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Vassiliev's Knot Invariants

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To my teacher I. M. Gelfand on the occasion of his 80th birthday

V. Vassiliev [V1, V2] defined a broad class of knot invariants using a kind of infinite-dimensional Alexander duality. Some comments on the main idea of V.Vassiliev are contained in § 1. The vector space V of Vassiliev's invariants has a filtration by natural numbers. We want to mention here several features of these invariants:

(1) The space of invariants of a fixed degree is finite-dimensional and there exists an a priori upper bound on its dimension. Moreover, this space is algorithmically computable. Unfortunately, the only known method to compute this space for a fixed degree takes a super-exponential time.
(2) For any of Vassiliev's invariants there exists a polynomial-time algorithm for computing this invariant for arbitrary knots.
(3) It is not hard to prove that if all Vassiliev's invariants for two knots coincide, then their (Alexander, Conway, Jones, Kaufmann, HOMFLY, etc.) polynomial invariants coincide. In a sense, Vassiliev's invariants are stronger than any invariant coming from the solution of the Yang-Baxter equation that can be deformed to the trivial solution. It seems likely that Vassiliev's invariants can distingush any two different knots.

Here the reader will find only a short exposition of the theory. We recommend a very detailed review by D. Bar-Natan [BN] containing most of what is written here and much more.

§0. Two formulas

Let $K : S^1 \hookrightarrow \mathbb{R}^3$ be a parametrized closed curve in $\mathbb{R}^3$. We shall write two different formulas for the simplest nontrivial Vassiliev invariant of the oriented knot $K(S^1)$. They arise from the perturbative Chern-Simons theory and from the (perturbative) Knizhnik-Zamolodchikov equation.

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First formula. Denote by $\omega(x)$ the closed 2-form $\frac{1}{8\pi}\varepsilon_{ijk}\frac{x^i dx^j \wedge dx^k}{|x|^3}$ on $\mathbb{R}^3\backslash\{0\}$ (the standard volume element on S^2 written in homogeneous coordinates). This form appears in the Gauss formula for the linking number of two nonintersecting oriented curves $L_1, L_2 \subset \mathbb{R}^3$:

$$\#(L_1, L_2) = \int_{x\in L_1, y\in L_2} \omega(x-y).$$

For the knot $K(S^1)$, where $S^1 = [0, 1]/\{0, 1\}$, the following sum

$$\int_{0<l_1<l_2<l_3<l_4<1} \omega(K(l_1) - K(l_3)) \wedge \omega(K(l_2) - K(l_4))$$
$$+ \int_{0<l_1<l_2<l_3<1,\, x\in\mathbb{R}^3\backslash K(S^1)} \omega(K(l_1) - x) \wedge \omega(K(l_2) - x) \wedge \omega(K(l_3) - x) - \frac{1}{24}$$

is an invariant, i.e., does not change when we continuously vary K in the class of embeddings.

The convergence of integrals above is almost clear (at least for real analytic knots). For example, for the second integral we can define a map from the integration domain to $(S^2)^3$ by sending (l_1, l_2, l_3, x) to the triple of directions from x to l_i. The image of this map is a real analytic subset of $(S^2)^3$ of dimension 6. The integral is equal to the part of volume of $(S^2)^3$ covered by this subset, counting, of course, multiplicities and orientations.

The invariance of the sum of 4-dimensional and 6-dimensional integrals above follows from Stokes's theorem and the following properties of the form ω:

(1) $\omega(\lambda x) = \mathrm{sgn}(\lambda)\omega(x)$ for any $\lambda \in \mathbb{R}^*$,
(2) $\int_{S^2} \omega(x) = 1$,
(3) $\int_{y\in\mathbb{R}^3\backslash\{x,z\}} \omega(x-y) \wedge \omega(y-z) = 0$ as a 1-form in variables $x \neq z \in \mathbb{R}^3$.

