0$ indexed by $i \in [[4, n]]$.

Case 1: there exists $i \in \{4, \ldots, n\}$ such that $\alpha_i \neq 0$. Then the equation

$$x_3 - x_3^p - \sum_{i=4}^{n} \alpha_i^p x_3^{p^2+p(i-4)} = 0$$

has at least p^2 distinct zeros, hence it has a zero β such that $\beta^p + \alpha_2 \notin \mathbb{F}_{p^2}^*$. From this and Example 14.1 applied as mentioned we get that $\mathrm{s}(\underline{\alpha}) > 0$.

Case 2: $\alpha_4 = \alpha_5 = \cdots = \alpha_n = 0$. If $\beta \in k$ is a zero of the equation $x_3 - x_3^p - \alpha_3 = 0$, then $\{\beta + i | i \in \mathbb{F}_p\}$ is the solution set of this equation. From this and Example 14.1, as $\beta^p + \alpha_2 \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p \Leftrightarrow \{\beta^p + i^p + \alpha_2 | i \in \mathbb{F}_p\} \subset \mathbb{I}_{p,p,p}^* = \mathbb{F}_{p^2}^*$ we get that

$$\mathrm{s}(\underline{\alpha}) = 0 \Leftrightarrow [\alpha_1 = -1 \text{ and } \beta^p + \alpha_2 \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p].$$

Based on the two cases we get a disjoint union decomposition

$$\Gamma_e = \sqcup_{\alpha \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p} \{(-1, -t^p + \alpha, -t^p + t, 0, 0, \ldots, 0) | t \in k\}$$

into $p^2 - p$ curves isomorphic to $\mathbb{A}_k^1$.

If $d \in \mathrm{EE}_n(k)$ is defined by

$$\big(x_1 - x_1^p x_2^{p-1}, x_2 - x_1^p x_2^p - x_3, x_3 - \sum_{i=4}^{n} x_i x_3^{p(i-3)}, x_4, x_5, \ldots, x_n\big) \in k[x_1, \ldots, x_n]^n$$

(cf. Example 14.1 applied to $m = p$, $q = 1$ and $\mathbb{I}_{p,p,1} = \mathbb{F}_p$), then a similar argument gives a disjoint union decomposition

$$\Gamma_d = \sqcup_{\alpha \in \mathbb{F}_p^*} \{(-1, -t + \alpha, t, 0, 0, \ldots, 0) | t \in k\}$$

into $p - 1$ curves isomorphic to $\mathbb{A}_k^1$.

19. Proof of Theorem 2.7

For $q \in \mathbb{N}^*$, we denote by F_{p^q} all functorial endomorphisms of linear algebraic group over $\mathrm{Spec}\, \mathbb{F}_{p^q}$ which for $\mathbf{GL}_{n,\mathbb{F}_{p^q}}$s is defined by the rule that maps each matrix into the matrix whose entries are raised to the p^q-th power. For Lang torsors we refer to [Bore2], Ch. V, Thm. 16.3 and Cor. 16.4.

The Lang torsor for $\mathbf{SL}_{2,k}$ over $\mathrm{Spec}\, \mathbb{F}_{p^q}$ is a morphism

$$\mathcal{L} : \mathbf{SL}_{2,k,\mathrm{s}} = \mathbf{SL}_{2,k} \to \mathbf{SL}_{2,k,\mathrm{t}} = \mathbf{SL}_{2,k}$$

defined by the following rule

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \mapsto \begin{bmatrix} w & x \\ y & z \end{bmatrix} \cdot F_{p^q}\left(\begin{bmatrix} w & x \\ y & z \end{bmatrix}\right)^{-1} = \begin{bmatrix} wz^{p^q} - xy^{p^q} & -wx^{p^q} + xw^{p^q} \\ yz^{p^q} - zy^{p^q} & -yx^{p^q} + zw^{p^q} \end{bmatrix}$$

on valued points; so $\mathcal{L}$ is a right $\mathbf{SL}_2(\mathbb{F}_{p^q})$-torsor under the right translation action of $\mathbf{SL}_2(\mathbb{F}_{p^q})$ on $\mathbf{SL}_{2,k,\mathrm{s}}$ (cf. [Bore2], Ch. V, Cor. 16.4).

Let $\star \in \{+, -\}$. Let $B^\star$ be the Borel subgroup scheme of $\mathbf{SL}_2$ of 2×2 matrices of determinant 1 that are upper triangular if $\star = +$ and are lower triangular if $\star = -$; we have a short exact sequence $1 \to \mathbb{U}^\star \to B^\star \to \mathbb{G}_m \to 1$ of group schemes over $\operatorname{Spec}\mathbb{Z}$, where $U^\star \cong \mathbb{G}_a$ is the unipotent radical of $B^\star$.

The product morphism

$$\mathcal{T} : U_k^3 := U_k^- \times_{\operatorname{Spec} k} U_k^+ \times_{\operatorname{Spec} k} U_k^- \cong \mathbb{A}_k^3 \to \mathbf{SL}_{2,k,\mathrm{t}}$$

is defined by the k-algebra homomorphism $\mathcal{T}^\# : k_{\mathrm{t}}[w,x,z,x^{-1}] \to k[x_1,x_2,x_3]$ that maps w, x, y and z to $1+x_2x_3$, x_2, $x_1+x_3+x_1x_2x_3$ and $1+x_1x_2$ (respectively). Thus $\mathcal{T}$ is birational; more precisely, it induces an isomorphism

$$U_k^3 \setminus D_k = \operatorname{Spec}(k[x_1,x_2,x_3,x_2^{-1}]) \cong \mathbf{SL}_{2,k,\mathrm{t}} \setminus B_k^- = \operatorname{Spec}(k_{\mathrm{t}}[w,x,z,x^{-1}]), \tag{18}$$

where $D := U^- \times_{\operatorname{Spec}\mathbb{Z}} 1_{U^+} \times_{\operatorname{Spec}\mathbb{Z}} U^-$ with 1_{U^+} as the identity element subgroup scheme of U^+. Hence $D_k = \mathcal{T}^{-1}(B_k^{-1})$ and the pullback $\mathcal{T}^*$ on divisor groups defined by $\mathcal{T}$ is bijective.

We view $\mathcal{T}$ and $\mathcal{L}$ as $\mathbb{G}_{a,k}$-invariant morphisms, the actions of $\mathbb{G}_{a,k}$ on U_k^3, $\mathbf{SL}_{2,k,\mathrm{t}}$ and $\mathbf{SL}_{2,k,\mathrm{s}}$ being defined by the identification $\mathbb{G}_{a,k} = U_k^-$ and the respective rules on valued points

$$\big(u,(u_1,u_2,u_3)\big) \mapsto \big(u\cdot u_1, u_2, u_3\cdot F_{p^q}(u)^{-1}\big), (u,x) \mapsto u\cdot x\cdot F_{p^q}(u)^{-1}, (u,x)\mapsto u\cdot x.$$

We consider the cartesian diagram of $\mathbb{G}_{a,k}$-varieties

$$\begin{tikzcd} \mathbb{X}_k^3 \arrow[r, "\mathcal{I}"] \arrow[d, "\mathcal{M}_3"] & \mathbf{SL}_{2,k,\mathrm{s}} \arrow[d, "\mathcal{L}"] \\ U_k^3 \arrow[r, "\mathcal{T}"] & \mathbf{SL}_{2,k,\mathrm{t}} \end{tikzcd}$$

which is defined over $\mathbb{F}_{p^q}$; so $\mathbb{X}_k^3$ has a canonical model $\mathbb{X}^3$ which is a $\mathbb{G}_{a,\mathbb{F}_{p^q}}$-variety over $\operatorname{Spec}\mathbb{F}_{p^q}$. Moreover, $\mathcal{M}_3 : \mathbb{X}_k^3 \to U_k^3$ is a right $\mathbf{SL}_2(\mathbb{F}_{p^q})$-torsor. From the rule defining $\mathcal{L}$ and the description of $\mathcal{T}^\#$ we get that

$$\mathbb{X}_k^3 = \operatorname{Spec} k[x_1,x_2,x_3,w,x,y,z]/(J),$$

where J is the ideal generated by $x_2x_3 - wz^{p^q} + xy^{p^q} - 1$, $x_2 + wx^{p^q} - xw^{p^q}$, $x_1x_2x_3 + x_1 + x_3 - yz^{p^q} + zy^{p^q}$, $x_1x_2 + yx^{p^q} - zw^{p^q} - 1$, and $wz - yz - 1$.

We have a disjoint union decomposition

$$\mathbb{B}_k^2 := \mathcal{L}^{-1}(B_k^-) = \sqcup_{i=1}^{p^q+1} B_k^- \tau_i$$

where $\tau_1, \ldots, \tau_{p^q+1} \in \mathbf{SL}_2(\mathbb{F}_{p^q})$ are representatives of the quotient set

$$B^-(\mathbb{F}_{p^q})\backslash \mathbf{SL}_2(\mathbb{F}_{p^q}).$$

For each $i \in [[1, p^q+1]]$ we have a disjoint union decomposition

$$B_k^-\tau_i \cap \mathcal{L}^{-1}\big(\mathcal{T}(D_k)\big) = B_k^-\tau_i \cap \mathcal{L}^{-1}(U_k^-) = \sqcup_{j=1}^{p^q-1} U_k^- \varpi_j \tau_i$$

where $\varpi_1, \ldots, \varpi_{p^q-1} \in B^-(\mathbb{F}_{p^q})$ are representatives of the quotient set

$$U^-(\mathbb{F}_{p^q})\backslash B^-(\mathbb{F}_{p^q}).$$

E.g., if we choose $\{\tau_1, \ldots, \tau_{p^q+1}\} = \{\begin{bmatrix} \alpha & 1 \\ -1 & 0 \end{bmatrix} | \alpha \in \mathbb{F}_{p^q}\} \cup \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}$ and $\{\varpi_1, \ldots, \varpi_{p^q-1}\} = \{\begin{bmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{bmatrix} | \beta \in \mathbb{F}^*_{p^q}\}$, it follows that $\mathbb{B}^2_k$ is the closed subscheme of $\mathbf{SL}_{2,k}$ defined by the equation $w^{p^q}x = wx^{p^q}$ and one computes that

$$\mathbb{G}_{a,k}\backslash[\mathbf{SL}_{2,k,\mathrm{s}} \setminus \mathbb{B}^2_k] \cong \mathbb{A}^2_k \setminus [\{(0,0)\} \cup \{(x_1, x_2) | (x_1 : x_2) \in \mathbb{P}^1(\mathbb{F}_{p^q})\}].$$

From the last two disjoint union decompositions we get that for each $i \in [[1, p^q + 1]]$ we have a disjoint union decomposition into irreducible divisors

$$\mathcal{I}^{-1}(B^-_k \tau_i) = \sqcup_{j=1}^{p^q-1} \mathcal{I}^{-1}(U^-_k \varpi_j \tau_i)$$

that are canonically isomorphic to the fiber product

$$D_k \times_{\mathcal{T}|D_k, U^-_k, \mathcal{L}|U^-_k} U^-_k \cong \mathbb{A}^2_k.$$

Here $\mathcal{T}|D_k : D_k \to U^-_k$ and $\mathcal{L}|U^-_k : U^-_k \to U^-_k$ are restrictions of $\mathcal{T}$ and $\mathcal{L}$ (respectively) with codomains replaced by U^-_k and the product morphism $\mathcal{T}|D_k$ is a trivial line bundle. From these disjoint union decompositions and canonical isomorphisms and the existence of the Isomorphism (18), we get that the pullback $\mathcal{I}^*$ on divisor groups defined by $\mathcal{I}$ is injective. From this and the identity $\mathcal{O}(\mathbf{SL}_{2,k,\mathrm{s}})^* = k^*$ implied by the existence of $\mathcal{T}$, we get that $\mathcal{O}(\mathbb{X}^3_k)^* = k^*$.

As $\mathcal{M}_3 : \mathbb{X}^3_k \to U^3_k$ is a right $\mathbf{SL}_2(\mathbb{F}_{p^q})$-torsor, by taking quotients through the p-Sylow subgroup $H^\star := U^\star(\mathbb{F}_{p^q})$ of $\mathbf{SL}_2(\mathbb{F}_{p^q})$, we get a finite étale morphism

$$\mathbb{Y}^{3,\star}_k := \mathbb{X}^3_k/H^\star \to \mathbb{A}^3_k \tag{19}$$

of degree $p^{2q} - 1$ with $\mathcal{O}(\mathbb{Y}^{3,\star}_k)^* = k^*$ and a birational morphism $\mathcal{I}/H^\star : \mathbb{Y}^{3,\star}_k \to \mathbf{SL}_{2,k,\mathrm{s}}/H^\star$, both defined over $\mathbb{F}_{p^q}$. To show that $\mathbf{SL}_{2,k,\mathrm{s}}/H^\star$ is rational we can work only with $\star = -$ and this case follows from the fact that $\mathcal{T}$ is H^--invariant inducing a birational morphism

$$U^3_k/H^- = U^-_k \times_{\operatorname{Spec} k} U^+_k \times_{\operatorname{Spec} k} (U^-_k/H^-) \to \mathbf{SL}_{2,k,\mathrm{s}}/H^-;$$

note that $(U^-_k/H^-) \cong U^-_k \cong \mathbb{G}_{a,k}$ (see [SGA3-2], Exp. XVII, Props. 2.2 and 4.1.1).

Taking quotients through $G^\star := B^\star(\mathbb{F}_{p^q})$, we similarly get a finite étale morphism

$$\mathbb{W}^{3,\star}_k := \mathbb{X}^3_k/G^\star \to \mathbb{A}^3_k \tag{20}$$

of degree $p^q + 1$ with $\mathcal{O}(\mathbb{W}^{3,\star}_k)^* = k^*$ and a birational morphism $\mathcal{I}/G^\star : \mathbb{W}^{3,\star}_k \to \mathbf{SL}_{2,k,\mathrm{s}}/G^\star$, both defined over $\mathbb{F}_{p^q}$. To show that $\mathbb{W}^{3,\star}_k$ is rational we can work only with $\mathbf{SL}_{2,k,\mathrm{s}}/G^\star$ and assume that $\star = +$. As the product morphism $U^-_k \times_{\operatorname{Spec} k} B^+_k \to \mathbf{SL}_{2,k}$ is G^+-invariant and an open embedding, it induces an open embedding

$$U^-_k \times_{\operatorname{Spec} k} (B^+_k/G^+) \to \mathbf{SL}_{2,k,\mathrm{s}}/G^+.$$

The Lang torsor for B^+_k over $\operatorname{Spec} \mathbb{F}_{p^q}$ induces an isomorphism $B^+_k/G^+ \cong B^+_k \cong \mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{G}_{m,k}$. From the last two sentences we get that $\mathbf{SL}_{2,k,\mathrm{s}}/G^+$ is birational to $\mathbb{A}^2_k \times_{\operatorname{Spec} k} \mathbb{G}_{m,k}$ and thus it is rational.

The morphism $\mathcal{M}_3$ and Morphisms (19) and (20) already give Theorem 2.7(1) for $n=3$. To get it for $n=2$, we recall that our varieties come with $\mathbb{G}_{a,k}$-actions by which we can quotient and define

$$\mathbb{E}_k := \mathcal{M}_3^{-1}(D_k) = \mathcal{M}_3^{-1}\big(\mathcal{T}^{-1}(B_k^-)\big) = (\mathcal{T}\circ\mathcal{M}_3)^{-1}(B_k^-) = (\mathcal{L}\circ\mathcal{I})^{-1}(B_k^-)$$
$$= \mathcal{I}^{-1}\big(\mathcal{L}^{-1}(B_k^-)\big) = \sqcup_{i\in[[1,p^q+1]]}\mathcal{I}^{-1}(B_k^-\tau_i) = \sqcup_{(i,j)\in[[1,p^q+1]]\times[[1,p^q-1]]}\mathcal{I}^{-1}(U_k^-\varpi_j\tau_i).$$

The morphism $\mathcal{I}$ induces an isomorphism

$$\mathbb{X}_k^3\setminus\mathbb{E}_k \cong \mathbf{SL}_{2,k,\mathrm{s}}\setminus\mathbb{B}_k^2. \tag{21}$$

We refer to $\mathbb{E}_k$ and $\mathbb{B}_k^2$ as the exceptional divisors.

Let

$$U_k^2 := \mathbb{G}_{a,k}\backslash U_k^3 \cong U_k^3/V_k^1 \cong \mathbb{G}_{a,k}^2,$$

where V_k^1 is the connected smooth subgroup of U_k^3 defined by the two equations $F_{p^q}(u_1)u_3 = u_2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$; recall that valued points of U_k^3 are triples (u_1,u_2,u_3) and $\mathbb{G}_{a,k}$ acts on U_k^3 by multiplication by elements of V_k^1. The quotient morphisms $U_k^3\to U_k^2$ and $\mathbf{SL}_{2,k,\mathrm{s}}\to\mathbb{G}_{a,k}\backslash\mathbf{SL}_{2,k,\mathrm{s}}\cong[A_k^2\setminus\{(0,0)\}]$ are geometric quotients and principal fiber bundles in the sense of [MFK], Ch. 0, Defs. 0.6 and 0.10.

We have a natural dominant morphism $\mathbb{G}_{a,k}\backslash[\mathbf{SL}_{2,k,\mathrm{s}}\setminus\mathbb{B}_k^2]\to U_k^2$ defined by $\mathcal{M}_3$ via the Isomorphism (21) and we define $\mathbb{X}_k^2$ to be the normalization of U_k^2 in the field of fractions of $\mathbb{G}_{a,k}\backslash[\mathbf{SL}_{2,k,\mathrm{s}}\setminus\mathbb{B}_k^2]$. We have a natural finite morphism $\mathcal{M}_2:\mathbb{X}_k^2\to U_k^2$ and $\mathcal{O}(\mathbb{X}_k^2)$ is the k-subalgebra of $\mathcal{O}(\mathbb{X}_k^3)$ of $\mathbb{G}_{a,k}$-invariant functions. Thus the inclusion between k-subalgebras of $\mathbb{G}_{a,k}$-invariant functions induced by $\mathcal{I}$

$$\mathcal{O}(\mathbb{G}_{a,k}\backslash\mathbf{SL}_{2,k,\mathrm{s}}) = \mathcal{O}(\mathbf{SL}_{2,k,\mathrm{s}})^{\mathbb{G}_{a,k}} \to \mathcal{O}(\mathbb{X}_k^2) = \mathcal{O}(\mathbb{X}_k^3)^{\mathbb{G}_{a,k}}$$

defines a birational morphism

$$\mathcal{M}_2:\mathbb{X}_k^2\to\operatorname{Spec}\big(\mathcal{O}(\mathbb{G}_{a,k}\backslash\mathbf{SL}_{2,k,\mathrm{s}})\big)\cong\operatorname{Spec}\mathcal{O}\big(\mathbb{A}_k^2\setminus\{(0,0)\}\big)\cong\mathbb{A}_k^2; \tag{22}$$

more precisely, from Zariski's Main Theorem we get that $\mathbb{G}_{a,k}\backslash[\mathbf{SL}_{2,k,\mathrm{s}}\setminus\mathbb{B}_k^2]$ is an open subvariety of $\mathbb{X}_k^2$. Thus the natural morphism

$$\mathbb{X}_k^3\to U_k^3\times_{U_k^2}\mathbb{X}_k^2 \tag{23}$$

between normal integral finite schemes over U_k^3 is birational and therefore it is a $\mathbb{G}_{a,k}$-invariant isomorphism. This implies that the morphism

$$\mathbb{X}_k^3\to\mathbb{X}_k^2$$

is a $\mathbb{G}_{a,k}$-torsor. As the $\mathbb{G}_{a,k}$-action on $\mathbb{X}_k^3$ commutes with the right action of $\mathbf{SL}_2(\mathbb{F}_{p^q})$, $\mathbf{SL}_2(\mathbb{F}_{p^q})$ acts on $\mathbb{X}_k^2$ and the Isomorphism (23) is $\mathbf{SL}_2(\mathbb{F}_{p^q})$-invariant. Thus $\mathbb{X}_k^2\to U_k^2$ is a right $\mathbf{SL}_2(\mathbb{F}_{p^q})$-torsor and we get morphisms

$$\mathbb{Y}_k^{2,\star} := \mathbb{X}_k^2/H^\star\to U_k^2 \ \text{ and } \ \mathbb{W}_k^{2,\star} := \mathbb{X}_k^2/G^\star\to U_k^2. \tag{24}$$

The Isomorphism (23) induces isomorphisms

$$\mathbb{Y}_k^{3,\star}\to U_k^3\times_{U_k^2}\mathbb{Y}_k^{2,\star} \ \text{ and } \ \mathbb{W}_k^3\to U_k^3\times_{U_k^2}\mathbb{W}_k^{2,\star}$$

which imply that $\mathbb{Y}_k^{2,\star}$ and $\mathbb{W}_k^{2,\star}$ are rational with group of units equal to k^* and that the morphisms

$$\mathbb{Y}_k^{3,\star}\to\mathbb{Y}_k^{2,\star} \ \text{ and } \ \mathbb{W}_k^{3,\star}\to\mathbb{W}_k^{2,\star}$$

are $\mathbb{G}_{a,k}$-torsors.

Theorem 2.7(1) for $n=2$ follows from the existence of Morphisms (22) and (24).

Assume $p^q = 2$. Then $\mathbf{SL}_2(\mathbb{F}_2)$ is the symmetric group S_3 and it has a normal cyclic subgroup C_3 of order 3 generated by

$$\tau := \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

We choose the τ_is such that we have $\{\tau_i | i \in [[1,3]]\} = C_3$. Let

$$\mathbb{V}_k^3 := \mathbb{X}_k^3 / C_3 \text{ and } \mathbb{V}_k^2 := \mathbb{X}_k^2 / C_3;$$

they have canonical models $\mathbb{V}^3$ and $\mathbb{V}^2$ (respectively) over $\operatorname{Spec} \mathbb{F}_2$. Moreover, $\mathbb{V}^3$ is a $\mathbb{G}_{a,\mathbb{F}_2}$-variety. The morphisms $\mathcal{M}_3$ and $\mathcal{M}_2$ factor through finite Galois covers

$$\mathbb{V}_k^3 \to \mathbb{A}_k^3 = \operatorname{Spec}(k[x_1, x_2, x_3]) \text{ and } \mathbb{V}_k^2 \to \mathbb{A}_k^2 \tag{25}$$

(respectively) of degree 2 and as above we argue that we have an isomorphism $\mathbb{V}_k^3 \to U_k^3 \times_{U_k^2} \mathbb{V}_k^2$ and that $\mathbb{V}_k^3 \to \mathbb{V}_k^2$ is the geometric quotient of the $\mathbb{G}_{a,k}$-action on $\mathbb{V}_k^3$. As C_3 permutes the irreducible components of the exceptional divisors, we get that the image of $\mathbb{E}_k$ in $\mathbb{V}_k^3$ is an irreducible divisor $\mathbb{J}_k$ given by the equation $x_2 = 0$; thus x_2 is a prime element of $A := \mathcal{O}(\mathbb{V}_k^3)$. As $\mathcal{I}^{-1}(U_k) = U_k \times_{\operatorname{Spec} k} 1_{U_k^+} \times_{\operatorname{Spec} k} U_k \cong \mathbb{A}_k^2$ is an irreducible component of $\mathbb{E}_k$, A/Ax_2 is a polynomial k-algebra in 2 indeterminates. Moreover, Isomorphism (21) induces an isomorphism

$$\operatorname{Spec}(A_{x_2}) = \mathbb{V}_k^3 \setminus \mathbb{J}_k \cong (\mathbf{SL}_{2,k} \setminus \mathbb{B}_k^2)/C_3 = \operatorname{Spec}(k_{\mathrm{s}}[w,x,z]_v)/C_3$$

with $v := wx(w+x)$, where the last identity follows from the facts that B^- is defined by the equation $x = 0$, $B^-\tau$ is defined by the equation $w = 0$ and $B^-\tau^2$ is defined by the equation $w + x = 0$. The k-automorphism of $k_{\mathrm{s}}[w,x,z]_v$ defined by the right translation action of $\tau^{-1} = \tau^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ on $\mathbf{SL}_{2,k}$, i.e., by the rule $\begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} w+x & w \\ y+z & y \end{bmatrix}$, maps w to $x+w$, x to w, $w+x$ to x, z to $y = (wz-1)x^{-1}$, y to $y+z$, and $y+z$ to z. Thus $A_{x_2} = k_{\mathrm{s}}[w,x,z]_v^{C_3}$ is the normalization of its k-subalgebra $B := k_{\mathrm{s}}[\omega_1, \omega_2, t, u, v]_v$, where $\omega_1 := y^2+yz+z^2$, $\omega_2 := yz(y+z)$, $t := x^3 + x^2w + w^3$, and $u := w^2 + wx + x^2$ (u and v are Dickson invariants, see [Di], Sect. 1, Thm.). Let $\omega := w(x+w)z + x \in A_{x_2}$. The identity

$$u^3 + t^2 + tv + v^2 = 0 \tag{26}$$

implies that we have a k-algebra homomorphism

$$\sigma_\omega : C := k[t,u,v,\omega]/(u^3 + t^2 + tv + v^2) \to A_{x_2}.$$

As the left-hand side of Equation (26) is the sum of two relatively prime homogeneous polynomials of degrees 3 and 2, $\operatorname{Spec} C$ is an integral rational hypersurface in $\mathbb{A}_k^4$ (see, [Sh], Sect. 3.4, Exc. 5 or [Ram], Probl. 11.115); concretely, the rule $(x_1,x_2,x_3) \mapsto \big(x_1(x_1^2+x_1x_2+x_2^2), x_2(x_1^2+x_1x_2+x_2^2), x_1^2+x_1x_2+x_2^2, x_3)\big)$ defines a birational morphism $\mathbb{A}_k^3 \to \operatorname{Spec} C$.[17] As C is an integral domain, σ_ω is injective by reasons of dimension. As we have $[k(w,x,z) : \operatorname{Frac}(C)] = [k(x,w) : k(t,u,v)]$,

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[17]Let $(m,n) \in [\mathbb{N}^* \setminus \{1\}]^2$. Let $f \in R := \mathbb{F}_{p^q}[x_1,...,x_n]$ be an irreducible homogeneous polynomial of degree m. The hypersurface $W := \operatorname{Spec}\big(R[x_{n+1}]/(x_{n+1}^{m+1} - f)\big)$ in $\mathbb{A}_{\mathbb{F}_{p^q}}^{n+1}$ is factorial by [Sa1], Thm. or [Sa2], Ch. 1, Thm. 8.1 with $\mathcal{O}(W)^* = \mathbb{F}_{p^q}^*$ and it is regular iff the zero locus $f = 0$ is. As the rule $(x_1,\ldots,x_n) \mapsto (x_1 f, \ldots, x_n f, f)$ defines a birational morphism $\mathbb{A}_{\mathbb{F}_{p^q}}^n \to W$ which induces an isomorphism between principal opens defined by inverting f and x_{n+1}, W has the same zeta function as $\mathbb{A}_{\mathbb{F}_{p^q}}^n$.

$[k(w,x,z) : \mathrm{Frac}(A_{x_2})] = 3$, $[k(x,w) : k(u,v)] = 6$ by [Di], Sect. 1, Thm., and $[k(t,u,v) : k(u,v)] = 2$, we conclude that $\mathrm{Frac}(\sigma_\omega)$ is an isomorphism. Thus $\mathbb{V}_k^3$ is rational. Similar arguments give that the inclusion $k[t,u,v]_v \to \mathcal{O}(\mathbb{V}_k^2)_{x_2}$ is birational and that $\mathbb{V}_k^2$ is rational.

Theorem 2.7(2) follows from the existence of Morphisms (25).

Assume $p^q = 3$. The group $\mathbf{SL}_2(\mathbb{F}_3)$ has order 24 and a unique normal subgroup Q_8 of order 8 (it is the quaternion group). Let $\mathbb{V}_k^3 := \mathbb{X}_k^3/Q_8$ and $\mathbb{V}_k^2 := \mathbb{X}^2/Q_8$; they have canonical models $\mathbb{V}^3$ and $\mathbb{V}^2$ (respectively) which are $\mathbb{G}_{a,\mathbb{F}_3}$-varieties over $\mathrm{Spec}\,\mathbb{F}_3$. The morphisms $\mathcal{M}_3$ and $\mathcal{M}_2$ factor through finite Galois covers

$$\mathbb{V}_k^3 \to \mathbb{A}_k^3 \ \text{ and } \ \mathbb{V}_k^2 \to \mathbb{A}_k^2 \tag{27}$$

of degree 3 and as above we have an isomorphism $\mathbb{V}_k^3 \to U_k^3 \times_{U_k^2} \mathbb{V}_k^2$ and $\mathbb{V}_k^3 \to \mathbb{V}_k^2$ is the geometric quotient of the $\mathbb{G}_{a,k}$-action on $\mathbb{V}_k^3$.

Lüroth's analog theorem for surfaces (see [Za], Sect. 1) applied to the separable composite morphism $B_k^+ \to U_k^-\backslash\mathbf{SL}_{2,k} \to U_k^-\backslash\mathbf{SL}_{2,k}/Q_8$ we get that $U_k^-\backslash\mathbf{SL}_{2,k}/Q_8$ is rational, hence $\mathbb{V}_k^2$ is also rational. As $\mathbb{V}_k^3 \cong U_k^3 \times_{U_k^2} \mathbb{V}_k^2$, $\mathbb{V}_k^3$ is also rational.

Theorem 2.7(3) follows from the existence of Morphisms (27).

Theorem 2.7(4) follows from constructions. Thus Theorem 2.7 holds.

Remark 19.1. We can assume that τ_1 is the identity element of $\mathbf{SL}_2(\mathbb{F}_{p^q})$. Let $\mathbb{A}^0 := \mathrm{Spec}\,\mathbb{Z}$. To simplify the notation, let $\zeta(\surd)$ be the logarithm of the zeta function of a variety $\surd$ over $\mathbb{F}_{p^q}$. If a variety W over k has a well defined (natural) model $\surd$ over $\mathbb{F}_{p^q}$, let $\zeta(W) := \zeta(\surd)$. For $l \in \mathbb{N}$, let

$$\zeta_{l,p^q} := \zeta(\mathbb{A}^l_{\mathbb{F}_{p^q}}).$$

Based on the mentioned canonical isomorphisms and Isomorphism (21) and their well defined models over $\mathbb{F}_{p^q}$, we compute

$$\begin{aligned}\zeta(\mathbb{X}^3) = \zeta(\mathbb{X}_k^3 \setminus \mathbb{E}_k) + \zeta(\mathbb{E}_k) &= \zeta\big(\mathbf{SL}_{2,k,\mathrm{s}} \setminus (B_k^- \sqcup_{i=2}^{p^q+1} B_k^-\tau_i)\big) + \zeta(\mathbb{E}_k)\\ = \zeta(\mathbf{SL}_{2,k,\mathrm{s}} \setminus B_k^-) - p^q\zeta(B_k^-) + \zeta(\mathbb{E}_k) &= \zeta(U_k^3 \setminus D_k) - p^q\zeta(B_k^-) + \zeta(\mathbb{E}_k) =\\ = \zeta_{3,p^q} - \zeta_{2,p^q} - p^q(\zeta_{2,p^q} - \zeta_{1,p^q}) &+ (p^q+1)(p^q-1)\zeta_{2,p^q}\\ &= \zeta_{3,p^q} + (p^{2q} - p^q - 2)\zeta_{2,p^q} + p^q\zeta_{1,p^q}.\end{aligned}$$

Thus

$$\zeta(\mathbb{X}^2) = \zeta_{2,p^q} + (p^{2q} - p^q - 2)\zeta_{1,p^q} + p^q\zeta_{0,p^q}.$$

We compute the zeta function of the canonical model $\mathbb{Y}^{3,-}$ of $\mathbb{Y}_k^{3,-}$. As U^- is normal in B^-, via right translations, we have $B^-H^- = B^-$ and $U^-\varpi_j\tau_1 H^- = U^-\varpi_j\tau_1$ for each $j \in [[1, p^q-1]]$, with $U^-v_j\tau_1/H^- \cong U^-/v_jH^-v_j^{-1} \cong U^-/H^- \cong U^- \cong \mathbb{A}_k^2$. As the normalizer of U_k^- (equivalently, of any non-identity element of $U^-(k)$) in $\mathbf{SL}_{2,k}$ is B_k^-, H^- acts transitively on the set $\{B^-\tau_i | i \in [[2, q+1]]\}$. Thus the image of the exceptional divisor $\mathbb{B}_k^2$ (resp. $\mathbb{E}_k$) in $\mathbf{SL}_{2,k}/H^-$ (resp. in $\mathbb{Y}_k^{3,-}$) is the disjoint union of 2 (resp. of $2(p^q-1)$) irreducible divisors isomorphic to $B_k^-/H^- \cong \mathbb{A}_k^1 \times_{\mathrm{Spec}\,k} \mathbb{G}_{m,k}$ (resp. to $\mathbb{A}_k^2$) through isomorphisms defined over $\mathbb{F}_{p^q}$, and we similarly compute

$$\zeta(\mathbb{Y}^{3,-}) = \zeta_{3,p^q} - \zeta_{2,p^q} - (\zeta_{2,p^q} - \zeta_{1,p^q}) + 2(p^q-1)\zeta_{2,p^q} = \zeta_{3,p^q} + (2p^q-4)\zeta_{2,p^q} + \zeta_{1,p^q},$$

$$\zeta(\mathbb{Y}^{2,-}) = \zeta_{2,p^q} + (2p^q-4)\zeta_{2,p^q} + \zeta_{0,p^q}.$$

In particular, if $p^q = 2$, then we have

$$\zeta(\mathbb{X}^2) = \zeta_{2,2} + 2\zeta_{0,2} \quad \text{and} \quad \zeta(\mathbb{Y}^{2,-}) = \zeta_{2,2} + \zeta_{0,2}.$$

Based on analogous computations over finite field extensions of $\mathbb{F}_{p^q} = \mathbb{F}_2$, neither $\mathbb{X}_k^2$ nor $\mathbb{Y}_k^{2,-}$ has an open subvariety isomorphic to $\mathbb{A}_k^2$. [Argument: if $\mathbb{A}_k^2$ is an open subvariety of either $\mathbb{X}_k^2$ or $\mathbb{Y}_k^{2,-}$, then we can assume that it is defined over a finite field, and this would contradict the zeta function computation as the complement is either empty or of pure codimension 1 by [SGA2], Exp. V, Ex. 3.4.]

Remark 19.2. As B_k^- is defined by the equation $x = 0$, from Equation (18) and [Ma1], Ch. 7, Thm. 20.2 we get that $\mathbf{SL}_{2,k}$ is factorial. Thus $\mathbb{X}_k^3$ is factorial iff $\mathcal{I}^*$ is a bijection on divisor groups and hence, based on the Isomorphism (21) and the formulas for the exceptional divisors, iff $p^q = 2$.

For $p^q = 2$, as $H^\star$ is a 2-group and $\mathbb{X}_k^3$ is factorial, we get that $\mathbb{Y}_k^{3,\star}$ is also factorial; similarly, as there exist no non-trivial homomorphisms $\mathbb{G}_{a,k} \to \mathbb{G}_{m,k}$ (see [SGA3-2], Exp. XVII, Prop. 2.4i)), we get that $\mathbb{X}_k^2$ and $\mathbb{Y}_k^{2,\star}$ are factorial. From this, end of Remark 19.1, and [Ru], Thm. 2 and Lem. 1.8, we get that there exists no separable dominant morphism $\mathbb{A}_k^2 \to \mathbb{X}_k^2$ or $\mathbb{A}_k^2 \to \mathbb{Y}_k^{2,-}$. Hence also, there exists no separable dominant morphisms $\mathbb{A}_k^N \to \mathbb{X}_k^3$ or $\mathbb{A}_k^N \to \mathbb{Y}_k^{3,-}$ with $N \geq 3$ an integer.

