VIRASORO ALGEBRA AND TEICHMÜLLER SPACES

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The supposition of the existence of a connection between the objects mentioned in the title of this note was stated by Manin [1]. It will be shown here that the Teichmüller space is (infinitesimally) the space of double cosets of the "Virasoro group" with respect to the analogs of compact and discrete subgroups, and the action of the Virasoro algebra on the determinant sheaves will be described. Similar results were obtained independently by A. A. Beilinson and V. V. Shekhtman. I thank A. A. Beilinson for helpful discussions.

1. Infinite-Dimensional Space of Moduli

All manifolds will be algebraic over $\mathbf{C}$.

We denote by $M_{g,1^k}$ the set of equivalence classes of objects of the form (C, a complete nonsingular curve of genus g, $p \in C$, a point; t, the k-jet of a parameter at p), where the parameter at the point t is an element of the completed local ring $\hat{\mathcal{O}}_p$, having a zero of first order at p.

For $k \geq 1$, $M_{g,1^k}$ is the set of points of a smooth algebraic variety, the space of moduli, and the Kodaira–Spencer map $T_{(c,p,t)}(M_{g,1^k}) \to H^1(C, \tau_c \otimes \mathcal{I}_p^{k+1})$, where τ_c is the tangent bundle, $\mathcal{I}_p$ is the sheaf of ideals of p, is an isomorphism.

Passing to the limit as $k \to \infty$, we get an infinite-dimensional smooth manifold $M_{g,1^\infty} = \varprojlim M_{g,1^k}$, whose points will be triplexes X (C being a curve, p being a point, t being a parameter at p). The tangent space $T_X(M_{g,1^\infty})$ is equal to

$$\varprojlim H^1(C, \tau_c \otimes \mathcal{I}_p^k) = \varprojlim \mathbf{C}((t))\frac{\partial}{\partial t} / t^k\mathbf{C}[[t]]\frac{\partial}{\partial t} + D = \mathbf{C}((t))\frac{\partial}{\partial t}/D,$$

where D is the space of Laurent series at p of vector fields which extend to $C \setminus \{p\}$. Thus, we get a map φ from $V = \mathbf{C}((t))\partial/\partial t$ to the space of vector fields on $M_{g,1^\infty}$.

Proposition. φ is a homomorphism of Lie algebras.

It is easy to see that the subalgebra $V_{-1} = \mathbf{C}[[t]]\partial/\partial t$ acts trasitively along fibers of the projection $M_{g,1^\infty} \to M_g$, and thus we get the assertion formulated at the beginning of the note.

2. Determinants of Infinite-Dimensional Spaces (Formal Version)

For a finite-dimensional vector space S we set $\det(S) = \Lambda^{\dim S}(S)$ which is a one-dimensional space.

Let T be a vector space over the field k with the topology of the space of Laurent series. In $T \approx k((t))$ two classes of subspaces are singled out: a) compact subspaces $K \subset T$ such that $\dim(k[[t]]/K \cap k[[t]]) < \infty$, $\dim(K/K \cap k[[t]]) < \infty$; b) discrete subspaces $D \subset T$ such that for some, and hence for any compact K, $h_K{}^0(D) = \dim(K \cap D) < \infty$, $h_K{}^1(D) = \dim(T/K + D) < \infty$. These definitions are independent of the choice of isomorphism of T with k((t)).

The Grassmanian $Gr(T) = \{D \subset T$, D being discrete$\}$ is covered by affine coordinate charts: to the decomposition $T = D \oplus K$ into a direct sum corresponds the open cell $\mathcal{U}_{D,K} = \{D' \subset T, \exists A: D \to K, D' = \{x \oplus Ax\}\}$. The Lie algebra $\mathfrak{gl}$ (T) of continuous endomorphisms of T acts by vector