Second formula. Introduce coordinates t, z in $\mathbb{R}^3$, $t = x_1 \in \mathbb{R}$, $z = x_2 + ix_3 \in \mathbb{C}$. Suppose that our knot is such that $t \circ K$ is a Morse function on $S^1 \subset \mathbb{R}^2$. Let us consider the knot $K(S^1)$ as a graph of multivalued functions $\mathbb{R}^1 \longrightarrow \mathbb{C}^1$. The following sum of 2-dimensional and 0-dimensional integrals

$$\frac{1}{4\pi^2}\int_{t_1<t_2} \sum_{\{z,z'\}} \frac{dz_1 - dz_1'}{z_1 - z_1'} \wedge \frac{dz_2 - dz_2'}{z_2 - z_2'} \prod_{\substack{4\text{ points}\\(t_*, z_*)}} \frac{\text{orientation of the knot}}{\text{orientation arising from } t}$$
$$+ \frac{1}{48}(\text{ number of critical points of } t \circ K) - \frac{1}{24},$$

where the sum is taken over all choices of two pairs of points (t_i, z_i), (t_i, z_i'), $i = 1, 2$, on $K(S^1)$ such that points of the first pair alternate with points of the second pair, is again a knot invariant. The proof uses Stokes' theorem and the identity

$$\frac{dz_1 - dz_2}{z_1 - z_2} \wedge \frac{dz_2 - dz_3}{z_2 - z_3} + \frac{dz_2 - dz_3}{z_2 - z_3} \wedge \frac{dz_3 - dz_1}{z_3 - z_1} + \frac{dz_3 - dz_1}{z_3 - z_1} \wedge \frac{dz_1 - dz_2}{z_1 - z_2} = 0.$$

Both formulas give the same integer number.

§1. Vassiliev's invariants

It is clear that knot invariants (with values in some abelian group) are the same as locally constant functions on the space of embeddings of the standard circle S^1 into the 3-dimensional Euclidean space $\mathbb{R}^3$, or, equivalently, zero-cohomology classes.

Let us consider the space of embeddings as the complement in the infinite-dimensional vector space of all mappings from S^1 to $\mathbb{R}^3$ to the closed subspace of maps with self-intersections or with singular image. We intersect both spaces, the space of knots and its complement, with an appropriate generic family of finite-dimensional vector spaces of growing dimensions. For example, the spaces of trigonometric polynomial maps of fixed degrees will do. Then we can apply the usual Alexander duality. Of course, we can generalize it to the case of embeddings of an arbitrary manifold into Euclidean space of arbitrary dimension.

The main techical invention of V. Vassiliev is a very simple simplicial resolution of singularities of the space of nonembeddings, which allows us to compute the homology groups with closed support. This technique can be applied to a very broad class of situations, and in good cases it gives a complete description of the weak homotopy type of some functional spaces. The case of knot invariants turns out to be marginal. The spectral sequence arising in Vassiliev's approach does not converge well. The zero-degree part of its limit is a certain countable-dimensional subspace in the continuum-dimensional space of all cohomology classes.

Let us fix a nonnegative integer n. We want to define invariants of degree strictly less than n.

For any knot $K : S^1 \hookrightarrow \mathbb{R}^3$ and any family of nonintersecting balls B_1, $B_2, \dots, B_n \subset \mathbb{R}^3$ such that the intersection of any ball with $K(S^1)$ is the standard one (see Figure 1 on the next page), one can construct 2^n knots. These knots will be labeled by the sequences of $+1$ and -1 of length n. The knot $K_{\varepsilon_1, \dots, \varepsilon_n}$ is obtained from $K = K(S^1)$ by replacing, for each i, $1 \le i \le n$, such that $\varepsilon_i = -1$, the part of the knot in the interior of B_i by another standard sample (see Figure 2 on the next page). Of course, $K_{1,1,\dots,1}$ is the initial knot K.

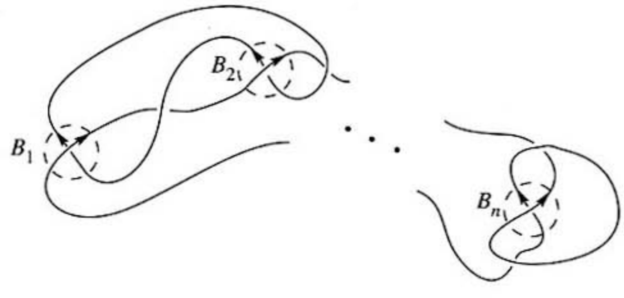

FIGURE 1

FIGURE 2

FIGURE 3

Let Φ be a knot invariant with the values in an abelian group A (for example, $A = \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \dots$).