Remark 19.3. Assume $p^q = 2$; so $-1 = 1$. Let $t, u, v \in k_{\mathrm{s}}[w,x,z]_v$ be the invariant polynomials that show up in Equation (26). Viewing the isomorphism $(\mathcal{T}_v^\#)^{-1} : k[x_1,x_2,x_3]_{x_2} \to k_{\mathrm{t}}[w,x,y,z]_v/(wz+xy+1)$ and the localization with respect to v of the k-algebra homomorphism $\mathcal{L}^\# : k_{\mathrm{t}}[w,x,y,z]/(wx+yz+1) \to k_{\mathrm{s}}[w,x,y,z]/(wx+yz+1)$ that defines $\mathcal{L}$ as identifications, then, as $xy = zw+1$, we get that x_1, x_2, and x_3 are equal to $\frac{(wz+1)x+zw^2+1}{v}$, the second Dickson invariant v, and $\frac{wz^2+\frac{(wz+1)^2}{x}+1}{v}$ (respectively). Solving for z (resp. z^2) in terms of x_1 (resp. x_3), x and w, we get that $z = \frac{x+x_1x_2+1}{w^2+wx}$ and $z^2 = \frac{x(x_2x_3+1+x^{-1})}{w^2+wx}$. From these identities we get that $\frac{(x+x_1x_2+1)^2}{w^2+wx} = x(x_2x_3+1+x^{-1})$, hence $(x+x_1x_2+1)^2 = x_2(x_2x_3+1+x^{-1})$ or $x^3+(x_1^2x_2^2+x_2^2x_3+x_2+1)x+x_2 = 0$. Based on this and $u = w^2+wx+x^2 = \frac{x_2}{x}+x^2$, the first Dickson invariant is $u = x_1^2x_2^2+x_2^2x_3+x_2+1$. The finite étale covers $\mathbb{Y}^{3,+} \to \mathbb{A}_{\mathbb{F}_2}^3$ and $\mathbb{Y}^{3,-} \to \mathbb{A}_{\mathbb{F}_2}^3$ are defined by the normalization of $\mathbb{F}_2[x_1,x_2,x_3]$ in the field extensions $\mathbb{F}_2(x_1,x_2,x_3) \to \mathbb{F}_2(x_1,x_2,x_3)[w]/(w^3+uw+v)$ and $\mathbb{F}_2(x_1,x_2,x_3) \to \mathbb{F}_2(x_1,x_2,x_3)[x]/(x^3+ux+v)$ (respectively). Equation (26) implies that the finite Galois cover $\mathbb{V}^3 \to \mathbb{A}_{\mathbb{F}_2}^3$ is defined by the normalization of $\mathbb{F}_2[x_1,x_2,x_3]$ in the field extension $\mathbb{F}_2(x_1,x_2,x_3) \to \mathbb{F}_2(x_1,x_2,x_3)[t]/(t^2+vt+v^2+u^3)$. Hence

$$\mathbb{V}^3 \cong \operatorname{Spec} \mathbb{F}_2[x_1,x_2,x_3][t_v]/(t_v^2+t_v+1+u_v)$$

with $u_v := \frac{u^3+v+1}{v^2} \in \mathbb{F}_2[x_1,x_2,x_3]$ and indeterminate $t_v := \frac{t+1}{v}$.

20. On finite étale covers of affine spaces and applications

There exist many finite étale morphisms $\Phi : X \to \mathbb{A}_k^2$ with X a connected affine smooth surface over k which either is isomorphic to $\mathbb{A}_k^2$ (e.g., $\Phi := d \times 1_{A_k^1}$ with $d \in \mathrm{EE}_1(k)$) or does not have an open subvariety isomorphic to $\mathbb{A}_k^2$ (e.g., $k^* \subsetneq \mathcal{O}(X)^*$ or is the pullback via a projection $\mathbb{A}_k^2 \to \mathbb{A}_k^1$ of a finite étale cover $C \to \mathbb{A}_k^1$ with C a non-rational curve). E.g., we have the following general result.

Fact 20.1. *Let $l \in \mathbb{N}^*$. If $p \geq 5$ (resp. $p \leq 3$) we assume that $l \geq p$ (resp. $l \geq p+1$). Then there exists a non-Galois finite étale cover $c_n : \mathbb{H}^n \to \mathbb{A}_k^n$ of degree l.*

Proof. We can assume that $n=1$ as we can take $c_{1+i} := c_1 \times 1_{\mathbb{A}^i_k}$ for $i \in \mathbb{N}^*$. Let $C \to \mathbb{A}^1_k$ be a finite Galois cover of Galois group G. If G has a non-normal subgroup H of index l, then the finite étale cover $C/H \to \mathbb{A}^1_k$ is non-Galois of degree l. We can take G to be the alternating group A_l (resp. symmetric group S_l) by [Abh], Sect. 13 First (resp. Second) Cor. As the subgroup A_{l-1} of A_l (resp. S_{l-1} of S_l) is non-normal, by choosing H to be the subgroup A_{l-1} (resp. S_{l-1}) the fact holds. □

If $m \in \mathbb{N}^*$, $f_1(x,y,z) \in k[x,y,z]$ is a monic polynomial in x of degree m and $g_1(y,z) \in k[y,z]$, then for $A := k[x,y,z]/\big(f_1(x^p,y,z)[x+g_1(y,z)]-1\big)$ we have $k^* \subsetneq A^*$ and the projection on the last two coordinates defines a finite étale morphism $\operatorname{Spec} A \to \mathbb{A}^2_k$ of degree $pm+1$.

To provide examples of finite étale morphisms $\Phi : X \to \mathbb{A}^2_k$ with $\mathbb{A}^2_k \subsetneq X$ we recall that for $(f,g) \in \Theta_k$ and a finite additive subgroup G of k that contains the zeros of $f(t)$, if $m \in \mathbb{N}$ is such that $|G| = pm$ and $h_G(t) := \prod_{\alpha \in G, \alpha f(\alpha) \neq 0}(t-\alpha)$, then the product $tf(t)h_G(t)$ is a monic additive polynomial (e.g., see [Go], Ch. 1, Thm. 1.2.1); being also separable, we have $(fh_G, g) \in \Theta^1_{k,m}$.

Proposition 20.2. *Assume $n \geq 2$. Let $(f,g) \in \Theta_k$ with $\deg(f) = n-1$. Let $m \in \mathbb{N}^*$ and $h(t) \in k[t]$ be such that $(fh,g) \in \Theta^1_{k,m}$. Let $q \in \mathbb{N}^*$ be such that $q > \lfloor \frac{m}{n} \rfloor$ and let $(\beta_1, \ldots, \beta_q) \in k^{q-1} \times k^*$. Then the following properties hold.*

(1) *The rule on valued points $(x,y,z) \mapsto \big(y, h(x)z + \sum_{i=1}^q \beta_i z^{pi}\big)$ defines a finite étale cover $\Phi_{f,g,h,q} : \mathbb{S}^2_{f,g,k} \to \mathbb{A}^2_k$ of degree pnq and hence the ring $k[x,y,z]/\big(xf(x)+yg(y)z\big)$ has a p-basis.*

(2) *The composite of the open embedding $\imath_{f,g,k}$ of Section 2 (resp. $\jmath_{f,g,k;\delta}$ of Proposition 13.1 if $g(0) \neq 0$) with the morphism $\Phi_{f,g,h,q}$ is a surjective endomorphism $e_{f,g,h,q} \in \mathrm{EE}_2(k)$ (resp. $d_{f,g,h,q} \in \mathrm{EE}_2(k)$). For $\star \in \{e,d\}$ we have $\deg(\star_{f,g,h,q}) = pnq$, $X_\star \cong \mathbb{S}_{f,g,k}$, and $\rho_{\text{ét}}(\star_{f,g,h,q}) = \rho(\star_{f,g,h,q}) = (n-1)\mathrm{z}(tg)$.*

Proof. It is easy to see that $\Phi_{f,g,h,q}$ is generically étale. Hence it suffices to show that for each $(\alpha,\beta) \in k^2$, the solution set $\mathcal{J}_{\alpha,\beta}$ in k^3 of the system of equations

$$y-\alpha = h(x)z + \Big[\sum_{i=1}^q \beta_i z^{pi}\Big] - \beta = xf(x) + yg(y)z = 0$$

has pnq solutions. As $(fh,g) \in \Theta^1_{k,m}$, there exists $(\alpha_0, \alpha_1, \ldots, \alpha_{m-1}) \in k^* \times k^{m-1}$ such that $tf(t)h(t) = \alpha_0 t + [\sum_{i=1}^{m-1} \alpha_i t^{pi}] + t^{pm}$. So $[tf(t)h(t)]' = \alpha_0 \in k^*$ and $tf(t)$ and $h(t)$ do not have a common root. If $\alpha g(\alpha) = 0$, then $xf(x) = 0$, $h(x) \neq 0$, and

$$\mathcal{J}_{\alpha,\beta} = \left\{(x,\alpha,z) \in k^3 \middle| xf(x) = 0 = -\beta + h(x)z + \sum_{i=1}^q \beta_i z^{pi}\right\}$$

has pnq elements as all fibers of the surjective function $\mathcal{J}_{\alpha,\beta} \to \{x \in k | xf(x) = 0\}$ have cardinality pq. If $\alpha g(\alpha) \neq 0$, then

$$\mathcal{J}_{\alpha,\beta} = \left\{\Big(x, \alpha, \frac{xf(x)}{-\alpha g(\alpha)}\Big) \in k^3 \middle| \frac{x^{pm} + \alpha_0 x + \sum_{i=1}^{m-1} \alpha_i x^{pi}}{-\alpha g(\alpha)} + \sum_{i=1}^q \frac{\beta_i x^{pi} f(x)^{pi}}{[-\alpha g(\alpha)]^{pi}} = \beta\right\}.$$

has pnq elements as $\alpha_0 \in k^*$ and $\deg\big(x^{pq} f(x)^{pq}\big) = pnq > pm$. Thus part (1) holds.

Part (2) follows from part (1) and Section 2 (resp. Proposition 13.1); the surjectivity part follows from the fact that for each $(\alpha,\beta) \in k^2$, the solution set $\mathcal{J}_{\alpha,\beta}$ is not contained in $\mathbb{D}^1_{f,g,k} = \mathbb{S}^2_{f,g,k} \setminus \mathrm{Im}(\imath_{f,g,k})$ (resp. $\mathbb{S}^2_{f,g,k} \setminus \mathrm{Im}(\jmath_{f,g,k;\delta})$) described in Section 2 (resp. the proof of Proposition 13.1). $\square$

Example 20.3. Assume p does not divide $n-1$. Thus for each monic polynomial $g(y) \in k[y]$ the surface $\mathbb{S}^2_{t^{n-1}-1,g,k}$ is smooth over $\operatorname{Spec} k$ and there exists $(l,m) \in (\mathbb{N}^*)^2$ such that $pm = l(n-1)+1$. For $h(t) := \sum_{i=0}^{l-1} t^{(n-1)i} \in k[t]$ we have $(t^n - t)h(t) = t^{pm} - t$. Let $q \in \mathbb{N}^*$ be such that $q > \lfloor \frac{m}{n} \rfloor$. The rule on valued points

$$(x,y,z) \mapsto \big(y, h(x)z + z^{pq}\big)$$

defines a finite étale morphism $\Phi_{n,m,q,g,k} : \mathbb{S}^2_{t^{n-1}-1,g,k} \to \mathbb{A}^2_k$ of degree pnq by Proposition 20.2(1). Let $e = e_{n,m,q,g,k} := \Phi_{n,m,q,g,k} \circ \imath_{t^{n-1}-1,g,k} \in \mathrm{EE}_2(k)$; therefore $\deg(e) = pnq$, $X_e = \mathbb{S}^2_{t^{n-1}-1,g,k}$ and $\rho_{\text{ét}}(e) = \rho(e) = (n-1)\mathrm{z}(tg)$ by Proposition 20.2(2). Moreover, $d_{n,m,q,k} := \Phi_{n,m,q,1,k} \circ \mathrm{Inv}_{\mathbb{S}^2_{t^{n-1}-1,1,k}} \circ \imath_{t^{n-1}-1,1,k} \in \mathrm{EE}_2(k)$ has same listed invariants as $e_{n,m,q,1,k}$, where $\mathrm{Inv}_{\mathbb{S}^2_{t^{n-1}-1,1,k}}$ is the standard involution of $\mathbb{S}^2_{t^{n-1}-1,1,k}$ defined by the rule $(x,y,z) \mapsto (x,z,y)$.

E.g., if $p=3$, then $d_{2,1,1,k}$ is defined by the rule $(x,y) \mapsto (x - x^2y, y + y^2x + y^3)$.

Let $l \in \mathbb{N}$. We take g such that $\mathrm{z}(tg) = l$. As $\mathbb{D}^1_{f,g,k}$ is the zero locus $f(x) = yg(y) = 0$ (see Section 2), N_e is the zero locus $yg(y) = 0$ and hence it is smooth and isomorphic to l copies of $\mathbb{A}^1_k$. In particular, if $l = 1$ (i.e., if $g(t) = t^s$ with $s \in \mathbb{N}$), then $N_e \cong \mathbb{A}^1_k$ is smooth and integral and therefore the Nollet–Xavier Conjecture over $\mathbb{C}$ (see [Je], Sect. 1 and [NX], Quest. 6) becomes false over k even for $n=2$.

Remark 20.4. We refer to Example 14.1 with $mq > 2$. As we have $\rho(e) - \rho_{\text{ét}}(e) = 1$, $Y_e := X_e \setminus X_e^{\text{ét}} \in \mathrm{Irr}(\Gamma_{\psi_e})$. The union $\mathbb{S}^2_e := \mathbb{S}_{Y_e} = \mathbb{A}^2_{k,\mathrm{s}} \cup Y_e$ is an affine open subvariety $\operatorname{Spec} A_{Y_e}$ of X_e (see Corollary 11.4(2)) which is smooth over $\operatorname{Spec} k$ (as ψ_e is regular) and for which $Y_e \cong \mathbb{A}^1_k$ (see Corollary 11.4(1)) and A_{Y_e} does not have a p-basis (see Corollary 11.8(2)). In particular, there exists no étale morphism from $\mathbb{S}^2_e$ to affine surfaces over k that have p-bases such as $\mathbb{S}^2_{f,g,k}$ with $(f,g) \in \Theta_k$ by Proposition 20.2(1) if $\deg(f) \geq 1$ and by $\mathbb{S}^2_{1,g,k} \cong \mathbb{A}^2_k$ if $\deg(f) = 0$ (i.e., if $f = 1$).

Corollary 20.5. *For $s \in \mathbb{N}^*$ we consider the set*

$$\mathbb{O}_{p,s} := \{\rho(e) | e \in \mathrm{EE}_2(k), \rho(e) = \rho_{\text{ét}}(e), \deg(e) = ps\}.$$

Then the following properties hold.

(1) *If s is even, then $\mathbb{O}_{p,s} = \mathbb{N}$.*

(2) *If s odd, then $(ps-1)\mathbb{N} \subset \mathbb{O}_{p,s}$ and for each divisor j of s such that $p \nmid j-1$ we have $(j-1)\mathbb{N} \subset \mathbb{O}_{p,s}$ (e.g., $(2s-1)\mathbb{N} \subset \mathbb{O}_{2,s}$, and $2\mathbb{N} \subset \mathbb{O}_{p,3}$ if $p \geq 3$ and $3|s$).*

Proof. As there exists $d \in \mathrm{EE}_2(k)$ which is finite étale of degree ps, $0 \in \mathbb{O}_{p,s}$. The inclusion $(ps-1)\mathbb{N}^* \subset \mathbb{O}_{p,s}$ follows from the existence of the finite étale endomorphisms $c_{t^{ps-1}-1,g,k} \in \mathrm{EE}_2(k)$ (see Section 2).

If s is even, in Example 20.3 we take $(n,l,m,q) = (2, p-1, 1, s/2)$; for $e := e_{2,1,q,g,k}$ we have $\deg(e) = ps$ and $\rho(e) = \rho_{\text{ét}}(e) = \mathrm{z}(tg) \in \mathbb{N}^*$, hence $\mathbb{N}^* \subset \mathbb{O}_{p,s}$.

We are left to prove that if s and j are as in part (2), then $(j-1)\mathbb{N}^* \subset \mathbb{O}_{p,s}$; we have $j \geq 3$. In Example 20.3 we take $(n,q) = (j, \frac{s}{j})$ and the unique $m \in [[1, j-2]]$ such that there exists $l \in \mathbb{N}$ with $pm = l(j-1)+1$. For $e := e_{j,m,\frac{s}{j},g,k}$ we have $\deg(e) = ps$ and $\rho(e) = \rho_{\text{ét}}(e) = (j-1)\mathrm{z}(tg) \in (j-1)\mathbb{N}^*$, hence $(j-1)\mathbb{N}^* \subset \mathbb{O}_{p,s}$. $\square$

Example 20.6. For $i \in \{1,2\}$, let $(n_i, m_i) \in \mathbb{N}^2$ with $n_i \geq 2$. Let $(f_i, t^{m_i}) \in \Theta_K$ with $\deg(f_i) = n_i - 1$. The open embedding $\imath_{f_i,g_i,K} : \mathbb{A}^2_K \to \mathbb{S}^2_{f_i,t^{m_i},K}$ is such that the complement $\mathbb{S}^2_{f_i,t^{m_i},K} \setminus \mathrm{Im}(\imath_{f_i,g_i,K})$ is a disjoint union of n_i curves isomorphic to $\mathbb{A}^1_K$. Thus the rank $\rho_{\mathbb{S}^2_{f_i,t^{m_i},K}} = n_i - 1$ of the Néron–Severi group of $\mathbb{S}^2_{f_i,t^{m_i},K}$ is independent of m_i. Writing $f_i(t) = \prod_{j=1}^{n_i-1}(t - \alpha_{i,j})$ with $(\alpha_{i,1}, \ldots, \alpha_{i,n_i-1}) \in (K^*)^{n_i-1}$, for $j \in [[1, n_i - 1]]$ let $f_{i,j}(t) := t - \alpha_{i,j}$. From Theorem 7.2(2) we get that $T_{\mathbb{S}^2_{f_i,t^{m_i},K}}$ is trivial. If $m_1 \neq m_2$, then $\mathbb{S}_{t-1,t^{m_1},K} \not\cong \mathbb{S}_{t-1,t^{m_2},K}$ by [GG], Thm. B(i). We have an affine open cover $\mathbb{S}^2_{f_i,t^{m_i},K} = \cup_{j=1}^{n_i-1} \mathrm{Im}(\imath_{f_{i,j},t^{m_i};f,K})$ (see Section 2 for notation) with each $\mathrm{Im}(\imath_{f_{i,j},t^{m_i};f,K}) \cong \mathbb{S}_{t-1,t^{m_i},K}$; it consists of $n_i - 1$ spheres (resp. almost spheres) if $m_i = 0$ (resp. $m_i > 0$).

Case 1: char$(K) = 0$. Let $\beta_i := \sum_{j=1}^{n_i-1} \alpha_{i,j}$. If $\beta_i = 0$ let $g_i := f_i$ and if $\beta_i \neq 0$ let $g_i(t) := (t + \frac{\beta_i}{n_i}) f_i(t + \frac{\beta_i}{n_i}) \in K[t]$; the coefficient of t^{n_i-1} in g_i is 0. If $(n_1, m_1) \neq (n_2, m_2)$ with $m_1 m_2 > 0$, then $\mathbb{S}_{f_1,t^{m_1},K} \not\cong \mathbb{S}_{f_2,t^{m_2},K}$ by [M-L2], Thm. 2 over $\mathbb{C}$. If $(n_1, m_1) = (n_2, m_2)$, then $\mathbb{S}_{f_1,t^{m_1},K} \cong \mathbb{S}_{f_2,t^{m_2},K}$ iff either $n_1 = 1$ or $n_1 > 1$ and there exists $\gamma \in K^*$ such that $g_2(t) = \gamma^{-n_1} g_1(\gamma t)$ by [M-L2], Thm. 2 over $\mathbb{C}$ if $m_1 > 0$ and by [BD], Thm. 5.4.5(1) if $m_1 = 0$.

Case 2: char$(K) = p$. Let $e_i \in \mathrm{EE}_2(k)$ be such that $X_{e_i} \cong \mathbb{S}^2_{f_i,t^{m_i},K}$ by Proposition 20.2(2) applied to $\imath_{f_i,t^{m_i},K}$; we have $\rho(e_i) = \rho_{\text{ét}}(e_i) = n_i - 1$ and $\big(\rho_1(e_i), \rho_2(e_i)\big)$ is $(n_i, 0)$ if $m_i = 0$ and is $(0, n_i)$ if $m_i > 0$.

Example 20.7. Let $f(t) \in k[t]$ be monic separable of degree $l \in \mathbb{N}^*$; so $(f, t^m) \in \Theta_k$ for each $m \in \mathbb{N}^*$. Let $(m_1, m_2) \in (\mathbb{N}^*)^2$. For $i \in \{1,2\}$, let $e_i \in \mathrm{EE}_2(k)$ be such that $X_{e_i} \cong \mathbb{S}^2_{f,t^{m_i},k}$ by Proposition 20.2(2) applied to $\imath_{f,t^{m_i},k}$; we have $\rho(e_i) = \rho_{\text{ét}}(e_i) = \rho_2(e_i) = l$ (the last equality by Example 20.6). If $m_1 \neq m_2$, then $X_{e_1} \not\cong X_{e_2}$ by [GG], Thm. B(i).

Example 20.8. Let $l \in \mathbb{N}^* \setminus \{1\}$. Let $\mathcal{P}_l(K^*)$ be the set of all finite subsets of K^* of cardinality l. For $\mathcal{B} \in \mathcal{P}_l(K^*)$ let $f_{\mathcal{B}}(t) := \prod_{\alpha \in \mathcal{B}}(t - \alpha)$. For $a \in \mathrm{GA}_1(K)$, let $a(\mathcal{B}) := \{a(\alpha) | \alpha \in \mathcal{B}\}$. For $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{P}_l(k^*)$, we have $\mathbb{S}^2_{f_{\mathcal{B}_1},1,k} \cong \mathbb{S}^2_{f_{\mathcal{B}_2},1,k}$ iff there exists $a \in \mathrm{GA}_1(k)$ such that $\mathcal{B}_2 = a(\mathcal{B}_1)$ by [BD], Thm. 5.4.5(1). Let $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{P}_l(K^*)$ be such that $\mathcal{C}_2 \neq a(\mathcal{C}_1)$ for all $a \in \mathrm{GA}_1(K)$. If $\mathrm{char}(K) = p$, then for $i \in \{1,2\}$, let $e_i \in \mathrm{EE}_2(k)$ be such that $X_{e_i} \cong \mathbb{S}^2_{f_{\mathcal{C}_i},1,k}$ by Proposition 20.2(2) applied to $\imath_{f_{\mathcal{C}_i},1,k}$; we have $X_{e_1} \not\cong X_{e_2}$ and $\rho(e_i) = \rho_{\text{ét}}(e_i) = \rho_1(e_i) = l$ (the last equality by Example 20.6).

Example 20.9. Let $(f, g) \in \Theta_K$. Let $(g_1, g_2) \in K[t]^2$ be such that there exists $g_3(t) \in K[t]$ with $g_1(t) g_3(t) = t g(t)$. Let $\mathbb{S}^2_{f,g;g_1,g_2,K} := \mathrm{Spec}\big(K[x,y,z]/(F)\big)$ where

$$F(x,y,z) := x f(x) + y g(y)[g_1(y) z - x g_2(x)] \in K[x,y,z].$$

The zero locus $F = F_x = F_y = F_z = 0$ in $\mathbb{A}^3_K$ is contained in the zero locus $y g(y) g_1(y) = x f(x) = f(x) + x f'(x) - y g(y)[g_2(x) + x g_2'(x)]$ and thus, as $g_1(t) | t g(t)$, is contained in the zero locus $x f(x) = f(x) + x f'(x) = y g(y) = 0$ which is empty as $t f(t)$ is separable. Hence $\mathbb{S}^2_{f,g;g_1,g_2,K}$ is smooth. The rule

$$(x, y) \mapsto \big(x y g(y), y, -x f(x y g(y)) + x g_3(y) g_2(x y g(y))\big)$$

defines an open embedding $\imath_{f,g;g_1,g_2,K} : \mathbb{A}^2_K \to \mathbb{S}^2_{f,g;g_1,g_2,K}$. Clearly, F is not in general of the form $x \tilde{f}(x) + y \tilde{g}(y) z$ with $(\tilde{f}, \tilde{g}) \in \Theta_K$ but even then it can happen that $\mathbb{S}^2_{f,g;g_1,g_2,K}$ is isomorphic to an open subvariety of $\mathbb{S}^2_{\tilde{f},\tilde{g},K}$ for some $(\tilde{f}, \tilde{g}) \in \Theta_K$.

If $\deg(f)=1$, $g(t)=t^q$ with $q\in\mathbb{N}$ and $\deg(g_2)=0$, then the ind-group scheme $\mathrm{Aut}(\mathbb{S}^2_{f,g;g_1,g_2,K})$ is a semidirect product of the ind-unipotent group scheme $K[t]$ and the cyclic group of order 2 by [Cr], Cor. 4.3; in such a case $\mathbb{S}^2_{f,g;g_1,g_2,K}$ is not isomorphic to any affine surface with a non-trivial $\mathbb{G}_{m,K}$-action.

As a concrete example, we take $f(t)=t+1$, $g(t)=1$, $g_1(t)=-t$, $g_2(t)=2$. Thus $F(x,y,z)=x^2+x+y(2x-yz)=(x+y+1)(x+y)+y[y(-z-1)-1]$ and hence $\mathbb{S}^2_{f,g;g_1,g_2,K}\cong\mathrm{Spec}\,\big(K[x,y,z]/x^2+x+y(yz-1)\big)$. If $\mathrm{char}(K)=2$, then $\mathbb{S}^2_{t+1,1;-t,2,K}=\mathbb{S}^2_{t+1,-t,K}\cong\mathbb{S}^2_{t-1,t,K}$ is not isomorphic to $\mathbb{S}^2_K$ by [GG], Thm. B(i). If $\mathrm{char}(K)\neq 2$, then $\mathbb{S}^2_{t+1,1;-t,2,K}\not\cong\mathbb{S}^2_{t-1,t^m,K}$ for all $m\in\mathbb{N}$ by the prior paragraph.

The open embedding $\jmath_{t-1,t+1,K;1}:\mathbb{A}^2_K\to\mathbb{S}^2_{t-1,t+1,K}$ (see Proposition 13.1) is defined by the rule $(x,y)\mapsto\big(h(yh+1),yh,-x-h^2\big)$, where $h:=1+xy\in K[x,y]$. We have a disjoint union decomposition $\mathbb{S}^2_{t-1,t+1,K}\setminus\mathrm{Im}(\jmath_{t-1,t+1,K;1})=Y_1\sqcup Y_2$, where Y_1 is the zero locus $x=z=0$ and Y_2 is the zero locus $x-1=y+1=0$ (see Cases 1 and 2 of the proof of Proposition 13.1). Let $R:=K[x,y,z]/\big(x^2-x+y(y+1)z\big)$. By definition, $\mathbb{S}^2_{t-1,t+1,K}=\mathrm{Spec}\,R$. So we can identify R with the K-subalgebra $K[h(yh+1),yh,-x-h^2]$ of $K[x,y]$. Under this identification, we write $\mathbb{S}^2_{Y_i}:=\mathbb{S}^2_{t-1,t+1,K}\setminus Y_{3-i}=\mathrm{Spec}\,S_i$ for $i\in\{1,2\}$. We have

$$S_1=R_{h(yh+1)-1}\cap R_{yh+1}=K[h,yh,x]=K[x,h,yh]\cong K[w,x,y']/(w^2-w+xy')$$

under the substitution $(w,y'):=(h,-yh)$. Thus $\mathbb{S}^2_{Y_1}\cong\mathbb{S}^2_K$. Similarly,

$$S_2=R_{x+h^2}\cap R_{h(yh+1)}=K[y,yh,x+h^2]$$

$$=K[y,y^2x,x+2xy+x^2y^2]\cong K[w,y,z]/\big(F(w,y,z')\big)$$

under the substitution $(w,z'):=(y^2x,x+2xy+x^2y^2)$. Thus $\mathbb{S}^2_{t+1,1;-t,2,K}\cong\mathbb{S}^2_{Y_2}$; so $Cl(\mathbb{S}^2_{t+1,1;-t,2,K})\cong\mathbb{Z}$ and $\mathbb{S}^2_{t+1,1;-t,2,K}$ is isomorphic to an open subvariety of $\mathbb{S}^2_{t-1,t+1,K}$; thus, as $\{(\tilde f,\tilde g)\in\Theta_K|Cl(\mathbb{S}^2_{\tilde f,\tilde g,K})\cong\mathbb{Z}\}=\{(t-\alpha,t^m)|\alpha\in K^*,m\in\mathbb{N}\}$, for $\mathrm{char}(K)\neq 2$ we have $\mathbb{S}^2_{t+1,1;-t,2,K}\not\cong\mathbb{S}^2_{\tilde f,\tilde g,K}$ for all $(\tilde f,\tilde g)\in\Theta_K$ by the prior paragraph and reasons of isomorphisms classes of divisor class groups.

Assume $\mathrm{char}(K)=p$. Let $e\in\mathrm{EE}_2(K)$ be such that $\imath_e=\imath_{t-1,t+1,K}$ (see Proposition 20.2(2)). Then $\{\mathbb{S}^2_{Y_1},\mathbb{S}^2_{Y_2}\}$ is the canonical open cover of $X_e=\mathbb{S}^2_{t-1,t+1,K}$ defined by e (see Definition 11.9). From the non-isomorphisms of the last two paragraphs we get that $\rho_1(e)=\rho_2(e)=1$ and $\rho(e)=\rho_{\text{ét}}(e)=2$; if $p\neq 2$, then $\mathbb{S}^2_{Y_1}$ is an almost sphere that did not show up priorly in the paper.

Example 20.10. Let $q\in\mathbb{N}^*\setminus\{1\}$. Let $\Sigma:\mathbb{S}^2_{t-1,t^{q-1},k}\to\mathbb{P}^1_k$ the the $\mathbb{A}^1$ fibration defined by the rule rule $(x,y,z)\mapsto(-x:y^q)$ or $(z:x-1)$ (whichever is defined; if both are defined, then they are equal). For each standard affine pseudo-plane $\mathbb{S}^2_{t-1,t^{q-1},k;\Sigma,P}$ of $\mathbb{S}^2_{t-1,t^{q-1},k}$ of type q in the sense of Definition 11.6(3), there exist étale morphisms $\mathbb{S}^2_{t-1,t^{q-1},k;\Sigma,P}\to\mathbb{A}^2_k$ that are composites of the open embedding $\mathbb{S}^2_{t-1,t^{q-1},k;\Sigma,P}\to\mathbb{S}^2_{t-1,t^{q-1},k}$ with finite étale covers $\mathbb{S}^2_{t-1,t^{q-1},k;\Sigma,P}\to\mathbb{A}^2_k$ as in Proposition 20.2(2). For instance, if $P=(0:1)\in\mathbb{P}^1_k(k)$, then $\Sigma^{-1}(P)$ is the zero locus $x=z=0$ and the action of $\mathbb{G}_{m,k}$ on $\mathbb{S}^2_{t-1,t^{q-1},k}$ defined by the rule $\alpha\cdot(x,y,z)=(x,\alpha y,\alpha^{-q}z)$ for $\alpha\in k^*$ restricts to an action of $\mathbb{G}_{m,k}$ on $\mathbb{S}^2_{t-1,t^q,k;\Sigma,P}$; so $\mathbb{S}^2_{f,t^q,k;\Sigma,P}$ is of type (q,q) in the sense of [MM1], Def. 2.1 (3) (cf. [MM2], Thm. 1.1). The case $q\geq 3$ (resp. $q=2$) is in contrast to the situation in characteristic zero in which, with $\mathrm{char}(K) = 0$, for an affine pseudo-planes X over K with non-trivial $\mathbb{G}_m$ action of type (q, r) with $r \in \mathbb{N}^*$ there exists no étale morphism from X to $\mathbb{A}_K^2$ if $r \neq 2$ and $d \nmid r-2$ by cite [Mi4], Thm. 3.6 (resp. the classical Jacobian Conjecure for $n = 2$ implies that there exists no étale morphism from X to $\mathbb{A}_K^2$ even if either $r = 2$ or $d \mid r-2$; cf. proof of [Mi4], Thm. 3.3).

21. Differential equations and applications

In this section we generalize [No], Ex. (6.10) with $(a, b) \in (k^*)^2$ and the recent work and differential equation of Lang (see [Lang-J], Props. 4.5 and 4.8) to all $n \geq 2$ in order to get more examples of endomorphisms $e \in \mathrm{EE}_n(k)$ that have interesting invariants and properties. We begin with a general lemma on determinants.

Lemma 21.1. *Let R be an arbitrary commutative $\mathbb{Z}$-algebra. For $i \in [[1, n]]$ let $(\alpha_i, \beta_i, \gamma_i) \in R^3$. Let $\triangle$ be the diagonal $n \times n$ matrix whose i, i entry is α_i for $i \in [[1, n]]$. Let $\triangle_1$ be the $n \times n$ matrix whose i, j entry is $\beta_i\gamma_j$ for each $i, j \in [[1, n]]$. Then the following identity holds*

$$\det(\triangle + \triangle_1) = (\prod_{i=1}^{n} \alpha_i) + \sum_{i=1}^{n} (\beta_i\gamma_i \prod_{j\in[[1,n]\setminus\{i\}} \alpha_j) \in R.$$

Proof. Note that $\triangle_1$ has rank at most 1. For a subset $\mathbb{O} \in \mathcal{P}_f([[1, n]) \setminus \{[[1, n]]\}$, let $\triangle_{1,\mathbb{O}}$ be the square matrix obtained from $\triangle_1$ by removing all i-th rows and columns with $i \in \mathbb{O}$; if $n - 2 \geq |\mathbb{O}|$, then $\det(\triangle_{1,\mathbb{O}}) = 0$. By convention $\det(\triangle_{1,[[1,n]]}) := 1$.

We compute

$$\det(\triangle + \triangle_1) = \sum_{\mathbb{O}\in\mathcal{P}_f([[1,n])} \big(\det(\triangle_{1,\mathbb{O}}) \prod_{i\in\mathbb{O}} \alpha_i\big)$$

$$= \sum_{\mathbb{O}\in\mathcal{P}_f([[1,n]]),|\mathbb{O}|\in\{n-1,n\}} \big(\det(\triangle_{1,\mathbb{O}}) \prod_{i\in\mathbb{O}} \alpha_i\big) = (\prod_{i=1}^{n} \alpha_i) + \sum_{i=1}^{n} (\beta_i\gamma_i \prod_{j\in[[1,n]]\setminus\{i\}} \alpha_j).$$

□

Corollary 21.2. *Let $n \geq 2$. Let $(l_1, \ldots, l_n) \in (\mathbb{N}^*)^n$, $(f_1, \ldots, f_n) \in k[t]^n$ and $L \in k[x_1^p, \ldots, x_n^p]$. Let $w := L(x_1, \ldots, x_n) \prod_{i=1}^n x_i^{l_i} \in k[x_1, \ldots, x_n]$. Then the determinant of the Jacobian matrix $J\big(x_1f_1(w), \ldots, x_nf_n(w)\big)(x_1, \ldots, x_n)$ is*

$$\prod_{i=1}^{n} f_i(w) + \sum_{i=1}^{n} l_iwf_1(w)\cdots f_{i-1}(w)f_i'(w)f_{i+1}(w)\cdots f_n(w).$$

Proof. Let $(\alpha_i, \beta_i, \gamma_i) := \big(f_i(w), x_iwf_i'(w), l_ix_i^{-1}\big) \in (k[x_1, \ldots, x_n][\frac{1}{\prod_{i=1}^n} x_i])^3$ for $i \in [[1, n]]$. Then $\frac{\partial x_if_i(w)}{\partial x_j}$ is $\beta_i\gamma_j$ if $j \neq i$ and is $\alpha_i + \beta_i\gamma_i$ if $j = i$. Thus the corollary follows from Lemma 21.1 applied over $R = k[x_1, \ldots, x_n][\frac{1}{\prod_{i=1}^n} x_i]$ to the Jacobian matrix $J\big(x_1f_1(w), \ldots, x_nf_n(w)\big)(x_1, \ldots, x_n) = \triangle + \triangle_1$. □

We have the following generalization of [Lang-J], Prop. 4.8.

