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Connected components of the moduli spaces of Abelian differentials with prescribed singularities

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Abstract. Consider the moduli space of pairs (C, ω) where C is a smooth compact complex curve of a given genus and ω is a holomorphic 1-form on C with a given list of multiplicities of zeroes. We describe connected components of this space. This classification is important in the study of dynamics of interval exchange transformations and billiards in rational polygons, and in the study of geometry of translation surfaces.

Contents

1. Introduction

1.1. Stratification of the moduli space of Abelian differentials. For integer $g \geq 2$ we define the space $\mathcal{H}_g$ as the moduli space of pairs (C, ω) where C is a smooth compact complex curve of genus g and ω is a holomorphic 1-form on C (i.e. an Abelian differential) which is not equal identically to zero. Obviously, $\mathcal{H}_g$ is a complex algebraic orbifold (in other words, a smooth stack) of dimension $4g-3$. It is fibered over the moduli space $\mathcal{M}_g$

of curves with the fiber over $[C] \in \mathcal{M}_g$ equal to the punctured vector space $\Gamma(C, \Omega^1_C) \setminus \{0\}$ (modulo the action of a finite group $Aut(C)$).

Orbifold $\mathcal{H}_g$ is naturally stratified by the multiplicities of zeroes of ω. Let $k_1, \dots, k_n$ be a sequence of positive integers, $n \geq 1$ with the sum $\sum_i k_i$ equal to $2g-2$. We denote by $\mathcal{H}(k_1, \dots, k_n)$ the subspace of $\mathcal{H}$ consisting of equivalence classes of pairs (C, ω) where ω has exactly n zeroes and their multiplicities are equal to $k_1, \dots, k_n$ (for some ordering of zeroes). Our notation is symmetric, $\mathcal{H}(k_1, \dots, k_n) = \mathcal{H}(k_{\pi(1)}, \dots, k_{\pi(n)})$ for any permutation $\pi \in \mathfrak{S}_n$. One has then

$$\mathcal{H}_g = \bigsqcup_{\substack{n,\, (k_1,\dots,k_n) \\ k_1 \leq \cdots \leq k_n \\ k_1+\cdots+k_n = 2g-2}} \mathcal{H}(k_1, \dots, k_n).$$

Thus, we have a stratification of the moduli space $\mathcal{H}_g$. It is well-known that each stratum $\mathcal{H}(k_1, \dots, k_n)$ is an algebraic orbifold of dimension

$$\dim_{\mathbb{C}} \mathcal{H}(k_1, \dots, k_n) = 2g + n - 1 \tag{1}$$

(see [11], [17], [19]). Moreover, it carries a natural holomorphic affine structure. Here is the description of this structure.

With any pair (C, ω) we associate an element $[\omega] \in H^1(C, Zeroes(\omega); \mathbb{C})$, the cohomology class of pair $(C, Zeroes(\omega))$ represented by closed complex-valued 1-form ω. Locally near each point x of $\mathcal{H}(k_1, \dots, k_n)$ we can identify cohomology spaces $H^1(C, Zeroes(\omega); \mathbb{C})$ with each other using the Gauss–Manin connection. (For points $x = (C, \omega)$ with nontrivial symmetry we would need to pass first to a finite covering of the neighborhood of x). Thus, we obtain (locally) a *period mapping* from $\mathcal{H}(k_1, \dots, k_n)$ to a domain in a complex vector space. It is well known that this mapping is holomorphic and locally one-to-one. The pullback of the tautological affine structure on $H^1(C, Zeroes(\omega); \mathbb{C})$ gives an affine structure on $\mathcal{H}(k_1, \dots, k_n)$. (See also [6] for a related construction concerning smooth closed 1-forms.)

In general, the strata $\mathcal{H}(k_1, \dots, k_n)$ are not fiber bundles over the moduli space of curves $\mathcal{M}_g$. For example, the dimension of the stratum $\mathcal{H}(2g-2)$ for $g \geq 2$ equals $2g$, while dimension of the moduli space of curves $\mathcal{M}_g$ equals $3g-3$ which is strictly larger than $2g$ for $g \geq 4$.

The goal of this paper is to describe the set of connected components of all strata $\mathcal{H}(k_1, \dots, k_n)$. Surprisingly, we found that the answer is quite complicated, some strata have up to 3 connected components. The full description of the connected components of strata is given in Sect. 2.3. This result was announced in the paper [7].

Remark 1. For any *sequence* $(k_1, \dots, k_n)$ of positive integers $k_i \geq 1$ such that $\sum_i k_i = 2g-2$ we define $\mathcal{H}^{num}(k_1, \dots, k_n)$ the moduli space of Abelian differentials on curves with *numbered* zeroes such that the first zero has multiplicity k_1 etc. Orbifold $\mathcal{H}^{num}(k_1, \dots, k_n)$ is a finite covering of $\mathcal{H}(k_1, \dots, k_n)$. One can show that preimage of any connected component of $\mathcal{H}(k_1, \ldots, k_n)$ in $\mathcal{H}^{num}(k_1, \ldots, k_n)$ is connected, i.e. the classification of connected components is essentially identical in both cases, no matter whether the zeroes are numbered or not.

1.2. Applications to interval exchange transformations. The motivation for our study came from dynamical systems, namely from the theory of so called interval exchange transformations.

First of all, there is an alternative description of $\mathcal{H}_g$ in terms of differential geometry. Outside of zeroes of an Abelian differential ω one can chose locally a complex coordinate z in such way that $\omega = dz$. This coordinate is defined up to a constant, $z = z' + const$, so it determines a natural flat metric $|dz^2|$ on the Riemann surface C punctured at zeroes of ω. At zero of ω of multiplicity k_i the metric has a conical singularity with the cone angle $2\pi(k_i + 1)$. This flat metric has trivial holonomy in the group $SO(2)$: a parallel transport of a vector tangent to the Riemann surface C along any closed path avoiding conical singularities brings the vector back to itself. Thus, choosing a tangent direction at any nonsingular point we can extend it using the parallel transport to all other nonsingular points, getting a smooth distribution on the punctured Riemann surface. This distribution is integrable: it defines a foliation with singularities at the conical points. The oriented foliation defined by the positive real direction x in coordinate $z = x + iy$ is called *horizontal*; the oriented foliation defined by the positive purely imaginary direction y is called *vertical*. At a conical point with a cone angle $2\pi(k_i + 1)$ one gets $k_i + 1$ horizontal (vertical) directions.

Conversely, a flat structure with trivial $SO(2)$-holonomy having several cone type singularities plus a choice of, say, horizontal direction uniquely determines a complex structure on the surface, and an Abelian differential in this complex structure.

