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RATIONAL CONFORMAL FIELD THEORY AND INVARIANTS OF 3-DIMENSIONAL MANIFOLDS

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Abstract : 3-dimensional topological field theory is defined by arbitrary RCFT.

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Recently E. Witten [W] defined 3-dimensional field theory with Chern-Simons action using 2-dimensional conformal gauge theory. Here I give a general construction of 3-D theories from arbitrary rational conformal field theory and describe also the converse construction.

After a very brief introduction to RCFT in Section 1, I give a series of definitions of different types of theories in section 2-4. The main constructions are in Sections 5 and 6. In Section 8 there are several remarks about central extensions and in Section 9, other remarks. The contents of Sections 3 and 7 are not used in the rest.

Convention : in Section 2-8 all manifolds are compact, smooth and oriented. Changing of orientation is denoted by a bar. Sign $\cup$ denotes a disjoint sum.

1. Rational Conformal Field Theory (RCFT). (More detailed reviews are in [MS] and [G]).

Examples of such theories are the scaling limits of exactly solvable 2-dimensional lattice models at the critical temperature.

RCFT is a field theory on all compact surfaces with a complex structure and with a finite set A of primary fields. As any field theory it is defined by its Green functions.

Choose some complex coordinates $m_1,...,m_{3g-3}$, on an open domain in the space of Riemanian surfaces of genus g. Let some family of nonintersecting disks D_i, $i = 1 ...n$. in surfaces be given together with a family of identifications $D_i \tilde{\rightarrow} \{z\in\mathbb{C} \mid |z| \leq 1\}$, analytically depending on m. So, on each surface from our family there are fields $\varphi(z_i,\bar{z}_i)$, where $|z_i|\leq 1$. Let $<\varphi(z_1,\bar{z}_1) ... \varphi_n(z_n,\bar{z}_n)>$ be the Green function of the theory, where $\varphi_1, ...,\varphi_n\in A$. The property which defines the RCFT is the assumption that

$$<\varphi_1(z_1,\bar{z}_1) ... \varphi_n(z_n,\bar{z}_n)> = Z^{-1} . \sum_{i=1}^{N} f_i \bar{g}_i$$

where f_i, g_i are analytic functions in z., m. and Z is a partition function,

$$Z = \sum_{j=1}^{M} F_j \bar{G}_j \in \mathbb{R}_+, \text{ where}$$

F_j, G_j are analytic functions in m. only. Numbers N, M and functions f., g. depend on genus g and on $\varphi_1, ...,\varphi_n$.

In invariant terms, there are some finite dimensional vector bundles over domain in some complex manifold with flat connection, flat pseudohermitean metric and some holomorphic section ξ. The numerator and the denominator in the above formula are then squares of the length (ξ,ξ) of the section ξ. Those bundles are called Friedan-Schenker bundles. These bundles are not unique. They are defined only up to $\otimes$-multiplication by a 1-dimensional flat bundle with flat hermitian metric and with any non-zero holomorphic section. So, really we have canonical flat bundles with fibres = (complex projective spaces) on the space of modules of the next data : (S-Riemanian surface ; $D_1,...,D_n \subset S$ - coordinate disks ; $x_i \in D_i$). In other words, we have a system of projective representations of fundamental groups of such ∞-dimensional spaces, which are Teichmüller groups.

Caution ! In what follows I suppose, for the sake of simplicity, that all representations are vectors (not projective). In section 7 there are some explanations for the of validity of this simplification.

2. Modular functor.

There is a purely topological part of the theory namely representations of the modular groups. Those representations are satisfied by some glueing properties, which are summarized in the definition of the modular functor.

I will give some variant of axioms which is equivalent to those in [G]. There are the following three types of data and conditions :

Data I :	1.	A finite set A	(fields)
	2.	involution $\delta : A \to A$	(changing of orientation)
	3.	element $\underline{0} \in A$	(vacuum)
	4.	numbers h_α, $\alpha \in A$	(dimensions of the fields).

Conditions I :

1. $\delta\,\underline{0} = \underline{0}$
2. $h_{\delta\alpha} = h_\alpha$
3. $h_{\underline{0}} = 0$

A preliminary definition :

A coloured surface is an oriented surface S with a set of distinguished points on each component of boundary, $I \subset \partial S$, $I \cong \pi_0(\partial S)$, and colouring $c : I \to A$.

Data II . 1. Finite dimensional complex vector spaces $V(S,c)$ depending naturally on a coloured surface.

This means, that the group $\pi_0(\mathrm{Aut}(S, \text{orientation}, I, c{:}I\rightarrow A))$ acts on V.

2. Changing of orientation :

There are fixed isomorphisms $V(S,c)^* \xrightarrow{\sim} V(\bar{S}, \delta \circ c)$

For any $i \in I$ there is a fixed element t_i in π_0 (Aut(...))

- twist around the component of boundary containing i in the positive direction (it is well defined because S is oriented), see fig.1.

Conditions II.

1. see fig.2.
2. t_i acts on $V(..)$ by multiplication on $\exp(2\pi\sqrt{-1}\, h_{c(i)})$

Data III. 1. If $S = S_1 \cup S_2$ and $c = c_1 \cup c_2$ then there is a fixed isomorphism $V(S.c) \xrightarrow{\sim} V(S_1,c_1) \otimes V(S_2,c_2)$.

