XI Solomon Lefschetz Memorial Lecture Series: Hodge structures in non-commutative geometry. (Notes by Ernesto Lupercio)

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ABSTRACT. Traditionally, Hodge structures are associated with complex projective varieties. In my expository lectures I discussed a non-commutative generalization of Hodge structures in deformation quantization and in derived algebraic geometry.

1. Lecture 1. September 8th, 2005

1.1. This talk deals with some relations between algebraic geometry and non-commutative geometry, in particular we explore the generalization of Hodge structures to the non-commutative realm.

1.2. Hodge Structures. Given a smooth projective variety X over $\mathbb{C}$ we have a naturally defined pure Hodge structure (HS) on its cohomology, namely:

- $H^n(X, \mathbb{C}) = \bigoplus_{p+q=n, \ p,q \geq 0} H^{p,q}(X)$ by considering $H^{p,q}$ to be the cohomology represented by forms that locally can be written as

$$\sum a_{i_1,\ldots,i_p;j_1,\ldots,j_q} dz_{i_1} \wedge dz_{i_2} \wedge \ldots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge \ldots \wedge d\bar{z}_{j_q}$$

- $H^n(X, \mathbb{C})$ is the complexification $H^n(X, \mathbb{Z}) \otimes \mathbb{C}$ of a lattice of finite rank.
- $H^{p,q} = \overline{H^{q,p}}$.

1.3. To have this Hodge structure (of weight n) is the same as having the decreasing filtration

$$F^p H^n := \bigoplus_{p' \geqslant p, \ p'+q'=n} H^{p',q'}$$

for we have $H^{p,q} = F^p H^n \cap \overline{F^q} H^n$.

1.4. What makes this Hodge structure nice is that whenever we have a family X_t of varieties algebraically dependent on a parameter t we obtain a bundle of cohomologies with a flat (Gauss-Manin) connection

$$H_t = H^n(X_t, \mathbb{C})$$

over the space of parameters, and F_t^p is a *holomorphic* subbundle (even though $H_t^{p,q}$ is not). Deligne developed a great theory of mixed Hodge structures that generalizes this for any variety perhaps singular or non-compact.

1.5. Non-commutative geometry. Non-commutative geometry (NCG) has been developed by Alain Connes [**3**] with applications regarding foliations, fractals and quantum spaces in mind, but not algebraic geometry. In fact it remains unknown what a good notion of non-commutative complex manifold is.

1.6. There is a calculus associated to NC spaces. Suppose $\mathbf{k}$ is a field and for the first talk the field will always be $\mathbb{C}$, only in the second talk finite fields become relevant. Let A be a unital, associative algebra over $\mathbf{k}$. An idea of Connes is to mimic topology, namely forms and the de Rham differential in this framework. We define the Hochschild complex $C_\bullet(A, A)$ of A as a negatively graded complex (for we want to have all differentials of degree $+1$),

$$\xrightarrow{\partial} A \otimes A \otimes A \otimes A \xrightarrow{\partial} A \otimes A \otimes A \xrightarrow{\partial} A \otimes A \xrightarrow{\partial} A,$$

where $A^{\otimes k}$ lives on degree $-k + 1$. The differential ∂ is given by

$$\partial(a_0 \otimes \cdots \otimes a_n) = a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n - a_0 \otimes a_1 a_2 \otimes \cdots \otimes a_n$$

$$+ \ldots + (-1)^{n-1} a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

This formula is more natural when we write terms cyclically:

(1.6.1)

$$
\begin{array}{ccccc}
 & & a_0 & & \\
 & \otimes & & \otimes & \\
a_n & & & & a_1 \\
\otimes & & & & \otimes \\
\vdots & & & & \vdots \\
 & \otimes & & \otimes & \\
 & & a_i & &
\end{array}
$$

for $a_0 \otimes \cdots \otimes a_n$. It is very easy to verify that $\partial^2 = 0$.

1.7. Homology of the Hochschild complex have an abstract meaning

$$\operatorname{Ker} \partial / \operatorname{Im} \partial = \operatorname{Tor}_\bullet^{A \otimes_k A^{\mathrm{op}} - \mathrm{mod}}(A, A).$$

1.8. An idea in NC geometry is that as A replaces a commutative space the Hochschild homology of A replaces in turn the complex of differential forms.

THEOREM 1.8.1 (Hochschild-Konstant-Rosenberg, 1961, [**6**]). *Let X be a smooth affine algebraic variety, then if $A = \mathcal{O}(X)$ we have*

$$HH_i(X) := H^{-i}(C_\bullet(A, A); \partial) \cong \Omega^i(X)$$

where $\Omega^i(X)$ is the space of i-forms on X.

The proof is very easy: consider the diagonal embedding $X \xrightarrow{\Delta} X \times X$ and by remembering that the normal bundle of Δ is the tangent bundle of X we have

$$HH_\bullet(X) = \operatorname{Tor}_\bullet^{\mathrm{Quasi-coherent}(X \times X)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$$

this together with a local calculation gives the result.

1.9. The Hochschild-Konstant-Rosenberg theorem motivates us to think of $HH_i(A)$ as a space of differential forms of degre i on a non-commutative space.

Note that if A is non-commutative we have

$$H^0(C_\bullet(A, A); \partial) = A/[A, A].$$

Also, for commutative $A = \mathcal{O}(X)$, if given an element $a_0 \otimes \cdots \otimes a_n$ in $C_\bullet(A, A)$ the corresponding form is given by $\frac{1}{n!} a_0 da_1 \wedge \ldots \wedge da_n$.

1.10. There is a reduced version of the complex $C_\bullet^{\mathrm{red}}(A, A)$ with the same cohomology obtained by reducing modulo constants all but the first factor

$$\longrightarrow A \otimes A/(\mathbf{k} \cdot 1) \otimes A/(\mathbf{k} \cdot 1) \longrightarrow A \otimes A/(\mathbf{k} \cdot 1) \longrightarrow A.$$

1.11. Connes main observation is that we can write a formula for an additional differential B on $C_\bullet(A, A)$ of degreee -1, inducing a differential on $HH_\bullet(A)$ that generalizes the de Rham differential:

$$B(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_\sigma (-1)^\sigma 1 \otimes a_{\sigma(0)} \otimes \cdots \otimes a_{\sigma(n)}$$

where $\sigma \in \mathbb{Z}/(n+1)\mathbb{Z}$ runs over all cyclic permutations. It is easy to verify that

$$B^2 = 0, \quad B\partial + \partial B = 0, \quad \partial^2 = 0,$$

which we depict pictorially as

$$\cdots \underset{\partial}{\overset{B}{\rightleftarrows}} A \otimes A/1 \otimes A/1 \underset{\partial}{\overset{B}{\rightleftarrows}} A \otimes A/1 \underset{\partial}{\overset{B}{\rightleftarrows}} A$$

that by taking cohomology gives us a complex $(\mathrm{Ker}\, \partial/\mathrm{Im}\, \partial; B)$. A naive definition on the de Rham cohomology in this context is the homology of this complex $\mathrm{Ker}\, B/\mathrm{Im}\, B$.

1.12. We can do better by defining the negative cyclic complex $C_\bullet^-(A)$, which is formally a projective limit (here u is a formal variable, $\deg(u) = +2$):

$$C_\bullet^- := (C_\bullet^{\mathrm{red}}(A, A)[[u]]; \partial + uB) = \varprojlim_N (C_\bullet^{\mathrm{red}}(A, A)[u]/u^N; \partial + uB).$$

1.13. We define the periodic complex as an inductive limit

$$C_\bullet^{\mathrm{per}} := (C_\bullet^{\mathrm{red}}(A, A)((u)); \partial + uB) = \varinjlim_i (u^{-i} C_\bullet^{\mathrm{red}}(A, A)[[u]]; \partial + uB).$$

This turns out to be a $\mathbf{k}((u))$-module and this implies that u induces a sort of Bott periodicity. The resulting cohomology groups called (even, odd) periodic cyclic homology and are written (respectively)

$$HP_{\mathrm{even}}(A), \qquad HP_{\mathrm{odd}}(A).$$

This is the desired replacement for de Rham cohomology.

