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Stability and Pinching

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§ 0 - Terminology and examples.

The word "stability" is common in the theory of dynamical systems and foliations where one speaks of the *structural stability* of *diffeomorphisms* and *flows* on smooth manifolds W. Besides (1-dimensional) flows there are examples of higher dimensional *foliations* f_0 on W which have the property of the structural stability : *every foliation* f *on* W *whose tangent bundle* $T(f) \subset T(W)$ *is sufficiently close to* $T(f_0)$ *can be obtained from* f_0 *by a homeomorphism* $h : W \to W$. *Moreover, this* h *which brings* f *to* f_0 *can be chosen close to the identity in the sence that if* $T(f)$ *is* ε*-close to* $T(f_0)$ *in the* C^1*-topology then* h *is uniformly* δ*-close to* Id *where* $\delta \to 0$ *for* $\varepsilon \to 0$.

The simplest example of a structurally stable foliation f_0 on a compact manifold W is where all leaves are *closed* (i.e. compact without boundaries) simply connected manifolds. Here one can do even better by relaxing the C^1-closeness between f and f_0 to the uniform (i.e. C^0) closeness and yet obtaining even nicer h, which is a diffeomorphism C^1-close to the identity.

The stability of foliations with compact leaves follows from the *Reeb stability theorem* which concerns individual leaves of a foliation rather than the totality of them. The theorem says that *if such a leaf* V_0 *is closed and simply connected then every other leaf* V *passing*

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through a point $w \in W$ *near* V_0 *is also closed simply connected and, moreover, it lies* C^1-*near* V_0, i.e. can be obtained from V_0 by a diffeomorphism of W which is C^1-close to the identity. (See [Law], [Fuks], [Rei] for a futher discussion and references concerning Reeb's theorem).

There is a slightly stronger version of the above stability of L_0 which comes nearer to the Riemannian pinching problem. Namely it is not necessary to assume that V is a leaf of any foliation at all. All we need is a closeness of the tangent bundle of V to that of V_0. Namely, we just have a closed submanifold $V_0 \subset W$ and another submanifold $V \subset W$. Here V may have selfintersections and one should think of V as an abstract manifold immersed into W. As our discussion is local around V_0, we restrict our attention to a (small) tubular neighbourhood W_0 of V_0 in W. Thus we may assume that $W = W_0$ has a structure of a vector bundle over V_0 where V_0 is embedded into W as the zero section. Finally, it is conveinient to have some Riemannian metric on W. We take and fix such a metric on W and to make the life easier we assume this metric is complete.

Now, with this metric, we have the horizontal subbundle $T^0 \subset T(W)$ which consist of the vectors normal to the fishers of the implied projection $W \to V_0$ and the closeness of the tangent bundle $T(V) \subset T(W)$ to $T(V_0)$ is measured by the "angles" between $T_v(V) \subset T_v(W)$ and T^0_v for $v \in V$.

With these preparations we now state the relevant (and fairly obvious) version of Reeb's theorem.

0.1. Elliptic prestability.
LEMMA : *Let* $\varepsilon : W \to \mathbb{R}_+$ *be a continuous function which vanishes on* $V_0 \subset W$ *and let* $V \subset W$ *be a connected submanifold, such that* $T_v(V)$ *is* ε-*close to* T^0 *at every point* $v \in V$. (Notice that this condition says nothing what-so-ever about the points v where ε is large but it does require $T_v(V)$ to be close to T^0_v at the points $v \in V$ which lie near V_0). *If* V_0 *is a closed simply connected manifold* (i.e. V_0 is topologically elliptic) *and if the induced metric in* V *is geodesicly complete, then* V *is a closed submanifold lying near* V_0, *provided there is a single point* $v \in V$ *which is close to* V_0.

Remark on Thurston's stability. Thurston (see [Thu] and also [Sto]) has generalized Reeb's stability theorem by replacing the condition $\pi_1(V_0) = 0$ by the following, weaker one, *every* *linear representation of the fundamental group* $\pi_1(V_0)$ (i.e. a homomorphism $\pi_1 \to GL_N(\mathbb{R})$ *is trivial.*

However this generalization does not apply to the above prestability lemma. In fact, if the fundamental group $\pi_1(V_0)$ is non trivial, one has the universal covering $V \to V_0 \subset W$ giving a counter example to the conclusion of the lemma. Moreover, if $\pi_1(V_0)$ is infinite, one can sometimes perturb the above map $V \to W$ (coming from the universal covering $V \to V_0$, such that the perturbed $V \subset W$ still satisfies the assumptions of the prestability lemma (i.e. $T(V)$ remains ε-close to $T(V_0)$) but some points $v \in V$ drift rather far away from V_0. Such a perturbation is easy to construct if the bundle $W \to V_0$ admits a non-zero section. Otherwise, *if the Euler class* of $W \to V_0$ *does not vanish* (and there is no non-zero section $V_0 \to W$) then the above perturbation is not always possible.
(This is definitely impossible if the fundamental group $\pi_1(V_0)$ is amenable).

0.2 Hyperbolic stability. It is quite remarkable and surprising that the stability may show up in foliations with non-compact leaves. Here we deal with a foliated manifold W which is also equipped with a Riemannian metric and so one can speak of the angles (or distances) between n-dimensional linear subspaces in $T_w(W)$, $w \in W$ and the tangent space $S_w \subset T_w(W)$ to the leaf through $w \in W$, where n equals the dimensions of the leaves.

Now *the prestability of the foliation relative to the chosen Riemannian metric* means that there exist an $\varepsilon > 0$, such that every immersed n-dimensional manifold $V \looparrowright W$ whose induced metric is complete and whose tangent spaces $T_v(V) \subset T(W)$ are ε-close to the tangent spaces $S_v \subset T_v(W)$ for all $v \in V$, is δ-close to the covering of some leaf V', where $\delta \to 0$ for $\varepsilon \to 0$. This means that the immersion $V \looparrowright W$ admits a δ-perturbation in the uniform C^1-topology, such that the perturbed map of V to W factors through a covering map $V \to V'$.

Notice that if W is compact, then the prestability property does not depend on the choice of a Riemannian metric. Another useful remark concerns prestability properties in W_1 and W_2, where the two manifolds are connected by a covering map $W_1 \to W_2$ which preserves the foliations and the metrics in question. In this case the prestability of the foliation in W_2 immediately implies that in W_1. Furthermore, if the *test manifold* $V \looparrowright W_2$ is simply connected and so can be lifted to W_1, then prestability in W_2 for V follows from that for the lift $\tilde{V} \looparrowright W_1$.

0.3. Geodesic flow for $K < 0$. This is the basic example of prestability. Here W is the unit tangent bundle of a complete Riemannian manifold X with strictly negative sectional curvature,

$$K(X) \leq \kappa < 0.$$

The geodesics in X form a 1-dimensional foliation of $W = UT(X)$ and the metric in W naturally comes from that in X. The prestability of this geodesic foliation is the famous theorem due to M. Morse and (in the modern rendition) to Anosov, called the *hyperbolic shadowing lemma* : it claims that *quasi-leaf* V is "shadowed" by an actual leaf of the foliation. (Our formulation of the shadowing lemma also includes the Morse-Anosov *closing lemma* which claims that every geodesic can be approximated by a periodic one).

0.4. Weakly contracting foliation. Denote by $G_t : W \to W, t \in \mathbb{R}$, the action of the geodesic flow on $W = UT(X)$ and say that two points w_1 and w_2 are *weakly* ω-*asymptotic* if there is a path γ in W between w_1 and w_2, such that

$$\limsup_{t \to +\infty} \text{ length } G_t(\gamma) < \infty.$$

This clearly is an equivalence relation on W where the equivalence classes are called *weakly contracted leaves* while the partition into these leaves is called *the weakly contracting foliation* F_{cont}. If $K(X) \leq \kappa < 0$, this is indeed a foliation whose leaves V have dimension $n = \dim X$. in fact, the map $V \to X$ on each leaf V induced by the projection $W \to X$ is a local diffeomorphism, and moreover, a covering map of the leaf V onto X.

Stability of contraction. *The weakly contracting foliation* F_{cont} *is prestable.*

This is a standard component of the hyperbolic package of Morse-Anosov. (see [Man]).

Recall that F_{cont} goes along with three other foliations on W. First we have the expanding twin of F_{cont}, called F_{exp} which corresponds to the α-*asymptotics,*

$$\limsup_{t \to -\infty} \text{ length } G_t(\gamma) < \infty.$$

In other words, F_{exp} is weakly contracting for the reversed flow G_{-t}. Since the reversed flow is isomorphic to G_t via the involution $T(X) \leftrightarrow T(X)$ for $\tau \leftrightarrow -\tau$, the foliation F_{exp} is isomorphic to F_{cont} by this involution.

The remaining two foliations are the strongly contracting f_{cont} and the strongly expanding f_{exp}, where the first is defined via the convergence

$$\limsup_{t \to +\infty} \text{ length } G_t(\gamma) \to 0 ,$$

and f_{exp} is obtained in the same manner for $t \to -\infty$. The foliations f_{cont} and f_{exp} have $(n-1)$-dimensional leaves. Projections of these leaves to X (or rather, their lifts to the universal covering $\tilde{X} \to X$) are called *horospheses*. Accordingly f_{cont} and f_{exp} are called *horospherical foliations*. Despite their hyperbolic origin these foliations should be regarded as *parabolic* objects. They are not prestable unlike their bigger brothers F_{cont} and F_{exp}.

To get a clear picture of all these foliations one may assume that X is simply connected. Then the leaves of F_{cont} correspond to the points of *ideal boundary* $\partial_\infty X$ as follows. Every point $y \in \partial_\infty X$ is given, by definition, by a family of ω-asymptotic geodesics in X which converge (in an appropriate topology in $X \cup \partial_\infty X$) to y for $t \to \infty$. These goedesics form a 1-dimensional foliation of X, denoted F^y, and their unit tangent vectors define a section $X \to W = UT(X)$. The image of this section is the leaf of F_{cont} corresponding to y.

The unit tangent field g on X represented by the above section $X \to W$ is *gradient* and the function h on X (define up to an additive constant) for which

$$-\operatorname{grad} h = g$$

is called the *horofunction* corresponding to y. (One thinks of h as the distance function $h : x \mapsto \operatorname{dist}(y, x)$). The levels $\{h(x) = c\}_{c \in \mathbb{R}}$ are called *horospheres*. Their lifts to W are the leaves of f_{cont} sitting inside the leaf of F_{cont} defined by $g : X \to W$. Here is the picture of who is who.

