Deformation quantization of Poisson manifolds

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Foreword

Here is the final version of the e-print *Deformation quantization of Poisson manifolds I* [33] posted on the web archive as `q-alg/9709040`. The changes that have been made are mostly cosmetic, I have just corrected few mistakes and tried to make clear links between several lemmas and theorems proven in the paper, and also straightened out some proofs.

Here follows a guide to a short and definitely not complete additional bibliography reflecting further development of the subject.

First of all, I have to mention the work of Dmitry Tamarkin (see [44] and a nice exposition in [22]), which gave a radically new approach to the formality theorem. One of main ideas is to consider the Lie algebras T_{poly} and D_{poly} not just as dg Lie algebras, but as homotopy Gerstenhaber algebras, which explains naturally the cup-product on the tangent space. A very important related issue here is the so called Deligne conjecture which says that on the Hochschild complex of an arbitrary associative algebra there is a natural action of the dg operad of chains of the little discs operad. The Deligne conjecture has now several proofs, see e.g. [36, 37], and a generalization to higher dimensions in [26]. Unfortunately, up to now, it is not clear how to extract explicit formulas from Tamarkin's work, or even how to compare it with the formality morphism from [33]. Tamarkin's proof is based on the Etingof-Kazhdan theorem about quantizations of Lie bialgebras, which is in a sense more complicated (and less explicit) than the Formality theorem itself! It seems that the Etingof-Kazhdan theorem is a "degree zero" part of a more general not yet established result of the formality of the differential graded Lie algebra controlling deformations of the symmetric algebra $Sym(V)$ of a vector space, considered as an associative and coassociative bialgebra. On this Lie algebra there should be an action of the operad of chains of little 3-dimensional cube operad and its formality should be considered as a natural generalization of the Formality theorem from [33]. Up to now there is no explicit complex of a "reasonable size", controlling deformations of bialgebras, see [39] for some recent attempts.

In [34] I have tried to perform a shortcut in Tamarkin's proof avoiding the reference to Etingof-Kazhdan's result. Also I proposed a new formality morphism with complex coefficients, different from the one in [33]. Conjecturally the new morphism behaves in a better way than the old one with respect to the arithmetic nature of the coefficients (weights of graphs) and should coincide with Tamarkin's quasi- isomorphism up to homotopy.

In [45] another generalization of the Formality theorem was proposed. Namely, one should consider not only the cohomological Hochschild complex, but also the homological Hochschild complex which is a module in certain sense over the cohomological one. Related colored operad here consists of configurations of disjoint discs in a cylinder with two marked points on both boundary components. This is important for the study of traces in deformation

quantization, see [15] for an approach to the quantization with traces.

The program of identifying graphs in the formality morphism with Feynman diagrams for a topological sigma-model (announced in [33]) was performed by Alberto S. Cattaneo and Giovanni Felder in a series of papers, see [7, 8].

In [4] it is established a formality of the dg Lie algebra which is a global Dolbeault complex for holomorphic polyvector fields on a given Calabi-Yau manifold X. Morally, together with the Formality theorem of [33], this should mean that the extended moduli space of triangulated categories is smooth in a formal neighborhood of the derived category of coherent sheaves on X.

In [9] an alternative way for the passage from local to global case in the Formality theorem was described, see also an appendix in [35].

In [35] I proposed a way to use results of [33] in the case of algebraic varieties. It seems that for rational Poisson varieties deformation quantization is truly canonical in a very strong sense. For example, I believe that for arbitrary field $\mathbf{k}$ of characteristic zero there exists certain canonical isomorphism between the automorphism group of the $\mathbf{k}$-algebra of polynomial differential operators on an affine n-dimensional space over $\mathbf{k}$, and the group of polynomial symplectomorphisms of the standard symplectic $2n$-dimensional affine space over $\mathbf{k}$. This is very surprising because the corresponding Lie algebras of derivations are not at all isomorphic.

Finally, repeating myself a bit, I comment on today's state of the topics listed in Section 0.2 in [33]:

1) The comparison with other deformation schemes is not yet performed.

2) This is still a wishful thinking.

3) See conjectures in [34], and also [35].

4) This is not done yet, results from [4] should be used as an intermediate step.

5) Done by Cattaneo and Felder.

6) Not yet completed, see conjectures in [45].

7) In [35] there is a recipe for a canonical quantization for quadratic brackets, see also the new conjecture from above about an isomorphism between two automorphisms groups.

0 Introduction

In this paper it is proven that any finite-dimensional Poisson manifold can be canonically quantized (in the sense of deformation quantization). Informally, it means that the set of equivalence classes of associative algebras close to algebras of functions on manifolds is in one-to-one correspondence with the set of equivalence classes of Poisson manifolds modulo diffeomorphisms. This is a corollary of a more general statement, which I proposed around 1993-1994 (the *Formality conjecture*, see [30, 43]).

For a long time the Formality conjecture resisted all approaches. The solution presented here uses in a essential way ideas of string theory. Our formulas can be viewed as a perturbation series for a topological two-dimensional quantum field theory coupled with gravity.

0.1 Content of the paper

Section 1: an elementary introduction to the deformation quantization, and precise formulation of the main statement concerning Poisson manifolds.

Section 2: an explicit formula for the deformation quantization written in coordinates.

Section 3: an introduction to the deformation theory in general, in terms of differential graded Lie algebras. The material of this section is basically standard.

Section 4: a geometric reformulation of the theory introduced in the previous section, in terms of odd vector fields on formal supermanifolds. In particular, we introduce convenient notions of an L_∞-morphism and of a quasi-isomorphism, which gives us a tool to identify deformation theories related with two differential graded Lie algebras. Also in this section we state our main result, which is an existence of a quasi-isomorphism between the Hochschild complex of the algebra of polynomials, and the graded Lie algebra of polyvector fields on affine space.

Section 5: tools for the explicit construction of the quasi- isomorphism mentioned above. We define compactified configuration spaces related with the Lobachevsky plane, a class of admissible graphs, differential polynomials on polyvector fields related with graphs, and integrals over configuration spaces. Technically the same constructions were used in generalizations of the perturbative Chern-Simons theory several years ago (see [29]). Compactifications of the configuration spaces are close relatives of Fulton-MacPherson compactifications of configuration spaces in algebraic geometry (see [16]).

Section 6: it is proven that the machinery introduced in the previous section gives a quasi-isomorphism and establishes the Formality conjecture for affine spaces. The proof is essentially an application of the Stokes formula, and a general result of vanishing of certain integral associated with a collection of rational functions on a complex algebraic variety.

Section 7: results of Section 6 are extended to the case of general manifolds. In order to do this we recall basic ideas of formal geometry of I. Gelfand and D. Kazhdan, and the language of superconnections. In order to pass from the affine space to general manifolds we have to find a non-linear cocycle of the Lie algebra of formal vector fields. It turns out that such a cocycle can be almost directly constructed from our explicit formulas. In the course of the proof we calculate several integrals and check their vanishing. Also, we introduce a general notion of direct image for certain bundles of supermanifolds.

Section 8: we describe an additional structure present in the deformation theory of associative algebras, the cup-product on the tangent bundle to the super moduli space. The isomorphism constructed in Sections 6 and 7 is compatible with this structure. One of new results is the validity of Duflo-Kirillov formulas for Lie algebras in general rigid tensor categories, in particular for Lie superalgebras. Another application is an equality of two cup-products in the context of algebraic geometry.

0.2 What is not here

Here is a list of further topics which are not touched in this paper, but are worth to mention.

1) the comparison of the formality with various other known constructions of star-products, the most notorious one are by De Wilde-Lecomte and by Fedosov for the case of symplectic manifolds (see [11, 14]), and by Etingof-Kazhdan for Poisson-Lie groups (see [13]),

2) a reformulation of the Formality conjecture as an existence of a natural construction of a triangulated category starting from an odd symplectic supermanifold,

3) a study of the arithmetic nature of coefficients in our formulas, and of the possibility to extend main results for algebraic varieties over arbitrary field of characteristic zero,

4) an application to the Mirror Symmetry, which was the original motivation for the Formality conjecture (see [32]),

5) a reformulation via a Lagrangian for a quantum field theory (from [1]) which seems to give our formulas as the perturbation expansion,

6) a version of the formality morphism for cyclic homology,

7) a canonical quantization of quadratic brackets, and more generally of algebraic Poisson manifolds.

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1 Deformation quantization

1.1 Star-products

Let $A = \Gamma(X, \mathscr{O}_X)$ be the algebra over $\mathbb{R}$ of smooth functions on a finite-dimensional C^∞-manifold X. A star-product on A (see [5]) is an *associative* $\mathbb{R}[[\hbar]]$-linear product on $A[[\hbar]]$ given by the following formula for $f, g \in A \subset A[[\hbar]]$:

$$(f,g) \mapsto f \star g = fg + \hbar B_1(f,g) + \hbar^2 B_2(f,g) + \dots \in A[[\hbar]],$$

where $\hbar$ is the formal variable, and B_i are bidifferential operators (i.e. bilinear maps $A \times A \longrightarrow A$ which are differential operators with respect to each argument of globally bounded order). The product of arbitrary elements of $A[[\hbar]]$ is defined by the condition of linearity over $\mathbb{R}[[\hbar]]$ and $\hbar$-adic continuity:

$$\left(\sum_{n\geq 0} f_n \hbar^n\right) \star \left(\sum_{n\geq 0} g_n \hbar^n\right) := \sum_{k,l\geq 0} f_k g_l \hbar^{k+l} + \sum_{k,l\geq 0, m\geq 1} B_m(f_k, g_l)\hbar^{k+l+m} .$$

There is a natural gauge group acting on star-products. This group consists of automorphisms of $A[[\hbar]]$ considered as an $\mathbb{R}[[\hbar]]$-module (i.e. linear transformations $A \longrightarrow A$ parametrized by $\hbar$), of the following form:

$$f \mapsto f + \hbar D_1(f) + \hbar^2 D_2(f) + \dots, \text{ for } f \in A \subset A[[\hbar]],$$

$$\sum_{n\geq 0} f_n \hbar^n \mapsto \sum_{n\geq 0} f_n \hbar^n + \sum_{n\geq 0, m\geq 1} D_m(f_n)\hbar^{n+m}, \text{ for general element } f(\hbar) = \sum_{n\geq 0} f_n \hbar^n \in A[[\hbar]],$$

where $D_i : A \longrightarrow A$ are differential operators. If $D(\hbar) = 1 + \sum_{m\geq 1} D_m \hbar^m$ is such an automorphism, it acts on the set of star- products as

$$\star \mapsto \star', \ f(\hbar) \star' g(\hbar) := D(\hbar)\big(D(\hbar)^{-1}(f(\hbar)) \star D(\hbar)^{-1}(g(\hbar))\big), \ f(\hbar), g(\hbar) \in A[[\hbar]] .$$

We are interested in star-products up to gauge equivalence.

1.2 First approximation: Poisson structures

It follows from the associativity of $\star$ that the bilinear map $B_1 : A \times A \longrightarrow A$ satisfies the equation

$$fB_1(g,h) - B_1(fg,h) + B_1(f,gh) - B_1(f,g)h = 0,$$

i.e. the linear map $\widetilde{B}_1 : A \otimes A \longrightarrow A$ associated with B_1 as $\widetilde{B}_1(f \otimes g) := B_1(f,g)$, is a 2-cocycle in the cohomological Hochschild complex of algebra A (the definition of this complex is given in Section 3.4.2).

Let us decompose B_1 into the sum of the symmetric part and of the anti-symmetric part:

$$B_1 = B_1^+ + B_1^-,\ B_1^+(f,g) = B_1^+(g,f),\ B_1^-(f,g) = -B_1^-(g,f)\ .$$

Gauge transformations

$$B_1 \mapsto B_1',\ \ B_1'(f,g) = B_1(f,g) - fD_1(g) + D_1(fg) - D_1(f)g$$

where D_1 is an arbitrary differential operator, affect only the symmetric part of B_1, i.e. $B_1^- = (B_1')^-$. One can show that the symmetric part B_1^+ can be killed by a gauge transformation (and it is a coboundary in the Hochschild complex).

Also one can show that the skew-symmetric part B_1^- is a derivation with respect to both functions f and g. Thus, B_1^- comes from a bi-vector field α on X:

$$B_1^-(f,g) = \langle \alpha, df \otimes dg \rangle,\ \ \alpha \in \Gamma(X, \wedge^2 T_X) \subset \Gamma(X, T_X \otimes T_X)\ .$$

Analogous fact in algebraic geometry is that the second Hochschild cohomology group of the algebra of functions on a smooth affine algebraic variety is naturally isomorphic to the space of bi-vector fields (see [25] and also Section 4.6.1.1).

The second term $O(\hbar^2)$ in the associativity equation $f \star (g \star h) = (f \star g) \star h$ implies that α gives a Poisson structure on X,

$$\forall f,g,h\ \ \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0,$$

$$\text{where } \{f,g\} := \frac{f \star g - g \star f}{\hbar}_{|\hbar=0} = 2B_1^-(f,g) = 2\langle \alpha, df \otimes dg \rangle\ .$$

In other words, $[\alpha,\alpha] = 0 \in \Gamma(X, \wedge^3 T_X)$, where the bracket is the Schouten-Nijenhuis bracket on polyvector fields (see Section 4.6.1 for the definition of this bracket).

Thus, gauge equivalence classes of star-products modulo $O(\hbar^2)$ are classified by Poisson structures on X. *A priori* it is not clear whether there exists a star-product with the first term equal to a given Poisson structure, and whether there exists a preferred choice of an equivalence class of star-products. We show in this paper that there is a canonical construction of an equivalence class of star-products for any Poisson manifold.

1.3 Description of quantizations

Theorem 1.1 *The set of gauge equivalence classes of star products on a smooth manifold X can be naturally identified with the set of equivalence classes of Poisson structures depending formally on $\hbar$:*

$$\alpha = \alpha(\hbar) = \alpha_1\hbar + \alpha_2\hbar^2 + \dots \in \Gamma(X, \wedge^2 T_X)[[\hbar]],\ [\alpha,\alpha] = 0 \in \Gamma(X, \wedge^3 T_X)[[\hbar]]$$

modulo the action of the group of formal paths in the diffeomorphism group of X, starting at the identity diffeomorphism.

Any given Poisson structure $\alpha_{(0)}$ gives a path $\alpha(\hbar) := \alpha_{(0)} \cdot \hbar$ and by the Theorem from above, a canonical gauge equivalence class of star products. We will not give a proof of this theorem, as it is an immediate corollary of the Main Theorem of this paper in Section 4.6.2 and a general result from deformation theory (see Section 4.4).

1.4 Examples

1.4.1 Moyal product

The simplest example of a deformation quantization is the Moyal product for the Poisson structure on $\mathbb{R}^d$ with constant coefficients:

$$\alpha = \sum_{i,j} \alpha^{ij} \partial_i \wedge \partial_j, \ \alpha^{ij} = -\alpha^{ji} \in \mathbb{R}$$

where $\partial_i = \partial/\partial x^i$ is the partial derivative in the direction of coordinate x^i, $i = 1, \ldots, d$. The formula for the Moyal product is

$$f \star g = fg + \hbar \sum_{i,j} \alpha^{ij} \partial_i(f) \partial_j(g) + \frac{\hbar^2}{2} \sum_{i,j,k,l} \alpha^{ij} \alpha^{kl} \partial_i \partial_k(f) \partial_j \partial_l(g) + \cdots =$$

$$= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \sum_{i_1, \ldots, i_n; j_1, \ldots j_n} \prod_{k=1}^{n} \alpha^{i_k j_k} \left(\prod_{k=1}^{n} \partial_{i_k} \right) (f) \times \left(\prod_{k=1}^{n} \partial_{j_k} \right) (g) \ .$$

Here and later symbol $\times$ denotes the usual product.

1.4.2 Deformation quantization up to the second order

Let $\alpha = \sum_{i,j} \alpha^{ij} \partial_i \wedge \partial_j$ be a Poisson bracket with variable coefficients in an open domain of $\mathbb{R}^d$ (i.e. α^{ij} is not a constant, but a function of coordinates), then the following formula gives an associative product modulo $O(\hbar^3)$:

$$f \star g = fg + \hbar \sum_{i,j} \alpha^{ij} \partial_i(f) \partial_j(g) + \frac{\hbar^2}{2} \sum_{i,j,k,l} \alpha^{ij} \alpha^{kl} \partial_i \partial_k(f) \partial_j \partial_l(g) +$$

$$+ \frac{\hbar^2}{3} \left(\sum_{i,j,k,l} \alpha^{ij} \partial_j(\alpha^{kl}) \left(\partial_i \partial_k(f) \partial_l(g) - \partial_k(f) \partial_i \partial_l(g) \right) \right) + O(\hbar^3)$$

The associativity up to the second order means that for any 3 functions f, g, h one has

$$(f \star g) \star h = f \star (g \star h) + O(\hbar^3) \ .$$

1.5 Remarks

In general, one should consider bidifferential operators B_i with complex coefficients, as we expect to associate by quantization self-adjoint operators in a Hilbert space to real-valued classical observables. In this paper we deal with purely formal algebraic properties of the deformation quantization and work mainly over the field $\mathbb{R}$ of real numbers.

Also, it is not clear whether the natural physical counterpart for the "deformation quantization" for general Poisson brackets is the usual quantum mechanics. Definitely it is true for the case of non-degenerate brackets, i.e. for symplectic manifolds, but our results show that in general a topological open string theory is more relevant.

2 Explicit universal formula

Here we propose a formula for the star-product for arbitrary Poisson structure α in an open domain of the standard coordinate space $\mathbb{R}^d$. Terms of our formula modulo $O(\hbar^3)$ are the same as in the previous section, plus a gauge-trivial term of order $O(\hbar^2)$, symmetric in f and g. Terms of the formula are certain universal polydifferential operators applied to coefficients of the bi-vector field α and to functions f, g. All indices corresponding to coordinates in the formula appear once as lower indices and once as upper indices, i.e. the formula is invariant under affine transformations of $\mathbb{R}^d$.

In order to describe terms proportional to $\hbar^n$ for any integer $n \geq 0$, we introduce a special class G_n of oriented labeled graphs.

All graphs considered in this paper are finite, *oriented* (i.e. every edge carries an orientation), have no multiple edges and no loops. Such objects we will call here simply graphs without adding adjectives.

Definition 2.1 *A graph Γ is a pair (V_Γ, E_Γ) of two finite sets such that E_Γ is a subset of $(V_\Gamma \times V_\Gamma) \setminus V_\Gamma$.*

Elements of V_Γ are vertices of Γ, elements of E_Γ are edges of Γ. If $e = (v_1, v_2) \in E_\Gamma \subseteq V_\Gamma \times V_\Gamma$ is an edge then we say that e starts at v_1 and ends at v_2.

For any integer $n \geq 0$ we define certain set G_n of labeled graphs. We say that Γ (with some additioanl labels) belongs to G_n if

1) Γ has $n+2$ vertices and $2n$ edges,

2) the set vertices V_Γ is $\{1, \ldots, n\} \sqcup \{L, R\}$, where L, R are just two symbols (capital roman letters, mean Left and Right),

3) edges of Γ are labeled by symbols $e_1^1, e_1^2, e_2^1, e_2^2, \ldots, e_n^1, e_n^2$,

4) for every $k \in \{1, \ldots, n\}$ edges labeled by e_k^1 and e_k^2 start at the vertex k.

Obviously, set G_n is finite, it has $\big(n(n+1)\big)^n$ elements for $n \geq 1$ and 1 element for $n = 0$.

With every labeled graph $\Gamma \in G_n$ we associate a bidifferential operator

$$B_{\Gamma,\alpha} : A \times A \longrightarrow A, \quad A = C^\infty(\mathscr{V}), \ \mathscr{V} \text{ is an open domain in } \mathbb{R}^d$$

which depends on bi-vector field $\alpha \in \Gamma(\mathscr{V}, \wedge^2 T_{\mathscr{V}})$, not necessarily a Poisson one. We show one example, from which the general rule should be clear. In Figure 1, one has $n = 3$ and the list of edges is

$$\big(e_1^1, e_1^2, e_2^1, e_2^2, e_3^1, e_3^2\big) = \big((1,L), (1,R), (2,R), (2,3), (3,L), (3,R)\big) \ .$$

In the picture of Γ we put independent indices $1 \leq i_1, \ldots, i_6 \leq d$ on edges, instead of labels e_*^*. The operator $B_{\Gamma,\alpha}$ corresponding to this graph is

$$(f, g) \mapsto \sum_{i_1, \ldots, i_6} \alpha^{i_1 i_2} \alpha^{i_3 i_4} \partial_{i_4}(\alpha^{i_5 i_6}) \partial_{i_1} \partial_{i_5}(f) \partial_{i_2} \partial_{i_3} \partial_{i_6}(g) \ .$$

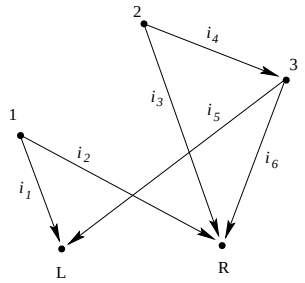

Figure 1: An example of a graph.

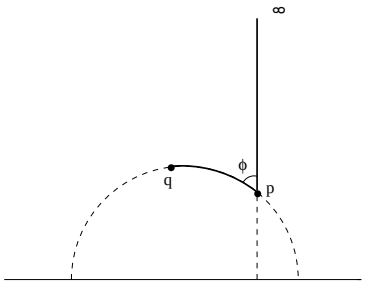

Figure 2: Angle ϕ^h.

The general formula for the operator $B_{\Gamma,\alpha}$ is

$$B_{\Gamma,\alpha}(f,g) := \sum_{I:E_\Gamma \longrightarrow \{1,\ldots,d\}} \left[\prod_{k=1}^{n} \left(\prod_{e\in E_\Gamma,\, e=(*,k)} \partial_{I(e)} \right) \alpha^{I(e_k^1)I(e_k^2)} \right] \times$$
$$\times \left(\prod_{e\in E_\Gamma,\, e=(*,L)} \partial_{I(e)} \right) f \times \left(\prod_{e\in E_\Gamma,\, e=(*,R)} \partial_{I(e)} \right) g \ .$$

In the next step we associate a weight $W_\Gamma \in \mathbb{R}$ with each graph $\Gamma \in G_n$. In order to define it we need an elementary construction from hyperbolic geometry.