1.14. Let us consider some examples. When $A = C^\infty(X)$ is considered as a nuclear Fréchet algebra, and if we interpret the symbol $\otimes$ as the topological tensor product then we have the canonical isomorphisms:

$$HP_{\text{even}}(A) \cong H^0(X, \mathbb{C}) \oplus H^2(X, \mathbb{C}) \oplus \cdots$$

$$HP_{\text{odd}}(A) \cong H^1(X, \mathbb{C}) \oplus H^3(X\, \mathbb{C}) \oplus \cdots$$

THEOREM 1.14.1 (Feigin-Tsygan, [**4**]). *If X is a singular affine algebraic variety and X_{top} its underlying topological space then*

$$HP_{\text{even}}(A) \cong H^{\text{even}}(X_{\text{top}}, \mathbb{C})$$

and

$$HP_{\text{odd}}(A) \cong H^{\text{odd}}(X_{\text{top}}, \mathbb{C})$$

(that are finite dimensional).

There is a natural lattice $H^\bullet(X, \mathbb{Z})$ but we will see later that the "correct" lattice should be slightly different.

1.15. Everything we said before can be defined for a differential graded algebra (dga) rather than only for an algebra A. Recall that a dga (A, d) consists of

- $A = \bigoplus_{n \in \mathbb{Z}} A^n$ a graded algebra with a graded product

$$A^{n_1} \otimes A^{n_2} \longrightarrow A^{n_1 + n_2}.$$

- $d_A \colon A^n \longrightarrow A^{n+1}$ a differential satisfying the graded Leibniz rule.

For example given a manifold X on has the de Rham dga $(\Omega^\bullet(X); d)$.

1.16. The definition of the degree in for $C_\bullet(A, A)$ is given by

$$\deg(a_1 \otimes \cdots \otimes a_n) := 1 - n + \sum_i \deg(a_i).$$

It is not hard to see that

$$\operatorname{rank}(HP_\bullet(A)) \leqslant \operatorname{rank}(HH_\bullet(A)),$$

and therefore, if the rank of the Hochschild homology is finite so is the rank of the periodic cyclic homology.

1.17. Hodge filtration on $HP_\bullet(A)$. We define $F^n HP_{\text{even}}(A)$ as the classes represented by sequences $\gamma_i \in C_i(A, A)$, $i \in 2\mathbb{Z}$ (namely $\sum \gamma_i u^{i/2}$) such that $i \geq 2n$. Similarly we define $F^{n+1/2} HP_{\text{odd}}(A)$.

1.18. We have an interesting instance of this situation in ordinary topology $A = C^\infty(X)$. Here we have:

$$HP_{\text{even}}(A) = H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X) \oplus \cdots$$

and $F^0 = HP_{\text{even}}(A)$, $F^1 = H^2(X) \oplus H^4(X) \oplus H^6(X) \oplus \cdots$ and so on. In non-commutative geometry this filtration is the best you can do for, there is no individual cohomologies.

1.19. Consider X an algebraic variety (not necessarily affine). Weibel [**15**] gave a sheaf-theoretic definition of $HH_\bullet(X)$ and $HP_\bullet(X)$. Namely, if X is given by affine open charts

$$X = \bigcup_{1 \leqslant i \leqslant r} U_i,$$

we obtain not an algebra, but a cosimplicial algebra

$$\mathcal{A}_k := \oplus_{(i_0, \ldots, i_k)} \mathcal{O}(U_{i_0} \cap \ldots \cap U_{i_k}),$$

whose total complex $\mathrm{Tot}(C_\bullet(\mathcal{A}_\bullet)) = C_\bullet(X)$ still has two differentials B and ∂ as before. In fact

$$H^\bullet(C_\bullet(X), \partial) = \mathrm{Tor}_\bullet^{\mathrm{Quasi-coherent}(X \times X)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta),$$

that is graded in both positive and negative degrees.

Weibel observed that one can recover a tilted version of the Hodge diamond in this manner. For smooth projective X one has

$$HP_\bullet(X) := HP_\bullet(\mathcal{A}_\bullet) = H^\bullet(X)$$

and the filtration we defined becomes

$$F^i(HP_\bullet) = \bigoplus_{p=i+n/2} F^p H^n(X), \quad i \in \frac{1}{2}\mathbb{Z},$$

reshuffling thus the usual Hodge filtration.

Observe that in this example we have:

$$H^{p,q} \subset F^{\frac{p-q}{2}}.$$

1.20. In general one can directly repalce an algebraic variety by a dga using a theorem by Bondal and van den Bergh. For example, if X is a smooth projective variety X and let $\mathcal{E}$ be a sufficiently "large" bundle over X. You may take for instance $\mathcal{E} = \mathcal{O}(0) + \mathcal{O}(1) + \cdots + \mathcal{O}(N)$ for sufficiently large N (usually $N = \dim X$ suffices). Take the algebra $\mathcal{A}$ to be

$$\mathcal{A} := (\Gamma(X, \mathrm{End}(\mathcal{E}) \otimes \Omega^{0,1}); \bar{\partial}).$$

Then one can show that one can repeat the previous constructions obtaining the corresponding filtration.

1.21. Take X_{alg} to be a smooth algebraic variety over $\mathbb{C}$ and let X_{C^∞} be its underlying smooth manifold. Consider the natural map

$$X_{C^\infty} \longrightarrow X_{\mathrm{alg}}.$$

This map induces an isomorphism

$$HP_\bullet(X_{C^\infty}) \longleftarrow HP_\bullet(X_{\mathrm{alg}}).$$

This isomorphism is compatible with the Hodge filtrations but the filtrations are *different*.

1.22. The next important ingredients are the integer lattices. Notice that $H^\bullet(X, \mathbb{C})$ has a natural integer lattice $H^\bullet(X, \mathbb{Z})$, which allows us to speak of periods, for example. There is also another lattice commensurable with $H^\bullet(X, \mathbb{Z})$, namely, the topological K-theory $K^\bullet_{\text{top}}(X) := K^{\text{even}}(X_{C^\infty}) \oplus K^{\text{odd}}(X_{C^\infty})$.

Let A be an algebra, then we get $K_0(A)$ by considering the projective modules over A. There is a Chern character map

$$K_0(A) \xrightarrow{\quad \text{ch} \quad} F^0(HP_{\text{even}}(A)) \lhook\joinrel\longrightarrow HP_{\text{even}}(A)$$
$$\searrow \qquad \nearrow$$
$$HC_0^-(A)$$

Here it may be appropriate to recall that $HP_\bullet(A)$ is a Morita invariant and that therefore we can replace A by $A \otimes \text{Mat}_{n \times n}$ for $\text{Mat}_{n \times n}$ a matrix algebra.

If $\pi \in A$ is a projector (namely $\pi^2 = \pi$) we have explicitly

$$\text{ch}(\pi) = \pi - \frac{2!}{1!}(\pi - 1/2) \otimes \pi \otimes \pi \cdot u + \frac{4!}{2!}(\pi - 1/2) \otimes \pi \otimes \pi \otimes \pi \otimes \pi \cdot u^2 + \dots.$$

There is a similar story for $K_1(A) \longrightarrow HP_{\text{odd}}(A)$, and also for higher K-theory.