If $\partial S = L \cup \bar{L} \cup$ (rest) and distinguished points on L and on $\bar{L}$ coincide, then identifying canonically L and $\bar{L}$ as manifolds without orientation, we can obtain a new oriented surface $S' = S/_{L\equiv\bar{L}}$. In this situation there is a fixed isomorphism :

2. $V(S',c') \xrightarrow{\sim} \bigoplus_{\alpha\in A} V(S, c'\cup\alpha \text{ on } L \cup \delta\,\alpha \text{ on } \bar{L})$

Conditions III. Isomorphisms in 1. and 2. are natural, compatible with all above isomorphisms in the second group of axioms, and with natural commutativity, associativity and distributivity axioms (see [DM],[MS]).

This is a full list of axioms.

Any surface of genus $g > 1$ can be cut in pantalons $= S^2 \setminus$ three open disks. Let $V_{\alpha\beta\gamma}$ denote the vector space associated with standard pantalons with colours α,β,γ (see fig.3).

If we choose the cutting of surface into pantalons then we can compute V(S) :

$$V(S) = \bigoplus_{\substack{\text{coloring} \\ \text{of lines}}} \ \bigotimes_{\text{pantalons}} V \ldots$$

S o V(S) is the partition function of a finite spin system with "vector-space-valued" weights.

A modular functor is defined by the above number of data (≡ representations of all Teichmüller groups, which satisfied an infinite number of conditions).

G. Moore and N. Seiberg [MS] have shown that the above structure is defined by a finite number of generators (vector spaces $V_{\alpha\beta\gamma}$) and by a finite number of relations.

3. Classical CFT.

There is a very simple analogue to the previous system of axioms - a "limit" of modular functors when h_α -s tends to 0 and surfaces shrink to graphs.

Let Γ be a finite group. In the situation considered A will be the set of equivalence classes of irreducible representations of Γ, which we denote by $\hat{\Gamma}$. For every $\alpha \in A$ let us fix some representation V_α from this class.

The involution $\sigma : A \rightarrow A$ is the passing to the dual representation.

$0 \in A$ is the trivial representation.

A graph is a finite 1-dimensional CW-complex, G = (V,E), where V is the set of vertices, E is the set of edges.

Let $\vec{E}$ denote the set of edges with orientations. Let $\bar{\delta}$ be the involution on $\vec{E}$, which changes the orientation of each edge.

We call a configuration any map, $a : \vec{E} \rightarrow A$ with the property $\delta \circ a = a \circ \bar{\delta}$ (they are the analogue of colouring from the above definitions).

The weight of the configuration is given by

$$W(a) = \bigotimes_{v \in V} \operatorname{Inv} \left(\bigotimes_{\xrightarrow[e]{} \bullet v} V_{a(e)} \right)$$

where Inv ( ) denotes the subspace of Γ-invariants.

In our case the "partition-function" - vector space is

$$Z(G) = \bigoplus_{\text{all configurations } a} W(a)$$

Proposition : Z(G) is homotopy invariant.

Main part of proof : Let us check that if we apply the surgery to the graph G as in fig.4, then Z(G) is canonically isomorphic to Z(G'). By locality it is sufficient to consider the case of parts of graphs, shown in fig.4, with arbitrary boundary conditions.

Let $V_L = \bigotimes_{e\in \vec{E}_L} V_{a(e)}$, $V_R = \bigotimes_{e\in \vec{E}_R} V_{a(e)}$.

Then

$$Z(G,a) = \bigoplus_{\alpha\in A} (\mathrm{Inv}(V_L \otimes V_\alpha) \otimes \mathrm{Inv}(V_\alpha^* \otimes V_R)) =$$

$$= \bigoplus_{\alpha\in \Gamma} (\mathrm{Hom}_\Gamma (V_L^*, V_\alpha) \otimes \mathrm{Hom}_\Gamma (V_\alpha, V_R),$$

$$Z(G',a) = \mathrm{Inv}\,(V_L \otimes V_R) = \mathrm{Hom}_\Gamma\,(V_L^*, V_R).$$

It is evident that the natural map from Z(G,a) to Z(G',a) is an isomorphism. ♦

So the group Out (free group) acts on Z(G) for a connected graph G, because G is K (free group,1).

There is a simple formula for Z : $Z(G) \cong \mathbb{C}^{H^1(G,F)}$ = the space of functions on the finite set $H^1(G,\Gamma)$.

General case : There is 1-1 correspondence between homotopy-invariant vector-space-valued theories on graphs and the abelian rigid semisimple ⊗-categories (see [MS], [DS]. In general, Z(G) is a commutative algebra with the action of Out (Free group).

4. Topological theories (in 3 dimensions).

Suppose the following data are given :

Data : 1. Finite dimensional vector spaces V(S), depending naturally on closed oriented surfaces S. Again, the group π_0 (Aut (S, orientation)) acts on V(S).

2. The isomorphisms $V(\bar{S}) \xrightarrow{\sim} V(S)^*$, $V(\bigcup_{i\in I} S_i) \xrightarrow{\sim} \bigotimes_{\alpha\in I} V(S_i)$ are fixed.

3. For any 3-dimensional manifold M there is a fixed vector $Z_M \in V(\partial M)$.

Conditions.