Let $p,q,\ p\neq q$ be two points in the upper half-plane $\mathscr{H} = \{z \in \mathbb{C} |\ Im(z) > 0\}$ endowed with the Lobachevsky metric. We denote by $\phi^h(p,q) \in \mathbb{R}/2\pi\mathbb{Z}$ the angle at p formed by two lines, $l(p,q)$ and $l(p,\infty)$ passing through p and q, and through p and the point ∞ on the absolute. The direction of the measurement of the angle is counterclockwise from $l(p,\infty)$ to $l(p,q)$. In the notation $\phi^h(p,q)$ letter h is for *harmonic* (see Figure 2).

An easy planimetry shows that one can express angle $\phi^h(p,q)$ in terms of complex numbers:

$$\phi^h(p,q) = Arg((q-p)/(q-\overline{p})) = \frac{1}{2i} Log\left(\frac{(q-p)(\overline{q}-p)}{(q-\overline{p})(\overline{q}-\overline{p})} \right) \ .$$

Superscript h in the notation ϕ^h refers to the fact that $\phi^h(p,q)$ is harmonic function in both variables $p,q \in \mathscr{H}$. Function $\phi^h(p,q)$ can be defined by continuity also in the case $p,q \in \mathscr{H} \sqcup \mathbb{R},\ p\neq q$.

Denote by $\mathscr{H}_n$ the space of configurations of n numbered pairwise distinct points on $\mathscr{H}$:

$$\mathscr{H}_n = \{(p_1, \dots, p_n) | p_k \in \mathscr{H},\ p_k \neq p_l \text{ for } k \neq l\} .$$

$\mathscr{H}_n \subset \mathbb{C}^n$ is a non-compact smooth $2n$-dimensional manifold. We introduce orientation on $\mathscr{H}_n$ using the natural complex structure on it.

If $\Gamma \in G_n$ is a graph as above, and $(p_1, \dots, p_n) \in \mathscr{H}_n$ is a configuration of points, then we draw a copy of Γ on the plane $\mathbb{R}^2 \simeq \mathbb{C}$ by assigning point $p_k \in \mathscr{H}$ to the vertex k, $1 \leq k \leq n$, point $0 \in \mathbb{R} \subset \mathbb{C}$ to the vertex L, and point $1 \in \mathbb{R} \subset \mathbb{C}$ to the vertex R. Each edge should be drawn as a line interval in hyperbolic geometry. Every edge e of the graph Γ defines an ordered pair (p,q) of points on $\mathscr{H} \sqcup \mathbb{R}$, thus an angle $\phi_e^h := \phi^h(p,q)$. If points p_i move around, we get a function ϕ_e^h on $\mathscr{H}_n$ with values in $\mathbb{R}/2\pi\mathbb{Z}$.

We define the weight of Γ as

$$w_\Gamma := \frac{1}{n!(2\pi)^{2n}} \int\limits_{\mathscr{H}_n} \bigwedge_{i=1}^{n} (d\phi^h_{e_k^1} \wedge d\phi^h_{e_k^2}) .$$

Lemma 2.2 *The integral in the definition of w_Γ is absolutely convergent.*

This lemma is a particular case of a more general statement proven in Section 6 (see the last sentence in Section 6.2).

Theorem 2.3 *Let α be a Poisson bi-vector field in a domain of $\mathbb{R}^d$. The formula*

$$f \star g := \sum_{n=0}^{\infty} \hbar^n \sum_{\Gamma \in G_n} w_\Gamma B_{\Gamma,\alpha}(f,g)$$

defines an associative product. If we change coordinates, we obtain a gauge equivalent star-product.

The proof of this theorem is in a sense elementary, it uses only the Stokes formula and combinatorics of admissible graphs. We will not give here the proof of this theorem as it is a corollary of a general result proven in Section 6.

3 Deformation theory via differential graded Lie algebras

3.1 Tensor categories *Super* and *Graded*

Here we make a comment about the terminology. This comment looks a bit pedantic, but it could help in the struggle with signs in formulas.

The main idea of algebraic geometry is to replace spaces by commutative associative rings (at least locally). One can further generalize this considering commutative associative algebras in general tensor categories (see [10]). In this way one can imitate many constructions from algebra and differential geometry.

The fundamental example is supermathematics, i.e. mathematics in the tensor category $Super^{\mathbf{k}}$ of super vector spaces over a field $\mathbf{k}$ of characteristic zero (see Chapter 3 in [38]). The category $Super^{\mathbf{k}}$ is the category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces over $\mathbf{k}$ (representations of the group $\mathbb{Z}/2\mathbb{Z}$) endowed with the standard tensor product, with the standard associativity

functor, and with a modified commutativity functor (the Koszul rule of signs). We denote by Π the usual functor $Super^{\mathbf{k}} \longrightarrow Super^{\mathbf{k}}$ changing the parity. It is given on objects by the formula $\Pi V = V \otimes \mathbf{k}^{0|1}$. In the sequel we will consider the standard tensor category $Vect^{\mathbf{k}}$ of vector spaces over $\mathbf{k}$ as the full subcategory of $Super^{\mathbf{k}}$ consisting of pure even spaces.

The basic tensor category which appears everywhere in topology and homological algebra is a full subcategory of the tensor category of $\mathbb{Z}$-graded super vector spaces. Objects of this category are infinite sums $\mathscr{E} = \oplus_{n\in\mathbb{Z}} \mathscr{E}^{(n)}$ such that $\mathscr{E}^{(n)}$ is pure even for even n, and pure odd for odd n. We will slightly abuse the language calling this category also the category of graded vector spaces, and denote it simply by $Graded^{\mathbf{k}}$. We denote by $\mathscr{E}^n$ the usual $\mathbf{k}$-vector space underlying the graded component $\mathscr{E}^{(n)}$. The super vector space obtained if we forget about $\mathbb{Z}$-grading on $\mathscr{E} \in Objects(Graded^{\mathbf{k}})$ is $\bigoplus_{n\in\mathbb{Z}} \Pi^n(\mathscr{E}^n)$.

Analogously, we will speak about graded manifolds. They are defined as supermanifolds endowed with $\mathbb{Z}$-grading on the sheaf of functions obeying the same conditions on the parity as above.

The shift functor $[1] : Graded^{\mathbf{k}} \longrightarrow Graded^{\mathbf{k}}$ (acting from the right) is defined as the tensor product with graded space $\mathbf{k}[1]$ where $\mathbf{k}[1]^{-1} \simeq \mathbf{k}$, $\mathbf{k}[1]^{\neq -1} = 0$. Its powers are denoted by $[n]$, $n \in \mathbb{Z}$. Thus, for graded space $\mathscr{E}$ we have

$$\mathscr{E} = \bigoplus_{n\in\mathbb{Z}} \mathscr{E}^n[-n] \ .$$

Almost all results in the present paper formulated for graded manifolds, graded Lie algebras etc., hold also for supermanifolds, super Lie algebras etc.

3.2 Maurer-Cartan equation in differential graded Lie algebras

This part is essentially standard (see [21, 23, 41]).

Let $\mathbf{g}$ be a differential graded Lie algebra over field $\mathbf{k}$ of characteristic zero. Below we recall the list of structures and axioms:

$$\mathbf{g} = \bigoplus_{k\in\mathbb{Z}} \mathbf{g}^k[-k], \quad [\ ,\ ] : \mathbf{g}^k \otimes \mathbf{g}^l \longrightarrow \mathbf{g}^{k+l}, \quad d : \mathbf{g}^k \longrightarrow \mathbf{g}^{k+1},$$

$$d(d(\gamma)) = 0,\ d[\gamma_1,\gamma_2] = [d\gamma_1,\gamma_2] + (-1)^{\overline{\gamma_1}}[\gamma_1, d\gamma_2],\ [\gamma_2,\gamma_1] = -(-1)^{\overline{\gamma_1}\cdot\overline{\gamma_2}}[\gamma_1,\gamma_2],$$

$$[\gamma_1,[\gamma_2,\gamma_3]] + (-1)^{\overline{\gamma_3}\cdot(\overline{\gamma_1}+\overline{\gamma_2})}[\gamma_3,[\gamma_1,\gamma_2]] + (-1)^{\overline{\gamma_1}\cdot(\overline{\gamma_2}+\overline{\gamma_3})}[\gamma_2,[\gamma_3,\gamma_1]] = 0 \ .$$

In formulas above symbols $\overline{\gamma_i} \in \mathbb{Z}$ mean the degrees of homogeneous elements γ_i, i.e. $\gamma_i \in \mathbf{g}^{\overline{\gamma_i}}$.

In other words, $\mathbf{g}$ is a Lie algebra in the tensor category of complexes of vector spaces over $\mathbf{k}$. If we forget about the differential and the grading on $\mathbf{g}$, we obtain a Lie superalgebra.

We associate with $\mathbf{g}$ a functor $Def_{\mathbf{g}}$ on the category of finite-dimensional commutative associative algebras over $\mathbf{k}$, with values in the category of sets. First of all, let us assume that $\mathbf{g}$ is a nilpotent Lie superalgebra. We define set $\mathscr{MC}(\mathbf{g})$ (the set of solutions of the Maurer-Cartan equation modulo the gauge equivalence) by the formula

$$\mathscr{MC}(\mathbf{g}) := \left\{ \gamma \in \mathbf{g}^1 \,|\, d\gamma + \frac{1}{2}[\gamma,\gamma] = 0 \right\} / \Gamma^0$$

where Γ^0 is the nilpotent group associated with the nilpotent Lie algebra $\mathbf{g}^0$. The group Γ acts by affine transformations of the vector space $\mathbf{g}^1$. The action of Γ^0 is defined by the exponentiation of the infinitesimal action of its Lie algebra:

$$\alpha \in \mathbf{g}^0 \mapsto (\dot{\gamma} = d\alpha + [\alpha, \gamma]) \ .$$

Now we are ready to introduce functor $Def_{\mathbf{g}}$. Technically, it is convenient to define this functor on the category of finite-dimensional nilpotent commutative associative algebras *without* unit. Let $\mathbf{m}$ be such an algebra, $\mathbf{m}^{dim(\mathbf{m})+1} = 0$. The functor is given (on objects) by the formula

$$Def_{\mathbf{g}}(\mathbf{m}) = \mathscr{MC}(\mathbf{g} \otimes \mathbf{m}) \ .$$

In the conventional approach $\mathbf{m}$ is the maximal ideal in a finite-dimensional Artin algebra *with* unit

$$\mathbf{m}' := \mathbf{m} \oplus \mathbf{k} \cdot 1 \ .$$

In general, one can think about commutative associative algebras without unit as about objects dual to spaces with base points. Algebra corresponding to a space with base point is the algebra of functions vanishing at the base point.

One can extend the definition of the deformation functor to algebras with linear topology which are projective limits of nilpotent finite-dimensional algebras. For example, in the deformation quantization we use the following algebra over $\mathbb{R}$:

$$\mathbf{m} := \hbar\mathbb{R}[[\hbar]] = \lim_{\leftarrow} \left(\hbar\mathbb{R}[\hbar]/\hbar^k\mathbb{R}[\hbar] \right) \ \text{ as } \ \mathrm{k} \to \infty \ .$$

3.3 Remark

Several authors, following a suggestion of P. Deligne, stressed that the set $Def_{\mathbf{g}}(\mathbf{m})$ should be considered as the set of equivalence classes of objects of certain groupoid naturally associated with $\mathbf{g}(\mathbf{m})$. Almost always in deformation theory, differential graded Lie algebras are supported in non-negative degrees, $\mathbf{g}^{<0} = 0$. Our principal example in the present paper, the shifted Hochschild complex (see the next subsection), has a non-trivial component in degree -1, when it is considered as a graded Lie algebra. The set $Def_{\mathbf{g}}(\mathbf{m})$ in such a case has a natural structure of the set of equivalence classes for a 2-groupoid. In general, if one considers differential graded Lie algebras with components in negative degrees, one meets immediately polycategories and nilpotent homotopy types. Still, it is only a half of the story because one can not say anything about $\mathbf{g}^{\geq 3}$ using this language. Maybe, the better way is to extend the definition of the deformation functor to the category of differential graded nilpotent commutative associative algebras, see the last remark in Section 4.5.2.

3.4 Examples

There are many standard examples of differential graded Lie algebras and related moduli problems.

3.4.1 Tangent complex

Let X be a complex manifold. Define $\mathbf{g}$ over $\mathbb{C}$ as

$$\mathbf{g} = \bigoplus_{k\in\mathbb{Z}} \mathbf{g}^k[-k];\ \ \mathbf{g}^k = \Gamma(X, \Omega_X^{0,k} \otimes T_X^{1,0})\ \text{ for } k \geq 0,\ \ \mathbf{g}^{<0} = 0$$

with the differential equal to $\overline{\partial}$, and the Lie bracket coming from the cup-product on $\overline{\partial}$-forms and the usual Lie bracket on holomorphic vector fields.

The deformation functor related with $\mathbf{g}$ is the usual deformation functor for complex structures on X. The set $Def_{\mathbf{g}}(\mathbf{m})$ can be naturally identified with the set of equivalence classes of analytic spaces $\widetilde{X}$ endowed with a flat map $p : \widetilde{X} \longrightarrow Spec(\mathbf{m}')$, and an identification $i : \widetilde{X} \times_{Spec(\mathbf{m}')} Spec(\mathbb{C}) \simeq X$ of the special fiber of p with X.

3.4.2 Hochschild complex

Let A be an associative algebra over field $\mathbf{k}$ of characteristic zero. The graded space of Hochschild cochains of A with coefficients in A considered as a bimodule over itself is

$$C^\bullet(A,A) := \bigoplus_{k\geq 0} C^k(A,A)[-k],\ C^k(A,A) := Hom_{Vect^{\mathbf{k}}}(A^{\otimes k}, A)\ .$$

We define graded vector space $\mathbf{g}$ over $\mathbf{k}$ by formula $\mathbf{g} := C^\bullet(A,A)[1]$. Thus, we have

$$\mathbf{g} = \bigoplus_{k\in\mathbb{Z}} \mathbf{g}^k[-k];\ \ \mathbf{g}^k := Hom(A^{\otimes(k+1)}, A)\ \text{ for } k \geq -1,\ \ \mathbf{g}^{<(-1)} = 0\ .$$

The differential in $\mathbf{g}$ is shifted by 1 the usual differential in the Hochschild complex, and the Lie bracket is the Gerstenhaber bracket. The explicit formulas for the differential and for the bracket are:

$$\begin{aligned}(d\Phi)(a_0 \otimes \cdots \otimes a_{k+1}) =& a_0 \cdot \Phi(a_1 \otimes \cdots \otimes a_{k+1}) \\ &- \sum_{i=0}^{k} (-1)^i \Phi(a_0 \otimes \cdots \otimes (a_i \cdot a_{i+1}) \otimes \cdots \otimes a_{k+1}) \\ &+ (-1)^k \Phi(a_0 \otimes \cdots \otimes a_k) \cdot a_{k+1}, \quad \Phi \in \mathbf{g}^k,\end{aligned}$$

and

$$[\Phi_1, \Phi_2] = \Phi_1 \circ \Phi_2 - (-1)^{k_1 k_2} \Phi_2 \circ \Phi_1, \quad \Phi_i \in \mathbf{g}^{k_i},$$

where the (non-associative) product $\circ$ is defined as

$$(\Phi_1 \circ \Phi_2)(a_0 \otimes \cdots \otimes a_{k_1+k_2}) =$$

$$\sum_{i=0}^{k_1} (-1)^{ik_2} \Phi_1(a_0 \otimes \cdots \otimes a_{i-1} \otimes (\Phi_2(a_i \otimes \cdots \otimes a_{i+k_2})) \otimes a_{i+k_2+1} \otimes \cdots \otimes a_{k_1+k_2})\ .$$

We would like to give here also an abstract definition of the differential and of the bracket on $\mathbf{g}$. Let F denote the free coassociative graded coalgebra with counit cogenerated by the graded vector space $A[1]$:

$$F = \bigoplus_{n\geq 1} \otimes^n (A[1])\ .$$

Graded Lie algebra $\mathbf{g}$ is the Lie algebra of coderivations of F in the tensor category $Graded^{\mathbf{k}}$. The associative product on A gives an element $m_A \in \mathbf{g}^1,\ m_A : A \otimes A \longrightarrow A$ satisfying the equation $[m_A, m_A] = 0$. The differential d in $\mathbf{g}$ is defined as $ad(m_A)$.

Again, the deformation functor related to $\mathbf{g}$ is equivalent to the usual deformation functor for algebraic structures. Associative products on A correspond to solutions of the Maurer-Cartan equation in $\mathbf{g}$. The set $Def_{\mathbf{g}}(\mathbf{m})$ is naturally identified with the set of equivalence classes of pairs $(\widetilde{A}, i)$ where $\widetilde{A}$ is an associative algebra over $\mathbf{m}' = \mathbf{m} \oplus \mathbf{k} \cdot 1$ such that $\widetilde{A}$ is free as an $\mathbf{m}'$-module, and i an isomorphism of $\mathbf{k}$-algebras $\widetilde{A} \otimes_{\mathbf{m}'} \mathbf{k} \simeq A$.

The cohomology of the Hochschild complex are

$$HH^k(A,A) = Ext^k_{A-mod-A}(A,A),$$

the Ext-groups in the abelian category of bimodules over A. The Hochschild complex *without* shift by 1 also has a meaning in deformation theory, it also has a canonical structure of differential graded Lie algebra, and it controls deformations of A as a bimodule.

4 Homotopy Lie algebras and quasi-isomorphisms

In this section we introduce a language convenient for the homotopy theory of differential graded Lie algebras and for the deformation theory. The ground field $\mathbf{k}$ for linear algebra in our discussion is an arbitrary field of characteristic zero, unless specified.

4.1 Formal manifolds

Let V be a vector space. We denote by $C(V)$ the cofree cocommutative coassociative coalgebra without counit cogenerated by V:

$$C(V) = \bigoplus_{n \geq 1} \left(\otimes^n V\right)^{\Sigma_n} \subset \bigoplus_{n \geq 1} \left(\otimes^n V\right) .$$

Intuitively, we think about $C(V)$ as about an object corresponding to a formal manifold, possibly infinite-dimensional, with base point:

$$\left(V_{formal}, \text{ base point}\right) := \left(\text{ Formal neighborhood of zero in } V, 0\right) .$$

The reason for this is that if V is finite-dimensional then $C(V)^*$ (the dual space to $C(V)$) is the algebra of formal power series on V vanishing at the origin.

Definition 4.1 *A formal pointed manifold M is an object corresponding to a coalgebra $\mathscr{C}$ which is isomorphic to $C(V)$ for some vector space V.*

The specific isomorphism between $\mathscr{C}$ and $C(V)$ is *not* considered as a part of data. Nevertheless, the vector space V can be reconstructed from M as the space of primitive elements in coalgebra $\mathscr{C}$. Here for a nonunital coalgebra $A = C(V)$ we define primitive elements as solutions of the equation $\Delta(a) = 0$ where $\Delta : A \longrightarrow A \otimes A$ is the coproduct on A.

Speaking geometrically, V is the tangent space to M at the base point. A choice of an isomorphism between $\mathscr{C}$ and $C(V)$ can be considered as a choice of an affine structure on M.

If V_1 and V_2 are two vector spaces then a map f between corresponding formal pointed manifolds is defined as a homomorphism of coalgebras (a kind of the pushforward map on distribution-valued densities supported at zero)

$$f_* : C(V_1) \longrightarrow C(V_2) \ .$$

By the universal property of cofree coalgebras any such homomorphism is uniquely specified by a linear map

$$C(V_1) \longrightarrow V_2$$

which is the composition of f_* with the canonical projection $C(V_2) \longrightarrow V_2$. Homogeneous components of this map,

$$f^{(n)} : \left(\otimes^n(V_1)\right)^{\Sigma_n} \longrightarrow V_2, \ \ n \geq 1$$

can be considered as Taylor coefficients of f. More precisely, Taylor coefficients are defined as symmetric polylinear maps

$$\partial^n f : \otimes^n(V_1) \longrightarrow V_2, \ \ \partial^n f(v_1 \cdot \cdots \cdot v_n) := \frac{\partial^n}{\partial t_1 \ldots \partial t_n}_{|t_1=\cdots=t_n=0} (f(t_1 v_1 + \cdots + t_n v_n)) \ .$$

Map $\partial^n f$ goes through the quotient $Sym^n(V_1) := \left(\otimes^n V_1\right)_{\Sigma_n}$. Linear map $f^{(n)}$ coincides with $\partial^n f$ after the identification of the *subspace* $\left(\otimes^n V_1\right)^{\Sigma_n} \subset \otimes^n V_1$ with the *quotient* space $Sym^n(V_1)$.

As in the usual calculus, there is the inverse mapping theorem: non-linear map f is invertible iff its first Taylor coefficient $f^{(1)} : V_1 \longrightarrow V_2$ is invertible.

Analogous definitions and statements can be made in other tensor categories, including $Super^{\mathbf{k}}$ and $Graded^{\mathbf{k}}$.

The reader can ask why we speak about base points for formal manifolds, as such manifolds have only one geometric point. The reason is that later we will consider formal graded manifolds depending on formal parameters. In such a situation the choice of the base point is a non-trivial part of the structure.

4.2 Pre-L_∞-morphisms

Let $\mathbf{g}_1$ and $\mathbf{g}_2$ be two graded vector spaces.