1.23. If $A = C^\infty(X)$ then the image of $K_0(A) = K^0_{\text{top}}(X)$ is up to torsion

$$H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \cdot 2\pi i \oplus H^4(X, \mathbb{Z}) \cdot (2\pi i)^2 \oplus \cdots.$$

We have of course $K^0_{\text{top}}(X) \otimes \mathbb{Q} = H^{\text{even}}(X, \mathbb{Q})$ but the lattice is different and so Bott periodicity is broken. In order to restore it we must rescale the odd degree part of the lattice by a factor of $\sqrt{2\pi i}$, and then we obtain

$$H^1(X, \mathbb{Z}) \cdot \sqrt{2\pi i} \oplus H^3(X, \mathbb{Z}) \cdot (\sqrt{2\pi i})^3 \oplus \cdots$$

We call this new lattice the *non-commutative integral cohomology*

$$H^\bullet_{\text{NC}}(X, \mathbb{Z}) \subset HP_\bullet(C^\infty(X)).$$

PROPOSITION 1.23.1. *For $A = C^\infty(X)$ the image up to torsion of*

$$\text{ch} : K_n(A) \longrightarrow HP_{(n \mod 2)}(A)$$

is

$$(2\pi i)^{n/2} H^{(n \mod 2)}_{\text{NC}}(X, \mathbb{Z}).$$

1.24. We are ready to formulate one of the *main problems in non-commutative geometry.* Let A be a dga over $\mathbb{C}$. Define a nuclear, Frechét algebra A_{C^∞} satisfying Bott periodicity $K_i(A_{C^\infty}) \cong K_{i+2}(A_{C^\infty}), i \geq 0$ together with an algebra homomorphism $A \to A_{C^\infty}$ satisfying:

- The homomorphism $A \to A_{C^\infty}$ induces an isomorphism

$$HP_\bullet(A) \cong HP_\bullet(A_{C^\infty}),$$

- ch : $K_n(A_{C^\infty}) \longrightarrow HP_\bullet(A_{C^\infty})$ is a lattice, i.e. when we tensor with $\mathbb{C}$ we obtain an isomorphism

$$K_n(A_{C^\infty}) \otimes_\mathbb{Q} \mathbb{C} \xrightarrow{\cong} HP_{(n \mod 2)}(A_{C^\infty}).$$

1.25. Consider for example the case of a commutative algebra A. Every such algebra is an inductive limit of finitely generated algebras

$$A = \varinjlim A_n$$

where each A_n can be thought of as a singular affine variety. In general $HP_\bullet(A) \neq \varinjlim HP_\bullet(A_n)$, but the right hand side is a better definition for $HP_\bullet(A)$. In this case we find that the lattice we are looking for is simply

$$\varinjlim K_{\mathrm{top}}^\bullet(\operatorname{Spec} A_n(\mathbb{C})).$$

1.26. We will attempt now to explain some non-commutative examples that are close to the commutative realm, and are obtained by a procedure called *deformation quantization*. Let us consider first a C^∞ non-commutative space.

Let T_θ^2 be the non-commutative torus (for $\theta \in \mathbb{R}$) so that $C^\infty(T_\theta^2)$ is precisely all the expressions of the form

$$\sum_{n,m \in \mathbb{Z}} a_{n,m} \hat{z}_1^n \hat{z}_2^m, \quad a_{n,m} \in \mathbb{C},$$

such that for all k we have

$$a_{m,n} = O((1 + |n| + |m|)^{-k}),$$

and

$$\hat{z}_1 \hat{z}_2 = e^{i\theta} \hat{z}_2 \hat{z}_1.$$

For $\theta \in 2\pi\mathbb{Z}$ we get the usual commutative torus.

1.27. We will also consider some non-commutative *algebraic* spaces obtained by deformation quantization. Start by taking a smooth affine algebraic variety X and a bi-vector field $\alpha \in \Gamma(X, \bigwedge^2 TX)$. Define, as is usual, the bracket by

$$\{f, g\} := \langle \alpha, df \wedge dg \rangle.$$

Field α defines a Poisson structure iff the bracket satisfies the Jacobi identity. We will call α *admissible at infinity* is there exists a smooth projective variety $\bar{X} \supset X$ and a divisor $\bar{X}_\infty = \bar{X} - X$ so that α extends to $\bar{X}$ and the ideal sheaf $\mathcal{I}_{\bar{X}_\infty}$ is a Poisson ideal (closed under brackets).

1.28. The simplest instance of this is when $X = \mathbb{C}^n$, $\bar{X} = \mathbb{C}P^n$ and the admissibility condition for $\alpha = \sum_{i,j} \alpha_{i,j} \partial_i \wedge \partial_j$ reads $\deg(\alpha_{ij}) \leqslant 2$.

1.29. We have the following [**10**]:

THEOREM 1.29.1. *If X satisfies*

$$H^1(\bar{X}, \mathcal{O}) = H^2(\bar{X}, \mathcal{O}) = 0,$$

(e.g. X is a rational variety) then there exists a canonical filtered algebra $A_\hbar$ over $\mathbb{C}[[\hbar]]$ (actually a free module over $\mathbb{C}[[\hbar]]$) that gives a $*$-deformation quantization, and when we equal the deformation parameter to 0 we get back $\mathcal{O}(X)$.

While the explicit formulas are very complicated the algebra $A_\hbar$ is *completely canonical*.

1.30.   This theorem raises the interesting issue of comparison of parameters. Take for example the case $X = \mathbb{C}^2$ and $\alpha = xy\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$. Here we can guess that

$$A_\hbar \cong \mathbb{C}[[\hbar]]\langle \hat{X}, \hat{Y} \rangle / \hat{X}\hat{Y} = e^\hbar \hat{Y}\hat{X}.$$

On the other hand the explicit formula for $A_\hbar$ involves infinitely many graphs and even for this simple example it is impossible to get the explicit parameter $e^\hbar$. A priori one only knows that just certain universal series $q(\hbar) = 1 + \dots$ should appear with $\hat{X}\hat{Y} = q(\hbar)\hat{Y}\hat{X}$.

1.31.   A slightly more elaborate example is furnished by considering (Sklyanin) *elliptic algebras*. Here we take $\bar{X} = \mathbb{C}P^2$ and $\alpha = p(x,y)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ with $\deg(p) = 3$. The divisor $\bar{X}_\infty \subset \mathbb{C}P^2$ in this case is a cubic curve. We take $X = \bar{X} - \bar{X}_\infty$, which is an affine algebraic surface and since it has a symplectic structure it also has a Poisson structure α.

Its quantum algebra $A_\hbar$ depends on an elliptic curve E and a shift $x \mapsto x + x_0$ on E. The question is then: How to relate E and x_0 to the parameters α and $\hbar$?

Again there is only one reasonable guess. Start with the bi-vector field α and obtain a 2-form α^{-1} on $\mathbb{C}P^2$ with a first order pole at $\bar{X}_\infty$. Our guess is that $E = \bar{X}_\infty$. Taking residues we obtain a holomorphic 1-form $\mathrm{Res}(\alpha^{-1}) \in \Omega^1(E)$. The inverse of this 1-form is a vector field $(\mathrm{Res}(\alpha^{-1}))^{-1}$ on E. Finally:

$$x_0 = \exp\left(\frac{\hbar}{\mathrm{Res}(\alpha^{-1})}\right),$$

but to prove this directly seems to be quite challenging.

1.32.   It is a remarkable fact that this comparison of parameters problem can be solved by considering the Hodge structures.