DEFINITION. Φ is an invariant of degree less than n if for all K and $B_1, \dots, B_n$ as above the following equality holds:

$$\sum_{\varepsilon_1, \dots, \varepsilon_n} \varepsilon_1 \cdots \varepsilon_n \Phi(K_{\varepsilon_1, \dots, \varepsilon_n}) = 0.$$

Now we show that, for example, Jones invariants are contained in Vassiliev's invariants. Recall that the Jones invariant

(1) takes values in the group $\mathbb{Z}[t, t^{-1}]$,
(2) is defined for links, and
(3) satisfies the skein relation (see Figure 3).

THEOREM 1.1 (Birman-Lin). *The kth coefficient in the Taylor expansion of the Jones invariant at $t = 1$ is a Vassiliev's invariant of degree less than $(k+1)$.*

PROOF. The skein relation degenerates at $t = 1$ to the relation $\Phi(K_0) = \Phi(K_1)$. This fact easily implies that

$$\sum_{\varepsilon_1, \dots, \varepsilon_{k+1}} \varepsilon_1 \cdots \varepsilon_{k+1} \Phi(K_{\varepsilon_1, \dots, \varepsilon_{k+1}})$$

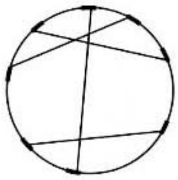

FIGURE 4

belongs to the $(k+1)$st degree of the ideal $(t-1)\mathbb{Z}[t, t^{-1}]$. Hence the kth Taylor coefficient of this alternating sum is zero. □

This proof can be generalized immediately to an arbitrary complex analytic family of solutions of the quantum Yang-Baxter equation containing the trivial solution. So, Vassiliev invariants are at least as strong as different kinds of polynomial invariants.

For $k = 0, 1, \ldots$ denote by V_k the vector space of $\mathbb{Q}$-valued invariants of degree less than $k+1$. Denote by $V = \bigcup_k V_k$ the space of all invariants of finite degree. It is clear that $V_0 \subset V_1 \subset V_2 \subset \cdots$ is a growing family of vector spaces. First of all, let us prove that all V_k are finite-dimensional.

LEMMA 1.1. $\dim(V_k/V_{k-1}) \le (2k-1)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1)$.

PROOF. For an invariant $\Phi \in V_k$ and any data $K, B_1, \ldots, B_k$ as above consider the value of the alternating sum

$$\nabla^k(\Phi) := \sum_{\varepsilon_1, \ldots, \varepsilon_k} \varepsilon_1 \cdots \varepsilon_k \Phi(K_{\varepsilon_1, \ldots, \varepsilon_k}).$$

We claim that this number does not change if we change the knot allowing self-intersections of the knot outside the union of balls B_i. It follows directly from the equation for invariants of degree less than $(k+1)$.

Hence these numbers depend only on k pairs of intervals on S^1, which are preimages of balls B_i under the map K. It is clear that these numbers are invariant under permutations of indices $1, \ldots, k$ and under the homotopy of families of intervals. We can connect two points on both components of $K^{-1}(B_i)$ by lines and obtain a finite family of chords (see Figure 4). Each family of k chords can be obtained in this way for some $K, B_1, \ldots, B_n$. So, we obtain a function $\nabla^k(\Phi)$ on families of k chords with all ends distinct. By definition, $\nabla^k(\Phi) = 0$ if and only if $\Phi \in V_{k-1}$. A simple computation shows that if we replace S^1 by $\mathbb{R}^1$, there are $(2k-1)!!$ topologically distinct families of pairs of points. It is clear that it gives an upper bound for the circle. □

Moreover, there are some linear relations between values of $\nabla^k(\Phi)$.

For a family S of $k-1$ chords, a point p on the circle S^1 different from all endpoints of chords in S, and a chord $a \in S$ we can construct four new familes S_1, S_2, S_3, S_4 of k chords. They are obtained by adding to S

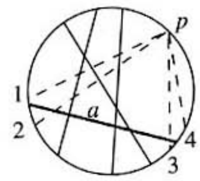

FIGURE 5

a new chord connecting p with a point on S^1 from the left and from the right side near both endpoints of a (see Figure 5).