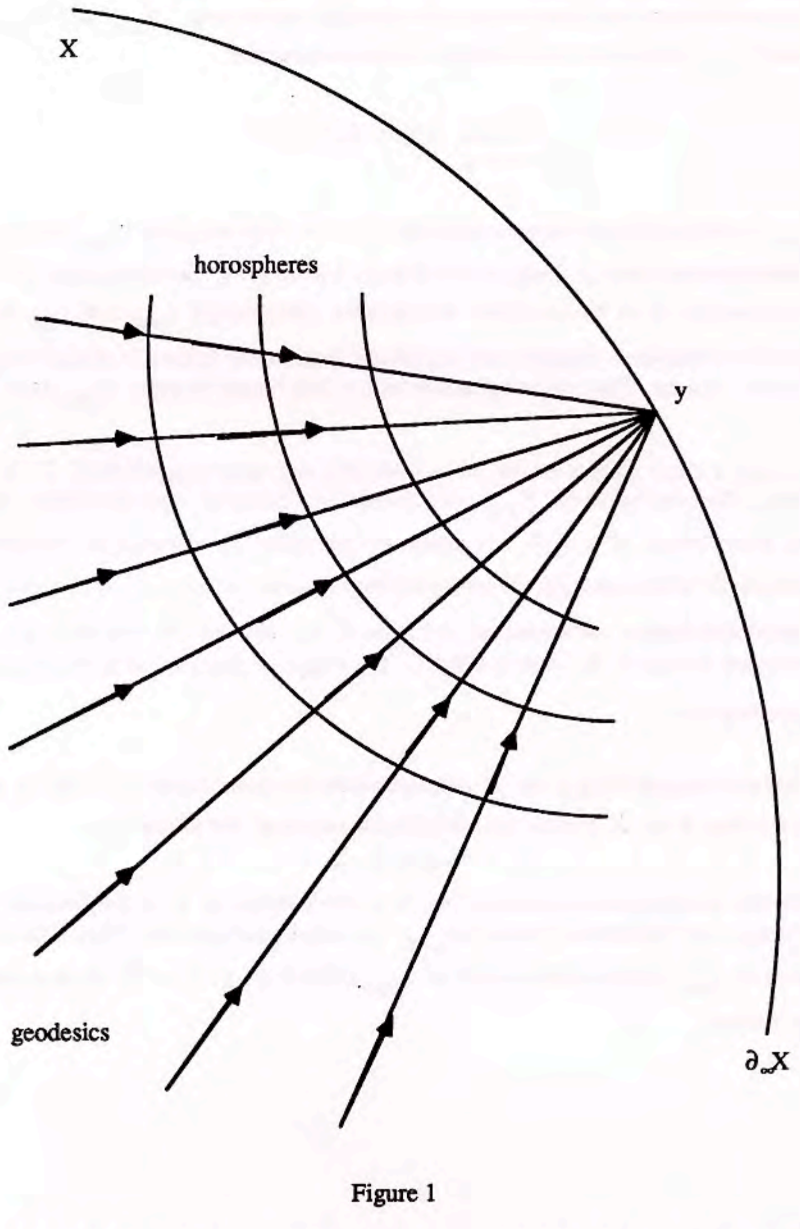

Figure 1

To complete the discussion we recall that the 1-dimensional foliation of W into the orbits of the geodesic flow equals the intersection of F_{cont} and F_{exp} and then all five foliations can be (locally) seen together in Fig. 2 below.

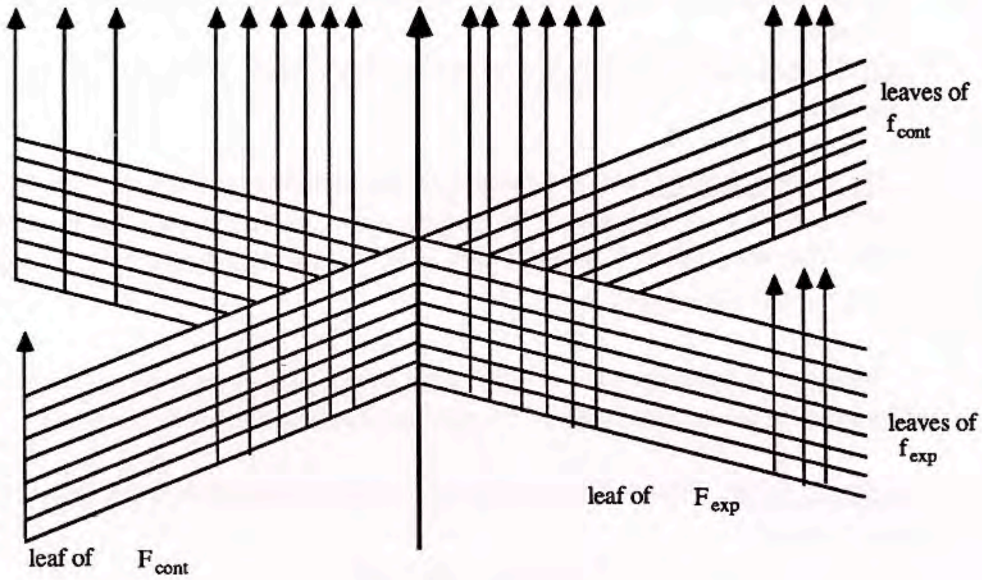

Figure 2

0.5. Pinching conditions. In what follows $V = (V, g)$ is a manifold with a Riemannian metric g. The *classical pinching condition* refers to the sectional curvature K of V (thought of as a function on the Grassmann bundle Gr_2V) and reads

$$\kappa_- \leq K(V) \leq \kappa_+ . \qquad (-+)$$

Here κ_- and κ_+ are given real numbers and one should distinguish the following three cases.

(1) Elliptic pinching, where $\kappa_- > 0$ and $\kappa_+ = (1+\varepsilon)\kappa_-$ for $\varepsilon \geq 0$.

(2) Hyperbolic pinching, where $\kappa_+ < 0$ and $\kappa_- = (1+\varepsilon)\kappa_1$ for $\varepsilon \geq 0$.

(3) Parabolic pinching, where $\kappa_- = -\varepsilon$ and $\kappa_+ = +\varepsilon$ for $\varepsilon \geq 0$.

In geometric terms, the condition $(-+)$ says that the (second order) infinitesimal geometry of (V, g) is ε-close to one of the following three standard geometries.

(1)' Elliptic case : the geometry of the sphere S^n of constant positive curvature.

(2)' Hyperbolic case : the geometry of the hyperbolic space H^n with constant negative curvature.

(3)' Parabolic case : the geometry of the Euclidean space $\mathbb{R}^n$ (which has zero curvature).

The pinching problem can be broadly speaking formulated as follows. Does the pinching condition (– +) imply that V is *globally close* to the corresponding standard manifold ? For example, does V admit a metric g_0 of constant curvature, such that $g \to g_0$ for $\varepsilon \to 0$? If not, what happens then ?

The answer to the first question is "Yes" in the elliptic case, where there is a variety of results making the answer more specific. The most famous among these is the following.

0.6. Sphere theorem. (Berger-Klingenberg). *If a closed simply connected Riemannian manifold* V *has*

$$\frac{\kappa}{4} < K(V) < \kappa, \text{ for } \kappa > 0,$$

then V *is homeomorphic to* S^n, n = dim V (See [Ch-Eb] for the proof).

This result is sharp as the *complex projective space* has κ varying in the *closed* interval [1/4, 1].

The pinching problem has the negative answer in the hyperbolic and parabolic cases. Yet one can achieve positive results by restricting the class of manifolds V in question, e.g. by imposing some bounds on the volume and/or on the diameter of V. (See $[\mathrm{Gro}]_1$ [Bus-Kar], [Gr-Th]).

0.7. Pinching according to Cheeger. Fix some Riemannian manifold (V_0, g_0), called a *model*, and let us compare a given manifold (V, g) with V_0 as follows. Suppose that for each point $v \in V$ there exists a diffeomorphism of a (small) neighbourhood $U \subset V$ of v into V_0, say $\varphi : U \to V_0$, such that the induced metric $\varphi^*(g_0)$ on U is ε-close to $g \mid U$ in the C^i-topology, i.e.

$$\left\| \varphi^*(g_0) - g \right\|_{C^i} \leq \varepsilon \qquad (*)$$

where the C^i-norm of the difference $\varphi^*(g_0) - g$ is measured with respect to g. To be specific, we use the g-exponential coordinates at v and define the norm as the supremum of the absolute values of the partial derivatives of the components of the quadratic form $\varphi^*(g_0) - g$ in these coordinates.

If we can achieve (*) for every $v \in V$, we say that V is *infinitesimally ε-close to* V *up to order* i, and write

$$\mathrm{dist}_{C^i}(V, V_0) \le \varepsilon. \qquad (**).$$

Notice that the classical pinching condition (roughly) translates to this language by

$$\mathrm{dist}_{C^2}(V, V_0) \le \varepsilon,$$

where V_0 has constant curvature.

Cheeger (following Rauch) has initiated in his paper $[\mathrm{Che}]_1$ the study of the case, where V_0 is a compact simply connected *symmetric space*. He has proven in certain cases that the "pinching" condition

$$\mathrm{dist}_{C^3}(V, V_0) \le \varepsilon$$

implies, for small enough $\varepsilon > 0$, that V is diffeomorphic to V_0. In fact his argument gives a metric g_0^* on V such that (V, g_0^*) is isometric to (V_0, g_0^*) and such that $\|g - g_0^*\|_{C^2} \to 0$ for $\varepsilon \to 0$. (See $[\mathrm{Che}]_1$ and § 1 in the present paper for the further discussion).

We shall explain later on (see § 1) how one can use Cheeger's results in order to study the more general *elliptic pinching problem* where V_0 is a homogeneous Riemannian manifold, $V = G/H$, for a *compact semisimple* Lie group G.

Pinching and stability. It is worthwhile to indicate at this point that there is a close similarity between the (pre)stability of foliations and the pinching problem. In fact, one can think of the latter as a special case of the former where the relevant foliation $\mathcal{F}$ is a certain huge universal object, such that every Riemannian manifold appears as a leaf of $\mathcal{F}$.

Remark on symmetries. An important specific feature of the pinching problem is the existence of non-trivial symmetries (i.e. isometries and/or Killing fields) of the model space V_0. In fact, the symmetries of V_0 play the crucial role in the study of the pinching (stability), while the metric g_0 itself is somewhat less important. For example if V_0 has no symmetries at all, then the pinching problem becomes rather trivial (See § 1 for more symmetry discussion).

The presence of symmetries brings along yet another kind of ***stability***, this time *for fixed points of group actions*. Here there are (at least) three somewhat different types of phenomena which go under the banner of stability.

(1) We have a group G acting on some space X, such that the action fixes a point $x_0 \in X$. Then we slightly perturb the action and ask ourselves if the new action still has a fixed point, say x_1, and whether x_1 lies near x_0.

(2) We have the same setting as above, we assume the new action does have a fixed point x_1 close to x_0 and we want to know if the new action (near x_1) is equivalent to the old (near x_0) in an appropriate category.

(3) We have an action of G on X such that some point x_0 is almost fixed (in an appropriate sense) under the action. Then we look for an actual fixed point x_1 near x_0.

If G is a compact group smoothly acting on a (finite or infinite dimensional) manifold, then the above problems can be successfully approached by using an appropriate averaging process on G. This possibility to average underlies most of the pinching (stability) results in the elliptic case.

More interesting and difficult problems appear when G is a non-compact semisimple group, and where the above stability problems have no universal solution. (The only known general method is the unitary trick of H. Weyl but this applies only in a limited number of examples). In fact different semi-simple groups may display different degrees of stability, as is seen, for example, in ***Kazhdan's*** T-*property* which corresponds to (3) for isometric actions on the unit sphere in the Hilbert space. (See [dlH-V] more about T-property).

In the remaining sections of this paper we briefly discuss various occurrences of the stability and pinching phenomena in differential geometry. The results we present are, for the most part, not new and we do not provide the detailed proofs. (These can be found in the papers cited in our list of references). What may be new and interesting for non-experts is an exposition of the stability/pinching ***philosophy*** which lies behind the basic results and methods in the field and which is very rarely (if ever) presented in print. (This common and unfortunate fact of the lack of an adequate presentation of basic ideas and motivations of almost any mathematical theory is, probably, due to the binary nature of mathematical perception : either you have no inkling of an idea or, once you have understood it, this very idea appears so embarrasingly obvious that you feel reluctant to say it aloud ; moreover, once your mind switches from the state of darkness to the light, all memory of the dark state is erased and it becomes impossible to conceive the existence of another mind for which idea appears non-obvious).

§ 1. Elliptic pinching.

We start our discussion with the following elementary.

1.1. Euclidean ancestor. *Let* V *be a smooth closed hypersurface in* $\mathbb{R}^{n+1}$ *where principal curvature* λ_i *are pinched between two positive numbers,*

$$\kappa \leq \lambda_i \leq (1+\varepsilon)\kappa \quad , i = 1, \dots, n, \qquad (*)$$

Then V *is* δ*-close to a round sphere* S^n *in* $\mathbb{R}^{n+1}$ *of radius* κ^{-1} *where* $\delta \to 0$ *as* $\varepsilon \to 0$.

This is well known and obvious. Moreover, if $n \geq 2$ one may allow a non-closed immersed hypersurface whose induced metric is complete. (We suggest to the reader to look at what happens for $n = 1$). Furthermore, one has a pretty good idea of the dependence of δ on ε. One can easily see in this regard that V is pinched (i.e. contained) between two round concentric spheres in $\mathbb{R}^{n+1}$ of radii κ^{-1} and $((1+\varepsilon)\kappa)^{-1}$. This pinching is sharp and has a pleasant feature of being independent of n, which allows (by a simple argument) an extension of this result to hypersurfaces in the infinite dimensional Hilbert space in place of $\mathbb{R}^{n+1}$.