1.33.   Let us consider X to be either a C^∞ or an affine algebraic variety and A to be $C^\infty(X)$ (respectively $\mathcal{O}(X)$). The theory of deformation quantization implies that all nearby non-commutative algebras and related objects (such as $HP_\bullet$, $HH_\bullet$, etc.) can be computed semi-classically. In particular nearby algebras are given by Poisson bi-vector fields α. Also $C_\bullet(A_\hbar, A_\hbar)$ is quasi-isomorphic to the negatively graded complex $(\Omega^{-i}(X), \mathcal{L}_\alpha)$ where the differential is $\mathcal{L}_\alpha = [\iota_\alpha, d]$. If you want to see this over $\mathbb{C}[[\hbar]]$ simply consider the differential $\mathcal{L}_{\hbar\alpha}$. We just described what Brylinski calls *Poisson cohomology*. The differential B in this case is simply the usual de Rham differential $B = d$. We would like to consider now $HP_\bullet(A_\hbar)$. This is computed by the complex

$$\left(\bigoplus \Omega^i(X)[i][[\hbar]]((u)), ud + \hbar\mathcal{L}_\alpha\right),$$

which is the sum of infinitely many copies of some finite dimensional complex. Namely $HP_\bullet(A_\hbar)\hat{\otimes}_{\mathbb{C}[[\hbar]]}\mathbb{C}((\hbar))$ is $\mathbb{C}((\hbar))$ tensored with the finite dimensional cohomology of the $\mathbb{Z}/2$-graded complex:

$$\Omega^N \underset{\mathcal{L}_\alpha}{\overset{d}{\rightleftarrows}} \Omega^{N-1} \underset{\mathcal{L}_\alpha}{\overset{d}{\rightleftarrows}} \cdots \underset{\mathcal{L}_\alpha}{\overset{d}{\rightleftarrows}} \Omega^1 \underset{\mathcal{L}_\alpha}{\overset{d}{\rightleftarrows}} \Omega^0$$

1.34. We claim that the cohomology of this complex is $H^\bullet(X)$. The reason for this is really simple, for we have that

$$\exp(\iota_\alpha) d \exp(-\iota_\alpha) = d + [\iota_\alpha, d] + \frac{1}{2!}[\iota_\alpha, [\iota_\alpha, d]] + \ldots = d + \mathcal{L}_\alpha.$$

Here we used the fact that $[\alpha, \alpha] = 0$ to conclude that only the first two terms survive.

1.35. Let's turn our attention to the lattice. Our definition uses $K_n(A)$ but we may get at this lattice by using the Gauss-Manin connection. If we have a family of algebras A_t depending on some parameter t, Ezra Getzler [**5**] defined a flat connection on the bundle HP_t over the parameter space. This allows us to start with the lattice $\oplus_k H^k(X, (2\pi i)^{k/2} \cdot \mathbb{Z})$ (up to torsion) at $t := \hbar = 0$ in our situation. The parallel transport for the Gauss-Manin connection comes from the above identification of periodic complexes given by the conjugation by $\exp(\hbar \iota_\alpha)$.

1.36. To compute the filtration we will assume that X is symplectic, and therefore α is non-degenerate. Again we set $\dim(X) = N = 2n$, and $\omega = \alpha^{-1}$ is a closed 2-form. We also set $\hbar := 1$ The following theorem is perhaps well known but in any case is very simple:

THEOREM 1.36.1. *For (X, ω) a symplectic manifold the Hodge filtration is given by*

- HP^{even}:
$$F^{n-k/2} = e^\omega(H^0 \oplus \cdots \oplus H^k), \quad k \in 2\mathbb{Z}$$
$$e^\omega H^0 \subset e^\omega(H^0 \oplus H^2) \subset \cdots$$

- HP^{odd}:
$$F^{n-k/2} = e^\omega(H^1 \oplus \cdots \oplus H^k), \quad k \in 2\mathbb{Z}+1$$
$$e^\omega H^1 \subset e^\omega(H^1 \oplus H^3) \subset \cdots$$

Notice that this is *not* the usual Hodge filtration coming from topology: $H^{2n} \subset H^{2n} \oplus H^{2n-2} \subset \cdots$

PROOF. Consider the $\mathbb{Z}/2$-graded complex

$$\Omega^{2n} \underset{\mathcal{L}_\alpha}{\overset{d}{\rightleftarrows}} \Omega^{2n-1} \underset{\mathcal{L}_\alpha}{\overset{d}{\rightleftarrows}} \cdots \underset{\mathcal{L}_\alpha}{\overset{d}{\rightleftarrows}} \Omega^1 \underset{\mathcal{L}_\alpha}{\overset{d}{\rightleftarrows}} \Omega^0$$

where Ω^k lives in $F^{k/2}$.

The differential is not compatible with the filtration, nevertheless after the conjugation by $\exp(\iota_\alpha)$ we can use instead the complex

$$\Omega^{2n} \xleftarrow{d} \Omega^{2n-1} \xleftarrow{d} \cdots \xleftarrow{d} \Omega^0$$

and we would like to understand what happens to the filtration.

Let $*$ be the Hodge operator with respect to ω. Under the Fourier transform $*$ the original $\mathbb{Z}/2$-graded complex becomes

$$\Omega^0 \underset{d}{\overset{\mathcal{L}_\alpha}{\rightleftarrows}} \Omega^1 \underset{d}{\overset{\mathcal{L}_\alpha}{\rightleftarrows}} \cdots \underset{d}{\overset{\mathcal{L}_\alpha}{\rightleftarrows}} \Omega^{2n-1} \underset{d}{\overset{\mathcal{L}_\alpha}{\rightleftarrows}} \Omega^{2n}$$

where the filtration has been reversed.

The important remark here is that now $e^{\iota\alpha}$ does preserve this filtration, transforming the last complex into the complex

$$\Omega^0 \xrightarrow{\ d\ } \Omega^1 \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \Omega^{2n}$$

deg	n	$n - 1/2$	$\cdots$	0

Finally, all that remains to be seen is that

$$e^{\omega\wedge\cdot} = e^{\iota\alpha} * e^{-\iota\alpha}.$$

$\square$

1.37. This theorem is related to the *Lefschetz decomposition formula*[1] for Kähler manifolds. In fact we have obtained

$$HP_\bullet(X_{\hbar\alpha}) = H^\bullet(X),$$

furthermore we have

$$\lim_{\hbar \to 0} F^i HP_\bullet(X_{\hbar\alpha}) = F^i HP_\bullet(X)$$

if and only if we have a Lefschetz decomposition for the symplectic manifold.

If (X, ω) is Kähler compact we define the multiplication operator

$$\omega\wedge : H^\bullet(X) \longrightarrow H^{\bullet+2}(X),$$

which is clearly nilpotent. The Lefschetz decomposition corresponds to the decomposition into Jordan blocks for this operator. In the case at hand the Lesfchetz decomposition becomes the Hodge decomposition of a non-commutative space.

1.38. Consider T_θ^2 the non-commutative torus. A result of Mark Rieffel [**12**] states that T_θ^2 is Morita equivalent to $T_{\theta'}^2$ if and only if

$$\theta' = \frac{a\theta + b}{c\theta + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

If you consider in this case

$$HP_{\mathrm{even}}(T_\theta^2) = H^0 \oplus H^2 \longleftarrow K_0(T_\theta^2)$$

you can see that $K_0(T_\theta^2)$ contains the semigroup of bona fide projective modules, producing a half-plane in the lattice bounded by a line of slope $\theta/(2\pi)$, which from our point of view can be identified with the Hodge filtration F^1. This helps to clarify the meaning of Rieffel's theorem. In this example we get an interesting filtration only for HP^{even}, and nothing for HP^{odd}.

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[1]Lefschetz influence in mathematics is clearly so large that is would be hard to give a talk in his honor without having the opportunity to mention his name at many points.

1.39. Consider an elliptic curve:

$$E = \mathbb{C}/\left(\mathbb{Z} + \tau\mathbb{Z}\right), \quad \Im(\tau) > 0.$$

Here

$$HP^{\mathrm{even}}(E) = H^0(T^2) \oplus H^2(T^2),$$

$$HP^{\mathrm{odd}}(E) = H^1(T^2, \mathbb{C}) \supset H^{1,0}(E) \ \{\text{terms in the Hodge filtration}\}.$$

This becomes in terms of generators

$$\mathbb{C} \otimes (\mathbb{Z}e_0 \oplus \mathbb{Z}e_1) \supset \mathbb{C} \cdot (e_0 + \tau e_1).$$

While the corresponding situation in T_θ^2 can be written as

$$\mathbb{C} \otimes (\mathbb{Z}\tilde{e}_0 + \mathbb{Z}\tilde{e}_1) \supset \mathbb{C} \cdot (\tilde{e}_0 + \frac{\theta}{2\pi}\tilde{e}_1).$$

All this suggests that non-commutative tori are limits off elliptic curves as $\tau \to \mathbb{R}$. Also E can be seen a quotient of a 1-dimensional complex torus $\mathbb{C}^\times$, while T_θ^2 plays the role of a *real* circle modulo a θ-rotation.