LEMMA 1.2. *For any invariant $\Phi \in V_k$ its higher derivative $\nabla^k(\Phi)$ satisfies the following relations:*

(1) *for any family S of $k-1$ arcs, p, and a as above,*

$$\sum_{i=1}^{4}(-1)^i \nabla^k(\Phi)(S_i) = 0;$$

(2) *if the family S of k chords contains a chord that does not intersect other chords, then $\nabla^k(\Phi)(S) = 0$.*

PROOF. To obtain the first relation consider the situation when there exists a ball intersecting our knot in three intervals parallel to coordinate axes. Let the endpoints of a and p lie on these intervals. There are eight topologically different possible arrangements of such intervals in a ball. Our relation contains 16 terms. One can check easily that each configuration arises twice with opposite signs. The second relation is simpler: one can consider the situation when the knot consists of two spatially separated parts connected by two strings. The spinning of this pair of strings does not change the topological type of the knot. □

Vassiliev's invariants are closed under the multiplication $V_k \times V_l \subset V_{k+l}$. This fact follows easily from the definition and a generalization of the Leibniz formula to difference derivatives.

§2. Algebra of diagrams and the main theorem

It will be convenient for us to cut the circle at some point and to obtain a family of nonintersecting 2-element subsets of a (horizontal) line $\mathbb{R}^1$. We connect two points belonging to one subset by an arc above the line.

We shall use now notations from the previous lemma.

DEFINITION. For each $k = 0, 1, \ldots, \mathscr{A}_k$ is the vector space over $\mathbb{Q}$ generated by homotopy classes of families of k arcs with distinct ends on the line $\mathbb{R}^1$ modulo relations

(1) $\sum_{i=1}^{4}(-1)^i S_i = 0$,

(2) if S contains an arc that does not intersect other arcs, then $S = 0$.

The first three spaces are: $\mathscr{A}_0 = \mathbb{Q}$, $\mathscr{A}_1 = 0$, $\mathscr{A}_2 = \mathbb{Q}$. Lemma 1.2 means that V_k/V_{k-1} is a subspace of the space $\mathscr{A}_k^* = Hom(\mathscr{A}_k, \mathbb{Q})$.

REMARK. D. Bar-Natan denotes our $\mathscr{A}$ by $\mathscr{A}^r$ and our $\hat{\mathscr{A}}$ (see the end of this section) by $\mathscr{A}$.

We shall prove the following

THEOREM 2.1. $V_k/V_{k-1} = \mathscr{A}_k^*$.

This theorem means that differentials for zero cohomology groups of the space of knots in higher terms of Vassiliev's spectral sequence are trivial up to the torsion.

The proof of Theorem 2.1 requires some preparations.

LEMMA-DEFINITION. *For two families of arcs such that all endpoints of S_1 are smaller than all endpoints of S_2 (as real numbers) the formula $S_1 \times S_2 = S_1 \cup S_2$ defines the structure of associative algebra on $\mathscr{A} := \bigoplus_{k=0}^{\infty} \mathscr{A}_k$.*

We have to verify that the product is compatible with the additive relations between families of arcs. It is almost evident.

A slightly less trivial fact is that any family of chords on the circle S^1 defines an element of $\mathscr{A}$.

LEMMA 2.1. *If S is a family of k chords with distinct ends on S^1 and $p_1, p_2 \in S^1$ are two points different from the endpoints of S, then two families of arcs S_1, S_2 on $\mathbb{R}^1$ obtained by deleting p_1 or p_2 from the circle, define the same element of $\mathscr{A}_k$.*

PROOF. We can proceed step by step. Let S' is a family of arcs on $\mathbb{R}^1$. Let $x \in \mathbb{R}^1$ be the minimal endpoint of S' and $\{x, y\} \in S'$ be the corresponding arc. Consider the sum of relations (1) over all arcs $a \in S' \backslash \{x, y\}$ for the triple

$$(\text{point } p, \ \text{arc } a, \ \text{family of arcs } S' \backslash \{x, y\}).$$

It is easy to see that this sum is equal to the difference $S' - S''$, where S'' is obtained from S' by replacing x with a point on $\mathbb{R}^1$ right to all endpoints of S'. □

COROLLARY. *$\mathscr{A}$ is a commutative algebra.*

Let us consider the standard Euclidean space $\mathbb{R}^3$ as the product of the real line $\mathbb{R}^1$ with the complex line $\mathbb{C}^1$. We denote the corresponding coordinates by t, z, $t \in \mathbb{R}$, $z \in \mathbb{C}$. A knot K is called a Morse knot if the function $(t \circ K)$ is a Morse function on S^1. As before, for any n we can define the notion of an invariant of Morse knots of degree less than n.