Definition 4.2 *A pre-L_∞-morphism $\mathscr{F}$ from $\mathbf{g}_1$ to $\mathbf{g}_2$ is a map of formal pointed graded manifolds*

$$\mathscr{F} : \left((\mathbf{g}_1[1])_{formal}, 0\right) \longrightarrow \left((\mathbf{g}_2[1])_{formal}, 0\right) \ .$$

Map $\mathscr{F}$ is defined by its Taylor coefficients which are linear maps $\partial^n \mathscr{F}$ of graded vector spaces:

$$\partial^1 \mathscr{F} : \mathbf{g}_1 \longrightarrow \mathbf{g}_2$$
$$\partial^2 \mathscr{F} : \wedge^2(\mathbf{g}_1) \longrightarrow \mathbf{g}_2[-1]$$
$$\partial^3 \mathscr{F} : \wedge^3(\mathbf{g}_1) \longrightarrow \mathbf{g}_2[-2]$$
$$\ldots$$

Here we use the natural isomorphism $Sym^n(\mathbf{g}_1[1]) \simeq \big(\wedge^n(\mathbf{g}_1)\big)[n]$. In plain terms, we have a collection of linear maps between ordinary vector spaces

$$\mathscr{F}_{(k_1,\dots,k_n)} : \mathbf{g}_1^{k_1} \otimes \cdots \otimes \mathbf{g}_1^{k_n} \longrightarrow \mathbf{g}_2^{k_1+\cdots+k_n+(1-n)}$$

with the symmetry property

$$\mathscr{F}_{(k_1,\dots,k_n)}(\gamma_1 \otimes \cdots \otimes \gamma_n) = -(-1)^{k_i k_{i+1}} \mathscr{F}_{(k_1,\dots,k_{i+1},k_i,\dots,k_n)}(\gamma_1 \otimes \cdots \otimes \gamma_{i+1} \otimes \gamma_i \otimes \cdots \otimes \gamma_n) \ .$$

One can write (slightly abusing notations)

$$\partial^n \mathscr{F}(\gamma_1 \wedge \cdots \wedge \gamma_n) = \mathscr{F}_{(k_1,\dots,k_n)}(\gamma_1 \otimes \cdots \otimes \gamma_n)$$

for $\gamma_i \in \mathbf{g}_1^{k_i}$, $i = 1, \dots, n$.

In the sequel we will denote $\partial^n \mathscr{F}$ simply by $\mathscr{F}_n$.

4.3 L_∞-algebras and L_∞-morphisms

Suppose that we have an odd vector field Q of degree $+1$ (with respect to $\mathbb{Z}$-garding) on formal graded manifold $(\mathbf{g}[1]_{formal}, 0)$ such that the Taylor series for coefficients of Q has terms of polynomial degree 1 and 2 only (i.e. linear and quadratic terms). The first Taylor coefficient Q_1 gives a linear map $\mathbf{g} \longrightarrow \mathbf{g}$ of degree $+1$ (or, better, a map $\mathbf{g} \longrightarrow \mathbf{g}[1]$). The second coefficient $Q_2 : \wedge^2\mathbf{g} \longrightarrow \mathbf{g}$ gives a skew-symmetric bilinear operation of degree 0 on $\mathbf{g}$.

It is easy to see that if $[Q,Q]_{super} = 2Q^2 = 0$ then $\mathbf{g}$ is a differential graded Lie algebra, with differential Q_1 and the bracket Q_2, and vice versa.

In [1], supermanifolds endowed with an odd vector field Q such that $[Q,Q]_{super} = 0$, are called Q- manifolds. By analogy, we can speak about formal graded pointed Q-manifolds.

Definition 4.3 *An L_∞-algebra is a pair $(\mathbf{g}, Q)$ where $\mathbf{g}$ is a graded vector space and Q is a coderivation of degree $+1$ on the graded coalgebra $C(\mathbf{g}[1])$ such that $Q^2 = 0$.*

Other names for L_∞-algebras are "(strong) homotopy Lie algebras" and "Sugawara algebras" (see e.g. [24]).

Usually we will denote L_∞-algebra $(\mathbf{g}, Q)$ simply by $\mathbf{g}$.

The structure of an L_∞-algebra on a graded vector space $\mathbf{g}$ is given by the infinite sequence of Taylor coefficients Q_i of the odd vector field Q (coderivation of $C(\mathbf{g}[1])$):

$$Q_1 : \mathbf{g} \longrightarrow \mathbf{g}[1]$$

$$Q_2 : \wedge^2(\mathbf{g}) \longrightarrow \mathbf{g}$$

$$Q_3 : \wedge^3(\mathbf{g}) \longrightarrow \mathbf{g}[-1]$$

$$\dots$$

The condition $Q^2 = 0$ can be translated into an infinite sequence of quadratic constraints on polylinear maps Q_i. First of these constraints means that Q_1 is the differential of the graded space $\mathbf{g}$. Thus, $(\mathbf{g}, Q_1)$ is a complex of vector spaces over $\mathbf{k}$. The second constraint means that Q_2 is a skew-symmetric bilinear operation on $\mathbf{g}$, for which Q_1 satisfies the Leibniz rule. The third constraint means that Q_2 satisfies the Jacobi identity up to homotopy given by

Q_3, etc. As we have seen, a differential graded Lie algebra is the same as an L_∞-algebra with $Q_3 = Q_4 = \cdots = 0$.

Nevertheless, we recommend to return to the geometric point of view and think in terms of formal graded Q-manifolds. This naturally leads to the following

Definition 4.4 *An L_∞-morphism between two L_∞-algebras $\mathbf{g}_1$ and $\mathbf{g}_2$ is a pre-L_∞-morphism $\mathscr{F}$ such that the associated morphism $\mathscr{F}_* : C(\mathbf{g}_1[1]) \longrightarrow C(\mathbf{g}_2[1])$ of graded cocommutative coalgebras, is compatible with coderivations.*

In geometric terms, an L_∞-morphism gives a Q-equivariant map between two formal graded manifolds with base points.

For the case of differential graded Lie algebras a pre-L_∞-morphism $\mathscr{F}$ is an L_∞-morphism iff it satisfies the following equation for any $n = 1, 2 \ldots$ and homogeneous elements $\gamma_i \in \mathbf{g}_1$:

$$d\mathscr{F}_n(\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n) - \sum_{i=1}^{n} \pm \mathscr{F}_n(\gamma_1 \wedge \cdots \wedge d\gamma_i \wedge \cdots \wedge \gamma_n)$$

$$= \frac{1}{2} \sum_{k,l \geq 1,\, k+l=n} \frac{1}{k!l!} \sum_{\sigma \in \Sigma_n} \pm [\mathscr{F}_k(\gamma_{\sigma_1} \wedge \cdots \wedge \gamma_{\sigma_k}), \mathscr{F}_l(\gamma_{\sigma_{k+1}} \wedge \cdots \wedge \gamma_{\sigma_n})]$$

$$+ \sum_{i<j} \pm \mathscr{F}_{n-1}([\gamma_i, \gamma_j] \wedge \gamma_1 \wedge \cdots \wedge \hat{\gamma}_i \wedge \cdots \wedge \hat{\gamma}_j \wedge \cdots \wedge \gamma_n) \ .$$

Here are first two equations in the explicit form:

$$d\mathscr{F}_1(\gamma_1) = \mathscr{F}_1(d\gamma_1)\,,$$

$$d\mathscr{F}_2(\gamma_1 \wedge \gamma_2) - \mathscr{F}_2(d\gamma_1 \wedge \gamma_2) - (-1)^{\overline{\gamma_1}} \mathscr{F}_2(\gamma_1 \wedge d\gamma_2) = \mathscr{F}_1([\gamma_1, \gamma_2]) - [\mathscr{F}_1(\gamma_1), \mathscr{F}_1(\gamma_2)] \ .$$

We see that $\mathscr{F}_1$ is a morphism of complexes. The same is true for the case of general L_∞-algebras. The graded space $\mathbf{g}$ for an L_∞-algebra $(\mathbf{g}, Q)$ can be considered as the tensor product of $\mathbf{k}[-1]$ with the tangent space to the corresponding formal graded manifold at the base point. The differential Q_1 on $\mathbf{g}$ comes from the action of Q on the manifold.

Let us assume that $\mathbf{g}_1$ and $\mathbf{g}_2$ are differential graded Lie algebras, and $\mathscr{F}$ is an L_∞-morphism from $\mathbf{g}_1$ to $\mathbf{g}_2$. Any solution $\gamma \in \mathbf{g}_1^1 \otimes \mathbf{m}$ of the Maurer-Cartan equation where $\mathbf{m}$ is a nilpotent non-unital algebra, produces a solution of the Maurer-Cartan equation in $\mathbf{g}_2^1 \otimes \mathbf{m}$:

$$d\gamma + \frac{1}{2}[\gamma, \gamma] = 0 \Longrightarrow d\widetilde{\gamma} + \frac{1}{2}[\widetilde{\gamma}, \widetilde{\gamma}] = 0 \quad \text{where } \widetilde{\gamma} = \sum_{n=1}^{\infty} \frac{1}{n!} \mathscr{F}_n(\gamma \wedge \cdots \wedge \gamma) \in \mathbf{g}_2^1 \otimes \mathbf{m} \ .$$

The same formula is applicable to solutions of the Maurer-Cartan equation depending formally on parameter $\hbar$:

$$\gamma(\hbar) = \gamma_1 \hbar + \gamma_2 \hbar^2 + \cdots \in \mathbf{g}_1^1[[\hbar]],$$

$$d\gamma(\hbar) + \frac{1}{2}[\gamma(\hbar), \gamma(\hbar)] = 0 \Longrightarrow d\widetilde{\gamma(\hbar)} + \frac{1}{2}[\widetilde{\gamma(\hbar)}, \widetilde{\gamma(\hbar)}] = 0 \ .$$

The reason why it works is that the Maurer-Cartan equation in any differential graded Lie algebra $\mathbf{g}$ can be understood as the collection of equations for the subscheme of zeroes of Q in formal manifold $\mathbf{g}[1]_{formal}$:

$$d\gamma+\frac{1}{2}[\gamma,\gamma]=0 \Longleftrightarrow Q_{|\gamma}=0$$

L_∞-morphisms map zeroes of Q to zeroes of Q because they commute with Q. We will see in Section 4.5.2 that L_∞-morphisms induce natural transformations of deformation functors.

4.4 Quasi-isomorphisms

L_∞-morphisms generalize usual morphisms of differential graded Lie algebras. In particular, the first Taylor coefficient of an L_∞-morphism from $\mathbf{g}_1$ to $\mathbf{g}_2$ is a morphism of complexes $(\mathbf{g}_1,Q_1^{(\mathbf{g}_1)})\longrightarrow(\mathbf{g}_2,Q_1^{(\mathbf{g}_2)})$ where $Q_1^{(\mathbf{g}_i)}$ are the first Taylor coefficients of vector fields $Q^{(\mathbf{g}_i)}$ (which we denoted before simply by Q).

Definition 4.5 *A quasi-isomorphism between L_∞-algebras $\mathbf{g}_1,\mathbf{g}_2$ is an L_∞-morphism $\mathscr{F}$ such that the first component $\mathscr{F}_1$ induces isomorphism between cohomology groups of complexes $(\mathbf{g}_1,Q_1^{(\mathbf{g}_1)})$ and $(\mathbf{g}_2,Q_1^{(\mathbf{g}_2)})$.*

Similarly, we can define quasi-isomorphisms for formal graded pointed Q-manifolds, as maps inducing isomorphisms of cohomology groups of tangent spaces at base points (endowed with differentials which are linearizations of the vector field Q).

The essence of the homotopy/deformation theory is contained in the following

Theorem 4.6 *Let $\mathbf{g}_1,\mathbf{g}_2$ be two L_∞-algebras and $\mathscr{F}$ be an L_∞-morphism from $\mathbf{g}_1$ to $\mathbf{g}_2$. Assume that $\mathscr{F}$ is a quasi-isomorphism. Then there exists an L_∞-morphism from $\mathbf{g}_2$ to $\mathbf{g}_1$ inducing the inverse isomorphism between cohomology of complexes $(\mathbf{g}_i,Q_1^{(\mathbf{g}_i)})$ $i=1,2$. Also, for the case of differential graded algebras, L_∞-morphism $\mathscr{F}$ induces an isomorphism between deformation functors associated with $\mathbf{g}_i$.*

The first part of this theorem shows that if $\mathbf{g}_1$ is quasi-isomorphic to $\mathbf{g}_2$ then $\mathbf{g}_2$ is quasi-isomorphic to $\mathbf{g}_1$, i.e. we get an equivalence relation.

The isomorphism between deformation functors at the second part of the theorem is given by the formula from the last part of Section 4.3.

This theorem is essentially standard (see related results in [21, 23, 41]). Our approach consists in the translation of all relevant notions to the geometric language of formal graded pointed Q-manifolds.

4.5 A sketch of the proof of Theorem 4.6

4.5.1 Homotopy classification of L_∞-algebras

Any complex of vector spaces can be decomposed into the direct sum of a complex with trivial differential and a contractible complex. There is an analogous decomposition in the non-linear case.

Definition 4.7 *An L_∞-algebra $(\mathbf{g}, Q)$ is called minimal if the first Taylor coefficient Q_1 of the coderivation Q vanishes.*

The property of being minimal is invariant under L_∞-isomorphisms. Thus, one can speak about minimal formal graded pointed Q-manifolds.

Definition 4.8 *An L_∞-algebra $(\mathbf{g}, Q)$ is called linear contractible if higher Taylor coefficients $Q_{\geq 2}$ vanish and the differential Q_1 has trivial cohomology.*

The property of being linear contractible is not L_∞-invariant. One can call formal graded pointed Q-manifold *contractible* iff the corresponding differential graded coalgebra is L_∞-isomorphic to a linear contractible one.

Lemma 4.9 *Any L_∞-algebra $(\mathbf{g}, Q)$ is L_∞-isomorphic to the direct sum of a minimal and of a linear contractible L_∞-algebras.*

This lemma says that there exists an affine structure on a formal graded pointed manifold in which the odd vector field Q has the form of a direct sum of a minimal and a linear contractible one. This affine structure can be constructed by induction in the degree of the Taylor expansion. The base of the induction is the decomposition of the complex $(\mathbf{g}, Q_1)$ into the direct sum of a complex with vanishing differential and a complex with trivial cohomology. We leave details of the proof of the lemma to the reader. *Q.E.D.*

As a side remark, we mention analogy between this lemma and a theorem from singularity theory (see, for example, the beginning of 11.1 in [2]): for every germ f of analytic function at critical point one can find local coordinates $(x^1, \ldots, x^k, y^1, \ldots, y^l)$ such that $f = constant + Q_2(x) + Q_{\geq 3}(y)$ where Q_2 is a nondegenerate quadratic form in x and $Q_{\geq 3}(y)$ is a germ of a function in y such that its Taylor expansion at $y = 0$ starts at terms of degree at least 3.

Let $\mathbf{g}$ be an L_∞-algebra and $\mathbf{g}^{min}$ be a minimal L_∞-algebra as in the previous lemma. Then there are two L_∞-morphisms (projection and inclusion)

$$(\mathbf{g}[1]_{formal}, 0) \longrightarrow (\mathbf{g}^{min}[1]_{formal}, 0), \ (\mathbf{g}^{min}[1]_{formal}, 0) \longrightarrow (\mathbf{g}[1]_{formal}, 0)$$

which are both quasi-isomorphisms. From this follows that if

$$(\mathbf{g}_1[1]_{formal}, 0) \longrightarrow (\mathbf{g}_2[1]_{formal}, 0)$$

is a quasi-isomorphism then there exists a quasi-isomorphism

$$(\mathbf{g}_1^{min}[1]_{formal}, 0) \longrightarrow (\mathbf{g}_2^{min}[1]_{formal}, 0) \ .$$

Any quasi-isomorphism between minimal L_∞-algebras is invertible, because it induces an isomorphism of spaces of cogenerators (the inverse mapping theorem mentioned at the end of Section 4.1). Thus, we proved the first part of the theorem. Also, we see that the set equivalence classes of L_∞-algebras up to quasi-isomorphisms can be naturally identified with the set of equivalence classes of minimal L_∞-algebras up to L_∞-isomorphisms.

4.5.2 Deformation functors at fixed points of Q

The deformation functor can be defined in terms of a formal graded Q-manifold M with base point (denoted by 0). The set of solutions of the Maurer-Cartan equation with coefficients in a finite-dimensional nilpotent non-unital algebra $\mathbf{m}$ is defined as the set of $\mathbf{m}$-points of the formal scheme of zeroes of Q:

$$Maps\left(\left(Spec(\mathbf{m}\oplus\mathbf{k}\cdot 1),\text{base point}\right),\left(Zeroes(Q),0\right)\right)$$

$$\subset Maps\left(\left(Spec(\mathbf{m}\oplus\mathbf{k}\cdot 1),\text{base point}\right),\left(M,0\right)\right)\ .$$

In terms of the coalgebra $\mathscr{C}$ corresponding to M this set is equal to the set of homomorphisms of coalgebras $\mathbf{m}^*\longrightarrow\mathscr{C}$ with the image annihilated by Q. Another way to say this is to introduce a *global* (i.e. not formal) pointed Q-manifold of maps from $\left(Spec(\mathbf{m}\oplus\mathbf{k}\cdot 1),\text{base point}\right)$ to $(M,0)$ and consider zeroes of the global vector field Q on it.

Two solutions p_0 and p_1 of the Maurer-Cartan equation are called gauge equivalent iff there exists (parametrized by $Spec(\mathbf{m}\oplus\mathbf{k}\cdot 1)$) polynomial family of odd vector fields $\xi(t)$ on M (of degree -1 with respect to $\mathbb{Z}$-grading) and a polynomial solution of the equation

$$\frac{dp(t)}{dt}=\left([Q,\xi(t)]_{super}\right)_{|p(t)},\ p(0)=p_0,\ p(1)=p_1,$$

where $p(t)$ is a polynomial family of $\mathbf{m}$-points of formal graded manifold M with base point.

In terms of L_∞-algebras, the set of polynomial paths $\{p(t)\}$ is naturally identified with $\mathbf{g}^1\otimes\mathbf{m}\otimes\mathbf{k}[t]$. Vector fields $\xi(t)$ depending polynomially on t are not necessarily vanishing at the base point 0.

One can check that the gauge equivalence defined above is indeed an equivalence relation, i.e. it is transitive. For formal graded pointed manifold M we define set $Def_M(\mathbf{m})$ as the set of gauge equivalence classes of solutions of the Maurer-Cartan equation. The correspondence $\mathbf{m}\mapsto Def_M(\mathbf{m})$ extends naturally to a functor denoted also by Def_M. Analogously, for L_∞-algebra $\mathbf{g}$ we denote by $Def_{\mathbf{g}}$ the corresponding deformation functor.

One can easily prove the following properties:

1) for a differential graded Lie algebra $\mathbf{g}$ the deformation functor defined as above for $(\mathbf{g}[1]_{formal},0)$, is naturally equivalent to the deformation functor defined in Section 3.2,

2) any L_∞-morphism gives a natural transformation of functors,

3) the functor $Def_{\mathbf{g}_1\oplus\mathbf{g}_2}$ corresponding to the direct sum of two L_∞-algebras, is naturally equivalent to the product of functors $Def_{\mathbf{g}_1}\times Def_{\mathbf{g}_2}$,

4) the deformation functor for a linear contractible L_∞-algebra $\mathbf{g}$ is trivial, $Def_{\mathbf{g}}(\mathbf{m})$ is a one-element set for every $\mathbf{m}$.

Properties 2)-4) are just trivial, and 1) is easy. It follows from properties 1)-4) that if an L_∞-morphism of differential graded Lie algebras is a quasi-isomorphism, then it induces an isomorphism of deformation functors. Theorem 4.6 is proven. *Q.E.D.*

We would like to notice here that in the definition of the deformation functor one can consider just a formal pointed *super* Q-manifold $(M,0)$ (i.e. not a graded one), and $\mathbf{m}$ could be a finite-dimensional nilpotent differential super commutative associative non-unital algebra.

4.6 Formality

4.6.1 Two differential graded Lie algebras

Let X be a smooth manifold. We associate with it two differential graded Lie algebras over $\mathbb{R}$. The first differential graded Lie algebra $D_{poly}(X)$ is a subalgebra of the shifted Hochschild complex of the algebra A of functions on X (see Section 3.4.2). The space $D^n_{poly}(X),\ n \geq -1$ consists of Hochschild cochains $A^{\otimes(n+1)} \longrightarrow A$ given by polydifferential operators. In local coordinates (x^i) any element of D^n_{poly} can be written as

$$f_0 \otimes \cdots \otimes f_n \mapsto \sum_{(I_0,\ldots,I_n)} C^{I_0,\ldots,I_n}(x) \cdot \partial_{I_0}(f_0) \ldots \partial_{I_n}(f_n)$$

where the sum is finite, I_k denote multi-indices, ∂_{I_k} denote corresponding partial derivatives, and f_k and $C^{I_0,\ldots,I_n}$ are functions in (x^i).

The second differential graded Lie algebra, $T_{poly}(X)$ is the graded Lie algebra of polyvector fields on X:

$$T^n_{poly}(X) = \Gamma(X, \wedge^{n+1} T_X),\ n \geq -1$$

endowed with the standard Schouten-Nijenhuis bracket and with the differential $d := 0$. We remind here the formula for this bracket:

$$\text{for } k,l \geq 0 \quad [\xi_0 \wedge \cdots \wedge \xi_k, \eta_0 \wedge \cdots \wedge \eta_l] =$$

$$= \sum_{i=0}^{k} \sum_{j=0}^{l} (-1)^{i+j+k} [\xi_i, \eta_j] \wedge \xi_0 \wedge \cdots \wedge \xi_{i-1} \wedge \xi_{i+1} \wedge \cdots \wedge \xi_k \wedge \eta_0 \wedge \cdots \wedge \eta_{j-1} \wedge \eta_{j+1} \wedge \cdots \wedge \eta_l,$$

where $\xi_i, \eta_j \in \Gamma(X, T_X)$,

$$\text{for } k \geq 0 \ \ [\xi_0 \wedge \cdots \wedge \xi_k, h] =$$

$$= \sum_{i=0}^{k} (-1)^i \xi_i(h) \cdot \left(\xi_0 \wedge \cdots \wedge \xi_{i-1} \wedge \xi_{i+1} \wedge \cdots \wedge \xi_k\right),\ h \in \Gamma(X, \mathscr{O}_X),\ \xi_i \in \Gamma(X, T_X)\ .$$

In local coordinates $(x^1, \ldots, x^d)$, if one replaces $\partial/\partial x^i$ by odd variables ψ_i and writes polyvector fields as functions in $(x^1, \ldots, x^d | \psi_1, \ldots, \psi_d)$, the bracket is

$$[\gamma_1, \gamma_2] = \gamma_1 \bullet \gamma_2 - (-1)^{k_1 k_2} \gamma \bullet \gamma_1$$

where we introduce the following notation:

$$\gamma_1 \bullet \gamma_2 := \sum_{i=1}^{d} \frac{\partial \gamma_1}{\partial \psi_i} \frac{\partial \gamma_2}{\partial x^i},\ \gamma_i \in T^{k_i}(\mathbb{R}^d)\ .$$

4.6.1.1. *A map from $T_{poly}(X)$ to $D_{poly}(X)$*

We have an evident map $\mathscr{U}_1^{(0)} : T_{poly}(X) \longrightarrow D_{poly}(X)$. It is defined, for $n \geq 0$, by:

$$\mathscr{U}_1^{(0)} : (\xi_0 \wedge \cdots \wedge \xi_n) \mapsto \left(f_0 \otimes \cdots \otimes f_n \mapsto \frac{1}{(n+1)!} \sum_{\sigma \in \Sigma_{n+1}} sgn(\sigma) \prod_{i=0}^{n} \xi_{\sigma_i}(f_i) \right),$$

and for $h \in \Gamma(X, \mathscr{O}_X)$ by:

$$h \mapsto \big(1 \mapsto h\big).$$

Theorem 4.10 *$\mathscr{U}_1^{(0)}$ is a quasi-isomorphism of complexes.*

This is a version of Hochschild-Kostant-Rosenberg theorem which says that for a smooth affine algebraic variety Y over a field $\mathbf{k}$ of characteristic zero, the Hochschild cohomology of algebra $\mathscr{O}(Y)$ coincides with the space $\oplus_{k\geq 0}\Gamma(X,\wedge^k T_Y)[-k]$ of algebraic polyvector fields on Y (see [25]). Analogous statement for C^∞ manifolds seems to be well known, although we were not able to find it in the literature (e.g. in [6] a similar statement was proven for Hochschild *cohomology*). In any case, we give here a proof.