1.40. I shall finish this lecture with one final puzzle. In the previous example the Hodge filtrations do not fit. In the elliptic curve the interesting Hodge structure detects parameters in odd cohomology while in the non-commutative torus the Hodge filtration detects parameters in even cohomology.

A reasonable guess for the solution of this puzzle is that one should tensor by a simple super-algebra (discovered by Kapustin in the Landau-Ginzburg model) given by

$$A = \mathbb{C}[\xi]/(\xi^2 = 1)$$

with ξ odd. Here

$$HP_\bullet(A) = \mathbb{C}^{0|1}.$$

The question is: How does this super-algebra naturally arise from the limiting process $\tau \to \mathbb{R}$ sending an elliptic curve to a foliation?

2. Lecture 2. September 9th, 2005.

2.1. Basic Derived Algebraic Geometry. This field started by A. Bondal and M. Kapranov in Moscow around 1990. Derived algebraic geometry is much simpler that algebraic geometry. While algebraic geometry starts with commutative rings and builds up spectra via the Zariski topology and the theory of sheafs, in derived algebraic geometry there is no room for many of these concepts and the whole theory becomes simpler.

2.2. Let us start by commenting on the algebraization of the notion of space. If we begin with a (topological) space X, first one can produce an algebra $A = \mathcal{O}(X)$, its algebra of functions. Next we assign an abelian category to this algebra, the abelian category $A - \mathbf{mod}$ of A-modules.

At every stage we *insist* in thinking of the space as the remaining object: The category $A - \mathbf{mod}$ *is* the space. The abelian category $A - \mathbf{mod}$ has a nice subfamily, that of *vector bundles*, namely finitely generated projective modules. Recall that projective modules are images of $(n \times n)$-matrices $\pi : A^n \to A^n$ satisfying $\pi^2 = \pi$.

The final step consists in producing from the category $A - \mathbf{mod}$ a triangulated category $D(A - \mathbf{mod})$ that goes by the name of the *derived category*.

$$X \mapsto \mathcal{O}(X) = A \mapsto A - \mathbf{mod} \mapsto D(A - \mathbf{mod}).$$

2.3.   While the abelian category $A - \mathbf{mod}$ is nice we are still forced to keep track of whether a fnctor is left-exact or right-exact, etc. This is greatly simplified in the derived category $D(A - \mathbf{mod})$.

The derived category $D(A - \mathbf{mod})$ is built upof ininite $\mathbb{Z}$-graded complexes of free A-modules and considering homotopies.

2.4.   We take one step further and consider the subcategory $\mathcal{C}_X \subset D(A - \mathbf{mod})$ of *perfect complexes*. A perfect complex is a finite length complex of finitely generated projective A-modules (vector bundles). All above can be generalized to dg algebras.

2.5.   We are ready to make an important definitions:

DEFINITION 2.5.1. Let k be a field. A k-linear space X is a small triangulated category $\mathcal{C}_X$ that is Karoubi closed (namely all projectors split), enriched by complexes of k-vector spaces. In particular for any two objects $\mathcal{E}$ and $\mathcal{F}$ we are given a complex $\mathrm{Hom}_{\mathcal{C}_X}^{\bullet}(\mathcal{E}, \mathcal{F})$ and this satisfies:

$$\mathrm{Hom}_{\mathcal{C}_X}(\mathcal{E}, \mathcal{F}) = H^0(\mathrm{Hom}_{\mathcal{C}_X}^{\bullet}(\mathcal{E}, \mathcal{F})).$$

DEFINITION 2.5.2. A k-linear space X is algebraic if $\mathcal{C}_X$ has a generator (with respect to cones and direct sums).

2.6.   The following holds:

PROPOSITION 2.6.1. *The category $\mathcal{C}_X$ has a generator if and only if there exists a dga A over k such that*

$$\mathcal{C}_X \cong \mathrm{Perfect}(A - \mathbf{mod}).$$

This proposition allows us to forget about categories and consider simply a dga (modulo a reasonable definition of derived Morita equivalence.)

2.7.   There is a nice relation with the notion of scheme:

THEOREM 2.7.1 (Bondal, Van den Bergh [**2**]). *Let X be a scheme of finite type over k, then $\mathcal{C}_X$ has a generator.*

The moral of the story in derived algebraic geometry is that *all spaces are affine.*

2.8.   The following example is due to Beilinson [**1**]. Consider $X = \mathbb{C}P^n$. Then

$$D^b(\mathrm{Coherent}(X)) = \mathrm{Perfect}(X) = \mathrm{Perfect}(A - \mathbf{mod}),$$

where $A = \mathrm{End}(\mathcal{O}(0) \oplus \cdots \oplus \mathcal{O}(n))$. A finite complex of finite-dimensional representations of A is the same as a finite complex of vector bundles over $X = \mathbb{C}P^n$.

2.9. We make a few more definitions.

DEFINITION 2.9.1. An algebraic k-linear space X is compact if for ever pair of objects $\mathcal{E}$ and $\mathcal{F}$ in $\mathcal{C}_X$ we have that

$$\sum_{i \in \mathbb{Z}} \text{rank Hom}(\mathcal{E}, \mathcal{F}[i]) < \infty.$$

In the language of the dga (A, d_A) this is equivalent to:

$$\sum_{i \in \mathbb{Z}} \text{rank } H^i(A, d_A) < \infty.$$

DEFINITION 2.9.2. We say that an algebraic k-linear space X is smooth if

$$A \in \text{Perfect}(A \otimes A^{\text{op}} - \mathbf{mod}).$$

DEFINITION 2.9.3. (a version of Bondal-Kapranov's) X is saturated if it is smooth and compact.

This is a good replacement for the notion of smooth projective variety.

2.10. The following concern the moduli of saturated spaces:

PROPOSITION 2.10.1 (Finiteness Property). *The moduli space of all sturated k-linear spaces X modulo isomorphisms can be written as a countable disjoint union of schemes of finite type:*

$$\coprod_{i \in I} S_i / \sim$$

modulo al algebraic equivalence relation.

2.11. Operations with saturated spaces. We have several basic operations inherited from the operations in algebras:

(i) Given a space X we can produce its opposite space X^{op} by sending the algebra A to its opposite A^{op}.

(ii) Given two spaces X and Y we can define their tensor product $X \otimes Y$ by multiplication of their coresponding dga's $A_X \otimes A_Y$.

(iii) Given X, Y we define the category $\text{Map}(X, Y) := A_X^{\text{op}} \otimes A_Y - \mathbf{mod}$.

(iv) There is a nice notion of gluing absent in algebraic geometry. Given $f : X \to Y$ (namely a $A_Y - A_X$-bimodule M_f) construct a new algebra $A_{X \cup_f Y}$ by considering upper triangular matrices of the form

$$\begin{pmatrix} a_x & m_f \\ 0 & a_y \end{pmatrix},$$

with $a_x \in A_X$, $a_y \in A_Y$ and $m_f \in M_f$.

Beilinson's theorem can be interpreted as stating that $\mathbb{C}P^n$ is obtained by gluing $n + 1$ points. This doesn't sound very geometric at first. An interesting outcome is that we get an unexpected action of the braid group in such decompositions of $\mathbb{C}P^n$.