Now we want to relate the pinching problem for submanifolds to that for foliations. To do this we take the unit tangent bundle $UT(\mathbb{R}^{n+1})$ for W where the points $w \in UT(\mathbb{R}^{n+1})$ are interpreted as oriented hyperplanes in $T(\mathbb{R}^{n+1})$. Then every oriented hypersurface V naturally lifts to a hypersurface $\tilde{V} \subset W$ for $v \mapsto T_v(V) \subset T_v(\mathbb{R}^{n+1})$. This lift in particular applies to the round spheres of radius κ^{-1} in $\mathbb{R}^{n+1}$ and the lifted spheres foliate W. Now, the pinching condition (*) translates to the ε-closeness of $T(\tilde{V}) \subset T(W)$ to the tangent bundle of the spherical foliation and so the stability of spheres in $\mathbb{R}^{n+1}$ (with respect to the condition $\lambda_i = \kappa$) follows for $n \geq 2$ from Reeb's stability (or elliptic prestability) in § 0. (Notice, that the foliation approach does not yield the above mentioned refined version of pinching between the concentric spheres).

Let us generalize the above to the case of oriented n-dimensional submanifolds V in $\mathbb{R}^n$ for $N \geq n + 1$. To simplify the notation we use $\kappa = 1$ and then formulate the curvature pinching condition as follows, at each point $v \in V$ there exists a unit normal vector $v_0(v)$ such that the principal curvatures in the direction of v_0 (i.e. the eigenvalues of the shape operator corresponding to v_0) are pinched between 1 and $1 + \varepsilon$, while the eigenvalues (curvatures) corresponding to unit normal vectors at v orthogonal to v_0 are bounded in the absolute values by ε. Here again we claim that V *is close to some round unit* n-*dimensional sphere in* $\mathbb{R}^N$. The proof follows as earlier from Reeb's stability applied to the oriented Grassmann bundle $Gr_n\mathbb{R}^N$ foliated by the tangential lifts of unit spheres $S^n \subset \mathbb{R}^N$ to $Gr_n\mathbb{R}^N$.

The next step in generality is where the round sphere S^n is replaced by an arbitrary n-dimensional submanifold $S \subset \mathbb{R}^N$. We express the C^i-*pinching* condition by saying that for each $v \in V$ there exist an isometry of $\mathbb{R}^N$ which moves v close to S, such that some neighbourhood $U \subset V$ of v become ε-close to S (or rather, to a neighbourhood in S of some point $s = s(v) \in S$) in the C^i-sense. This means, there is a diffeomorphism of $\mathbb{R}^N$ C^i-close to the identity (with the measure of this closeness given by ε) which moves U onto some neighbourhood in S. What we want to obtain as a conclusion is the existence of an isometry moving all of V close to S. The foliated space W one can use in this case consists of the germs of the n-dimensional submanifolds in $\mathbb{R}^N$ and the pertinent foliations is defined on the subset $W^S \subset W$ which can be moved into S by an isometry of $\mathbb{R}^N$. The leaves of this foliation are the tautological lifts to $W^S \subset W$ of the copies of S obtained by isometric moves of S in $\mathbb{R}^N$.

Before stating a positive result we indicate the following.

Counter-example. Let $S \subset \mathbb{R}^3$ be a C^∞-smooth space curve which contains a straight segment somewhere in the middle, see Fig. 3.

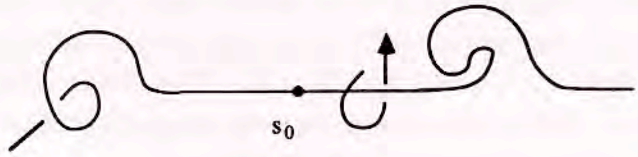

Figure 3

Then such S admits non-isometric deformations which are locally isometric. For example one can rotate the parts of S positioned to the right of s_0 (See Fig. 3) using the straight segment as the axis without moving anything to the left of s_0. The deformed curve is locally ε-close to S for $\varepsilon = 0$ (and any $i = 1, 2, \ldots$, you wish) but it can not be isometrically moved close to S as a whole.

The source of the problem stems from the fact different germs of S have different isotropy subgroups in Iso $\mathbb{R}^3$. For example, the isotropy of a straight segment inside S consists of the rotations around this segment, while the isotropy of every non-flat piece of S is the trivial group.

The above problem does not appear in the (most interesting) case where S is an orbit of some point $x \in \mathbb{R}^N$ under a connected subgroup $H \subset \text{Iso}\,\mathbb{R}^N$. Here one does not have to invoke the (infinite dimensional) space W but may rather use the manifold W_j for some $j \geq 1$ which is defined by induction as follows, $W_1 = \text{Gr}_n \mathbb{R}^N$, $W_2 = \text{Gr}_n W_1, \ldots$

Now, if i and j are large enough compared to N, the pinching immediately follows from Reeb's stability theorem, provided S is compact and simply connected.

Remarks (a) It may seem that by passing from $\mathbb{R}^N$ to W_j one looses some potentially useful information since not every n-dimensional submanifold V in W_j appears as the lift of some submanifold in $\mathbb{R}^N$. Yet nothing is lost as our V in question is everywhere locally close (up to a Euclidean motion) to some leaf of your foliation and this leaf does come from $\mathbb{R}^N$. It follows that if the first project V to $\mathbb{R}^N$ and then lift this projection to W_j, the resulting manifold, say $V' \subset W$, will lie close to V and thus will still satisfy the pinching condition.

(b) The manifold $V \subset W_j$ does not have to lie in the region $W_j^S \subset W_j$ which is foliated by the lifts of the isometric translates of S. However, W_j^S is a smooth submanifold in W_i and the pinching condition forces $V \subset W_i$ to lie close to W_i^S. Therefore we may (normally) project V to W_i^S and reduce our study to that of submanifolds in a (now everywhere) foliated space.

(c) P.D.E versus O.D.E. On the bottom of our proof (by the appeal to Reeb) of the above pinching results lies the standard property of continuity of solutions of O.D.E. (and thus of leaves of foliations) of the initial data and of the coefficients of the (ordinary differential) equations in question. A basic fact here is the *Gronwall inequality* which appears under the name of the *Rauch comparison theorem* in the context of Riemannian manifolds and the curvature pinching problems. There are similar inequalities (a priori estimates) for certain classes of (elliptic) *partial* differential equations which may lead to P.D.E-versions of pinching theorems. For example, instead of pinching the principal curvatures λ_i of $V \subset \mathbb{R}^{n+1}$ one could only restric the *mean curavature* by

$$n\,\kappa \leq \sum_{i=1}^{n} \lambda_i \leq (1+\varepsilon)\, n\,\kappa$$

and then ask if this makes V close to a round sphere. One knows in this regard that the answer may be "yes" or "no" depending on extra conditions imposed on V. In any case, the P.D.E-pinching is significantly deeper than the O.D.E-case. Yet, it seems that the best solution of O.D.E -pinching is significantly deeper thant the O.D.E-case. Yet, it seems that the best solution of O.D.E-pinching problems must eventually rely on P.D.E-techniques.

Elliptic pinching in G-manifolds. Let us replace $\mathbb{R}^N$ of the previous discussion by an arbitrary complete Riemannian manifold X acted upon by an isometry group G which replaces Iso $\mathbb{R}^N$ we had before. Then we take an n-dimensional orbit $S \subset X$ of some connected subgroup $H \subset G$ and we formulate the pinching problem for n-dimensional submanifolds $V \subset X$ as earlier. Here again, the problem admits the positive solution if the order i of the pinching is sufficiently high and if S is simply connected.

A remark on non-isometric actions. Let us briefly discuss what goes wrong if the action of G on X is non-isometric and can not be made isometric by a change of the metric in X. This happens, for example, if the isotropy subgroup of some point $x \in X$ is non-compact. What is more serious is a possible non-compactness of the isotropy subgroups of germs of S, since it leads to non-compactness of the set of the transformations $g \in G$ moving some neighbourhood $U \subset V$ close to S such that a given point $v \in U$ goes close to a fixed $s \in S$. If this happens, the measure of the C^i-closeness between moved U and S may become ambiguous. All this does not preclude the study of pinching and stability but rather indicates the presence of more subtle (non-elliptic) phenomena.

1.2. Elliptic Riemannian pinching. We follow the set-up of §0 with a given model (V_0, g_0) of dimension n and then we study manifolds V which have dist $(V, V_0) \leq \varepsilon$ for some i = 2, 3, ... and small $\varepsilon > 0$ (compare 0.7). In what follows we do not care much for the precision of our results, and assume i as large as we want (depending on n and/or on the model) and then we start speaking of V and V_0 being infinitesimally ε-close. We do not assume V a priori compact, only complete, and we want the compactness to follow from some (ellipticity) properties of the model.

Here are some conditions of this type which insure the compactness of V and provide a universal upper bound on the diameter of V.

(1) Ricci $V_0 \geq \delta_0 > 0$. Then clearly Ricci $V \geq \delta = \delta(V_0) > 0$ for small ε and so V is compact with

$$\text{Diam } V \leq \delta^{-1/2}\ \pi/n-1.$$

(2) V_0 *is a closed homogeneous manifold where every geodesic contains a pair of conjugate points*. Then there is a number $\ell_0 > 0$, such that every geodesic segment in V_0 of length $> \ell_0$ can be shortened by some small homotopy keeping the ends of this segment fixed. Now consider a geodesic segment γ in V of a fixed length ℓ. If ε is small enough we can find a "corresponding" segment γ_0 in V_0 of length ℓ such that the geometry of V_0 near γ_0 is close to that of V near γ. To construct γ_0 we take a diffeomorphism which establishes the infinitesimal closeness (see relation (*) in § 0.7) at one of the ends of γ, say $\varphi : U \to V_0$, for a neighbourhood $U \subset V$ of the end $v \in V$ of γ, and then take the geodesic of length ℓ issuing from $\varphi(v) \in V_0$ in the direction of $D\varphi(\dot\gamma_v)$, where $\dot\gamma_v \in T_v(V)$ is the unit tangent vector to γ at v and $D\varphi$ denotes the differential of φ. This differential at v defines together with the parallel transports along γ and γ_0 an isomorphism ψ between the normal bundles $N(\gamma)$ in V and $N(\gamma_0)$ in V_0. Then these bundles are exponentially mapped into V and V_0 correspondingly and the induced metrics, say $\tilde g$ on $N(\gamma)$ and $\tilde g_0$ on $N(\gamma_0)$ are brought close together by ψ. Namely, for every ρ-neighbourhood $\tilde U \subset N(\gamma)$ of $\gamma \subset N(\gamma)$ with a fixed ρ, the difference $\tilde g - \psi^*(\tilde g_0)$ goes to zero on $\tilde U$ (along with $i-2$ derivatives) as $\varepsilon \to 0$.

Now every small $\tilde g_0$-shortening deformation of γ_0 induces (via ψ) that of γ and our claim follows.

Probably it is not hard to replace the homogeneity of V_0 by the weaker condition of *global extendability of every local isometry* of V, but without such a condition one may (?) have counter examples.

(3) Let us generalize the above by considering an arbitrary *compact simply connected homogeneous manifold* V_0. Now we need not only a correspondence between geodesics in V and V_0 but between arbitrary paths which are related by means of their Frenet equations. Then also we have a correspondence between families of paths in V and V_0. In particular, every homotopy of a geodesic segment γ_0 in V_0 induces a homotopy of the corresponding segment γ in V as homotopies are families of paths. Now every geodesic segment γ_0 in V_0 of length $\ell > \text{Diam}\, V_0$ can be homotoped to a shorter path and the corresponding homotopy in V also shortens γ for small enough ε. Since $\gamma \subset V$ can be chosen arbitrary and then shortened with the shortening of the corresponding γ_0 in V_0, we see that he diameter of V is bounded by

$$\text{Diam}\, V \leq \ell \underset{\varepsilon \to 0}{\longrightarrow} \text{Diam}\, V_0 .$$

Here as earlier one can often remove the homogeneity condition on V_0 with some necessary precaution.