1. Usual properties of isomorphisms in Data 2 , naturality under diffeomorphisms.

2. $Z_{\bigcup_{i\in I} M_i} = \bigotimes_{i\in I} Z_{M_i}$

If $\partial M = S \cup \bar{S} \cup$ (the rest) , there exists a new manifold $M' = M/_{S \equiv \bar{S}}$, such that the vector spaces $V(\partial M)$ and $V(\partial M')$ are $V(S) \otimes V(S)^* \otimes V$ (the rest) and V(the rest) correspondingly, and there exists a natural map $p : V(\partial M) \to V(\partial M')$.

3. $p(Z_M) = Z_{M'}$.

The definition is complete.

It follows from the axioms that in the case $\partial M = \emptyset$ the partition functions Z_M are complex numbers, which are invariants of diffeomorphism type.

5. From Modular functor to Topological theory.

Suppose that we have a modular functor V(S, colouring). We shall construct the corresponding topological theory. We take for spaces V(S) the same spaces as in MF. We must construct only vectors Z_M. For the sake of simplicity M is assumed to be closed.

Let us choose any Morse function f on M (i.e. a function with only quadratic critical points). We shall construct a family of linear maps $V(f^{-1}(t)) \to V(f^{-1}(t'))$ where $t < t'$ are regular values of f by "induction" in t. If $t \to \pm\infty$ then $f^{-1}(t) = \emptyset$, $V(f^{-1}(t)) = \mathbb{C}$ and the map $V(f^{-1}(-\infty)) \to V(f^{-1}(+\infty))$ by definition will be the multiplication by Z_M. These maps form a representation of ordered set ($\mathbb{R} \setminus \{$critical values $\}, \leq$).

When there are no critical values between t and t' then there is a fixed diffeotopy-type identification $f^{-1}(t) \xrightarrow{\sim} f^{-1}(t')$ because we can choose a metric on M and identify fibres of f using the gradient flow. So, by MF axioms we have the identification $V(f^{-1}(t)) \xrightarrow{\sim} V(f^{-1}(t'))$.

Crossing a critical value :

By locality of V(S) it is sufficient to construct a map $V(...) \to V(...)$ near each critical point of any index with any boundary condition (see fig.5).

It is easy to see that in the cases 0,3 V(left picture) $\xrightarrow{\sim}$ V(right picture) $\xrightarrow{\sim} \mathbb{C}$. In the case 1, if $\alpha \neq \underline{0}$ or $\beta \neq \underline{0}$ then V(left picture) $\xrightarrow{\sim} 0$, analogously to the case 2, so the map is unique.

If in cases 1,2 $\alpha = \beta = \underline{0}$ then V(left picture) = V(right picture) = $\mathbb{C}$ and Hom (V(left picture), V (right picture)) = $\mathbb{C}$.

So, to define the above maps it is sufficient to choose 4 complex numbers λ_0, λ_1, λ_2, λ_3. We want to choose the numbers $\lambda_0,...,\lambda_3$ in such a way that the "partition function" Z_M does not depend on the choice of the Morse function.

If we have two Morse functions f_0 and f_1 then there is a family of function in general position f_s, $s \in [0,1]$ joining f_0 and f_1. For some values s the function f_s is not a Morse function and has a singularity of type A_2 (by Arnold's classification) :

$f_s(X,Y,Z) = X^3 + (s-s_{cr}) X \pm Y^2 \pm Z^2 + \text{const.}$, where X,Y,Z are some local coordinates on the manifold.

If $s > s_{cr}$ then there are no critical points near the point $x = y = z = 0$, so locally the level sets of the function f_s near the value $f_s(0,0,0)$ look like in fig.6. Boundary conditions are zero by conditions II.1 in Section 2.

Consider now the case $s < s_{cr}$. According to the above formula, three possibilities for the evolution of the level set of f_s are those shown in the movies on fig. 7-9. Each of them gives an equation on λ's. Fig 7 gives an equation $\lambda_2.\lambda_3 = 1$, Fig.9 gives an equation $\lambda_0.\lambda_1 = 1$.

We comment on the third equation, see fig. 8.
Using decomposition of V(...) into direct sum , the modular functor defines the sequence of maps , the composition of which is just the multiplication by some number which we call μ. The equation in question is $\lambda_1 . \lambda_2 . \mu = 1$.

Hence the topological theory exists only for those MF, for which $\mu \neq 0$.

So, we have three equations on four variables, and there is a 1-parameter family of topological theories. It turns out that all these theories are equivalent. Indeed, if (λ'_i) is another solution, then $\frac{\lambda'_i}{\lambda_i} = (v)^{(-1)^i}$ for some $v \in \mathbb{C}^*$ so that $\frac{Z'(M)}{Z(M)} = v^{\chi(M)} = 1$, where $\chi(M)$ is the Euler characteristic of M, which is always zero for three-dimensional manifolds.

This is a well defined construction, except for the condition that μ, which will be $Z(S^3)$ is not zero. I think that the positivity of this number follows from other properties of RCFT (perhaps unitarity).

6. From topological theory to MF.

Suppose we have a topological theory. Denote by ϑ the vector space $V(S^2)$. We shall define on ϑ the structure of a commutative and associative algebra with 1.