Notice that the cohomology of a gluing is the direct sum of the cohomologies of the building blocks.

2.12. Duality theory. The story is again exceedingly simple for saturated spaces. There is a canonical Serre functor

$$S_X \in \mathrm{Map}(X, X)$$

satisfying

$$\mathrm{Hom}(\mathcal{E}, \mathcal{F})^* = \mathrm{Hom}(\mathcal{F}, S_X(\mathcal{E}))^*$$

In terms of the dga A_X we have that

$$S_X^{-1} = R\mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(A, A \otimes A^{\mathrm{op}}).$$

In the commutative case this reads

$$S_X = K_X[\dim X] \otimes \cdot\ ,$$

where $K_X = \Omega^{\dim X}$.

2.13. It seems to be the case that there is a basic family of objects from which (almost) everything can be glued up: Calabi-Yau spaces.

DEFINITION 2.13.1. A *Calabi-Yau saturated space* of dimension N is a saturated space X where the Serre functor S_X is the shifting functor $[N]$.

There are reflections of various concepts in commutative algebraic geometry such as positivity and negativity of curvature in the context of separated spaces. Here we should warn the reades that sometimes it is impossible to reconstruct the commutative manifold from its saturated space: several manifolds produce the same saturated space. Think for example of Fourier-Mukai transforms.

2.14. We also have a $\mathbb{Z}/2$-graded version of this theory. We require all complexes and shift funstors to be 2-periodic.

2.15. Examples of saturated spaces.
- Schemes. They form a natural family of saturated spaces.
- Deligne-Mumford stacks that look locally like X with a finite group Γ acting X. By considering (locally) the algebra $A = \mathcal{O}(X) \rtimes k[\Gamma]$ we can see immediately that they also furnish examples of saturated spaces.
- Quantum projective varieties. Suppose we start by considering an ample line bundle L over a smooth projective variety X. Say we have α_X a bi-vector field defined over $L - \mathbf{0}$ the complement of the zero section. We assume that α_X is invariant under $\mathbb{G}_m = \mathrm{GL}_1$. Deformation quantization implies that we can obtain a saturated non-commutive space over $k((\hbar))$.
- Landau-Ginzburg models. This is a $\mathbb{Z}/2$-graded example. Here we are given a map

$$f : X \longrightarrow \mathbb{A}^1, \quad f \in \mathcal{O}(X), \quad f \not\equiv 0,$$

where X is a smooth non-compact variety and $\mathbb{A}^1$ is the affine line. The idea comes from the B-model. The category $\mathcal{C}_{(X,f)}$ consists of *matrix factorizations*. In the affine case the objects are super-vector bundles $\mathcal{E} = \mathcal{E}^{\mathrm{even}} \oplus \mathcal{E}^{\mathrm{odd}}$ over X, together with a differential $d_{\mathcal{E}}$ such that

$$d_{\mathcal{E}}^2 = f \cdot \mathrm{Id}.$$

In local coordinates we are looking for a pair of matrices (A_{ij}) and (B_{ij}) so that

$$A \cdot B = f \cdot \mathrm{Id}.$$

We define $\mathrm{Hom}((\mathcal{E}, d_\mathcal{E}), (\tilde{\mathcal{E}}, d_{\tilde{\mathcal{E}}}))$ to be the complex of $\mathbb{Z}/2$-graded spaces $\mathrm{Hom}_{\mathcal{O}(X)}(\mathcal{E}, \tilde{\mathcal{E}})$ with differential

$$d(\phi) = \phi \cdot d_\mathcal{E} - d_{\tilde{\mathcal{E}}} \cdot \phi.$$

It is very easy to verify that $d^2 = 0$.

The generalization of the definition of $\mathcal{C}_{(X,f)}$ to the global case is due to Orlov [**11**]. Consider $Z = f^{-1}(0)$ as a (possibly nilpotent) subscheme of X. Then

$$\mathcal{C}_{(X,f)} := D^b(\mathrm{Coherent}(Z))/\mathrm{Perfect}(Z).$$

We expect $\mathcal{C}_{(X,f)}$ to be saturated if and only if $X_0 = \mathrm{Critical}(f) \cap f^{-1}(0)$ is compact. This is undoubtedly an important new class of triangulated categories.

- A beautiful final example is obtained by starting with a C^∞ compact symplectic manifold (X, ω), with very large symplectic form $[\omega] \gg 0$ (this can be arranged by replacing ω by $\lambda\omega$ with λ big.) The *Fukaya category* $\mathcal{F}(X, \omega)$ is defined by taking as its objects Lagrangian subamanifolds and as its arrows holomorphic disks with Lagrangian boundary conditions (but the precise definition is not so simple.)

  Paul Seidel [**13**] has proposed an argument showing that in many circumstances $\mathcal{F}(X, \omega)$ is saturated. This is a manifestation of *Mirror Symmetry* that says that $\mathcal{F}(X, \omega)$ is equivalent to $\mathcal{C}_{(X,\omega)^\vee}$, where $(X, \omega)^\vee$ is the mirror dual to (X, ω). While originally mirror symmetry was defined only in the Calabi-Yau case now we expect that the mirror dual to a general symplectic manifold will be dual to a category of Landau-Ginzburg type. In any case in many known examples the category is glued out of Calabi-Yau pieces.

2.16. In derived algebraic geometry there are a bit more spaces but much more identifications and symmetries that in ordinary algebraic geometry. For instance, when X is Calabi-Yau then $\mathrm{Aut}(\mathcal{C}_X)$ is huge and certainly much bigger that $\mathrm{Aut}(X)$. Another example: there are two different $K3$ surfaces X, X' that have the same $\mathcal{C}_X = \mathcal{C}_{X'}$, but in fact X and X' need not be diffeomorphic. Again think of the symmetries furnished by the Fourier-Mukai transform.

2.17. Cohomology. Let us return to the subject of cohomology . First, let us make an important remark. If A is a saturated dga then its Hochschild homology $H_\bullet(A, A) := H^\bullet(C_\bullet(A, A))$ is of *finite rank*, and the rank of the periodic cyclic homology is bounded by

$$\mathrm{rank} HP_\bullet(A) \leqslant \mathrm{rank} H_\bullet(A, A).$$

In the case in which A is a commutative space we have $HP_\bullet(A) \cong H^\bullet_{\mathrm{deRham}}(X)$ and $H_\bullet(A, A) \cong H^\bullet_{\mathrm{Hodge}}(X)$.

2.18. This motivates the following definition:

DEFINITION 2.18.1. For a saturated space X over k the Hodge to de Rham spectral sequence collapses if

$$\mathrm{rank}\, H_\bullet(A, A) = \mathrm{rank}\, HP_\bullet(A).$$

This happens if and only if for all $N \geqslant 1$ we have that $H^{\bullet}(C_{\bullet}^{\mathrm{red}}(A, A)[u]/u^N, \partial + uB)$ is a free flat $k[u]/u^N$-module.

2.19. The Degeneration Conjecture: For any saturated X the Hodge to de Rham spectral sequence collapses.[2]

This conjecture is true for commutative spaces, for quantum projective schemes and for Landau-Ginzburg models (X, f).

There are two types of proofs. The first uses Kähler geometry and resolution of singularities. This method seems very hard to generalize. The second method of proof uses finite characteristic and Frobenius homomorphisms, this is known and the *Deligne-Illusie method* and we expect it to work in general.

2.20. Let us assume this conjecture from now on. We have then a vector bundle over Spec $k[[u]]$ and we will call H_u the fiber. The sections of this bundle are $HC_{\bullet}^{-}(A)$. This bundle carries a canonical connection ∇ with a first or second order pole at $u = 0$.