Define a sequence of invariants of Morse knots with values in $\mathscr{A}_n \otimes \mathbb{C}$.

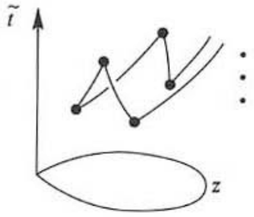

FIGURE 6

DEFINITION. $Z_0(K) = 1 \in \mathbb{Q} = \mathscr{A}_0 \subset \mathscr{A}_0 \otimes \mathbb{C}$. For $n > 0$

$$Z_n(K) = \int\limits_{\substack{\text{noncritical} \\ t_1 < t_2 < \cdots < t_n}} \cdots \int \sum_{(z_i, z_i')} (\text{the corresponding element of } \mathscr{A}_n) \times \prod_{i=1}^{n} \frac{d\log(z_i - z_i')}{2\pi\sqrt{-1}} (-1)^{\varepsilon},$$

where the summation is taken over all choices of nonordered pairs of distinct points on the sections $t = t_i$ of the knot K, ε denotes the number of points (t_i, z_i), (t_i, z_i') with the orientation in the negative direction. The element of $\mathscr{A}_n$ corresponding to n pairs of points on $K(S^1)$ is well defined by Lemma 2.1.

THEOREM 2.2. (1) *The integral defining* $Z_n(K)$ *is absolutely convergent,*

(2) $Z_n(K)$ *is invariant under the homotopy in the class of Morse knots,*

(3) $Z_n(K)$ *is an invariant of degree less than* $(n+1)$.

PROOF. For any Morse knot there exists a homeomorphism $\tilde{t}(t)$ of $\mathbb{R}$ that is a diffeomorphism everywhere outside the set of critical values of the map $(t \circ K)$ and that transforms the image of the knot to the union of finitely many lines transversal to horizontal planes $\{t, z \mid t \text{ is fixed}\}$ (see Figure 6). One can see that all possible divergences arise at the domain where a short interval (z_i, z_i') arises. If there are no other endpoints of other intervals in the small part of the knot connecting these two points, then the integral is zero by the relation (2). In all other cases the integral can be estimated above by

$$\text{const} \int\limits_{0 < x_1 < x_2 < \cdots < x_k < 1} \cdots \int dx_1 \frac{dx_2}{x_2} \frac{dx_3}{x_3} \cdots \frac{dx_k}{x_k}.$$

The last integral is absolutely convergent. Hence the first part of the theorem is proved.

Using the same estimates, one can verify that $Z_n(K)$ is a continuous function of Morse knot K. Now it is sufficient to verify that this function is locally constant for the variation of a Morse knot with all distinct critical values. In this situation we can use Stokes' formula since we integrate a

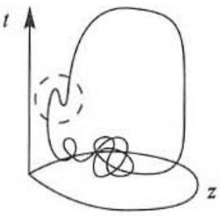

FIGURE 7

closed form over a manifold with the corners. The reader can check that the last relation from § 0 and relation (1) in the definition of $\mathcal{A}_n$ imply the homotopy invariance of $Z_n(K)$.

Part (3) is very simple. Using part (2) proved just now we can consider only the situation when all $(n+1)$ balls have nonintersecting images under the projection onto the vertical axis $\mathbb{R}$. Then the vanishing of the corresponding alternating sum is evident. □

The integral formula above arose from the attempts to understand the remarkable work of V. Drinfel'd [D1] on quasi-Hopf algebras. It follows more or less explicitly from another work of Drinfel'd [D2] that $Z_n(K) \in \mathcal{A}_n \subset \mathcal{A}_n \otimes \mathbb{C}$. We do not know any direct geometric proof of this fact.