An Abelian differential ω defines also two smooth closed real-valued 1-forms $\omega_v = Re(\omega)$ and $\omega_h = Im(\omega)$ on C considered as a smooth oriented two-dimensional surface M^2. The vertical and horizontal foliations described above are the kernel foliations of the 1-forms ω_v and ω_h correspondingly.

Conversely, let M^2 be a compact smooth oriented surface of genus g with a pair of closed 1-forms ω_v, ω_h such that $\omega_v \wedge \omega_h > 0$ everywhere on M^2 outside of a finite set. Then there is a unique point $[(C, \omega)] \in \mathcal{H}_g$ producing such M^2 with forms ω_v, ω_h.

There is a non-holomorphic continuous action on $\mathcal{H}_g$ of the group $GL(2, \mathbb{R})_+$ (the group of matrices with positive determinants). In terms of pairs of 1-forms $(\omega_v, \omega_h) = (Re(\omega), Im(\omega))$ this action is given simply by linear transformations

$$(\omega_v, \omega_h) \mapsto (a\omega_v + b\omega_h, c\omega_v + d\omega_h).$$

Later in this text we shall use all descriptions of $\mathcal{H}_g$: the algebro-geometric one, the one in terms of flat surfaces with a choice of the horizontal direction, and the one in terms of pairs of measured oriented foliations.

It was proved by H. Masur, see [11] and by W.A. Veech (see [17]) that for a generic (with respect to the Lebesgue measure) point of any stratum $\mathcal{H}(k_1, \ldots, k_n)$ the horizontal foliation (and also the vertical one) is uniquely ergodic. Let us take any interval I on the surface M^2 transversal to the vertical foliation, with the canonical induced length element. The first return map $T : I \longrightarrow I$ (defined almost everywhere on I) is an interval exchange map, i.e. a one-to-one map with finitely many discontinuity points such that the derivative of T is equal almost everywhere to $+1$. The interval exchange map is parametrized by the number m of maximal open subintervals $(I_i)_{i=1,\ldots,m}$ of continuity of the transformation T, by the sequence of lengths of these subintervals $\lambda_1, \ldots, \lambda_m$ where $\lambda_i > 0$, $i = 1, \ldots, m$, and by a permutation $\pi \in \mathfrak{S}_m$ describing the order in which intervals $T(I_i)$ are placed in I: the k-th interval is sent to the place $\pi(k)$. It follows from the unique ergodicity that the permutation π is *irreducible*, which means in our context that $\forall k = 1, \ldots, m-1$ we have $\pi(\{1, \ldots, k\}) \neq \{1, \ldots, k\}$.

Conversely, for any interval exchange map T one can construct an Abelian differential ω and a horizontal interval I on a complex curve C such that the first return map to I along the vertical foliation of ω is the given map T, see [11], [17]. Though the Abelian differential ω is not uniquely determined by the interval exchange map, the collection of multiplicities of zeroes $(k_1, \ldots, k_n)$ of ω and even the connected component of the moduli space $\mathcal{H}(k_1, \ldots, k_n)$ containing point $[(C, \omega)]$ are uniquely determined by the permutation π, see [11], [17]. Thus one may decompose the set of irreducible permutations into groups called *extended Rauzy classes* corresponding to connected components of the strata $\mathcal{H}(k_1, \ldots, k_n)$.

The application of our result to the theory of interval exchange maps is based on the corollary of the fundamental theorem of H. Masur [11] and W. Veech [17] which we present in the next section. The corollary is as follows: dynamical properties of a generic interval exchange map depend only on the extended Rauzy class of the permutation of subintervals. Genericity is understood here with respect to the Lebesgue measure on the space $\mathbb{R}_+^m$ parameterizing lengths $(\lambda_i)_{1\le i\le m}$ of subintervals under exchange.

Actually, the extended Rauzy classes can be defined in purely combinatorial terms, see Appendix 5.4 for details. Thus the problem of the description of the extended Rauzy classes, and hence, of the description of connected components of the strata of Abelian differentials, is purely combinatorial. However, it seems to be very hard to solve it directly. Still, for small genera the problem is tractable. W. Veech showed in [19] that the stratum $\mathcal{H}(4)$ has two connected components. P. Arnoux proved that the stratum $\mathcal{H}(6)$ has three connected components.

In the present paper we give a classification of extended Rauzy classes using not only combinatorics but also tools of algebraic geometry, topology and of dynamical systems.

1.3. Ergodic components of the Teichmüller geodesic flow. There is a natural immersion of the moduli space of Abelian differentials into the moduli space of holomorphic quadratic differentials: we associate to an Abelian differential its square. With every quadratic differential we can again associate a flat metric with conical singularities. The group $GL(2,\mathbb{R})_+$ acts naturally on this larger moduli space as well; this action leaves the immersed moduli space of Abelian differentials invariant, moreover, on the immersed subspace it coincides with the action defined in the previous section. This action preserves the natural stratification of the moduli space of quadratic differentials by multiplicities of zeroes.

The action of the diagonal subgroup of $SL(2,\mathbb{R}) \subset GL(2,\mathbb{R})_+$ on the moduli space of quadratic differentials can be naturally identified with the geodesic flow on the moduli space of curves for Teichmüller metric (which is piecewise real-analytic Finsler metric on $\mathcal{M}_g$). Group $SL(2,\mathbb{R})$ preserves the hypersurface in the moduli space of quadratic differentials consisting of those ones for which the associated flat metric has the total area equal to 1.

Numerous important results in the theory of interval exchange maps, of measured foliations, of billiards in rational polygons, of dynamics on translation surfaces are based on the following fundamental observation by H. Masur [11] and W. Veech [17]:

Theorem (H. Masur; W. Veech). *The Teichmüller geodesic flow acts ergodically on every connected component of every stratum of the moduli space of quadratic differentials with total area equal to* 1*; the corresponding invariant measure on the stratum is a finite Lebesgue equivalent measure.*

Thus our classification of connected components of the strata of Abelian differentials gives the classification of ergodic components of the Teichmüller geodesic flow on the strata of squares of Abelian differentials in the moduli space of quadratic differentials.

The complete classification of connected components of strata of quadratic differentials is in progress (see an announcement in [8]). For example, the stratum of those quadratic differentials on a curve of genus $g = 4$, which cannot be represented as a square of an Abelian differential, and which have a single zero of degree 12, has two connected components, but at the moment a topological invariant which would distinguish representatives of these two connected components is not known yet.

In general, it seems to be very interesting to describe invariant submanifolds (closures of orbits, invariant measures) for the action of $GL(2,\mathbb{R})_+$ on the moduli spaces. Connected components of the strata are only the simplest invariant submanifolds, there are many others. For example the Teichmüller disks of Veech curves form the smallest possible invariant submanifolds.