Consider the 3-manifold $M = B^3 \setminus \text{int.}(B_1^3 \cup B_2^3)$, (see fig.10). The boundary ∂M of this manifold is $S^2 \cup \bar{S}^2 \cup \bar{S}^2$, so Z_M is an element of $V(S^2) \otimes V(\bar{S}^2) \otimes V(\bar{S}^2) = \vartheta \otimes \vartheta^* \otimes \vartheta^*$. The tensor Z_M defines a linear operator $\vartheta \otimes \vartheta \rightarrow \vartheta$. It is clear that this operator gives on ϑ a structure of algebra with unit Z_{B^3}.

Algebra ϑ acts on V(S) for any connected S : the action is defined through the manifold $M = S \times [0,1]$ - int B^3, (see fig.11). Boundary ∂M is equal to $\bar{S}^2 \cup \bar{S} \cup S$, vector Z_M defined an operator $\vartheta \otimes V(S) \to V(S)$.

ϑ is a finite dimensional algebra over $\mathbb{C}$ so that , if ϑ has no nilpotents then $\vartheta \cong \mathbb{C} \oplus \ldots \oplus \mathbb{C}$. It is easy to see that our theory is a direct sum of the theories over Spec ϑ so, for simplicity, we can assume that $\vartheta \cong \mathbb{C}$.

Let $\mathcal{A}$ be the vector space V (standard torus). $\mathcal{A}$ is also a commutative algebra (it is a Verlinde algebra). The manifold M, defining multiplication in A is $(B^2 \setminus \text{int} (B_1{}^2 \cup B_2{}^2)) \times S^1$, where B^2 are disks (see fig.12).

Again, we suppose additionally that A is semisimple. Let A be Spec. $\mathcal{A}$. There is an involution on A, corresponding to the antipodal involution of the torus $\mathbb{R}^2/\mathbb{Z}^2$.

If we have a family of disjoint circles $\bigcup_{i \in I} L_i \subset S$ then V(S) has a structure of $\mathcal{A}^{\otimes I}$ - module : the corresponding manifold is $S \times [0,1] \bigcup_{i \in I} \mathcal{U}_\varepsilon (L_i \times \{t\})$, where $t \in (0,1)$ is an arbitrary point, and by $\mathcal{U}_\varepsilon(\ )$ we denote ε-neighbourhood, ε is sufficiently small (see fig.13).

Spec $\mathcal{A}^{\otimes I}$ is A^I -set of colouring of the lines, as in section 2, and we have a decomposition into the direct sum $V(S) = \bigoplus_{\text{colouring}} V(S, \text{colouring})$, where summands are fibres V(S) as $\mathcal{A}^{\otimes I}$ -module.

The last and the hardest part is to construct the vector spaces corresponding to surfaces with boundaries and to find a $\otimes$-decomposition of V(S, colouring).

Let $P \in \mathcal{A} \otimes \mathcal{A}$ be an element Z_M where M = annulus x circle. It is easy to see that $P = \sum_{\alpha \in A} c_\alpha \cdot p_\alpha \otimes p_\alpha$, where p_α are canonical projectors in $\mathcal{A}$, c_α are non zero complex numbers.

Suppose that we have two disjoint circles L_1, L_2 in S with some identifications $L_1 \cong L_2 \cong S^1$. We can construct a new surface S' by cutting S along L_1 and L_2 and glueing again, mixing the borders. Denote by $T_{S'S}$ the operator from V(S) to V(S') obtained by the manifold pictured on fig. 14. Then $T_{SS'} \circ T_{S'S} = P$ where P acts on V(S) as the element of $\mathcal{A} \otimes \mathcal{A}$.

Consider now the case of any number of nonintersecting circles L_i, $i \in I$ in a surface S with identifications $L_i \cong S^1$ and choose some colour $\alpha \in A$. We can again cut S and glue in another way and obtain new surfaces S'. There is a family of identifications between some V(S',α on all lines) and V(S", α on all lines) by operators $C_\alpha^{-1/2}\, T_{S''S'}$. It is easy to see that any closed sequence of such identifications gives the identity operator.

There are different ways of reconstructing the vector spaces V(surface with boundary, colouring). One way is to define them in terms of V(surface with torus without disk glued to each component of the boundary) (see fig. 15). Another way is to introduce the structure of associative algebra on $V(S \bigcup_{\partial S} \overline{S'}$, colouring on $\partial S)$ and to prove that this algebra is a matrix algebra of some vector space which will be V(S, colouring). Careful considerations show that this does not give a modular functor but a super-modular functor : the set of colours is a graded set, $A = A_{even} \cup A_{odd}$, and vector spaces V(S, colouring) are super-vector spaces purely even or odd corresponding to the parity of colouring. We will change square roots $C_\alpha^{-1/2}$ by $(-C_\alpha^{-1/2})$ for $\alpha \in A_{odd}$ in the definition of identifications (see above) and will obtain a modular functor.

7. Homotopical theories.

Here I shall describe a type of field theories which corresponds to the classical CFT in sect.3 and gives a trivial series of topological theories.

HT is defined over all finite CW-complexes. The list of axioms is obtained from axioms of TT by changing notions "oriented manifolds with boundary" and "diffeomorphisms" by "finite CW-pairs (X,Y)" and "homotopy equivalences". It follows from the axioms that each V(X) is a vector space with nondegenerate scalar product.