1.3 The collapsing problem. The above diameter discussion has nothing specifically Riemannian about it and could have been conducted in the language of (compact and simply connected !) leaves of foliations. A new non-trivial point which emerges in the abstract Riemannian framework (and which does not easily appear for submanifolds of a given manifold) is the existence of *collapse* : one may have a sequence of manifolds (V_i, g_i) which infinitesimally converge to (V_0, g_0) at every point, but nevertheless go further and further from V_0 globally as they have the volumes $\to 0$,

$$\text{Vol}\,(V_i, g_i) \to 0, i \to \infty.$$

By a result of Cheeger, (see $[\text{Che}]_2$) this volume collapse is equivalent in our case (of bounded diameter and curvature) to the collapse of the injectivity radii of V_i.

In fact the injectivity radius of an arbitrary closed manifold V at a given point $v \in V$ satisfies

$$C_1(\text{Vol}\, V)^n \leq (\text{In}_j\, \text{Rad}_v\, V)^n \leq C_2\, \text{Vol}\, V$$

where C_1 and C_2 are positive constants depending on $n = \dim V$, Diam V and $\sup |K(V)|$.

If the collapse does not occur and the manifolds V in question have $\text{Inj Rad}\, V \geq \rho > 0$ then the pinching problem is easily solvable whenever the diameter of V is bounded by a fixed constant. Namely one has the following result which is implicit in the work of Cheeger (see $[\text{Che}]_1$, $[\text{Che}]_2$).

Non-collapsed pinching. *Let* (V_0, g_0) *be an n-dimensional locally homogeneous manifold and let* di *and* $v\ell$ *be positive constant. Then there exists a positive* ε *(depending on* V_0, di *and* $v\ell$*) such every closed manifold* V *with*

$$\text{Diam } V \leq \text{di} \quad \text{and} \quad \text{Vol } V \geq v\ell,$$

which is infinitesimally ε*-close to* V_0, i.e.

$$\text{dist}_{C^i}(V, V_0) \leq \varepsilon \text{ for some } i \geq 2,$$

admits a C^{i-1}*-smooth metric* $\tilde{g}_0$ *with the following two properties,*

(1) $\tilde{g}_0$ *is infinitesimally isometric to* g_0 *with order* i *at every point* $v \in V$, i.e. *there exists a diffeomorphism of a small neighbourhood* $U \subset V$ *of* v *into* V_0, say $\tilde{\varphi} : U \to V_0$ such that $\tilde{g}_0 - \tilde{\varphi}^*(g_0)$ vanishes at v with the derivatives of orders $1, 2, \ldots, i-1$.

(2) $\tilde{g}_0$ *is* δ*-close to* g *in the* C^{i-1}*-topology where* $\delta \to 0$ *for* $\varepsilon \to 0$.

Remarks.

(a) The metric $\tilde{g}_0$ is somewhat better than stated. Namely if $i = 2$ its curvature tensor R is almost everywhere defined and bounded. In general, for $i \geq 2$ this applies to covariant derivatives of R of order $i - 2$. Furthermore, at almost all points $\tilde{g}_0$ is infinitesimally isometric to g_0 with order i.

(b) The condition (1) implies for large i (about $\frac{3}{2}n$) that $\tilde{g}_0$ is locally isometric to g_0. In fact in many cases this i do not have to be very large. For example, if V_0 has *constant sectional curvature*, then $i = 2$ is enough and if V_0 locally symmetric then $i = 3$ will do. (Compare p. 165 in $[\text{Gro}]_2$ and [K - T - V].

Let us indicate a relation of the non-collapsed pinching with the pinching of submanifolds in the Hilbert space. We assume that the manifolds V in question have Inj Rad $V \geq \rho$ and we take a standard smooth function on $[0, \rho]$ which equals 1 near zero and vanishes near ρ. Then this function is extended to $[0, \infty]$ be zero and denoted p(d), $d \geq 0$. Now we assign to each $v \in V$ the function p(dist (v, ·) on V. Thus we map V into the Hilbert space $\mathbb{R}^\infty$ of functions on V and similarly we map $V_0 \to \mathbb{R}^\infty$ using $p(\text{dist}_{V_0}(V_0, \cdot))$. If Vol V = Vol V_0, and V_0 is simply connected then we are in a position to use the elliptic pinching in $\mathbb{R}^\infty$ and to conclude that (V, g) is globally close (e.g. diffeomorphic) to (V_0, g_0). The same applies if Vol V is ε-close to Vol V as one can acheive the equality by rescaling g, but the general case requires a more complicated reduction (of the Riemannian pinching) to Reeb's stability theorem.

Instead of the distance function one can sometimes successfully use more sophisticated analytic means, such as the *heat kernel* or *eigenfunctions* of the Laplace operator on V. The latter works for example, if V_0 equals the standard unit sphere S^n and V is simply connected. Then the first n + 1 eigenfunctions (orthogonal to const) nicely place V into $\mathbb{R}^{n+1}$ where the image lies close to the unit sphere for small enough ε.

1.4. Prevention of collapse. If the manifold V_0 has non-trivial Euler characteristic then the Euler-Gauss-Bonnet integrant Ω is nowhere zero (here we assume V_0 is homogeneous) and so the same is true for V if $\text{dist}_{C^2}(V, V_0)$ is sufficiently small. It follows that $\chi(V) \neq 0$ in this case and since $\chi(V)$ is an integer we conclude that

$$\int_V |\Omega_g|\, dv = |\chi(V)| \geq 1.$$

Thus Vol V ≥ $(\sup |\Omega_g|)^{-1} > 0$ and so the (volume) collapse is impossible. This exactly where Cheeger's criterion Volume collapse ⇐ Inj. rad collapse, becomes especially useful. For example it allows us to apply (following Cheeger) the non-collapsed pinching proposition to compact simply connected homogeneous spaces V_0 with $\chi(V_0) \neq 0$ or similarly to those which have a non-vanishing Pontryagin number.

1.5. Examples of collapse. The cyclic group $\mathbb{Z}_i$ may freely and isometrically act on S^n for every odd n. Thus we obtain a collapsing sequence $V_i = S^n/\mathbb{Z}_i$, $i \to 0$ whose members are infinitesimally identical to $V_0 = S^n$.

It requires some effort (see below) to take care of this rather simple minded collapse coming from free group actions, but nothing unexpected happens along the road. But the following example may (and should) give a painful shock to an unprepared mind.

Berger's construction. Let the 2-torus $\mathbb{T}^2$ freely and isometricly act on a compact manifold $\tilde{V}$ and let S^1 be some circle (group) in $\mathbb{T}^2$. One can always construct a sequence of circles $S^1_\varepsilon \subset \mathbb{T}^2$, for $\varepsilon = \varepsilon_j \to 0, j \to \infty$, (i.e. of compact 1-dimensional subgroups whose Lie algebras (i.e. the tangents at the identity id ∈ $\mathbb{T}^2$) converge to that of $S^1 = S^1_0$ and such that all S_ε for $\varepsilon > 0$ are different from S_0. Then necessarily, length $S_\varepsilon \to \infty$ for $\varepsilon \to 0$. All this has nothing to do so far with $\tilde{V}$, but now we consider the manifolds $V_\varepsilon = \tilde{V}/S^1_\varepsilon$. Since $\mathrm{Lie}(S^1_\varepsilon) \to \mathrm{Lie}(S^1_0)$ the local geometry of V_ε converges to that of V_0, i.e.

$\mathrm{dist}_{C^i}(V_\varepsilon, V_0) \underset{\varepsilon \to 0}{\to} 0$, for every i = 2, 3, ... But since length $S^1_\varepsilon \to \infty$ we have

$$\mathrm{Vol}\, V_\varepsilon = \mathrm{Vol}\, \tilde{V}/\mathrm{Vol}\, S^1_\varepsilon \to 0.$$

Notice that if $\tilde{V}$ is simply connected then so are the manifolds V_ε and this collapse can not be dismissed on the topological basis as in the previous case. In fact one may have a situation where all V_ε are (accidentally) diffeomorphic to V_0. This happens, for example, for $\tilde{V} = S^3 \times S^3$ with the double Hopf action of $\mathbb{T}^2$ and with $V_0 = S^2 \times S^3$. On the other hand, nothing of this kind ever happens to the products of the spheres of dimension ≥ 3 and to explain this distinction Cheeger has devised the following.

1.7. Collapse prevention criterion. *If the (compact homogeneous) manifold* V_0 *is 2-connected (i.e. the first and the second homotopy groups are zero), then every* V *which is infinitesimally sufficiently close to* V_0 *has*

$$\mathrm{Vol}\, V \geq C(\mathrm{ord}\, \pi_1(V))^{-1},$$

for some positive constant C *depending on* V_0 (See [Che]$_1$ where this criterion is stated for *symmetric* spaces V_0 in geometric rather than topological terms).

Sketch of the proof. Since $\pi_1(V_0) = \pi_2(V_0) = 0$, there exist a polyhedron $\Sigma \subset V_0$ of codimension ≥ 3 such that the complement is diffeomorphic to the unit Euclidean Ball. We think of such ball as the union of unit straight segments using from the center and then the complement $V_0 - \Sigma$ becomes a union of paths in V_0 issuing from some point in V_0. Then we take the corresponding paths in V and thus obtain a continuous maps, $V_0 - \Sigma$ to V, denoted α. This map is locally diffeomorphic away from Σ i.e. on the domain $\Omega_\delta \subset V_0 - \Sigma$ consisting of the points which are δ-far from Σ, where $\delta \to 0$ for $\varepsilon \to 0$. Moreover, the map α is almost isometric on Ω_δ,

$$\|g_0 - \alpha^*(g)\| \mid \Omega_\delta \leq h$$

where $h \to 0$ for $\varepsilon \to 0$.

Now, we want to use the codimension inequality codim $\Sigma \geq 3$ in order to extend the map α to all of V_0. We observe, that every short loop in Ω_δ, where "short" means of length $\leq$ const δ, can be filled in by a small disk in Ω_δ, also of size about δ. (This property may be not satisfied without some extra precautions but we can always slightly modify Ω_δ to have if fulfilled). In other words Ω_δ is *uniformly locally simply connected*.

First we look at the extension problem at a point $v = \alpha(v_0)$ for some $v_0 \in V$ under the assumption that the injectivity radius $\text{Inj Rad}_v(V) \geq \rho > 0$ for some fixed ρ which is much greater than δ and we try to extend α to a continuous map in the ball $B(\delta') \subset V_0$ around v_0 of radius δ' slightly greater than δ, say $\delta' = 10\delta$. If v_0 lies far from the boundary of Ω_δ and $B(\delta')$ is contained in Ω_δ there is nothing to do as α is already defined on $B(\delta)$. The interesting case is where v_0 lies outside Ω_δ, say in Σ. Since codim $\Sigma \geq 2$ the intersection $B(\delta'') \cap \Omega_\delta$ is connected (or can be made such by a little adjustment). It follows that every isometry $B(\delta') \cap \Omega_\delta \to V_0$ uniquely extends to an isometry $B(\delta') \to V_0$. Now, as our manifold V is ε-close to V_0 and has $\text{Inj Rad}_v(V)$ much greater than δ', the geometry of V around the α-image of $B(\delta') \cap \Omega_\delta$ is approximately equal to that in V_0. It follows that α extends to $B(\delta')$ by an almost isometric diffeomorphism.

Now, in the general case, where the injectivity radius of V at v may become arbitrarilly small we consider the exponential map at v of the ρ-ball $B_\rho \subset T_v(V)$ to V and we give this ball the metric induced from the metric of V. The radius ρ here must be much greater than δ', but it also must be not too big in order for the exponential map to be an immersions on B_ρ. In fact, this map looks very much like a covering map. For example, every short (i.e. much shorter than ρ) loop in V at v can be lifted to B_ρ provided it can be filled in by a small (compared to ρ) disc in V. Using this and the uniform simply connectedness of Ω_δ one can show that the map α on $B(\delta') \cap \Omega_\delta$ can be lifted to a continuous map $\tilde{\alpha} : B(\delta') \cap \Omega_\delta \to B_\rho$ where the surrounding geometry of B_ρ is ε-close to the geometry of V_0. By the previous case the lifted map extends to $B(\delta')$ and this extension is then projected back to V_0. This operation is performed at all points in V_0 (or rather in Σ) and with some effort the local extensions can be made coherent at different points. Thus we obtain the desired map $\beta : V_0 \to V$ which can be also made locally diffeomorphic at all points and, moreover, having the norm $\|g_0 - \beta^*(g)\|_{C^{i-1}}$ as small (for $\varepsilon \to 0$) as one wishes. This map β is, in particular, a covering map, say k-sheeted covering for $k = \text{ord}\, \pi_1(V)$ and so the volume of V is close to $\text{Vol}\, V_0 / k$ which is more than was needed to prove.