One can use a submanifold invariant under the action of $GL(2,\mathbb{R})_+$ to produce other invariant submanifolds in higher genera applying some fixed ramified covering construction to all pairs (C,ω) constituting the initial invariant submanifold. In Sect. 2.1 we use a particular case of this construction to define some special connected components of some strata.

2. Formulation of results

2.1. Hyperelliptic components. First of all, we introduce the moduli spaces of meromorphic quadratic differentials.

Definition 1. *For integer $g \geq 0$ and collection $(l_1, \dots, l_n)$, $n \geq 1$ such that $l_j \geq -1$, $l_j \neq 0$ for all j and $\sum_j l_j = 4g - 4$, denote by $\mathcal{Q}(l_1, \dots, l_n)$ the moduli space of pairs (C, ϕ) where C is a smooth compact complex curve of genus g and ϕ is a meromorphic quadratic differential on C with zeroes of orders l_j (simple poles if $l_j = -1$) such that ϕ is not equal to the square of an Abelian differential.*

It is known (see [19]) that $\mathcal{Q}(l_1, \dots, l_n)$ is a complex algebraic orbifold of dimension

$$\dim_{\mathbb{C}} \mathcal{Q}(l_1, \dots, l_n) = 2g + n - 2. \tag{2}$$

Sometimes we shall use “exponential” notation to denote multiple zeroes (simple poles) of the same degree, for example $\mathcal{Q}(-1^5, 1) := \mathcal{Q}(-1, -1, -1, -1, -1, 1)$. The condition that ϕ is not a square is automatically satisfied if at least one of parameters l_j is odd.

One can canonically associate with every meromorphic quadratic differential (C, ϕ) another connected curve C' with an Abelian differential ω on it. Namely, C' is the unique double covering of C (maybe ramified at singularities of ϕ), such that the pullback of ϕ is a square of an Abelian differential ω. We have automatically $\sigma^*(\omega) = -\omega$ where σ is the involution on C' interchanging points in the generic fiber over C. Curve C' is connected because of the condition that ϕ is not a square of an Abelian differential.

Thus, we obtain a map from the stratum $\mathcal{Q}(l_1, \dots, l_n)$ of meromorphic quadratic differentials to the stratum $\mathcal{H}(k_1, \dots, k_m)$ of Abelian differentials, where numbers (k_i) are obtained from (l_j) by the following rule: to each even $l_j > 0$ we associate a pair of zeroes of ω of orders $(l_j/2, l_j/2)$ in the list (k_i), to each odd $l_j > 0$ we associate one zero of order $l_j + 1$, and associate nothing to simple poles (e.g. to $l_j = -1$).

Lemma 1. *The canonical map described above*

$$\mathcal{Q}(l_1, \dots, l_n) \to \mathcal{H}(k_1, \dots, k_m)$$

is an immersion.

Proof. Denote as above by C' the double covering of C with Abelian differential ω and involution σ.

Consider the induced involution

$$\sigma^* : H^1(C', Zeroes(\omega); \mathbb{C}) \to H^1(C', Zeroes(\omega); \mathbb{C}).$$

It defines decomposition $H^1(C', Zeroes(\omega); \mathbb{C}) \simeq V_1 \oplus V_{-1}$ of the first cohomology into the direct sum of subspaces invariant and anti invariant under the involution σ^*. By construction $[\omega] \in V_{-1}$. Thus, we obtain (locally) a mapping from $\mathcal{Q}(l_1, \dots, l_n)$ to a domain in the complex vector space $V_{-1} \subseteq H^1(C', Zeroes(\omega); \mathbb{C})$. It is well known that this mapping is holomorphic and locally one-to-one. Since the space $\mathcal{H}(k_1, \dots, k_m)$ is locally identified with $H^1(C', Zeroes(\omega); \mathbb{C})$ by means of the period mapping, this completes the proof of lemma. □

The following two series of maps of this kind would play a special role for us:

$$\begin{array}{ll} & \mathcal{Q}(-1^{2g'+1}, 2g'-3) \to \mathcal{H}(2g'-2) \\ (3) & \\ & \mathcal{Q}(-1^{2g'+2}, 2g'-2) \to \mathcal{H}(g'-1, g'-1), \end{array}$$

where $g' \geq 2$ in both cases. In both cases curve C is rational (i.e. $g = 0$), and hence curve C' is hyperelliptic of genus g'. In these two cases the dimension of the image stratum of Abelian differentials coincides with the dimension of the original stratum of meromorphic quadratic differentials. Indeed, formula (2) gives

$$\dim_{\mathbb{C}} \mathcal{Q}(-1^{2g'+1}, 2g'-3) = 2 \cdot 0 + (2g'+2) - 2 = 2g'$$
$$\dim_{\mathbb{C}} \mathcal{Q}(-1^{2g'+2}, 2g'-2) = 2 \cdot 0 + (2g'+3) - 2 = 2g'+1,$$

while formula (1) gives the following dimensions of the image strata:

$$\dim_{\mathbb{C}} \mathcal{H}(2g'-2) = 2g'+1-1 = 2g'$$
$$\dim_{\mathbb{C}} \mathcal{H}(g'-1, g'-1) = 2g'+2-1 = 2g'+1.$$

Remark 2. We have constructed a map $\mathcal{Q}(l_1, \dots, l_n) \to \mathcal{H}(k_1, \dots, k_m)$ using certain canonical double covering $C' \to C$. Choosing some other (ramified) covering of some fixed type one can construct some other (local) maps between moduli spaces of quadratic or Abelian differentials. The reader can find a detailed description of all maps of this kind between moduli spaces of *quadratic differentials*, which give coincidence of dimensions, in paper [8].

Before returning to maps (3) which are of a particular interest for us we need to prove the following statement.

Proposition 1. *In the case $g = 0$ every stratum $\mathcal{Q}(l_1, \dots, l_n)$ of meromorphic quadratic differentials is nonempty and connected.*

Proof. For any divisor on $\mathbb{C}P^1$ with given multiplicities the corresponding meromorphic quadratic differential exists and is unique up to a non-zero scalar. Thus, we have

$$\mathcal{Q}(l_1, ..., l_n)/\mathbb{C}^* \cong \big((\mathbb{C}P^1)^n \backslash diagonals\big)/\big(PSL(2, \mathbb{C}) \times (\text{finite symmetry group})\big).$$

Therefore the orbifold $\mathcal{Q}(l_1, \dots, l_n)$ is nonempty and connected. □

Lemma 1, the observation on coincidence of dimensions of the corresponding strata in (3), together with Proposition 1 justify the following definition.