Proof: First of all, one can immediately check that the image of $\mathscr{U}_1^{(0)}$ is annihilated by the differential in $D_{poly}(X)$, i.e. that $\mathscr{U}_1^{(0)}$ is a morphism of complexes.

Complex $D_{poly}(X)$ is filtered by the total degree of polydifferential operators. Complex $T_{poly}(X)$ endowed with zero differential also carries a very simple filtration (just by degrees), such that $\mathscr{U}_1^{(0)}$ is compatible with filtrations. We claim that

$$Gr\big(\mathscr{U}_1^{(0)}\big) : Gr\big(T_{poly}(X)\big) \longrightarrow Gr\big(D_{poly}(X)\big)$$

is a quasi-isomorphism. In the graded complex $Gr\big(D_{poly}(X)\big)$ associated with the filtered complex $D_{poly}(X)$ all components are sections of some natural vector bundles on X, and the differential is A-linear, $A = C^\infty(X)$. The same is true by trivial reasons for $T_{poly}(X)$. Thus, we have to check that the map $Gr\big(\mathscr{U}_1^{(0)}\big)$ is a quasi-isomorphism fiberwise.

Let x be a point of X and T be the tangent space at x. Principal symbols of polydifferential operators at x lie in vector spaces

$$Sym(T)\otimes\cdots\otimes Sym(T) \quad (n \text{ times},\ n\geq 0)$$

where $Sym(T)$ is the free polynomial algebra generated by T. It is convenient here to identify $Sym(T)$ with the cofree cocommutative coassociative coalgebra *with* counit cogenerated by T:

$$\mathscr{C} := C(T)\oplus(\mathbf{k}\cdot 1)^* \ .$$

$Sym(T)$ is naturally isomorphic to the algebra of differential operators on T with constant coefficients. If D is such an operator then it defines a continuous linear functional on the algebra of formal power series at $0\in T$:

$$f \mapsto \big(D(f)\big)(0)$$

i.e. an element of coalgebra $\mathscr{C}$.

We denote by Δ the coproduct in coalgebra $\mathscr{C}$. It is easy to see that differential in the complex $Gr\big(D_{poly}(X)\big)$ in the fiber at x is the following:

$$d : \otimes^{n+1}\mathscr{C} \longrightarrow \otimes^{n+2}\mathscr{C},$$

$$d = 1^*\otimes id_{\otimes^{n+1}\mathscr{C}} - \sum_{i=0}^{n}(-1)^i\, id\otimes\cdots\otimes\Delta_i\otimes\cdots\otimes id + (-1)^n id_{\otimes^{n+1}\mathscr{C}}\otimes 1^*$$

where Δ_i is coproduct Δ applied to the i-th argument.

Lemma 4.11 *Let $\mathscr{C}$ be the cofree cocommutative coassociative coalgebra with counit cogenerated by a finite-dimensional vector space T. Then the natural homomorphism of complexes*

$$\left(\wedge^{n+1}T, \text{differential} = 0\right) \longrightarrow \left(\otimes^{n+1}\mathscr{C}, \text{differential as above}\right)$$

is a quasi-isomorphism.

What we consider is one of the standard complexes in homological algebra. One of possible proofs is the following:
Proof: let us decompose complex $\left(\otimes^{n+1}\mathscr{C}\right)$ into the infinite direct sum of subcomplexes consisting of tensors of fixed total degrees (homogeneous components with respect to the action of the Euler vector field on T). Our statement means in particular that for only finitely many degrees these subcomplexes have non-trivial cohomology. Thus, the statement of the lemma is true iff the analogous statement holds when infinite sums are replaced by infinite products in the decomposition of $\left(\otimes^{n+1}\mathscr{C}\right)$. Components of the completed complex are spaces $Hom(A^{\otimes(n+1)}, \mathbf{k})$ where A is the algebra of polynomial functions on T. It is easy to see that the completed complex calculates groups $Ext^{n+1}_{A-mod}(\mathbf{k},\mathbf{k}) = \wedge^{n+1}T$, where 1-dimensional space $\mathbf{k}$ is considered as A-module (via values of polynomial at $0 \in T$) and has a resolution

$$\ldots \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0 \longrightarrow \ldots$$

by free A-modules. *Q.E.D.*

As a side remark, we notice that the statement of the lemma holds also if one replaces $\mathscr{C}$ by $C(T)$ (i.e. the free coalgebra without counit) and removes terms with 1^* from the differential. In the language of Hochschild cochains it means that the subcomplex of *reduced* cochains is quasi-isomorphic to the total Hochschild complex.

The lemma implies that $gr\left(\mathscr{U}_1^{(0)}\right)$ is an isomorphism fiberwise. Applying the standard argument with spectral sequences we obtain the proof of the theorem. *Q.E.D.*

4.6.2 Main theorem

Unfortunately, map $\mathscr{U}_1^{(0)}$ does not commute with Lie brackets, the Schouten-Nijenhuis bracket does not go to the Gerstenhaber bracket. We claim that this defect can be cured:

MAIN THEOREM *There exists an L_∞-morphism $\mathscr{U}$ from $T_{poly}(X)$ to $D_{poly}(X)$ such that $\mathscr{U}_1 = \mathscr{U}_1^{(0)}$*

In other words, this theorem says that $T_{poly}(X)$ and $D_{poly}(X)$ are quasi-isomorphic differential graded Lie algebras. In analogous situation in rational homotopy theory (see [42]), a differential graded commutative algebra is called formal if it is quasi-isomorphic to its cohomology algebra endowed with zero differential. This explains the title of Section 4.6.

The quasi-isomorphism $\mathscr{U}$ in the theorem is not canonical. We will construct explicitly a family of quasi-isomorphisms parametrized in certain sense by a contractible space. It means that our construction is canonical up to (higher) homotopies.

Solutions of the Maurer-Cartan equation in $T_{poly}(X)$ are exactly Poisson structures on X:

$$\alpha \in T^1_{poly}(X) = \Gamma(X, \wedge^2 T_X),\ [\alpha,\alpha] = 0\ .$$

Any such α defines also a solution formally depending on $\hbar$,

$$\gamma(\hbar) := \alpha \cdot \hbar \in T^1_{poly}(X)[[\hbar]]\,,\ \ [\gamma(\hbar),\gamma(\hbar)] = 0\ .$$

The gauge group action is the action of the diffeomorphism group by conjugation. Solutions of the Maurer-Cartan equation in $D_{poly}(X)$ formally depending on $\hbar$ are star-products. Thus, we obtain as a corollary that any Poisson structure on X gives a canonical equivalence class of star-products, and the Theorem 1.1.

The rest of the paper is devoted to the proof of the Main Theorem, and to the discussion of various applications, corollaries and extensions. In Section 5 we will make some preparations for the universal formula (Section 6) for an L_∞-morphism from $T_{poly}(X)$ to $D_{poly}(X)$ in the case of flat space $X = \mathbb{R}^d$. In Section 7 we extend our construction to general manifolds.

4.6.3 Non-uniqueness

There are other natural quasi-isomorphisms between $T_{poly}(X)$ and $D_{poly}(X)$ which differ essentially from the quasi-isomorphism $\mathscr{U}$ constructed in Sections 6 and 7, i.e. not even homotopic in a natural sense to $\mathscr{U}$. By *homotopy* here we mean the following. L_∞-morphisms from one L_∞-algebra to another can be identified with fixed points of Q on infinite-dimensional supermanifold of maps. Mimicking constructions and definitions from Section 4.5.2 one can define an equivalence relation (homotopy equivalence) on the set of L_∞-morphisms.

Firstly, the multiplicative group $\mathbb{R}^\times$ acts by automorphisms of $T_{poly}(X)$, multiplying elements $\gamma \in T_{poly}(X)^k$ by λ^k for $\lambda \in \mathbb{R}^\times$. Composing these automorphisms with $\mathscr{U}$ one get a one-parameter family of quasi-isomorphisms. Secondly, in [30] we constructed an exotic infinitesimal L_∞-automorphism of $T_{poly}(X)$ for the case $X = \mathbb{R}^d$ which probably extends to general manifolds. In particular, this exotic automorphism produces a vector field on the "space of Poisson structures". The evolution with respect to time t is described by the following non-linear partial differential equation:

$$\frac{d\alpha}{dt} := \sum_{i,j,k,l,m,k',l',m'} \frac{\partial^3 \alpha^{ij}}{\partial x^k \partial x^l \partial x^m} \frac{\partial \alpha^{kk'}}{\partial x^{l'}} \frac{\partial \alpha^{ll'}}{\partial x^{m'}} \frac{\partial \alpha^{mm'}}{\partial x^{k'}} (\partial_i \wedge \partial_j)$$

where $\alpha = \sum_{i,j} \alpha^{ij}(x) \partial_i \wedge \partial_j$ is a bi-vector field on $\mathbb{R}^d$.

A priori we can guarantee the existence of a solution of the evolution only for small times and real-analytic initial data. One can show that:

1) this evolution preserves the class of (real-analytic) Poisson structures,

2) if two Poisson structures are conjugate by a real-analytic diffeomorphism then the same will hold after the evolution.

Thus, our evolution operator is essentially intrinsic and does not depend on the choice of coordinates.

Combining it with the action of $\mathbb{R}^\times$ as above we see that the Lie algebra $aff(1,\mathbb{R})$ of infinitesimal affine transformations of the line $\mathbb{R}^1$ acts non-trivially on the space of homotopy classes of quasi- isomorphisms between $T_{poly}(X)$ and $D_{poly}(X)$. Maybe, there are other exotic L_∞-automorphisms, this possibility is not ruled out. It is not clear whether our quasi-isomorphism $\mathscr{U}$ is better than others.

5 Configuration spaces and their compactifications

5.1 Definitions

Let n, m be non-negative integers satisfying the inequality $2n+m \geq 2$. We denote by $Conf_{n,m}$ the product of the configuration space of the upper half-plane with the configuration space of the real line:

$$Conf_{n,m} = \{(p_1, \ldots, p_n; q_1, \ldots, q_m) | p_i \in \mathscr{H}, q_j \in \mathbb{R}, p_{i_1} \neq p_{i_2} \text{ for } i_1 \neq i_2,\ q_{j_1} \neq q_{j_2} \text{ for } j_1 \neq j_2\}.$$

$Conf_{n,m}$ is a smooth manifold of dimension $2n+m$. The group $G^{(1)}$ of holomorphic transformations of $\mathbb{CP}^1$ preserving the upper half-plane and the point ∞, acts on $Conf_{n,m}$. This group is a 2-dimensional connected Lie group, isomorphic to the group of orientation-preserving affine transformations of the real line:

$$G^{(1)} = \{z \mapsto az+b | a, b \in \mathbb{R}, a > 0\}\ .$$

It follows from the condition $2n+m \geq 2$ that the action of $G^{(1)}$ on $Conf_{n,m}$ is free. The quotient space $C_{n,m} := Conf_{n,m}/G^{(1)}$ is a manifold of dimension $2n+m-2$. If $P = (p_1, \ldots, p_n; q_1, \ldots, q_m)$ is a point of $Conf_{n,m}$ then we denote by $[P]$ the corresponding point of $C_{n,m}$.

Analogously, we introduce simpler spaces $Conf_n$ and C_n for any $n \geq 2$:

$$Conf_n := \{(p_1, \ldots, p_n) | p_i \in \mathbb{C},\ p_i \neq p_j \text{ for } i \neq j\},$$
$$C_n = Conf_n/G^{(2)},\ dim(C_n) = 2n-3,$$

where $G^{(2)}$ is a 3-dimensional Lie group,

$$G^{(2)} = \{z \mapsto az+b | a \in \mathbb{R}, b \in \mathbb{C}, a > 0\}\ .$$

We will construct compactifications $\overline{C}_{n,m}$ of $C_{n,m}$ (and compactifications $\overline{C}_n$ of C_n) which are smooth manifolds with corners.

We remind that a manifold with corners (of dimension d) is defined analogously to a usual manifold with boundary, with the only difference that the manifold with corners looks locally as an open part of closed simplicial cone $(\mathbb{R}_{\geq 0})^d$. For example, the closed hypercube $[0,1]^d$ is a manifold with corners. There is a natural smooth stratification by faces of any manifold with corners.

First of all, we give one of possible formal definitions of the compactification $\overline{C}_n$ where $n \geq 2$. With any point $[(p_1, \ldots, p_n)]$ of C_n we associate a collection of $n(n-1)$ angles with values in $\mathbb{R}/2\pi\mathbb{Z}$:

$$(Arg(p_i - p_j))_{i \neq j}$$

and $n^2(n-1)^2$ ratios of distances:

$$(|p_i - p_j|/|p_k - p_l|)_{i \neq j, k \neq l}$$

It is easy to see that we obtain an embedding of C_n into the manifold $(\mathbb{R}/2\pi\mathbb{Z})^{n(n-1)} \times \mathbb{R}_{>0}^{n^2(n-1)^2}$. The space $\overline{C}_n$ is defined as the compactification of the image of this embedding in larger manifold

$$(\mathbb{R}/2\pi\mathbb{Z})^{n(n-1)} \times [0, +\infty]^{n^2(n-1)^2}$$

For the space $C_{n,m}$ we use first its embedding to C_{2n+m} which is defined on the level of configuration spaces as

$$(p_1,\dots,p_n;q_1,\dots,q_m)\mapsto(p_1,\dots,p_n,\overline{p}_1,\dots,\overline{p}_n,q_1,\dots,q_m)$$

and then compactify the image in $\overline{C}_{2n+m}$. The result is by definition the compactified space $\overline{C}_{n,m}$.

One can show that open strata of $\overline{C}_{n,m}$ are naturally isomorphic to products of manifolds of type $C_{n',m'}$ and $C_{n'}$. In the next subsection we will describe explicitly $\overline{C}_{n,m}$ as a manifold with corners.

There is a natural action of the permutation group Σ_n on C_n, and also of $\Sigma_n\times\Sigma_m$ on $C_{n,m}$. This gives us a possibility to define spaces C_A and $C_{A,B}$ for finite sets A,B such that $\#A\geq 2$ or $2\#A+\#B\geq 2$ respectively. If $A'\hookrightarrow A$ and $B'\hookrightarrow B$ are inclusions of sets then there are natural fibrations (forgetting maps) $C_A\longrightarrow C_{A'}$ and $C_{A,B}\longrightarrow C_{A',B'}$.

5.2 Looking through a magnifying glass

From the definition of the compactification given in the previous subsection it is not clear what is exactly a point of the compactified space. We are going to explain an intuitive idea underlying a direct construction of the compactification $\overline{C}_{n,m}$ as a manifold with corners. For more formal treatment of compactifications of configuration spaces we refer the reader to [16] (for the case of smooth algebraic varieties).

Let us try to look through a magnifying glass, or better through a microscope with arbitrary magnification, on different parts of the picture formed by points on $\mathscr{H}\cup\mathbb{R}\subset\mathbb{C}$, and by the line $\mathbb{R}\subset\mathbb{C}$. Here we use *Euclidean* geometry on $\mathbb{C}\simeq\mathbb{R}^2$ instead of Lobachevsky geometry.

Before doing this let us first consider the case of a configuration on $\mathbb{R}^2\simeq\mathbb{C}$, i.e. without the horizontal line $\mathbb{R}\subset\mathbb{C}$. We say that the configuration $(p_1,\dots,p_n)$ is in *standard position* iff

1) the diameter of the set $\{p_1,\dots,p_n\}$ is equal to 1, and,

2) the center of the minimal circle containing $\{p_1,\dots,p_n\}$ is $0\in\mathbb{C}$.

It is clear that any configuration of n pairwise distinct points in the case $n\geq 2$ can be uniquely put to standard position by a unique element of group $G^{(2)}$. The set of configurations in standard position gives a continuous section s^{cont} of the natural projection map $Conf_n\longrightarrow C_n$.

For a configuration in standard position there could be several domains where we will need magnification in order to see details. These domains are those where at least two points of the configuration come too close to each other.

After an appropriate magnification of any such domain we again get a stable configuration (i.e.the number of points there is at least 2). Then we can put it again in standard position and repeat the procedure.

In such a way we get an oriented tree T with one root, and leaves numbered from 1 to n. For example, the configuration in Figure 3 gives the tree in Figure 4.

For every vertex of tree T except leaves, we denote by $Star(v)$ the set of edges starting at v. For example, in the figure from above the set $Star(root)$ has three elements, and sets $Star(v)$ for other three vertices all have two elements.

Points in C_n close to one which we consider, can be parametrized by the following data:

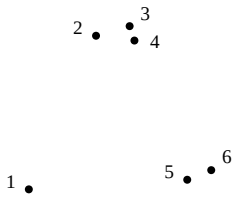

Figure 3: Configuration of points close to the boundary of the compactified configuration space.

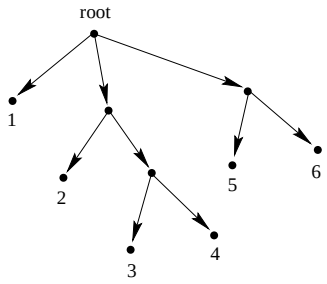

Figure 4: Tree corresponding the limiting point in the configuration space.

a) for each vertex v of T except leaves, a stable configuration c_v in the standard position of points labeled by the set $Star(v)$,

b) for each vertex v except leaves and the root of the tree, the scale $s_v > 0$ with which we should put a copy of c_v instead of the corresponding point $p_v \in \mathbb{C}$ on stable configuration c_u where $u \in V_T$ is such that $(u,v) \in E_T$.

More precisely, we act on the configuration c_v by the element $(z \mapsto s_v z + p_v)$ of $G^{(2)}$.

Numbers s_v are small but positive. The compactification $\overline{C}_n$ is achieved by formally permitting some of scales s_v to be equal to 0.

In this way we get a compact topological manifold with corners, with strata C_T labeled by trees T (with leaves numbered from 1 to n). Each stratum C_T is canonically isomorphic to the product $\prod_v C_{Star(v)}$ over all vertices v except leaves. In the description as above points of C_T correspond to collections of configurations with *all* scales s_v equal to zero. Let us repeat: as a set $\overline{C}_n$ coincides with

$$\bigsqcup_{\text{trees } T} \prod_{v \in V_T \setminus \{\text{leaves}\}} C_{Star(v)} \ .$$

In order to introduce a smooth structure on $\overline{C}_n$, we should choose a Σ_n-equivariant *smooth* section s^{smooth} of the projection map $Conf_n \longrightarrow C_n$ instead of the section s^{cont} given by configurations in standard position. Local coordinates on $\overline{C}_n$ near a given point lying in stratum C_T are scales $s_v \in \mathbb{R}_{\geq 0}$ close to zero and local coordinates in manifolds $C_{Star(v)}$ for all $v \in V_T \setminus \{\text{leaves}\}$. The resulting structure of a smooth manifold with corners does not depend on the choice of section s^{smooth}.

The case of configurations of points on $\mathscr{H} \cup \mathbb{R}$ is not much harder. First of all, we say that a finite non-empty set S of points on $\mathscr{H} \cup \mathbb{R}$ is in *standard position* iff

1) the projection of the convex hull of S to the horizontal line $\mathbb{R} \subset \mathbb{C} \simeq \mathbb{R}^2$ is either the

Figure 5: Space $C_{1,1}$ homeomorphic to an interval.

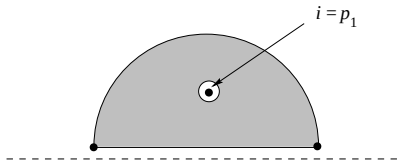

Figure 6: Space $\overline{C}_{2,0}$.

one-point set $\{0\}$, or it is an interval with the center at 0,

2) the maximum of the diameter of S and of the distance from S to $\mathbb{R}$ is equal to 1.

It is easy to see that for $2n+m \geq 2$ (the stable case) any configuration of n points on $\mathscr{H}$ and m points on $\mathbb{R}$ can be put uniquely in standard position by an element of $G^{(1)}$. In order to get a smooth structure, we repeat the same arguments as for the case of manifolds C_n.

Domains where we will need magnification in order to see details, are now of two types. The first case is when at least two points of the configuration come too close to each other. We want to know whether what we see is a single point or a collection of several points. The second possibility is when a point on $\mathscr{H}$ comes too close to $\mathbb{R}$. Here we want also to decide whether what we see is a point (or points) on $\mathscr{H}$ or on $\mathbb{R}$.

If the domain which we want to magnify is close to $\mathbb{R}$, then after magnification we get again a stable configuration which we can put into the standard position. If the domain is inside $\mathscr{H}$, then after magnification we get a picture without the horizontal line in it, and we are back in the situation concerning $\overline{C}_{n'}$ for $n' \leq n$.