1.6. Taking care of $\pi_1(V)$. The deck transformation group of the covering map $\rho : V_0 \to V$ acts on V_0 by almost isometric transformations and these individually, can be approximated by isometries of V_0 by elementary (stability) properties of isometries of compact manifolds. The resulting map $\pi_1(V) \to \mathrm{Iso}\, V_0$ is not, however, a homomorphism, but the error can be made arbitrarilly small for $\varepsilon \to 0$. Then one can use a theorem of Grove, Karcher and Ruh (See [G-K-R] where the model $V_0 = S^n$ is considered) which allows us to perturb an almost homomorphism to an actual homomorphism with the allowed error (harboured by "almost") independent of ord π_1. (This result is yet another stability theorem for compact groups).

Now we bring everything together and obtain the following version of Cheeger's pinching theorem.

Stable elliptic models. *Let* $V_0 = (V_0, g_0)$ *be a compact homogeneous Riemannian manifold (model) with* $\pi_1(V_0) = \pi_2(V_0) = 0$. *Then there exists an* $\varepsilon > 0$, *such that every closed Riemannian manifold* $V = (V, g)$ *(infinitesimally) having*

$$\mathrm{dist}_{C^i}(V, V_0) \le \varepsilon \quad \text{for some } i >> \frac{3}{2}n$$

admits a metric $\tilde{g}_0$ *which is locally isometric to* g_0 *and such that* $\tilde{g}_0 \to g$ *in* C^{i-1}-*topology for* $\varepsilon \to 0$.

Remark. If $V_0 = S^n$, then one may take $i = 2$ by the theorem of Grove-Karcher-Ruh and if V is symmetric $i = 3$ suffices.

1.8. Pinching with compact semisimple models. We want now to understand what happens if the collapse prevention does not work, for example if $V_0 = S^2 \times S^3$ and the ordinary stability is destroyed by Berger's example. We assume here, that the isometry group G of V_0 is semisimple (as well as compact and transitive on V_0). Notice that this group can be described in *local* terms on V. Namely we pick up a frame f_0 at some point $v_0 \in V_0$ and then G can be identified with the set F_0 of all frames obtained by infinitesimal isometries (of high enough order) moving v_0 to other points in V_0. Now if we have a manifold V infinitesimally close to V_0 we can distinguish a certain set of frames f on V which are infinitesimally allmost isometric to f_0 and whose totality F is a manifold with a (naturally constracted) metric which is infinitesimally close to the (homogeneous) manifold F_0.

Since F_0 is (diffeomorphic to) a semisimple Lie group a finite covering $\tilde{F}_0$ of F_0 has $\pi_1(\tilde{F}) = \pi_2(\tilde{F}) = 0$ and so $\tilde{F}_0$ is a stable elliptic model. It follows that the original metric in F can be perturbed to a new metric which makes F locally homogeneous (and locally isometric to F_0). In fact F is homogeneous of the form $\tilde{F}_0/\Gamma$ for a finite group Γ and it is also acted upon by the isotropy subgroup H. (Recall that $V_0 = G/H$, where H can be identified with the set of our distinguished frames at v_0). This action is not a priori isometric for the new (homogeneous) metric in F, but by an appropriate generalization of the above mentioned Grove-Karcher-Ruh theorem this property can be achieved by a small perturbation. (Some difficulty here stems from the fact that the order of Γ may be arbitrarilly large for $\pi_1(V)$ large and so the almost action of H is defined on manifolds with possibly collapsing geometries. This problem does not appear if we impose an a priori bound on $\pi_1(V)$ but in a general case one must exercise some caution. Yet, it seems nothing goes wrong in the end).

The conclusion of this discussion is as follows.

Quasistability of V_0. If (V, g) is infinitesimally ε-close to a compact homogeneous manifold (V_0, g_0) with Iso V_0 semi-simple, then V admits a locally homogeneous metric g' provided $\varepsilon > 0$ is small enough (depending on V_0). (Notice that g' does not have to be locally isometric to g_0 but, infinitesimally, $g' \to g_0$ for $\varepsilon \to 0$).

Remark on non-complete locally homogeneous models. It may be interesting to look at non-complete locally homogeneous models V_0 which do not extend to complete manifolds. For example, let $\tilde{V}$ come as earlier with $\mathbb{T}^2$ action and take a copy of $\mathbb{R}$ in $\mathbb{T}^2$. Then the manifold f = $\tilde{V}/\mathbb{R}$ is defined locally but makes no global sense anymore. Here every V infinitesimally close to this V_0 must be of the type $\tilde{V}/S^1_\varepsilon$ for the circles S^1_ε approximating our $\mathbb{R}$ in T^2. (If there is a non-zero fundamental group $\pi_1(V)$ one may also have V diffeomorphic to $\Gamma \backslash \tilde{V} / S^1_\varepsilon$ for a finite group Γ essentially equal to $\pi_1(V)$).

Remark on the role of $\pi_1(V)$. In all situations we have met so far the group $\pi_1(V)$ was just a nuisance as it was making the injectivity radius of V small. On the other hand, this group acts on the universal covering $\tilde{V} \to V$ and so adds to the symmetry of $\tilde{V}$. Thus one may eventually use (not fight) $\pi_1(V)$ in order to improve pinching results for large π_1 (compare [Im-Ruh]).

Non-homogeneous models. If V_0 has no local isometries at all, then for every $v \in V$ the image of v under our ε-isometry $\varphi : U \to V_0$ is uniquely defined up to an arbitrarily small error. This trivially implies the existence of a global almost isometric map $V \to V_0$. A more interesting picture emerges if V_0 does have some symmetries. In fact it seems reasonable to fix an isometry group G acting on V_0 and also to prescribe a map $b : V \to V_0/G$. Now the infinitesimal closeness of V to V_0 is expressed in terms of those maps $\varphi : U \to V_0$ which project to $b|U$ when we pass to quotient. (In the comogeneous case V_0/G is a single point and b is the constant map). Probably, most of our elliptic pinching discussion generalizes to the non-homogeneous case in this setting. Then one may think of some induction theorems relating the stability of V_0 to those of the fiber and the vase of the fibration $V_0 \to V_0/G$, at least in the base where this projection is a fibration.

Remark on the P.D.E. approach. All our discussion was based on the O.D.E techniques introduced by Rauch and developed by Cheeger. Yet, it seems more natural to construct a locally homogeneous metric on V by modifying the original g using some kind of a *heat flow* which would dissolve the non-homogeneity (as this is assumed ε-smalle to start with). Such approach was consistently applied by Ruh (see $[Ruh]_1$, $[Ruh]_2$, $[Ruh]_3$) and also Min-O and Ruh (see $[Mi\text{-}Ruh]_2$ [Min]). Also some geometric techniques inspired by P.D.E are used by Im Hof and Ruh [Im-Ru] in their :

Diff-sphere theorem. *If the sectional curvatures of a simply connected* V *are pinched between* 0,68 *and* 1 *then* V *is diffeomorphic to* S^n (You bet 0,68 is not the final solution).

Remarks on the infinite dimensional manifolds. The 0.D.E. technique has one important advantage over P.D.E. as they apply in the infinite dimensional case. For example, the $\frac{1}{4}$-pinching (sphere) theorem goes to the infinite dimension without any modification in the proof. In the general case we should be aware of many non-equivalent (for $n = \infty$) ways to measure the infinitesimal distance between V and V_0, where not all measurements lead to interesting (and yet correct) results.

Non-Riemannian pinching. What should be the pinching (stability) philosophy for non-Riemannian structures, e.g. for pseudo-Riemannian metrics g ? If the (homogeneous) model (V_0, g_0) has not a too big isometry group G, i.e. the isotropy subgroup H of $v_0 \in V$ is *compact,* then one can reduce the pinching problem to the Riemannian case. (Sometimes one needs a slightly stronger technical condition of "stable compactness" of H). On the other hand, if H is non-compact, the set of the ε-isometries $\varphi : U \to V_0$ also becomes non-compact, (even if $\varphi(v) = v_0$ is fixed) and it becomes unclear what to do. Probably, one should start with cases where H is a small, (e.g. a 1-dimensional) group. An interesting geometric structure with 1-dimensional non-compactness of H is the conformal structure. Can one ever make good sense of (V, g) being almost conformally equivalent to (V_0, g_0), such that this notions would be invariant under the conformal changes of the metrics g and g_0 ? If "yes" what are good models (V_0, g_0) to look at ? (The obvious model $V_0 = S^n$ appears the most difficult for pinching).

The simplest (and most fundamental) case of the homogeneous pinching problem is that where the isotropy group H is trivial. Then the corresponding structure on V_0 can be reduced to a frame of vector fields, say $X_1^0, \dots, X_n^0$, such that they generate the Lie algebra isomorphic to Lie(G). In particular

$$\left[X_i^0, X_j^0\right] = \sum_{k=1}^{n} a_{ijk}^0 X_k^0$$

where a_{ijk}^0 are constant on V_0 (and equal the structure constants of Lie(G) in some basis). Now the (pinched) structure on V is given by independent fields X_i, such that the coefficients a_{ijk} in the decomposition

$$[X_i, X_j] = a_{ijk} X_k$$

are *almost constant* and then $(V, X_1, \dots, X_n)$ is called according to Ruh (see $[\text{Ruh}]_3$) an *almost Lie group*. The importance of this notion for the general pinching problem stems from the fact that every homogeneous pinching problem (with non-trivial H) reduces to the above case of H = id by passing to an appropriate frame bundle (compare 1.8).

<u>Remark on removing C^i</u>. One can remove the smoothness assumptions on V in the Riemannian pinching problem by using neighbourhoods $U_0 \subset V_0$ mapped into V (rather than $U \subset V$ mapped to V_0). For example, in the case of a homogeneous V_0 we may fix some neighbourhood $U_0 \subset V_0$ of a fixed point $v_0 \in V_0$ and then say that V is *locally* ε-close to V_0 if for every $v \in V$ there exists a C^1-map $\varphi_0 : U_0 \to V$, such that $\varphi_0(v_0) = v$ and the induced metric $\varphi_0^*(g)$ is uniformly (i.e. C^0) ε-close to g_0. In this setting there is little gain in using C^i for $i > 0$ since we do not allow U_0 to depend on V. Most of our C^i-pinching results can be made C^0 in this modified sense.

§ 3 - Parabolic pinching and almost flat Riemannian manifolds.

Let the model manifold V_0 be $\mathbb{R}^n$ with the standard Riemannian metric g_0. Then every compact (V, g) can be brought infinitesimally close to V_0 if we multiply g by a large constant. This excludes any meaningful stability (pinching) conclusion unless we assume some bound on the size of V. The simplest such bound is

$$\text{Diam } V \le \text{const},$$

and if this is assumed, then the infinitesimal closeness

$$\text{dist}_{C^2}(V, V_0) \le \varepsilon,$$

makes the global geometry as much Euclidean as is conceivably possible. Namely, *there exists a positive constant* $\varepsilon = \varepsilon_n$, *such that if* V is *ε-flat*, i.e.

$$|K(V)| \, (\text{Dim } V)^2 \le \varepsilon,$$

then V *admits a locally homogeneous metric* g' *which converges to* g *for* $\varepsilon \to 0$.

The above metric g' is not, in general, locally flat (Euclidean) but it is an immediate neighbour of a flat metric in the following sense : *the universal covering* $(\tilde{V}, \tilde{g}')$ *admits a free transitive isometric action of a nilpotent Lie group* N, *such that the intersection of the deck transformation group* Γ $(=\pi_1(V))$ *with* N *has finite index in* Γ. *Moreover, all of* Γ *is contained in the group of* (N-affine) *transformations generated by N and by automorphisms of* N.