It is easy to see that the set of complete invariants of Morse knot K consists of the topological type of the knot and the number of critical points of the map $(t \circ K)$. Let us consider the sequence of invariants for the special noncompact Morse knot K_0

$$K_0 : \mathbb{R}^1 \longrightarrow \mathbb{R} \times \mathbb{C}, \qquad K_0(x) = (x^3 - x, x).$$

Define an element $Z(K_0) \in \bar{\mathcal{A}} := \prod \mathcal{A}_n$ to be the sum $\sum_{k=0}^{\infty} Z_n(K_0)$. This element is invertible because the series for it starts from 1. One can replace here K_0 by any compact trivial Morse knot with four critical points.

THEOREM 2.3. *For any Morse knot K the element*

$$\tilde{Z}(K) := \sum_{k=0}^{\infty} Z_k(K) \times Z(K_0)^{-\frac{1}{2}\text{ number of critical points of }(t\circ K)} \in \bar{\mathcal{A}}$$

depends only on the topological type of K. For any n the first $(n+1)$ components $\tilde{Z}_0(K), \dots, \tilde{Z}_n(K)$ of $\tilde{Z}(K)$ together give the universal $\mathbb{Q}$-valued Vassiliev invariant of degree less than $(n+1)$.

PROOF. Consider the Morse knot such that its top part looks like the union of the middle part of K_0 with a vertical line far away from this curved part, and with a convex arc connecting these two parts (see Figure 7). The value of normalized invariant $\tilde{Z}(K)$ does not change when we replace the curved part on the top by a straight line. So, $\tilde{Z}(K)$ is a knot invariant. $\tilde{Z}_k(K)$, $k \leq n$, are invariants of degree less than $(n+1)$ because they are linear combinations of invariants $Z_k(K)$, $k \leq n$. The map $\nabla^n(\tilde{Z}_n) : \mathcal{A}_n \longrightarrow \mathcal{A}_n$ is the identity map. □

Theorem 2.1 follows immediately from Theorem 2.3. We have an explicit isomorphism

$$V_n \simeq \bigoplus_{k=0}^{n} \mathscr{A}_k^* .$$

Define a comultiplication $\Delta : \mathscr{A}_n \longrightarrow \bigoplus_{k+l=n} \mathscr{A}_k \otimes \mathscr{A}_l$ by the formula

$$\Delta(S) = \sum_{S' \subset S} S' \otimes (S \backslash S') .$$

A simple computation shows that this comultiplication is well defined and compatible with the algebra structure on $\mathscr{A}$. By the usual structure theorem (Milnor-Moore theorem) we conclude that $\mathscr{A}$ is canonically isomorphic to the symmetric algebra of the graded vector space of primitive elments

$$P = \mathrm{Prim}(\mathscr{A}) := \{a \in \mathscr{A} \mid \Delta(a) = 1 \otimes a + a \otimes 1\} .$$

The value of the universal knot invariant $\tilde{Z}(K)$ is always a group-like element of $\bar{\mathscr{A}}$, $\Delta(\tilde{Z}(K)) = \tilde{Z}(K) \otimes \tilde{Z}(K)$. Hence there is a space of algebraically independent invariants $\bigoplus P_k^*$ and all other rational-valued Vassiliev's invariants are polynomials in it.

D. Bar-Natan has calculated vector spaces P_k for $k \leq 8$ using a computer. The list of dimensions of these spaces is $0, 0, 1, 1, 2, 3, 5, 8, 12$ for $k = 0, 1, 2, 3, 4, 5, 6, 7, 8$ respectively. The computation time grows superexponentially, and the computation of P_9 seems to be quite hard.

We define a modified invariant of knot by the formula $\tilde{Z}_{\mathrm{mod}}(K) = \tilde{Z}(K) \times Z(K_0)$. One can check that again $\tilde{Z}_{\mathrm{mod}}(K)$ is a group-like element, and the modified primitive invariant $\log(\tilde{Z}_{\mathrm{mod}}(K)) \in \bar{P}$ is an additive invariant with the respect to the usual addition of knots. We can introduce the structure of the Hopf algebra on the space V of Vassiliev's invariants with the comultiplication coming from the addition of knots. Hopf algebra $\mathscr{A}$ is in a sense dual to V.

There are two obvious involutions on the set of knot types, corresponding to the change of the orientation of the Euclidean space $\mathbb{R}^3$, and of the knot S^1. The corresponding involutions on $\mathscr{A}$ are multiplication by $(-1)^{\mathrm{degree}}$ and the change of the orientation of the line $\mathbb{R}^1$.