It is instructional to draw low-dimensional spaces $C_{n,m}$. The simplest one, $C_{1,0} = \overline{C}_{1,0}$ is just a point. The space $C_{0,2} = \overline{C}_{0,2}$ is a two-element set. The space $C_{1,1}$ is an open interval, and its closure $\overline{C}_{1,1}$ is a closed interval (the real line $\mathbb{R} \subset \mathbb{C}$ is dashed in Figure 5).

The space $C_{2,0}$ is diffeomorphic to $\mathscr{H} \setminus \{0+1\cdot i\}$. The reason is that by action of $G^{(1)}$ we can put point p_1 to the position $i = \sqrt{-1} \in \mathscr{H}$. The closure $\overline{C}_{2,0}$ can be drawn as in Figure 6 or as in Figure 7.

Forgetting maps (see the end of Section 5.1) extend naturally to smooth maps of compactified spaces.

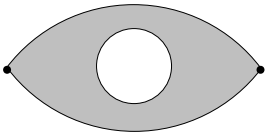

Figure 7: The Eye.

5.2.1 Boundary strata

We give here the list of all strata in $\overline{C}_{A,B}$ of codimension 1:

S1) points $p_i \in \mathscr{H}$ for $i \in S \subseteq A$ where $\#S \geq 2$, move close to each other but far from $\mathbb{R}$,

S2) points $p_i \in \mathscr{H}$ for $i \in S \subseteq A$ and points $q_j \in \mathbb{R}$ for $j \in S' \subseteq B$ where $2\#S + \#S' \geq 2$, all move close to each other and to $\mathbb{R}$, with at least one point left outside S and S', i.e. $\#S + \#S' \leq \#A + \#B - 1$.

The stratum of type S1 is

$$\partial_S \overline{C}_{A,B} \simeq C_S \times C_{(A\setminus S)\sqcup\{pt\},B}$$

where $\{pt\}$ is a one-element set, whose element represents the cluster $(p_i)_{i\in S}$ of points in $\mathscr{H}$. Analogously, the stratum of type S2 is

$$\partial_{S,S'} \overline{C}_{A,B} \simeq C_{S,S'} \times C_{A\setminus S,(B\setminus S')\sqcup\{pt\}} \ .$$

6 Universal formula

In this section we propose a formula for an L_∞-morphism $T_{poly}(\mathbb{R}^d) \longrightarrow D_{poly}(\mathbb{R}^d)$ generalizing a formula for the star-product in Section 2. In order to write it we need to make some preparations.

6.1 Admissible graphs

Definition 6.1 *Admissible graph Γ is a graph with labels such that*
1) the set of vertices V_Γ is $\{1,\dots,n\} \sqcup \{\overline{1},\dots,\overline{m}\}$ where $n,m \in \mathbb{Z}_{\geq 0}$, $2n+m-2 \geq 0$; vertices from the set $\{1,\dots,n\}$ are called vertices of the first type, vertices from $\{\overline{1},\dots,\overline{m}\}$ are called vertices of the second type,
2) every edge $(v_1,v_2) \in E_\Gamma$ starts at a vertex of first type, $v_1 \in \{1,\dots,n\}$,
3) for every vertex $k \in \{1,\dots,n\}$ of the first type, the set of edges

$$Star(k) := \{(v_1,v_2) \in E_\Gamma | \, v_1 = k\}$$

starting from k, is labeled by symbols $(e_k^1,\dots,e_k^{\#Star(k)})$.

Labeled graphs considered in Section 2 are exactly (after the identifications $L = \overline{1},\ R = \overline{2}$) admissible graphs such that m is equal to 2, and the number of edges starting at every vertex of first type is also equal to 2.

6.2 Differential forms on configuration spaces

The space $\overline{C}_{2,0}$ (the Eye) is homotopy equivalent to the standard circle $S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$. Moreover, one of its boundary components, the space $C_2 = \overline{C}_2$, is naturally identified with the standard circle S^1. The other component of the boundary is the union of two closed intervals (copies of $\overline{C}_{1,1}$) with identified end points.

Definition 6.2 *An angle map is a smooth map $\phi : \overline{C}_{2,0} \longrightarrow \mathbb{R}/2\pi\mathbb{Z} \simeq S^1$ such that the restriction of ϕ to $C_2 \simeq S^1$ is the angle measured in the anti-clockwise direction from the vertical line, and ϕ maps the whole upper interval $\overline{C}_{1,1} \simeq [0,1]$ of the Eye, to a point in S^1.*

We will denote $\phi([(x,y)])$ simply by $\phi(x,y)$ where $x,y \in \mathscr{H} \sqcup \mathbb{R},\ x \neq y$. It follows from the definition that $d\phi(x,y) = 0$ if x stays in $\mathbb{R}$.

For example, the special map ϕ^h used in the formula in Section 2, is an angle map. In the rest of the paper we can use any ϕ, not necessarily harmonic.

We are now prepared for the analytic part of the universal formula. Let Γ be an admissible graph with n vertices of the first type, m vertices of the second type and with $2n+m-2$ edges. We define the weight of graph Γ by the following formula:

$$W_\Gamma := \prod_{k=1}^{n} \frac{1}{(\#Star(k))!} \frac{1}{(2\pi)^{2n+m-2}} \int\limits_{\overline{C}^+_{n,m}} \bigwedge_{e \in E_\Gamma} d\phi_e \ .$$

Let us explain what is written here. The domain of integration $\overline{C}^+_{n,m}$ is a connected component of $\overline{C}_{n,m}$ which is the closure of configurations for which points q_j, $1 \leq j \leq m$ on $\mathbb{R}$ are placed in the increasing order:

$$q_1 < \cdots < q_m \ .$$

The orientation of $Conf_{n,m}$ is the product of the standard orientation on the coordinate space $\mathbb{R}^m \supset \{(q_1,\ldots,q_m) | \, q_j \in \mathbb{R}\}$, with the product of standard orientations on the plane $\mathbb{R}^2$ (for points $p_i \in \mathscr{H} \subset \mathbb{R}^2$). The group $G^{(1)}$ is even-dimensional and naturally oriented because it acts freely and transitively on complex manifold $\mathscr{H}$. Thus, the quotient space $C_{n,m} = Conf_{n,m}/G^{(1)}$ carries again a natural orientation.

Every edge e of Γ defines a map from $\overline{C}_{n,m}$ to $\overline{C}_{2,0}$ or to $\overline{C}_{1,1} \subset \overline{C}_{2,0}$ (the forgetting map). Here we consider inclusion $\overline{C}_{1,1}$ in $\overline{C}_{2,0}$ as the *lower* interval of the Eye. The pullback of the function ϕ by the map $\overline{C}_{n,m} \longrightarrow \overline{C}_{2,0}$ corresponding to edge e is denoted by ϕ_e.

Finally, the ordering in the wedge product of 1-forms $d\phi_e$ is fixed by enumeration of the set of sources of edges and by the enumeration of the set of edges with a given source.

The integral giving W_Γ is absolutely convergent because it is an integral of a smooth differential form over a compact manifold with corners.

6.3 Pre-L_∞-morphisms associated with graphs

For any admissible graph Γ with n vertices of the first type, m vertices of the second type, and $2n+m-2+l$ edges where $l \in \mathbb{Z}$, we define a linear map

$$\mathscr{U}_\Gamma : \otimes^n T_{poly}(\mathbb{R}^d) \longrightarrow D_{poly}(\mathbb{R}^d)[1+l-n].$$

This map has only one non-zero graded component $(\mathscr{U}_\Gamma)_{(k_1,\ldots,k_n)}$ where $k_i = \#Star(i) - 1$, $i = 1,\ldots,n$. If $l = 0$ then from $\mathscr{U}_\Gamma$ after anti-symmetrization we obtain a pre-L_∞-morphism.

Let $\gamma_1,\ldots,\gamma_n$ be polyvector fields on $\mathbb{R}^d$ of degrees $(k_1+1),\ldots,(k_n+1)$, and $f_1,\ldots,f_m$ be functions on $\mathbb{R}^d$. We are going to write a formula for function Φ on $\mathbb{R}^n$:

$$\Phi := (\mathscr{U}_\Gamma(\gamma_1 \otimes \cdots \otimes \gamma_n))\,(f_1 \otimes \cdots \otimes f_m) \ .$$

The formula for Φ is the sum over all configurations of indices running from 1 to d, labeled by E_Γ:

$$\Phi = \sum_{I : E_\Gamma \longrightarrow \{1,\ldots,d\}} \Phi_I \ ,$$

where Φ_I is the product over all $n+m$ vertices of Γ of certain partial derivatives of functions f_j and of coefficients of γ_i.

Namely, with each vertex i, $1 \leq i \leq n$ of the first type we associate a function ψ_i on $\mathbb{R}^d$ which is a coefficient of the polyvector field γ_i:

$$\psi_i = \langle \gamma_i, dx^{I(e_i^1)} \otimes \cdots \otimes dx^{I(e_i^{k_i+1})} \rangle \ .$$

Here we use the identification of polyvector fields with skew-symmetric tensor fields as

$$\xi_1 \wedge \cdots \wedge \xi_{k+1} \longrightarrow \sum_{\sigma \in \Sigma_{k+1}} sgn(\sigma)\, \xi_{\sigma_1} \otimes \cdots \otimes \xi_{\sigma_{k+1}} \in \Gamma(\mathbb{R}^d, T^{\otimes(k+1)}) \ .$$

For each vertex $\overline{j}$ of second type the associated function $\psi_{\overline{j}}$ is defined as f_j.

Now, at each vertex of graph Γ we put a function on $\mathbb{R}^d$ (i.e. ψ_i or $\psi_{\overline{j}}$). Also, on edges of graph Γ there are indices $I(e)$ which label coordinates in $\mathbb{R}^d$. In the next step we put into each vertex v instead of function ψ_v its partial derivative

$$\left(\prod_{e \in E_\Gamma, e=(*,v)} \partial_{I(e)} \right) \psi_v,$$

and then take the product over all vertices v of Γ. The result is by definition the summand Φ_I.

Construction of the function Φ from the graph Γ, polyvector fields γ_i and functions f_j, is invariant under the action of the group of affine transformations of $\mathbb{R}^d$ because we contract upper and lower indices.

6.4 Main Theorem for $X = \mathbb{R}^d$, and the proof

We define a pre-L_∞-morphism $\mathscr{U} : T_{poly}(\mathbb{R}^d) \longrightarrow D_{poly}(\mathbb{R}^d)$ by the formula for its n-th Taylor coefficient $\mathscr{U}_n$, $n \geq 1$ considered as a skew- symmetric polylinear map (see Section 4.2) from $\otimes^n T_{poly}(\mathbb{R}^d)$ to $D_{poly}(\mathbb{R}^d)[1-n]$:

$$\mathscr{U}_n := \sum_{m \geq 0} \sum_{\Gamma \in G_{n,m}} W_\Gamma \times \mathscr{U}_\Gamma \ .$$

Here $G_{n,m}$ denotes the set of all admissible graphs with n vertices of the first type, m vertices in the second group and $2n+m-2$ edges, where $n \geq 1$, $m \geq 0$ (and automatically $2n+m-2 \geq 0$).

Theorem 6.3 *$\mathscr{U}$ is an L_∞-morphism, and also a quasi-isomorphism.*

Proof: the condition that $\mathscr{U}$ is an L_∞-morhism (see Sections 4.3 and 3.4.2) can be written explicitly as

$$f_1 \cdot (\mathscr{U}_n(\gamma_1 \wedge \cdots \wedge \gamma_n))(f_2 \otimes \cdots \otimes f_m) \pm (\mathscr{U}_n(\gamma_1 \wedge \cdots \wedge \gamma_n))(f_1 \otimes \cdots \otimes f_{m-1}) \cdot f_m +$$

$$+ \sum_{i=1}^{m-1} \pm (\mathscr{U}_n(\gamma_1 \wedge \cdots \wedge \gamma_n))(f_1 \otimes \cdots \otimes (f_i f_{i+1}) \otimes \cdots \otimes f_m) +$$

$$+\sum_{i\neq j}\pm\left(\mathscr{U}_{n-1}([\gamma_i,\gamma_j]\wedge\gamma_1\wedge\cdots\wedge\gamma_n)\right)(f_1\otimes\cdots\otimes f_m)+$$

$$+\frac{1}{2}\sum_{k,l\geq 1,\ k+l=n}\frac{1}{k!l!}\sum_{\sigma\in\Sigma_n}\pm\left[\mathscr{U}_k(\gamma_{\sigma_1}\wedge\cdots\wedge\gamma_{\sigma_k}),\mathscr{U}_l(\gamma_{\sigma_{k+1}}\wedge\cdots\wedge\gamma_{\sigma_n})\right](f_1\otimes\cdots\otimes f_m)=0\ .$$

Here γ_i are polyvector fields, f_i are functions, $\mathscr{U}_n$ are homogeneous components of $\mathscr{U}$ (see Section 4.1). There is a way to rewrite this formula. Namely, we define $\mathscr{U}_0$ as the map $\otimes^0(T_{poly}(\mathbb{R}^d))\longrightarrow D_{poly}(\mathbb{R}^d)[1]$ which maps the generator 1 of $\mathbb{R}\simeq\otimes^0(T_{poly}(\mathbb{R}^d))$ to the product $m_A\in D^1_{poly}(\mathbb{R}^d)$ in the algebra $A:=C^\infty(\mathbb{R}^d)$. Here $m_A: f_1\otimes f_2\mapsto f_1f_2$ is considered as a bidifferential operator.

The condition from above for $\mathscr{U}$ to be an L_∞-morphism is equivalent to the following one:

$$\sum_{i\neq j}\pm\left(\mathscr{U}_{n-1}((\gamma_i\bullet\gamma_j)\wedge\gamma_1\wedge\cdots\wedge\gamma_n)\right)(f_1\otimes\cdots\otimes f_m)+$$

$$+\sum_{k,l\geq 0,\ k+l=n}\frac{1}{k!l!}\sum_{\sigma\in\Sigma_n}\pm\left(\mathscr{U}_k(\gamma_{\sigma_1}\wedge\cdots\wedge\gamma_{\sigma_k})\circ\mathscr{U}_l(\gamma_{\sigma_{k+1}}\wedge\cdots\wedge\gamma_{\sigma_n})\right)(f_1\otimes\cdots\otimes f_m)=0\ .$$

Here we use all polylinear maps $\mathscr{U}_n$ including case $n=0$, and definitions of brackets in D_{poly} and T_{poly} via operations $\circ$ (see Section 3.4.2) and $\bullet$ (see Section 4.6.1). We denote the l.h.s. of the expression above by (F).

$\mathscr{U}+\mathscr{U}_0$ is not a pre-L_∞-morphism because it maps 0 to a non-zero point m_A. Still the equation $(F)=0$ makes sense and means that the map $(\mathscr{U}+\mathscr{U}_0)$ from formal Q-manifold $T_{poly}(\mathbb{R}^d)[1])_{formal}$ to the formal neighborhood of point m_A in the graded vector space $D_{poly}(\mathbb{R}^d)[1]$ is Q-equivariant, where the odd vector field Q on the target is purely quadratic and comes from the bracket on $D_{poly}(\mathbb{R}^d)$, forgetting the differential.

Also, the term $\mathscr{U}_0$ comes from the unique graph Γ_0 which was missing in the definition of $\mathscr{U}$. Namely, Γ_0 has $n=0$ vertices of the first type, $m=2$ vertices of the second type, and no edges at all. It is easy to see that $W_{\Gamma_0}=1$ and $\mathscr{U}_{\Gamma_0}=\mathscr{U}_0$.

We consider the expression (F) simultaneously for all possible dimensions d. It is clear that one can write (F) as a linear combination

$$\sum_\Gamma c_\Gamma\cdot\mathscr{U}_\Gamma\big(\gamma_1\otimes\cdots\otimes\gamma_n\big)(f_1\otimes\cdots\otimes f_m)$$

of expressions $\mathscr{U}_\Gamma$ for admissible graphs Γ with n vertices of the first type, m vertices of the second type, and $2n+m-3$ edges where $n\geq 0, m\geq 0, 2n+m-3\geq 0$. We assume that $c_\Gamma=\pm c_{\Gamma'}$ if graph Γ' is obtained from Γ by a renumeration of vertices of first type and by a relabeling of edges in sets $Star(v)$ (see Section 6.5 where we discuss signs).

Coefficients c_Γ of this linear combination are equal to certain sums with signs of weights $\mathscr{W}_{\Gamma'}$ associated with some other graphs Γ', and of products of two such weights. In particular, numbers c_Γ do not depend on the dimension d in our problem. Perhaps it is better to use here the language of rigid tensor categories, but we will not do it.

We want to check that c_Γ vanishes for each Γ.

The idea is to identify c_Γ with the integral over the boundary $\partial\overline{C}_{n,m}$ of the closed differential form constructed from Γ as in Section 6.2, with the only difference that now we consider

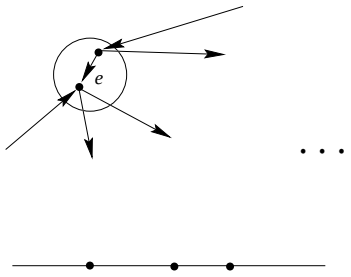

Figure 8: Term corresponding to the operation $\bullet$.

graphs with $2n+m-3$ edges. The Stokes formula gives the vanishing:

$$\int\limits_{\partial\overline{C}_{n,m}} \bigwedge_{e\in E_\Gamma} d\phi_e = \int\limits_{\overline{C}_{n,m}} d\left(\bigwedge_{e\in E_\Gamma} d\phi_e\right) = 0 \ .$$

We are going to calculate integrals of the form $\wedge_{e\in E_\Gamma} d\phi_e$ restricted to all possible boundary strata of $\partial\overline{C}_{n,m}$, and prove that the total integral as above is equal to c_Γ. In Section 5.2.1 we have listed two groups of boundary strata, denoted by S1 and S2 and labeled by sets or pairs of sets. Thus,

$$0 = \int\limits_{\partial\overline{C}_{n,m}} \bigwedge_{e\in E_\Gamma} d\phi_e = \sum_S \int\limits_{\partial_S\overline{C}_{n,m}} \bigwedge_{e\in E_\Gamma} d\phi_e + \sum_{S,S'} \int\limits_{\partial_{S,S'}\overline{C}_{n,m}} \bigwedge_{e\in E_\Gamma} d\phi_e \ .$$

6.4.1 Case S1

Points $p_i \in \mathscr{H}$ for i from subset $S \subset \{1,\dots,n\}$ where $\#S \geq 2$, move close to each other. The integral over the stratum $\partial_S\overline{C}_{n,m}$ is equal to the product of an integral over $C_{n_1,m}$ with an integral over C_{n_2} where $n_2 := \#S$, $n_1 := n-n_2+1$. The integral vanishes by dimensional reasons unless the number of edges of Γ connecting vertices from S is equal to $2n_2-3$.

There are several possibilities:

6.4.1.1. *First subcase of S1: $n_2 = 2$ (Figure 8)*
In this subcase two vertices from S_1 are connected exactly by one edge, which we denote by e. The integral over C_2 here gives number ± 1 (after division by 2π coming from the formula for weights W_Γ). The total integral over the boundary stratum is equal to the integral of a new graph Γ_1 obtained from Γ by the contraction of edge e. It is easy to see (up to a sign) that this term corresponds to the first line in our expression (F), the one where the operation $\bullet$ on polyvector fields appears.

6.4.1.2. *Second subcase of S1: $n_2 \geq 3$ (Figure 9)*

This is the most non-trivial case. The integral corresponding to this boundary stratum vanishes because the integral of any product of $2n_2-3$ angle forms over C_{n_2} where $n_2 \geq 3$ vanishes, as is proven later in Section 6.6.

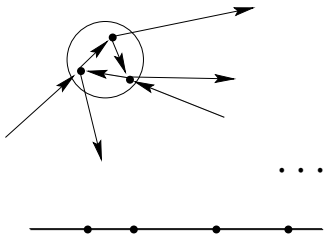

Figure 9: Many points collapse together inside $\mathscr{H}$.

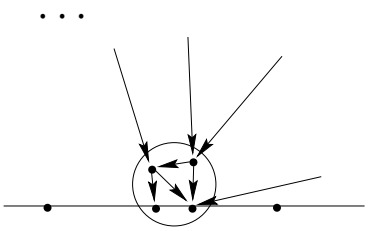

Figure 10: Many points collapse on $\mathbb{R}$, no bad edges.

6.4.2 Case S2

Points p_i for $i \in S_1 \subset \{1,\dots,n\}$ and points q_j for $\overline{j} \in S_2 \subset \{\overline{1},\dots,\overline{m}\}$ move close to each other and to the horizontal line $\mathbb{R}$. The condition is that $2n_2+m_2-2 \geq 0$ and $n_2+m_2 \leq n+m-1$ where $n_2 := \#S_1$, $m_2 := \#S_2$. The corresponding stratum is isomorphic to $C_{n_1,m_1} \times C_{n_2,m_2}$ where $n_1 := n-n_2$, $m_1 = m-m_2+1$. The integral of this stratum decomposes into the product of two integrals. It vanishes if the number of edges of Γ connecting vertices from $S_1 \sqcup S_2$ is not equal to $2n_2+m_2-2$.

6.4.2.1. *First subcase of S2: no bad edges (Figure 10)*
In this subcase we assume that there is no edge (i,j) in Γ such that $i \in S_1, j \in \{1,\dots,n\} \setminus S_1$.

The integral over the boundary stratum is equal to the product $W_{\Gamma_1} \times W_{\Gamma_2}$ where Γ_2 is the restriction of Γ to the subset $S_1 \sqcup S_2 \subset \{1,\dots,n\} \sqcup \{\overline{1},\dots,\overline{m}\} = V_\Gamma$, and Γ_1 is obtained by the contraction of all vertices in this set to a new vertex of the second type. Our condition guarantees that Γ_1 is an admissible graph. This corresponds to the second line in (F), where the product $\circ$ on polydifferential operators appears.

6.4.2.2. *Second subcase of S2: there is a bad edge (Figure 11)*
Now we assume that there is an edge (i,j) in Γ such that $i \in S_1, j \in \{1,\dots,n\} \setminus S_1$. In this case the integral is zero because of the condition $d\phi(x,y) = 0$ if x stays on the line $\mathbb{R}$.

The reader can wonder about what happens if after the collapsing the graph will have multiple edges. Such terms do not appear in (F). Nevertheless, we ingore them because in this case the differential form which we integrate vanishes as it contains as a factor the square of a 1-form.