Some preliminaries : call the space S the homotopy finite if 1) $\# \pi_0(S) < \infty$, 2) $\forall s \in S$ $\forall n \geqslant 1$ $\# \pi_n(S,s) < \infty$, 3) $\exists N$ $\forall n > N$ $\forall s \in S$ $\pi_n(S,s) = 0$. Let us denote by $\chi(S)$ the homotopy Euler characteristic :

$$\chi(S) = \sum_{\alpha \in \pi_0(S)} \prod_{n=1}^{\infty} (\# \pi_n(S, s_\alpha))^{(-1)^n} ,$$

for some $s_\alpha \in$ component α. If S is not empty then $\chi(S)$ is a positive rational number.

If $F \to E \to B$ is a Serre bundle with connected base B then a) if two of the spaces (F,E,B) are homotopy finite, then the third one also is, b) in this case $\chi(E) = \chi(B)\,\chi(F)$, (from the homotopy exact sequence).

Choose some nonempty homotopy finite space S. It is easy to see that for any finite CW-complex X the space S^X of continuous maps from X to S is again homotopy finite. We can define HT by the next data :

1) V(X) is a vector space with basis e_α, where $\alpha \in \pi_0(S^X)$,
2) the scalar product is $(e_\alpha, e_\beta) = \chi\,(\text{component } \alpha)^{-1}\, \delta_{\alpha\beta}$
3) for pair $Y \subset X$

$$Z_X = \sum_{\alpha \in \pi_0(S^Y)} \chi(\text{space of maps } X \to S, \text{ which restriction on Y belongs to } \alpha).\ e_\alpha.$$

As in the previous section we can introduce a structure of commutative associative algebra on $V(X \times S^1)$ for any X. This algebra is exactly represented by symmetric matrices with real elements (moreover, rational elements), so it is semsimple. Let $A_X =$ Spec $V(X \times S^1)$ be the set of colours. This set is a finite set with Gal $(\bar{\mathbb{Q}} / \mathbb{Q})$-action and it is used in computing global spaces V(...) from the local one when we cut along X.

Classical CFT is obtained from any HT with $\pi_0(S) = \cdot$, $\pi_1(S) = \Gamma$.

Of course, any HT gives a topological theory. We cannot obtain known MF in this way, even if we modify the definition of TT considering oriented tangent bundle, because TT leads to the representations of modular groups with finite images. Representations in known MF have all infinite images when (genus of S) ≥ 2.

From another point of view HT are good field theories because Z_x are positive numbers and we can go to thermodynamic limit in situations like the one below :

X is a finite CW-complex, $H_1(X,\mathbb{Z}) = \mathbb{Z}^d$.

Consider N^d-fold Galois cover X_N of X defined by the map $\pi_1(X) \to H_1(X,\mathbb{Z}) = \mathbb{Z}^d \to (\mathbb{Z}/_{N\mathbb{Z}})^d$. Then Z_{X_N} is a partition function for some d-dimensional lattice system and there exists $\lim\limits_{N\to\infty} (N^{-d} \log Z_{X_N})$.

8. Remarks on central charge and central extensions.

In each CFT there arise certain central extensions of different groups (diffeomorphism group of circle, modular groups), which are dependent on the number $c \in \mathbb{C}$ - central charge. In Rational CFT this c is rational [MA]. If c is an integer, then projective representations of modular groups are actually linear representations.

I shall now describe a certain universal central extension for topological theories (T.T.). T.T. is in other words a $\otimes$-representation of the $\otimes$-category C_3, whose objects are oriented 2-dimensional closed manifolds and whose morphisms $\mathrm{Mor}_{c_3}(S_1, S_2)$ are equivalence classes of 3-dimensional manifolds M with $\partial M = \bar{S}_1 \cup S_2$, $\otimes$-structure in this category is a disjoint union (analogously to the Segal's definition of conformal theories).

Let us define a new category $\tilde{C}_3$ with the same objects as in C_3 and with new morphisms.

First, choose (not canonically !) for each surface S a 3-dimensional manifold M_s with $\partial M_s = S$. Let $\mathrm{Mor'}_{c_3}(S_1, S_2)$ be the set of equivalence classes of pairs (M^3,N^4), where $\partial M^3 = \bar{S}_1 \cup S_2$, $\partial N^4 = M^3 \underset{S_1}{\cup} M_{S_1} \underset{S_2}{\cup} \bar{M}_{S_2}$, see fig. 16.

Define a function f on pairs (M^3,N^4) , $(M^3,\tilde{N}^4)$ with common M^3 :

$f(N^4,\tilde{N}^4) = P_1 (N^4 \underset{\partial N^4}{\cup} \overline{\tilde{N}^4})$ - is the first Pontryagin number of a closed oriented 4-manifold.

<u>Proposition</u>. $f(N^4,\tilde{N}^4) + f(\tilde{N}^4,\tilde{\tilde{N}}^4) = f(N^4,\tilde{\tilde{N}}^4)$, in other words f(,) is a cocycle.