Proposition 21.3. *Let $(l_1, \ldots, l_n) \in [[1, \ldots, p-1]]^n$, $L \in k_{\mathrm{s}}[x_1^p, \ldots, x_n^p]$ and $\alpha \in k^*$. Let $(f_1, \ldots, f_n) \in k[t]^n$ be such that the product $\prod_{i=1}^n f_i \neq 0$ is a non-zero solution of the differential equation*

$$\Big(t \prod_{i=1}^{n} f_i^{l_i}\Big)' = \alpha \prod_{i=1}^{n} f_i^{l_i-1}. \tag{28}$$

By defining $w := L\prod_{i=1}^n x_i^{l_i}$*, the rule*

$$(x_1,\ldots,x_n)\mapsto \big(x_1f_1(w),\ldots,x_nf_n(w)\big)$$

defines an étale endomorphism $e^{l_1,\ldots,l_n}_{f_1,\ldots,f_n;L}\in \mathrm{EE}_n(k)$ *of Jacobian determinant* α*.*

Proof. Dividing Equation (28) by $\prod_{i=1}^n f_i^{l_i-1}$, based on Corollary 21.2 we get that $\det\big(J(x_1f_1(w),\ldots,x_nf_n(w))(x_1,\ldots,x_n)\big)$ is equal to

$$\prod_{i=1}^n f_i(w)+\sum_{i=1}^n l_iwf_1(w)\cdots f_{i-1}(w)f_i'(w)f_{i+1}(w)\cdots f_n(w)=\alpha,$$

from which the proposition follows. $\square$

For each non-zero solution $(f_1,\ldots,f_n)\in k[t]^n$ of Equation (28), the polynomial $t\prod_{i=1}^n f_i(t)\in k[t]$ is separable; but only in the case $l_1=\cdots=l_n$ we are able to get a nice description of the solution set of Equation (28) as follows.

Example 21.4. Let $(l,\alpha)\in[[1,p-1]]\times k^*$. Assume that $l_1=\cdots=l_n=l$. Let $s\in[[1,p-1]]$ be such that $ls\equiv -1 \pmod p$. By defining $f_0:=\prod_{i=1}^n f_i\in k[t]$, Equation (28) is equivalent to the differential equation

$$f_0+ltf_0'=\alpha. \tag{29}$$

As for each $m\in\mathbb{N}^*$, we have $x^m+lt(x^m)'=(1+lm)x^m$, the solution set of Equation (29) is $\alpha+t^sk[t^p]$. Thus from Proposition 21.3 we get that for every n-tuple $(f_1,\ldots,f_n)\in(k_{\mathrm{s}}[t])^n$ with $\prod_{i=1}^n f_i\in\alpha+t^sk[t^p]$ and $L\in k_{\mathrm{s}}[x_1^p,\ldots,x_n^p]$, by defining $z:=\prod_{i=1}^n x_i$ and $w:=z^lL$, the rule

$$(x_1,\ldots,x_n)\mapsto\big(x_1f_1(w),\ldots,x_nf_n(w)\big)$$

defines an étale endomorphism $e^l_{f_1,\ldots,f_n;L}\in\mathrm{EE}_n(k)$ of Jacobian determinant α.

We generalizes the case $g=1$ of Example 13.2 by elaborating on the case $(l,\alpha,L)=(1,1,1)$ of Example 21.4.

Example 21.5. For $m\in\mathbb{N}^*$ and $(\alpha_1,\ldots,\alpha_m)\in k^{m-1}\times k^*$ we define polynomials $f(t):=1+\sum_{i=1}^m\alpha_it^{pi-1}$ and $F(t):=tf(t)$; we have $F'(t)=1$ and thus F (or f) is separable. Assume $n\geq 2$ and let $z:=\prod_{i=1}^n x_i\in k_{\mathrm{s}}[x_1,\ldots,x_n]$. Let $f_1(t),\ldots,f_n\in k[t]$ be such that $f_1(0)=\cdots=f_n(0)=1$ and $\prod_{i=1}^n f_i(t)=f(t)$. Let

$$\mathbb{K}^n_{f_1,\ldots,f_n}:=\mathrm{Spec}\,\big(k[x_0,x_1,\ldots,x_n]/(F(x_0)-z)\big).$$

The projection

$$\pi_0:\mathbb{K}^n_{f_1,\ldots,f_n}\to\mathbb{A}^n_{k,\mathrm{t}}$$

on the last n coordinates is a finite étale morphism of degree pm.

Let $\imath_{f_1,\ldots,f_n}:\mathbb{A}^n_{k,\mathrm{s}}\to\mathbb{K}^n_{f_1,\ldots,f_n}$ be the morphism defined by the rule

$$(x_1,\ldots,x_n)\mapsto\big(z,x_1f_1(z),\ldots,x_nf_n(z)\big).$$

Clearly, $\imath_{f_1,\ldots,f_n}$ is birational; to show that it is in fact an open embedding, it suffices to show that for each $(\beta_0,\ldots,\beta_n)\in k^{n+1}$, the system $\mathcal{S}$ of equations

$$z-\beta_0=x_1f_1(z)-\beta_1=\cdots=x_nf_n(z)-\beta_n$$

in the indeterminates $x_1,\ldots,x_n$ has at most one solution in k^n. If $f(\beta_0)\neq 0$, then the only solution of $\mathcal{S}$ is $\big(\frac{\beta_1}{f_1(\beta_0)},\ldots,\frac{\beta_n}{f_n(\beta_0)}\big)$. Assume now that $f(\beta_0)=0$; thus $\beta_0\neq 0$. As the roots of f are distinct, there exists a unique $j\in[[1,n]]$ such that $f_j(\beta)=0$. Thus for $i\in[[1,n]]\setminus\{j\}$ we have $x_i=\frac{\beta_i}{f_i(\beta_0)}$. If there exists $i\in[[1,n]]\setminus\{j\}$ such that $\beta_i=0$, then $x_i=0$, hence $z=0$ which contradicts $\beta_0\neq 0$, therefore in such a case $\mathcal{S}$ inconsistent. If for each $i\in[[1,n]]\setminus\{j\}$ we have $\beta_i\neq 0$, then $x_j=\beta_0\prod_{i\in[[1,n]]\setminus\{j\}}\frac{f_i(\beta_0)}{\beta_i}$ and $\mathcal{S}$ has a unique solution. We conclude that $\imath_{f_1,\ldots,f_n}$ is an open embedding and that

$$\mathbb{E}^{n-1}_{f_1,\ldots,f_n}:=\mathbb{K}^n_{f_1,\ldots,f_n}\setminus \mathrm{Im}(\imath_{f_1,\ldots,f_n})=\cup_{j\in[[1,n]]}\cup_{\beta\in k,f_j(\beta)=0}\cup_{i\in[[1,n]]\setminus\{j\}}\mathbb{E}^{n-1}_{\beta,i}$$

where $\mathbb{E}^{n-1}_{\beta,i}$ is the closed subvariety of $\mathbb{K}^n_{f_1,\ldots,f_n}$ (or of $\mathbb{A}^{n+1}_k$) defined by the equations $x_0-\beta=x_i=0$; we have $\mathbb{E}^{n-1}_{\beta,i}\cong\mathbb{A}^{n-1}_k$. It follows that

$$|\mathrm{Irr}(\mathbb{E}^{n-1}_{f_1,\ldots,f_n})|=\sum_{j=1}^n \deg(f_j)(n-1)=(n-1)(pm-1). \tag{30}$$

Thus

$$e_{f_1,\ldots,f_n}:=\pi_0\circ\imath_{f_1,\ldots,f_n}\in \mathrm{EE}_n(k)$$

is defined by the n-tuple $\big(x_1f_1(z),x_2f_2(z),\ldots,x_nf_n(z)\big)\in(k_{\mathrm{s}}[x_1,\ldots,x_n])^n$ and we have $\deg(e_{f_1,\ldots,f_n})=pm$ and $X_{e_{f_1,\ldots,f_n}}\cong\mathbb{K}^n_{f_1,\ldots,f_n}$. Also, $e_{f_1,\ldots,f_n}$ becomes finite étale after inverting $z\in k_{\mathrm{t}}[x_1,\ldots,x_n]$. For each point P of the zero locus $z=0$, we have $(0,P)\in\mathrm{Im}(\imath_{f_1,\ldots,f_n})$, hence $P\in\mathrm{Im}(e_{f_1,\ldots,f_n})$. The last two sentences imply that $e_{f_1,\ldots,f_n}$ is surjective. So $\varphi(e_{f_1,\ldots,f_n})=\iota(e_{f_1,\ldots,f_n})=0$. Note that $e_{f_1,\ldots,f_n}$ is $e^1_{f_1,\ldots,f_n;1}$ of Example 21.4 and has 1 point at infinity modulo p in the sense of Definition 6.6(5) iff $n=2$ and $p\mid\deg(f_1)\deg(f_2)$.

If $n\geq 3$, then, for distinct elements i_1,i_2,j of $[[1,n]]$ and $\beta\in k$ with $f_j(\beta)=0$, the intersection $\mathbb{E}^{n-1}_{\beta,i_1}\cap\mathbb{E}^{n-1}_{\beta,i_2}\cong\mathbb{A}^{n-2}_k$ is non-empty, in contrast to Corollary 11.4(1).

We consider the particular case $n=2$. Let $(g,h):=(f_1,f_2)$. Equation (30) gives $\rho_{\text{ét}}(e_{g,h})=\rho(e_{g,h})=pm-1$. From the description of $\mathbb{E}^1_{g,h}$ we get that $e^{-1}_{g,h}(0,0)=\{(0,0)\}$ and that for each $\beta\in k^*$ we have $e^{-1}_{g,h}(0,\beta)=\{\big(\frac{zh(z)}{\beta},\frac{\beta}{h(z)}\big)|z\in k,\, g(z)=0\}$ and $e^{-1}_{g,h}(\beta,0)=\{\big(\frac{\beta}{g(z)},\frac{zg(z)}{\beta}\big)|z\in k,\, h(z)=0\}$. If $1\notin\{g,h\}$, then $\mathcal{F}(e_{g,h})=\{1,\deg(g)+1,\deg(h)+1\}$. If $1\in\{g,h\}$, then $\mathcal{F}(e_{g,h})=\{1\}$. Also, from Proposition 4.3(1) to (3) we get that for $\star\in\{\mathrm{l},\mathrm{r}\}$ we have

$$2\min\{\deg(g),\deg(h)\}+1\leq\pi_{\mathrm{i}}(e_{g,h})\leq\pi_\star(e_{g,h})=2\max\{\deg(g),\deg(h)\}+1;$$

so $\pi_{\mathrm{l}}(e_{g,h})=\pi_{\mathrm{r}}(e_{g,h})\geq 2\lceil\frac{pm-1}{2}\rceil+1$. E.g., if p is odd and $\deg(g)=\deg(h)=\frac{pm-1}{2}$, then $\pi_{\mathrm{i}}(e_{g,h})=\pi_{\mathrm{i}}(e_{g,h})=\pi_{\mathrm{l}}(e_{g,h})=pm$. If $1\notin\{g,h\}$ let $\epsilon:=2$ and if $1\in\{g,h\}$ let $\epsilon:=1$; Proposition 4.3(3) also gives that $\max\{\deg(g),\deg(h)\}+\epsilon\leq\pi_{\mathrm{i}}(e_{g,h})$. As N_e is the zero locus $z=0$ we have $\nu(e)=2$.

To give a concrete general example independent of any product decomposition for $n=2$, in this paragraph we assume that $p=2o+1$ is odd. Let $r\in\mathbb{N}$. We take $g(t)=1+f(t)$, $h(t)=1-f(t)$, where $f(t):=\sum_{i=0}^r\beta_it^{pi+o}\in k[t]$ with $\beta_0,\ldots,\beta_{r-1}\in k$ and $\beta_r\in k^*$. Then $\deg(g)=\deg(h)=pr+o$, and

$$tg(t)h(t)=t-t^p\Big(\sum_{i=0}^r\beta_it^{pi}\Big)^2=t+\sum_{i=1}^{2r+1}\alpha_it^{pi};$$

so $\alpha_1=-\beta_0^2,\alpha_2=-2\beta_0\beta_1,\ldots,\alpha_{2r+1}=-\beta_r^2$ are quadratic forms in $\beta_0,\ldots,\beta_r$ and $m=2r+1$. Denoting $e_f:=e_{1+f,1-f}$, we get that $\deg(e_f)=pm=p(2r+1)$ and $\mathcal{F}(e_f)=\{1,pr+o+1\}=\{1,\frac{p+1}{2}+pr\}$.

Remark 21.6. **(1)** Referring to Example 21.5, $e_{f_1,\ldots,f_n} \in \mathrm{EE}_n(k)\setminus\mathrm{GA}_n(k)$ fixes all points $(\alpha_1,\ldots,\alpha_n) \in k^n$ with $\prod_{i=1}^n \alpha_i = 0$' so it maps injectively each hyperplane $x_i = 0$ onto a smooth closed hypersurface, which is in contrast to the char$(K) = 0$ situation: if $d \in \mathrm{EE}_2(K)$ is injective on one line of $\mathbb{A}^2_{K,\mathrm{s}}$ (resp. maps one line to a smooth curve), then $d \in \mathrm{GA}_2(K)$ by [Gw], Thm. 1.1 (resp. [YD], Thm. 2.4).

(2) Referring to $e_{f,g}$ of Example 21.5, if $1 \notin \{g,h\}$ and $p \nmid m$ then $e_{f,g}$ is not a p-morphism by Remark 6.7(6).

Next we generalize [Lang-J], Prop. 4.5, Case $a \equiv b \equiv 1 \pmod p$ to all $n \geq 2$ by elaborating on the case $(l,\alpha,L) = (1,1,\prod_{i=1}^n x_i^{pq_i})$ with $(q_1,\ldots,q_n) \in \mathbb{N}^n$ of Example 21.4.

Example 21.7. Let $n \geq 2$, $m \in \mathbb{N}^*$, $(\alpha_1,\ldots,\alpha_m) \in k^{m-1} \times k^*$, $f(t)$, $F(t)$, $f_1(t),\ldots,f_n(t)$, and $z = \prod_{i=1}^n x_i$ be as in Example 21.5. Recall that $e_{f_1,\ldots,f_n}$ is defined by the n-tuple $(g_1,\ldots,g_n)$, where $g_i(x_1,\ldots,x_n) := x_i f_i(z)$ for $i \in [[1,n]]$.

Let $(q_1,\ldots,q_n) \in \mathbb{N}^n$. Let $L := \prod_{i=1}^n x_i^{pq_i} \in k_{\mathrm{s}}[x_1^p,\ldots,x_n^p]$. For $i \in [[1,n]]$ let $m_i := pq_i + 1$. Let $w := \prod_{i=1}^n x_i^{m_i}$; we have divisibilities $z|w|z^{\max\{m_1,\ldots,m_n\}}$ and $w = zL$. For $i \in [[1,n]]$ let $h_i(x_1,\ldots,x_n) := x_i f_i(w)$.

Let $e^{q_1,\ldots,q_n}_{f_1,\ldots,f_n} \in \mathrm{End}_n(k)$ be defined by the n-tuple $(h_1,\ldots,h_n) \in (k_{\mathrm{s}}[x_1,\ldots,x_n])^n$; as $e^{q_1,\ldots,q_n}_{f_1,\ldots,f_n}$ is $e^1_{f_1,\ldots,f_n;L}$ of Example 21.4, we have $e^{q_1,\ldots,q_n}_{f_1,\ldots,f_n} \in \mathrm{EE}_n(k)$. Note that $e_{f_1,\ldots,f_n} = e^{0,\ldots,0}_{f_1,\ldots,f_n}$.

As in the case of $e_{f_1,\ldots,f_n}$ (see Example 21.5), it is easy to see that $e^{q_1,\ldots,q_n}_{f_1,\ldots,f_n}$ becomes finite étale after inverting z (equivalently, w).

Let

$$\mathbb{K}^{n;q_1,\ldots,q_n}_{f_1,\ldots,f_n} := \mathrm{Spec}\,\big(k[x_0,x_1,\ldots,x_n]/(F(x_0)\prod_{i=1}^n f_i(x_0)^{pq_i} - w)\big).$$

The projection

$$\pi_0 : \mathbb{K}^{n;q_1,\ldots,q_n}_{f_1,\ldots,f_n} \to \mathbb{A}^n_{k,\mathrm{t}}$$

on the last n coordinates is a finite flat morphism of degree $pm + \sum_{i=1}^n pq_i \deg(f_i)$. Let $\imath^{q_1,\ldots,q_n}_{f_1,\ldots,f_n} : \mathbb{A}^n_{k,\mathrm{s}} \to \mathbb{K}^{n;q_1,\ldots,q_n}_{f_1,\ldots,f_n}$ be the morphism defined by the rule

$$(x_1,\ldots,x_n) \mapsto \big(w, h_1(x_1,\ldots,x_n),\ldots,h_n(x_1,\ldots,x_n)\big) = \big(w, x_1 f_1(w),\ldots,x_n f_n(w)\big).$$

Clearly, $\imath^{q_1,\ldots,q_n}_{f_1,\ldots,f_n}$ is birational. As $e^{q_1,\ldots,q_n}_{f_1,\ldots,f_n} = \pi_0 \circ \imath^{q_1,\ldots,q_n}_{f_1,\ldots,f_n}$, it follows that $\imath^{q_1,\ldots,q_n}_{f_1,\ldots,f_n}$ is quasi-finite, $\deg(e^{q_1,\ldots,q_n}_{f_1,\ldots,f_n}) = pm + \sum_{i=1}^n pq_i \deg(f_i)$ and $\mathbb{K}^{n;q_1,\ldots,q_n}_{f_1,\ldots,f_n}$ is a flat model of $X_{e^{q_1,\ldots,q_n}_{f_1,\ldots,f_n}}$. As in Example 21.5 we argue that $\mathbb{K}^{n;q_1,\ldots,q_n}_{f_1,\ldots,f_n} \setminus \mathrm{Im}(\imath^{q_1,\ldots,q_n}_{f_1,\ldots,f_n}) = \mathbb{E}^{n-1}_{f_1,\ldots,f_n}$ is independent on $(q_1,\ldots,q_n)$ and therefore $\rho(e) \geq (n-1)(pm-1)$ by Equation (30) and Remark 11.7(1). Moreover, $\imath^{q_1,\ldots,q_n}_{f_1,\ldots,f_n}$ is injective iff $q_1 = \cdots = q_n = 0$.

Example 21.8. We take $q_1 = \cdots = q_{n-1} = 0$ and $q := q_n \in \mathbb{N}^*$ and assume that $\deg(f_n) \geq 1$; let $\alpha_n \in k^*$ be such that $f_n(\alpha_n) = 0$. We have

$$pm + pq(pm-1) \geq \deg(e^{0,\ldots,0,q}_{f_1,\ldots,f_n}) = pm + pq\deg(f_n) \geq p(m+q).$$

Using the substitution $v := x_0 - \alpha_n$, it follows that

$$\mathbb{K}^{n;0,\ldots,0,q}_{f_1,\ldots,f_n} = \mathrm{Spec}\,\big(k[v,x_1,\ldots,x_n]/(v^{pq+1}g(v) - x_n^{pq+1}\prod_{i=1}^{n-1} x_i)\big),$$

with $g(v) \in k[v]$ such that $g(0) \neq 0$ and $\deg(g) = p[m + q\deg(f_n) - q] - 1$.

Let $\mathcal{R}$ be the normalization of the local ring of $\mathbb{K}^{n;0,\ldots,0,q}_{f_1,\ldots,f_n}$ at the point $(0,\ldots,0) \in \mathbb{K}^{n;0,\ldots,0,q}_{f_1,\ldots,f_n}(k)$; it is a localization of the $k_{\mathrm{t}}[x_1,\ldots,x_n]$-algebra $\mathcal{O}(X_{e^{0,\ldots,0,q}_{f_1,\ldots,f_n}})$. Let

$$R_{n,q,k} := k[[w,x_1,\ldots,x_n]]/(w^{pq+1} - \prod_{i=1}^{n-1} x_i).$$

As $g(0) \neq 0$ it is easy to see that we have a homomorphism $\mathcal{R} \to R_{n,q,k}$ of $k_{\mathrm{t}}[x_1,\ldots,x_n]$-algebras that maps $v + \big(v^{pq+1}g(v) - x_n^{pq+1}\prod_{i=1}^{n-1} x_i\big)$ into a unit times $wx_n + (w^{pq+1} - \prod_{i=1}^{n-1} x_i)$ and this implies that the completion of $\mathcal{R}$ at a suitable maximal ideal of it is isomorphic as a $k[[x_1,\ldots,x_n]]$-algebra to $R_{n,q,k}$. Thus for $n \geq 3$ we get that $\mathcal{R}$ is a non-regular ring and hence $\psi_{e^{0,\ldots,0,q}_{f_1,\ldots,f_n}}$ is a non-regular Jacobian variety. If $n = 2$, we only get that $\psi_{e^{0,q}_{f_1,f_2}}$ is non-étale and one can easily check that it is in fact regular; moreover, if $\deg(f_2) = 1$, i.e., $f_2(t) = t - \alpha_2$, then $\mathbb{K}^{2;0,q}_{f_1,t-\alpha_2}$ is isomorphic to $\mathbb{S}^2_{pq+1,pq+1,(t+\alpha_2)f_1(t+\alpha_2),1,k}$ of Example 13.4.

Remark 21.9. The additive version of Example 21.5 uses sums instead of products. If $n \geq 1$ and $f_1(t),\ldots,f_n(t) \in k[t]$ are such that $\sum_{i=1}^n f_i(t) = \sum_{i=0}^m \alpha_i t^{pi}$ with $m \in \mathbb{N}^*$ and $(\alpha_1,\ldots,\alpha_m) \in k^{m-1} \times k^*$, then, by defining $z := \sum_{i=1}^n x_i$, the n-tuple $\big(x_1 + f_1(z),\ldots,x_n + f_n(z)\big)$ defines $e^{\mathrm{a}}_{f_1,\ldots,f_n} \in \mathrm{EE}_n(k)$ of Jacobian matrix

$$\begin{bmatrix} 1+f_1'(z) & f_1'(z) & \cdots & f_1'(z) \\ \cdots & \cdots & \cdots & \cdots \\ f_n'(z) & f_n'(z) & \cdots & 1+f_n'(z) \end{bmatrix}$$

that has determinant 1 as one can easily check based on the fact that for each column, its entries add up to 1. For $(\beta_1,\ldots,\beta_n) \in k^n$, the fiber $(e^{\mathrm{a}}_{f_1,\ldots,f_n})^{-1}(\beta_1,\ldots,\beta_n)$ has pm elements as it equals

$$\{\big(\beta_1 - f_1(\delta),\ldots,\beta_n - f_n(\delta)\big)|\delta \in k, \delta + \sum_{i=0}^m \alpha_i \delta^{pi} = \sum_{i=1}^n \beta_i\}.$$

Thus $\deg(e^{\mathrm{a}}_{f_1,\ldots,f_n}) = pm$ and $e^{\mathrm{a}}_{f_1,\ldots,f_n}$ is finite étale; hence $\psi_{e^{\mathrm{a}}_{f_1,\ldots,f_n}} = e^{\mathrm{a}}_{f_1,\ldots,f_n}$ and $\rho_{\text{ét}}(e^{\mathrm{a}}_{f_1,\ldots,f_n}) = \rho(e^{\mathrm{a}}_{f_1,\ldots,f_n}) = \varphi(e^{\mathrm{a}}_{f_1,\ldots,f_n}) = \iota(e^{\mathrm{a}}_{f_1,\ldots,f_n}) = 0$.

Remark 21.10. In Example 21.5 we take $n = 2$ and $\deg(f_1) \leq \deg(f_2)$; the pair $(g_1,g_2) := \big(x_1f_1(x_1x_2), x_2f_2(x_1x_2)\big)$ defines $e_{f_1,f_2} \in \mathrm{EE}_2(k)$ with $\deg(e_{f_1,f_2}) = pm$ and inequalities

$$\deg(g_1) = 2\deg(f_1) + 1 \leq pm \leq 2\deg(f_2) + 1 = \deg(g_2) = 2pm - \deg(g_1);$$

note that $m \in \mathbb{N}^*$ and $\deg(g_1) \in [[1,pm]] \cap (2\mathbb{N}^* - 1)$ are arbitrary.

Similarly, in Example 21.7 we take $n = 2$ with $\deg(f_1) \leq \deg(f_2)$. The pair $(h_1,h_2) = \big(x_1f_1(x_1^{pq_1+1}x_2^{pq_2+1}), x_2f_2(x_1^{pq_1+1}x_2^{pq_2+1})\big)$ defines $e^{q_1,q_2}_{f_1,f_2} \in \mathrm{EE}_2(k)$ with $\deg(e^{q_1,q_2}_{f_1,f_2}) = pm + pq_1\deg(f_1) + pq_2\deg(f_2)$ and we have inequalities

$$\deg(h_1) = (pq_1 + pq_2 + 2)\deg(f_1) + 1 \leq \deg(e^{q_1,q_2}_{f_1,f_2}) \leq (pq_1 + pq_2 + 2)\deg(f_2) + 1$$
$$= \deg(h_2) = 2\deg(e^{q_1,q_2}_{f_1,f_2}) - \deg(h_1) + p(q_1 - q_2)\big(\deg(f_2) - \deg(f_1)\big).$$

If p, q_1 and q_2 are odd, then $\deg(h_1)$ and $\deg(h_1)$ are even.

Let $(q,s) \in (\mathbb{N}^* \setminus \{1\}) \times \mathbb{N}^*$ and $l \in [[ps, 2ps-1]]$. The pair $(x_1 + x_2^q, x_2 + x_1^{ps})$ defines $d \in \mathrm{EE}_2(k)$ which is finite with $\deg(d) = pqs$ by Fact 15.1(2). If $a \in \mathrm{GA}_2(K)$ is defined by the pair $(x_1, x_2 + x_1^l)$, then $e := ad \in \mathrm{EE}_2(k)$ is finite with $\deg(e) = pqs$.

Moreover, ae is defined by the pair $(g_3, g_4) := \big(x_1 + x_2^q, x_2 + x_1^{ps} + (x_1 + x_2^q)^l\big)$ for which we have inequalities

$$\deg(g_3) = q < pqs = \deg(e) \le ql = \deg(g_4) \le 2pqs - q;$$

if $q \in 2\mathbb{N}^*$, $l = 2ps - 1$ and $m = qs$ we get an even analogue of the particular $\deg(g_1)|m$ of the first paragraph.

The examples of the last three paragraphs are in contrast to the $\mathrm{char}(K) = 0$ situation: if $c \in \mathrm{EE}_2(K)$ is defined by a pair $(g, h) \in K_{\mathrm{s}}[x_1, x_2]$ with $\deg(g) \le \deg(h)$, then $\deg(c) \le \deg(g)$ by [Zh], Introd., Thm. or [Kat], Thm. 1.

22. Additional examples and properties

Example 22.1. Let $m \ge 2$ and $q \ge 1$ be integers with $\mathrm{char}(K) \nmid q$. The rule $(x, y) \mapsto x + x^m y^q$ defines a smooth morphism $e : \mathbb{A}^2_K \to \mathbb{A}^1_K$ which is not a projection as $|\mathrm{Irr}\big(e^{-1}(0)\big)| \ge 2$; if $q = 1$, then for each $\alpha \in K^*$, $e^{-1}(\alpha) \cong \mathbb{A}^1_K \setminus \{0\}$.

Example 22.1 leads to the following definition that is used in connection to Conjecture 1.3 in Section 25.

Definition 22.2. *Let $n \in \mathbb{N}^*$, $e \in \mathrm{EE}_n(K)$, and $m \in [[0, n]]$. We say that e has type (resp. strong type or weak type) $\le m$ if there exists a smooth morphism (resp. a projection or a morphism) $\pi : \mathbb{A}^n_{K,\mathrm{s}} \to \mathbb{A}^m_K$ such that the quasi-finite morphism $\pi \times e : \mathbb{A}^n_{K,\mathrm{s}} \to \mathbb{A}^m_{K,\mathrm{t}} \times_{\mathrm{Spec}\, K} \mathbb{A}^n_K$ is birational to the Zariski closure $\overline{\mathrm{Im}(\pi \times e)}$ of its image and has the property that for each irreducible component Y of the closed subvariety $(\pi \times e)^{-1}\big(\mathrm{Sing}(\overline{\mathrm{Im}(\pi \times e)}\big)$ of $\mathbb{A}^n_{K,\mathrm{s}}$ of dimension $n - 1$ we have $e^{-1}\big(e(Y)\big) = Y$. We say that e has type (resp. strong type or weak type) m if it has type (resp. strong type weak type) $\le m$ but, if $m \ge 1$, does not have type (resp. strong type or weak type) $\le m - 1$.*

Example 22.3. Let $e \in \mathrm{EE}_n(K)$. Then e has (weak or strong) type 0 iff $e \in \mathrm{GA}_n(K)$ by Lemma 6.1(4). If via $e^\#$, the $K_{\mathrm{t}}[x_1, \dots, x_t]$-algebra $K_{\mathrm{s}}[x_1, \dots, x_t]$ is generated by 1 element, then e has weak type ≤ 1.

Lemma 22.4. *Each $e \in \mathrm{EE}_n(K)$ has weak type ≤ 2.*

Proof. Let $z \in A_e$ be such that the approximation $Z_e = \mathrm{Spec}\, B_e$ of X_e defined by the $K_{\mathrm{t}}[x_1, \dots, x_n]$-subalgebra B_e of A_e generated by z is a model of X_e. Let Y be the smallest closed subvariety of $\mathbb{A}^n_{K,\mathrm{t}}$ such that the birational morphism $X_e \to Z_e$ is an isomorphism above $Z_e \setminus (Z_e \times_{\mathbb{A}^n_{K,\mathrm{t}}} Y)$. Let $l \in \mathbb{N}$ be such that Y has l irreducible components of dimension $n - 1$. Let $O_1, \dots, O_l$ be the local rings of $\mathbb{A}^n_{K,\mathrm{t}}$ at the generic points of these s irreducible components. For $i \in [[1, l]]$, the normalization $O_{i,\mathrm{s}}$ of O_i in $K_{\mathrm{s}}(x_1, \dots, x_n)$ is a free O_i-module. Denoting by $\widehat{O}_i$ the completion of O_i, the tensor product $O_{i,\mathrm{s}} \otimes_{O_i} \widehat{O}_i$ is a product $\widehat{O}_{i,1} \times \widehat{O}_{i,2}$ such that the factors of $\widehat{O}_{i,1}$ (resp. $\widehat{O}_{i,2}$) are completions of local rings of X_e at points in $\mathrm{Im}(\imath_e)$ (resp. at points in $X_e \setminus \mathrm{Im}(\imath_e)$). Thus $\widehat{O}_{i,1}$ is an étale $\widehat{O}_i$-algebra. As the residue field $\mathcal{K}_i$ of $\widehat{O}_i$ is infinite, each étale $\mathcal{K}_i$-algebra is generated by 1 element. Hence there exists $y_i \in \widehat{O}_{i,1}$ that generates the $\widehat{O}_i$-algebra $\widehat{O}_{i,1}$. From the theory of non-equivalent valuations of $\mathrm{Frac}(K_{\mathrm{s}}[x_1, \dots, x_n])$ we get that there exists $y \in \mathrm{Frac}(K_{\mathrm{s}}[x_1, \dots, x_n])$ such that that $y - y_i$ belongs to Jacobson radical of $\widehat{O}_{i,1}$ for each $i \in [[1, l]]$. As $K_{\mathrm{s}}[x_1, \dots, x_n]$ is a unique factorization domain, we can assume that $y \in K_{\mathrm{s}}[x_1, \dots, x_n]$. If C_e is the B_e-subalgebra of $K_{\mathrm{s}}[x_1, \dots, x_n]$ generated by y, it follows that the morphism

$\mathbb{A}^n_{K,\mathrm{s}} \to C_e$ induces an open embedding outside a closed subvariety W of $\mathbb{A}^n_{K,\mathrm{s}}$ of codimension at least 2. As for the morphism $\pi : \mathbb{A}^n_{K,\mathrm{s}} \to \mathbb{A}^2_K$ defined by the rule $(x_1, \ldots, x_n) \mapsto (y, z)$, $\pi \times e : \mathbb{A}^n_{K,\mathrm{s}} \to \mathbb{A}^2_K \times_{\mathrm{Spec}\, K} \mathbb{A}^n_{K,\mathrm{t}}$ is a locally closed embedding outside W it follows that e has weak type ≤ 2. □

Example 22.5. For $m, l \in \mathbb{N}^*$ with l prime to p, we consider the morphism

$$\Sigma_{m,l} : \mathbb{G}_{m,k} = \mathrm{Spec}\,(k[t, t^{-1}]) \to \mathbb{A}^1_k$$

defined by the k-algebra homomorphism $k[x_1] \to k[t, t^{-1}]$ that maps x_1 to $t^{-l} - t^{pm}$; it is defined over $\mathbb{F}_p$. As $d(t^{-l} - t^{pm}) = -lt^{-l-1}dt \in k[t, t^{-1}]^* dt$, we get that $\Sigma_{m,l}$ is étale. As for $\alpha \in k$, the equation $\beta^{-l} - \alpha - \beta^{pm} = 0$ with $\beta \in k^*$ is equivalent to the equation $\beta^{pm+l} + \alpha\beta^l - 1 = 0$ with $\beta \in k$ and as the polynomial

$$f(t) = t^{pm+l} + \alpha t^l - 1 = t^l(t^{pm} + \alpha) - 1$$

is separable because for its derivative $f'(t) = lt^{l-1}(t^{pm} + \alpha)$ we have an identity $f(t) - l^{-1}tf'(t) = -1$, it follows that $\Sigma_{m,l}$ is finite étale of degree $pm + l$. Also, $\Sigma_{m,l}$ is induced by the endomorphism of $\mathbb{P}^1_k$ defined on valued points by the rule

$$(x_0 : x_1) \mapsto (-x_0^{pm+l} + x_1^{pm+l} : x_0^l x_1^{pm}).$$

Therefore, for $l = 1$, we denote simply $\Sigma_m := \Sigma_{m,1}$, and we conclude that Σ_m extends to a surjective non-finite étale morphism

$$\Sigma^+_m : \mathbb{A}^1_k \to \mathbb{P}^1_k$$

of degree $pm + 1$ which is finite over the 'finite part' open subvariety $\mathbb{A}^1_k$ of $\mathbb{P}^1_k$. We have $(\Sigma^+_m)^{-1}(1 : 0) = 0$ and Σ_1 is the morphism Σ of Section 2. Moreover, Σ^+_m extends to a finite flat endomorphism $\Sigma^{\mathrm{proj}}_m : \mathbb{P}^1_k \to \mathbb{P}^1_k$.

Example 22.6. If $p = 2o+1$ is odd, then the pair $(x + x^{o+1}y^o + x^p, y - x^o y^{o+1} - y^p)$ defines an endomorphism $e \in \mathrm{EE}_2(k)$ with $\pi(e) \leq p$ and Jacobian matrix

$$\begin{bmatrix} 1 + (o+1)x^o y^o & o x^{o+1} y^{o-1} \\ -o x^{o-1} y^{o+1} & 1 - (o+1)x^o y^o \end{bmatrix}.$$

Using the indeterminate $z := \dfrac{y}{x}$, so $y = xz$, the system

$$x + x^{o+1}(x^o + y^o) - x_1 = 0 = y - y^{o+1}(x^o + y^o) - x_2$$

gives $-z^{o+1}(x - x_1) = y - x_2 = zx - x_2$, hence $x = \frac{x_1 z^{o+1} + x_2}{z^{o+1} + z}$, $y = \frac{x_1 z^{o+1} + x_2}{z^o + 1}$, and $x - x_1 = \frac{-x_1 z + x_2}{z^{o+1} + z}$. Substituting these expressions in the equation $x - x_1 = -x^{o+1}(x^o + y^o) = -x^p(z^o + 1)$ we get that

$$\frac{-x_1 z + x_2}{z^{o+1} + z} = -(z^o + 1)\frac{(x_1 z^{o+1} + x_2)^p}{z^p (z^o + 1)^p} = -\frac{(x_1 z^{o+1} + x_2)^p}{z^p (z^o + 1)^{p-1}}.$$

So the $k_{\mathrm{t}}(x_1, x_2)$-algebra $k_{\mathrm{s}}(x_1, x_2)$ is isomorphic to $k_{\mathrm{t}}(x_1, x_2)[z]/\big(h(z)\big)$, where

$$h(z) = h(x_1, x_2, z) := x_1^p z^{p(o+1)} + z^{p-1}(z^o + 1)^{p-2}(-x_1 z + x_2) + x_2^p \in k_{\mathrm{t}}[x_1, x_2][z].$$

Thus $z \in (A_e)_{x_1}$, $z^{-1} \in (A_e)_{x_2}$ and $\deg(e) = p(o+1)$. The group

$$G := \{\alpha \in k | \alpha^{p-1} = 1\} \times \{\alpha \in k | \alpha^o = 1\}$$

of order $o(p-1) = 2o^2$ acts on the morphism $X_e \to \mathbb{A}^2_{k,\mathrm{t}}$ defined by ψ_e via the following rule at the level of k-algebras: $(\alpha, \beta) \cdot (x_1, x_2, z) = (\alpha x_1, \alpha\beta x_2, \beta z)$. The group G is cyclic iff $o = 1$.