Thus, we see that we have exhausted all possibilities and get contributions of all terms in the formula (F). We just proved that $c_\Gamma = 0$ for any Γ, and that $\mathscr{U}$ is an L_∞-morphism.

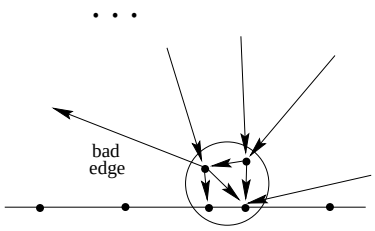

Figure 11: Many points collapse on $\mathbb{R}$, with a bad edge.

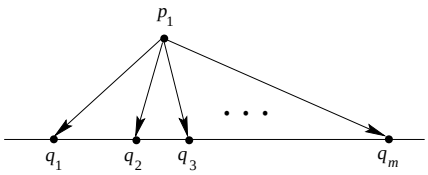

Figure 12: A tree with one vertex in $\mathscr{H}$.

6.4.3 We finish the proof of Theorem 6.3

In order to check that $\mathscr{U}$ it is a quasi-isomorphism, we should show that its component $\mathscr{U}_1$ coincides with $\mathscr{U}_1^{(0)}$ introduced in 4.6.1.1. It follows from definitions that every admissible graph with $n=1$ vertex of first type and $m \geq 0$ vertices of the second type, and with m edges, is the tree in Figure 12.

The integral corresponding to this graph is $(2\pi)^m/m!$. The map $\widetilde{U}_\Gamma$ from polyvector fields to polydifferential operators is the one which appears in paragraph 4.6.1.1:

$$\xi_1 \wedge \cdots \wedge \xi_m \longrightarrow \frac{1}{m!} \sum_{\sigma \in \Sigma_m} sgn(\sigma) \cdot \xi_{\sigma_1} \otimes \cdots \otimes \xi_{\sigma_m}, \;\; \xi_i \in \Gamma(\mathbb{R}^d, T) \; .$$

Theorem 6.3 is proven. *Q.E.D.*

6.4.4 Comparison with the formula from Section 2

The weight w_Γ defined in Section 2 differ from W_Γ defined in Section 6.2 by the factor $2^n/n!$. On the other hand, the bidifferential operator $B_{\Gamma,\alpha}(f,g)$ is 2^{-n} times $\mathscr{U}_\Gamma(\alpha \wedge \cdots \wedge \alpha)(f \otimes g)$. The inverse factorial $1/n!$ appears in the Taylor series (see the end of Section 4.3). Thus, we obtain the formula from Section 2.

6.5 Grading, orientations, factorials, signs

Taylor coefficients of $\mathscr{U} + \mathscr{U}_0$ are maps of graded spaces

$$Sym^n((\bigoplus_{k \geq 0} \Gamma(\mathbb{R}^d, \wedge^k T)[-k])[2]) \longrightarrow (\underline{Hom}(A[1]^{\otimes m}, A[1]))[1]$$

where $\underline{Hom}$ denotes the internal Hom in the tensor category $Graded^{\mathbf{k}}$. We denote the expression from above by (E). First of all, in the expression (E) each polyvector field $\gamma_i \in$

$\Gamma(\mathbb{R}^d, \wedge^{k_i} T)$ appears with the shift $2-k_i$. In our formula for $\mathscr{U}$ the same γ_i gives k_i edges of the graph, and thus k_i 1-forms which we have to integrate. Also, it gives 2 dimensions for the integration domain $\overline{C}_{n,m}$. Secondly, every function $f_j \in A$ appears with shift 1 in (E) and gives 1 dimension to the integration domain. We are left with two shifts by 1 in (E) which are accounted for 2 dimensions of the group $G^{(1)}$. From this it is clear that our formula for $\mathscr{U}$ is compatible with $\mathbb{Z}$-grading.

Moreover, it is also clear that things responsible for various signs in our formulas:
1) the orientation of $\overline{C}_{n,m}$,
2) the order in which we multiply 1-forms $d\phi_e$,
3) $\mathbb{Z}$-gradings of vector spaces in (E),
are naturally decomposed into pairs. This implies that the enumeration of the set of vertices of Γ, and also the enumeration of edges in sets $Star(v)$ for vertices v of the first type are not really used. Thus, we see that $\mathscr{U}_n$ is skew-symmetric.

Inverse factorials $1/(\#Star(v)!)$ kill the summation over enumerations of sets $Star(v)$. The inverse factorial $1/n!$ in the final formula does not appear because we consider higher derivatives which are already multiplied by $n!$.

The last thing to check is that in our derivation of the fact that $\mathscr{U}$ is an L_∞-morphism using the Stokes formula we did not loose anywhere a sign. This is a bit hard to explain. How, for example, can one compare the standard orientation on $\mathbb{C}$ with shifts by 2 in (E)? As a hint to the reader we would like to mention that it is very convenient to "place" the resulting expression

$$\Phi := (\mathscr{U}_\Gamma(\gamma_1 \otimes \cdots \otimes \gamma_n))\,(f_1 \otimes \cdots \otimes f_m)$$

at the point ∞ on the absolute.

6.6 Vanishing of integrals over $C_n,\ n \geq 3$

In this subsection we consider the space C_n of $G^{(2)}$- equivalence classes of configurations of points on the Euclidean plane. Every two indices $i, j,\ i \neq j,\ 1 \leq i, j \leq n$ give a forgetting map $C_n \longrightarrow C_2 \simeq S^1$. We denote by $d\phi_{i,j}$ the closed 1-form on C_n which is the pullback of the standard 1-form $d(angle)$ on the circle. We use the same notation for the pullback of this form to $Conf_n$.

Lemma 6.4 *Let $n \geq 3$ be an integer. The integral over C_n of the product of any $2n-3 = dim(C_n)$ closed 1-forms $d\phi_{i_\alpha,j_\alpha}$, $\alpha = 1,\ldots,2n-3$, is equal to zero.*

Proof: First of all, we identify C_n with the subset C'_n of $Conf_n$ consisting of configurations such that the point p_{i_1} is $0 \in \mathbb{C}$ and p_{j_1} is on the unit circle $S^1 \subset \mathbb{C}$. Also, we rewrite the form which we integrate as

$$\bigwedge_{\alpha=1}^{2n-3} d\phi_{i_\alpha,j_\alpha} = d\phi_{i_1,j_1} \wedge \bigwedge_{\alpha=2}^{2n-3} d(\phi_{i_\alpha,j_\alpha} - \phi_{i_1,j_1})\ .$$

Let us map the space C'_n onto the space $C''_n \subset Conf_n$ consisting of configurations with $p_{i_1} = 0$ and $p_{j_1} = 1$, applying rotations with the center at 0. Differential forms $d(\phi_{i_\alpha,j_\alpha} - \phi_{i_1,j_1})$ on C'_n are pullbacks of differential forms $d\phi_{i_\alpha,j_\alpha}$ on C''_n. The integral of a product of $2n-3$ closed 1-forms $d\phi_{i_\alpha,j_\alpha},\ \alpha = 1,\ldots,2n-3$ over C'_n is equal to $\pm 2\pi$ times the integral of the product $2n-4$ closed 1-forms $d\phi_{i_\alpha,j_\alpha},\ \alpha = 2,\ldots,2n-3$ over C''_n.

The space C_n'' is a complex manifold. We are calculating an absolutely converging integral of the type

$$\int\limits_{C_n''} \prod_\alpha dArg(Z_\alpha)$$

where Z_α are holomorphic invertible functions on C_n'' (differences between complex coordinates of points of the configuration). We claim that it is zero, because of the general result proven in Section 6.6.1. *Q.E.D.*

6.6.1 A trick using logarithms

Theorem 6.5 *Let X be a complex algebraic variety of dimension $N \geq 1$, and $Z_1, \ldots, Z_{2N}$ be rational functions on X, not equal identically to zero. Let U be any Zariski open subset of X such that functions Z_α are defined and non-vanishing on U, and U consists of smooth points. Then the integral*

$$\int\limits_{U(\mathbb{C})} \wedge_{\alpha=1}^{2N} d(Arg\, Z_\alpha)$$

is absolutely convergent, and equal to zero.

This result seems to be new, although the main trick used in the proof is well-known. A. Goncharov told me that he also came to the same result in his study of mixed Tate motives.
Proof: First of all, we claim that the differential form $\wedge_{\alpha=1}^{2N} dArg(Z_\alpha)$ on $U(\mathbb{C})$ coincides with the form $\wedge_{\alpha=1}^{2N} dLog\,|Z_\alpha|$ (this is the trick).

We can replace $dArg(Z_\alpha)$ by the linear combination of a holomorphic and an anti-holomorphic form

$$\frac{1}{2i}\left(d(Log\, Z_\alpha) - d(Log\, \overline{Z}_\alpha)\right) \ .$$

Thus, the form which we integrate over $U(\mathbb{C})$ is a sum of products of holomorphic and of anti-holomorphic forms. The summand corresponding to a product of a non-equal number of holomorphic and of anti-holomorphic forms, vanishes identically because $U(\mathbb{C})$ is a complex manifold. The conclusion is that the number of anti-holomorphic factors in non-vanishing summands is the same for all of them, it coincides with the complex dimension N of $U(\mathbb{C})$. The same products of holomorphic and of anti-holomorphic forms survive in the product

$$\bigwedge_{\alpha=1}^{2N} d\,Log\,|Z_\alpha| = \bigwedge_{\alpha=1}^{2N} \frac{1}{2}\left(d(Log\, Z_\alpha) + d(Log\, \overline{Z}_\alpha)\right) \ .$$

Let us choose a compactification $\overline{U}$ of U such that $\overline{U} \setminus U$ is a divisor with normal crossings. If ϕ is a smooth differential form on $U(\mathbb{C})$ such that coefficients of ϕ are locally integrable on $\overline{U}(\mathbb{C})$, then we denote by $\mathscr{I}(\phi)$ corresponding differential form on $\overline{U}(\mathbb{C})$ with coefficients in the space of distributions.

Lemma 6.6 *Let ω be a form on $U(\mathbb{C})$ which is a linear combination of products of functions $Log\,|Z_\alpha|$ and of 1-forms $d\,Log\,|Z_\alpha|$ where $Z_\alpha \in \mathscr{O}^\times(U)$ are regular invertible functions on U. Then coefficients of ω and of $d\omega$ are locally L^1 functions on $\overline{U}(\mathbb{C})$. Moreover, $\mathscr{I}(d\omega) = d(\mathscr{I}(\omega))$. Also, the integral $\int\limits_{U(\mathbb{C})} \omega$ is absolutely convergent and equal to the integral $\int\limits_{\overline{U}(\mathbb{C})} \mathscr{I}(\omega)$.*

The lemma is an elementary exercise in the theory of distributions, after passing to local coordinates on $\overline{U}(\mathbb{C})$. We leave details of the proof to the reader. Also, the statement of the lemma remains true without the condition that $\overline{U} \setminus U$ is a divisor with normal crossings. *Q.E.D.*

The vanishing of the integral in the theorem is clear now by the Stokes formula:

$$\int\limits_{U(\mathbb{C})} \bigwedge_{\alpha=1}^{2N} d\,Arg\,(Z_\alpha) = \int\limits_{U(\mathbb{C})} \bigwedge_{\alpha=1}^{2N} d\,Log\,|Z_\alpha| = \int\limits_{\overline{U}(\mathbb{C})} \mathscr{I}\left(d\left(Log\,|Z_1| \bigwedge_{\alpha=2}^{2N} d\,Log\,|Z_\alpha|\right)\right) =$$

$$= \int\limits_{\overline{U}(\mathbb{C})} d\left(\mathscr{I}\left(Log\,|Z_1| \bigwedge_{\alpha=2}^{2N} d\,Log\,|Z_\alpha|\right)\right) = 0\ . \qquad Q.E.D.$$

In fact, the convergence and the vanishing of the integral $\int\limits_{U(\mathbb{C})} \wedge_{\alpha=1}^{2N} d\,Log\,|Z_\alpha|$ is a purely geometric fact. Namely, the image of $U(\mathbb{C})$ in $\mathbb{R}^{2N}$ under the map $x \mapsto (Log\,|Z_1(x)|, \ldots, Log\,|Z_{2N}(x)|)$ has finite volume and every non-critical point in this image appears zero times, when points in the pre-image are counted with signs arising from the comparison of canonical orientations on $U(\mathbb{C})$ and $\mathbb{R}^{2N}$.

6.6.2 Remark

The vanishing of the integral in Lemma 6.4 has higher-dimensional analogue which is crucial in the perturbative Chern-Simons theory in the dimension 3, and its generalizations to dimensions ≥ 4 (see [29]). However, the vanishing of integrals in dimensions ≥ 3 follows from a much simpler fact which is the existence of a geometric involution making the integral to be equal to minus itself. In the present paper we will use several times similar arguments involving involutions.

7 Formality conjecture for general manifolds

In this section we establish the formality conjecture for general manifolds, not only for open domains in $\mathbb{R}^d$. It turns out that that essentially all the work has been done already. The only new analytic result is vanishing of certain integrals over configuration spaces, analogous to Lemma 6.4.

One can treat $\mathbb{R}^d_{formal}$, the formal completion of vector space $\mathbb{R}^d$ at zero, in many respects as usual manifold. In particular, we can define differential graded Lie algebras $D_{poly}(\mathbb{R}^d_{formal})$ and $T_{poly}(\mathbb{R}^d_{formal})$. The Lie algebra $W_d := Vect(\mathbb{R}^d_{formal})$ is the standard Lie algebra of formal vector fields. We consider W_d as a differential graded Lie algebra (with the trivial grading and the differential equal to 0). There are natural homomorphisms of differential graded Lie algebras:

$$m_T : W_d \longrightarrow T_{poly}(\mathbb{R}^d_{formal}),\ \ m_D : W_d \longrightarrow D_{poly}(\mathbb{R}^d_{formal}),$$

because vector fields can be considered as polyvector fields and as differential operators.

We will use the following properties of the quasi-isomorphism $\mathscr{U}$ from Section 6.4:

P1) $\mathscr{U}$ can be defined for $\mathbb{R}^d_{formal}$ as well,

P2) for any $\xi \in W_d$ we have the equality

$$\mathscr{U}_1(m_T(\xi)) = m_D(\mathscr{U}_1(\xi)),$$

P3) $\mathscr{U}$ is $GL(d,\mathbb{R})$-equivariant,
P4) for any $k \geq 2$, $\xi_1, \ldots, \xi_k \in W_d$ we have the equality

$$\mathscr{U}_k(m_T(\xi_1) \otimes \cdots \otimes m_T(\xi_k)) = 0,$$

P5) for any $k \geq 2$, $\xi \in gl(d,\mathbb{R}) \subset W_d$, and for any $\eta_2, \ldots, \eta_k \in T_{poly}(\mathbb{R}^d_{formal})$ we have

$$\mathscr{U}_k(m_T(\xi) \otimes \eta_2 \otimes \cdots \otimes \eta_k) = 0 \ .$$

We will construct quasi-isomorphisms from $T_{poly}(X)$ to $D_{poly}(X)$ for arbitrary d-dimensional manifold X using only properties P1–P5 of the map $\mathscr{U}$. Properties P1, P2 and P3 are evident, and the properties P4, P5 will be established later in paragraphs 7.3.1.1 and 7.3.3.1.

It will be convenient to use in this section the geometric language of formal graded manifolds, instead of the algebraic language of L_∞-algebras. Let us fix the dimension $d \in \mathbb{N}$. We introduce three formal graded Q-manifolds *without* base points:

$$\mathscr{T}, \mathscr{D}, \mathscr{W} \ .$$

These formal graded Q-manifolds are obtained in the usual way from differential graded Lie algebras $T_{poly}(R^d_{formal})$, $D_{poly}(R^d_{formal})$ and W_d forgetting base points.

In Sections 7.1 and 7.2, we present two general geometric constructions, which will used in Section 7.3 for the proof of formality of $D_{poly}(X)$.

7.1 Formal geometry (in the sense of I. Gelfand and D. Kazhdan)

Let X be a smooth manifold of dimension d. We associate with X two infinite-dimensional manifolds, X^{coor} and X^{aff}. The manifold X^{coor} consists of pairs (x, f) where x is a point of X and f is an infinite germ of a coordinate system on X at x,

$$f : (\mathbb{R}^d_{formal}, 0) \hookrightarrow (X, x) \ .$$

We consider X^{coor} as a projective limit of finite-dimensional manifolds (spaces of finite germs of coordinate systems). There is an action on X^{coor} of the (pro-Lie) group G_d of formal diffeomorphisms of $\mathbb{R}^d$ preserving base point 0. The natural projection map $X^{coor} \longrightarrow X$ is a principal G_d-bundle.

The manifold X^{aff} is defined as the quotient space $X^{coor}/GL(d,\mathbb{R})$. It can be thought as the space of formal affine structures at points of X. The main reason to introduce X^{aff} is that fibers of the natural projection map $X^{aff} \longrightarrow X$ are contractible.

The Lie algebra of the group G_d is a subalgebra of codimension d in W_d. It consists of formal vector fields vanishing at zero. Thus, $Lie(G_d)$ acts on X^{coor}. It is easy to see that in fact the whole Lie algebra W_d acts on X^{coor} and is isomorphic to the tangent space to X^{coor} at each point. Formally, the infinite-dimensional manifold X^{coor} looks as a principal homogeneous space of the non-existent group with the Lie algebra W_d.

The main idea of formal geometry (se [17]) is to replace d-dimensional manifolds by "principal homogeneous spaces" of W_d. Differential-geometric constructions on X^{coor} can be obtained from Lie-algebraic constructions for W_d. For a while we will work only with X^{coor}, and then at the end return to X^{aff}. In terms of Lie algebras it corresponds to the difference between absolute and relative cohomology.

7.2 Flat connections and Q-equivariant maps

Let M be a C^∞-manifold (or a complex analytic manifold, or an algebraic manifold, or a projective limit of manifolds,...). Denote by ΠTM the supermanifold which is the total space of the tangent bundle of M endowed with the reversed parity. Functions on the ΠTM are differential forms on M. The de Rham differential d_M on forms can be considered as an odd vector field on ΠTM with the square equal to 0. Thus, ΠTM is a Q-manifold. It seems that the accurate notation for ΠTM considered as a graded manifold should be $T[1]M$ (the total space of the graded vector bundle $T_M[1]$ considered as a graded manifold).

Let $N \longrightarrow M$ be a bundle over a manifold M whose fibers are manifolds, or vector spaces, etc., endowed with a flat connection ∇. Denote by E the pullback of this bundle to $B := \Pi TM$. The connection ∇ gives a lift of the vector field $Q_B := d_M$ on B to the vector field Q_E on E. This can be done for arbitrary connection, and only for flat connection the identity $[Q_E, Q_E] = 0$ holds.

A generalization of a (non-linear) bundle with a flat connection is a Q-equivariant bundle whose total space and the base are Q-manifolds. In the case of graded vector bundles over $T[1]M$ this notion was introduced Quillen under the name of a superconnection (see [40]). A generalization of the notion of a covariantly flat morphism from one bundle to another is the notion of a Q-equivariant map.

Definition 7.1 *A flat family over Q-manifold B is a pair $(p : E \longrightarrow B, \sigma)$ where $p : E \longrightarrow B$ is a Q-equivariant bundle whose fibers are formal manifolds, and a $\sigma : B \longrightarrow E$ is a Q-equivariant section of this bundle.*

In the case $B = \{point\}$ a flat family over B is the same a formal Q-manifold with base point. It is clear that flat families over a given Q-manifold form a category.

We apologize for the terminology. More precise name for "flat families" would be "flat families of pointed formal manifolds", but it is too long.

One can define analogously flat graded families over graded Q-manifolds.

We refer the reader to a discussion of further examples of Q-manifolds in [31].

7.3 Flat families in deformation quantization

Let us return to our concrete situation. We construct in this section two flat families over ΠTX (where X is a d-dimensional manifold), and a morphism between them. This will be done in several steps.

7.3.1 Flat families over $\mathscr{W}$

The first bundle over $\mathscr{W}$ is trivial as a Q-equivariant bundle,

$$\mathscr{T} \times \mathscr{W} \longrightarrow \mathscr{W}$$

but with a non-trivial section $\sigma_{\mathscr{T}}$. This section is not the zero section, but the graph of the Q-equivariant map $\mathscr{W} \longrightarrow \mathscr{T}$ coming from the homomorphism of differential graded Lie algebras $m_T : W_d \longrightarrow T_{poly}(\mathbb{R}^d_{formal})$. Analogously, the second bundle is the trivial Q-equivariant bundle

$$\mathscr{D} \times \mathscr{W} \longrightarrow \mathscr{W}$$

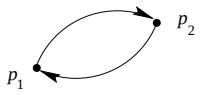

Figure 13: The only graph for property P4.

with the section $\sigma_{\mathscr{D}}$ coming from the homomorphism $m_D : W_d \longrightarrow D_{poly}(\mathbb{R}^d_{formal})$.

Formulas from Section 6.4 give a Q-equivariant map $\mathscr{U} : \mathscr{T} \longrightarrow \mathscr{D}$.

Lemma 7.2 *The morphism $(\mathscr{U} \times id_{\mathscr{W}}) : \mathscr{T} \times \mathscr{W} \longrightarrow \mathscr{D} \times \mathscr{W}$ is a morphism of flat families over $\mathscr{W}$.*

Proof: We have to check that $(\mathscr{U} \times id_{\mathscr{W}})$ maps one section to another, i.e. that

$$(\mathscr{U} \times id_{\mathscr{W}}) \circ \sigma_{\mathscr{T}} = \sigma_D \in Maps(\mathscr{W}, \mathscr{D} \times \mathscr{W}) \ .$$

We compare Taylor coefficients. The linear part $\mathscr{U}_1$ of $\mathscr{U}$ maps a vector field (considered as a polyvector field) to itself, considered as a differential operator (property P2). Components $\mathscr{U}_k(\xi_1, \ldots, \xi_k)$ for $k \geq 2,\ \xi_i \in T^0(\mathbb{R}^d) = \Gamma(\mathbb{R}^d, T)$ vanish, which is the property P4. *Q.E.D.*

7.3.1.1. *Proof of the property P4*

Graphs appearing in the calculation of $\mathscr{U}_k(\xi_1, \ldots, \xi_k)$ have k edges, k vertices of the first type, and m vertices of the second type, where

$$2k + m - 2 = k \ .$$

Thus, there are no such graphs for $k \geq 3$ as m is non-negative. The only interesting case is $k = 2, m = 0$ which is represented in Figure 13.