Definition 2. *By hyperelliptic components we call the following connected components of the following strata of Abelian differentials on compact complex curves of genera $g \geq 2$:*

The connected component $\mathcal{H}^{hyp}(2g-2)$ of the stratum $\mathcal{H}(2g-2)$ consisting of Abelian differentials on hyperelliptic curves of genus g corresponding to the orbifold $\mathcal{Q}(-1^{2g+1}, 2g-3)$;

The connected component $\mathcal{H}^{hyp}(g-1, g-1)$ of $\mathcal{H}(g-1, g-1)$ corresponding to the orbifold $\mathcal{Q}(-1^{2g+2}, 2g-2)$.

Remark 3. Points of $\mathcal{H}^{hyp}(2g-2)$ (respectively of $\mathcal{H}^{hyp}(g-1, g-1)$) are Abelian differentials on hyperelliptic curves of genus g which have a single zero of multiplicity $2g-2$ invariant under the hyperelliptic involution (respectively a pair of zeroes of orders $g-1$ symmetric to each other with respect to the hyperelliptic involution).

Note that if an Abelian differential on a hyperelliptic curve has a single zero of order $2g-2$ then this zero is necessarily invariant under the hyperelliptic involution σ, because $\sigma^*(\omega) = -\omega$ for any Abelian differential ω. Therefore, this Abelian differential belongs to the component $\mathcal{H}^{hyp}(2g-2)$. However, if an Abelian differential ω has two zeroes of degrees $g-1$, there are two possibilities: the zeroes might be interchanged by the hyperelliptic involution, and they might be invariant under the hyperelliptic involution. In the first case the Abelian differential belongs to the component $\mathcal{H}^{hyp}(g-1, g-1)$, while in the second case it does not.

2.2. Parity of a spin structure: a definition.

Definition 3. *A* spin structure *on a smooth compact complex curve C is a choice of a half of the canonical class, i.e. of an element $\alpha \in Pic(C)$ such that*

$$2\alpha = K_C := -c_1(T_C).$$

The parity of the spin structure *is the residue modulo 2 of the dimension*

$$dim\, \Gamma(C, L) = dim\, H^0(C, L)$$

for line bundle L with $c_1(L) = \alpha$.

On a curve of genus $g \geq 1$ there are 2^{2g} different spin structures among which $2^{2g-1} + 2^{g-1}$ are even and $2^{2g-1} - 2^{g-1}$ are odd. It follows from the results of M. Atiyah [1] and D. Mumford [14] that the parity of a spin structure is invariant under continuous deformations.

Let ω be an Abelian differential with *even* multiplicities of zeroes, $k_i = 2l_i$ for all i, $i = 1, \dots, n$. The divisor of zeroes of ω

$$\text{Zeroes}(\omega) = 2l_1 P_1 + \cdots + 2l_n P_n$$

represents the canonical class K_C. Thus, we have a canonical spin structure on C defined by

$$\alpha_\omega := [l_1 P_1 + \cdots + l_n P_n] \in Pic(C).$$

By continuity the parity of this spin structure is constant on each connected component of stratum $\mathcal{H}(2l_1, \ldots, 2l_n)$.

Definition 4. *We say that a connected component of $\mathcal{H}(2l_1, \ldots, 2l_n)$ has even or odd spin structure depending on whether α_ω is even or odd, where ω belongs to the corresponding connected component.*

In Sect. 3.1 we present an equivalent definition of the parity of spin structure in terms of elementary differential topology.

2.3. Main results. First of all, we describe connected components of strata in the "stable range" when the genus of the curve is sufficiently large.

Theorem 1. *All connected components of any stratum of Abelian differentials on a curve of genus $g \geq 4$ are described by the following list:*

The stratum $\mathcal{H}(2g-2)$ has three connected components: the hyperelliptic one, $\mathcal{H}^{hyp}(2g-2)$, and two other components: $\mathcal{H}^{even}(2g-2)$ and $\mathcal{H}^{odd}(2g-2)$ corresponding to even and odd spin structures.

The stratum $\mathcal{H}(2l, 2l)$, $l \geq 2$ has three connected components: the hyperelliptic one, $\mathcal{H}^{hyp}(2l, 2l)$, and two other components: $\mathcal{H}^{even}(2l, 2l)$ and $\mathcal{H}^{odd}(2l, 2l)$.

All the other strata of the form $\mathcal{H}(2l_1, \ldots, 2l_n)$, where all $l_i \geq 1$, have two connected components: $\mathcal{H}^{even}(2l_1, \ldots, 2l_n)$ and $\mathcal{H}^{odd}(2l_1, \ldots, 2l_n)$, corresponding to even and odd spin structures.

The strata $\mathcal{H}(2l-1, 2l-1)$, $l \geq 2$, have two connected components; one of them: $\mathcal{H}^{hyp}(2l-1, 2l-1)$ is hyperelliptic; the other $\mathcal{H}^{nonhyp}(2l-1, 2l-1)$ is not.

All the other strata of Abelian differentials on the curves of genera $g \geq 4$ are nonempty and connected.

Finally we consider the list of connected components in the case of small genera $1 \leq g \leq 3$, where some components are missing in comparison with the general case.

Theorem 2. *The moduli space of Abelian differentials on a curve of genus $g = 2$ contains two strata: $\mathcal{H}(1, 1)$ and $\mathcal{H}(2)$. Each of them is connected and coincides with its hyperelliptic component.*

Each of the strata $\mathcal{H}(2, 2)$, $\mathcal{H}(4)$ of the moduli space of Abelian differentials on a curve of genus $g = 3$ has two connected components: the hyperelliptic one, and one having odd spin structure. The other strata are connected for genus $g = 3$.

Parities of spin structures for hyperelliptic strata are calculated in the Appendix A.4, Corollary 5.

Theorems 1 and 2 were announced in [7].

2.4. Plan of the proof. We possess two invariants of connected components: the components could be either hyperelliptic or not, and in the case of even multiplicities the associated spin structure could be either even or odd. We show that these invariants classify the connected components. The maximal number of connected components is 3, and it is achieved for the strata $\mathcal{H}(2g-2)$ for $g \geq 4$. We call the stratum $\mathcal{H}(2g-2)$ *minimal*.

Our plan of the proof is the following:

In Sect. 3 we give an alternative description of the parity of the spin structure defined by an Abelian differential having zeroes of even degrees. For a special class of Abelian differentials introduced in Sect. 4 this description in terms of differential topology will make the computation of the parity of the spin structure especially easy.

The subset of points $[(C, \omega)]$ whose horizontal foliation has only *closed leaves*, is dense in every stratum. In Sect. 4.1 we consider Abelian differentials only of this type. We propose a combinatorial way to represent such Abelian differentials by diagrams, and it is particularly convenient for the minimal stratum. In Sect. 4.1 we establish a criterion for diagrams selecting the ones associated to Abelian differentials. We call corresponding diagrams *realizable*. Also in Sect. 4.1 we describe diagrams corresponding to hyperelliptic Abelian differentials.