The appearence of nilpotent groups here is not accidental but due to the fact that (the isomorphism classes of) nilpotent Lie algebras may lie infinitely close to the (isomorphism class of) Abelian algebra. (Recall that the space of isomorphism classes of n-dimensional Lie algebras is obtained by first taking the space of the Lie algebra structures on $\mathbb{R}^n$ given by the tables of structure constants in a fixed basis and then dividing by the group GL(n) which changes the basis. The resulting quotient space is non-Haussdorf at the point corresponding to the Abelian algebra. Moreover, the closure of this point contains all isomorphism classes of nilpotent algebras of dimension n).

The proof of the above ε-flat theorem is more difficult than in the elliptic case as one can not make Inj Rad $V \ge \rho > 0$ without making the diameter unduly large. Here one has to use both the O.D.E. and P.D.E. techniques (see [Bus-Bar], $[\text{Ruh}]_1$). It seems (I thought about the matter many years ago and dit not check all details in the course of writing this article) that these techniques can be combined with the elliptic pinching in order to yield the following conclusion.

Let the model (V_0, g_0) *be a complete simply connected homogeneous manifold whose isometry group* G *contains an Abelian normal subgroup* $A = \mathbb{R}^n$ *with a compact quotient group. Then for every constant* di *there exists* $\varepsilon = \varepsilon(V_0, \mathrm{di}) > 0$ *such that whenever a compact manifold* (V, g) *with* Diam $V \leq \mathrm{di}$ *is* C^i-*infinitesimally* ε-*close to* V_0 *for some* $i >> \frac{3}{2} n$, *there is a locally homogeneous metric* g' *on* V *which eventually converges to* g *as* $\varepsilon \to 0$.

3.1. Conjecture. The above conclusion remain true for *all* locally homogeneous models (V_0, g_0).

Remarks on collapse. Since we bound Diam V, the only source of difficulty comes from the collapse of V which means "no lower bound on Vol V". Now-a-days the structure of the collapse is understood pretty well (see [Ch-Gr], [Fu], [Ch-Fu-Gr]) and so the above conjecture appears not at all difficult.

Among special cases of this conjecture one can distinguish the following three possibilities.

1) Nilpotent (parabolic) case where V_0 admits a free transitive action of a nilpotent group N. Here one can probably apply the almost flat techniques used in the proof of the case $N = \mathbb{R}^N$ and one may even obtain more precise information concerning the nature of the locally homogeneous metric g' (for example if $N \neq \mathbb{R}^n$, then g' can *not* be locally flat).

2) Solvable case. Here V_0 equals a solvable Lie group G with a left invariant metric g_0. This mediates between parabolicity (if G is nilpotent or, say, unimodular) and hyperbolicity (where the metric g_0 has negative sectional curvature). For most models ($V_0 = G, g_0$) the (compact !) manifold V does not exist at all. It seems easy to show that V (of bounded diameter and infinitesimally close to V_0) may not exist unless V_0 admits a discrete group Γ of isometries with compact quotient V/Γ (of course, this remark has nothing to do with the solvability of G). On the other hand one has a clear picture of the (bounded diameter) collapse of solvmanifolds, which are quotients of G by discrete subgroups $\Gamma \subset G$, and one believes that the (possible) collapse of (V, g) must have essentially the same pattern at least for *generic* G-invariant metrics g_0 where the isometry group of (V_0, g_0) is not by far greater than G.

3) Symmetric case. If the space V_0 is symmetric it isometrically splits into three factors, $V_0 = V_0^+ \times \mathbb{R}^k \times V_0^-$, where V_0^+ is compact while V_0^- has negative Ricci curvature and non-positive sectional curvature. The part $V_0^+ \times \mathbb{R}^k$ has already been discussed and the (hyperbolic) remainder V_0^- will be handled in the next §. It seems our conjecture is practically solved in this case.

Quantitative almost flatness. The known estimates on $\varepsilon = \varepsilon_n$ in the ε-flat theorem are extremely poor and unrealistic ; one wonders if those can be improved. Moreover, one would like to have a sharp estimate of this kind with a description of the extremal cases. For example, what is the minimal value of $|K(V)|(\mathrm{Diam}\ V)^2$) in the class of all closed *simply connected* manifold V ? Does this value depend on $n = \dim V$? If not, is there an infinite dimensional generalization ?

It seems that $|K|\ (\mathrm{Diam})^2$ is not a right choice from the quantitative point of view and one should look for better invariants expressing the same (qualitative) idea but better behaving on the space of manifolds (V, g) in the quantitative sense.

§ 3 - Hyperbolic stability and pinching

We start our final discussion with an explicit definition of stability of a (locally) homogeneous Riemannian (model) manifold (V_0, g_0).

The manifold (V_0, g_0) (or the metric g_0) is called *stable* , if there exists $i = 2, 3$ and $\varepsilon > 0$, such that every complete manifold (V, g) which is infinitesimally close to V_0 with order i, i.e. $\mathrm{dist}_{C^i}(V, V_0) \le \varepsilon$ (see 0.7) admits a metric g_0^* locally isometric to g_0 such that
$g_0^* \to g$ (say in C^{i-2}-topology) for $\varepsilon \to 0$.

We say that (V_0, g_0) is *quasistable* if V admits a metric g_0^* which instead of being locally isometric to g_0 is only required to be ***locally homogeneous*** (and infinitesimally close to g_0).

We know (see § 1) that there exist *stable* compact homogeneous spaces (e.g. $V_0 = S^n$) and also quasistable but not stable ones (e.g. $S^2 \times S^3$). On the other hand, we have not seen yet a single example of a *non-compact* stable model. (Our model $V_0 = \mathbb{R}^n$ in § 2 is, of course non-compact, but there we needed a bound on Diam V which is not allowed by the above definition of stability).

Example. The hyperbolic plane $V_0 = H^2$, where g_0 has constant negative curvature, is stable.

Proof. Every complete surface (V, g) with a metric g of strictlty negative curvature $K \leq -\kappa \leq 0$ admits a *conformally equivalent* metric g_0^* of constant negative curvature. Futhermore if the curvature of g is everywhere close to that of g_0^* and g_0^* must be uniformly close to g by the Schwartz lemma, and if we have the higher derivatives of g under control we can insure the closeness in C^i-topologies for $i > 0$.

The above model probably remains stable (or quasistable) when multiplied by a compact homogeneous model, i.e. $H^2 \times V_0$ is stable (respectively quasistable) if and only if V_0' is such, where V_0' is a compact simply connected homogeneous manifold. But further non-compact (quasi) stable examples look hard (if at all possible) to come by. Here are some examples.

3.1. Non-stability of H^n and $H^{2n}_{\mathbb{C}}$. Let us explain why these hyperbolic spaces are unstable. The metric g_0 in H^n can be written in some (horospherical) coordinates $t, x_1, \dots x_{n-1}$, as follows,

$$g_0 = dt^2 + e^t\left(\sum_{i=1}^{n-1} d\,x_i^2\right).$$

Now take some constants $\lambda_i = 1 + \varepsilon_i$ and let

$$g_\varepsilon = dt^2 + \sum_{i=1}^{n-1} e^{t\lambda_i} d\,x_i^2.$$

If $\varepsilon_i \to 0$ then g_ε becomes infinitesimally inifinitely close to g_0. For example, the sectional curvature of g_ε uniformely converges to $1 = K(g_0)$. Yet, this g_ε is globally quite far away from g_0 unless $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-1}$. (Notice that this equality is automatic for $n = 2$ which makes H^2 special). In fact, (V, g_ε) is not quasi-isometric to (H^n, g_0) for non-equal λ_i (see [Gr-Pa]). In particular, there is no uniformly bi-Lipshitz diffeomorphism between (V, g_ε) and (H^n, g_0) and so for no $\varphi : V \to H^n$ the norm $\|g_\varepsilon - \varphi^*(g_0)\|_g$ is bounded by a small constant.

To get some idea why an apparantly small change of g_0 may so drastically affect the global geometry, we look not at (H^n, g_0) itself but on the *cusp* -manifold $(C^n, \bar{g}_0)$ obtained by dividing H^n by the group $\mathbb{Z}^{n-1}$ acting by translations on $\mathbb{R}^{n-1}$ with the coordinates $x_1, \ldots, x_{n-1}$, i.e.

$$(t, x_1, \ldots, x_n) \overset{z}{\mapsto} (t, x_1 + z_1 \ldots, x_{n-1} + z_{n-1})$$

for all $z = (z_1, \ldots, z_{n-1}) \in \mathbb{Z}^{n-1}$.

Topologically this C^n is $\mathbb{R} \times \mathbb{T}^{n-1}$ where the geometry of the torus $T_t^{n-1} = t \times \mathbb{T}^{n-1}$ is given by $\bar{g}'_t = e^t g'_0$, where g'_0 is the standard flat metric on $\mathbb{T}^{n-1} = \mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$. Now if we pass to g_ε with non equal $\lambda_i = 1 + \varepsilon_i$, then the torus T_t^{n-1} for very large t (or very small $t \to -\infty$) becomes the product of circles of greatly varying lengths and its shape become very different from any torus $(T_t^{n-1}, e^t g'_0)$. This shows that $(C^n, \bar{g}_\varepsilon)$ is not uniformly bi-Lipschitz homeomorphic to $(C^n, \bar{g}_0)$ and makes one believe that the same is true without any
$\mathbb{Z}^{n-1}$ (though the actual proof of non-quasi-isometry between (H^n, g_0) and (H^n, g_ε) goes along different lines, compare $[\mathrm{Pan}]_1$).

Notice that every metric g_ε is homogeneous for some solvable group depending on λ_i. Namely, if we denote by $A_\lambda : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ the linear operator given by the diagonal matrix with the entries λ_i, then we see that the metric $\sum_{i=1}^{n-1} e^{t\lambda_i} d x_i^2$ on $\mathbb{R}^{n-1}$ is obtained from the standard metric $g_0' = \sum_{i=1}^{n-1} d x_i^2$ by applying the t-th power of A_λ. Then we can write

$$g_\varepsilon = dt^2 + A_\lambda^t g_0'$$

and see that g_ε is invariant under the solvable group S obtained as the semidirect product of $\mathbb{R}$ by $\mathbb{R}^{n-1}$ with the help of the automorphism A_λ of $\mathbb{R}^{n-1}$.

Now we can extend the construction of g_ε to the case of the complex hyperbolic space $H_{\mathbb{C}}^{2n}$ which can be given (using horospherical coordinates) a structure of a solvable group obtained as a semidirect product of $\mathbb{R}$ by the *Heisenberg* group N^{2n-1} (See[Gr-Pa]). This is a nilpotent Lie group with a (standard) one-parameter family of dilations $A^t : N^{2n-1} \to N^{2n-1}$ and the complex hyperbolic metric g_0 is

$$g_0 = dt^2 + A^t g_0'$$

for a certain left invariant metric g_0' on N. Now, as in the real case one can slightly perturb A^t to another one-parameter group of automorphism, say $A_\lambda^t : N^{2n-1} \to N^{2n-1}$ and then the metrics $dt^2 + A_\lambda^t g_0'$ provides a counterexample to the stability of $(H_{\mathbb{C}}^n, g_0)$ if the automorphisms A_λ converge to A without being *conformal* for t he *Carnot* metric on N^{2n-1} (see $[\text{Pan}]_2$, $[\text{Pan}]_3$), where one can learn more about this story).

Remark. The (non-stable) perturbation of H^n and $H_{\mathbb{C}}^{2n}$ indicated above was locally homogeneous, but a simple modification with *variable* λ_i gives us non-homogeneous perturbations which show that these hyperbolic spaces are *not even quasistable*. Moreover, using the polar coordinates instead of the horospherical ones, one can produce perturbations g_ε with *compact supports* which are arbitrarily *close* to g_0 infinitesimally but arbitrarily *far* from g_0 in the global (quasi-isometry) sense.