It is convenient to also use x and x^{-1} (or y and y^{-1}) as indeterminates generating the field extension $k_{\mathrm{t}}(x_1,x_2)\to k_{\mathrm{s}}(x,y)$. The system of equations $x+x^{o+1}y^o+x^p-x_1=0=y-x^oy^{o+1}-y^p-x_2$ gives $y^o=\frac{x_1-x-x^p}{x^{o+1}}=\frac{x_1}{x^{o+1}}-\frac{1}{x^o}-x^o$ from the first equation. From this and the equation $y(1-x^oy^o-y^{2o})=x_2$ we get that

$$y\Big(1-\frac{x_1}{x}+1+x^{2o}-\frac{x_1^2}{x^{2o+2}}-\frac{1}{x^{2o}}-x^{2o}-2+\frac{2x_1}{x}+2\frac{x_1}{x^{2o+1}}\Big)=x_2$$

and hence $y\big(\frac{x_1}{x}-\frac{1}{x^{2o}}+2\frac{x_1}{x^{2o+1}}-\frac{x_1^2}{x^{2o+2}}\big)=y\frac{-x_1^2+2x_1x-x^2+x_1x^p}{x^{2o+2}}=x_2$. So

$$y=\frac{x_2x^{2o+2}}{-x_1^2+2x_1x-x^2+x_1x^p}.$$

The o-th power of last expression equalized to the prior expression $y^o=\frac{x_1-x-x^p}{x^{o+1}}$ gives $x_2^ox^{p(o+1)}-(x_1-x-x^p)(-x_1^2+2x_1x-x^2+x_1x^p)^o=0$ in $k_{\mathrm{s}}(x,y)$. By defining

$$g(x_1,x_2,x):=x_2^ox^{p(o+1)}-(x_1-x-x^p)(-x_1^2+2x_1x-x^2+x_1x^p)^o\in k_{\mathrm{t}}[x_1,x_2][x],$$

we get that the $k_{\mathrm{t}}(x,y)$-algebra $k_{\mathrm{s}}(x,y)$ is isomorphic to $k_{\mathrm{t}}(x_1,x_2)[x]/\big(g(x_1,x_2,x)\big)$ as well. As $2o=-1$ in k, we compute that $g_x(x_1,x_2,x)$ is

$$(-x_1^2+2x_1x-x^2+x_1x^p)^{o-1}[-x_1^2+2x_1x-x^2+x_1x^p-o(x_1-x-x^p)(2x_1-2x)]$$
$$=(-x_1^2+2x_1x-x^2+x_1x^p)^{o-1}[-x_1^2+2x_1x-x^2+x_1x^p+(x_1-x)^2-x_1x^p+x^{p+1}]$$
$$=x^{p+1}(-x_1^2+2x_1x-x^2+x_1x^p)^{o-1}.$$

Working with y instead of x, a similar computation shows that for the polynomial $f(x_1,x_2,y):=x_1^oy^{p(o+1)}-(-x_2+y-y^p)(x_2^2-2x_2y+y^2+x_2y^p)^o\in k_{\mathrm{t}}[x_1,x_2][y]$ we have $f(x_1,x_2,y)=0$ in $k_{\mathrm{s}}(x,y)$ and $f_y(x_1,x_2,y)=-y^{p+1}(x_2^2-2x_2y+y^2+x_2y^p)^o$.

We show that $e^{-1}(0,0)=\{(\alpha,0)|\alpha\in k,\alpha=-\alpha^p\}\cup\{(0,\beta)|\beta\in k,\beta=\beta^p\}$. It suffices to show that the assumption that the system of equations $1+x^o(y^o+x^o)=0=1-y^o(x^o+y^o)$ has a solution in k^2 leads to a contradiction. By subtracting the two equations we get that $(x^o+y^o)^2=0$, so $x^o+y^o=0$, and by substituting in the first equation we get that $1=0$, a contradiction. Thus $|e^{-1}(0,0)|=2p-1$.

We show that for $(\alpha,\beta)\in k^2$ with $\alpha\beta(\alpha^o+\beta^o)\neq 0$ we have $|e^{-1}(\alpha,\beta)|=p(o+1)$. The coefficient of $x^{p(o+1)}$ in $g(x_1,x_2,x)$ is $x_1^o+x_2^o$, hence $g(\alpha,\beta,x)\in k[x]$ has degree $p(o+1)$ and we have $g(\alpha,\beta,0)=(-1)^{o+1}\alpha^p\neq 0$. From these and the expressions of $g_x(\alpha,\beta,0)$ it follows that $g(\alpha,\beta,x)$ is separable. Hence

$$e^{-1}(\alpha,\beta)=\{(x,\frac{\beta x^{2o+2}}{-\alpha^2+2\alpha x-x^2+\alpha x^p})|g(\alpha,\beta,x)=0\}$$

has $p(o+1)$ elements.

Let $(\alpha,\beta)\in k^2$ be such that $\alpha^o+\beta^o=0\neq\alpha\beta$. In this case $g(\alpha,\beta,x)$ has degree $po+2$ and a similar argument shows that it is separable and $|e^{-1}(\alpha,\beta)|=po+2$.

If $o\geq 2$ let $(\alpha,\beta)\in k^2\setminus\{(0,0)\}$ be such that $\alpha\beta=0$. By the similarity of g and f and their partial derivatives, we can assume that $\alpha\neq 0$ and $\beta=0$. We know that $e^{-1}(\alpha,0)$ is the solution set in $k^*\times k$ of the system of equations

$$y^o-\frac{\alpha-x-x^p}{x^{o+1}}=y(-\alpha^2+2\alpha x-x^2+\alpha x^p)=0.$$

As $-\alpha^2+2\alpha x-x^2+\alpha x^p+\alpha(\alpha-x-x^p)=x(\alpha-x)$ and $\alpha\neq 0$, the polynomials $-\alpha^2+2\alpha x-x^2+\alpha x^p$ and $\alpha-x-x^p$ are separable of degree p and do not have a common root. Therefore the fiber

$$e^{-1}(\alpha,0)=\Big\{(\delta,0)|\delta\in k,\delta+\delta^p=\alpha\}\cup\{\delta,\sqrt[o]{\frac{\alpha-\delta-\delta^p}{\delta^{o+1}}}\Big|\alpha^2-2\alpha\delta+\delta^2=\alpha\delta^p\Big\}$$

has $po+p=p(o+1)$ elements.

We conclude that, regardless of the value of o, N_e is the zero locus $x_1^o+x_2^o=0$, hence $\nu(e)=o$, and $\mathcal{F}(e)=\{2p-1,po+2\}$.

If $o=1$, i.e., $p=3$, then $g(x_1,x_2,x)=(x_1+x_2)x^6-x^5+(x_1-1)x^3+x_1^3=0$, N_e is the zero locus $x_1+x_2=0$, $\nu(e)=1$, $\mathcal{F}(e)=\{5\}$ and $\deg(e)=6$; hence $\rho_{\text{ét}}(e)=\rho(e)=1$ and ψ_e is regular, being in fact finite étale of degree 6 by either Example 10.3 or by remarking that the rules $(x,y)\mapsto(x-y,y)$, $(x,y)\mapsto(x+y,y)$ and $(x,y)\mapsto(y,x)$ define automorphisms a, b and c (respectively) in $\mathrm{GA}_2(k)$ with the property that $bea=cd_{1,1,k}c$ (see Example 20.3 for $d_{1,1,k}$); thus $X_e\cong\mathbb{S}_k^2$.

From now on we assume that $o\geq 2$ (i.e., if $p\geq 5$). For $\gamma\in\mathbb{I}:=\{\alpha\in k|\alpha^o=-1\}\subset\mathbb{F}_p$, let ℓ_γ be the line given by the equation $x_1-\gamma x_2=0$ and let $\widehat{O}_\gamma$ be the completion of the local ring O_γ of $\mathbb{A}^2_{k,\mathrm{t}}$ at the generic point of ℓ_γ. As these lines are permuted transitively by G, the O_γs are also permuted transitively by G.

Each irreducible component of $X_e\setminus\mathbb{A}^2_{k,\mathrm{s}}$ maps onto one line ℓ_γ with $\gamma\in\mathbb{I}$. So to show that $\rho(e)=o$ it suffices to show that there exists a unique irreducible component Y_γ of $X_e\setminus\mathbb{A}^2_{k,\mathrm{s}}$ that maps onto ℓ_γ and it maps onto ℓ_γ isomorphically.

As $(x_1x_2)^{-1}\in\widehat{O}_\gamma$, $A_e\times_{k[x_1,x_2]}\widehat{O}_\gamma$ is the normalization of $\widehat{O}_\gamma[z]/\big(h(z)\big)$ in $\mathrm{Frac}(\widehat{O}_\gamma)[z]/\big(h(z)\big)$ and we show that it is a product $\widehat{O}_{\gamma,1}\times\widehat{O}_{\gamma,2}$ with $\widehat{O}_\gamma\to\widehat{O}_{\gamma,1}$ finite étale of degree $po+2$ and $\widehat{O}_\gamma\to\widehat{O}_{\gamma,2}$ a finite flat totally ramified homomorphism of degree $p-2$. We compute

$$\begin{aligned}h_z(x_1,x_2,z)&=z^{p-2}(z^o+1)^{p-3}[-(z^o+1)(-x_1z+x_2)+z^o(-x_1z+x_2)-x_1z(z^o+1)]\\&=z^{p-2}(z^o+1)^{p-3}(-x_2-x_1z^{o+1}).\end{aligned}$$

Therefore, after inverting x_1x_2, the multiple roots of $h(z)$ occur precisely when $z^o+1=x_1z^{o+1}+x_2=0$; so above the open subset of ℓ_γ defined by $x_1x_2\neq 0$, the only multiple root of $h(z)$ is $z=\gamma^{-1}$ with multiplicity $p-1$. For the uniformizer $\pi_\gamma:=x_2-\gamma^{-1}x_1\in k_{\mathrm{t}}[x_1,x_2]$ of $\widehat{O}_\gamma$ and the substitution $t:=z-\gamma^{-1}$, we compute

$$h(x_1,x_2,t):=h(x_1,x_2,t+\gamma^{-1})=\pi_\gamma^p+x_1^p\Big[\sum_{i=1}^{o+1}\binom{o+1}{i}t^{pi}\gamma^{-p(o+1-i)}\Big]$$

$$+(t+\gamma^{-1})^{p-1}\Big[\sum_{i=1}^{o}\binom{o}{i}t^i\gamma^{-(o-i)}\Big]^{p-2}(-x_1t+\pi_\gamma).$$

Under the additional substitution $v:=\pi_\gamma^{-1}t$, we can write

$$h(x_1,x_2,v):=h(x_1,x_2,\pi_\gamma v)=\pi_\gamma^{p-1}(\pi_\gamma+u_1v^{p-2}+u_2v^{p-1}),$$

with $u_1,u_2\in\widehat{O}_\gamma[v]\cap\widehat{O}_\gamma[[v]]^*$. As the normalization of $\widehat{O}_\gamma$ in

$$\mathrm{Frac}(\widehat{O}_\gamma)[[v]]/(\pi_\gamma+u_1v^{p-2}+u_2v^{p-1})$$

has a factor which is a finite flat totally ramified $\widehat{O}_\gamma$-algebra $\widehat{O}_{\gamma,2}$ of degree $p-2$ and has a factor which is a an integral étale $\widehat{O}_\gamma$-algebra of degree at least $po-2$ (recall that $|e^{-1}(\alpha,\beta)|=po+2$ if $(\alpha,\beta)\in\ell_\gamma(k)\setminus\{(0,0)\}$), we conclude that $A_e\times_{k[x_1,x_2]}\widehat{O}_\gamma$

is a product $\widehat{O}_{\gamma,1} \times \widehat{O}_{\gamma,2}$ of the type mentioned. Hence $\rho(e) = o$ and $\rho_{\text{ét}}(e) = 0$. As the finite morphism $Y_\gamma \to \ell_\gamma$ is birational, it is an isomorphism.

As $\rho(e) - \rho_{\text{ét}}(e) = o$, from Corollary 11.8(6) it follows that the singular locus of X_e has at most o points. If X_e has a non-regular k-valued point P that maps to a point in the set $\{(\alpha,\beta) \in (k^*)^2 | \alpha^o + \beta^o = 0\}$ on which G acts freely, then the G-orbit of P has $o(p-1)$ points and thus X_e has at least $o(p-1)$ non-regular points, a contradiction. From Criterion 10.7(3) we get that X_e is actually regular.

Example 22.7. We have the following variant of Example 22.6 with smaller geometric degrees. If $p = 2o+1$ is odd, then the pair

$$\bigl(x + y^o(x^{o+1} + y^{o+1}), y - x^o(x^{o+1} + y^{o+1})\bigr)$$

defines an endomorphism $e \in \mathrm{EE}_2(k)$ with $\pi(e) \le p$. Using the same substitution $z := \frac{y}{x}$ we have the analogous expressions $x = \frac{x_2 z^o + x_1}{z^{o+1}+1}$ and $y = \frac{x_2 z^{o+1} + x_1 z}{z^{o+1}+1}$ and polynomial $h(z) = h(x_1,x_2,z) := x_2^p z^{po} + (-x_1 z + x_2)(z^{o+1}+1)^{p-2} + x_1^p \in k_{\mathrm{t}}[x_1,x_2][z]$ whose degree in z is po, the coefficient of z^{po} being $-x_1 + x_2^p$; thus $\deg(e) = po$.

As we have $po + o = 2o(o+1) = (p-1)(o+1) = \frac{p^2-1}{2}$ it follows that the cyclic group $G := \{\alpha \in k | \alpha^{po+o} = 1\}$ acts on the morphism $X_e \to \mathbb{A}^2_{k,\mathrm{t}}$ defined by ψ_e via the following rule at the level of k-algebras: $\alpha \cdot (x_1,x_2,z) := (\alpha^{\frac{1}{p}} x_1, \alpha x_2, \alpha^{\frac{p-1}{p}} z)$.

Arguments similar to the ones of Example 22.6 give that e becomes a finite étale morphism after inverting $x_1^{o+1} + x_2^{o+1}$, that $|e^{-1}(0,0)| = 1$, that for $(\alpha,\beta) \in (k^*)^2$ with $\alpha^{o+1} + \beta^{o+1} = 0$ we have $|e^{-1}(\alpha,\beta)| = po - p + 2$, that $\mathcal{F}(e) = \{1, po-p+2\}$, and that for $o \ge 2$ we have $\rho_{\text{ét}}(e) = 0$, $\rho(e) = o+1$, and X_e is regular.

If $o = 1$, so $p = 3$, then, using the substitution $t := x^2 + y^2$ and noting that $[x + y(x^2+y^2)]^2 + [y - x(x^2+y^2)]^2 = t + t^3$, we easily get that $\mathcal{F}(e) = \{1,2\}$ and in fact that for $(\alpha,\beta) \in k^2$ the solution set of the system of 2 polynomial equations

$$x + y(x^2+y^2) - \alpha = y - x(x^2+y^2) - \beta$$

is $\{(0,0)\}$ if $\alpha = \beta = 0$, has 2 elements if $\alpha^2 + \beta^2 = 0 \ne \alpha\beta$, and for $\alpha^2 + \beta^2 \ne 0$ is

$$\Bigl\{\Bigl(\frac{\alpha - \beta t}{t^2+1}, \frac{\beta + \alpha t}{t^2+1}\Bigr)\Big| t \in k,\ t^3 + t = \alpha^2 + \beta^2\Bigr\}.$$

The morphism $X_e \to \mathbb{A}^2_{k,\mathrm{t}}$ is the projection on the first two coordinates

$$\mathrm{Spec}\,\bigl(k[x_1,x_2,t]/(t^3 + t - x_1^2 - x_2^2)\bigr) \to \mathrm{Spec}\,(k[x_1,x_2])$$

and thus it is a finite Galois cover isomorphic to $c_{t^2+1,t,k}$ of Section 2. For each $i \in k$ with $i^2 = -1$ there exists a unique irreducible component (affine line) of $X_e \setminus \mathbb{A}^2_{k,\mathrm{s}}$ that maps isomorphically onto the line $x_1 + ix_2 = 0$ of $\mathbb{A}^2_{k,\mathrm{t}}$; so $\rho_{\text{ét}}(e) = \rho(e) = 2$.

Example 22.8. For $\star \in \{\mathrm{s},\mathrm{t}\}$ let $I_\star$ be the ideal (x_1,x_2) of $k_\star[x_1,x_2]$. Let $e \in \mathrm{EE}_2(k)$ be defined by a pair $(x_1 + g_1, x_2 + g_2) \in (k_{\mathrm{s}}[x_1,x_2])^2$ with $(g_1,g_2) \in I_{\mathrm{s}}^2$. We assume that there exist $\alpha \in k$ and $h \in k[t]t^2$ such that $t + g_1\bigl(t,h(t)\bigr)$ and $h(t) + g_2\bigl(t,h(t)\bigr)$ have α as a common root with the same multiplicity $m \in \mathbb{N}^*$; thus $\alpha \ne 0$. If C is the irreducible curve of $\mathbb{A}^2_{k,\mathrm{s}}$ which is the zero locus $x_2 - h(x_1) = 0$, then the Zariski closure $\overline{e(C)}$ of $e(C)$ in $\mathbb{A}^2_{k,\mathrm{t}}$ is the zero locus of an irreducible polynomial $f(x_1,x_2) \in k_{\mathrm{t}}[x_1,x_2]$ such that $f\bigl(t+g_1(t,h(t)), h(t)+g_2(t,h(t))\bigr) = 0$ for all $t \in k$. Let $(\beta_1,\beta_2) \in k^2$ be such that $f(x_1,x_2) - \beta_1 x_1 - \beta_2 x_2 \in I_{\mathrm{t}}^2$. We have $\beta_1 = 0$ as β_1 is the coefficient of t in $f\bigl(t+g_1(t,h(t)), h(t)+g_2(t,h(t))\bigr) = 0$. We have $\beta_2 = 0$ as there exists $\gamma \in k^*$ such that $f\bigl(t + g_1(t,h(t)), h(t) + g_2(t,h(t))\bigr) - \gamma\beta_2(t-\alpha)^m \in$

$k[t](t-\alpha)^{m+1}$. Thus $(0,0) \in \mathrm{Sing}(\overline{e(C)})$. If $d \in \mathrm{EE}_2(k)$ is such that $C \in \mathrm{Irr}(N_d)$ with $N_d \subset \mathbb{A}^2_{k,\mathrm{t}} = \mathbb{A}^2_{k,\mathrm{s}}$, then $\overline{e(C)} \in \mathrm{Irr}(N_{ed})$ is a singular irreducible component.

As a concrete example, if $p = 2o+1$ is odd, $(g_1, g_2) = (x_1^{o+1}x_2^o - 2x_1^{p(o+1)}, -x_1^o x_2^{o+1})$, $h(t) = t^{p+1}$ and $\alpha \in \{\delta \in k | \delta^{o(p+2)} = 1\}$, then $t + g_1\big(t, h(t)\big) = t\big(1 - t^{o(p+2)}\big)$ and $h(t) + g_2\big(t, h(t)\big) = t^{p+1}\big(1 - t^{o(p+2)}\big)$ and hence $m = 1$ is the same multiplicity.

Proposition 22.9. *Let $m, q \in \mathbb{N}^*$. Let $\mathbb{U} \subset \{1, p, 2p, 3p, \ldots, (m-1)p\}$ be a subset with $0 < |\mathbb{U}| \leq p^q$. Then there exists $e \in \mathrm{BE}_2(k)$ with $\deg(e) = pm$ and $\mathcal{F}(e) = \mathbb{U}$ (so e is surjective) which is defined over $\mathbb{F}_{p^q}$.*

Proof. We write $\mathbb{U} = \{r_1, \ldots, r_j\}$ with $r_1 < \cdots < r_j$; so $j \in [[1, m]]$. As $p^q \geq j$, we consider j distinct elements $\alpha_1, \ldots, \alpha_j$ of $\mathbb{F}_{p^q}$. Let $f_1(t), \ldots, f_m(t) \in \mathbb{F}_{p^q}[t] \setminus \{0\}$ be subject to the following property.

(♯) If $\mathbb{I}_i := \{\alpha \in k | f_i(\alpha) = 0\}$ for $i \in [[1, m]]$, then we have a chain of inclusions

$$\mathbb{I}_1 \subset \mathbb{I}_2 \subset \cdots \subset \mathbb{I}_m = \{\alpha_i | i \in [[1, j]]\}$$

defined recursively as follows. We define $\mathbb{I}_1 := \emptyset$ if $1 \notin \mathbb{U}$ and $\mathbb{I}_1 := \{\alpha_1\}$ if $1 \in \mathbb{U}$. For $s \in [[1, m-1]]$, $\mathbb{I}_{s+1} \setminus \mathbb{I}_s$ is empty if $ps \notin \mathbb{U}$ and is $\{\alpha_o\}$ if $ps = r_o \in \mathbb{U}$.

The elementary p-morphism $e \in \text{p-Mor}_2(k)$ defined by the pair

$$\big(x, y + \sum_{i=1}^m f_i(x) y^{pi}\big)$$

is defined over $\mathbb{F}_{p^q}$. Property (♯) implies that for each $(\alpha, \beta) \in k^2$ the fiber $e^{-1}(\alpha, \beta)$ has pm points if $f_m(\alpha) \neq 0$ and has r_o points if $\alpha = \alpha_o$, r_o being the degree of the polynomial $y + \sum_{i=1}^m f_i(\alpha_l) y^{pi} \in \mathbb{F}_{p^q}[y]$. Thus we have $\mathcal{F}(e) = \mathbb{U}$. Note that if we chose $f_1(t), \ldots, f_m(t) \in \mathbb{F}_{p^q}[t^p] \setminus \{0\}$, then $e \in \mathrm{BE}_2(k)$. □

Example 22.10. Generalizing the endomorphisms in $\mathrm{EE}_2(k)$ that are defined by the rules $(x, y) \mapsto \big(x + g(x)^p, y + y^p f(x)^p\big)$ (see Example 12.1) or $\big(x, y + \sum_{i=1}^m f_i(x) y^{pi}\big)$ (see Proposition 22.9) or that are as in Example 21.7 for $(n, q_1, f_1) = (2, 1, 1)$, let $e \in \mathrm{EE}_2(k)$ be defined by a pair $(g_1, g_2) \in k[x_1] \times k[x_1, x_2]$. So $e = d_1 d_2$, with $d_2 := \mathrm{e}\big(x_1, g_2(x_1, x_2)\big) \in \mathrm{EE}_2(k)$ surjective and $d_1 := \mathrm{e}\big(g_1(x_1), x_2\big) \in \mathrm{EE}_2(k)$ finite, and there exist pairs $(\alpha_1, \alpha_2) \in (k^*)^2$ and $(g, h) \in k[x_1] \times k[x_1, x_2]$ such that $(g_1, g_2) = \big(\alpha_1 x_1 + g(x_1^p), -\alpha_2 x_2 - h(x_1, x_2^p)\big)$. Both d_1 and d_2 have strong type ≤ 1 in the sense of Definition 22.2 and $(e, d_1, d_2) \in \text{p-Mor}_2(k)^3$. As $\deg(d_1) = \max\{1, p\deg(g)\}$ and $\deg(d_2) = \max\{1, pm\}$, where m is the degree of h in x_2, Conjecture 2.10 holds for e. As A_e is the normalization of $k[x_1, x_2]$ in $\mathrm{Frac}(A_e) = \mathrm{Frac}\big(k[x_1, x_2][y]/(\alpha_2 y + h(x_1, y^p) + x_2)\big)$, to study it, using the substitution $x_2' := x_2 + h(x_1, 0)$, we can assume that $h(x_1, 0) = 0$; so $m \neq 0$. If $m = -\infty$, then d_2 is an isomorphism; so we can assume $m \geq 1$. We write

$$h(x_1, x_2) = \sum_{i=1}^m f_i(x_1) x_2^i$$

with $(f_1, \ldots, f_m) \in k[x_1]^m$, $f_m \neq 0$. Substituting $z := f_m(x_1) y$, A_e is the normalization of the $k[x_1, x_2]$-algebra $B_e := k[x_1, x_2][z]/\big(h(z)\big)$, where

$$h(z) := z^{pm} + \sum_{i=1}^{m-1} f_i(x_1) f_m(x_1)^{pm-1-pi} z^{pi} + \alpha_2 f_m(x_1)^{pm-2} z + f_m(x_1)^{pm-1} x_2.$$

Multiplying $h(z)$ by $f_m(x_1)x_2^{pm-1}$ and substituting $w := f_m(x_1)x_2z^{-1}$, A_e is the normalization of the $k[x_1,x_2]$-algebra

$$B'_e := k[x_1,x_2][w]/\big(w^{pm}+\alpha_2w^{pm-1}+\sum_{i=1}^{m} f_i(x_1)x_2^{pi-1}w^{p(m-i)}\big).$$

If m is a power of p and $h(x_1,x_2)$ is additive in x_2 (i.e., and $f_i(x_1)=0$ if $i\in[[1,m]]$ is not a power of p), then $\mathbb{G}_{\mathrm{a},k}$ is a subgroup scheme of the group ind-scheme of automorphisms of the morphism $X_e\to\mathbb{A}^2_{k,\mathrm{t}}$ defined by ψ_e: at the level of k-algebras, they are parametrized by $\beta\in k$ via the rule

$$\beta\cdot(x_1,x_2,y)=\big(x_1,x_2-\alpha_2\beta-\sum_{i=1}^{m} f_i(x_1)\beta^{pi},y+\beta\big),$$

where we identify y with $y+(\alpha_2y+h(x_1,y^p)+x_2)\in\mathrm{Frac}(A_e)$.

Assume that $m=1$. For $p=2$ we have $A_e\cong B_e\cong B'_e$ and e and d_2 are étale with $0<\rho_{\text{ét}}(e)=\rho(e)=\rho_{\text{ét}}(d_2)=\rho(d_2)=\mathrm{z}(f_m)$ while for $p>2$ we have $\rho(e)\geq\mathrm{z}(f_m)$ and A_e is regular by Lemma 5.1(4).

If $m=2$, then A_e is regular by Lemma 5.2.

If $m\geq3$, then A_e is regular in many cases including the following ones: (i) f_m is a separable polynomial by Lemma 5.3; (ii) m is a power of p and $h(x_1,x_2)$ is additive in x_2 by Lemma 5.2 or as a direct consequence of the existence of the $\mathbb{G}_{\mathrm{a},k}$ group scheme of automorphisms; (iii) Criterion 10.7(3) or Lemma 5.1 applies.

As e is finite étale above the complement of the union of lines of $\mathbb{A}^2_{k,\mathrm{t}}$ given by equation $x_1=\gamma$ with $\gamma\in k$ such that $f_m(\gamma)=0$, $\nu(e)=\mathrm{z}(f_m)$.

Example 22.11. We refer to Example 22.10 with $p=2$, $m=2$, and $h(x_1,x_2)=x_1x_2^2$; so $f_1(x_1)=0$, $f_2(x_1)=x_1$, $q=1$, and $h(z)=z^4+x_1^2z+x_1^3x_2$. Denoting also by z the element $z+(h(z))\in k_{\mathrm{t}}[x_1,x_2][z]/\big(h(z)\big)\subset A_e$ and introducing new indeterminates and elements $w:=\frac{z^2}{x_1}$, $v:=\frac{w^3}{x_1}+1$, and $t:=v+x_2w$, we get that A_e is also the normalization of either one of the three $k_{\mathrm{t}}[x_1,x_2]$-algebras $k_{\mathrm{t}}[x_1,x_2][w]/(w^4+wx_1+x_1^2x_2^2)$, $k_{\mathrm{t}}[x_1,x_2][v]/(v^4+v^3+x_1^2x_2^6)$, and $k_{\mathrm{t}}[x_1,x_2][t]/(t^4+t^3+x_1x_2^3)$, but with only one generator we cannot get better than this (cf. Corollary 25.6 below). We have $z,w,v,t\in A_e$ satisfying $z=w^2+x_1x_2$ and $v=t^2$. As $w^3+x_1x_2w+x_1t+x_1=t^2+t+x_2w=wt+w^2x_2+x_1x_2^2=0$, we have a natural $k_{\mathrm{t}}[x_1,x_2]$-algebra homomorphism

$$\tilde{\imath}_e: k_{\mathrm{t}}[x_1,x_2][w,t]/(w^3+x_1x_2w+x_1t+x_1,t^2+t+x_2w,wt+w^2x_2+x_1x_2^2)\to A_e.$$

The domain B_e of $\tilde{\imath}_e$ is a $k_{\mathrm{t}}[x_1,x_2]$-module generated by $\{1,w,w^2,t\}$ and by tensoring with $k_{\mathrm{t}}(x_1,x_2)$ it follows that it is free of rank 4. By computing the ranks of the 3×4 Jacobian matrices at points of $\operatorname{Spec}B_e$ we get that it is a smooth k-algebra. It follows that $\tilde{\imath}_e$ is an isomorphism and we reobtain the fact that A_e is regular. Either from this last isomorphism or from the proof of Lemma 5.2 applied with $p=m=2$ we get that $\rho_{\text{ét}}(e)=0$, $\rho(e)=1$. Hence $Y:=X_e\setminus X_e^{\text{ét}}=X_e\setminus\mathrm{Im}(\imath_e)$ is irreducible and its multiplicity in $\psi_e^{-1}\big(\psi_e(Y)\big)$ is 3 and (see Corollary 11.8(1) and (2) and its notation) $A_Y=A_e$ does not have a 2-basis.

Example 22.12. Let $\big((f_1,g_1),(f_2,g_2)\big)\in\Theta^2_K$. We have a complete intersection

$$\mathbb{S}^2_{f_1,g_1;f_2,g_2,K}:=\operatorname{Spec}\big(K[w,x,y,z]/(x_1f_1(x)+yg_1(y)z,wf_2(w)+xg_2(y)\big)$$

in $\mathbb{A}^4_K$. The projection $\mathbb{A}^4_K \to \mathbb{A}^3_K$ on the last three coordinates induces a finite flat morphism $\phi : \mathbb{S}^2_{f_1,g_1;f_2,g_2,K} \to \mathbb{S}^2_{f_1,g_1,K}$. Let $g := g_1g_2 \in K[t]$. The cartesian diagram

$$\begin{tikzcd}
\mathbb{S}^2_{f_2,g,K} \arrow[r, "\jmath_{f_1,g_1;f_2,g_2,K}"] \arrow[d, "\pi"] & \mathbb{S}^2_{f_1,g_1;f_2,g_2,K} \arrow[d, "\phi"] \\
\mathbb{A}^2_K \arrow[r, "\imath_{f_1,g_1,K}"] & \mathbb{S}^2_{f_1,g_1,K}
\end{tikzcd}$$

is defined by the identification $\mathbb{S}^2_{f_2,g,K} = \mathrm{Spec}\,\big(K[w,x,y]/(wf_2(w)+xyg(y))\big)$ with π induced by the projection on the last two variables and gives an open embedding

$$\imath_{f_1,g_1;f_2,g_2,K} := \jmath_{f_1,g_1;f_2,g_2,K} \circ \imath_{f_2,g} : \mathbb{A}^2_K \to \mathbb{S}^2_{f_1,g_1;f_2,g_2,K}$$

defined by the rule $(x,y) \mapsto \big(xyg(y), Fyg_1(y), y, -Ff_1(Fyg_1(y)\big)$ for the polynomial $F(x,y) := -xf_2\big(xyg(y)\big) \in K[x,y]$. The flatness of ϕ and the integrality of $\mathbb{S}^2_{f_2,g,K}$ imply that $\mathbb{S}^2_{f_1,g_1;f_2,g_2,K}$ is integral.

Note that if for $i \in \{1,2\}$ we have $\deg(f_i) \geq 1$ and there exists $m_i \in \mathbb{N}^*$ such that $g_i(t) = t^{m_i}$, then the resulting surfaces $\mathbb{S}^2_{f_1,g_1;f_2,g_2,K}$ form a subclass of the class of double Danielewski surfaces introduced and studied in [GS], Sect. 3 (they are integral by [GS], Lem. 3.3 and [GS], Thm. 3.11 lists their isomorphism classes).

Assume $\mathrm{char}(K) = p$. Let $m \in \mathbb{N}^*$. We assume that $(f_2, g_2) \in \Theta^1_{k,m}$. Thus ϕ is a finite étale cover of degree pm. Let $\Phi := \Phi_{f_1,g_1,h_1,q_1} : \mathbb{S}^2_{f_1,g_1,K} \to \mathbb{A}^2_K$ be a finite étale cover as in Proposition 20.2(1). The composite $e := \Phi \circ \phi \circ \imath_{f_1,g_1;f_2,g_2,K} \in \mathrm{EE}_2(K)$ is such that $X_e = \mathbb{S}^2_{f_1,g_1;f_2,g_2,K}$ and p^2 divides $\deg(e) = pm\deg(\Phi)$; we have $\mathrm{edim}(X_e) \leq 4$ and the equality holds in general.

Example 22.13. Let $s \in \mathbb{N}^* \setminus \{1\}$. Let $\big((f_1,g_1),\ldots,(f_s,g_s)\big) \in \Theta^s_K$. Let

$$\mathbb{S}^{s+1}_{(f_1,g_1;\ldots;f_s,g_s),K} := \mathrm{Spec}\,\big(K[x_1,\ldots,x_s,y,z_1,\ldots,z_s]/(x_if_i(x_i)+yg_i(y)z_i|i \in [[1,s]])\big);$$

it's a complete intersection in $\mathbb{A}^{2s+1}_K$ of dimension $s+1$ equipped with a finite flat morphism $\mathbb{S}^{s+1}_{(f_1,g_1;\ldots;f_s,g_s),K} \to \mathbb{A}^{s+1}$ defined by the projection $\mathbb{A}^{2s+1} \to \mathbb{A}^{s+1}$ on the last $s+1$ coordinates. One can use the open embeddings $\imath_{f_i,g_i,K}$ and, when $\deg(f_i) \geq 1$ and $g_i(0) \neq 0$, $\jmath_{f_i,g_i,K;\delta_i}$ to get open embeddings $\mathbb{A}^{s+1}_K \to \mathbb{S}^{s+1}_{(f_1,g_1;\ldots;f_s,g_s),K}$. E.g., using only $\imath_{f_i,g_i,K}$, we have an open embedding

$$\imath_{(f_1,g_1;\ldots;f_s,g_s),K} : \mathbb{A}^{s+1}_K \to \mathbb{S}^{s+1}_{(f_1,g_1;\ldots;f_s,g_s),K}$$

defined by the rule

$$(x_1,\ldots,x_s,y) \mapsto \big(x_1yg_1(y),\ldots,x_syg_s(y),y,-x_1f_1(x_1yg_1(y)),\ldots,-x_sf_s(x_1yg_s(y))\big).$$

If $\mathrm{char}(K) = p$ and $n_i := \deg(f_i) - 1 \geq 1$ for each $i \in [[1,s]]$, then the proof of Proposition 20.2(1) applies to show that $\mathbb{S}^{s+1}_{(f_1,g_1;\ldots;f_s,g_s),K}$ is a finite étale cover of $\mathbb{A}^{s+1}_n$ of degree $p^s\prod_{i=1}^s q_in_i$, where for each $i \in [[1,s]]$ the integer $q_i \in \mathbb{N}^*$ is associated to (f_i,g_i) as in Proposition 20.2.

Proposition 22.14. *For each $l \in [[1,\lfloor\frac{pm-1}{2}\rfloor]]$ there exists $e \in \mathrm{EE}_2(k)$ with $\deg(e) = pm$, $\varphi(e) = 0$, and $\mathcal{F}(e) = \{1, l{+}1, pm{-}l\}$. Moreover, we can chose e to be defined over each finite field $\mathbb{F}_{p^q}$ that contains l zeros of the equation $z^{pm-1} - 1 = 0$ (e.g., we can take $q \in \mathbb{N}^*$ such that $pm-1$ divides p^q-1), or over each $\mathbb{F}_{p^q}$ with $q \in \mathbb{N}^*$ such that $p^q \geq l+1$ if $l \in [[1,m-1]]$, or over $\mathbb{F}_p$ if l divides $pm-1$.*

Proof. We apply Example 21.5 for $n = 2$ as follows. If $g \in k[t]$ is such that $\deg(g) = l$, then $\deg(h) = pm-l-1$, $\deg(e_{g,h}) = pm$ and $\mathcal{F}(e_{g,h}) = \{1, l+1, pm-l\}$. If $tg(t)h(t) = t - t^{pm}$, then we can choose g and h and hence also $e_{g,h}$ to be defined over each finite field $\mathbb{F}_{p^q}$ which contains l zeros of the equation $z^{pm-1} - 1 = 0$ (e.g., we can take $q \in \mathbb{N}^*$ such that $pm - 1$ divides $p^q - 1$).