We complete Sect. 4 by introducing a surgery ("bubbling a handle") which allows us to construct an Abelian differential in the minimal stratum in genus $g+1$ from an Abelian differential from the minimal stratum in genus g. This surgery can be applied to any Abelian differential; however, when the horizontal foliation of an Abelian differential has only closed leaves, one can apply the surgery in such way that the horizontal foliation of the resulting Abelian differential also has only closed leaves. In this particular case the surgery can be described in terms of diagrams. Also we describe how the parity of the spin structure changes under the surgery.

In Sect. 5 we prove the classification theorem. First we prove it for the minimal stratum. In Sect. 5.1 we study possible transformations of realizable diagrams representing points in the minimal stratum preserving the connected component. We prove by induction in genus $g \geq 2$ that the classification of connected components of the minimal stratum $\mathcal{H}(2g-2)$ is as in Theorems 1 and 2. We have to note that a surgery used in the step of induction ("tearing off a handle") is based on combinatorial Lemma 20 from Appendix A.3 concerning extended Rauzy classes.

In Sect. 5.2 we study the topology of the adjacency of strata, and prove that the number of connected components in every stratum adjacent to the minimal stratum is bounded above by the number of connected components of the minimal stratum. More precisely, we identify the set of such components with a quotient of the set $\pi_0(\mathcal{H}(2g-2))$.

In Sect. 5.3 we prove that any Abelian differential which does not belong to the minimal stratum, can be degenerated to a differential with less zeroes. Thus, by induction we prove that any connected component of any stratum is adjacent to the minimal stratum, opening the way to apply results of Sect. 5.2.

Using another class of transformations of diagrams we prove in Sect. 5.4 that in certain cases two connected components of a stratum adjacent to two given different components of the minimal stratum coincide.

Using the previous results we prove the *upper* bound on the number of connected components of every stratum. On the other hand, topological invariants plus a realization construction (see the end of Sect. 5.4) give a *lower* bound on the number of components. These two bounds coincide, thus we obtain the main result.

Although we shall not do it explicitly in the present paper, one can easily modify the proof for the case of numbered zeroes and obtain essentially the same classification of connected components.

3. Spin structure determined by an Abelian differential

In this section we give an alternative description of the spin structure determined by an Abelian differential with zeroes of even orders on a closed complex curve.

3.1. Spin structure: topological definition. We begin by recalling the topological definition of the spin structure on a Riemann surface (see [13], [1]). Let M_g^2 be a Riemann surface of genus g, and let P be the S^1-bundle of directions of non-zero tangent vectors to M_g^2. A *spin structure* on M_g^2 is a double-covering $Q \to P$ whose restriction to each fiber of P is isomorphic to the standard double covering $S^1 \overset{\mathbb{Z}/2}{\to} S^1$.

Since the structure group of the covering $Q \to P$ is just $\mathbb{Z}/2$, the spin structures are in the one-to-one correspondence with the $\mathbb{Z}/2$-valued linear functions on $H_1(P; \mathbb{Z}/2)$, having nonzero value on the cycle representing the fiber S^1 of P. Thus, spin structures are classified by a coset of $H^1(M_g^2; \mathbb{Z}/2)$ in $H^1(P; \mathbb{Z}/2)$.

In [5] D. Johnson associates to every spin structure $\xi \in H^1(P; \mathbb{Z}/2)$ on a Riemann surface a $\mathbb{Z}/2$-valued quadratic form Ω_ξ on $H_1(M_g^2; \mathbb{Z}/2)$, and shows, that the parity of the spin structure ξ coincides with the Arf-invariant of Ω_ξ. We present briefly a sketch of the construction from [5].

First of all, there is a canonical lifting $c \mapsto \tilde{c}$, $H_1(M_g^2; \mathbb{Z}/2) \to H_1(P; \mathbb{Z}/2)$ (a map of sets) defined in the following way. Having a cycle $c \in H_1(M_g^2; \mathbb{Z}/2)$ one can represent it by a collection of simple closed oriented curves $c = \sum_{i=1}^m [\alpha_i]$. Let $[\vec{\alpha_i}]$ be the cycle in $H_1(P; \mathbb{Z}/2)$ represented by the *framed curve* in P consisting of positive tangent directions to α_i. Let $z \in H_1(P; \mathbb{Z}/2)$ be the homology class represented by the fiber S^1. The lifting is defined as

$$c \mapsto \tilde{c} := \sum_{i=1}^{m} [\vec{\alpha_i}] + mz.$$

According to [5] the map is well-defined. The map obeys the following relation

$$\widetilde{a+b} = \tilde{a} + \tilde{b} + (a \cdot b)z$$

where $(a \cdot b)$ is the intersection index of cycles a and b. Notice that the lifting is *not* a homomorphism of groups.

A $\mathbb{Z}/2$-valued quadratic form Ω on $H_1(M_g^2; \mathbb{Z}/2)$ with the associated bilinear form $(a, b) \mapsto a \cdot b$ is any function $\Omega : H_1(M_g^2; \mathbb{Z}/2) \to \mathbb{Z}/2$ such that

$$\Omega(a+b) = \Omega(a) + \Omega(b) + a \cdot b.$$

Having a spin structure $\xi \in H^1(P; \mathbb{Z}/2)$ one associates to it the following quadratic form Ω_ξ on $H_1(M_g^2; \mathbb{Z}/2)$:

$$\Omega_\xi(a) \overset{def}{:=} \langle \xi, \tilde{a} \rangle.$$

Given a symplectic basis $a_i, b_i \in H_1(M_g^2; \mathbb{Z})$ the *Arf-invariant* of a quadratic form Ω_ξ is determined as

$$\Phi(\Omega_\xi) \overset{def}{:=} \sum_{i=1}^{g} \Omega_\xi(a_i)\Omega_\xi(b_i) (\text{mod } 2).$$

It is proved in [5] that the parity of the spin structure ξ coincides with the Arf-invariant of Ω_ξ.

3.2. Spin structure determined by an Abelian differential. Consider an Abelian differential ω having zeroes of even degrees $(2l_1, \dots, 2l_n)$ on a Riemann surface M_g^2. It determines a flat structure on M_g^2 with cone-type singularities. Recall, that this flat metric has trivial holonomy. In particular, outside of finite number of singularities (corresponding to zeroes of ω) we have a well-defined horizontal direction. Consider a smooth simple closed oriented curve α on M_g^2 which does not contain any zeroes of ω. The flat structure allows us to determine the index $ind_\alpha \in \mathbb{Z}$ of the field tangent to the curve; ind_α coincides with the degree of the corresponding Gauss map: the total change of the angle between the vector tangent to the curve, and the vector tangent to the horizontal foliation is equal to $2\pi \cdot ind_\alpha$.