By our construction, $\mathscr{U}_2$ restricted to vector fields is equal to the non-trivial quadratic map

$$\xi \longmapsto \sum_{i,j=1}^{d} \partial_i(\xi^j)\partial_j(\xi^i) \in \Gamma(\mathbb{R}^d, \mathscr{O}),\ \ \xi = \sum_i \xi^i \partial_i \in \Gamma(\mathbb{R}^d, T)$$

with the weight

$$\int\limits_{C_{2,0}} d\phi_{(12)} d\phi_{(21)} = \int\limits_{\mathscr{H}\setminus\{z_0\}} d\phi(z, z_0) \wedge d\phi(z_0, z)$$

where z_0 is an arbitrary point of $\mathscr{H}$.

Lemma 7.3 *For arbitrary angle map the integral $\int\limits_{\mathscr{H}\setminus\{z_0\}} d\phi(z, z_0) \wedge d\phi(z_0, z)$ is equal to zero.*

Proof: We have a map $\overline{C}_{2,0} \longrightarrow S^1 \times S^1,\ [(x,y)] \mapsto (\phi(x,y), \phi(y,x))$. We calculate the integral of the pullback of the standard volume element on two-dimensional torus. It is easy to see that the integral does not depend on the choice of map $\phi : \overline{C}_{2,0} \longrightarrow S^1$. The reason is that the image of the boundary of the integration domain $\partial\overline{C}_{2,0}$ in $S^1 \times S^1$ cancels with the reflected copy of itself under the involution $(\phi_1, \phi_2) \mapsto (\phi_2, \phi_1)$ of the torus $S^1 \times S^1$. Let us assume that $\phi = \phi^h$ and $z_0 = 0 + 1 \cdot i$. The integral vanishes because the involution $z \mapsto -\overline{z}$ reverses the orientation of $\mathscr{H}$ and preserves the form $d\phi(z, z_0) \wedge d\phi(z_0, z)$. *Q.E.D.*

7.3.2 Flat families over $\Pi T(X^{coor})$

If X is a d-dimensional manifold, then there is a natural map of Q-manifolds (the Maurer-Cartan form)

$$\Pi T(X^{coor}) \longrightarrow \mathscr{W} \ .$$

It follows from following general reasons. If G is a Lie group, then it acts freely by left translations on itself, and also on ΠTG. The quotient Q-manifold $\Pi TG/G$ is equal to $\Pi\mathbf{g}$ where $\mathbf{g} = Lie(G)$. Thus, we have a Q-equivariant map

$$\Pi TG \longrightarrow \Pi\mathbf{g} \ .$$

Analogous construction works for any principal homogeneous space over G. We apply it to X^{coor} considered as a principal homogeneous space for a non-existent group with the Lie algebra $\mathbf{g} = W_d$.

The pullbacks of flat families of formal manifolds over $\mathscr{W}$ constructed in Section 7.3.1, are two flat families over $\Pi T(X^{coor})$. As Q-equivariant bundles these families are trivial bundles

$$\mathscr{T} \times \Pi T(X^{coor}) \longrightarrow \Pi T(X^{coor}), \ \ \mathscr{D} \times \Pi T(X^{coor}) \longrightarrow \Pi T(X^{coor}) \ .$$

Pullbacks of sections $\sigma_{\mathscr{T}}$ and $\sigma_{\mathscr{D}}$ gives sections in the bundles above. These sections we denote again by $\sigma_{\mathscr{T}}$ and $\sigma_{\mathscr{D}}$. The pullback of the morphism $\mathscr{U} \times id_{\mathscr{W}}$ is also a morphism of flat families.

7.3.3 Flat families over $\Pi T(X^{aff})$

Recall that X^{aff} is the quotient space of X^{coor} by the action of $GL(d,\mathbb{R})$. Thus, from functorial properties of operation ΠT ($= \underline{Maps}(\mathbb{R}^{0|1}, \cdot)$) follows that $\Pi T(X^{aff})$ is the quotient of Q-manifold $\Pi T(X^{coor})$ by the action of Q-group $\Pi T(GL(d,\mathbb{R}))$. We will construct an action of $\Pi T(GL(d,\mathbb{R}))$ on flat families $\mathscr{T} \times \Pi T(X^{coor})$ and $\mathscr{D} \times \Pi T(X^{coor})$ over $\Pi T(X^{coor})$. We claim that the morphism between these families is invariant under the action of $\Pi T(GL(d,\mathbb{R}))$. Flat families over $\Pi T(X^{aff})$ will be defined as quotient families. The morphism between them will be the quotient morphism.

The action of $\Pi T(GL(d,\mathbb{R}))$ on $\mathscr{T}$ and on $\mathscr{W}$ is defined as follows. First of all, if G is a Lie group with the Lie algebra $\mathbf{g}$, then ΠTG acts Q-equivariantly on Q-manifold $\Pi\mathbf{g}$, via the identification $\Pi\mathbf{g} = \Pi TG/G$. Analogously, if $\mathbf{g}$ is a subalgebra of a larger Lie algebra $\mathbf{g}_1$, and an action of G on $\mathbf{g}_1$ is given in a way compatible with the inclusion $\mathbf{g} \hookrightarrow \mathbf{g}_1$, then ΠTG acts on $\Pi\mathbf{g}_1$. We apply this construction to the case $G = GL(n,\mathbb{R})$ and $\mathbf{g}_1 = T_{poly}(\mathbb{R}^d_{formal})$ or $\mathbf{g}_1 = D_{poly}(\mathbb{R}^d_{formal})$.

One can check easily that sections $\sigma_{\mathscr{T}}$ and $\sigma_{\mathscr{D}}$ over $\Pi T(X^{coor})$ are $\Pi T(GL(d,\mathbb{R}))$-equivariant. Thus, we get two flat families over $\Pi T(X^{aff})$.

The last thing we have to check is that the morphism $\mathscr{U} \times id_{\Pi T(X^{coor})}$ of flat families

$$\mathscr{T} \times \Pi T(X^{coor}) \longrightarrow \mathscr{D} \times \Pi T(X^{coor})$$

is $\Pi T(GL(d,\mathbb{R}))$-equivariant. After the translation of the problem to the language of Lie algebras, we see that we should check that $\mathscr{U}$ is $GL(d,\mathbb{R})$-invariant (property P3, that is clear by our construction), and that if we substitute an element of $gl(d,\mathbb{R}) \subset W_d$ in $\mathscr{U}_{\geq 2}$, we get zero (property P5, see 7.3.3.1).

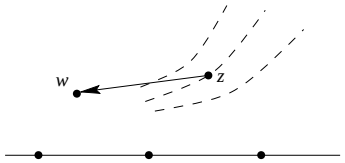

Figure 14: Level sets for function $\varphi(z,w)$ for fixed w (dashed lines).

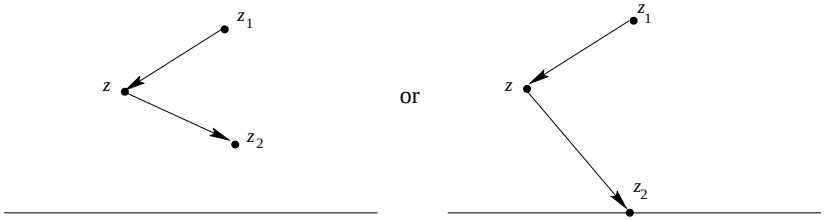

Figure 15: Two graphs for property P5.

CONCLUSION *We constructed two flat families over $\Pi T(X^{aff})$ and a morphism between them. Fibers of these families are isomorphic to $\mathscr{T}$ and to $\mathscr{D}$.*

7.3.3.1. Property P5
This is again reduces to the calculation of an integral. Let v be a vertex of Γ to which we put an element of $gl(d,\mathbb{R})$. There is exactly one edge starting at v because we put a vector field here. If there are no edges ending at v, then the integral is zero because the domain of integration is foliated by lines along which all forms vanish. These lines are level sets of the function $\phi(z,w)$ where $w \in \mathscr{H} \sqcup \mathbb{R}$ is fixed and z is the point on $\mathscr{H}$ corresponding to v (see Figure 14).

If there are at least 2 edges ending at v, then the corresponding polydifferential operator is equal to zero, because second derivatives of coefficients of a linear vector field vanish.

The only relevant case is when there is only one edge starting at v, and only one edge ending there. If these two edges connect our vertex with the same vertex of Γ, then the vanishing follows from Lemma 7.3. If our vertex is connected with two different vertices as in Figure. 15 then we apply the following two lemmas:

Lemma 7.4 *Let $z_1 \neq z_2 \in \mathscr{H}$ be two distinct points on $\mathscr{H}$. Then the integral*

$$\int\limits_{z \in \mathscr{H} \setminus \{z_1, z_2\}} d\phi(z_1,z) \wedge d\phi(z,z_2)$$

vanishes.

Lemma 7.5 *Let $z_1 \in \mathscr{H}$, $z_2 \in \mathbb{R}$ be two points on $\mathscr{H} \sqcup \mathbb{R}$. Then the integral*

$$\int\limits_{z \in \mathscr{H} \setminus \{z_1, z_2\}} d\phi(z_1,z) \wedge d\phi(z,z_2)$$

vanishes.

Proof: One can prove analogously to Lemma 7.3 that the integral does not depend on the choice of an angle map, and also on points z_1, z_2. In the case of $\phi = \phi^h$ and both points z_1, z_2 are pure imaginary, the vanishing follows from the anti-symmetry of the integral under the involution $z \mapsto -\overline{z}$. *Q.E.D.*

7.3.4 Flat families over X

Let us choose a section s^{aff} of the bundle $X^{aff} \longrightarrow X$. Such section always exists because fibers of this bundle are contractible. For example, any torsion-free connection ∇ on the tangent bundle to X gives a section $X \longrightarrow X^{aff}$. Namely, the exponential map for ∇ gives an identification of a neighborhood of each point $x \in X$ with a neighborhood of zero in the vector space T_xX, i.e. an affine structure on X near x, and a point of X^{aff} over $x \in X$.

The section s^{aff} defines a map of formal graded Q-manifolds $\Pi TX \longrightarrow \Pi T(X^{aff})$. After taking the pullback we get two flat families $\mathscr{T}_{s^{aff}}$ and $\mathscr{D}_{s^{aff}}$ over ΠTX and an morphism $m_{s^{aff}}$ from one to another.

We claim that these two flat families admit definitions independent of s^{aff}. Only the morphism $m_{s^{aff}}$ depends on s^{aff}.

Namely, let us consider infinite-dimensional bundles of differential graded Lie algebras $jets_\infty T_{poly}$ and $jets_\infty D_{poly}$ over X whose fibers at $x \in X$ are spaces of infinite jets of polyvector fields or polydifferential operators at x respectively. These two bundles carry natural flat connections (in the usual sense, not as in Section 7.2) as any bundle of infinite jets. Thus, we have two flat families (in generalized sense) over ΠTX.

Lemma 7.6 *Flat families $\mathscr{T}_{s^{aff}}$ and $\mathscr{D}_{s^{aff}}$ are canonically isomorphic to flat families described just above.*

Proof: it follows from definitions that pullbacks of bundles $jets_\infty T_{poly}$ and $jets_\infty D_{poly}$ from X to X^{coor} are canonically trivialized. The Maurer-Cartan 1-forms on X^{coor} with values in graded Lie algebras $T_{poly}(\mathbb{R}^d_{formal})$ or $D_{poly}(\mathbb{R}^d_{formal})$ come from pullbacks of flat connections on bundles of infinite jets. Thus, we identified our flat families over $\Pi T(X^{coor})$ with pullbacks. The same is true for X^{aff}. *Q.E.D.*

7.3.5 Passing to global sections

If in general $(p : E \longrightarrow B, \sigma)$ is a flat family, then one can make a new formal pointed Q-manifold:

$$\big(\Gamma(E \longrightarrow B)_{formal}, \sigma\big) \ .$$

This is an infinite-dimensional formal super manifold, the formal completion of the space of sections of the bundle $E \longrightarrow B$ at the point σ. The structure of Q-manifold on $\Gamma(E \longrightarrow B)$ is evident because the Lie supergroup $\mathbb{R}^{0|1}$ acts on $E \longrightarrow B$.

Lemma 7.7 *Formally completed spaces of global sections of flat families $\mathscr{T}_{s^{aff}}$ and $\mathscr{D}_{s^{aff}}$ a naturally quasi-isomorphic to $T_{poly}(X)$ and $D_{poly}(X)$ respectively.*

Proof: It is well-known that if $E \longrightarrow X$ is a vector bundle then de Rham cohomology of X with coefficients in formally flat infinite-dimensional bundle $jets_\infty E$ are concentrated in degree 0 and canonically isomorphic to the vector space $\Gamma(X,E)$. Moreover, the natural homomorphism of complexes

$$\big(\Gamma(X,E)[0], \text{differential} = 0\big) \longrightarrow \big(\Omega^*(X, jets_\infty(E)), \text{de Rham differential}\big)$$

is quasi-isomorphism.

Using this fact, the lemma from the previous subsection, and appropriate filtrations (for spectral sequences) one sees that that the natural Q-equivariant map from the formal Q-manifold $(T_{poly}(X)_{formal}[1], 0)$ to $(\Gamma(\mathscr{T}_{s^{aff}} \longrightarrow T[1]X)_{formal}, \sigma_{\mathscr{T}})$ (and analogous map for D_{poly}) is a quasi-isomorphism. *Q.E.D.*

It follows from the lemma above and the result of paragraph 4.6.1.1 that we have a chain of quasi-isomorphisms

$$T_{poly}(X)[1]_{formal} \longrightarrow \Gamma(\mathscr{T}_{s^{aff}} \longrightarrow T[1]X)_{formal} \longrightarrow \cdots$$
$$\cdots \longrightarrow \Gamma(\mathscr{D}_{s^{aff}} \longrightarrow T[1]X)_{formal} \longleftarrow T_{poly}(X)[1]_{formal} \ .$$

Thus, differential graded Lie algebras $T_{poly}(X)$ and $D_{poly}(X)$ are quasi-isomorphic. The Main Theorem stated in Section 4.6.2. is proven. *Q.E.D.*

The space of sections of the bundle $X^{aff} \longrightarrow X$ is contractible. From this fact one can conclude that the quasi-isomorphism constructed above is well-defined homotopically.

8 Cup-products

8.1 Cup-products on tangent cohomology

The differential graded Lie algebras T_{poly}, D_{poly} and (more generally) shifted by $[1]$ Hochschild complexes of arbitrary associative algebras, all carry an additional structure. We do not know at the moment a definition, it should be something close to so called homotopy Gerstenhaber algebras (see [18, 19]), although definitely not precisely this. At least, a visible part of this structure is a commutative associative product of degree $+2$ on cohomology of the tangent space to any solution of the Maurer-Cartan equation. Namely, if $\mathbf{g}$ is one of differential graded Lie algebras listed above and $\gamma \in (\mathbf{g} \otimes \mathbf{m})^1$ satisfies $d\gamma + \frac{1}{2}[\gamma,\gamma] = 0$ where $\mathbf{m}$ is a finite- dimensional nilpotent non-unital differential graded commutative associative algebra, the tangent space T_γ is defined as complex $\mathbf{g} \otimes \mathbf{m}[1]$ endowed with the differential $d + [\gamma, \cdot]$. Cohomology space H_γ of this differential is a graded module over graded algebra $H(\mathbf{m})$ (the cohomology space of $\mathbf{m}$ as a complex). If γ_1 and γ_2 are two gauge equivalent solutions, then H_{γ_1} and H_{γ_2} are (non-canonically) isomorphic $\mathbf{m}$-modules.

We define now cup-products for all three differential graded Lie algebras listed at the beginning of this section. For $T_{poly}(X)$ the cup-product is defined as the usual cup-product of polyvector fields (see Section 4.6.1). One can check directly that this cup-product is compatible with the differential $d + [\gamma, \cdot]$, and is a graded commutative associative product. For the Hochschild complex of an associative algebra A the cup-product on H_γ is defined in a more

tricky way. It is defined on the complex by the formula

$$(t_1 \cup t_2)(a_0 \otimes \cdots \otimes a_n) :=$$
$$\sum_{0 \le k_1 \le k_2 \le k_3 \le k_4 \le n} \pm \gamma^{n-(k_2-k_1+k_4-k_3)}(a_0 \otimes \ldots \otimes t_1(a_{k_1} \otimes \ldots) \otimes a_{k_2} \otimes \cdots$$
$$\cdots \otimes t_2(a_{k_3} \otimes \ldots) \otimes a_{k_4} \otimes \ldots),$$

where $\gamma^l \in Hom(A^{\otimes(l+1)}, A) \otimes (\mathbf{k}[0] \cdot 1 \oplus \mathbf{m})^{1-l}$ is homogeneous component of $(\gamma + m_A \otimes 1)$.

It is not a trivial check that the cup-product on the Hochschild complex is compatible with differentials, and also is commutative, associative and gauge-equivariant on the level of cohomology. Formally, we will not use this fact. The proof is a direct calculation with Hochschild cochains. Even if one replaces formulas by appropriate pictures the calculation is still quite long, about 4-5 pages of tiny drawings. Alternatively, there is a simple abstract explanation using the interpretation of the deformation theory related with the shifted Hochschild complex as a deformation theory of triangulated categories (or, better, A_∞-categories, see [32]).

We define the cup-product for $D_{poly}(X)$ by the restriction of formulas for the cup-product in $C^\bullet(A,A)$.

8.2 Compatibility of $\mathscr{U}$ with cup-products

Theorem 8.1 *The quasi-isomorphism $\mathscr{U}$ constructed in Section 6 maps the cup-product for $T_{poly}(X)$ to the cup-product for $D_{poly}(X)$.*

Sketch of the proof: we translate the statement of the theorem to the language of graphs and integrals. The tangent map is given by integrals where one of vertices of the first type is marked. This is the vertex where we put a representative t for the tangent element $[t] \in H_\gamma$. We put copies of γ (which is a polyvector field with values in $\mathbf{m}$) into all other vertices of the first type. The rule which we just described follows directly from the Leibniz formula applied to the Taylor series for $\mathscr{U}$.

Now we are interested in the behavior of the tangent map with respect to a bilinear operation on the tangent space. It means that we have now *two* marked vertices of the first type.

The statement of the theorem is an identity between two expressions, corresponding to cup products for $T_{poly}(X)$ and $D_{poly}(X)$ respectively.

8.2.1 Pictures for the cup-product in polyvector fields

We claim that the side of identity with the cup-product for the case $T_{poly}(X)$, corresponds to pictures where two points (say, p_1, p_2) where we put representatives of elements of H_γ which we want to multiply, are infinitely close points on $\mathscr{H}$. Precisely, this means that we integrate products of copies of form $d\phi$ over preimages P_α of some point α in $\mathbb{R}/2\pi\mathbb{Z} \simeq C_2 \subset \overline{C}_{2,0}$ with respect to the forgetting map

$$\overline{C}_{n,m} \longrightarrow \overline{C}_{2,0} \ .$$

It is easy to see that P_α has codimension 2 in $\overline{C}_{n,m}$ and contains no strata C_T of codimension 2. It implies that as a singular chain P_α is equal to the sum of closures of non-compact hypersurfaces

$$P_\alpha \cap \partial_S(\overline{C}_{n,m}),\ P_\alpha \cap \partial_{S_1,S_2}(\overline{C}_{n,m})$$

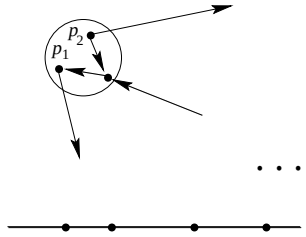

Figure 16: A priori picture for terms for the cup-product in T_{poly}.

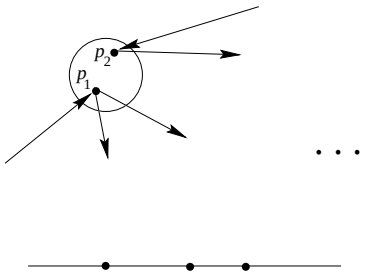

Figure 17: Non-zero terms for the cup-product in T_{poly}.

in boundary strata of $\overline{C}_{n,m}$. It is easy to see that intersections $P_\alpha \cap \partial_{S_1,S_2}(\overline{C}_{n,m})$ are empty, and intersection $P_\alpha \cap \partial_S(\overline{C}_{n,m})$ is non-empty iff $S \supseteq \{1,2\}$. In general pictures which can contribute potentially with a non-zero weight are something like the one in Figure 16.

In other words, we have a collision of several points in $\mathscr{H}$ including both points p_1 and p_2. Points p_1 and p_2 should not be connected by an edge because otherwise the integral vanishes (remember that the direction from p_1 to p_2 is fixed). Also, if $\#S \geq 3$ then the integral vanishes by lemma from 6.6. The only non-trivial case which is left is when $S = \{1,2\}$ and points p_1, p_2 are not connected. Figure 17 represents a non-vanishing terms corresponding to the cup- product in $T_{poly}(X)$.

8.2.2 Pictures for the cup-product in the Hochschild complex

The cup-product for $D_{poly}(X)$ is given by pictures where these two points are separated and infinitely close to $\mathbb{R}$. Again, the precise definition is that we integrate products of copies of $d\phi$ over the preimage $P_{0,1}$ of the point $[(0,1)] \in \overline{C}_{0,2} \subset \overline{C}_{2,0}$. Analysis analogous to the one from the previous subsection shows that $P_{0,1}$ does not intersect any boundary stratum of $\overline{C}_{n,m}$. Thus, as a chain of codimension 2 this preimage $P_{0,1}$ coincides with the union of closures of strata C_T of codimension 2 such that $C_T \subseteq P_{0,1}$. It is easy to see that any such stratum gives pictures like the one in Figure 18 where there is no arrow going from circled regions outside (as in Figure 11), and we get exactly the cup-product in the tangent cohomology of the Hochschild complex as was described above.

. . .