In the $\mathbb{Z}$-graded case we have a $\mathbb{G}_m$-action,

$$\lambda \in k^{\times}, \quad u \mapsto \lambda^2 u,$$

defining the connection. The monodromy of the connection is 1 on $HP^{\mathrm{even}}(A)$ and -1 on $HP^{\mathrm{odd}}(A)$. In this case the connection has a first order pole at $u = 0$ and the spectrum of the residue of the connection is $\frac{1}{2}\mathbb{Z}$.

The $\mathbb{Z}/2$-graded case is even nicer, for the connection can be written in a universal way with an explicit but complicated formula containing the sum of five terms (see [**9**]). There is a reason for this connection to exists, and we explain it in two steps.

- Recall that if you have a family of algebras A_t over a parameter space you get a flat connection on the bundle $HP_{\bullet}(A_t)$ whose formulas tend to be very complicated.
- Consider the moduli stack of $\mathbb{Z}/2$-graded spaces. We have an action of $\mathbb{G}_m$:

$$(A, d_A) \mapsto (A, \lambda d_A).$$

  This corresponds in string theory to the renormalization group flow. The fixed points of this action contain $\mathbb{Z}$-graded spaces (but there are also the elements of fractional charge, and the quasihomogeneous singularities.) This corresponds to a scaling $u \mapsto \lambda u$ and therefore produces the desired connection. In this case we have a second order pole at $u = 0$.

2.21. The basic idea is that the connection ∇ replaces the Hodge filtration.

Notice that a vector space together with a $\mathbb{Z}$-filtration is the same as a vector bundle over $k[[u]]$ together with a connection with first order pole at $u = 0$ and with trivial monodromy. Of course, our connection is more complicated but it is generalizing the notion of filtration.

---

[2]See the great work of Kaledin that has appeared since, [**7, 8**].

2.22. We will use now the Chern character

$$\mathrm{ch} : K_0(X) \longrightarrow \{\text{covariantly constant sections of the bundle } H_u\}.$$

If in particular we consider the Chern class of $\mathrm{id}_X \in \mathcal{C}_{X \times X^{\mathrm{op}}}$ we have that $\mathrm{ch}(\mathrm{id}_X)$ is a covariantly constant pairing

$$H_u \overset{\cong}{\longrightarrow} H^*_{-u}$$

that is non-degenerate at $u = 0$.

2.23. Let us now describe the construction of an algebraic model for a string theory of type IIB.

Let X be a saturated algebraic space together with:

- A Calabi-Yau structure (this exists if the Serre functor is isomorphic to a shift functor)[3] To be precise a Calabi-Yau structure is a section $\Omega_u \in HC_\bullet^-(A) = \Gamma(H_u)$ such that as $\Omega_{u=0}$ is an element in Hochschild homology of X, that in turn gives a functional on $H_\bullet(A, d_A)$ making it into a Frobenius algebra.
- A trivialization of H_u compatible with the pairing

$$H_u \otimes H_{-u} \longrightarrow k$$

  and so the pairing becomes constant.

If the main conjecture is true and the Hodge to de Rham degeneration holds the from such X we can construct a 2-dimensional topological quantum field theory (TQFT). The state space $\mathcal{H}$ of the theory will be $\mathcal{H} = H_0$. As a part of the structure one gets maps

$$\mathcal{H}^{\otimes n} \longrightarrow H^\bullet(\bar{\mathcal{M}}_{g,n}, k).$$

We will not describe the whole (purely algebraic) construction here, but we shall just say that it provides solutions to holomorphic anomaly equations.

It seems to be the case that when we apply this procedure to the Fukaya category we recover the usual Gromov-Witten invariants for a symplectic manifold. It is very interesting to point out that the passage to stable curves is dictated by both, the degeneration of the spectral sequence, and the trivialization of the bundle. This fact was my main motivation for the Degeneration Conjecture.

In the $\mathbb{Z}$-graded case a Calabi-Yau structure requires a volume element Ω and a splitting of the non-commutative Hodge filtration compatible with the Poncaré pairing.

2.24. We can make a important definition.

DEFINITION 2.24.1. A non-commutative *pure Hodge Structure* over $\mathbb{C}$ is a holomorphic super vector bundle H_u over $D = \{|u| \leqslant 1, \ u \in \mathbb{C}\}$ with connection ∇ outside of $u = 0$, with a second order pole and a regular singularity[4] together with a covariantly constant finitely generated $\mathbb{Z}/2$-graded abelian group K_u^{top} for $u \neq 0$ such that $K_u^{\mathrm{top}} \otimes \mathbb{C} = H_u$.

---

[3]In a sense a Calabi-Yau structure *is* more or less a choice of isomorphism between Serre's functor and a shift functor.

[4]A regular singularity we mean that covariantly constant sections grow only polynomialy. Therefore under a meromorphic gauge transformation we end up with a first order pole.

2.25.   Recall that we have a pairing $H_u \cong H_{-u}^*$.

DEFINITION 2.25.1. A polarization on H_u is a covariantly constant pairing

$$K_u^{\text{top}} \otimes K_{-u}^{\text{top}} \longrightarrow \mathbb{Z}.$$

Morally this pairing should arise from the Chern character of an endofunctor.

2.26.   The positivity condition will be defined depending on a radius $R > 0$. For this we forget the $\mathbb{Z}/2$-grading. Consider the antiholomorphic involution

$$\sigma_R : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$$

$$\sigma_R(u) = -\frac{R^2}{\bar{u}}.$$

and consider H_u, $|u| \leqslant R$ and extend to $\mathbb{C}P^1$ by

$$H_{\sigma(u)} \cong \bar{H}_u^*,$$

on $|u| = R$ use the pairing and the complex conjugation arising from the lattice. The resulting bundle is denoted by H^{global}.

DEFINITION 2.26.1. We say that *the positivity condition at $R > 0$ holds* if H^{global} is holomorphically trivial.

If we abuse the letter H and write $H := \Gamma(\mathbb{C}P^1, H^{\text{global}})$ this space inherits a hermitian positive scalar product. In this setting this is the notion of polarization.

A recent result of C. Sabbah indicates that in the Landau-Ginzburg model this holds for sufficiently small R, so we have some sort of hidden parameters of scale.

2.27. The Non-commutative Hodge Conjecture.   Let X be a saturated space. Consider the map

$$\mathbb{Q} \otimes \text{Image}(K_0(\mathcal{C}_X) \longrightarrow \Gamma(H_u)) \to \mathbb{Q} \otimes \text{Hom}_{\text{NC−Hodge−structures}}(\mathbf{1}, H(X)).$$

The conjecture says that this map is an isomorphism.

The existence of a polarization in addition to the Hodge conjecture imply that the image of $K_0(\mathcal{C}_X)$ is

$$K_0(\mathcal{C}_X)/\text{numerical equivalence},$$

where numerical equivalence is the kernel of a pairing

$$\langle, \rangle : K_0 \otimes K_0 \longrightarrow \mathbb{R},$$

given by

$$\langle \mathcal{E}, \mathcal{F} \rangle = \chi(R\text{Hom}(\mathcal{E}, \mathcal{F})).$$

This pairing is neither symmetric nor anti-symmetric, so *a priori* it could have left and right kernels, but the Serre functor ensures us that they coincide.

2.28.   We can now go on á la Grothendieck and define a category of non-commutative pure motives. Consider X a saturated space over k. We define now

$$\underline{\text{Hom}}(X, Y) = \mathbb{Q} \otimes K_0(\text{Map}(X, Y))/\text{numerical equivalence}.$$

Ordinarily one takes algebraic cycles of all possible dimensions on the product of two varieties. In our situation we must be careful to add direct summands. This should be equivalent to a category of representations of the projective limit of some reductive algebraic groups over k. This non-commutative motivic Galois group responsible for such representations that is much larger than usual because of the $\mathbb{Z}/2$-gradings.