The spin structure $\xi \in H^1(P; \mathbb{Z}/2)$ determined by ω has the following property:

$$\langle \xi, \widetilde{[\alpha]} \rangle + 1 \equiv \langle \xi, [\vec{\alpha}] \rangle = ind_\alpha (\text{mod } 2).$$

This property can be considered as a topological definition of the spin structure determined by an Abelian differential. It gives also the following effective way to compute the parity $\varphi(\omega)$ of the spin structure defined by ω: choose oriented smooth paths $(\alpha_i, \beta_i)_{i=1,\dots,g}$ representing a symplectic basis of $H_1(M_g^2, \mathbb{Z}/2)$. Then

$$\begin{aligned}
\varphi(\omega) := \Phi(\Omega_\xi) &= \sum_{i=1}^{g} \Omega_\xi([\alpha_i]) \cdot \Omega_\xi([\beta_i]) \pmod 2 = \\
&= \sum_{i=1}^{g} \langle \xi, \widetilde{[\alpha_i]} \rangle \cdot \langle \xi, \widetilde{[\beta_i]} \rangle \pmod 2 = \\
&= \sum_{i=1}^{g} (ind_{\alpha_i} + 1)(ind_{\beta_i} + 1) \pmod 2.
\end{aligned} \tag{4}$$

In particular, using this definition it is easy to calculate the parity of the spin structure given any permutation from the corresponding Rauzy class.

We complete this section with the following obvious statements.

Lemma 2. *Let α be a smooth simple closed oriented curve everywhere transversal to the horizontal (vertical) foliation. Then $ind_\alpha = 0$. Let α be a closed regular leaf of the horizontal (vertical) foliation. Then $ind_\alpha = 0$.*

Lemma 3. *The spin structure of an Abelian differential on a surface of genus one is always odd.*

Proof. An Abelian differential ω on a surface of genus one defines a flat metric on the torus. One can represent a symplectic basis of cycles on this flat torus by a pair of closed geodesics α, β. By Lemma 2 we get $ind_\alpha = ind_\beta = 0$. Thus, formula 4 gives the following value for the parity $\varphi(\omega)$ of the spin-structure defined by ω

$$\varphi(\omega) = (ind_\alpha + 1)(ind_\beta + 1) \pmod 2 = 1.$$

□

4. Preparation of a surgery toolkit

4.1. Separatrix diagrams. In this section we consider a special class of Abelian differentials. Namely, we assume that all leaves of the horizontal foliation are either closed or connect critical points (a leaf joining two critical points is called a *saddle connection* or a *separatrix*). Later we will be saying simply that the horizontal foliation has only closed leaves. The square of an Abelian differential having this property is a particular case of Jenkins–Strebel quadratic differential, see [16].

Lemma 4. *Abelian differentials whose horizontal (vertical) foliations have only closed leaves form a dense subset in arbitrary stratum $\mathcal{H}(k_1, \dots, k_n)$.*

Proof. We prove the statement for horizontal foliations; for vertical foliations it is completely analogous. First of all, using the period mapping one concludes immediately that points $[(C, \omega)]$ with rational periods of $\omega_h := Im(\omega)$ are dense in arbitrary stratum. We claim that for these points the horizontal foliation (given by the kernel of ω_h) has only closed leaves. The reason is that in this case the integration of ω_h gives a smooth proper map $\pi : C \to \mathbb{R}/\frac{1}{N}\mathbb{Z} \simeq S^1$ where $N \in \mathbb{N}$ is a common denominator of periods of ω_h. Also we have the equality $\omega_h = \pi^*(dy)$ where y denotes the standard coordinate on the real line $\mathbb{R}$. All leaves of the horizontal foliations belong to fibers of π, therefore are either closed or connect critical points. □

We will associate with each Abelian differential (C, ω) whose horizontal foliation has only closed leaves a combinatorial data called *separatrix diagram*.

We start with an informal explanation. Consider the union of all saddle connections for the horizontal foliation, and add all critical points (zeroes of ω). We obtain a finite oriented graph Γ. Orientation on the edges comes from the canonical orientation of the horizontal foliation. Moreover, graph Γ is drawn on an oriented surface, therefore it carries so called *ribbon structure* (even if we forget about the orientation of edges), i.e. on the star of each vertex v a cyclic order is given, namely the counterclockwise order in which edges are attached to v. The direction of edges attached to v alternates (between directions toward v and from v) as we follow the counterclockwise order.

It is well known that any finite ribbon graph Γ defines canonically (up to an isotopy) an oriented surface $S(\Gamma)$ with boundary. To obtain this surface we replace each edge of Γ by a thin oriented strip (rectangle) and glue these strips together using the cyclic order in each vertex of Γ. In our case surface $S(\Gamma)$ can be realized as a tubular ε-neighborhood (in the sense of transversal measure) of the union of all saddle connections for sufficiently small $\varepsilon > 0$.

The orientation of edges of Γ gives rise to the orientation of the boundary of $S(\Gamma)$. Notice that this orientation is *not* the same as the canonical orientation of the boundary of an oriented surface. Thus, connected components of the boundary of $S(\Gamma)$ are decomposed into two classes: positively and negatively oriented (positively when two orientations of the boundary components coincide and negatively, when they are different). The complement to the tubular ε-neighborhood of Γ is a finite disjoint union of open cylinders foliated by oriented circles. It gives a decomposition of the set of boundary circles $\pi_0(\partial(S(\Gamma)))$ into pairs of components having opposite signs of the orientation.

Now we are ready to give a formal definition:

Definition 5. *A separatrix diagram (or simply a diagram) is a finite oriented ribbon graph* Γ*, and a decomposition of the set of boundary components of* $S(\Gamma)$ *into pairs, such that*

(1) the orientation of edges at any vertex is alternated with respect to the cyclic order of edges at this vertex;
(2) there is one positively oriented and one negatively oriented boundary component in each pair.

Notice that ribbon graphs which appear as a part of the structure of a separatrix diagram are very special. Any vertex of such a graph has even degree, and the number of boundary components of the associated surface with boundary is even. Notice also, that in general the graph of a separatrix diagram is *not* planar.

Any separatrix diagram $(\Gamma, \mathit{pairing})$ defines a closed oriented surface together with an embedding of Γ (up to a homeomorphism) into this surface. Namely, we glue to the surface with boundary $S(\Gamma)$ standard oriented cylinders using the given pairing.

In pictures representing diagrams we encode the pairing on the set of boundary components painting corresponding domains in the picture by some colors (textures in the black-and-white text) in such a way that every color appears exactly twice. We will say also that paired components have the *same color*.