If l divides $pm - 1$, then we can choose $g(t) = 1 - t^l$ and $h(t) = \frac{1-t^{pm-1}}{1-t^l}$ in $\mathbb{F}_p[t]$.

If $l \in [[1, m-1]]$, let $\beta_1, \ldots, \beta_l \in k^*$ be distinct and we take

$$g(t) := \prod_{i=1}^{l} (1 + \beta_i t) \in k[t];$$

clearly $\deg(g) = l$. As the determinant of the generalized Vandermonde $l \times l$ matrix whose columns indexed by $s \in [[1, l]]$ are $(\beta_s^{-p}, \beta_s^{-2p}, \ldots, \beta_s^{-pl})$ is in k^* (being $\prod_{i=1}^{l} \beta_i^{-p}$ times a non-zero Vandermonde determinant), there exists a unique l-tuple $(\alpha_1, \ldots, \alpha_l) \in k^l$ such that $tg(t)$ divides $F(t) = t + (\sum_{i=1}^{l} \alpha_i t^{pi}) + \sum_{i=l+1}^{m} t^{pi}$. If $p^q \geq l + 1$, then by taking $\beta_1, \ldots, \beta_l \in \mathbb{F}_{p^q}^*$ we get that $g, h = \frac{F(t)}{tg(t)} \in \mathbb{F}_{p^q}[t]$. $\square$

Proposition 22.15. *Let $\mathcal{S}$ be an F_2-system over k such that $\pi(\mathcal{S}) \leq 2p$. Then $e_{\mathcal{S}}$ is surjective, i.e., $\varphi(e_{\mathcal{S}}) = 0$.*

Proof. We can assume that $e := e_{\mathcal{S}} \in \mathrm{EE}_2(k)$ is defined by a pair

$$(\ell_1 + \ell_3^p + g_1^p, \ell_2 + \ell_4^p + g_2^p) \in k[x_1, x_2]^2,$$

where ℓ_1 to ℓ_4 are linear forms and g_1 and g_2 are quadratic forms. As e is étale, ℓ_1 and ℓ_2 are linearly independent over k. If the system of homogeneous equations $g_1 = 0 = g_2$ has no solution in $\mathbb{P}^1_k$, then e is finite surjective and $\deg(e) = 4p^2$ by Fact 15.1(2). Hence we can assume that there exist linear forms ℓ_0, ℓ_5 and ℓ_6 such that $\ell_0 \neq 0$, $g_1 = \ell_0\ell_5$ and $g_2 = \ell_0\ell_6$. Let LCC_{s} (resp. LCC_{t}) denote a linear change of coordinates in the source (target) space. We consider two disjoint cases as follows, and we also compute $\deg(e)$ in all subcases except one.

Case 1: ℓ_5 and ℓ_6 are linearly dependent. Performing LLC_{t} we can arrange that $\ell_5 = 0$; so $g_1 = 0$. If $\ell_3 = 0$, then by performing LLC_{s} we can assume that $\ell_1 = x_1$ and $\ell_2 = x_2$; so for each $\alpha \in k$, the equation $x_2 + \ell_4^p(\alpha, x_2)^p + g_2(\alpha, x_2)^p = \beta$ in x_2 has solutions in k and $\deg(e)$ is the degree in x_2 of $x_2 + \ell_4^p(\alpha, x_2)^p + g_2(\alpha, x_2)^p$; so $\deg(e) \in \{1, p, 2p\}$. Thus we can assume that $\ell_3 \neq 0$. If $\ell_6 \neq 0$ and $\ell_3 \notin k^*\ell_0 \cup k^*\ell_6$, then e is finite surjective and $\deg(e) = 2p^2$ by Fact 15.1(2). Thus we can assume that $\ell_3 = \ell_0$. By performing LLC_{s} we can also assume that $\ell_1 = x_1$ and $\ell_3 \in \{x_1, -x_2\}$. If $\ell_3 = x_1$, then the system of equations $x_1 + x_1^p - \alpha = 0 = \ell_2 + \ell_4^p + x_1^p\ell_6^p - \beta = 0$ has solutions in k^2, so e is surjective, and $\deg(e)$ is p times the degree in x_2 of $\ell_2 + \ell_4^p + x_1^p\ell_6^p$; so $\deg(e) \in \{p, p^2, p^3\}$. If $\ell_3 == x_2$, then the system of equations $x_1 - x_2^p - \alpha = 0 = \ell_2 + \ell_4^p - x_2^p\ell_6^p - \beta = 0$ over k is equivalent to the equation $F_2(x_2) = \beta$ with $F_2 := \ell_3(x_2^p + \alpha, x_2) + \ell_4(x_2^p + \alpha, x_2)^p - x_2^p\ell_6(x_2^p + \alpha, x_2)^p \in k_{\mathrm{s}}[x_2]$, hence e is finite with $\deg(e) = \deg(F_2) \in \{1, p, 2p, p^2, p^2 + p\}$.

Case 2: ℓ_5 and ℓ_6 are linearly independent. Let $\pi_1 : \mathbb{A}^2_{k,\mathrm{t}} \to \mathbb{A}^1_{k,\mathrm{t}}$ be the first projection. We have $\varphi(e_{\mathcal{S}}) = 0$ iff for each $\alpha \in \mathbb{A}^1_{k,\mathrm{t}}(k)$, the étale morphism

$$e_\alpha : (\pi_1 e)^{-1}(\alpha) \to \pi_1^{-1}(\alpha) = \{\alpha\} \times_{\mathrm{Spec}\, k} \mathbb{A}^1_{k,\mathrm{t}} \cong \mathbb{A}^1_k$$

is surjective. By performing LLC_{s} we can assume that $\ell_0 = x_1$, $g_1 = x_1^2$ and $g_2 = x_1^p x_2^p$. The plane curve $(\pi_1 e)^{-1}(\alpha)$ is the zero locus $\ell_1 + \ell_3^p + x_1^{2p} - \alpha = 0$. Let $(\gamma_1, \gamma_2, \delta_1, \delta_2) \in k^4$ be such that $\ell_1 = \gamma_1 x_1 + \gamma_2 x_2$ and $\ell_3 = \delta_1 x_1 + \delta_2 x_2$.

If $\gamma_2 = \delta_2 = 0$, then $\gamma_1 \neq 0$ and $(\pi_1 e)^{-1}(\alpha)$ is isomorphic to $2p$ copies of $\mathbb{A}^1_k$ indexed by the roots of the equation $x_1^{2p} + \delta_1 x_1^p + \gamma_1 x_1 - \alpha = 0$ and for each such $\mathbb{A}^1_k$ copy the resulting étale morphism $\mathbb{A}^1_k \to \pi_1^{-1}(\alpha)$ is finite surjective by Lemma 10.5(3) of degree p for each $\alpha \in k$ outside a finite set; thus $\deg(e) = 2p^2$.

If $\gamma_2\delta_2 = 0 \neq \gamma_2 + \delta_2$, then $(\pi_1 e)^{-1}(\alpha) \cong \mathbb{A}^1_k$. The resulting étale morphism $(\pi_1 e)^{-1}(\alpha) \to \pi_1^{-1}(\alpha)$ is finite surjective by Lemma 10.5(3) of degree $\deg(e) = 2p^2 + p$ if $\gamma_2 \neq 0$ and of degree $\deg(e) = 3p^2$ if $\delta_2 \neq 0$.

If $\gamma_2\delta_2 \neq 0$, then $(\pi_1 e)^{-1}(\alpha)$ is either isomorphic to p copies of $\mathbb{A}^1_k$ or is a connected finite Galois cover of $\mathbb{A}^1_k$ of Galois group $\mathbb{Z}/p\mathbb{Z}$ and thus, as $\mathbb{P}^1_k$ is simply connected, it has 1 point at infinity; as $(\pi_1 e)^{-1}(\alpha)$ is a disjoint union of smooth affine curves with 1 point at infinity, e_α is surjective by Lemma 10.5(3). $\square$

23. Étale endomorphisms of $\mathbb{A}^2_k$ via line bundles

We have the following moduli interpretation of $\mathbb{S}^2$. With B^+ and B^- as in Section 19, the intersection

$$T := B^+ \cap B^-$$

is the maximal torus of $\mathbf{SL}_2$ formed by diagonal 2×2 matrices of determinant 1. The centralizer of T in $\mathbf{GL}_2$ is the maximal torus GT of $\mathbf{GL}_2$ formed by invertible diagonal 2×2 matrices. The moduli scheme $\mathbb{M}$ over $\operatorname{Spec}\mathbb{Z}$ that parametrizes 2×2 matrices that are projectors of rank 1, equivalently, have trace 1 and determinant 0 and hence can be put in the form $\begin{bmatrix} x & y \\ z & 1-x \end{bmatrix}$ with $x^2 - x + yz = 0$, is naturally identified with $\mathbb{S}^2$. The action by conjugation

$$\mathbf{GL}_2 \times_{\operatorname{Spec}(\mathbb{Z})} \mathbb{M} \to \mathbb{M}$$

is transitive. As the stabilizer (centralizer) of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is GT, we identify

$$\mathbf{SL}_2/T = \mathbf{GL}_2/GT \cong \mathbb{M} = \mathbb{S}^2.$$

If $wz - xy = 1$, then we compute

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix}^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z & -x \\ -y & w \end{bmatrix} = \begin{bmatrix} wz & -wx \\ yz & 1-wz \end{bmatrix}.$$

Thus the composite morphism $\mathbf{SL}_2 \to \mathbf{SL}_2/T \to \mathbb{S}^2$ is given by the following rule on valued points: $\begin{bmatrix} w & x \\ y & z \end{bmatrix} \mapsto (wz, -wx, yz)$.

Let $\mathcal{Q} : \mathbb{S}^2 = \mathbf{SL}_2/T \to \mathbf{SL}_2/B^+ \cong \mathbb{P}^1$ be the natural quotient morphism that defines a line bundle over $\mathbb{P}^1$ and that allows us to identify $\mathbb{D}^1 = \mathcal{Q}^{-1}(1:0)$. So, on valued points, $\mathcal{Q}$ is defined by the rule: (x, y, z) maps to either $(-x : y)$ or $(z : x-1)$ (whichever one is defined; if both are defined, then they are equal).

For $m \in \mathbb{N}^*$ we consider the cartesian diagram

$$\begin{tikzcd} \mathbb{A}^2_k \arrow[r, "\mathrm{nat}"] \arrow[d, "\Sigma_{m,2}"] & \mathbb{A}^1_k \arrow[d, "\Sigma_m^+"] \\ \mathbb{S}^2_k = \mathbf{SL}_{2,k}/T_k \arrow[r, "\mathcal{Q}_k"] & \mathbf{SL}_{2,k}/B^+_k \cong \mathbb{P}^1_k, \end{tikzcd}$$

where Σ_m^+ is as in Example 22.5 and we are using the fact that each line bundle over $\mathbb{A}_k^1$ is trivial; so nat : $\mathbb{A}_k^2 \to \mathbb{A}_k^1$ is the first projection and the diagram is defined over $\mathbb{F}_p$. Thus $\Sigma_{m,2}$ is surjective étale of degree $pm+1$ as so is Σ_m^+. We get that

$$\mathbb{A}_k^1 \cong \mathrm{nat}^{-1}(0) = \mathrm{nat}^{-1}\big((\Sigma_m^+)^{-1}(1:0)\big) = \Sigma_{m,2}^{-1}\big(\mathcal{Q}_k^{-1}(1:0)\big) = \Sigma_{m,2}^{-1}(\mathbb{D}_k^1)$$

maps via $\Sigma_{m,2}$ isomorphically onto $\mathbb{D}_k^1 = \mathcal{Q}_k^{-1}(1:0)$.

Remark 23.1. Using the definition of $\imath_{t-1,1,\mathbb{Z}}$ (see Section 2), the morphism

$$\Sigma_{m,2} : \mathbb{A}_k^2 \to \mathbb{S}_k^2$$

is, in suitable coordinates, given by the following rule on valued points

$$(x,y) \mapsto \big((x^{pm+1}+1)(-xy+1), x - x^2y, (x^{2pm} + 2x^{pm-1})(x^2y - x) + x^{pm} + y\big).$$

We have $\Sigma_{m,2}(\{0\} \times_{\operatorname{Spec} k} \mathbb{A}_k^1) = \mathbb{D}_k^1$.

Proposition 23.2. *Let $m \in \mathbb{N}$ and $(f,g) \in \Theta_k$ with $\deg(f) \geq 1$. Let $(\alpha,\beta) \in k^* \times k$ be such that $f(\alpha) = \beta g(\beta) = 0 \neq g(\beta) + \beta g'(\beta)$. Let $Y = Y_{\alpha,\beta} \in \operatorname{Irr}\big(\mathbb{S}_{f,g,k}^2 \setminus \operatorname{Im}(\imath_{f,g,k})\big)$ be the zero locus $x - \alpha = y - \beta = 0$ (see Section 2). For the open subvariety $\mathbb{S}_{f,g,k,Y}^2 := \operatorname{Im}(\imath_{f,g,k}) \cup Y$ of $\mathbb{S}_{f,g,k}^2$ there exists a non-finite surjective étale morphism $\Sigma_{f,g,m,Y} : \mathbb{A}_k^2 \to \mathbb{S}_{f,g,k,Y}^2$ of degree $pm+1$ with the following properties.*

(1) *It is defined over each subfield of k that contains all roots of fg.*

(2) *It is finite over $\operatorname{Im}(\imath_{f,g,k})$ and $\Sigma_{f,g,m,Y}^{-1}(Y) \cong \mathbb{A}_k^1$.*

(3) *It is the composite of an open embedding $\imath_{f,g,m,Y} : \mathbb{A}_k^2 \to \mathbb{T}_{f,g,m,Y}^2$ and a finite flat morphism $\mathbb{T}_{f,g,m,Y}^2 \to \mathbb{S}_{f,g,k}^2$ of degree $pm+1$, with $\mathbb{T}_{f,g,m,Y}^2$ a smooth surface over k which is a line bundle over $\mathbb{P}_k^1$ and for which $(\mathbb{T}_{f,g,m,Y}^2 \setminus \imath_{f,g,m,Y}) \cong \mathbb{A}_k^1$.*

Proof. The proofs of parts (1) and (2) are as in the case of $\mathbb{S}_k^2$, we only have to define $\mathcal{Q}_{f,g,k}$ that generalizes $\mathcal{Q}_k$. Let $\mathcal{Q}_{f,g,k} : \mathbb{S}_{f,g,k}^2 \to \mathbb{P}_k^1$ be defined by the rule: (x,y,z) maps to either $\big(-x : yg(y)\big)$ or $\big(z : f(x)\big)$ (whichever one is defined; if both are defined, then they are equal). As $g(\beta) + \beta g'(\beta) \neq 0$, Y is a connected component of $\mathcal{Q}_{f,g,k}^{-1}(1:0) = \operatorname{Spec}\big(k[x,y,z,x^{-1}]/(xf(x), yg(y))\big)$. Let $\mathcal{Q}_{f,g,k,Y} := \mathcal{Q}_{f,g,k}|\mathbb{S}_{f,g,k,Y}^2 : \mathbb{S}_{f,g,k,Y}^2 \to \mathbb{P}_k^1$; so $\mathcal{Q}_{f,g,k,Y}^{-1}(0:1) = Y$ and hence $\mathcal{Q}_{f,g,k,Y}$ is a line bundle by Theorem 11.2(2) applied to $q=1$. So parts (1) and (2) hold by taking $\Sigma_{f,g,m,Y}$ to be the pullback of Σ_m^+ via $\mathcal{Q}_{f,g,k,Y}$. Part (3) holds by taking $\mathbb{T}_{f,g,m,Y}^2 \to \mathbb{S}_{f,g,k}^2$ to be the pullback of the finite flat morphism $\Sigma_m^{\mathrm{proj}} : \mathbb{P}_k^1 \to \mathbb{P}_k^1$ of Example 22.5 via $\mathcal{Q}_{f,g,k,Y}$. $\square$

Remark 23.3. Let $\mathbb{T}_{f,g,m}^2 \to \mathbb{S}_{f,g,k}^2$ be the pullback of Σ_m^{proj} via the flat morphism $\mathcal{Q}_{f,g,k} : \mathbb{S}_{f,g,k}^2 \to \mathbb{P}_k^1$. Then $\mathbb{T}_{f,g,m,Y}^2$ is an open subvariety of $\mathbb{T}_{f,g,m}^2$ and the resulting morphism $\mathbb{T}_{f,g,m}^2 \setminus \mathbb{T}_{f,g,m,Y}^2 \to \mathbb{S}_{f,g,k}^2 \setminus \mathbb{S}_{f,g,k,Y}^2$ is an isomorphism between reduced affine varieties isomorphic to $\deg(f)\mathrm{z}(tg) - 1$ copies of $\mathbb{A}_k^1$.

Remark 23.4. Let $n := \deg(f) + 1 \geq 2$. The composite of $\Sigma_{f,g,m,Y}$ with the inclusion $\mathbb{S}_{f,g,k,Y}^2 \to \mathbb{S}_{f,g,k}^2$ and with a finite étale morphism $\Phi_{f,g,h,q} : \mathbb{S}_{f,g,k}^2 \to \mathbb{A}_k^2$ as in Proposition 20.2(1), is a surjective non-finite endomorphism $e = e_{f,g,m,h,q} \in \mathrm{EE}_2(k)$ of degree $pnq(pm+1)$ such that $X_e = \mathbb{T}_{f,g,m}^{2,\mathrm{n}}$ is the normalization of $\mathbb{T}_{f,g,m}^2$ and ψ_e factors through the composite of $\mathbb{T}_{f,g,m}^{2,\mathrm{n}} \to \mathbb{S}_{f,g,k}^2$ with $\Phi_{f,g,h,q}$; so $\mathbb{T}_{f,g,m}^2$ is a flat model of X_e. If $tg(t)$ is separable, then $\mathcal{Q}_{f,g,k}$ is smooth and thus $X_e = \mathbb{T}_{f,g,m}^2$ is smooth over $\mathbb{P}_k^1$ and hence regular. As $\mathbb{T}_{f,g,m}^2 \setminus \operatorname{Im}(\imath_e) = (\mathbb{T}_{f,g,m}^2 \setminus \mathbb{T}_{f,g,m,Y}) \cup$

$\big(\mathbb{T}^2_{f,g,m,Y}\setminus\mathrm{Im}(\imath_e)\big)$, we have $\rho(e)\geq \deg(f)\mathrm{z}(tg)$ by Remark 11.7(1) and the equality holds if $tg(t)$ is separable. As the étale locus of Σ_m^{proj} is $\mathbb{A}^1_k$, if $tg(t)$ is separable we get that $X_e^{\text{ét}}=\mathbb{A}^2_{k,s}$ and thus $\rho_{\text{ét}}(e)=0$.

Remark 23.5. Assume $\mathrm{char}(K)=0$. For $l\in\mathbb{N}$, we do not know if there exist étale morphisms $\mathbb{A}^2_K\to\mathbb{S}^2_{t-1,t^l,K}$ with finite complements (cf. [Mi4], Lem. 1.9(2) which proves that generically Galois étale morphisms $\mathbb{A}^2_K\to\mathbb{S}^2_{t-1,t^l,K}$ with finite complements do not exist).

24. Affine moduli schemes

Let $q\in\mathbb{N}^*$. Recall the affine scheme $End_{n,q}=\mathrm{Spec}\,(A_{n,q})$ over $\mathrm{Spec}\,\mathbb{Z}$ and its subsheaves $EE_{n,q}$ and $GA_{n,q}$ of sets defined by the rules on R-valued points:

$$End_{n,q}(R):=\{(f_1,\ldots,f_n)\in R[x_1,\ldots,x_n]^n|\pi(f_1,\ldots,f_n)\leq q\},$$

$$EE_{n,q}(R):=\{(f_1,\ldots,f_n)\in End_{n,q}(R)|\mathrm{e}(f_1,\ldots,f_n)\text{ is étale}\},$$

$$GA_{n,q}(R):=\{(f_1,\ldots,f_n)\in End_{n,q}(R)|\mathrm{e}(f_1,\ldots,f_n)\text{ is an automorphism}\}.$$

For $(l_1,\ldots,l_n)\in\mathbb{N}^n$ let $\deg(l_1,\ldots,l_n):=\sum_{i=1}^n l_i$. We endow the set

$$\mathbb{L}_{n,q}:=\{(l_1,\ldots,l_n)\in\mathbb{N}^n|\deg(l_1,\ldots,l_n)\leq q\}$$

with the lexicographic order: its smallest and greatest elements are the n-tuples $(0,\ldots,0)$ and $(q,0,\ldots,0)$ (respectively). Let $N(n,q):=|\mathbb{L}_{n,q}|\in\mathbb{N}^*$.

For $(f_1,\ldots,f_n)\in End_{n,q}(R)$ and $i\in[[1,n]]$ we write

$$f_i(x_1,\ldots,x_n)=\sum_{\lambda=(l_1,\ldots,l_n)\in\mathbb{L}_{n,q}}\alpha_\lambda^{(i)}\prod_{j=1}^n x_j^{l_j},$$

where each $\alpha_\lambda^{(i)}\in R$, and the rule

$$(f_1,\ldots,f_n)\mapsto\big(\alpha^{(1)}_{(0,\ldots,0)},\ldots,\alpha^{(1)}_{(q,0,\ldots,0)},\ldots,\alpha^{(n)}_{(0,\ldots,0)},\ldots,\alpha^{(n)}_{(q,0,\ldots,0)}\big)$$

allows us, by viewing $\alpha_\lambda^{(i)}$ as indeterminates, to identify

$$A_{n,q}=\mathbb{Z}[\alpha^{(1)}_{(0,\ldots,0)},\ldots,\alpha^{(n)}_{(q,\ldots,0)}]\ \text{ and }\ End_{n,q}=\mathbb{A}^{nN(n,q)}.$$

For $(f_1,\ldots,f_n)\in End_{n,q}(R)$ we have $(f_1,\ldots,f_n)\in\mathrm{EE}_{n,q}(R)$ iff its Jacobian determinant belongs to $R[x_1,\ldots,x_n]^*$.

There exists an $\mathbb{N}$-bigradation on each $A_{n,q}\otimes_{\mathbb{Z}}R$ that comes from the natural $\mathbb{G}_m$-actions on $\mathbb{Z}_{\mathrm{t}}[x_1,\ldots,x_n]$ and $\mathbb{Z}_{\mathrm{s}}[x_1,\ldots,x_n]$ defined by the rule $(t,x_i)\mapsto tx_i$ for each $i\in[[1,n]]$. So for $\lambda\in\mathbb{L}_{n,q}$, each indeterminate $\alpha_\lambda^{(i)}$ has bidegree $\big(\deg(\lambda),1\big)$.

We have truncation closed embeddings

$$\Xi_{n,q}:End_{n,q}\to End_{n,q+1}$$

that forget all coefficients of degree $q+1$, i.e., are defined by ring homomorphisms

$$\Xi_{n,q}^{\#}:A_{n,q+1}\to A_{n,q}$$

that are surjective and map each $\alpha_\lambda^{(i)}$ to itself if $\deg(\lambda)\leq q$ and to 0 if $\deg(\lambda)=q+1$. This allows us to define the projective limit A_n of the projective system $A_{n,q+1}\to A_{n,q}$ indexed by $q\in\mathbb{N}^*$ (in each bidegree the inverse limit stabilizes).

Let $SEE_{n,q}$ be the closed subscheme of $End_{n,q}$ defined by the following rule on R-valued points:

$$SEE_{n,q}(R) := \{(f_1, \dots, f_n) \in End_{n,q}(R) | \det\bigl(J(f_1, \dots, f_n)\bigr) = 1\}.$$

Proposition 24.1. *Let $l \in \mathbb{N}$. Let $SA_{n,q,l}$ be the closed subscheme of the product $SEE_{n,q} \times_{\operatorname{Spec} \mathbb{Z}} SEE_{n,q^{n-1}+l}$ whose R-valued points are pairs*

$$\bigl((f_1, \dots, f_n), (g_1, \dots, g_n)\bigr) \in SEE_{n,q}(R) \times SEE_{n,q^{n-1}+l}(R)$$

with the property that we have $\mathrm{e}(f_1, \dots, f_n)\mathrm{e}(g_1, \dots, g_n) = \mathrm{e}(x_1, \dots, x_n)$. If R is reduced, then the first projection induces an identity

$$SA_{n,q,l}(R) = GA_n(R) \cap SEE_{n,q}(R)$$

and thus the reduced scheme $SA_{n,q} := (SA_{n,q,l})_{\mathrm{red}}$ does not depend on $l \in \mathbb{N}$.

Proof. To show that such endomorphisms $\mathrm{e}(f_1, \dots, f_n)$ and $\mathrm{e}(g_1, \dots, g_n)$ are automorphisms we can assume that R is a finitely generated $\mathbb{Z}$-algebra. As $\mathrm{e}(g_1, \dots, g_n)$ is injective, a classical theorem of Ax–Grothendieck's implies that it is surjective (see [Ax], Thm., [Gro4], Prop. (10.4.11); see also [Bore1], Sect. 3). So $\mathrm{e}(g_1, \dots, g_n)$ is a bijective étale endomorphism of $\mathbb{A}^n_R$ and thus it is an isomorphism. This implies that $\mathrm{e}(f_1, \dots, f_n) = \mathrm{e}(g_1, \dots, g_n)^{-1}$ is also an automorphism.

Thus the first projections $SEE_{n,q}(R) \times SEE_{n,q^{n-1}+l}(R) \to SEE_{n,q}(R)$ induce functorial maps

$$\Upsilon_{n,q,l}(R) : SA_{n,q,l}(R) \to GA_n(R) \cap SEE_{n,q}(R)$$

which are injective for every R, and to end the proof it suffices to prove that $\Upsilon_{n,q,l}(R)$ are in fact surjective if R is reduced. To show this, we can assume that R is a finitely generated $\mathbb{Z}$-algebra, let $(f_1, \dots, f_n) \in GA_n(R) \cap SEE_{n,q}(R)$ and let $(g_1, \dots, g_n) \in GA_n(R)$ be such that $\mathrm{e}(g_1, \dots, g_n) = \mathrm{e}(f_1, \dots, f_n)^{-1}$. We are left to show that actually we have $(g_1, \dots, g_n) \in SEE_{n,q^{n-1}}(R)$. As R is reduced, by replacing it with fields that are direct factors of the total ring of fractions of R, to show that $(g_1, \dots, g_n) \in SEE_{n,q^{n-1}}(R)$ we can assume R is a field and in this case the relation follows from the proof of [BCW2], Cor. (1.6). □

Remark 24.2. The R-valued points rule

$$EE^*_{n,q}(R) := \{(f_1, \dots, f_n) \in \mathrm{EE}_{n,q} | \det\bigl(J(f_1, \dots, f_n)\bigr) \in R^*\}$$

defines a locally closed subscheme $EE^*_{n,q}$ of $End_{n,q}$. As the group monomorphism $R^* \subset R[x_1, \dots, x_n]^*$ is an isomorphism iff R is reduced, the functorial inclusion $EE^*_{n,q}(R) \subset EE_{n,q}(R)$ is an identity iff R is reduced. Note that

$$EE^*_{n,q} \cong \mathbb{G}_m \times_{\operatorname{Spec} \mathbb{Z}} SEE_{n,q}$$

(e.g., by multiplying x_1 in the source or the target by invertible scalars). If in Proposition 24.1 we replace SEE by EE^* we obtain affine schemes $GA^*_{n,q,l}$ over $\operatorname{Spec} \mathbb{Z}$ whose reductions do not depend on l and define $GA^*_{n,q}$; we have isomorphisms

$$GA^*_{n,q,l} \cong \mathbb{G}_m \times_{\operatorname{Spec} \mathbb{Z}} SA_{n,q,l} \text{ and } GA^*_{n,q} \cong \mathbb{G}_m \times_{\operatorname{Spec} \mathbb{Z}} SA_{n,q}.$$

Remark 24.3. The scheme $SA_{n,q,K}$ is not a priori reduced but its reduction is canonically identified with $(SA_{n,q,l,K})_{\mathrm{red}}$ for all $l \in \mathbb{N}$.

The injective maps $\Upsilon_{n,q,l}(R)$ define natural morphisms $SA_{n,q,l} \to SEE_{n,q}$ and thus also a natural morphism $SA_{n,q} \to SEE_{n,q}$.

Fact 24.4. *The morphism $SA_{n,q} \to SEE_{n,q}$ is a closed embedding.*

Proof. The image $S := \mathrm{Im}\big(\mathcal{O}(SEE_{n,q}) \to \mathcal{O}(SA_{n,q})\big)$ is reduced. As $\Upsilon_{n,q,l}(S)$ are isomorphisms, from [vdE], Part I, Ch. 1, Prop. 1.1.7 applied to the inclusion $S \to \mathcal{O}(SA_{n,q})$ we get that this inclusion is an isomorphism. $\square$

We view the injective maps $\Xi_{n,q}(R) : End_{n,q}(R) \to End_{n,q+1}(R)$ as inclusions and we also speak about $SA_{n,q}(R) \subset SA_{n,q+1}(R)$, $SEE_{n,q}(R) \subset SEE_{n,q+1}(R)$, etc. E.g., we identify $SA_{n,q,l}$ with a closed subscheme of $SA_{n,q,l+1}$ for all $l \in \mathbb{N}$.

We write $SEE_{n,q} = \mathrm{Spec}\,(B_{n,q})$, where $B_{n,q} = A_{n,q}/I_{n,q}$ is a quotient of $A_{n,q}$ compatible with the bigradings. As $\Xi^{\#}_{n,q}(I_{n,q+1}) \subset I_{n,q}$, the inverse limit $B_n = A_n/I_n$ of the projective system $B_{n,q+1} \to B_{n,q}$ indexed by $q \in \mathbb{N}^*$ is well-defined.

Let $\overline{\mathrm{Spec}\,(B_{n,q,\mathbb{Q}})}$ and $\overline{\mathrm{Spec}\,(B_{n,\mathbb{Q}})}$ be the schematic closures in $\mathrm{Spec}\,\big(B_{n,q,\mathbb{Z}_{(p)}}\big)$ and $\mathrm{Spec}\,\big(B_{n,\mathbb{Z}_{(p)}}\big)$ (respectively). The natural closed embedding

$$\mathrm{Spec}\,(B_{n,q}) \to \mathrm{Spec}\,(B_n)$$

induces a closed embedding

$$\mathfrak{C}_{n,q} : \overline{\mathrm{Spec}\,(B_{n,q,\mathbb{Q}})} \to \overline{\mathrm{Spec}\,(B_{n,\mathbb{Q}})} \cap \mathrm{Spec}\,\big(B_{n,q,\mathbb{Z}_{(p)}}\big)$$

which is not always an isomorphism, as the next example shows.

Example 24.5. The pair $(x_1 - x_1^p, x_2) \in \mathrm{EE}_{2,p}(\mathbb{F}_p)$ can be lifted to a formal pair $\big(x_1 - x_1^p, x_2(1 - px_1^{p-1})^{-1}\big)$ over $\mathbb{Z}_p$ whose Jacobian matrix has determinant 1, hence

$$(x_1 - x_1^p, x_2) \in [\overline{\mathrm{Spec}\,(B_{2,\mathbb{Q}})} \cap \mathrm{Spec}\,\big(B_{2,p,\mathbb{Z}_{(p)}}\big)](\mathbb{F}_p).$$

But if p is small, say $p = 2$ by [Wa2], Thm. 62 (based on [Moh], Sect. 6 and [Wa1], one could take $p \leq 97$), then

$$(x_1 - x_1^p, x_2) \notin \overline{\mathrm{Spec}\,(B_{2,p,\mathbb{Q}})}(\mathbb{F}_p)$$

which implies that $\mathfrak{C}_{2,p}$ is not an isomorphism. This disproves the equivalence used in [MR], Def. 3.3 within parentheses and implicitly disproves [MR], Conj. 3.4 if the first part (not within parentheses) of [MR], Def. 3.3 is used.[18]

The following general lemma helps in keeping track of q in mixed characteristic $(0, p)$ in the case when $q \in p\mathbb{N}^*$.

Lemma 24.6. *For $i \in [[1, n]]$ let $r_i \in \mathbb{N}$ and let $g_i(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$ be homogeneous of degree r_i. We consider the set $\mathcal{X}$ of n-tuples $(\alpha_1, \ldots, \alpha_n) \in (K^*)^n$ with the property that the homogeneous system of n equations*

$$g_1(x_1, \ldots, x_n) + \alpha_1 x_1^{r_1} = \cdots = g_n(x_1, \ldots, x_n) + \alpha_n x_n^{r_n} = 0 \tag{31}$$

has no solution in $\mathbb{P}^{n-1}(K)$. Then $\mathcal{X}$ is Zariski dense in $\mathbb{A}^n_K$.

Proof. If there exists $i \in [[1, n]]$ such that $r_i = 0$, so $g_i \in K$, by taking $\alpha_i \in K \setminus \{-g_i\}$ the lemma holds. Thus we can assume that we have $r_i \in \mathbb{N}^*$ for all $i \in [[1, n]]$.

For $i \in [[1, n]]$ let t_i be an indeterminate and let $\mathcal{H} \in K[t_1, \ldots, t_n]$ be the normalized resultant of the n homogeneous polynomials $x_i^{r_i} + t_i g_i(x_1, \ldots, x_n)$ (see [vdW], Ch. XI, Sect. 82, p. 14 or [Jo2], Sect. 2, Prop. 2.3 (i) or [Dem], Sect. 4, Def. 3); we have $\mathcal{H}(0, \ldots, 0) = 1$. Thus, as K is infinite, there exists $(\beta_1, \ldots, \beta_n) \in (K^*)^n$ such that $\mathcal{H}(\beta_1, \ldots, \beta_n) \neq 0$. Properties of resultants show that the system

[18] In [MR], Conj. 3.4, if the second part (within parentheses) of [MR], Def. 3.3 is used, is false in characteristic p and is only a reformulation of the Jacobian Conjecture in characteristic 0.

of homogeneous equations $x_i^{r_i}+\beta_i g_i(x_1,\ldots,x_n)=0$ indexed by $i\in[[1,n]]$ has no solution in $\mathbb{P}^{n-1}(K)$ (see [vdW], Ch. XI, Sect. 82, p. 15 or [Jo2], Sect. 1 or [Dem], Sect. 4, Sch.). Hence, if $\alpha_i:=\beta_i^{-1}$ for all $i\in[[1,n]]$, the System (31) has no solution in $\mathbb{P}^{n-1}(K)$, and thus we have an inclusion $\{(\beta_1^{-1},\ldots,\beta_n^{-1})\in(K^*)^n|\mathcal{H}(\beta_1,\ldots,\beta_n)\neq 0\}\subset\mathcal{X}$ from which the lemma follows. $\square$

Let $q_+:=p\lceil\frac{q}{p}\rceil\in p\mathbb{N}^*$; so $q\leq q_+$, and we have $q=q_+$ iff $q\in p\mathbb{N}^*$.