Non-stability of $H_{\mathbb{H}}^{4n}$. If one tries a similar perturbation of the quaternion hyperbolic spaces or of the Caylay plane one will run into a trouble as all dilations of the corresponding nilpotent groups are conformal (see $[\text{Pan}]_3$). However, there is a different deformation of the geometry of the quatornian hyperbolic space $H_{\mathbb{H}}^{4n}$ pointed out to me by P. Pansu. Here again, the (symmetric) metric on $H_{\mathbb{H}}^{4n}$ is left invariant for a solvable group structure on $H_{\mathbb{H}}^{4n}$. The solvable group S which appears here has a normal nilpotent subgroup N of dimension $4n - 1 = \dim H^{4n}_{\mathbb{H}} - 1$ as earlier and the quotient group is $\mathbb{R}$ which acts on N by the (above mentioned) dilations. Although we can not essentially deform the dilation we can deform the group N itself as this has a 3-dimensional center. The deformed group goes along with a dilation which gives a deformation of the group S. Then a left invariant metric on the deformed S gives the desired example of a non-stable deformation of the original metric on $H^{4n}_{\mathbb{H}}$.

Symmetric spaces of ran k ≥ 2. The above deformation construction may apply to every symmetric space V_0 of non-compact type (this was also pointed out to me by by P. Pansu). Namely, every such V_0 admits a structure of a solvable group S (corresponding to AN in the KAN decomposition of $G = \mathrm{Iso}V_0$) and whenever S is deformable one can deform the metric g_0 of V_0. Yet, I have not checked if such deformations in reality exists. In some cases they do not exist, e.g. if $V_0 = H^2 \times H^2$, and then there is a chance that (V_0, g_0) is stable.

Remark. One may think that one can enhance the desired stability by passing from the Riemannian manifolds to Lie groups and to almost Lie structures (see 1.8) but in practice this does not seem to help.

3.2. Hyperbolic pinching with bounded diameter and volume. Let (V, g) be a *compact* Riemannian manifold without boundary which is infinitesimally ε-close to a symmetric space (V_0, g_0) of non-compact type with no Euclidean factor. For example, if V_0 is the real hyperbolic space this closeness (with order i = 2) amounts to a pinching of the sectional curvature of (V, g) between -1 and $-1-\varepsilon$. One may think that the asumption of compactness of V may aid the stability. Yet, this does not work for $V_0 = H^n$, $n \geq 4$. Moreover, there exist closed manifolds V of a given dimension $n \geq 4$ which arbitrarily infinitesimally close to H^n (e.g. $-1-\varepsilon \geq K \geq -1$ for a given $\varepsilon > 0$) but yet not homotopy equivalent to manifolds of constant curvature (see [Gr-Th]), where the examples can be even made *conformally flat*). Furthermore, there are examples of pinched (V, g) which are *homeomorphic but not diffeomorphic* to manifolds of constant negative curvature, starting from n = 7 (see [Far-Jo]). It is unknown if there are similar examples for $V_0 = H^n_{\mathbb{C}}$ but they may exist, for all we know, even among *Kähler* manifolds V. On the other hand nothing of this kind ever happens if we bound the diameter of V because of the following.

Non-collapsing property. *Let* (V_0, g_0) *be a symmetric space of non-compact type with no flat factor* (i.e. Ricci $g_0 < 0$). *Then there exists an* $\varepsilon > 0$, *such that if* (V, g) *is infinitesimally* ε*-close to* (V_0, g_0) *with order* 3, *then the injectivity radius of* V *at some point* $v \in V$ *is away from zero* ,

$$\text{Inj. Rad}_v(V) \geq \rho > 0$$

for some universal $\rho = \rho(V_0)$ (e.g. for $\rho = \varepsilon$).

This trivially follows form the corresponding result of Kazhdan and Margulis for $V = V_0$ (see p.p. 123, 124 in [B-G-S]).

Remark. If $K(V_0) < 0$, then one can replace "order 3" by "order 2" and this is also likely in the general case.

The non-collapsing property allows us to use the non-collapsed pinching (theorem) in 1.3 and conclude that the above V *admits a metric* g_0^* *locally isometric to* g_0 *and close to* g, *whenever the above* ε *is small enough depending on Diam* V. (Compare $[\text{Mi-Ruh}]_1$).

One may expect more geometric juice if the bound on the diameter is replaced by that on the volume of V. However if $K(V_0) < 0$ (and hence, $K(V) < 0$) then for dim $V_0 \geq 4$ the bound on Vol V insures a bound on Diam V (see$[\text{Gro}]_3$) and so nothing new happens in this case. On the other hand, dim V = 3 is more interesting, as one may have closed 3-dimensional manifolds of uniformly bounded volume and with arbitrary large diameters (see $[\text{Gro}]_4$). Yet the *topological pinching* result still remains valid : *If* (V, g_0) *has* ε*-pinched negative curvature with small positive* ε *depending on* Vol V, *then* V *admits a metric* g_0^* *of constant negative curvature (which is new is not necessarily close to g).*

The proof here (for n = 3) is less boring than for $n \geq 4$, as it relies on Thurston's cusp closing procedure (see $[\text{Gro}]_4$).

Finally we indicate a recently announced result by Ye [Ye] where he removes the bound on Vol V by using a heat flow on the space of metrics which brings g to the desirable g_0^* of constant curvature.

3.3. Pinching at large. We enlisted above all known results (and counter examples) concerning the *stability* problem in the symmetric hyperbolic (i.e. Ricci < 0) case, but there is much more to the pinching than a mere stability, if one takes the following more liberal geometric (rather than analytic) point of view. We fix as earlier a homogeneous model (V_0, g_0) and an integer $i \geq 2$ and then we define the *pinching number* of $V = (V, g)$ *relative to* (V_0, g_0) as $\mathrm{pi}(V) = \mathrm{dist}_{C^i}(V, V_0)$, that is the lower bound of those ε which satisfy $\mathrm{dist}_{C^i}(V, V_0) \leq \varepsilon$ in the sense of (*) in 0.7. (The number pi should be thought of as a generalization of the ordinary pinching number sup K(V)/infK(V). This $\mathrm{pi} : (V, g) \mapsto \mathbb{R}_+$ seems an interesting function on the space of Riemannian manifolds and we want to learn its basic properties. In particular, we want to characterize the manifolds V having $\mathrm{pi}(V, g) \leq \varepsilon$ for given $\varepsilon \geq 0$ (which brings us near the stability problem for $\varepsilon \to 0$).

More specificly, given a smooth manifold V, we may try to evaluate $\inf_g \mathrm{pi}(V,g)$, which thus becomes a topological invariant of V. There is some information in this direction for the ordinary pinching (see [Gr-Th] and $[\mathrm{Gro}]_5$) but the general case has not been studied much so far.

3.4. Generalized pinching. Let us again look a the pinching/stability problem from an analytic view point. We want to scrutinize the *homogeneity* of g_0 of our model manifold. For every such g_0, one can easily construct a system of differential equations, on the space of metrics g on a given n-dimensional manifold, such that the solutions of this system, say $\mathcal{D}g = 0$ are exactly the metrics which are locally isometric to g_0 (in words, the equation $\mathcal{D}g = 0$ should say that g is infinitesimally isometric to g_0 at every point with some given order $i >> \frac{2}{2}n$). Notice that the equations which appears here are *totally integrable* modulo Diff V (compare pp. 163-168 in $[\mathrm{Gro}]_2$) and so their solutions are governed by the laws of O.D.E. (which does not at all exclude P.D.E. from their study). The stability/pinching problem amounts in this language to the study of the inequalities $\|\mathcal{D}g\| \leq \varepsilon$ for $\varepsilon > 0$.

Once this is spelled out, we can try an *arbitrary* P.D.E. system $\mathcal{D}g = 0$, where the only condition we need to set up the problem is the *invariance* of this system under the *diffeomorphisms* of V, in order to have the system defined on every smooth manifold. Here are some interesting suggestions.

(1) Study (V, g) where g is *almost* (up to small $\varepsilon > 0$) *infinitesimally homogeneous* of given order i (compare pp. 164, 165 in $[\mathrm{GRo}]_2$ and [K-T-V] . This is still appears the realm of O.D.E, but the next two example are definitely of the P.D.E. nature.

(2) Metrics g which are *almost Einstein* , i.e. the Ricci tensor of g is ε-close to λg for a given constant λ. (These have been recently studied by Min-Oo and by Ye).

(3) Metrics g on V are *almost Kähler*. This means, V admits an *almost parallel* 2-form ω, such that the pair (g, ω) is *almost Hermitian* in an obvious sense.

3.5. Stability of hyperbolic foliations. We could not produce enough convincing expamles of stable non-compact models and now we try to justify ourselves by looking at the prestability problem for foliations (compare 0.2) which seems easier than the Riemannian counterpart. Here we study (as in 0.2) a foliation on a Riemannian manifold W whose tangent (sub)bundle is denoted $S \subset T(W)$ and $N = T(W)/S$ stands for the normal bundle. If w and w' are two points in some leaf of the foliation and Π a path from w to w' in this leaf then there is a well defined linear map $A = A^{\Pi} : N_w \to N_{w'}$, called the *holonomy operator*, which depends only on the homotopy class of Π in the leaf. Then we write

$$w \geq \lambda w' | \Pi$$

for some $\lambda > 1$ if the norm of the operator A satisfies $\|A\| \geq \lambda$, i.e. A expands all vectors in N_w at least λ times. This is expressed in words by saying that w' lies *λ-down-stream* of w relative to Π or that w is λ-upstream of w'. The idea is that the "stream" moves different leaves further and further apart.

The example one should have here in mind is that of the weakly contracting foliation F_{cont} (see 0.2) where the down-stream motion is realized by the geodesic flow moving the leaves of F_{cont} apart from each other.

Now a foliation F is called *coexpanding* if there exists a down-stream flow (as explained below) of diffeomorphisms $d_t : W \to W$, such that F equals the weakly contracting foliation for this flow.

Down-stream flows. A flow is called *down-stream* if it preserves every leaf of F and if there exist numbers $\lambda > 1$ and $t_0 > 0$, such that $d_t(w)$ lies λ-down-stream of w for all $t \geq t_0$ and $w \in W$, where the implied path is the orbit of the flow from w to $w' = d_t(w)$.

Recall, that the weakly contracting property of F amounts to the fact every compact smooth curve C which is contained in a single leaf of F satisfies

$$\text{length } d_t(C) \leq \text{const} < \infty$$

for $t \geq 0$.

This definition of coexpansion is by no means the most general, but it is the easiest to spell out. The idea behind it is that starting from each w one can go arbitrarily far down-stream. Moreover, given two points w_1 and w_2 in a leaf one can move them simultaneously down-stream without making them too far apart in the leaf. What one needs is the following property of the points w_1' and w_2' obtained by λ-down stream motions of w_1 and w_2.

Let A' denote the holonomy operator $N_{w_1'} \to N_{w_2'}$ corresponding to some path obtained by λ-down-stream motion from a fixed path between w_1 and w_2 in the leaf these points are contained in (we imagine the points w_1 and w_2 being fixed while w_1' and w_2' go further and further down stream with $x \to \infty$). Then the norms of the operators A' and $(A')^{-1}$ are small relative to λ. (Probably, a fixed bound, something like $\|A'| + \|(A')^{-1}\| \leq 0.1\,\lambda$ will do).

Remark on uniformity. If W is noncompact, the above definition must be strenghtend by making all numerical invariant uniformly controlled on W. For example, one may insists that the implied metric on W has *bounded geometry*, i.e. $|K(W)| \leq \text{const} < 0$, $\text{Inj Rad}_w(W) \geq \rho > 0$, and that the relative curvatures of the leaves of F are also bounded. Then in the definition of the coexpanding property we must bound the C^0 and C^1-norm of the field defining the flow and insist on the uniform bound on the expansion of curves in the leaves. The resulting notion is called *uniform coexpanding*.

Shadowing proposition. *Every coexpanding foliation on a closed manifold* W *is prestable in the sense of 0.2. Namely, every complete immersed submanifold* $V \subset W$ *with* $T(V)$ *sufficiently close to* S *is shadowed by a (unique) leaf* V^* *of the foliaiton.*

Idea of the proof. The flow d_t almost preserve V and so there is a flow $d_t' : V \to V$ which is (in an obvious sense) close to d_t. Then the required manifold V^* is obtained as an appropriate limit of $d_t^{-1} d_t'(V) \subset W$ for $t \to \infty$.