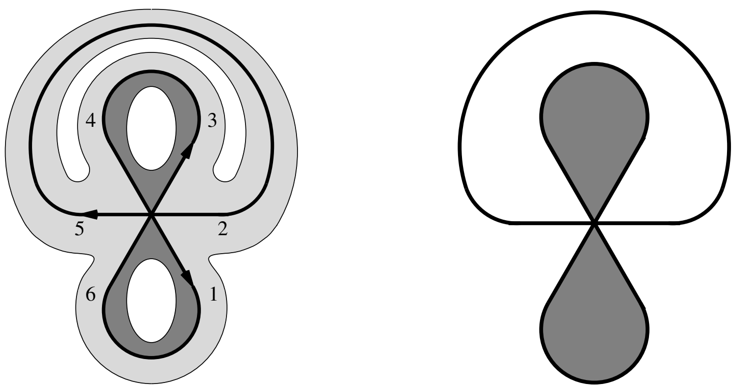

Fig. 1. An example of a separatrix diagram. A detailed picture on the left can be encoded by a schematic picture on the right

Example 1. The ribbon graph presented at Fig. 1 corresponds to the horizontal foliation of an Abelian differential on a surface of genus $g = 2$. The Abelian differential has a single zero of order 2. The ribbon graph has two pairs of boundary components.

Any separatrix diagram represents an orientable measured foliation with only closed leaves on a compact oriented surface without boundary. We say that a diagram is *realizable* if, moreover, this measured foliation can be chosen as the horizontal foliation of some Abelian differential. Lemma 5 below gives a criterion of realizability of a diagram.

Assign to each saddle connection a real variable standing for its "length". Now any boundary component is also endowed with a "length" obtained as sum of the "lengths" of all those saddle connections which belong to this component. If we want to glue flat cylinders to the boundary components, the lengths of the components in every pair should match each other. Thus for every two boundary components paired together (i.e. having the same color) we get a linear equation: "the length of the positively oriented component equals the length of the negatively oriented one".

Lemma 5. *A diagram is realizable if and only if the corresponding system of linear equations on "lengths" of saddle connections admits strictly positive solution.*

The proof is obvious.

Example 2. The diagram presented at Fig. 1 has three saddle connections, all of them are loops. Let p_{16}, p_{52}, p_{34} be their "lengths". There are two pairs of boundary components. The corresponding system of linear equations is as follows:

$$\begin{cases} p_{34} = p_{16} \\ p_{16} + p_{52} = p_{34} + p_{52}. \end{cases}$$

Here is a simple but important result which together with Lemma 4 shows that one can encode (not uniquely) connected components of strata by realizable separatrix diagrams.

Lemma 6. *Let the horizontal foliations of Abelian differentials ω_1, ω_2 have only closed leaves. If the corresponding separatrix diagrams are isomorphic, then both Abelian differentials belong to the same connected component of the same stratum of Abelian differentials.*

Proof. In this context it is convenient to think of an Abelian differential as of a flat surface with cone type singularities, with trivial holonomy and with a choice of a covariantly constant horizontal direction.

A family of Abelian differentials sharing the same diagram is parametrized by the collection of "horizontal" parameters representing the lengths of edges of the graph (i.e., the lengths of saddle connections) and by the collection of "vertical" parameters: heights of the cylinders, and twists used to paste them in. The vertical and the horizontal parameters are independent. There are no constraints on vertical parameters: the heights of the cylinders are arbitrary positive numbers; the twists are arbitrary angles. The horizontal parameters belong to a simplicial cone: they are presented by strictly positive solutions of a system of homogeneous linear equations described in Lemma 5. Thus the space of parameters is connected. □

Lemma 7. *Diagram of the horizontal foliation of Abelian differential $-\omega$ is obtained from the diagram of the horizontal foliation of Abelian differential ω by reversing the arrows (orientations of edges).*

The proof is obvious.

As a corollary, we obtain a necessary and sufficient condition for diagrams to represent a *hyperelliptic* Abelian differential from $\mathcal{H}(2g-2)$. First of all, such a diagram has one only vertex of valence $4g-2$. Consider a small neighborhood of the vertex of such graph; it is represented by $4g-2$ rays joined at the vertex which are organized in a cyclic order. There is a natural (local) involution of this neighborhood, the *central symmetry*, which fixes the vertex and sends each ray to the opposite one.

Lemma 8. *For any diagram with one vertex corresponding to a hyperelliptic Abelian differential $(C,\omega) \in \mathcal{H}(2g-2)$ the central symmetry extends to an involution of the ribbon graph interchanging any two paired boundary components. Also the number of cylinders in the diagram is equal to one plus the number of two-element orbits of the involution on the set of the edges of the graph (separatrix loops). Conversely, any diagram with one vertex and properties as above is realizable and represents a hyperelliptic Abelian differential.*

Proof. Hyperelliptic involution acts as a central symmetry near the unique zero of ω, also it transforms ω to $-\omega$. This implies the symmetry of the graph underlying the diagram. Also it shows that the decomposition of boundary components into pairs is also invariant under the involution. Let us prove that the involution preserves each pair. Suppose there is a pair of distinct cylinders which are interchanged by the involution. Change slightly the "height" of one of them. This corresponds to a continuous deformation of the vertical foliation, which leaves the horizontal foliation unchanged. The deformed Abelian differential is supposed to stay in the component $\mathcal{H}^{hyp}(2g-2)$ which leads to a contradiction, since the involution does not exist anymore.

Let us establish now the numerical property $n_c = n_2 + 1$ where n_c is the number of cylinders and n_2 is the number of two-element orbits as in lemma. The set of fixed points of the involution consists of a) the vertex of the diagram, b) the middle point on every involutive separatrix loop, c) two points in the interior of each cylinder. The total number of separatrix loops is equal to $2g-1$, therefore the number n_1 of loops invariant under the involution is equal to $2g-1-2n_2$. Hence, we have $1+n_1+2n_c = 2g+2(n_c-n_2)$ fixed points. On the other hand, the number of fixed points of a hyperelliptic involution is equal to $2g+2$ which implies that $n_c = n_2+1$.

Conversely, for a diagram with the properties listed in the lemma the realizability is obvious because we can assign to each separatrix loop the same length, which gives us a positive solution of the system of linear equations from Lemma 5. The corresponding surface carries canonically an involution with $2g+2$ fixed points, therefore by Hurwitz formula the quotient surface has genus zero and we are in the hyperelliptic case. □

Remark 4. Consider a realizable separatrix diagram corresponding to a *connected* closed surface, and forget the orientation of the edges. There are exactly two ways to orient again our graph (keeping the initial structure of the ribbon graph, and keeping the initial distribution of the boundary components into pairs) which lead to a realizable diagram: the initial way, and the opposite one. This is true even if the underlying graph of the diagram is not connected. According to Lemma 7 these two orientations correspond to Abelian differentials ω and $-\omega$. Note that Abelian differentials ω and $-\omega$ belong to the same stratum $\mathcal{H}(k_1, \dots, k_n)$; moreover, they belong to the same connected component since they can be joined inside the stratum by the continuous path $e^{i\theta}\omega$, $\theta \in [0; \pi]$. Thus, it follows from Lemmas 6 and 7 that both orientations of a *realizable* diagram represent Abelian differentials from the same connected component of the same stratum. Hence, if we care only about connected components of strata then in pictures of separatrix diagrams we can omit arrows (directions of edges).