Proof : It is easy to construct a manifold P^5 (it looks like pantalons) with boundary ∂P^5 equal to

$$(N^4 \cup_{\partial N^4} \tilde{N}^4) \cup (\tilde{N}^4 \cup_{M^3} \tilde{\tilde{N}}^4) \cup \overline{N^4 \cup_{M^3} \tilde{\tilde{N}}^4} .$$

The Pontryagin number of a boundary is zero, so we have proven the equality. ♦

Let us identify elements (M^3,N^4) and $(M^3,\tilde{N}^4)$ of $Mor'_{C_3}(S_1,S_2)$ when $f(N^4,\tilde{N}^4) = 0$. After this identification we obtain the category $\tilde{C}_3$, whose morphisms $Mor_{\tilde{C}_3}(S_1,S_2)$ are principal $\mathbb{Z}$-bundles over $Mor_{C_3}(S_1,S_2)$.

It is easy to check that this category $\tilde{C}_3$ is the universal central extension of $\otimes$-category C_3.

In the above construction it is essential that all 2- and 3-dimensional manifolds are bordant to zero. An interesting problem is to describe universal central extensions of analogous categories C_n .

It is also possible to give a description of the universal central extension using the language of modular functors (A. Beilinson). The modification of definition is the following : instead of the vector spaces $V(S,c)$ one has to consider local systems over Lagrangian Grassmanians of the real symplectic vector space $Ker\,(H^1(S,\mathbb{R}) \to H^1(\partial S,\mathbb{R}))$.

It is known that the fundamental group of any Lagrangian Grassmanian equals $\mathbb{Z}$. Canonical generator 1 of $\mathbb{Z}$ acts by multiplication on $\exp(2\pi \sqrt{-1}\, c)$ in the fibres of local systems arising in RCFT with central charge c.

One can form a tensor product of two RCFT. The central charge of the resulting RCFT then equals the sum of central charges of factors. Hence, for any RCFT some tensor power of it has an integer central charge (because $c \in \mathbb{Q}$), and so our projective representation are actually linear representations of modular groups and category C_3.

9. Concluding remarks and further questions.

The presence of an Hermitian scalar product in RCFT means that the local systems $V(\ldots)$ are defined over $\mathbb{R}$ (that is, for closed surfaces with complex structure $V(\overline{S}) = \overline{V(S)}$, where the bar denotes the complex conjugation).

It seems that $V(S)$ are cohomology groups of some algebraic varieties depending naturally on the curve S over any field of characteristic 0.

The full definition of RCFT means that the complex structure on the surface is similar to the 3-dimensional manifold with boundary S, because both structures give a vector in V(S).

A. Beilinson proposed a beautiful possibility for giving a rigorous sense to this analogy : both a 3-dimensional manifold and a complex structure, defines full ⊗-subcategory in Repr (π_1(S)). In 3-dimensional case it is the category of representations which can be continued on π_1(3-manifold), in case of complex structure it is the category of local systems on S which admits a complex variation of the Hodge structure (see [S]).

I have heard from B. Feigin that N. Reshetikhin and V. Turaev in Leningrad defined vector spaces V(S) for WZW model in terms of representation theory of quantum groups with parameter q = root of unity. In fact they did not define V(S) but matrix algebras End V(S).

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FIGURES CAPTIONS

1. A twist around a component of boundary.
2. Vector space for disk with arbitrary boundary condition.
3. The pantalons.
4. Surgery of a graph.
5. Level sets for 4 possible Morse critical points.
6. Level sets for f_s, $s > s_{cr}$.

7,8,9. Level sets for f_s, $s < s_{cr}$.

10. Ball minus two balls, defines multiplication in $\mathcal{O}$.
11. Ball inside surface $\times$ interval, defines action of $\mathcal{O}$ on V(S).
12. Pantalons $\times$ circle, defines multiplication in $\mathcal{A}$.
13. Surface $\times$ interval minus solid torus, action of $\mathcal{A}$.
14. Three-dimensional picture near L_1 and L_2 looks like the product of two-dimensional picture by the standard circle S^1.
15. Cutting and glueing, using standard surfaces with genus 2.
16. A representative for morphism in $\widetilde{C}_3$.

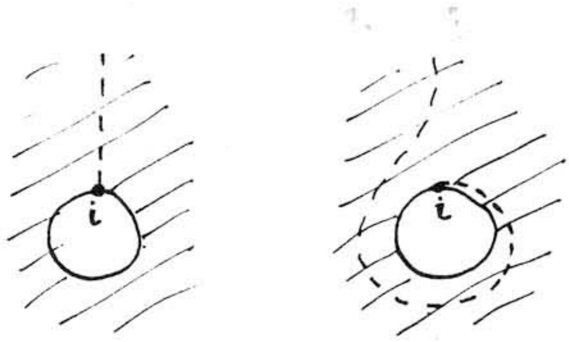


fig. 1

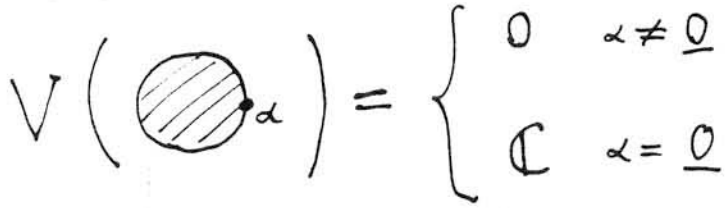
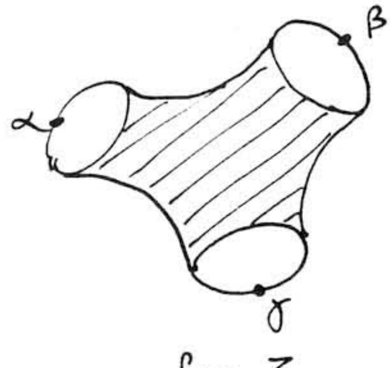