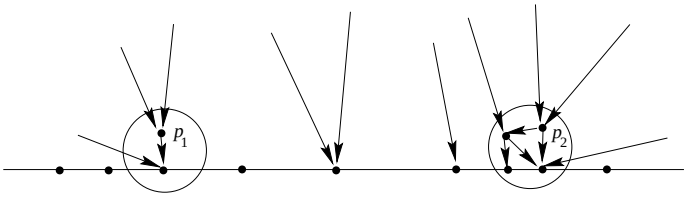

Figure 18: Cup-product in the Hochschild complex.

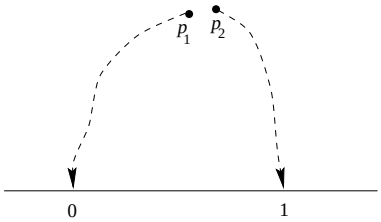

Figure 19: Path in the configuration space of two points in $\mathscr{H}$. Dashed lines are trajectories of two points.

8.2.3 Homotopy between two pictures

Choosing a path from one (limiting) configuration of two points on $\mathscr{H}$ to another configuration (see Figure 19), we see that two products coincide on the level of cohomology. *Q.E.D.*

8.3 First application: Duflo-Kirillov isomorphism

8.3.1 Quantization of the Kirillov-Poisson bracket

Let $\mathbf{g}$ be a *finite-dimensional* Lie algebra over $\mathbb{R}$. The dual space to $\mathbf{g}$ endowed with the Kirillov-Poisson bracket is naturally a Poisson manifold (see [28]). We remind here the formula for this bracket: if $p \in \mathbf{g}^*$ is a point and f, g are two functions on $\mathbf{g}$ then the value $\{f,g\}_{|p}$ is defined as $\langle p, [df_{|p}, dg_{|p}]\rangle$ where differentials of functions f, g at p are considered as elements of $\mathbf{g} \simeq (\mathbf{g}^*)^*$. One can consider $\mathbf{g}^*$ as an algebraic Poisson manifold because coefficients of the Kirillov-Poisson bracket are linear functions on $\mathbf{g}^*$.

Theorem 8.2 *The canonical quantization of the Poisson manifold $\mathbf{g}^*$ is isomorphic to the family of algebras $\mathscr{U}_{\hbar}(\mathbf{g})$ defined as universal enveloping algebras of $\mathbf{g}$ endowed with the bracket $\hbar[\ ,\ ]$.*

Proof: in Section 6.4 we have constructed a canonical star-product on the algebra of functions on arbitrary finite-dimensional affine space endowed with a Poisson structure. Therefore we obtain a canonical star-product on $C^\infty(\mathbf{g}^*)$. We claim that the product of any two *polynomials* on $\mathbf{g}^*$ is a polynomial in $\hbar$ with coefficients which are polynomials on $\mathbf{g}^*$. The reason is that the star-product is constructed in invariant way, using the contraction of indices. Let

us denote by $\beta \in \mathbf{g}^* \otimes \mathbf{g}^* \otimes \mathbf{g}$ the tensor giving the Lie bracket on $\mathbf{g}$. All non-zero natural operations $Sym^k(\mathbf{g}) \otimes Sym^l(\mathbf{g}) \longrightarrow Sym^m(\mathbf{g})$ which can be defined by contractions of indices with the tensor product of several copies of β, exist only for $m \leq k+l$, and for every given m there are only finitely many ways to contract indices. Thus, it makes sense to put $\overline{h}$ equal to 1 and obtain a product on $Sym(\mathbf{g}) = \oplus_{k \geq 0} Sym^k(\mathbf{g})$. We denote this product also by $\star$.

It is easy to see that for $\gamma_1, \gamma_2 \in \mathbf{g}$ the following identity holds:

$$\gamma_1 \star \gamma_2 - \gamma_2 \star \gamma_1 = [\gamma_1, \gamma_2] \ .$$

Moreover, the top component of $\star$-product which maps $Sym^k(\mathbf{g}) \otimes Sym^l(\mathbf{g})$ to $Sym^{k+l}(\mathbf{g})$, coincides with the standard commutative product on $Sym(\mathbf{g})$. From this two facts one concludes that there exists a unique isomorphism of algebras

$$I_{alg} : (\mathscr{U}\mathbf{g}, \cdot) \longrightarrow (Sym(\mathbf{g}), \star)$$

such that $I_{alg}(\gamma) = \gamma$ for $\gamma \in \mathbf{g}$, where $\cdot$ denotes the universal enveloping algebra of $\mathbf{g}$ with the standard product.

One can easily recover variable $\overline{h}$ in this description and get the statement of the theorem. *Q.E.D.*

Corollary 8.3 *The center of the universal enveloping algebra is canonically isomorphic as an algebra to the algebra* $\big(Sym(\mathbf{g})\big)^{\mathbf{g}}$ *of* $\mathbf{g}$*-invariant polynomials on* $\mathbf{g}^*$.

Proof: The center of $\mathscr{U}\mathbf{g}$ is 0-th cohomology for the (local) Hochschild complex of $\mathscr{U}\mathbf{g}$ endowed with the standard cup-product. The algebra $\big(Sym(\mathbf{g})\big)^{\mathbf{g}}$ is the 0-th cohomology of the algebra of polyvector fields on $\mathbf{g}^*$ endowed with the differential $[\alpha, \cdot]$ where α is the Kirillov-Poisson bracket. From Theorem 8.1, we conclude that applying the tangent map to $\mathscr{U}$ we get an isomorphism of algebras.

8.3.2 Three isomorphisms

In the proof of Theorem 8.2 we introduced an isomorphism I_{alg} of algebras.

We denote by I_{PBW} the isomorphism of vector spaces

$$Sym(\mathbf{g}) \longrightarrow \mathscr{U}\mathbf{g}$$

(subscript from the Poincaré-Birkhoff-Witt theorem), which is defined as

$$\gamma_1 \gamma_2 \ldots \gamma_n \longrightarrow \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \gamma_{\sigma_1} \cdot \gamma_{\sigma_2} \cdot \cdots \cdot \gamma_{\sigma_n} \ .$$

Analogously to arguments from above, one can see that the tangent map from polyvector fields on $\mathbf{g}^*$ to the Hochschild complex of the quantized algebra can be defined for $\overline{h} = 1$ and for polynomial coefficients. We denote by I_T its component which maps polynomial 0-vector fields on $\mathbf{g}^*$ (i.e. elements of $Sym(\mathbf{g})$) to 0-cochains of the Hochschild complex of the algebra $(Sym(\mathbf{g}), \star)$. Thus, I_T is an isomorphism of vector spaces

$$I_T : Sym(\mathbf{g}) \longrightarrow Sym(\mathbf{g})$$

and the restriction of I_T to the algebra of $ad(\mathbf{g})^*$-invariant polynomials on $\mathbf{g}^*$ is an isomorphism of algebras

$$Sym(\mathbf{g})^{\mathbf{g}} \longrightarrow Center((Sym(\mathbf{g}), \star)) \ .$$

Combining all facts from above we get a sequence of isomorphisms of vector spaces:

$$Sym(\mathbf{g}) \xrightarrow{I_T} Sym(\mathbf{g}) \xleftarrow{I_{alg}} \mathscr{U}\mathbf{g} \xleftarrow{I_{PBW}} Sym(\mathbf{g}) \ .$$

These isomorphisms are $ad(\mathbf{g})$-invariant. Thus, one get isomorphisms

$$(Sym(\mathbf{g}))^{\mathbf{g}} \xrightarrow{I_{T\,|\dots}} Center(Sym(\mathbf{g}), \star) \xleftarrow{I_{alg\,|\dots}} Center(\mathscr{U}\mathbf{g}) \xleftarrow{I_{PBW\,|\dots}} (Sym(\mathbf{g}))^{\mathbf{g}},$$

where the subscript $|\dots$ denotes the restriction to subspaces of $ad(\mathbf{g})$-invariants. Moreover, first two arrows are isomorphism of algebras. Thus, we have proved the following

Theorem 8.4 *The restriction of the map*

$$\left(I_{alg}\right)^{-1} \circ I_T : Sym(\mathbf{g}) \longrightarrow \mathscr{U}\mathbf{g}$$

to $(Sym(\mathbf{g}))^{\mathbf{g}}$ *is an isomorphism of algebras* $(Sym(\mathbf{g}))^{\mathbf{g}} \longrightarrow Center\,(\mathscr{U}\mathbf{g})$.

8.3.3 Automorphisms of $Sym(\mathbf{g})$

Let us calculate automorphisms I_T and $I_{alg} \circ I_{PBW}$ of the vector space $Sym(\mathbf{g})$. We claim that both these automorphisms are translation invariant operators on the space $Sym(\mathbf{g})$ of polynomials on $\mathbf{g}^*$.

The algebra of translation invariant operators on the space of polynomials on a vector space V is canonically isomorphic to the algebra of formal power series generated by V. Generators of this algebra acts as derivations along constant vector fields in V. Thus, any such operator can be seen as a formal power series at zero on the dual vector space V^*. We apply this formalism to the case $V = \mathbf{g}^*$.

Theorem 8.5 *Operators I_T and $I_{alg} \circ I_{PBW}$ respectively are translation invariant operators associated with formal power series $S_1(\gamma)$ and $S_2(\gamma)$ at zero in $\mathbf{g}$ of the form*

$$S_1(\gamma) = exp\left(\sum_{k\geq 1} c_{2k}^{(1)}\, Trace\,(ad(\gamma)^{2k})\right),\ S_2(\gamma) = exp\left(\sum_{k\geq 1} c_{2k}^{(2)}\, Trace\,(ad(\gamma)^{2k})\right)$$

where $c_2^{(1)}, c_4^{(1)}, \dots$ and $c_2^{(2)}, c_4^{(2)}, \dots$ are two infinite sequences of real numbers indexed by even natural numbers.

Proof: we will study separately two cases.

8.3.3.1. *Isomorphism I_T*

The isomorphism I_T is given by the sum over terms corresponding to admissible graphs Γ with no vertices of the second type, one special vertex v of the first type such that no edge start at v, and such that at any other vertex start two edges and ends no more than one edge. Vertex v is the marked vertex where we put an element of $Sym(\mathbf{g})$ considered as an element of tangent cohomology. At other vertices we put the Poisson-Kirillov bi-vector field on $\mathbf{g}^*$,

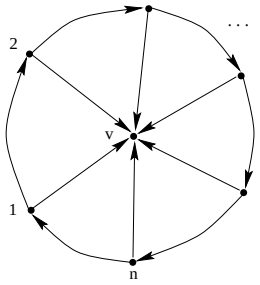

Figure 20: Wheel graph.

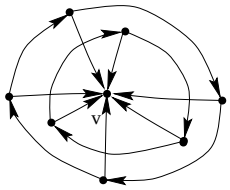

Figure 21: A union of wheels.

i.e. the tensor of commutator operation in $\mathbf{g}$. As the result we get 0-differential operator, i.e. an element of algebra $Sym(\mathbf{g})$.

It is easy to see that any such graph is isomorphic to a union of copies of "wheels" $Wh_n,\ n \geq 2$ represented in Figure 20 with identified central vertex v. Figure 21 shows a typical graph of the union.

In the integration we may assume that the point corresponding to v is fixed, say that it is $i \cdot 1 + 0 \in \mathscr{H}$, because group $G^{(1)}$ acts simply transitively on $\mathscr{H}$. First of all, the operator $Sym(\mathbf{g}) \longrightarrow Sym(\mathbf{g})$ corresponding to the individual wheel Wh_n is the differential operator on $\mathbf{g}^*$ with constant coefficients, and it corresponds to the polynomial $\gamma \mapsto Trace(ad(\gamma)^n)$ on $\mathbf{g}$. The operator corresponding to the joint of several wheels is the product of operators associated with individual wheels. Also, the integral corresponding to the joint is the product of integrals. Thus, with the help of symmetry factors, we conclude that the total operator is equal to the exponent of the sum of operators associated with wheels $Wh_n,\ n \geq 2$ with weights equal to corresponding integrals. By the symmetry argument used several times before ($z \mapsto -\overline{z}$), we see that integrals corresponding to wheels with odd n vanish. The first statement of Theorem 8.5 is proven. *Q.E.D.*

8.3.3.2. *Isomorphism $I_{alg} \circ I_{PBW}$*

The second case, for the operator $I_{alg} \circ I_{PBW}$, is a bit more tricky. Let us write a formula for this map:

$$I_{alg} \circ I_{PBW} : \gamma^n \mapsto \gamma \star \gamma \star \gamma \cdots \star \gamma \ \ (n \text{ copies of } \gamma) \ .$$

This formula defines the map unambiguously because elements $\gamma^n,\ \gamma \in \mathbf{g},\ n \geq 0$ generate $Sym(\mathbf{g})$ as a vector space.

In order to multiply several (say, m, where $m \geq 2$) elements of the quantized algebra we should put these elements at *m fixed* points in increasing order on $\mathbb{R}$ and take the sum over all possible graphs with m vertices of the second type of corresponding expressions

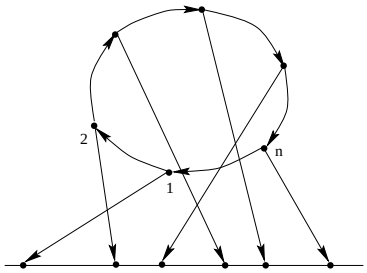

Figure 22: One of basic elements in the formula for $\gamma\star\cdots\star\gamma$.

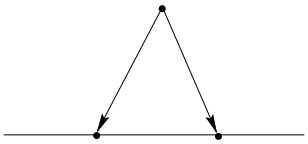

Figure 23: Another potential basic element, it vanishes for symmetry reasons.

with appropriate weights. The result does not depend on the position of fixed points on $\mathbb{R}$ because the star-product is associative. Moreover, if we calculate a power of a given element with respect to the $\star$-product, we can put all these points in arbitrary order. It follows that we can take an average over configurations of m points on $\mathbb{R}$ where each point is random, distributed independently from other points, with certain probability density on $\mathbb{R}$. We choose a probability distribution on $\mathbb{R}$ with a smooth symmetric (under transformation $x \mapsto -x$) density $\rho(x)$. We assume also that $\rho(x)dx$ is the restriction to $\mathbb{R} \simeq C_{1,1}$ of a smooth 1-form on $\overline{C}_{1,1} \simeq \{-\infty\} \sqcup \mathbb{R} \sqcup \{+\infty\}$. With probability 1 our m points will be pairwise distinct. One can check easily that the interchanging of order of integration (i.e. for the taking mean value from the probability theory side, and for the integration of differential forms over configuration spaces) is valid operation in our case.

The conclusion is that the m-th power of an element of quantized algebra can be calculated as a sum over all graphs with m vertices of the second type, with weights equal to integrals over configuration spaces where we integrate products of forms $d\phi$ and 1-forms $\rho(x_i)dx_i$ where x_i are points moving along $\mathbb{R}$.

The basic element of pictures in our case are "wheels without axles" (Figure 22) and the Λ-graph (Figure 23) which gives 0 for symmetry reasons. The typical total picture is something like (with $m = 10$) the one drawn in Figure 24.

Again, it is clear from all this that the operator $I_{alg} \circ I_{PBW}$ is a differential operator with constant coefficients on $Sym(\mathbf{g})$, equal to the exponent of the sum of operators corresponding

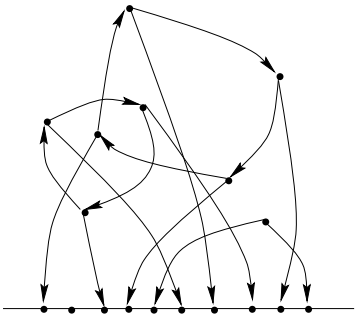

Figure 24: A term in the formula for $\gamma\star\cdots\star\gamma$.

to individual wheels. These operators are again proportional to operators associated with power series on $\mathbf{g}$

$$\gamma \longrightarrow Trace\,(ad(\gamma)^n)\ .$$

By the same symmetry reasons as above we see that integrals corresponding to odd n vanish. The second part of Theorem 8.5 is proven. *Q.E.D.*

8.3.4 Comparison with the Duflo-Kirillov isomorphism

For the case of semi-simple $\mathbf{g}$ there is so called Harish-Chandra isomorphism between algebras $\big(Sym(\mathbf{g})\big)^{\mathbf{g}}$ and $Center(\mathscr{U}\mathbf{g})$. A. Kirillov realized that there is a way to rewrite the Harish-Chandra isomorphism in a form which makes sense for arbitrary finite-dimensional Lie algebra, i.e. without using the Cartan and Borel subalgebras, the Weyl group etc. Later M. Duflo (see [12]) proved that the map proposed by Kirillov is an isomorphism for all finite-dimensional Lie algebras.

The explicit formula for the Duflo-Kirillov isomorphism is the following:

$$I_{DK} : \big(Sym(\mathbf{g})\big)^{\mathbf{g}} \simeq Center(\mathscr{U}(\mathbf{g})),\ \ I_{DK} = I_{PBW}{}_{|(Sym(\mathbf{g}))^{\mathbf{g}}} \circ I_{strange}{}_{|(Sym(\mathbf{g}))^{\mathbf{g}}},$$

where $I_{strange}$ is an invertible translation invariant operator on $Sym(\mathbf{g})$ associated with the following formal power series on $\mathbf{g}$ at zero, reminiscent of the square root of the Todd class:

$$\gamma \mapsto \exp\left(\sum_{k\geq 1} \frac{B_{2k}}{4k(2k)!}\, Trace\,(ad(\gamma)^{2k})\right)$$

where $B_2, B_4, \ldots$ are Bernoulli numbers. Formally, one can write the right-hand side as $det(q(ad(\gamma)))$ where

$$q(x) := \sqrt{\frac{e^{x/2} - e^{-x/2}}{x}}\ .$$

The fact that the Duflo-Kirillov isomorphism is an isomorphism of algebras is highly non-trivial. All proofs known before (see [12, 20]) used certain facts about finite-dimensional Lie algebras which follow only from the classification theory. In particular, the fact that the analogous isomorphism for Lie superalgebras is compatible with products, was not known.

We claim that our isomorphism coincides with the Duflo-Kirillov isomorphism. Let us sketch the argument. In fact, we claim that

$$I_{alg}^{-1} \circ I_T = I_{PBW} \circ I_{strange} \ .$$

If it is not true then we get a non-zero series $Err \in t^2\mathbb{R}[[t^2]]$ such that the translation invariant operator on $Sym(\mathbf{g})$ associated with $\gamma \mapsto I_{det(exp(ad\gamma)))}$ gives an *automorphism* of algebra $(Sym(\mathbf{g}))^{\mathbf{g}}$. Let $2k > 0$ be the degree of first non-vanishing term in the expansion of Err. Then it is easy to see that the operator on $Sym(\mathbf{g})$ associated with the polynomial $\gamma \mapsto Trace(ad(\gamma)^{2k}$ is a *derivation* when restricted to $(Sym(\mathbf{g}))^{\mathbf{g}}$. One can show that it is not true using Lie algebras $\mathbf{g} = gl(n)$ for large n. Thus, we get a contradiction and proved that $Err = 0$. *Q.E.D.*

As a remark we would like to mention that if one replaces series $q(x)$ above just by the inverse to the square root of the series related to the Todd class

$$\left(\frac{x}{1-e^{-x}}\right)^{-\frac{1}{2}}$$

then one still get an isomorphism of algebras. The reason is that the one-parameter group of automorphisms of $Sym(\mathbf{g})$ associated with series

$$\gamma \longrightarrow \exp(const \cdot Trace(ad(\gamma)))$$

preserves the structure of Poisson algebra on $\mathbf{g}^*$. This one-parameter group also acts by automorphisms of $\mathscr{U}\mathbf{g}$. It is analogous to the Tomita-Takesaki modular automorphism group for von Neumann algebras.

8.3.5 Results in rigid tensor categories

Many proofs from this paper can be transported to a more general context of rigid $\mathbb{Q}$-linear tensor categories (i.e. abelian symmetric monoidal categories with the duality functor imitating the behavior of finite-dimensional vector spaces). We will be very brief here.

First of all, one can formulate and prove the Poincaré-Birkhoff-Witt theorem in a great generality, in $\mathbb{Q}$-linear additive symmetric monoidal categories with infinite sums and kernels of projectors. For example, it holds in the category of A-modules where A is arbitrary commutative associative algebra over $\mathbb{Q}$. Thus, we can speak about universal enveloping algebras and the isomorphism I_{PBW}.

One can define Duflo-Kirillov morphism for a Lie algebra in a $\mathbf{k}$-linear rigid tensor category where $\mathbf{k}$ is a field of characteristic zero, because Bernoulli numbers are rational. Our result from 8.3.4 saying that it is a morphism of algebras, holds in this generality as well. It *does not* hold for infinite-dimensional Lie algebras because we use traces of products of operators in the adjoint representation.

In [27] a conjecture was made in the attempt to prove that that Duflo-Kirillov formulas give a morphism of algebras. It seems plausible our results can help one to prove this conjecture. Also, there is another related conjecture concerning two products in the algebra of chord diagrams (see [3]) which seems to be a corollary of our results.

8.4 Second application: algebras of Ext-s.

Let X be complex manifold, or a smooth algebraic variety of field $\mathbf{k}$ of characteristic zero. We associate with it two graded vector spaces. The first space $HT^\bullet(X)$ is the direct sum $\bigoplus_{k,l} H^k(X, \wedge^l T_X)[-k-l]$. The second space $HH^\bullet(X)$ is the space $\bigoplus_k Ext^k_{Coh(X\times X)}(\mathscr{O}_{diag}, \mathscr{O}_{diag})[-k]$ of Ext-groups in the category of coherent sheaves on $X \times X$ from the sheaf of functions on the diagonal to itself. The space $HH^\bullet(X)$ can be thought as the Hochschild cohomology of the space X. The reason is that the Hochschild cohomology of any algebra A can be also defined as $Ext^\bullet_{A-mod-A}(A,A)$ in the category of bimodules.

Both spaces, $HH^\bullet(X)$ and $HT^\bullet(X)$ carry natural products. For $HH^\bullet(X)$ it is the Yoneda composition, and for $HT^\bullet(X)$ it is the cup-product of cohomology and of polyvector fields.

CLAIM *Graded algebras $HH^\bullet(X)$ and $HT^\bullet(X)$ are canonically isomorphic. The isomorphism between them is functorial with respect to étale maps.*

This statement (important for the Mirror Symmetry, see [32]) is again a corollary of Theorem 8.1. Here we will not give the proof.

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