One can extend the definition of Vassiliev invariants and all constructions above to the case of framed knots (= embeddings of the standard solid torus into $\mathbb{R}^3$). We define the group $\hat{\mathscr{A}_k}$ in the same way as $\mathscr{A}_k$ with relation (2) omitted.

The universal $\bar{\mathbb{Q}}$-valued Vassilev's invariant for framed knots takes values in the space $\bar{\hat{\mathscr{A}}} := \prod \hat{\mathscr{A}_k}$. The integral has to be regularized near critical points. The algebra $\hat{\mathscr{A}}$ is a commutative cocommutative Hopf algebra canonically isomorphic to $\mathscr{A}[t]$, where t is the variable of degree 1 corresponding to the unique family of single arcs.

§3. Algebra $\hat{\mathscr{A}}$ and universal enveloping algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, and $t \in \mathfrak{g} \otimes \mathfrak{g}$ be a symmetric element invariant under the adjoint action, i.e.,

$$t \in S^2(\mathfrak{g})^{\mathfrak{g}} \subset S^2(\mathfrak{g}) \subset \mathfrak{g} \otimes \mathfrak{g}.$$

Choose any decomposition of the tensor t into the sum of squares of elements of $\mathfrak{g}$,

$$t = \sum g_i \otimes g_i, \qquad g_i \in \mathfrak{g}.$$

Then for any finite sequence of letters $i, j, k, \dots$ such that every letter appears twice we can construct an element of the universal enveloping algebra $U\mathfrak{g}$. For example, the element corresponding to the word $ijkjlilk$ is

$$\sum_{i,j,k,l} g_i g_j g_k g_j g_l g_i g_l g_k.$$

With such a word one can associate also a family of arcs on the line. The invariance of t implies that the relation (1) in the definition of the group $\hat{\mathscr{A}}$ maps to zero. Our construction gives a morphism of algebras $\hat{\mathscr{A}} \longrightarrow U\mathfrak{g}$, which does not depend on the choice of the decomposition of t. In general this map is not compatible with coproduct structures on $\hat{\mathscr{A}}$ and $U\mathfrak{g}$. Since the tensor t is invariant, the image of this map belongs to the subalgebra $(U\mathfrak{g})^{\mathfrak{g}}$ of invariants of $U\mathfrak{g}$ under the adjoint action of $\mathfrak{g}$. It is clear that $(U\mathfrak{g})^{\mathfrak{g}}$ is the center $ZU\mathfrak{g}$ of the algebra $U\mathfrak{g}$.

As a corollary, we obtain a map $\mathscr{A} \longrightarrow ZU\mathfrak{g}/\hat{t}ZU\mathfrak{g}$, where $\hat{t} = \sum g_i g_i$.

Any linear functional $\chi : ZU\mathfrak{g} \to \mathbb{C}$ gives an infinite sequence of $\mathbb{C}$-valued Vassiliev's invariants for ordinary and framed knots. For example, any finite-dimensional representation ρ of the Lie algebra $\mathfrak{g}$ gives the trace functional

$$\chi_\rho(x) = \mathrm{tr}(\rho(x)), \qquad x \in (U\mathfrak{g})^{\mathfrak{g}} \subset U\mathfrak{g}.$$

All invariants up to degree 7 computed by D. Bar-Natan are linear combinations of invariants arising from simple Lie algebras of types A, B, C, D. Using this construction one can obtain the estimate

$$\dim(V_n) > e^{c\sqrt{n}}, \qquad n \to +\infty,$$

for any positive constant $c < \pi\sqrt{2/3}$ (see [BN, Exercise 6.14]).

§4. Problem of orientation

The construction described above gives a lot of Vassiliev's invariants. Unfortunately, all such invariants constructed from semisimple Lie algebras cannot detect the change of orientation of knots.

The change of orientation corresponds to the passing to the dual representation. Any simple Lie algebra admits an automorphism (Cartan involution) acting as the conjugation on the set of irreducible representations.

At the moment it is not clear whether or not ($\mathbb{Q}$-valued) Vassiliev invariants can detect the change of orientation. All the standard polynomial invariants also cannot do it. Nevertheless, I hope that there are such invariants. Probably, the first one has degree 9. Recall that the first noninvertible knot was discovered only in 1964 and the simplest diagram for noninvertible knot has eight self-intersections. D. Bar-Natan conjectured that all invariants arise from the classical Lie algebras. His conjecture implies that Vassiliev invariants cannot detect noninvertibility.