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fields on Gr(T): in the domain $\mathcal{U}_{D,K}$ to the operator $F = \begin{pmatrix} F^{DD} & F^{DK} \\ F^{KD} & F^{KK} \end{pmatrix}$ corresponds the vector field

$$L_F A = F^{KD} + F^{KK} A - AF^{DD} - AF^{DK} A.$$

We fix a compact $K \subset T$. We define a line bundle $\det_K$ in the domain $\mathcal{U}_K = \{D \in \mathrm{Gr},\ D \cap K = 0\}$, as follows: to the subspace D corresponds $\det(T/(D + K))^{-1}$. If $K \supset K'$, then we can extend $\det_K$ to $\mathcal{U}_{K'} \supset \mathcal{U}_K$ by the formula $\det_K = \det_{K'} \otimes \det(K/K')$. It is easy to see that we get an invertible sheaf $\det_K$ on all of Gr(T), and between sheaves corresponding to different K, there exists a canonical isomorphism (up to multiplication by a constant factor).

Each decomposition $T = D \oplus K$ defines a 2-cocycle on $\mathfrak{gl}(T)$: $\omega_{D,K}(F_1 F_2) = \mathrm{Tr}(F_1^{DK} F_2^{KD} - F_2^{DK} F_1^{KD})$. In the domain $\mathcal{U}_{D,K}$ the bundle $\det_K$ is canonically trivialized, and we define the action of the central extension of $\mathfrak{gl}(T)$ by the cocycle $\omega_{D,K}$ by the formula $\mathfrak{L}_F = L_F + \mathrm{Tr}(F^{DK} A)$, $F \in \mathfrak{gl}$. In order to define the action of this extension on another domain $\mathcal{U}_{D',K'}$, we note that there exists a unique primitive $\alpha \in C^1(\mathfrak{gl}(T))$, $d\alpha = \omega_{D,K} - \omega_{D',K'}$, and in $\mathcal{U}_{D',K'}$ we set $\mathfrak{L}_F + \mathfrak{L}_F' + \alpha(F)$. It is easy to see that the action of the extension $\widetilde{\mathfrak{gl}}(T)$ on sections of the sheaf $\det_K$ is well defined.

3. Action of the Virasoro Algebra

We fix $j \in \mathbf{Z}$. Let $T = \mathbf{C}((t))dt^j$, $K = \mathbf{C}[[t]]dt^j$. We map $M_{g,1^\infty}$ into Gr(T) as follows: to the point $(C, p, t) \in M_{g,1^\infty}$ corresponds the subspace of Laurent series of j-differentials which extend to $C \setminus \{p\}$. It is easy to see that this is an imbedding, compatible with the action of $V \subset \mathfrak{gl}(T)$. The bundle $\lambda_j = \det_K$, lifted from Gr(T) to $M_{g,1^\infty}$, is the lift from M_g to $M_{g,1^\infty}$ of the standard determinant bundle, whose fiber at the point $(C, \text{a curve}) \in M_g$ is $\det(H^0(C, \tau_c^{\otimes -j})) \otimes (\det(H^1(C, \tau_c^{\otimes -j})))^{-1}$.

According to point 2 we get an action on λ_j of a one-dimensional extension of V, i.e., a Virasoro algebra. One can show that a central element c in the usual basis of $\mathrm{Vir} = V \oplus \mathbf{C}\cdot c$, $[t^{i+1}\partial/\partial t, t^{i+1}\partial/\partial t] = (j - i)t^{i+j-1}\partial/\partial t + \delta_{i,-j}(j^3 - j)/6\cdot c$ acts by multiplication by $c_j = 6j^2 - 6j + 1$.

COROLLARY. In the sheaf $\lambda_1^{-c_j} \otimes \lambda_j$ there is a flat connection which is the lift of the flat connection in the corresponding sheaf on M_g. Mumford proved (cf. [3]) that this sheaf is trivial.

4. Remarks

1) In points 1 and 3 one can assume that several points with local parameters are distinguished on the curve.

2) One can also replace the Lie algebra of vector fields by the algebra of currents (cf. [2]).

3) It would be interesting to investigate the representations of Vir which arise on the spaces of sections of the sheaves λ_j.

LITERATURE CITED

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