Proposition 24.7. *With $q,q_+\in\mathbb{N}^*$ as above, the following properties hold.*

(1) *Each $e\in\mathrm{EE}_{n,q}(K)$ (resp. $e\in SEE_{n,q}(K)$) can be deformed to an element $a\in GA_{n,1}(K)\subset GA_{n,q}(K)$ (resp. $a\in SA_{n,1}(K)\subset SA_{n,q}(K)$) using polynomials in $K_{\mathrm{s}}[x_1,\ldots,x_n,t]$ of degree at most $2q-1$ and of total degree in $x_1,\ldots,x_n$ at most q. Moreover, if $e\in GA_{n,q}(K)$ (resp. $e\in SA_{n,q}(K)$), then we can assume that the deformation, by evaluating t at arbitrary elements of K, defines elements of $GA_{n,q}(K)$ (resp. $SA_{n,q}(K)$).*

(2) *Each $d\in\mathrm{EE}_{n,q}(k)\subset\mathrm{EE}_{n,q_+}(k)$ (resp. $d\in SEE_{n,q}(k)\subset SEE_{n,q_+}(k)$) is the specialization of a finite endomorphism in $\mathrm{EE}_{n,q_+}\big(k(t)\big)$ (resp. in $SEE_{n,q_+}\big(k(t)\big)$) of geometric degree q_+^n.*

Proof. If e is defined by an n-tuple $(f_1,\ldots,f_n)\in K_{\mathrm{s}}[x_1,\ldots,x_n]^n$, then let

$$h_{i,t}=h_i(x_1,\ldots,x_n,t):=\frac{f_i(tx_1,\ldots,tx_n)+(t-1)f_i(0,\ldots,0)}{t}\in K_{\mathrm{s}}[x_1,\ldots,x_n,t];$$

its degree is at most $2q-1$ and its total degree in $x_1,\ldots,x_n$ is at most q. For a fixed $t_0\in K$ we get $e_{t_0}\in\mathrm{EE}_{n,q}(K)$ defined by the n-tuple $(h_{1,t_0},\ldots,h_{n,t_0})\in K_{\mathrm{s}}[x_1,\ldots,x_n]$, with $e_1=e$ and, as $\pi(e_0)=1$, with $e_0\in GA_{n,1}(K)\subset GA_{n,q}(K)$. If $e\in GA_{n,q}(K)$ (resp. $e\in SA_{n,q}(K)$), then we have $e_{t_0}\in GA_{n,q}(K)$ (resp. $e_{t_0}\in SA_{n,q}(K)$) for all $t_0\in K$. So part (1) holds.

For part (2), let d be defined by an n-tuple $(f_1,\ldots,f_n)\in k_{\mathrm{s}}[x_1,\ldots,x_n]$ and for each $i\in[[1,n]]$ let $g_i\in k_{\mathrm{s}}[x_1,\ldots,x_n]$ be 0 if $\deg(f_i)<q_+$ (e.g., if $q<q_+$) and be the homogeneous part of f_i of degree q_+ if $\deg(f_i)=q=q_+$. From Fact 15.1(2) and from Lemma 24.6 applied to $r_1=\cdots=r_n=q_+$ and $(g_1,\ldots,g_n)$ we get that there exists $(\alpha_1,\ldots,\alpha_n)\in(k^*)^n$ such that

$$d_1:=\mathrm{e}(f_1+\alpha_1x_1^{q_+},\ldots,f_1+\alpha_nx_n^{q_+})\in\mathrm{EE}_{n,q_+}(k)$$

is finite with $\deg(d_1)=q_+^n$. Then

$$E:=\mathrm{e}(f_1+t\alpha_1x_1^{q_+},\ldots,f_n+t\alpha_nx_n^{q_+})\in\mathrm{EE}_{n,q^+}(k[t])$$

specializes for $t=0$ to d and for $t=1$ to d_1. The last specialization implies that the generic fiber $E_{k(t)}\in\mathrm{EE}_{n,q_+}\big(k(t)\big)$ of E is also finite of degree q_+^n. Also, $d\in SEE_{n,q}(k)\Rightarrow E\in SEE_{n,q^+}(k[t])$. As $E_{k(t)}$ specializes to d, part (2) holds. $\square$

Remark 24.8. The existence of E of the proof of Proposition 24.7(2) disproves [MR], Conj. 5.3(2) for each integral k-algebra that contains $k[t]$ and has k as a quotient ring.

Corollary 24.9. *Let $q\in\mathbb{N}^*$. Then the affine schemes $SA_{n,q,K}$ and $SEE_{n,q,K}$ (hence also $EE_{n,q,K}^*$) over $\operatorname{Spec}K$ are connected.*

Proof. As $SA_{n,1,K}$ is connected (being the smooth group of affine transformations of K^n of determinant 1), this follows from Proposition 24.7(1). $\square$

Proposition 24.10. *Let $q \in \mathbb{N}^*$. Let $\deg_{p,q,n} \in \mathbb{N}^*$ (resp. $\deg^{\mathrm{f}}_{p,q,n} \in \mathbb{N}^*$) be the largest such that there exists an (resp. a finite) endomorphism $e \in SEE_{n,q}(k)$ such that $\deg(e) = \deg_{p,q,n}$ (resp. $\deg(e) = \deg^{\mathrm{f}}_{p,q,n}$). Then the following properties hold.*

(1) *We have inequalities $\deg^{\mathrm{f}}_{p,q,n} \le \deg_{p,q,n} \le q^n$.*

(2) *If $p \mid q$, then $\deg^{\mathrm{f}}_{p,q,n} = \deg_{p,q,n} = q^n$.*

(3) *If $p \nmid q$ and $q > p$, then $pq^{n-1}\lfloor \frac{q}{p} \rfloor \le \deg^{\mathrm{f}}_{p,q,n}$.*

(4) *There exists an open non-empty subscheme $SEE^{\mathrm{gen}}_{n,q,k}$ of $SEE_{n,q,k}$ such that $SEE^{\mathrm{gen}}_{n,q,k}(k) = \{e \in SEE_{n,q,k}(k) = SSE_{n,q}(k) | \deg(e) = \deg_{p,q,n}\}$.*

(5) *There exists a constructible subset $SEE^{\mathrm{f}}_{n,q,k}$ of $SEE_{n,q,k}$ formed by points P over whose residue fields $\kappa(P)$ we get finite étale endomorphisms of $\mathbb{A}^n_{\kappa(P)}$.*

(6) *There exists a constructible subset $SEE^{\mathrm{f-gen}}_{n,q,k}$ of $SEE_{n,q,k}$ contained in $SEE^{\mathrm{f}}_{n,q,k}$ and formed by points P over whose residue fields $\kappa(P)$ we get finite étale endomorphisms of $\mathbb{A}^n_{\kappa(P)}$ of degrees $\deg^{\mathrm{f}}_{p,q,n}$.*

(7) *If $p \mid q$, then $SEE^{\mathrm{f-gen}}_{n,q,k}$ is a constructible subset of $SEE^{\mathrm{gen}}_{n,q,k}$ which is Zariski dense in $(SEE_{n,q,k})_{\mathrm{red}}$.*

Proof. The inequality $\deg^{\mathrm{f}}_{p,q,n} \le \deg_{p,q,n}$ is clear and the inequality $\deg_{p,q,n} \le q^n$ follows from Fact 15.1(1); so part (1) holds.

If $p \mid q$, then $e := \mathrm{e}(x_1 - x_1^q, \ldots, x_n - x_n^q) \in SEE_{n,q}(k)$ is finite by Fact 15.1(2) with $\deg(e) = q^n$. From this and part (1) we get that part (2) holds.

If $p \nmid q$ with $q > p$, then $e := \mathrm{e}(x_1 - x_2^q, \ldots, x_{n-1} - x_n^q, x_n - x_1^{p\lfloor \frac{q}{p} \rfloor}) \in SEE_{n,q}(k)$ is finite by Fact 15.1(2) with $\deg(e) = pq^{n-1}\lfloor \frac{q}{p} \rfloor$, so part (3) holds.

We consider the universal étale endomorphism $e^{\mathrm{univ}}_{n,q,p} : \mathbb{A}^n_{SEE_{n,q,k},\mathrm{s}} \to \mathbb{A}^n_{SEE_{n,q,k},\mathrm{t}}$. From either [Gro4], Prop. (15.5.1)(i) or [Gro5], Prop. (18.2.8) we get that there exists an open subscheme $U_{n,q,k}$ of $\mathbb{A}^n_{SEE_{n,q,k},\mathrm{t}}$ whose k-valued points are those P for which $|(e^{\mathrm{univ}}_{n,q,p})^{-1}(P)| = \deg_{p,q,n}$; it is non-empty by the definition of $\deg_{p,q,n}$. As the projection $A^n_{SEE_{n,q,k},\mathrm{t}} \to SEE_{n,q,k}$ is flat, the image of $U_{n,q,k}$ under it is an open non-empty subscheme $SEE^{\mathrm{gen}}_{n,q,k}$ of $SEE_{n,q,k} = SEE_{n,q,k,\mathrm{t}}$ by [Gro3], Thm. (2.4.6). We have $SEE^{\mathrm{gen}}_{n,q,k}(k) = \{e \in SEE_{n,q,k}(k) = SSE_{n,q}(k) | \deg(e) = \deg_{p,q,n}\}$, so part (4) holds. Part (5) is a particular case of [Gro4], Prop. (9.6.1)(vi).

For part (6), if $\deg^{\mathrm{f}}_{p,q,n} = \deg_{p,q,n}$, then $SEE^{\mathrm{f,gen}}_{n,q,k} = SEE^{\mathrm{gen}}_{n,q,k} \cap SEE^{\mathrm{f}}_{n,q,k}$, so we can assume that $\deg^{\mathrm{f}}_{p,q,n} < \deg_{p,q,n}$. From either [Gro4], Prop. (15.5.1)(i) or [Gro5], Prop. (18.2.8) we get that there exists an open subscheme $U^{\mathrm{f}}_{n,q,k}$ of $\mathbb{A}^n_{SEE_{n,q,k},\mathrm{t}}$ whose k-valued points are those P for which $|(e^{\mathrm{univ}}_{n,q,p})^{-1}(P)| \ge \deg^{\mathrm{f}}_{p,q,n} + 1$; as in the previous paragraph we argue that its image in $SEE_{n,q,k} = SEE_{n,q,k,\mathrm{t}}$ is open, so it has a reduced closed complement $SEE^{\le \deg^{\mathrm{f}}_{p,q,n}}_{n,q,k}$. Repeating the arguments for the pullback of $e^{\mathrm{univ}}_{n,q,p}$ via the closed embedding $SEE^{\le \deg^{\mathrm{f}}_{p,q,n}}_{n,q,k} \to SE_{n,q,k}$, we get that there exists an open subscheme $SEE^{=\deg^{\mathrm{f}}_{p,q,n}}_{n,q,k}$ of $SEE^{\le \deg^{\mathrm{f}}_{p,q,n}}_{n,q,k}$ whose k-valued points are those P for which we get étale endomorphisms of $\mathbb{A}^n_k$ of degrees $\deg^{\mathrm{f}}_{p,q,n}$; as $SEE^{\mathrm{f,gen}}_{n,q,k} = SEE^{=\deg^{\mathrm{f}}_{p,q,n}}_{n,q,k} \cap SEE^{\mathrm{f}}_{n,q,k}$, part (6) holds.

For part (7), the inclusion $SEE^{\mathrm{f,gen}}_{n,q,k} \subset SEE^{\mathrm{gen}}_{n,q,k}$ follows from part (2) and the Zariski density follows from Proposition 24.7(2). $\square$

Example 24.11. Assume that $p = n = q = 2$. Then an arbitrary pair $(f_1, f_2) \in SEE_{2,2,k}(R)$ is of the form

$$f_i(x_1, x_2) = \alpha^{(i)}_{(0,0)} + \alpha^{(i)}_{(1,0)}x_1 + \alpha^{(i)}_{(0,1)}x_2 + \alpha^{(i)}_{(2,0)}x_1^2 + \alpha^{(i)}_{(1,1)}x_1x_2 + \alpha^{(i)}_{(0,2)}x_2^2$$

and, as $1 = -1$, one computes that $J(f_1, f_2)(x_1, x_2)$ is equal to

$$\alpha^{(1)}_{(1,0)}\alpha^{(2)}_{(0,1)}+\alpha^{(1)}_{(0,1)}\alpha^{(2)}_{(1,0)}+\big(\alpha^{(1)}_{(1,0)}\alpha^{(2)}_{(1,1)}+\alpha^{(1)}_{(1,1)}\alpha^{(2)}_{(1,0)}\big)x_1+\big(\alpha^{(1)}_{(1,1)}\alpha^{(2)}_{(0,1)}+\alpha^{(1)}_{(0,1)}\alpha^{(2)}_{(1,1)}\big)x_2.$$

Thus

$$A_{2,2} \otimes_{\mathbb{Z}} k = k[\alpha^{(i)}_{(0,0)}, \alpha^{(i)}_{(1,0)}, \alpha^{(i)}_{(0,1)}, \alpha^{(i)}_{(2,0)}, \alpha^{(i)}_{(1,1)}, \alpha^{(i)}_{(0,2)}|i \in \{1, 2\}]$$

and $\mathrm{Im}(I_{2,2} \to A_{2,2} \otimes_{\mathbb{Z}} k)$ is the ideal

$$\big(\alpha^{(1)}_{(1,0)}\alpha^{(2)}_{(0,1)} + \alpha^{(1)}_{(0,1)}\alpha^{(2)}_{(1,0)} + 1, \alpha^{(1)}_{(1,0)}\alpha^{(2)}_{(1,1)} + \alpha^{(1)}_{(1,1)}\alpha^{(2)}_{(1,0)}, \alpha^{(1)}_{(1,1)}\alpha^{(2)}_{(0,1)} + \alpha^{(1)}_{(0,1)}\alpha^{(2)}_{(1,1)}\big).$$

Therefore $B_{2,2} \otimes_{\mathbb{Z}} k = (A_{2,2}/I_{2,2}) \otimes_{\mathbb{Z}} k$ is isomorphic to a polynomial algebra in 6 indeterminates (they are $\alpha^{(i)}_{0,0}$, $\alpha^{(i)}_{2,0}$, and $\alpha^{(i)}_{0,2}$ with $i \in \{1, 2\}$) over the k-algebra

$$k[x, y, z, x', y', z']/(xx' + yy' + 1, xz' + zy', zx' + yz')$$

which is isomorphic to $k[x, y, x', y']/(xx' + yy' + 1)$ as $z' = z = 0$, i.e., $\alpha^{(1)}_{1,1} = \alpha^{(2)}_{1,1} = 0$. Thus $SEE_{2,2,k}$ is an affine smooth irreducible variety over k of dimension 9.

We show that $SA_{2,2,k}$ has dimension 6 and 3 irreducible components. As $SA_{2,1,k}$ has dimension 5 and is irreducible and as $SA_{2,1,k}$ acts on $SA_{2,2,k}$ via left or right translations, it suffices to show that the affine scheme W over k that involves the eight indeterminates y_1 to y_4 and z_1 to z_4 and is defined by the composite equation

$$\mathrm{e}(x_1+y_1x_1^2+y_2x_2^2, x_2+y_3x_1^2+y_4x_2^2)\mathrm{e}(x_1+z_1x_1^2+z_2x_2^2, x_2+z_3x_1^2+z_4x_2^2) = \mathrm{e}(x_1, x_2)$$

has dimension 1 and 3 irreducible components. This composite equation implies, as $1 = -1$, that we have $z_i = y_i$ for all $i \in [[1, 4]]$ and that

$$W \cong \mathrm{Spec}\,\big(k[y_1, y_2, y_3, y_4]/\big(y_1^3 + y_2y_3^2, y_4^3 + y_1^2y_3, y_1(y_2 + y_4)^2, y_4(y_1 + y_3)^2\big)\big)$$

is non-reduced. The irreducible components of W are given by the equations $y_1 = y_4 = y_2 = 0$, $y_1 = y_4 = y_3 = 0$, and $y_1 = y_2 = y_3 = y_4$.

As $SA_{2,2,0,k} \cong SA_{2,1,k} \times_{\mathrm{Spec}\,k} W$, $SA_{2,2,0,k}$ is also non-reduced.

Based on Remark 24.2 and the identities $\alpha^{(1)}_{1,1} = \alpha^{(2)}_{1,1} = 0$ we get that if $e \in \mathrm{EE}_2(k)$ is such that $\pi(e) = 2$, then $e \in \mathrm{BE}_2(k)$, and from the proof of [V], Thm. 2.4.1(b) we get that $\mathcal{F}(e) = \emptyset$, i.e., e is finite, and $\deg(e) \in \{2, 4\}$. Thus we have identities $SEE^{\mathrm{f}}_{2,2,k} = SEE_{2,2,k}$ and $SEE^{\mathrm{f-gen}}_{2,2,k} = SEE^{\mathrm{gen}}_{2,2,k}$ of constructible sets.

Example 24.12. Assume $p = 2$. The rule $(x, y) \mapsto (x, y + xy^2)$ defines $d \in \mathrm{EE}_2(k)$ with $\deg(d) = 2$, $\mathcal{F}(d) = \{1\}$. From Proposition 4.3(1) to (3) we get relations $\pi(d) \leq \pi_{\mathrm{r}}(d) = \pi_{\mathrm{l}}(d) = \pi_{\mathrm{i}}(d) = 3$. From the previous paragraph we get that $\pi(d) \neq 2$. As $d \notin \mathrm{GA}_2(k)$, we have $\pi(d) \neq 1$. We conclude that $\pi(d) = 3$.

For a $n \times n$ matrix $\square$ with entries in R and $i \in [[1, n]]$, let $\mathrm{tr}_i(\square) \in R$ be the trace of the $\binom{n}{i} \times \binom{n}{i}$ matrix which is the i-th exterior power $\bigwedge^i(\square)$ of $\square$. If 1_n is the identity matrix of size $n \times n$, then we have the following formula

$$\det(1_n + \square) = 1 + \sum_{i=1}^{n} \mathrm{tr}_i(\square) \tag{32}$$

which follows from the fact that $\Box$ is similar to an upper triangular matrix.

Theorem 24.13. *Let $q \in \mathbb{N}^*$. Then $SEE_{n,q,K}$ (equivalently, $EE^*_{n,q,K}$) is smooth over $\operatorname{Spec} K$ iff either $n = q = \operatorname{char}(K) = 2$ or $\min\{n, q\} = 1$, in which case it is also irreducible.*

Proof. If $\operatorname{char}(K) = 2$, then $SEE_{2,2,K}$ is smooth and irreducible (see Example 24.11). Clearly, $SEE_{n,1} = SA_{n,1} \cong \mathbf{SL}_n \times_{\operatorname{Spec}\mathbb{Z}} \mathbb{A}^n$. If $\operatorname{char}(K) = 0$, then

$$SEE_{1,q,K} \cong SEE_{1,1,K} = SA_{1,1,K} = \operatorname{Spec}\big(K[\alpha^{(1)}_{(0)}]\big) \cong \mathbb{A}^1_K.$$

If $\operatorname{char}(K) = p$, then

$$SEE_{1,q,K} = SEE_{1,p\lfloor\frac{q}{p}\rfloor,K} = \operatorname{Spec}\big(K[\alpha^{(1)}_{(pi)}|i \in [[0, \lfloor\frac{q}{p}\rfloor]]]\big) \cong \mathbb{A}_K^{\lfloor\frac{q}{p}\rfloor+1}.$$

We are left to prove that if $\min\{n, q\} \geq 2$ and for $n = q = 2$ we have $\operatorname{char}(K) \neq 2$, then $SEE_{n,q,K}$ is not smooth. It suffices to show that there exists $m \in \{2, 3\}$ and a $K[t]/(t^m)$-valued point P_m of $SEE_{n,q,K}$ which does not lift to a $K[t]/(t^{m+1})$-valued point of $EE^*_{n,q,K}$ (hence also of $SEE_{n,q,K}$).

We consider an n-tuple $(f_1, \ldots, f_n) \in \mathrm{EE}_{n,q,K}\big(K[t]/(t^{m+1})\big)$ whose reduction modulo $\big(t)/(t^{m+1}\big)$ is $(x_1, \ldots, x_n)$ and whose reduction modulo $\big(t^2)/(t^{m+1}\big)$ is a point $P_2 \in SEE_{n,q,K}\big(K[t]/(t^2)\big)$. For $i \in [[1, n]]$ we write

$$f_i(x_1, \ldots, x_n) = x_i + \sum_{j=1}^{m} t^j g_{i,j}(x_1, \ldots, x_n)$$

with each $g_{i,j} \in K_{\mathrm{s}}[x_1, \ldots, x_n]$ of degree at most q. Let $h_1, \ldots, h_m \in K_{\mathrm{s}}[x_1, \ldots, x_s]$ be such that

$$\det\big(J(f_1, \ldots, f_n)\big) = 1 + \sum_{j=1}^{m} t^j h_j(x_1, \ldots, x_n);$$

as $P_2 \in SEE_{n,q,K}(K[t]/(t^2))$, we have $h_1 = 0$. From Equation (32) applied to $\Box = \sum_{j=1}^m t^j J(g_{1j}, \ldots, g_{nj})$ we get identities

$$h_1(x_1, \ldots, x_n) = \operatorname{tr}_1\big(J(g_{1,1}, \ldots, g_{n,1})\big) = \sum_{i=1}^{n} \frac{\partial g_{i,1}}{\partial x_i}(x_1, \ldots, x_n), \tag{33}$$

$$h_2(x_1, \ldots, x_n) = \operatorname{tr}_2\big(J(g_{1,1}, \ldots, g_{n,1})\big) + \sum_{i=1}^{n} \frac{\partial g_{i,2}}{\partial x_i}(x_1, \ldots, x_n). \tag{34}$$

We show that we can choose $m \in \{2, 3\}$ and $g_{i,j}$ with $j \in [[1, m-1]]$ such that $h_1 = \cdots = h_{m-1} = 0$ but, no matter what $g_{1,m}, \ldots, g_{n,m}$ are, we have $h_m \notin K$: P_m is the reduction of $(f_1, \ldots, f_n)$ modulo $(t^m)/(t^{m+1})$.

We first consider three non-disjoint cases with $m = 2$ and $g_{i,1} = 0$ if $i \in [[3, n]]$.

Case 1: $\operatorname{char}(K) \nmid q$. We take $g_{1,1} = -q^{-1}x_1^q$ and $g_{2,1} = x_1^{q-1}x_2$. Equations (33) and (34) give that indeed $h_1 = 0$ but $h_2 = -x_1^{2q-2} + \sum_{i=1}^n \frac{\partial g_{i,2}}{\partial x_i}(x_1, \ldots, x_n) \notin K$ (each $\frac{\partial g_{i,2}}{\partial x_i}$ has degree at most $q-1$ and thus less than $2q-2$ as $q \geq 2$); with $m = 2$, P_2 has the desired property.

Case 2: $n \geq 3$. By taking $g_{1,1} = -x_1x_3^{q-1}$ and $g_{2,1} = x_2x_3^{q-1}$, this case is similar to the previous one: $h_2 = -x_3^{2q-2} + \sum_{i=1}^n \frac{\partial g_{i,2}}{\partial x_i}(x_1, \ldots, x_n) \notin K$.

Case 3: $\mathbf{char}(K) \nmid q-1$ and $q \geq 4$. By taking $g_{1,1} = -(q-1)^{-1}x_1^{q-1}$ and $g_{2,1} = x_1^{q-2}x_2$ it follows that we have relations $h_2 = -x_3^{2q-3} + \sum_{i=1}^n \frac{\partial g_{i,2}}{\partial x_i}(x_1, \ldots, x_n) \notin K$ (we have $2q-3 > q-1$ as $q \geq 4$).

The only case not covered by the first three ones is the following one.

Case 4: $n = 2$ and $q = \mathbf{char}(K) = 3$. We take $m = 3$, $g_{1,1} = x_1^2$, $g_{1,2} = 0$, $g_{2,1} = x_1x_2$ and $g_{2,2} = x_1^2x_2$. As

$$\det\bigl(J(x_1 + tx_1^2, x_2 + tx_1x_2 + t^2x_1^2x_2)\bigr) = 1 + 3tx_1 + 3t^2x_1^2 + 2t^3x_1^3 = 1 + 2t^3x_1^3,$$

$h_1 = h_2 = 0$ but $h_3 = 2x_1^3 + \frac{\partial g_{1,3}}{\partial x_1} + \frac{\partial g_{2,3}}{\partial x_2} \notin K$ as $\deg(g_{i,3}) \leq 3$ for $i \in \{1, 2\}$; so the reduction P_3 of $(f_1, \ldots, f_n)$ modulo $(t^3)/(t^4)$ has the desired property. $\square$

25. On Conjecture 2.10

Artin–Schreier equations define connected finite Galois covers of $\mathbb{A}_k^n$ of degree p. We show that [Mi1], Thms. 1.1 and 1.2 hold for affine schemes over $\operatorname{Spec}\mathbb{F}_p$.

Fact 25.1. *Let W be a connected $\mathbb{F}_p$-scheme such that $H^1(W, \mathcal{O}_W) = 0$ (e.g., W is affine). Then we have a functorial isomorphism of groups*

$$\text{(35)} \qquad \operatorname{cob}_W : \mathcal{O}(W)/\bigl(x^p - x | x \in \mathcal{O}(W)\bigr) \cong H^1_{\text{ét}}(W, \mathbb{Z}/p\mathbb{Z})$$

which for $f \in \mathcal{O}(W)$ maps $f + \bigl(x^p - x | x \in \mathcal{O}(W)\bigr)$ to the $\mathbb{Z}/p\mathbb{Z}$-torsor V over W defined by the homomorphism $\mathbb{Z}/p\mathbb{Z} \to \operatorname{Aut}_{\mathcal{O}_W-\text{algebra}}\bigl(\mathcal{O}_W[x]/(x^p - x - f)\bigr)$ that maps $1 + p\mathbb{Z}$ to the automorphism defined by $x \mapsto x + 1$.

Proof. The first cohomology coboundary of the short exact sequence

$$0 \to (\mathbb{Z}/p\mathbb{Z})_W \to \mathbb{G}_{a,W} \xrightarrow{x^p - x} \mathbb{G}_{a,W} \to 0$$

in the étale topos of W is the functorial isomorphism of Equation (35). The identification of $H^1_{\text{ét}}(W, \mathbb{Z}/p\mathbb{Z})$ with the non-abelian $H^1(W, \mathbb{Z}/p\mathbb{Z})$, defined as the set of isomorphism classes of $(\mathbb{Z}/p\mathbb{Z})_W$-torsors as in [Gi], Ch. III, Rm. 3.5.4, gives that V is the pullback

$$\begin{tikzcd} V \arrow[r] \arrow[d] & W \arrow[d, "\mathrm{s}_f"] \\ \mathbb{G}_{a,W} \arrow[r, "x^p - x"] & \mathbb{G}_{a,W} \end{tikzcd}$$

via the section s_f that defines f, from which the fact follows. $\square$

We also include a proof of the following likely known result.

Proposition 25.2. *There exists no connected finite étale cover X of $\mathbb{A}_k^n$ which is either of degree $m \in [[2, p-1]]$ or Galois of degree at least 2 and prime to p.*

Proof. We show that the assumption that there exists an $n \in \mathbb{N}^*$ such that $\mathbb{A}_k^n$ has a connected finite étale cover X of the type mentioned leads to a contradiction. We choose the smallest such n. If $n = 1$, the connected finite Galois cover X^+ of $\mathbb{A}_k^1$ generated by X has a degree $m^+ > 1$ which either divides $m!$ and hence $(p-1)!$ or is prime to p; so $X^+ \to \mathbb{A}_k^1$ is a non-trivial tamely ramified finite Galois cover of the curve $\mathbb{A}_k^1$ and this contradicts [SGA1], Exp. XIII, Cor. 2.12. If $n \geq 2$, from Bertini Theorem (see [Jo1], Part I, Thm. 6.3(4)) we get that there exists a hyperplane $\mathbb{H} \subset \mathbb{A}_k^n$ such that $X \times_{\mathbb{A}_k^n} \mathbb{H}$ is a connected finite étale cover of $\mathbb{H} \cong \mathbb{A}_k^{n-1}$ of the type mentioned, and this contradicts the minimality assumption. $\square$

One could view the following direct consequence of Proposition 25.2 as classical 'finite' and 'Galois field extension' contexts evidence for Conjecture 2.10.

Corollary 25.3. *Let $e \in \mathrm{EE}_n(k)$ be such that one of the following conditions holds:*

(i) *we have $\deg(e) < p$ and e is finite;*

(ii) *the field extension $k_{\mathrm{t}}(x_1, \ldots, x_n) \to k_{\mathrm{s}}(x_1, \ldots, x_n)$ induced by $e^{\#}$ is Galois with Galois group of order prime to p.*

Then we have $e \in \mathrm{GA}_n(k)$.

Proof. Based on Lemma 6.1(4), it suffices to show that $\deg(e) = 1$. If condition (i) holds, then $\deg(e) = 1$ by Proposition 25.2. Assume now that condition (ii) holds. If O_{t} is a local ring of $\mathbb{A}^n_{k,\mathrm{t}}$ which is a discrete valuation ring, from Lemma 6.1(2) we get that there exists a local ring O_{s} of $\mathbb{A}^n_{k,\mathrm{s}}$ which dominates O_{t} and hence it is a discrete valuation ring. As $\mathbb{A}^n_{k,\mathrm{s}}$ is an open subvariety of X_e by Zariski's Main Theorem, O_{s} is a local ring of X_e which is an étale O_{t}-algebra. From this, as the finite field extension $k_{\mathrm{t}}(x_1, \ldots, x_n) \to k_{\mathrm{s}}(x_1, \ldots, x_n)$ is Galois, we get that the morphism $X_e \to \mathbb{A}^n_{k,\mathrm{t}}$ is étale above all points of $\mathbb{A}^n_{k,\mathrm{t}}$ of dimension at most 1 and hence, based on the purity of the branch locus (see [SGA2], Exp. X, Thm. 3.4 (i)), it is étale. Thus $\deg(e) = 1$ by Proposition 25.2 applied to X_e. □

It is a standard piece of algebraic geometry to show that the classical results of Razar (see [Raz], Thm. 2) and Wright (see [Wr1], Thms. 3.2, 3.3 and 3.7) in characteristic zero follow also directly from Corollary 25.3.

Directly from Corollaries 24.10(1) and 25.3 we get the following result.

Corollary 25.4. *If $q \in \mathbb{N}^*$ is such that $q^n < p$, then $\deg^{\mathrm{f}}_{p,n,q} = 1$.*

Next we generalize the part of Example 22.10 that pertains to Conjecture 2.10.

Proposition 25.5. *If $e \in \mathrm{EE}_n(k)$ has weak type 1 in the sense of Definition 22.2, then Conjecture 2.10 holds for e.*

Proof. Let $\pi : \mathbb{A}^n_{k,\mathrm{s}} \to \mathbb{A}^1_k$ be a morphism such that the condition for weak type ≤ 1 of Definition 22.2 holds for $\pi \times e : \mathbb{A}^n_{k,\mathrm{s}} \to \mathbb{A}^1_k \times_{\mathrm{Spec}\, k} \mathbb{A}^n_{k,\mathrm{t}}$. The Zariski closure Z of $\mathrm{Im}(\pi \times e)$ in $\mathbb{A}^1_k \times_{\mathrm{Spec}\, k} \mathbb{A}^n_{k,\mathrm{t}}$ is an irreducible hypersurface in $\mathbb{A}^1_{k,\mathrm{s}} \times_{\mathrm{Spec}\, k} \mathbb{A}^n_k = \mathbb{A}^{n+1}_k$ and hence the zero locus of an irreducible polynomial $f(x_0, \ldots, x_n) \in k[x_0, \ldots, x_n]$ whose degree in x_0 is $\deg(e)$. Let $(g_1, \ldots, g_n) \in k_{\mathrm{s}}[x_1, \ldots, x_n]^n$ define e and let $g_0 \in k_{\mathrm{s}}[x_1, \ldots, x_n]$ define π. For all $(\alpha_1, \ldots, \alpha_n) \in k^n$ we have $f\big(g_0(\alpha_1, \ldots, \alpha_n), \ldots, g_n(\alpha_1, \ldots, \alpha_n)\big) = 0$. Taking partial derivatives we get that for each $i \in [[1, n]]$ we have

$$\sum_{j=0}^{n} f_{x_j}\big(g_0(\alpha_1, \ldots, \alpha_n), \ldots, g_n(\alpha_1, \ldots, \alpha_n)\big) g_{j,x_i}(\alpha_1, \ldots, \alpha_n) = 0.$$

Note that $g_0 + x_{n+1} \in k_{\mathrm{s}}[x_1, \ldots, x_{n+1}]$ defines a smooth morphism $\pi' : \mathbb{A}^{n+1}_{k,\mathrm{s}} \to \mathbb{A}^1_k$. Thus for $e' := e \times 1_{\mathbb{A}^1_k} \in \mathrm{EE}_{n+1}(k)$ we have $\deg(e') = \deg(e)$ and $\pi' \times e'$ is isomorphic to $(\pi \circ \pi_{n+1}) \times e'$, where $\pi_{n+1} : \mathbb{A}^{n+1}_{k,\mathrm{s}} \to \mathbb{A}^n_{k,\mathrm{s}}$ is the projection on the first n coordinates. It follows that e' has type ≤ 1. Hence by replacing (e, π) with (e', π') we can assume that π is smooth.

As the determinant of the Jacobian matrix $J\big(g_1(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots, x_n)\big)$ is in k^* and for each $(\alpha_1, \ldots, \alpha_n) \in k^n$ there exists an index $j \in [[1, n]]$ such that

$g_{0,x_j}(\alpha_1,\ldots,\alpha_n)\neq 0$, both systems of equations $f=f_{x_0}=\cdots=f_{x_n}=0$ and $f=f_{x_0}=0$ along $\mathrm{Im}(\pi\times e)$ define $\mathrm{Sing}(Z)\cap\mathrm{Im}(\pi\times e)$. Thus the zero locus of the image of $f_{x_0}\big(g_0(\alpha_1,\ldots,\alpha_n),\ldots,g_n(\alpha_1,\ldots,\alpha_n)\big)$ in $k_{\mathrm{s}}[x_1,\ldots,x_n]$ via the k-algebra homomorphism $k[x_0,\ldots,x_n]\to k[x_1,\ldots,x_n]$ that maps x_i to g_i for each $i\in[[0,n]]$ is an effective divisor $\sum_{i=1}^s q_iY_i$ of $\mathbb{A}^n_{k,\mathrm{s}}$, where $s\in\mathbb{N}$, $q_1,\ldots,q_s\in\mathbb{N}^*$ and $Y_1,\ldots,Y_s$ are distinct irreducible divisors, such that $e^{-1}\big(e(Y_i)\big)=Y_i$ for each $i\in[[1,s]]$. Thus for each $i\in[[1,s]]$ there exists an irreducible polynomial $h_i\in k_{\mathrm{t}}[x_1,\ldots,x_n]$ such that the zero locus $e^{\#}(h_i)=0$ is Y_i. It follows that there exists $\beta\in k^*$ such that $f_{x_0}-\beta\prod_{i=1}^s h_i^{q_i}$ is divisible by f and by reasons of degrees in x_0 it follows that $f_{x_0}=\beta\prod_{i=1}^s h_i^{q_i}$ has degree 0 in x_0. Thus $\deg(e)\in\{1\}\cup p\mathbb{N}^*$. □

Corollary 25.6. *Let $e\in\mathrm{EE}_n(k)$. Then the following properties hold.*

(1) *If the $k_{\mathrm{t}}[x_1,\ldots,x_t]$-algebra A_e is generated by 1 element, then ψ_e is étale and Conjecture 2.10 holds for e.*

(2) *If A_e does not have a p-bases, then the $k_{\mathrm{t}}[x_1,\ldots,x_t]$-algebra A_e is not generated by 1 element.*

(3) *If A_e is regular and does not have a p-bases, then* $\mathrm{ciedim}(X_e)=\infty$.

Proof. To prove part (1), let $\pi:\mathbb{A}^n_{k,\mathrm{s}}\to\mathbb{A}^1_k$ be a morphism defined by an element in $A_e\subset k_{\mathrm{s}}[x_1,\ldots,x_t]$ that generates the $k_{\mathrm{t}}[x_1,\ldots,x_t]$-algebra A_e. The Zariski closure of $\mathrm{Im}(\pi\times e)$ in $\mathbb{A}^1_k\times_{\mathrm{Spec}\,k}\mathbb{A}^n_{k,\mathrm{t}}$ is X_e itself; so e is of weak type ≤ 1 and Conjecture 2.10 holds for e by Proposition 25.5. To prove that ψ_e is étale we can replace e by $e\times 1_{\mathbb{A}^1_k}$ and thus (see proof of Proposition 25.5) we can assume that π is smooth. With its notations, the mentioned proof gives that $f_{x_0}=\beta\in k^*$, hence ψ_e is étale.

If A_e does not have a p-bases, then ψ_e is non-étale and hence part (2) follows from part (1). Part (3) follows from part (2) and Theorem 7.3(1) applied to $Y=X_e$. □

Corollary 25.7. *Let $e\in\mathrm{EE}_n(K)\setminus\mathrm{GA}_n(K)$. If either* $\mathrm{char}(K)=0$ *or Conjecture 2.10 does not hold for e, then e has weak type* 2.

Proof. Based on Lemma 22.4 and Example 22.3, it suffices to show that the assumption that e has weak type 1 leads to a contradiction; as $e\notin\mathrm{GA}_n(K)$, we have $\deg(e)>1$ and $n\geq 2$. We can assume that $\mathrm{char}(K)=0$ by Proposition 25.5.