§5. Relation with the perturbative Chern-Simons theory

Three-dimensional topological field theory with Chern-Simons action is now known to be solvable (see [W]). The corresponding Feynman integral is defined for integral values k of some parameter called level, and the value of this integral is an algebraic number. From the physical point of view there must exist an asymptotic formula for the value of the Feynman integral for large k in terms of the set of representations of the fundamental group of 3-manifolds into a compact Lie group (=critical points for the Chern-Simons action).

The perturbation theory was discussed in [GMM] for two-loop diagrams and was worked out completely by S. Axelrod and I. Singer [AS] and by myself. It turns out that gauge fields, ghost fields, and anti-ghost fields can be considered as differential forms of degrees $0, 1, 2$. Feynman diagrams (for knots in $\mathbb{R}^3$) are connected 3-valent graphs Γ with a fixed oriented simple closed cycle C in it and an *orientation* that could be defined in several ways (see below).

Feynman integral associated with a diagram Γ and a knot $K(S^1) \subset \mathbb{R}^3$ is

$$I_\Gamma(K) = \int\limits_{\substack{\text{straight embeddings} \\ (\Gamma, C) \hookrightarrow (\mathbb{R}^3, K(S^1))}} \prod_{\text{edges } \in \Gamma \setminus C} \omega(\text{edge}) ,$$

where the words "straight embeddings" denote embeddings of the graph Γ into $\mathbb{R}^3$ such that edges from C are parts of $K(S^1)$ with compatible orientations and other edges are straight line intervals in $\mathbb{R}^3$. The orientation of the graph Γ is a way to fix a sign in the definition of the corresponding integral. One can describe it as

1) the ordering on the set of vertices plus the choice of orientation of each edge from $\Gamma \setminus C$ modulo even permutations and an even number of changes, or, equivalently, as

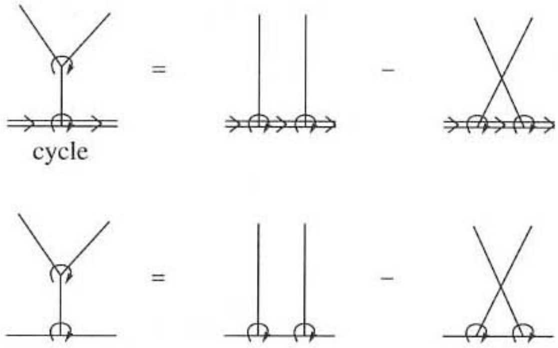

FIGURE 8

2) the choice of cyclic order on 3-element sets of edges coming to each vertex, modulo an even number of changes.

Of course, chord diagrams from Vassiliev's theory are examples of Feynman diagrams for CS theory. One can define a vector space generated by equivalence classes of Feynman diagrams with orientation modulo relations $(\Gamma, -\mathrm{or}) = -(\Gamma, \mathrm{or})$ and a kind of "Jacobi identity" (see Figure 8). D. Bar-Natan has proved that this space is canonically isomorphic to $\hat{\mathscr{A}}$.

The perturbative Chern-Simons theory for *all* compact Lie groups near the trivial representation gives the following series

$$1 + \sum_{\text{equivalence classes of } \Gamma} \frac{\Gamma}{\#\mathrm{Aut}(\Gamma)} I_\Gamma(K) \in \mathscr{A} \otimes \mathbb{R}.$$

The proof of the topological invariance of this formula is somewhat more technical than for the case with the Knizhnik-Zamolodchikov equation, but it uses only three properties of the form ω mentioned in Section 0.

Question. Are invariants arising from CS theory the same as invariants arising from the KZ equation?

The approach with the Gauss form has a straightforward generalization

1) to the case of higher-degree cohomology classes of the space of embeddings,

2) to the case of embeddings into $\mathbb{R}^n$ with $n \geq 3$. It implies collapsing of the spectral sequence of Vassiliev in the 3-dimensional case and "complete" computation of $H^*(\mathrm{Emb}(S^1, \mathbb{R}^n), \mathbb{Q})$ for $n \geq 4$.

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