fig. 2



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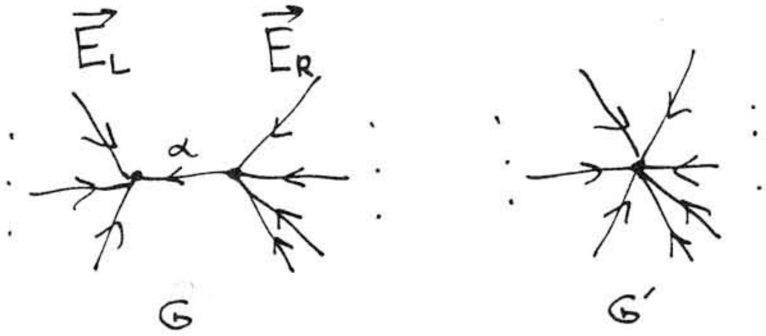

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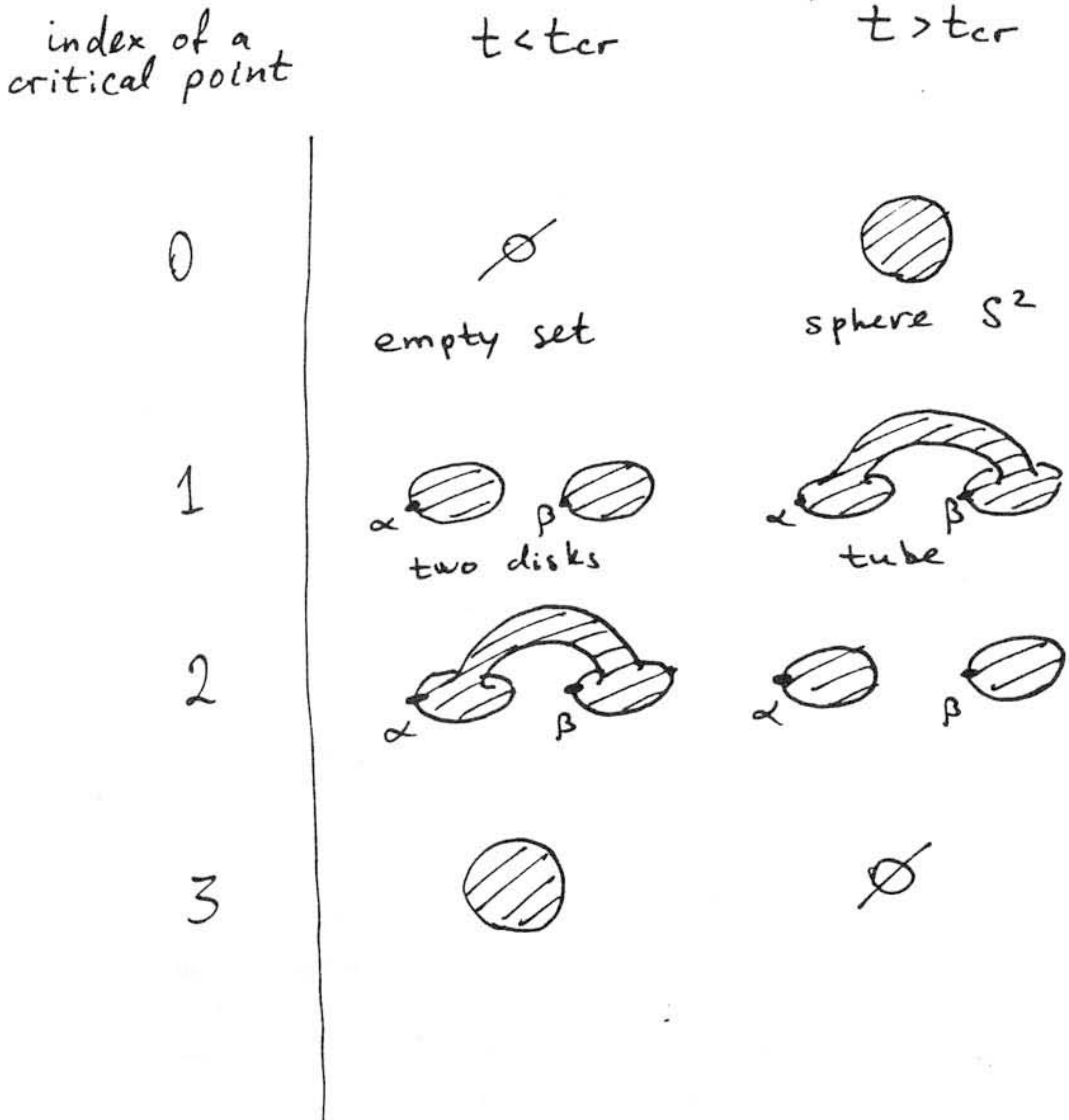

index of a critical point
t < tcr
t > tcr
0
empty set
sphere S2
1
α
β
two disks
α
β
tube
2
α
β
α
β
3