2.29. We can also discuss *mixed motives* in this context. They are a replacement of Voedvodski's triangulated category of mixed motives. We start again by considering saturated spaces X but now we want to define a new $\text{Hom}(X, Y)$ space as the K-theory spectrum of $\text{Map}(X, Y)$ (an infinite loop space.) We can canonically form the triangulated envelope. Notice that this construction contains ordinary mixed motives for usual varieties modulo the tensoring by $\mathbb{Z}(1)[2]$.

2.30. Crystalline cohomology and Euler functors. Consider an algebra A flat over $\mathbb{Z}_p$ (the p-adic integers) and saturated over $\mathbb{Z}_p$. We expect a canonical Frobenius isomorphism

$$\text{Fr}_p : H^\bullet(C_\bullet^{\text{red}}(A, A)((u)), \partial + uB) \xrightarrow{\cong} H^\bullet(C_\bullet^{\text{red}}(A, A)((u)), \partial + uB),$$

as $\mathbb{Z}_p((u))$-modules preserving connections.

Such isomorphism does exists in the commutative case, given a smooth X over $\mathbb{Z}_p$ we have

$$H^\bullet(X, \Omega_X; d) \cong H^\bullet(X, \Omega_X; pd)$$

for $p > \dim X$.

Using the holonomy of the connection $\nabla_{\frac{\partial}{\partial u}}$ we can go from u to pu and get an operator Fr_p with coefficients in $\mathbb{Q}_p$.

We can state

2.31. The non-commutative Weil conjecture. Let $\lambda_\alpha \in \text{Spec} F_p$ then

- $\lambda \in \bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$.
- For all $\ell \neq p$, then $|\lambda_\alpha|_\ell = 1$.
- $|\lambda_\alpha|_\mathbb{C} = 1$.

2.32. For the Landau-Ginzburg model $X = \mathbb{A}^1$ and $f = x^2$ we get that the cohomology is 1-dimensional, $\lambda \in \mathbb{Q}_p$ and

$$\lambda = \left(\frac{p-1}{2}\right)! (\mod p),$$

$$\lambda^4 = 1.$$

The periods for the Hodge structure are $\sqrt{2\pi}$. There is a reasonable hope for the existence of the Frobenius isomorphism (cf. the work of D. Kaledin.)

2.33. For every associative algebra A over $\mathbb{Z}/p$ there is a canonical linear map $H_0(A, A) \longrightarrow H_0(A, A)$ given by

$$a \mapsto a^p.$$

Recall that $H_0(A, A) = A/[A, A]$. There are two things to verify, that this map is well defined and that it is linear tThis is a very pleasant exercise). Moreover this map lifts to a map

$$H_0(A, A) \longrightarrow HC_0^-(A).$$

There is an explicit formula for this lift. For $p \geqslant 3$ we have

$$a \mapsto a^p + \sum_{\substack{n \text{ even}, \ p-3 \geqslant n \geqslant 2 \\ \sum i_\alpha = p}} (\text{coefficients}) a^{i_0} \otimes \cdots \otimes a^{i_n} u^{\frac{n-1}{2}} + \left(\frac{p-1}{2}\right)! \, a^{\otimes p} u^{\frac{p-1}{2}},$$

where the last coefficient is non-zero.

The formula for $p = 2$ reads:

$$a \mapsto a^2 + 1 \otimes a \otimes a \cdot u.$$

2.34. One can calculate various simple examples and this seems to suggest a potential mechanism for the degeneration of the Hodge-to-de Rham spectral sequence in characteristic $p > 0$. The situation is radically different for $p = 0$. This mechanism works as follows.

Let us consider polynomials in u:

$$H^\bullet(C_\bullet^{\mathrm{red}}(A, A)[u], \partial + uB).$$

There is no obvious spectral sequence in this case. What we have is a quasi-coherent sheaf over $\mathbb{A}^1$ with coordinate u.

In characteristic $p = 0$ we have that this sheaf vanishes when $u \neq 0$, namely $(C_\bullet^{\mathrm{red}}(A, A)[u, u^{-1}], \partial + uB)$ is acyclic (this is an early observation by Connes). So we have everything concentrated in an infinite dimensional stalk at $u = 0$, in the form of an infinite Jordan block, plus some finite Jordan blocks. It certainly looks like nothing resembling a vector bundle, it is very singular. The degeneration we seek would mean that we have no finite Jordan blocks. The situation is unfortunately quite involved.

In contrast, in finite characteristic p, it seems to be the case that the complex $(C_\bullet^{\mathrm{red}}(A, A)[u], \partial + uB)$ is actually a coherent sheaf, and it will look like a vector bundle if the desired spectral sequence degeneration occurs. The following conjecture explains why we obtain a vector bundle.

2.35. Conjecture. Let A be an algebra over $\mathbb{Z}_p$ flat over p. Let $A_0 = A \otimes \mathbb{Z}/p\mathbb{Z}$ over $\mathbb{Z}/p\mathbb{Z}$. Then $(C_\bullet^{\mathrm{red}}(A_0, A_0)[u, u^{-1}], \partial + uB)$ is canonically quasi-isomorphic to $(C_\bullet^{\mathrm{red}}(A_0, A_0)[u, u^{-1}], \partial)$ as $\mathbb{Z}/p[u, u^{-1}]$-modules.

In this conjecture there is no finiteness condition at all.

2.36. The reason for this is as follows. The complex $C_\bullet^{\mathrm{red}}(A, A)$ admits an obvious increasing filtration

$$\mathrm{Fil}_{\leqslant n} = A \otimes (A/1)^{\otimes \leqslant n-1} + 1 \otimes (A/1)^{\otimes n}.$$

Let $V := A/1[1]$, on $\mathrm{gr}(\mathrm{Fil}_n)$ we have that we can write $\partial + B$ as:

$$V^{\otimes n} \underset{1-\sigma}{\overset{1+\sigma+\cdots+\sigma^{n-1}}{\rightleftarrows}} V^{\otimes n}$$

where σ is the generator of $\mathbb{Z}/n\mathbb{Z}$ (this would be acyclic in characteristic 0). On the other hand ∂ is simply:

$$V^{\otimes n} \xrightarrow{1-\sigma} V^{\otimes n}.$$

For any free $\mathbb{Z}/p\mathbb{Z}$-module the above complex $\mathrm{gr}(\mathrm{Fil})$ with differential $\partial + B$ is acyclic if $(n, p) = 1$. At the same time if $n = kp$ the complex is canonically isomorphic to

$$V^{\otimes k} \xrightarrow{1-\sigma} V^{\otimes k}.$$

The hope is that some finite calculation of this sort could allow us to go deep into the spectral sequence and prove the desired degeneration.

2.37. We will finish making some remarks regarding the Weil conjecture. Let A be over $\mathbb{Z}$. It would be reasonable to hope the we can define local L-factors by

$$L_p(s) = \det\left(1 - \frac{\mathrm{Fr}_p}{p^s}\right)^{-1},$$

and we should get a sort of non-commutative L-function.

We define for a saturated space X an L-function:

$$L(X) = \prod L_p(s).$$

This $L(X)$ should satisfy

- The Riemann hypothesis. Namely its zeroes lie on $\Re(s) = \frac{1}{2}$.
- The Beilinson conjectures. They state that the vanishing order and leading coefficients at $s \in \frac{1}{2}\mathbb{Z}$, $s \leqslant 1/2$ are expressed via $K_{1-2s}(X)$ that should in turn be finite dimensional.

2.38. This L-function differs from the traditional L-function defined as a product over all points of a variety over finite fields $\mathbb{F}_q$ weighted by $1/q^s$. We can imagine an L-function defined on a saturated space X as the sum over objects of $\mathcal{C}_X$ weighted somehow. We do not know the exact form of these weights but we expect them to depend on certain stability condition. Such sums appear in string theory as sums over D-branes (for example in the calculation of the entropy of a black hole [**14**]). It is reasonable to imagine that there is a big non-commutative L-function one of whose limiting cases is arithmetic while the other is topological string theory.

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