Here one may need (depending on a specific kind of the coexpanding property used) some additional regularization of $V_t = d_t^{-1} d_t'(V)$ along the road. Also one may take some V_t^t for a fixed large t for a new submanifold, say V' , and then use the iteration, V", V"', ... → V^*.

Remarks (a) The stability of expanding is standard in O.D.E. (see [Man]) but the proof smells of P.D.E. Also notice that this argument works under somewhat more liberal assumptions than ours concerning the regularity, continuity and inversibility of d_t.

(b) If W is non-compact, one needs a uniform control (see the previous remark) of all quantities in question and then one obtains the *prestability of the uniformly coexpanding* foliations.

Examples. We have already mentioned the weakly contracting foliations of geodesic flows on manifolds of negative curvature. If we want to have a coexpanding foliation of dimenion 1 we should forfait the compactness of W and then we have plenty of these. For example, the reversed flow in Fig 1 (see (0.4) consisting of the geodesics eminating from a fixed point of the ideal boundary (of a manifolds with $K < -\kappa < 0$), is coexpanding.

Next, if we are given two coexpanding foliations, say F_1 on W_1 and F_2 and W_2, then $F_1 \times F_2$ on $W_1 \times W_2$ (where the leaves are Cartesian products of those in F_1 and F_2) is also coexpanding.

Finally, we consider a locally symmetric space X of non-compact type with no Euclidean (flat) factor. Then we take the space W' of all *maximal flats* which are isometric (and totally geodesic) copies of $\mathbb{R}^k \subset X$ for $k = \text{rank } X$. If $k = 1$, this is the same as the space of non-oriented geodesics, i.e. $UT(X)/\mathbb{Z}_2$, and as for $k = 1$ there is a finite cover $W \to W'$ whose every element w over w' is distinguished by a choice of a *Weyl chamber* in the flat w'. There is a natural action of $\mathbb{R}^k$ on W (generalizing the geodesic flow for $k = 1$) which displays definite hyperbolic features.

For example if we take a *regular* element $a \in \mathbb{R}^k$, i.e. being inside some Weyl chamber (where $\mathbb{R}^k$ is divided into chambers according to the corresponding structure in the flats), then there is a smooth weakly contracting foliation F^a_{cont} whose leaves contain the $\mathbb{R}^k$-orbits and the transformation $a : W \to W$ moves this foliation down stream, i.e. it coexpands the leaves. Thus F^a_{cont} is coexpanding and, consequently, a prestable foliation.

Cohyperbolic foliations. A foliation on W is called *cohyperbolic* if it is a transversal intersection of several coexpanding foliation on W (i.e. each leaf is a transversal intersection of coexpanding leaves). For example the geodesic flow foliation F on $W = UT(X)$ is cohyperbolic, namely $F = F_{cont} \cap F_{exp}$ if X has $K(X) \leq -\kappa < 0$. Similarly the $\mathbb{R}^k$-foliation F of the above chamber flow also is cohyperbolic as $F = F^a_{cont} \cap F^{a^{-1}}_{exp}$ for every regular $a \in \mathbb{R}^k$. Also notice that the Cartesian products and transversal intersections of cohyperbolic foliations are cohyperbolic.

Prestability theorem. *Every cohyperbolic foliation on a closed manifold* W *is prestable*. (If W is non-compact one must augment the cohyperbolicity by a uniformity requirement and the prestability does not suffer).

This result is standard in the theory of the dynamical systems (see[Man]) where it is usually stated in slightly different terms. The idea of the proof is as follows. We have our foliation F which is the interseciton of several coexpanding F_ν, $\nu = 1, 2, \ldots \eta$. Then every "almost leaf" $V \subset W$ can be slightly enlarged with every F_ν which makes $V = \bigcap_{\nu=1}^{\eta} V_\nu$. Then the proof of the shadowing proposition applies to these V_ν.

Remark. Our definition of cohyperbolicity seems unduly artificial and restrictive. A better definition should directly appeal to the set of hyperbolic *holonomy operators*. Recall, that the holonomy operator is the linear map $A : N_w \to N_{w'}$ associated to a path Π joining w and w' in some leaf. We call A *hyperbolic* if there exist orthogonal splittings $N_w = N^+_w \oplus N^-_w$ and $N_{w'} = N^+_{w'} \oplus N^-_{w'}$, such that A maps N^+_w to N^-_w, and N^-_w to $N^-_{w'}$ such that the norm of A on N^+_w is > 1 (A expands N^+_w) and on N^-_w this norm is < 1.

Example. Consider the standard action of the group $\Gamma = SL_m\mathbb{Z}$ on the torus $\mathbb{T}^m$ and let B be some (compact) manifold with $\pi_1(B) = \Gamma$. Then there is a natural torus fibration $W \to B$ with a horizontal foliation. Since the group Γ contains many hyperbolic automorphisms of $\mathbb{T}^m$, one may believe that this foliation is rather hyperbolic and prestable at least for large m. This is motivated by recent results by Hurder and Katok-Lewis on the stability of the action of Γ on $\mathbb{T}^n$.

Prestability of submanifolds in G-manifolds.

We use here the set-up indicated in the elliptic case in 1.1. Namely we take a complete Riemannian manifold acted upon by an isometry group G. In most examples G equals the full isometry group of X and is not mentioned explicitely. Here we take an n-dimensional orbit $S \subset X$ of some connected subgroups $H \subset G$ and we say that a given $V \subset X$ is *infinitesimally* ε*-close to* S *with order* i if for each point $v \in V$ there exists an isometry $g \in G$ which moves v ε-close to S such that some germ of V at v also becomes ε-close to S in some (fixed once and for ever) C^i*-norm*. We say that S is C^i*-stable* if the above closeness condition for a sufficiently small $\varepsilon > 0$ insures the existence of an isometry which brings all of V δ-close to S for $\delta \to 0$ with $\varepsilon \to 0$. We have seen in 1.1 that this is a special case of the prestability for foliations. On the other hand some of the above mentioned stability results for co-hyperbolic foliations can be reformulated in the G-language.

Example. let X be a symmetric space with $K(X) < 0$ and S be a geodesic (which *is* the orbit of a certain subgroup in G = Iso X). Then S is C^2-stable as follows from the prestability of the geodesic foliation for $K < 0$.

More generally, let Ricci $X < 0$ (i.e. X is a symmetric space of non-compact type with no Euclidian factor) and $S \subset X$ be the maximal flat. Then this S is also C^2-stable for a similar reason. Notice that here one may expect a stronger P.D.E.-kind of stability where the bound on the second fundamental form of V, i.e. $\|II(V)\| \le \varepsilon$ encoded into the C^2-closeness condition, is replaced by a weaker bound, say on the mean curvature of V, i.e. $\|\mathrm{Trace}\ II(V)\| \le \varepsilon$. It is easy to prove this kind of stability if we know a priori that V lies within bounded distance δ from some maximal flat V_0. In fact, a simple application of the maximum principle to the function $x \mapsto \mathrm{dist}\,(X, V_0)$ on X shows that this δ is bounded in terms of the mean curvature of V, i.e. $\delta = \delta(\varepsilon)$ for

$$\varepsilon = \|\mathrm{Mean.\ Curv}\| = \|\mathrm{Trace}\ II(V)\|$$

and $\delta(\varepsilon) \to 0$ for $\varepsilon \to 0$.

The most interesting example of stability known up to date is provided by the *quasi-isometric rigidity theorem* of Pansu (see $[\mathrm{Pan}]_3$). This theorem claims that every quasi-isometry (e.g. a uniformly bi-Lipshitz homeomorphism) of a quaternian hyperbolic space $Y = H^{4m}_{\mathbb{H}}$ into itself, say $f : Y \to Y$ lies within finite distance from some isometry f_0, i. e

$$\sup_{y \in Y} \mathrm{dist}\,(f_0(y), f(y)) < \infty .$$

Pansu also proved this rigidity for Y the hyperbolic Cayley plane. On the other hand, Margulis has conjectured such rigidity many years ago for symmetric spaces X where all irreducible factors have rank ≥ 2. In view of what we know, the conjecture should encompass those symmetric spaces Y whose isometry group G has Kazhdan's property T.

Now we consider $X = Y \times Y$ and take the diagonal of $Y \times Y$ for S. If some $V \subset X$ is C^1-infinitesimally close to this S, then it serves as a graph of a bi-Lipshitz homeomorphism $Y \to Y$ and so can be approximated by an isometric translate S' of S representing the graph of some isometry $Y \to Y$, whenever one can use Pansu's theorem or willing to accept the above conjecture. Then it is easy to see that S' necessarily lies δ-close to V where $\delta \to 0$ if the implied ε goes to zero.

Remark. Simple examples show that neither Pansu's theorem nor the above stability corollary hold true for the real and complex hyperbolic spaces Y.

Now, let X be an arbitrary symmetric space of non-compact type with Ricci $X < 0$ and let S be an H-orbit for $H \subset G = \text{iso } X$. For example one may take some totally geodesic submanifold of X for S. Then one believes that in many cases this S is stable. On the other hand there are certain counter examples which indicate the limit of the stability. Here are some of them.

Let X be the real hyperbolic space H^n and S a totally geodesic subspace of dimension m in the interval $2 \leq m \leq n-1$. Then this S is unstable. To see this we observe that every totally geodesic subspace $S \subset H^n$ hits the ideal boundary $\partial H^n = S^{n-1}$ by a round sphere $\Sigma = \partial S$ of dimension $m-1$. One can slightly perturb this sphere and then the new surface Σ' "bound" a submanifold $S' \subset X$ which only slightly non-geodesic. Yet it is very far from anything totally geodesic because $\Sigma' = \partial S'$ is not spherical.

Let us notice that the problem here is purely asymptotical. For example, if ∂V is *round* in $S^{n-1} = \partial H^n$, (e.g. V lies within bounded distance from some totally geodesic submanifold) then V lies δ-close to the totally geodesic submanifold S with $\partial S = \partial V$ where $\delta \to 0$ for $\varepsilon \to 0$, and where ε is (as usual) the measure of the flatness of V. Moreover, one can replace here the bound $\|II(V)\| \leq \varepsilon$ by $\|\text{Trace} II(V)\| \leq \varepsilon$ as we had done at previous occasions. (See $[\text{Gro}]_5$ for further P.D.E. stability results of this kind).

It seems a similar picture persists for all symmetric spaces of rank 1 (i.e. for the hyperbolic spaces over $\mathbb{C}$, $\mathbb{H}$ and the Cayley plane) and so the stability of submanifolds S needs rank $X \geq 2$ and $S \subset X$ must be maximal in a certain sense (e.g. admitting no totally geodesic submanifolds between S and $X \supset S$).

Linear stability problem. Let $X \to B$ be a Euclidean vector bundle over a Riemannian manifold B and let G act coherently on B and on X, where the action on B should be isometric and the action on X fiberwise linear and isometric. The images of sections $B \to X$ are submanifolds, X and so the following stability problem nicely fits into our framework. When can be an almost G-invariant section $B \to X$ approximated by a G-invariant one ?.

The examples we have in mind are those where G acts transitively on B and $X \to B$ is some natural bundle. For instance, one can take a symmetric space B and some tensor bundle which has invariant sections. One knows in this regard that many symmetric spaces carry invariant exterior forms which are necessarily harmonic. Then one may look at an almost invariant form and try to see what happens to its harmonic projection. Here it may be also interesting to replace B by another manifold B' which is infinitesimally close to B and then the invariant forms on B because almost invariant (or almost parallel) on B'. If B' is compact one may try to prove that the harmonic projection does not annihilate (some of) these forms. This would give a non-trivial topological information on B'. Notice, that if an invariant form ω on B is *characteristic* (i.e. represents via Chern-Weil a characteristic class of the tangent bundle with the relevant structure group) then its "shadow" w' on B' can be assumed *closed* and then the harmonic projection can not annihilate w'. Yet it is unclear how far this projection Pw' lies away form w in the uniform sense.

Our final question appears the simplest (to ask) of all. When (and how) can an almost left invariant tangent field on a (semi simple) Lie group be approximated by an invariant field ?

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