Let $\pi:\mathbb{A}^n_{K,\mathrm{s}}\to\mathbb{A}^1_K$ be a morphism such that the condition for weak type ≤ 1 of Definition 22.2 holds for $e\times\pi:\mathbb{A}^n_{K,\mathrm{s}}\to\mathbb{A}^1_K\times_{\mathrm{Spec}\,K}\mathbb{A}^n_{K,\mathrm{t}}$. Let R be a finitely generated regular $\mathbb{Z}$-subalgebra of K such that e and π are pullbacks of an endomorphism $e_R\in\mathrm{EE}_n(R)$ and a morphism $\pi_R:\mathbb{A}^n_R\to\mathbb{A}^1_R$ (respectively); thus e_R is universally open by [Gro4], Cor. (14.4.3). As $\pi\times e$ is a birational quasi-finite morphism, so is the fiber of $\pi_R\times e_R:\mathbb{A}^n_{R,\mathrm{s}}\to\mathbb{A}^1_R\times_{\mathrm{Spec}\,R}\mathbb{A}^n_{R,\mathrm{t}}$ over $\mathrm{Frac}(R)$. By shrinking $\mathrm{Spec}\,R$ we can assume that all fibers of $e_R\times\pi_R$ are birational quasi-finite by [Gro4], Prop. (9.6.1)(vii) and (x) and all geometric fibers of e_R have geometric degrees $\deg(e)$ by a direct application of [Gro4], Prop. (15.5.1)(i). By shrinking $\mathrm{Spec}\,R$ we can also assume that the Zariski closure Z of $\mathrm{Im}(\pi_R\times e_R)$ in $\mathbb{A}^1_R\times_{\mathrm{Spec}\,R}\mathbb{A}^n_{R,\mathrm{t}}$ has a singular locus $\mathrm{Sing}(Z)$ equal to the Zariski closure of the singular locus of $Z_{\mathrm{Frac}(R)}$ in Z and (see [Gro4], Thm. (9.7.7)) the geometric fibers of $Z\to\mathrm{Spec}\,R$ are integral varieties. Let $Y_{\mathrm{Frac}(R)}$ be the union of irreducible components of $(\pi_R\times e_R)^{-1}\big(\mathrm{Sing}(Z)\big)_{\mathrm{Frac}(R)}$ of dimension $n-1$. Let $W_{\mathrm{Frac}(R)}$ be the reduced closed subscheme of $\mathbb{A}^n_{\mathrm{Frac}(R),\mathrm{t}}$ such that $e^{-1}_{\mathrm{Frac}(R)}\big(W_{\mathrm{Frac}(R)}\big)=Y_{\mathrm{Frac}(R)}$; it exists as e has weak type 1. Let Y (resp. W) be the Zariski closure of $Y_{\mathrm{Frac}(R)}$

in $\mathbb{A}^n_{R,\mathrm{s}}$ (resp. of $W_{\mathrm{Frac}(R)}$ in $\mathbb{A}^n_{R,\mathrm{t}}$). By shrinking $\operatorname{Spec} R$ we can assume that the morphisms $Y \to \operatorname{Spec} R$ and $W \to \operatorname{Spec} R$ have equidimensional geometric fibers of dimension $n-1$, that $e_R^{-1}(W) = Y \subset (\pi_R \times e_R)^{-1}\bigl(\mathrm{Sing}(Z)\bigr)$, and that all fibers of $(\pi_R \times e_R)^{-1}\bigl(\mathrm{Sing}(Z)\bigr) \setminus Y \to \operatorname{Spec} R$ have dimension at most $n-2$. Thus the reductions of e_R modulo maximal ideals of R have weak type 1. Hence $\deg(e)$ is divisible by infinitely many primes p by Proposition 25.5, a contradiction. $\square$

26. On geometric degrees 3

Proposition 26.1. *Let $e \in \mathrm{EE}_n(K)$ be such that $\deg(e) = 3$. Let $R := K_{\mathrm{t}}[x_1, \ldots, x_n]$, $S := K_{\mathrm{s}}[x_1, \ldots, x_n]$. Let $(h_0, h_1) \in R^2$ be such that $N_e^{\text{n-ét}} = \operatorname{Spec} R/Rh_0$ and $N_e = \operatorname{Spec} R/Rh_1$ (so $h_0 \in K^*$ iff ψ_e is étale and $h_1 \in K^*$ iff $\imath_e$ is an isomorphism). Let $h \in R$ be such that $h_0 \mid h \mid h_1$. We consider the following conditions.*

($\dagger_h$) *The R_h-module $(A_e)_h/R_h$ is free (of rank 2).*

($\ddagger_h$) *Either $n \geq 3$ and there exists $(y_1, \ldots, y_n) \in \mathcal{A}_n(K)$ such that each irreducible factor of h is a polynomial in an indeterminate in $K \oplus_{i=1}^n Ky_i$ or $n = 2$.*

(S_3) *The field extension $\mathrm{Frac}(e^\#) : \mathrm{Frac}(R) \to \mathrm{Frac}(S)$ is non-Galois.*

($\perp_h$) *Let $h_2 \in R$ be such that $h = h_0 h_2$. For each irreducible component Z of the zero locus $h_2 = 0$, $e^{-1}(Z)$ is irreducible.*

Then the following properties hold.

(1) *If condition (S_3) does not hold, then $\mathrm{char}(K) = 3$.*

(2) *Assume (S_3) holds.*

(2.a) *The field extension $\mathrm{Frac}(e^\#)$ generates a Galois extension $\mathrm{Frac}(R) \to K_0$ of Galois group $G \cong S_3$.*

(2.b) *There exists a finite morphism*

$$\phi : Y_e = \operatorname{Spec} B_e \to A^n_{K,\mathrm{t}} = \operatorname{Spec} R$$

between integral normal varieties which is generically étale of degree 2 and such that the normalization $W_e = \operatorname{Spec} C_e$ of X_e in K_0 is the normalization of $X_e \times_{\mathbb{A}^n_{K,\mathrm{t}}} Y_e$ (so $K_0 = \mathrm{Frac}(C_e)$).

(2.c) *We have $\dim\bigl(\mathrm{Sing}(Y_e)\bigr) \leq n-2$ and the finite morphism $W_e \times_{Y_e} \mathrm{Reg}(Y_e) \to \mathrm{Reg}(Y_e)$ is Galois.*

(2.d) *The non-étale locus of ϕ is the zero locus $h_0 = 0$ in Y_e.*

(2.e) *The R-module B_e is free of rank 2.*

(2.f) *If $\mathrm{char}(K) \neq 2$, then $B_e \cong R[z]/(z^2 - h_0)$ and hence $h_0 \notin K^*$.*

(2.g) *If $\mathrm{char}(K) = 2$, then $(B_e)_{h_0} \cong R_{h_0}[z]/(z^2 + z + f_0)$ for a suitable $f_0 \in R_{h_0}$.*

(2.h) *If $\mathrm{char}(K) = 2$, then there exists a triple $(g_0, g_1, l) \in R^2 \times \mathbb{N}$ such that $g_1 \mid h_0^l$ and $B_e \cong R[z]/(z^2 + h_0 g_1 z + g_0)$.*

(2.i) *We have $\mathrm{ciedim}(Y_e) \leq n+1$.*

(2.j) *If $\dim\bigl(\mathrm{Sing}(Y_e)\bigr) \leq n-3$, then ϕ is étale.*

(3) *If ψ_e is étale and condition (S_3) holds, then $\mathrm{char}(K) = 2$.*

(4) *We have ($\ddagger_h$) $\Rightarrow$ ($\dagger_h$). Even more, if condition ($\ddagger_h$) holds then for each $m \in \mathbb{N}$, every finitely generated projective $R_h[x_{n+1}, \ldots, x_{n+m}]$-module is free.*

(5) *Assume condition ($\dagger_h$) holds. Then there exists a unimodular quadruple $(\alpha, \beta, \gamma, \delta) \in R_h^4$ such that by denoting $L(x,y) := \alpha x^3 + \beta x^2 y + \gamma x y^2 + \delta y^3 \in R_h[x,y]$,* *for each R_h-algebra A, the A-algebra $(A_e)_h \otimes_{R_h} A$ is generated by 1 element iff there exists $(u, v) \in A^2$ such that $L(u, v) \in A^*$.*

(6) *If conditions ($\dagger_h$) and ($\perp_h$) hold, then there exists a pair $(v_1, v_2) \in (A_e)_h^2$ such that $L(v_1, v_2) \in R_h^*$.*

(7) *If condition ($\perp_h$) holds and the R_h-algebra $(A_e)_h$ is generated by 1 element, then* $\mathrm{char}(K) = 3$.

(8) *If condition (S_3) holds, then we have inequalities* $\mathrm{ciedim}(\mathrm{Spec}\, R_h) \leq n+1$, $\mathrm{ciedim}\big(\mathrm{Spec}\,(B_e)_h\big) \leq n+2$ *if* $\mathrm{char}(K) = 2$, $\mathrm{ciedim}\big(\mathrm{Spec}\,(B_e)_h\big) \leq n+1$ *if* $\mathrm{char}(K) \neq 2$, $\mathrm{ciedim}\big(\mathrm{Spec}\,(A_e)_h\big) \leq 2n+2$ *and* $\mathrm{ciedim}\big(\mathrm{Spec}\,(C_e)_h\big) \leq 2n+2$.

(9) *Assume condition ($\dagger_h$) holds. We consider an R_h-algebra presentation $(A_e)_h = R_h[y_1, y_2]/I$ such that $\{1+h, y_1+I, y_2+I\}$ is an R_h-basis of $(A_e)_h$. Then the ideal I is generated by 3 polynomials of degree 2 in y_1 and y_2. Moreover, if condition ($\ddagger_h$) holds, then I is generated by 2 elements iff the $R_h[y_1, y_2]$-module $\mathrm{Ext}^1_{R_h[y_1,y_2]}(I, R_h[y_1, y_2])$ is cyclic, and in such a case, if condition (S_3) also holds, we have inequalities* $\mathrm{ciedim}\big(\mathrm{Spec}\,(A_e)_h\big) \leq n+3$ *and* $\mathrm{ciedim}\big(\mathrm{Spec}\,(C_e)_h\big) \leq n+3$.

(10) *Assume that* $\mathrm{char}(K) \neq 3$ *and condition (S_3) holds. Let*

$$< e_h > \in H^1_{\text{ét}}\big(\mathrm{Spec}\,(B_e)_h, (\mathbb{Z}/3\mathbb{Z})_K\big)$$

be the class that defines the Galois cover $\mathrm{Spec}\,(C_e)_h \to \mathrm{Spec}\,(B_e)_h$. *Let M_h be a finitely generated projective $(B_e)_h$-module of rank 1 whose class*

$$< M_h > \in H^1_{\text{ét}}\big(\mathrm{Spec}\,(B_e)_h, \mathbb{G}_{m,K}\big)$$

is the image of $< e_h >$ via the monomorphism $(\mathbb{Z}/3\mathbb{Z})_K \to \mathbb{G}_{m,K}$.

(10.a) *Assume that $M_h \cong (B_e)_h$. Then there exists $u \in (B_e)_h^*$ such that $(C_e)_h \cong (B_e)_h[t]/(t^3 - u)$ and we have inequalities* $\mathrm{ciedim}(\mathrm{Spec}\,(C_e)_h) \leq n+2$ *and* $\mathrm{ciedim}(\mathrm{Spec}\,(A_e)_h) \leq n+3$ *if* $\mathrm{char}(K) \neq 2$ *and an inequality* $\mathrm{ciedim}(\mathrm{Spec}\,(C_e)_h) \leq n+3$ *if* $\mathrm{char}(K) = 2$.

(10.b) *There exists a finitely projective B_e-module M of rank 1 such that the notations match, i.e., $M_h = M \otimes_{B_e} (B_e)_h$, iff there exists a closed subvariety Z of the zero locus $h = 0$ of Y_e and a line bundle over $Y_e \setminus Z$ such that $\dim(Z) \leq n-4$ and the line bundle restricts to line bundle over* $\mathrm{Spec}\,(B_e)_h \setminus Z$ *that defines $< M_h >$.*

(10.c) *If $\dim\big(\mathrm{Sing}(Y_e)\big) \leq n-3$, then the class $< e_h >$ is the image of a class $< e > \in H^1_{\text{ét}}\big(\mathrm{Spec}\, B_e, (\mathbb{Z}/3\mathbb{Z})_K\big)$ and hence there exists a finitely projective B_e-module M of rank 1 such that the notations match, i.e., $M_h = M \otimes_{B_e} (B_e)_h$.*

(11) *Assume that conditions ($\ddagger_h$) and (S_3) hold and* $\mathrm{char}(K) \neq 3$.

(11.a) *The $(R_e)_h$-module M_h is free of rank 2.*

(11.b) *The $(B_e)_h$-module M_h is generated by 2 elements and $M_h \oplus M_h^{\otimes 2} \cong (B_e)_h^2$.*

(11.c) *Assume* $\mathrm{char}(K) \neq 2$. *Then there exists $(\beta_z, \gamma_z, \delta_z) \in (R_e)_h^3$ such that $(\beta_z, \gamma_z) \in (R_e)_h^2$ is unimodular, the identity $\delta_z^2 + \beta_z \gamma_z = h$ holds, by defining*

$$L_2(x, y) := \gamma_z x^2 + 2\delta_z xy - \beta_z y^2 \in (A_e)_h[x, y],$$

we have $M_h \cong (B_e)_h$ iff there exists $(u, v) \in (B_e)_h^2$ such that $L_2(u, v) \in (B_e)_h^*$, and the cocycle that defines the class $< M_h >$ is the unit $z + \delta_z \in (B_e)^*_{h\beta_z\gamma_z}$.

(12) *Assume that* $\mathrm{char}(K) \neq 3$, *condition (S_3) holds, and there exists a finitely projective B_e-module M of rank 1 such that $M_h = M \otimes_{B_e} (B_e)_h$.*

(12.a) *The R_e-module M is free of rank 2.*

(12.b) *The B_e-module M is generated by 2 elements. If* $\dim\bigl(\mathrm{Sing}(Y_e)\bigr) \le n-3$, *then we can choose M such that* $M \oplus M^{\otimes 2} \cong B_e^2$.

(12.c) *Assume* $\mathrm{char}(K) \neq 2$. *Then there exists* $(\beta_z, \gamma_z, \delta_z) \in R_e^3$ *such that* $(\beta_z, \gamma_z) \in R_e^2$ *is unimodular, the identity* $\delta_z^2 + \beta_z\gamma_z = h$ *holds, for* $L_2(x,y)$ *as in part (11.c) we have* $M \cong B_e$ *iff there exists* $(u,v) \in B_e^2$ *such that* $L_2(u,v) \in B_e^*$, *and the cocycle that defines the class* $< M >$ *is the unit* $z + \delta_z \in (B_e)^*_{\beta_z\gamma_z}$.

Proof. Part (1) holds as we have $\mathrm{char}(K) \neq 0$ by [Raz], Thm. 2 and $\mathrm{char}(K) \neq p$ with p a prime different from 3 by Corollary 25.3.

Parts (2.a) and (2.b) are standard Galois theory. To show that the morphism $W_e \times_{Y_e} \mathrm{Reg}(Y_e) \to \mathrm{Reg}(Y_e)$ is Galois (equivalently, is étale) we can assume that $X_e \neq X_e^{\text{ét}}$; thus $h_0 \notin K$. Let $Y \in \mathrm{Irr}(N_e^{\mathrm{n-ét}})$ and let η be its generic point; so $\bigl(\psi_e^{-1}(\eta)\bigr)_{\mathrm{red}}$ consists of 2 points $\eta_1 \in \mathrm{Im}(\imath_e)$ and $\eta_2 \in X_e \setminus X_e^{\text{ét}} \subset X_e \setminus \mathrm{Im}(\imath_e)$. Thus $e^{-1}(Y)$ is irreducible and η_1 is ramified over η. Also $\bigl(\phi^{-1}(\eta)\bigr)_{\mathrm{red}}$ is one point η_0 of Y_e which is ramified over η. The inverse image of η_0 in W_e is at least 2 points and hence it is 3 points, thus $W_e \to Y_e$ is étale at these 3 points. As $\deg(e) = 3$ implies that the generic points of $\psi_e^{-1}\bigl(\psi_e(X_e^{\text{ét}} \setminus \mathrm{Im}(\imath)_e)\bigr)$ are contained in $X_e^{\text{ét}}$, it follows that $W_e \to Y_e$ is étale in codimension 1 and hence it is étale outside a codimension at least 2 closed subset of W_e. Thus, as the non-étale locus of $W_e \times_{Y_e} \mathrm{Reg}(Y_e) \to \mathrm{Reg}(Y_e)$ is either empty or of pure codimension 1 by the affineness of étale loci (see [Stacks1] or [GZ], Thm. 1), we get that it is empty. So $W_e \times_{Y_e} \mathrm{Reg}(Y_e) \to \mathrm{Reg}(Y_e)$ is a finite étale cover. As Y_e is normal, $\mathrm{Sing}(Y_e)$ has codimension at least 2 in Y_e. So part (2.c) holds. The images in $\mathbb{A}^n_{K,\mathrm{t}}$ of the non-étale loci of ϕ and $W_e \to \mathbb{A}^n_{K,\mathrm{t}}$ are $\mathrm{Spec}\, R/Rh_0$. From this and the ramification of ϕ at η_0s we get that part (2.d) holds.

For part (2.e), we have a direct sum decomposition $B_e = R \oplus B_1$ of R-modules by [Ho], Thm. 2. The R-module B_1 is torsionfree and the coherent $\mathcal{O}_{\mathrm{Spec}\, R}$-module it defines is normal (as B_e is normal) in the sense of Barth and hence it is reflexive by [H3], Prop. 1.6 and Def. before it. From this, as B_1 has rank 1, it follows that $B_1 \cong R$ by [H3], Prop. 1.9. Thus part (2.e) holds.

If $\mathrm{char}(K) \neq 2$, then we can identify $B_e = R[z]/(z^2 - \tilde{h}_0)$ with $\tilde{h}_0 \in R \setminus K$ a product of non-associated irreducible polynomials. The non-étale locus of ϕ is defined by $\tilde{h}_0$. From this and part (2.d) we get that the zero loci $h_0 = 0$ and $\tilde{h}_0 = 0$ in $\mathbb{A}^n_{K,\mathrm{t}}$ coincide. Thus we can assume that $\tilde{h}_0 = h_0$, hence part (2.f) holds.

Part (2.g) follows from part (2.d) and Fact 25.1.

To prove part (2.h), let $z \in B_e$ be such that $B_e = R \oplus Rz$. Thus we can identify $B_e = R[z]/(z^2 + f_1 z + g_0)$ with $(f_1, g_0) \in R^2$. We have $f_1 \in R^*_{h_0}$ by part (2.g). From part (2.d) we get that $h_0 \mid f_1$ in R, i.e., there exists $g_1 \in R$ such that $f_1 = h_0 g_1$. As $g_1 = f_1 h_0^{-1} \in R^*_{h_0}$, the existence of $l \in \mathbb{N}$ follows. Thus part (2.h) holds.

Part (2.i) follows from definitions and parts (2.f) and (2.h). Part (2.j) follows from part (2.i) and [SGA2], Exp. X, Thm. 3.4(ii).

Part (3) follows from part (2.f).

Based on Quillen–Suslin Theorem (e.g., see [Lang-S2], Ch. XXI, Sect. 3, Thm. 3.7) and [Roi], Prop. 2, to prove part (4) we can assume that $m = 0$. The case $n = 2$ follows from the fact that each vector bundle over $\mathrm{Spec}\, R_h$ extends to a vector bundle over $\mathrm{Spec}\, R = \mathbb{A}^2_{K,\mathrm{t}}$ by [SGA2], Exp. X, Lem. 3.5 and hence it is trivial by [Ses], Thm. The case $n \geq 3$ is proved in [Ga2], Thm. 2.1. Thus part (4) holds.

To prove part (5), we consider an R_h-basis $\{1, w_1, w_2\}$ of $(A_e)_h$. The A-algebra $(A_e)_h \otimes_{R_h} A$ is generated by 1 element iff there exists $(u, v) \in A^2$ such that A is generated by $w_1 \otimes u + w_2 \otimes v$ and iff, by writing

$$(w_1 \otimes u + w_2 \otimes v)^2 = 1 \otimes t_2 + w_1 \otimes u_2 + w_2 \otimes v_2$$

with $(t_2, v_2, w_2) \in A^3$, the matrix $\triangle_{u,v} := \begin{bmatrix} u & v \\ u_2 & v_2 \end{bmatrix}$ is invertible, i.e., is in $\mathbf{GL}_2(A)$. As u_2 and v_2 are homogeneous polynomials in u and v of degree 2 with coefficients in R_h that do not depend on A, there exists $(\alpha, \beta, \gamma, \delta) \in R_h^4$ such that by defining $L(x, y) := \alpha x^3 + \beta x^2 y + \gamma x y^2 + \delta y^3 \in R_h[x, y]$ we have $\det(\triangle_{u,v}) = L(u, v)$. As $(A_e)_h$ modulo each maximal ideal of R_h is generated by 1 element, $(\alpha, \beta, \gamma, \delta) \in R_h^4$ is unimodular. We conclude that part (5) holds.

If condition $(\perp_h)$ holds, then $e^{-1}(Y)$ is irreducible for each Y irreducible component of the zero locus $h = 0$ (for $Y \in \mathrm{Irr}(N_e^{\text{n-ét}})$ see the proof of part (2.c)). Hence the homomorphism $R_h \to (A_e)_h$ induces an isomorphism $R_h^* \to (A_e)_h^*$.

For part (6), the $(A_e)_h$-algebra $(A_e)_h \otimes_{R_h} (A_e)_h$ is a product $(A_e)_h \times A$ with A as an $(A_e)_h$-algebra which as an $(A_e)_h$-module is free of rank 2 and for which the $(A_e)_h$-module $A/(A_e)_h$ is free of rank 1. So the $(A_e)_h$-algebra $(A_e)_h \otimes_{R_h} (A_e)_h$ is generated by $(0, w_0) \in (A_e)_h \times A$ such that $A/(A_e)_h = (A_e)_h[w_0 + (A_e)_h]$. Thus there exists $(v_1, v_2) \in (A_e)_h$ such that $L(v_1, v_2) \in (A_e)_h^* = R_h^*$ and part (6) holds. To prove part (7), let $x_h \in A_e$ be such that its image in $(A_e)_h$ generates the R_h-algebra $(A_e)_h$. Let $\pi_h : \mathbb{A}^n_{K,\mathrm{s}} \to \mathbb{A}^1_K$ be defined by x_h. From the previous paragraph we get that e is of weak type 1 by Definition 22.2 applied to $\pi_h \times e$. From this and Corollary 25.7 we get that $\mathrm{char}(K) = 3$. Thus part (7) holds.

We have $\mathrm{ciedim}(\mathrm{Spec}\, R_h) \leq n+1$ as $R_h \cong K[x_1, \ldots, x_{n+1}]/(x_{n+1}h - 1)$. Based on this, the inequality $\mathrm{ciedim}\big(\mathrm{Spec}\,(B_e)_h\big) \leq n+2$ (resp. $\mathrm{ciedim}\big(\mathrm{Spec}\,(B_e)_h\big) \leq n+1$) follows from part (2.g) (resp. part (2.f) and the isomorphism $(B_e)_h \cong K[x_1, \ldots, x_{n+1}]/(x_{n+1}^2 h - 1)$). For $\triangleleft \in \{A, C\}$ the morphism $\mathrm{Spec}\,(\triangleleft_e)_h \to \mathrm{Spec}\, R_h$ is étale. So $T_{\mathrm{Spec}\,(\triangleleft_e)_h}$ is trivial, thus $\mathrm{ciedim}\big(\mathrm{Spec}\,(\triangleleft_e)_h\big) \leq 2n+2$ by Theorem 7.2(2) and (2.f). Hence part (8) holds.

For part (9), note that the ideal $I = \big(y_1^2 - \ell_1(y_1, y_2), y_1 y_2 - \ell_{12}(y_1, y_2), y_2^2 - \ell_2(y_1, y_2)\big)$ of $R_h[y_1, y_2]$ is defined by polynomials $\ell_1, \ell_{12}, \ell_2 \in R_h[y_1, y_2]$ of degree 1 (it is easy to see that the 3 polynomials cannot be of degree 0). As locally in the Zariski topology of $\mathrm{Spec}\, R_h$ the R_h-algebra $(A_e)_h$ is generated by 1 element (e.g., this follows from part (5)), locally in the Zariski topology of $\mathrm{Spec}\, R_h$ the ideal I is generated by 2 elements and hence, as $R_h[y_1, y_2]$ and $(A_e)_h$ are regular, the homological dimension of I is at most 1. Based on this, part (9) follows from [Ser1], Part II, Sect. 4, Cor.; note that if I is generated by 2 elements f and g, then $(A_e)_h \cong K[x_1, \ldots, x_{n+3}]/(x_{n+1}h - 1, f, g)$ and $(C_e)_h \cong K[x_1, \ldots, x_{n+3}]/(x_{n+1}^2 h - 1, f, g)$.

For part (10), as $\mathrm{char}(K) \neq 3$, there exists $w \in K_0$ that generates K_0 over $\mathrm{Frac}(B_e)$ and such that $u := w^3 \in \mathrm{Frac}(B_e)$ by Kummer theory; we can assume that $w \in C_e$ and that $u \in B_e$ is cubic free. As $\mathrm{Spec}\,(C_e)_h \to \mathrm{Spec}\,(B_e)_h$ is a finite Galois cover between regular varieties of degree 3, for each point $P \in \mathrm{Spec}\,(B_e)_h(K)$, there exists an affine open subvariety U_P of $\mathrm{Spec}\,(B_e)_h$ which contains P and an element $u_P \in \mathrm{Frac}(B_e)$ such that uu_P^3 is an invertible global function on U_P. The set $\{(U_P, uu_P^3) | P \in \mathrm{Spec}\,(B_e)_h(K)\}$ is a Cartier divisor that defines $< M_h >$. As $M_h \cong (B_e)_h$, we can assume that $u \in (B_e)_h^*$, hence $(C_e)_h \cong (B_e)_h[t]/(t^3 - u)$. If $\mathrm{char}(K) = 2$, then from this and part (2.g) we get that $\mathrm{ciedim}\big(\mathrm{Spec}\,(C_e)_h\big) \leq n+3$.

In this and the next two paragraphs we assume that $\mathrm{char}(K) \neq 2$. We have $\mathrm{ciedim}\big(\mathrm{Spec}\,(C_e)_h\big) \leq n+2$ as $(C_e)_h \cong K[x_1,\ldots,x_{n+2}]/(x_{n+1}^2 h-1, x_{n+2}^3-u)$.

As $\mathrm{char}(K) \neq 2$, we write $u = f+zg$ with $(f,g) \in R^2$ by part (2.f). Let $\tau \in G$ be the unique element of order 2 that fixes A_e. Then $\tau(z) = -z$, $x := w\tau(w) \in A_e$ and $v := u\tau(u) = f^2 - hg^2 \in R \cap R_h^*$. As $x^3 = v$, $\mathrm{Frac}(R) \to \mathrm{Frac}(A_e)$ is non-Galois, it follows that $x \in \mathrm{Frac}(R) \cap A_e = R$ and there exists $q \in \mathbb{N}$ such that $x|h^q$.

We have $\tau(w) = \frac{x}{w}$ and $\tau(u) = \frac{v}{u}$. As $u \in (B_e)_h^*$, $\{1, z, w, wz, w^2, w^2 z\}$ is an R_h-basis of $(C_e)_h$. As τ interchanges $Rw \oplus Rwz$ and $Rw^2 \oplus Rw^2 z$ and $\tau(R \oplus Rz) = R \oplus Rz$ and $(A_e)_h := \{x \in (C_e)_h | \tau(x) = x\}$, it follows that $\{1, w+\tau(w), wz+\tau(wz)\}$ is an R_h-basis of $(A_e)_h$ and condition $(\dagger_h)$ holds. We apply parts (5) and (9) to $(w_1, w_2) := \big(w+\tau(w), wz+\tau(wz)\big)$. As $w := \frac{w_1}{2} + \frac{w_2 z}{2h}$ generates the $(B_e)_h$-algebra $(C_e)_h$, we have $L(\frac{1}{2}, \frac{z}{2h}) \in (B_e)_h^*$. As $(C_e)_h \cong (B_e)_h[t]/(t^3-u)$ under the isomorphism that maps w to $t+(t^3-u)$, the $(B_e)_h[y_1,y_2]$-module

$$\mathrm{Ext}^1_{(B_e)_h[y_1,y_2]}(I \otimes_{R_h} \otimes (B_e)_h, (B_e)_h[y_1,y_2]\big) = \mathrm{Ext}^1_{R_h[y_1,y_2]}(I, R_h[y_1,y_2]) \otimes_{R_h} (B_e)_h$$
$$= \mathrm{Ext}^1_{R_h[y_1,y_2]}\big(I, R_h[y_1,y_2]\big) \otimes_{R_h[y_1,y_2]} (B_e)_h[y_1,y_2]$$

is cyclic isomorphic to $(B_e)_h[y_1,y_2]/I_z$ for some ideal I_z of $(B_e)_h[y_1,y_2]$. As there exists an $(B_e)_h[y_1,y_2]$-linear isomorphism $(B_e)_h[y_1,y_2]/I_z \cong (B_e)_h[y_1,y_2]/\tau(I_z)$, we have $\tau(I_z) = I_z$ and Galois descend implies that the $R_h[y_1,y_2]$-module $\mathrm{Ext}^1_{R_h[y_1,y_2]} \cong R_h[y_1,y_2]/\{y \in I_z | \tau(y) = y\}$ is cyclic. So $\mathrm{ciedim}\big(\mathrm{Spec}\,(A_e)_h\big) \leq n+3$ by part (9), and part (10.a) holds.

Part (10.b) follows from part (2.i) and [SGA2], Exp. XI, Thm. 3.13(ii). Part (10.c) follows from part (2.j): we can take $< e > \in H^1_{\text{ét}}\big(\mathrm{Spec}\,B_e, (\mathbb{Z}/3\mathbb{Z})_K\big)$ to define the Galois cover $\mathrm{Spec}\,C_e \to \mathrm{Spec}\,B_e$ and $< M > \in H^1_{\text{ét}}\big(\mathrm{Spec}\,B_e, \mathbb{G}_{m,K}\big)$ to be the image of $< e >$ via the monomorphism $(\mathbb{Z}/3\mathbb{Z})_K \to \mathbb{G}_{m,K}$.

Part (11.a) is a particular case of part (4). The first part of (11.b) follows from part (11.a). As M_h is generated by two elements by part (11.a), there exists a projective $(B_e)_h$-module $M_h^\vee$ such that $M_h \oplus M_h^\vee \cong (B_e)_h^2$. Taking determinants, we get that $M_h^\vee$ is the dual of M_h. As $< M_h >$ is a class of order 1 or 3, $M_h^{\otimes 3} \cong (B_e)_h$, hence $M_h^\vee \cong M_h^{\otimes 2}$ and part (11.b) holds. We identify $M_h = R_h^2$ as R_h-modules.

Assume $\mathrm{char}(K) \neq 2$. As $(B_e)_h = R_h[z]/(z^2-h)$, we have $M_h \cong (B_e)_h$ as $(B_e)_h$-modules iff there $w_h \in M_h$ such that $\{w_h, zw_h\}$ is an R_h-basis of M_h. Let $\{w_1, w_2\}$ be the standard basis of R_h^2. Let the matrix $\triangle_z = \begin{bmatrix} \alpha_z & \beta_z \\ \gamma_z & \delta_z \end{bmatrix} \in \mathbf{GL}_2(R_h)$ define the action of z on M_h. We have $\triangle_z^2 = \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix}$. As h is not a square in R_h, a simple computation gives identities $\alpha_z = -\delta_z$ and $\delta_z^2 + \beta_z\gamma_z = h$. The $(B_e)_h$-module M_h is projective iff the zero locus $\beta_z = \gamma_z = 0$ in $\mathrm{Spec}\,R_h$ is the empty set and iff (β_z, γ_z) is unimodular. For $(u,v) \in R_h$, by taking $w_h := uw_1 + vw_2$, $\{w_h, zw_h\}$ is an R_h-basis of M_h iff $\begin{bmatrix} u & v \\ -\delta_z u + \beta_z v & \gamma_z u + \delta_z v \end{bmatrix} \in \mathbf{GL}_2(R_h)$ and iff $L_2(u,v) \in R_h^*$. The cocycle part follows from the identities $(B_e)_{h\gamma_z} w_1 = (M_h)_{\gamma_z}$, $(B_e)_{h\beta_z} w_2 = (M_h)_{\beta_z}$ and $w_2 = \frac{z+\delta_z}{\gamma_z} w_1$. Thus part (11.c) holds.

The proof of part (12) is entirely the same as of part (11) once we note that the R-module M is free by Quillen–Suslin Theorem. $\square$

Remark 26.2. Assume $p = 2$. Let $R = k[x_1,\ldots,x_n]$. Conjecture 2.10 predicts that there exists no $e \in \mathrm{EE}_2(k)$ such that $\deg(e) = 3$, ψ_e is étale and condition (S_3)

holds for $K = k$. Based on [MW], Cor. 2.2(b_1) and Thm. 3.1 it suffices to show that there exists no $F \in \mathrm{Frac}(R)$ such that the following properties hold:

(i) The polynomials $t^3 + t + F, t^2 + Ft + 1 \in \mathrm{Frac}(R)[t]$ are irreducible and the normalizations of R in $\mathrm{Frac}(R)[t]/(t^3+t+F)$ and $\mathrm{Frac}(R)[t]/(t^2+Ft+1)$ are étale R-algebras A and B (respectively).

(ii) The affine variety $\mathrm{Spec}\, A$ contains an open subvariety isomorphic to $\mathbb{A}_k^n$.

Note that for $n \geq 2$ there exists $F \in \mathrm{Frac}(R)$ such that property (i) holds, property (ii) does not hold, $A^* = k^*$, $\mathrm{Spec}\,(A)$ is rational and moreover A is factorial. To check this we can assume that $n \in \{2, 3\}$ and this case follows from Theorem 2.7(1) and (2) and Remarks 19.1 and 19.2 for $n \in \{2, 3\}$) based on [MW], Cor. 2.2(b_1) and Thm. 3.1. Explicit Fs for $n = 3$ can be computed based on Remark 19.3.

27. Appendix: weak separations

Definition 27.1. *Let X be a normal scheme of finite type over a field $\mathcal{K}$. Let $\overline{\Delta}_X$ be the closure in $X \times_{\mathrm{Spec}\,\mathcal{K}} X$ of the image Δ_X of the diagonal immersion $X \to X \times_{\mathrm{Spec}\,\mathcal{K}} X$, and let $U \subset \overline{\Delta}_X$ be the (open) locus where both projections to X are quasi-finite. We say that X is weakly separated if $U = \Delta_X$.*

Note that U is open in $\overline{\Delta}_X$ by Chevalley's Semi-continuity Theorem (see [Gro4], Thm. (13.1.3)).

Proposition 27.2. *With notation of Definition 27.1, the following properties hold.*

(1) *Both projections $\pi_1 : U \to X$ and $\pi_2 : U \to X$ are local isomorphisms.*

(2) *The morphism $U \to X \times_{\mathrm{Spec}\,\mathcal{K}} X$ is an equivalence relation.*

Before proving this proposition, we first make a second definition and prove a lemma needed in the proof.