fig. 5

fig.6

fig.7

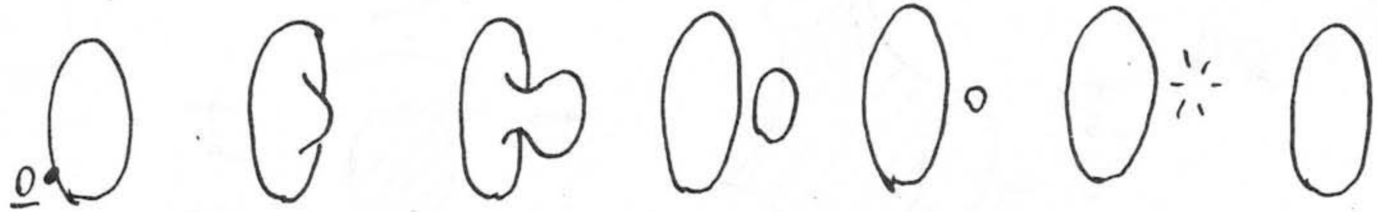
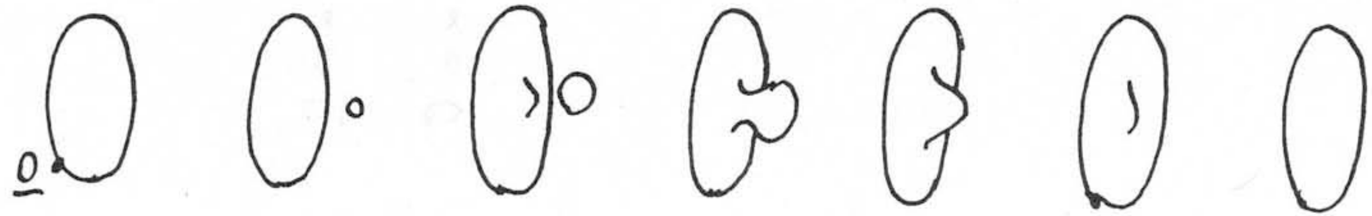


fig.8

fig.9



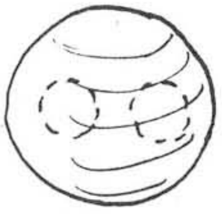


fig. 10

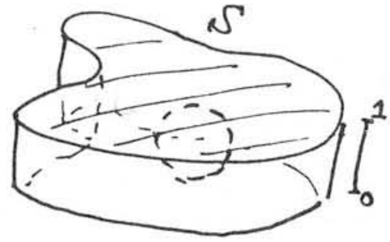


fig. 11

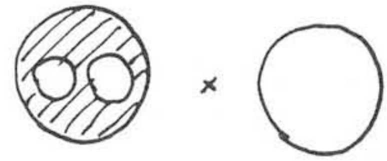


fig. 12

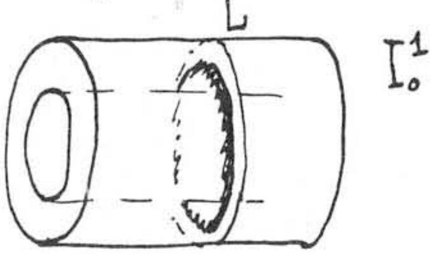


fig. 13

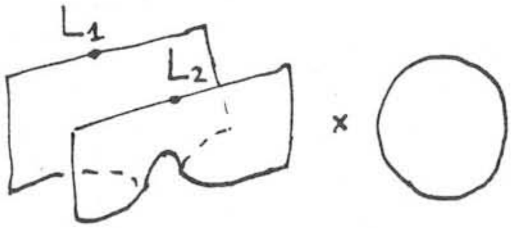


fig. 14

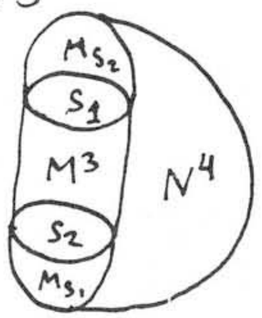


fig. 16

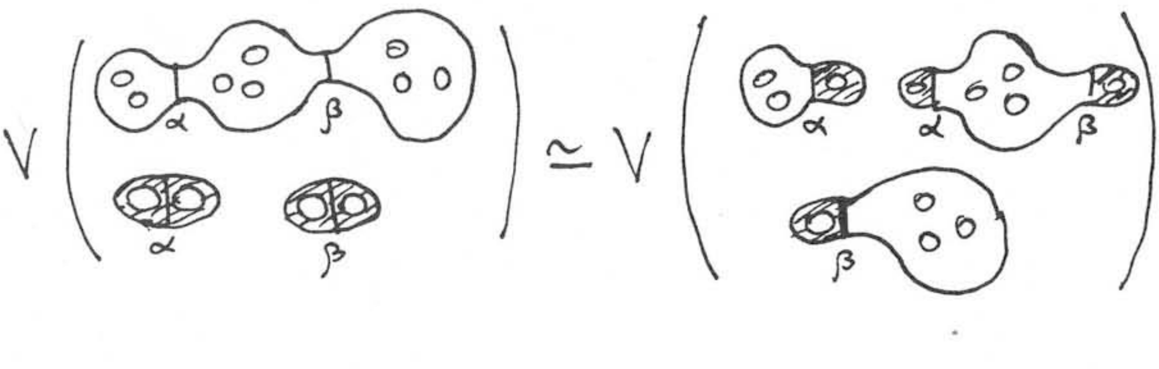


fig. 15