Definition 27.3. *Let X be a normal scheme of finite type over a field $\mathcal{K}$. By the weak separation of X we mean the quotient $X_{\mathrm{w}} := X/U$ of X by the equivalence relation U (quotient by an equivalence relation of big Zariski sheaves or quotient in the category of ringed spaces).*

Lemma 27.4. *Assume we have a commutative diagram* $\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & Y & \end{tikzcd}$ *of integral schemes of finite type over a field $\mathcal{K}$ with Y normal, f dominant quasi-finite and g birational. Then $h(X)$ is contained in an open subscheme Ω of Z such that the morphism $\Omega \to Y$ induced by g is a local isomorphism.*

Proof. It suffices to show that for each $x \in X$, the homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{Z,h(x)}$ of local rings induced by g is an isomorphism. To show this we can assume that X, Y and Z are affine and we can replace X and Z by their normalizations. Then X is open in the normalization W of Z in the field of fraction of X by Zariski's Main Theorem. By the going down property $h(X)$ is open in Z. So $h(X) \to Y$ is quasi-finite birational and thus an open embedding by Zariski's Main Theorem. $\square$

Proof of Proposition 27.2. Part (1) follows from Lemma 27.4 applied to $h = 1_U$. The relation defined by the monomorphism $U \to X \times_{\mathrm{Spec}\,\mathcal{K}} X$ is clearly reflexive and symmetric by definition. To show that it is transitive we can assume X is integral and we consider the factorization $h : U \times_{\pi_2,X,\pi_1} U \to \overline{\Delta}_X$ of the morphism $U \times_{\pi_2,X,\pi_1} U \to X \times_{\operatorname{Spec}\mathcal{K}} X$ defined by the following rule on valued points $(u,v) \mapsto \big(\pi_1(u), \pi_2(v)\big)$. From Lemma 27.4 applied to g equal to one of the two projections of $\overline{\Delta}_X$ on X it follows that h factors through an open subscheme Ω of $\overline{\Delta}_X$ with the property that both projections from it to X are local isomorphisms. Hence Ω is contained in U. Thus the relation is also transitive, so part (2) also holds.

Lemma 27.4 implies that the association $X \mapsto U$ is functorial with respect to dominant quasi-finite morphisms between integral normal schemes of finite type over $\mathcal{K}$. So dominant quasi-finite morphisms between integral normal schemes of finite type over $\mathcal{K}$ induce morphisms on weak separations. One verifies that $\overline{\Delta}_X$ is the inverse image under the morphism $X \times_{\operatorname{Spec}\mathcal{K}} X \to X_{\mathrm{w}} \times_{\operatorname{Spec}\mathcal{K}} X_{\mathrm{w}}$ of the diagonal closure $\overline{\Delta}_{X_{\mathrm{w}}}$ in $X_{\mathrm{w}} \times_{\operatorname{Spec}\mathcal{K}} X_{\mathrm{w}}$ of the image of the diagonal immersion $X_{\mathrm{w}} \to X_{\mathrm{w}} \times_{\operatorname{Spec}\mathcal{K}} X_{\mathrm{w}}$. It follows that weak separations are weakly separated.

Example 27.5. Let $d : C \to C$ be an endomorphism of a curve over K that permutes the generic points of its irreducible components. Then $d^{|\mathrm{Irr}(C)|!}$ fixes these generic points and hence induces dominant endomorphisms of its irreducible components. So to prove that $\iota(d) \in \mathbb{N}$ we can assume that C is irreducible. The weak separation C_{w} of C is the integral separated curve obtained from C by identifying the points whose local rings coincide inside the field of fractions of C. The functorial endomorphism $d_{\mathrm{w}} : C_{\mathrm{w}} \to C_{\mathrm{w}}$ induced by d is finite surjective by Lemma 10.5(3). By considering an open non-empty subset U of C_{w} isomorphic to its inverse image in C, we have $U \subset \mathrm{Im}(d^i)$ for all $i \in \mathbb{N}$. Thus $\iota(d) \le |C \setminus U|$.

For an endomorphism $e : X \to X$ of a variety over K and $P \in X(K)$ we consider the following condition.

($\mathrm{w}\natural_{e,P}$) *The condition* ($\flat_{e,P}$) *holds and for each* $n \in \mathbb{N}^*$ *the Zariski closure of the set* $\{e^m(P) | m \in \mathbb{N}, m \ge n, e^{-1}\big(e^m(P)\big) = \{e^{m-n}(P)\}\}$ *contains a non-empty open subvariety* Ω *of an irreducible component of positive dimension of the Zariski closure* $Y_{0,1}$ *of the orbit* $\{e^i(P) | i \in \mathbb{N}\}$.

Note that if ($\mathrm{w}\natural_{e,P}$) holds and $m \in \mathbb{N}^*$ is such that $e^m(P) \in \Omega$, then ($\mathrm{w}\natural_{d,e^m(P)}$) holds for which endomorphism $d : Y \to Y$ induced by e with Y a locally closed subvariety of X that contains $e^m(P)$. Clearly, ($\natural_{e,P}$)$\Rightarrow$ ($\mathrm{w}\natural_{e,P}$)$\Rightarrow$ ($\flat_{e,P}$) and hence the following proposition is a weaker form of Theorem 16.8(2).

Proposition 27.6. *Let* $e : X \to X$ *be a quasi-finite endomorphism of a variety over* K. *Then there exists no* $P \in X(K)$ *for which condition* ($\mathrm{w}\natural_{e,P}$) *holds.*

Proof. We show that the assumption that there exists $P \in X(K)$ for which condition ($\mathrm{w}\natural_{e,P}$) holds leads to a contradiction. We can assume that $Y_{0,1} = X$. Let $s \in \mathbb{N}^*$ be such that e^s fixes the generic points of irreducible components of the equidimensional variety $X^- := \cup_{Y \in \mathrm{Irr}(X), \dim(Y) > 0} Y$ by Lemma 16.14(1). Let $Z \in \mathrm{Irr}(X^-)$ and let $d : Z \setminus Z$ be induced by e^s. Let $l \in \mathbb{N}$ be the smallest such that $e^l(P) \in Z$. As ($\mathrm{w}\natural_{e,P}$) holds, so do ($\mathrm{w}\natural_{d,e^l(P)}$) and ($\flat_{d,e^m(P)}$). We choose Z such that each point $e^m(P)$ in some non-empty open subvariety of Z satisfies $d^{-1}\big(e^m(P)\big) = \{e^{m-s}(P)\}$. In particular, $d : Z \to Z$ is dominant (being quasi-finite with Z irreducible) and generically radicial, and the same holds for its normalization $d^{\mathrm{n}} : Z^{\mathrm{n}} \to Z^{\mathrm{n}}$. From Zariski's Main Theorem we get that the functorial weak separation $d^{\mathrm{n}}_{\mathrm{w}} : Z^{\mathrm{n}}_{\mathrm{w}} \to Z^{\mathrm{n}}_{\mathrm{w}}$ is a bijection. By considering an open non-empty subset

Ω of Z naturally isomorphic to $(d^{\mathrm{n}})^{-1}(U)$ and its image in $Z^{\mathrm{n}}_{\mathrm{w}}$, we have $U \subset \mathrm{Im}(d^i)$ for all $i \in \mathbb{N}$, which contradicts $(\flat_{d,e^m(P)})$. $\square$

Corollary 27.7. *Let $f : X \to Y$ be a birational morphism between irreducible varieties over K. Assume there exist endomorphisms $e : X \to X$ and $d : Y \to Y$ such that $d \circ f = f \circ e$. Let $Q \in Y(K)$ be such that its orbit $\{d^i(Q)|i \in \mathbb{N}\}$ is Zariski dense in Y. Let $i \in \mathbb{N}$ be the smallest such that there exists $P \in X(k)$ such that $f(P) = d^i(Q)$. If $(\mathrm{w}\natural_{d,Q})$ holds, then e is not quasi-finite.*

Proof. As $(\mathrm{w}\natural_{d,Q})$ holds, so does $(\mathrm{w}\natural_{d,P})$; so corollary holds by Proposition 27.6. $\square$

Remark 27.8. In the proof of Proposition 16.15, instead of its last paragraph, from Corollary 27.7 applied to the birational composite morphism $\Omega \to Z^{\mathrm{n}} \to Z$, the endomorphisms $e_\Omega : \Omega \to \Omega$ and $e_Z : Z \to Z$, and the point $e^m(P) \in Z(K)$, we get that that e_Ω is not quasi-fine, a contradiction.

28. Appendix: on pullbacks of Lang torsors

Let κ be a subfield of k such that the field extension $\kappa \to k$ is algebraic. Let $X = \mathrm{Spec}\, R$ be an integral affine scheme of finite type over $\mathrm{Spec}\, \kappa$ and of dimension at least 1. To construct Galois covers over X that are pullbacks of Lang torsors, we first prove the following general proposition.

Proposition 28.1. *Let $\mathcal{K}$ be a subfield of K such that the field extension $\mathcal{K} \to K$ is algebraic. Let $\mathcal{G}$ be a smooth connected linear algebraic group over $\mathcal{K}$. Let $R_u(\mathcal{G}_K)$ be the unipotent radical of $\mathcal{G}_K$. Then the following properties hold.*

(1) *Assume there exists $N \in \mathbb{N}^*$ and a dominant morphism $\Sigma : \mathbb{A}^N_K \to \mathcal{G}_K$ whose general fibers are geometrically integral. Then $\mathcal{G}_K/R_u(\mathcal{G}_K)$ is a simply connected semisimple group.*

(2) *Assume $R_u(\mathcal{G}_K)$ is defined over $\mathcal{K}$ (i.e., the unipotent radical $R_u(G)$ of G is defined) and $\mathcal{G}_K/R_u(\mathcal{G}_K)$ is a simply connected semisimple group. We also assume that $R_u(G)$ is $\mathcal{K}$-split (e.g., this holds if $\mathcal{K}$ is perfect) and $\mathcal{G}/R_u(\mathcal{G})$ is quasi-split (e.g., this holds if $\mathcal{K}$ is finite). Then there exists $N \in \mathbb{N}^*$ and a surjective morphism $\Sigma : \mathbb{A}^N_K \to \mathcal{G}_K$ whose general fibers are geometrically integral.*

Proof. To prove part (1), we can assume that $\mathcal{K} = K$. By composing Σ with the quotient morphism $\mathcal{G} \to \mathcal{G}/R_u(\mathcal{G})$, we can assume that $R_u(\mathcal{G})$ is trivial, i.e., $\mathcal{G}$ is a reductive group. By reasons of units, each morphism $\mathbb{A}^N_K \to \mathbb{G}_{m,K}$ is constant. So $\mathbb{G}_{m,K}$ is not a quotient of $\mathcal{G}$ and thus $\mathcal{G}$ is semisimple. If $\mathcal{G}^{\mathrm{sc}}$ is the simply connected semisimple group cover of $\mathcal{G}$, then its center is of multiplicative type (see [SGA3-3], Exp. XXII, Cor. 4.1.7), hence the kernel Ker of the central isogeny $\mathcal{G}^{\mathrm{sc}} \to \mathcal{G}$ is finite of multiplicative type. As $H^1_{\text{ét}}(\mathbb{A}^N_K, \boldsymbol{\mu}_{m,K}) =$ for each $m \in \mathbb{N}^*$ and Ker is a finite product of such finite group schemes $\boldsymbol{\mu}_{m,K}$ over $\mathrm{Spec}\, K$, the Ker-torsors on $\mathbb{A}^N_K$ are trivial. So Σ factor through $\mathcal{G}^{\mathrm{sc}}$ and thus, as its general fibers are geometrically integral, it follows that Ker is trivial, i.e., $\mathcal{G} = \mathcal{G}^{\mathrm{sc}}$; so part (1) holds.

To prove part (2), let $G \cong R_u(\mathcal{G}) \times G/R_u(G)$ be an isomorphism of schemes defined over $\mathrm{Spec}\, \mathcal{K}$ by [Ros2], Thm. 1. By very definition of a $\mathcal{K}$-split unipotent group, $R_u(G)$ has a composition series whose factors are $\mathbb{G}_{a,\mathcal{K}}$, hence there exists $m \in \mathbb{N}$ such that $R_u(\mathcal{G}) \cong \mathbb{A}^m_{\mathcal{K}}$. Based on the last two sentences, we can assume that $R_u(G)$ is trivial. As $\mathcal{G}$ is a simply connected semisimple group which is quasi-split, it has a Borel subgroup and hence it has a pair of opposite Borel subgroups $B^+_{\mathcal{G}}$ and

$B^-_{\mathcal{G}}$. For $* \in \{+,-\}$, let $U^*_{\mathcal{G}}$ be the unipotent radical of $B^*_{\mathcal{G}}$. From [CGP], Part I, Ch. II, Props. 2.2.9 and 2.1.10 and Cor. 2.2.5 we get that $U^*_{\mathcal{G}}$ is $\mathcal{K}$-split. Hence each product over $\operatorname{Spec}\mathcal{K}$ of a finite number of copies of either $U^+_{\mathcal{G}}$ or $U^-_{\mathcal{G}}$ is isomorphic to an affine space over $\mathcal{K}$. Let $\mathcal{G}'$ be the subgroup of $\mathcal{G}$ generated by $U^+_{\mathcal{G}}$ and $U^-_{\mathcal{G}}$.

To prove that $\mathcal{G}' = \mathcal{G}$, in this paragraph we assume that $\mathcal{K} = K$. As $\mathcal{G}$ is a product of simply connected semisimple groups that have simple adjoints by the classification theorem [SGA3-3], Exp. XXV, Thm. 1.1, we can assume that the adjoint group of $\mathcal{G}$ is simple. As $\mathcal{G}'$ is normalized by $U^+_{\mathcal{G}}$, $U^-_{\mathcal{G}}$ and by the maximal torus $B^+_{\mathcal{G}} \cap B^-_{\mathcal{G}}$ of $\mathcal{G}$, it is a normal subgroup of $\mathcal{G}$ and hence $\mathcal{G}' = \mathcal{G}$.

As $\mathcal{G} = \mathcal{G}'$, there exists a smallest $q \in \mathbb{N}$ such that we have a surjective product morphism $\Sigma : V_q \to \mathcal{G}$ with $V_q := \prod_{i=1}^q U_i$ where U_i is $U^-_{\mathcal{G}}$ if i is odd and is $U^+_{\mathcal{G}}$ if i is even. E.g., if $\mathcal{G} = \mathbf{SL}_{2,\mathcal{K}}$ then we have $q \le 5$ as one can check easily based on the analog of Equation (18) over $\mathcal{K}$. For $i \in [[1,n]]$, let 1_i be the identity element of U_i. An induction on $j \in [[1, ji]$ based on [Ros1], Thm. 10 gives that the product morphism $\Sigma_j : V_j \to \Sigma(V_j)$ induced by Σ has a rational section (called cross-section in loc. cit.), where $V_j := \prod_{i=1}^j U_i \times_{\operatorname{Spec}\mathcal{K}} \prod_{i=j+1}^q 1_i$. So the general fibers of $\Sigma = \Sigma_q$ are geometrically integral. $\square$

Let κ' be the algebraic closure of κ in $\operatorname{Frac}(R)$; let X' be a Zariski dense open subscheme of X which is geometrically irreducible over $\operatorname{Spec}\kappa'$. Let G^{ab} be the abelianization of an abstract group G. The following result refines [Kam], Thm. and Item 2 of p. 640 from several points of view (e.g., the case $p^q = 2$ is not excluded, when κ infinite we do not requite either $\mathcal{G}$ to have trivial unipotent radical or X to be geometrically integral over $\operatorname{Spec}\kappa$).

Theorem 28.2. *Let $q \in \mathbb{N}^*$. Let $\mathcal{G}$ be a smooth connected linear algebraic group over $\mathbb{F}_{p^q}$ such that $\mathcal{G}/R_u(\mathcal{G})$ is simply connected semisimple. Then there exists a Galois cover $Y = \operatorname{Spec} S \to X$ of Galois group $G := \mathcal{G}(\mathbb{F}_{p^q})$ such that S is an integral domain and κ' is algebraically closed in* $\operatorname{Frac}(S)$ *(hence, if $Y' := Y \times_X X'$, the Galois cover $Y' \to X'$ is geometrically irreducible over κ').*

Proof. Let F_{p^q} be as in the beginning of Section 19. For the Weil restriction of scalars $\mathcal{G}' := \operatorname{Res}_{\mathbb{F}_{p^q}/\mathbb{F}_p}\mathcal{G}$ (see [CGP], Part IV, App. A.5) we have $\mathcal{G}'(\mathbb{F}_p) = G = \mathcal{G}(\mathbb{F}_{p^q})$ and $\mathcal{G}'_k = \mathcal{G}^q_k$. Thus by replacing $\mathcal{G}$ with $\mathcal{G}'$ we can assume that $q = 1$. Let $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ be the Lang torsor; it is a G-torsor defined on valued points by the rule $\square \mapsto \square \cdot F_p(\square)^{-1}$. To prove the theorem we can assume that $|G| \ge 2$, i.e., that $\dim(\mathcal{G}) \ge 1$. Recall that $\mathcal{G}/R_u(\mathcal{G})$ is quasi-split (e.g., see [Bore2], Ch. V, Prop. 16.6) and $R_u(\mathcal{G})$ is $\mathbb{F}_p$-split (e.g., see [Bore2], Ch. V, Prop. 15.5 (ii)).

Following [Kam], we consider two disjoint cases as follows.

Case 1: κ is a finite field. Based on Proposition 28.1(2), there exists $N \in \mathbb{N}^*$ for which there exists a surjective morphism $\Sigma : U^N := \mathbb{A}^N_{\mathbb{F}_p} \to \mathcal{G}$ whose general fibers are geometrically integral. Thus the pullback $\mathcal{M}_N : \mathbb{X}^N \to U^N$ of $\mathcal{L}$ via Σ is a G-torsor with $\mathbb{X}^N$ geometrically irreducible over $\operatorname{Spec}\mathbb{F}_p$.

For each irreducible G-torsor $Y' \to X'$, let κ'' be the algebraic closure of κ in the field of fractions of Y'. The inclusion $\kappa' \subset \kappa''$ is a finite Galois extension and we have a surjective homomorphism $\mathcal{G}(\mathbb{F}_p)^{\mathrm{ab}} \to \operatorname{Gal}(\kappa''/\kappa')$. In particular, if m is the exponent of $\mathcal{G}(\mathbb{F}_p)^{\mathrm{ab}}$ and $\kappa' \to \kappa'_m$ is the field extension of degree m, for Y' to be geometrically irreducible over κ' it suffices that $Y'_{\kappa'_m}$ is irreducible.

Let $\kappa'_m \subset \kappa_+$ be a finite field extension. By a version of Chebotarev density theorem (cf. [Del], Thm. 3.5.3 in the case of finite monodromy), we can assume that κ_+ is large enough so that the following two properties hold: (i) for each conjugacy class conj of G there exist closed points of $\mathbb{A}^N_{\mathbb{F}_p}$ with residue field κ_+ and Frobenius conjugacy class (defined by $\mathcal{M}_N$) conj; (ii) the number of closed points of X' with residue field κ_+ is at least equal to the number of conjugacy classes of G. It follows that there exists a morphism $\Sigma_X : X \to U^N$ with prescribed values on certain κ_+-valued closed points, and using Jordan's Theorem (e.g., see [Ser2], Thm. 4' or 4 and [Kam], Lem. 1) as in [Kam], Sect. 2, the G-torsor $Y := \mathbb{X}^N \times_{\mathcal{M}_N, U^N, \Sigma_X} X \to X$ is such that the G-torsor $Y'_{\kappa_+} \to X'_{\kappa_+}$ is irreducible. Thus $Y'_{\kappa'_m}$ is also irreducible and the theorem holds in this case.

Case 2: κ is an infinite field. Let $m \in \mathbb{N}^*$ be such that there exists a closed embedding $c : X \to \mathbb{A}^m_\kappa$; so $c_N := c \times 1_{U^N_\kappa}$ is a closed embedding. We show that there exists a linear (not necessarily homogeneous) morphism $\Sigma_X : X \to U_\kappa$ such that the G-torsor $Y := \mathbb{X}^N_\kappa \times_{\mathcal{M}_{N,\kappa}, U^N_\kappa, \Sigma_X} X \to X$ is such that the induced torsor $Y' \to X'$ is geometrically irreducible over κ'. This is equivalent to cutting with a general subspace of codimension N of the affine space $\mathbb{A}^m_\kappa \times_{\mathrm{Spec}\,\kappa} U^N_\kappa = \mathbb{A}^{m+N}_\kappa$. As $\dim(X' \times_{\mathrm{Spec}\,\kappa'} \mathbb{X}^N_{\kappa'}) \geq N+1$, the existence of Σ_X follows from [Jo1], Part I, Cor. 6.7(3) applied (or from [Jo1], Part I, Thm. 6.3(4) applied N times) to $c_N \circ (1_X \times \mathcal{M}_{N,\kappa}) : X' \times_{\mathrm{Spec}\,\kappa'} \mathbb{X}^N_\kappa \to \mathbb{A}^m_{\kappa'} \times_{\mathrm{Spec}\,\kappa'} U^N_{\kappa'} = \mathbb{A}^{m+N}_{\kappa'}$. □

29. Appendix: open problems, questions, and conjectures

P1. For all integers $n \geq 2$ and $l \geq 0$ compute the smallest $\pi_{p,n,l} \in \{1\} \cup p\mathbb{N}^*$ such that there exists an F_n-system $\mathcal{S}$ over k with $\pi(\mathcal{S}) = \pi_{p,n,l}$ and $\mathrm{s}(\mathcal{S}) = l$. Clearly, $\pi_{p,n,1} = 1$ and the sequences $(\pi_{p,n,l})_{n\geq 2}$ are decreasing.

Remark 29.1. Each F_n-system $\mathcal{S}$ over k is equivalent to an F_m-system over k of the form $x_i + f_i^p = 0$ for $i \in [[1,m]]$, where $m \in \mathbb{N}^*$ depends only on n and $\pi(\mathcal{S})$ and the partial degree of $f_i \in k[x_1, \ldots, x_m]$ in x_j is at most 1 for all $i, j \in [[1,m]]$.

Example 29.2. (1) Clearly, for $l \in \mathbb{N}$ we have $\pi_{p,n,l} = 1$ iff $l = 1$ and (see [V], Thm. 2.4.1(b)) we have $\pi_{p,n,l} = p$ iff $l \in \{p^m | m \in [[1,n]]\}$. Thus $\pi_{p,n,0} \geq 2p$.

(2) We have $\pi_{p,2,0} \geq 3p$ by Proposition 22.15. The system of equations $x - x^p y^{p(p-1)} + 1 = y - xy^p - 1 = 0$ over k has no solutions by Example 14.1 applied to $(m,q) = (1,p)$ and it is equivalent to the F_2-system $x - x^p y^{p(p-1)} + 1 = y + y^p - x^p y^{p^2} - 1 = 0$ over k, hence $\pi_{p,2,0} \leq p^2 + p$, and also to the F_3-system $x - x^p y^{p(p-1)} + 1 = y + y^p - x^p z^p - 1 = 0 = z - y^p$ over k, hence $\pi_{p,3,0} \leq p^2$. Part (1) and the last two sentences imply that $\pi_{2,2,0} = 6$, $\pi_{3,2,0} \in \{9, 12\}$, and $\pi_{2,n,0} = 4$ and $\pi_{3,n,0} \in \{6, 9\}$ for $n \geq 3$.

P2. Identify properties of the finite sets $\mathcal{F}(e) \in \mathcal{P}_f(\mathbb{N})$ with $e \in \mathrm{BE}_2(k)$.

Remark 29.3. If $\mathcal{F}(e) \neq \emptyset$ for $e \in \mathrm{BE}_2(k)$, then the union $\mathcal{F}(e) \cup \{\deg(e)\}$ has a 'midpoint' $[\deg(e) + \min\{\mathcal{F}(e)\}]/2$ but in general it is not symmetric with respect to it (e.g., this is so if in Proposition 22.9 we have $1 \in \mathbb{U}$ and $|\mathbb{U}| \geq 2$).

P3. Describe the inclusions $\{\varphi(e^{\iota(e)}) | e \in \mathrm{BE}_2(k)\} \subset \{\varphi(e^{\iota(e)}) | e \in \mathrm{EE}_2(k)\}$ and $\{\iota(e) | e \in \mathrm{BE}_2(k)\} \subset \{\iota(e) | e \in \mathrm{EE}_2(k)\}$ and the set $\{\deg(e) | e \in \mathrm{BE}_2(k), \varphi(e) > 0\}$. In particular, does there exists $e \in \mathrm{BE}_2(k)$ with $\deg(e) = p$ and $\varphi(e) > 0$?

P4. For $n \in \mathbb{N}^*$ compute the smallest $\vartheta_p(n) \in \mathbb{N}^*$ for which there exists $e \in \mathrm{BE}_2(k)$ with $\varphi(e) = n$ and $\deg(e) = \vartheta_p(n)$. E.g., we have $\vartheta_2(4n-1) \leq 4n$ by Theorem 2.4(3).
P5. Classify the sequences $\big((\varphi(e^n)\big)_{n\in\mathbb{N}}$ with $e \in \mathrm{QF}_2(K)$.
P6. Let $d \in \mathrm{EE}_n(K)$. For $n \geq 2$, classify the sequences $\big(\rho_{\text{ét}}(d^n)\big)_{n\in\mathbb{N}}$ and $\big(\rho(d^n)\big)_{n\in\mathbb{N}}$, decide if $\rho_{\text{ét}}(X_d)$ is or is not equal to $\rho_{\text{ét}}(d)$ (cf. Theorem 7.3(4)). For $n = 2$, how do the invariants $\rho_1(d)$ or $\rho_2(d)$ vary with d when the isomorphism class X_d is fixed?

Example 29.4. If $e \in \mathrm{EE}_2(k)$, then for all $n, m \in \mathbb{N}^*$, the identity $e^{n+m} = e^n e^m$ induces a finite flat morphism $X_{e^{n+m}} \to X_{e^n}$ such that the inverse image of the open subvariety $\mathbb{A}^2_{k,\mathrm{s}}$ of X_{e^n} is X_{e^m} and therefore $\rho(e^{n+m}) \geq \rho(e^n) + \rho(e^m)$.

P7. Given $e \in \mathrm{EE}_n(K)$ and an open embedding $\imath : \mathbb{A}^n_K \to X_e$, is it true that $\mathrm{Im}(\imath) \subset X_e^{\text{ét}}$? If it is not true, does there exist $d \in \mathrm{EE}_n(K)$ such that $\imath_d$ is isomorphic to d?

Let X be an affine connected smooth surface over k that has an open subvariety isomorphic to $\mathbb{A}^2_k$. Does there exist $e \in \mathrm{EE}_2$ such that X is isomorphic to an open subvariety of X_e? If yes, can we actually choose $e \in \mathrm{EE}_2(k)$ such that $X \cong X_e$? If not, is it true that there exists a non-surjective étale endomorphism $d : X \to X$?
P8. Study other invariants of $e \in \mathrm{EE}_n(K)$ such as the reduced locally closed subvariety $N_l(e)$ with $l \in \mathcal{F}(e) \cup \{\deg(e)\}$ of Remark 10.2 or the isomorphism class of the Galois group of the finite Galois extension of $K_{\mathrm{t}}(x_1, \ldots, x_n)$ generated by $\mathrm{Frac}(e^{\#})$, or the number $\nu(e)$ of irreducible components of N_e.
P9. For $e \in \mathrm{EE}_2(k)$, describe relations such as inequalities or values of polynomial expressions between invariants of its iterates. E.g., the set $\{\frac{\rho(e)}{\deg(e)} | e \in \mathrm{EE}_2(k)\}$ is unbounded (see Theorem 2.4(2)).
P10. Use special polynomials as in [Wr2], Defs. 1.3 to 1.7 to provide more examples of moduli spaces of endomorphisms $e \in \mathrm{EE}_n(R)$ that have relevant properties. E.g., for integers $n, q \geq 2$ we have a closed subscheme $HEE^*_{n,q}$ of $EE^*_{n,q}$ whose R-valued points are those n-tuples of the form

$$(f_1, \ldots, f_n) = (x_1 + l_1^q, \ldots, x_n + l_n^q) \in R[x_1, \ldots, x_n]^n$$

with $l_1, \ldots, l_n$ linear forms. Of particular interest is to describe the set

$$\nabla_{n,q,\mathrm{char}(K)} := \{\deg\big(\mathrm{e}(f_1, \ldots, f_n)\big) | (f_1, \ldots, f_n) \in HEE^*_{n,q}(K)\}.$$

Example 29.5. **(1)** We have $\nabla_{2,q,0} = \{1\}$, i.e., $(HEE_{2,q,K})_{\mathrm{red}} \to SA^*_{2,q,K}$ is a closed embedding if $\mathrm{char}(K) = 0$ (see [vdE], Part I, Ch. 2, Sect. 2.1, Exc. 4 due to Furter).

(2) We have $\nabla_{n,1,p} = \{1, p, p^2, \ldots, p^n\}$ (the part '$\subset$' follows from [V], Thm. 2.4.1(b); the part '$\supset$' can be seen by taking $l_i \in \{0, x_i\}$ for all $i \in [[1, n]]$).

(3) For $t \in \mathbb{N}^*$ we have $\nabla_{2,p^t+1,p} = \nabla_{3,p^t+1,p} = \{1\}$, i.e., we have closed embeddings $(HEE_{2,1+p^t,k})_{\mathrm{red}} \to SA^*_{2,p^t+1,k}$ and $(HEE_{3,p^t+1,k})_{\mathrm{red}} \to SA^*_{3,p^t+1,k}$.

P11. Classify all $SEE_{n,q,K}$ that are irreducible. Compute the dimensions of $SEE_{n,q,K}$ and $SA^*_{n,q,K}$.
P12. Prove an analogue of Theorem 24.13 with SEE replaced by SA.
P13. Is it possible to define moduli schemes $BE_{n,q,\mathbb{F}_p}$ and $SBE_{n,q,\mathbb{F}_p}$?

P14. For each integer $q \geq 3$, classify all surfaces that are non-trivial torsors under the line bundle $\mathcal{O}_{\mathbb{P}^1_K}(-q)$ over $\mathbb{P}^1_K$. In particular, classify the isomorphism classes defined by $\mathbb{S}^2_{f,g,k,Y}$ or $\mathbb{T}^2_{f,g,m,Y}$ of Proposition 23.2.

Remark 29.6. Torsors under $\mathcal{O}_{\mathbb{P}^1_K}(-q)$ are parametrized by $H^1\big(\mathbb{P}^1_K, \mathcal{O}_{\mathbb{P}^1_K}(-q)\big)$. As $H^1\big(\mathbb{P}^1_K, \mathcal{O}_{\mathbb{P}^1_K}(-2)\big) \cong K$, the group of automorphisms of $\mathcal{O}_{\mathbb{P}^1_K}(-2)$ acts transitively on the non-trivial torsors under $\mathcal{O}_{\mathbb{P}^1_K}(-2)$ and thus, as schemes over $\operatorname{Spec} K$, they are all isomorphic to $\mathbb{S}^2_K$. By Serre duality for $\mathbb{P}^1_K$ (see [H2], Ch. III, Thm. 7.1(b)), for each non-zero class in $H^1\big(\mathbb{P}^1_K, \mathcal{O}_{\mathbb{P}^1_K}(-q)\big)$ there exists a homomorphism $\mathcal{O}_{\mathbb{P}^1_K}(-q) \to \mathcal{O}_{\mathbb{P}^1_K}(-2)$ that maps it to a non-zero class in $H^1\big(\mathbb{P}^1_K, \mathcal{O}_{\mathbb{P}^1_K}(-2)\big)$. Thus the non-trivial torsors under $\mathcal{O}_{\mathbb{P}^1_K}(-q)$ are affine as they are affine schemes over $\mathbb{S}^2_K$.

P15. Identify more smooth affine surfaces over K that have proper open subvarieties isomorphic to $\mathbb{A}^2_K$ and compute their embedding dimensions (in the set $\{3,4,5\}$, see Theorem 7.2(1)) and decide if they are or are not complete intersections. For $(n_1,n_2,n_3) \in \mathbb{N}^3$ with $n_1+n_2+n_3>0$ and $q \in \{3,4,5\}$ classify all $e \in \mathrm{EE}_2(k)$ with $\big(\rho_1(e), \rho_2(e), \rho(e)-\rho_{\text{ét}}(e)\big) = (n_1,n_2,n_3)$ and $\mathrm{edim}(X_e) = q$.

P16. Classify all Galois field extensions $\mathcal{Z}$ of $k(x,y)$ which are of degree p and unirational in a way that one can decide if $\mathcal{Z}$ is rational or if $\mathcal{Z}$ has the property that the spectrum X of the normalizations of $k[x,y]$ in $\mathcal{Z}$ has an affine open subvariety Y which is étale over $\mathbb{A}^2_k$ and: (i) isomorphic to $\mathbb{A}^2_k$; (ii) finite over $\mathbb{A}^2_k$; (iii) such that $\mathcal{O}(Y) = k^*$; (iv) factorial.

P17. If k is an algebraic closure of $\mathbb{F}_p$, does there exist $d_2 \in \mathrm{D}_2(k)$ which is generically étale with $\deg(d_2) > 1$ and $\iota(d_2) = \infty$?

P18. Do there exist affine regular varieties X over k with trivial tangent bundles but without p-bases? In particular, do there exist étale morphisms $\mathbf{SL}_{2,k} \to \mathbb{A}^3_k$?

P19. Let $e \in \mathrm{EE}_n(K)$. Is it true that $\mathrm{Sing}(X_e) \subset \cup_{(Y,Z)\in \mathrm{Irr}(X_e\setminus X_e^{\text{ét}})^2, Y\neq Z} Y \cap Z$? For $n=2$, if this is true then $\mathrm{Sing}(X_e) = \emptyset$ by Corollary 11.4(1), so X_e is regular. If $n=2$, then for $m \in \mathbb{N}$ with $p^m > \varphi(e)$ and a $d \in \mathrm{EE}_2(k)$ which is a finite étale cover of degree p^m, $de \in \mathrm{EE}_2(k)$ is surjective with $X_{ed} = X_e$. So there exists $e \in EE_2(k)$ with X_e singular iff there exists a surjective $e \in EE_2(k)$ with X_e singular.

P20. Let $e \in \mathrm{EE}_n(K)$. Is it true that, with the notations of Criterion 10.7, we have $\mathrm{Irr}(X_e \setminus X_e^{\text{ét}}) = \mathrm{Irr}_{\mathrm{t}}(X_e \setminus X_e^{\text{ét}})$?

P21. Describe the submonoid $\mathrm{MBE}_n(k)$ of $\mathrm{GA}_n(k)$ generated by $\mathrm{BE}_n(k)$. Describe the set $\infty_p(M) := \{\big(\infty_p(f), \infty_p(g)\big) | (f,g) \in k_{\mathrm{s}}[x_1,x_2], \mathrm{e}(f,g) \in M\} \subset (\mathbb{N}^*)^2$, where $M \in \{\mathrm{EE}_2(k), \mathrm{MBE}_2(k)\}$.

P22. If $n \geq 2$ and $m \in \mathbb{N}^*$, is the set $\{e \in \text{p-Mor}_n(k) \setminus \mathrm{BE}_n(k) | \deg(e) = pm\}$ non-empty? Note that if $n \geq 2$ and $e \in \mathrm{BE}_n(k)$ is non-surjective, then $e \notin \text{p-Mor}_n(k)$.

P23. If $e \in \mathrm{EE}_2(k)$ has 1 point at infinity modulo p and $a \in \mathrm{GA}_2(k)$, is it true that ea has 1 point at infinity modulo p?

P24. Is it true that there exists no $e \in \mathrm{EE}_n(k)$ with $2 < \deg(e) < p$ or at least that $\deg_{p,n,q} = 1$ if $q^n < p$? [An affirmative answer implies the classical Jacobian Conjecture. If such an e exists, then ψ_e is non-étale by Corollary 25.3.] Do there exist counterexamples to Conjecture 2.10 with $\deg(e) > p$? Do there exist examples with $\deg^f_{p,n,q} \neq \deg_{p,n,q}$ or with $SEE^{\text{f-gen}}_{n,q,k}$ not Zariski dense in $SEE_{n,q,k}$?

The following conjecture is our expectation of what [Zh], Introd., Thm. for $n=2$ would become for $n \geq 3$.

Conjecture 29.7. *Assume* $\mathrm{char}(K) = 0$ *and* $n \geq 3$. *If* $\mathrm{e}(g_1, \ldots, g_n) \in \mathrm{EE}_n(K)$, *then* $\deg\big(\mathrm{e}(g_1, \ldots, g_n)\big) \leq \prod_{i=1}^{n-1} \deg(g_i)$.

For $n \geq 3$, note that Conjecture 29.7 neither implies the classical Jacobian Conjecture for n nor is implied by [Fo2], Thm. 2. It is a weakening of The n-Degree Conjecture of Moskowicz (see [Mos], Conj. 3.1) which does imply the classical Jacobian Conjecture for n by [Mos], Prop. 3.8.

The next conjecture is a natural extrapolation of Corollaries 25.3 and 25.4, [Wa2], Thm. 62, Section 21, and moduli considerations.

Conjecture 29.8. *If* $n \geq 2$, $q \geq 3$ *and* $\sqrt[n]{p} \leq q < p$, *then* $\delta^{\mathrm{f}}_{p,n,q} = 1$.

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