4.2. Bubbling handles. In this section we describe a local surgery ("bubbling a handle") which modifies the surface in a small neighborhood of a chosen zero of the Abelian differential. Here it will be convenient to use "numbered" versions of moduli spaces (see Remark 1 from Introduction). Also here and later in Sect. 5 in order to alleviate notations we will denote a point $[(C, \omega)]$ of the moduli space simply by ω and will write slightly incorrectly $\omega \in \mathcal{H}^{num}(k_1, \dots, k_n)$.

Topologically the surgery corresponds to adding a handle to the surface. Metrically we choose a small disk centered at the chosen conical singularity, then we make some geodesic cuts inside the disk and paste in a small metric cylinder. Having started with an Abelian differential $\omega \in \mathcal{H}^{num}(k_1, \dots, k_{i-1}, k_i, k_{i+1}, \dots, k_n)$ we construct an Abelian differential $\hat{\omega} \in \mathcal{H}^{num}(k_1, \dots, k_{i-1}, k_i + 2, k_{i+1}, \dots, k_n)$, where the surface was modified in the neighborhood of the zero P_i of multiplicity k_i. The surgery depends on one discrete and on two complex parameters.

Revising this paper we decided to replace the initial version of the surgery, by the more general one described recently in [3], called there the "figure eight construction". Here we present briefly this latter construction consisting of two steps.

Breaking up a zero. We first describe how one can break up a zero P_i of multiplicity k of an Abelian differential into two zeroes of multiplicities k', k'', where $k' + k'' = k$, by a local surgery. In fact, we will need this construction also in a slightly more general case when parameter k'' is equal to zero.

Consider a metric disk of a small radius ε centered at the point P_i, i.e. the set of points Q of the surface such that Euclidean distance from Q to the point P_i is less than or equal to ε. We suppose that $\varepsilon > 0$ is chosen small enough, so that the ε-disk does not contain any other conical points of the metric; we assume also, that the disk which we defined in the metric sense is homeomorphic to a topological disk. Then, metrically our disk has a structure of a regular cone with a cone angle $2\pi(k_i + 1)$; here k_i is the multiplicity of the zero P_i. Now cut the chosen disk (cone) out of the surface. We shall modify the flat metric inside it preserving the metric at the boundary, and then paste the modified disk (cone) back into the surface.

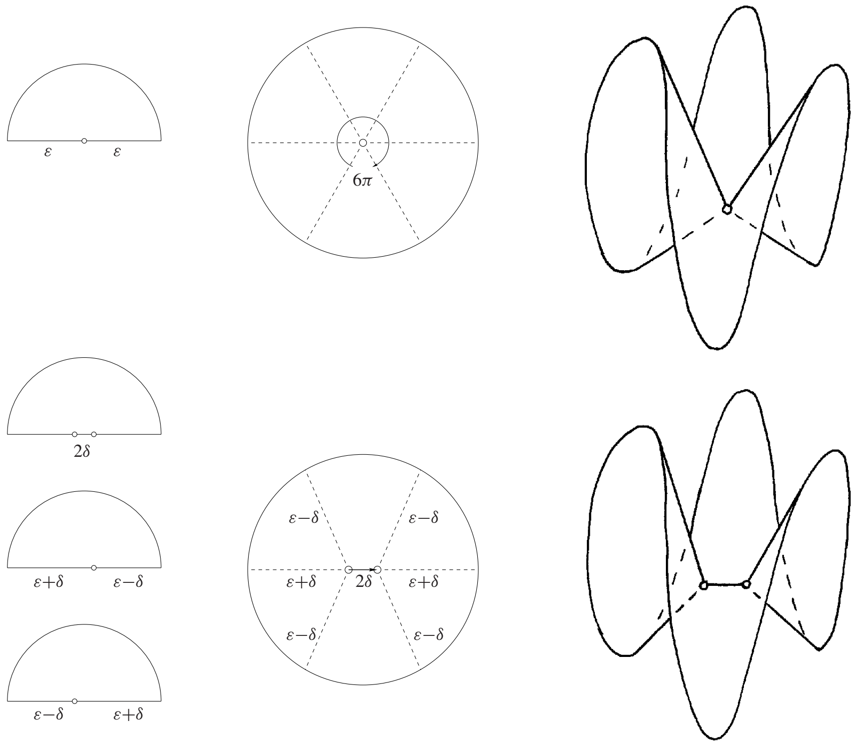

Fig. 2. Breaking up a zero into two zeroes (after [3])

Our cone can be glued from $2(k_i + 1)$ copies of standard metric half-disks of the radius ε, see the picture at the top of Fig. 2. Choose some small δ, where $0 < \delta < \varepsilon$ and change the way of gluing the half-disks as indicated on the bottom picture of Fig. 2. As patterns we still use the standard metric half-disks, but we move slightly the marked points on their diameters. Now we use two special half-disks; they have two marked points on the diameter at the distance δ from the center of the half disk. Each of the remaining $2k_i$ half-disks has a single marked point at the distance δ from the center of the half-disk. We are alternating the half-disks with the marked point moved to the right and to the left from the center. The picture shows that all the lengths along identifications are matching; gluing the half-disks in this latter way we obtain a topological disk with a flat metric; now the flat metric has two cone-type singularities with the cone angles $2\pi(k' + 1)$ and $2\pi(k'' + 1)$, where $k' + k'' = k_i$, and $k', k'' \in \mathbb{Z}_+$.

By convention we denote the multiplicities of the newborn zeroes in such way that $k' \geq k''$. Here $2k'$ and $2k''$ are the numbers of half-disks with one marked point glued in between the distinguished pair of half-disks with two marked points.

By technical reasons it would be convenient to include into consideration the trivial case, when k'' is equal to zero. In this latter case we, actually, do not change the metric at all; we just mark a point P'' at the distance 2δ from the point $P_i = P'$.

Note that a small tubular neighborhood of the boundary of the initial cone is isometric to the corresponding tubular neighborhood of the boundary of the resulting object. Thus we can paste it back into the surface. Pasting it back we can turn it by any angle φ, where $0 \leq \varphi < 2\pi(k_i + 1)$.

We described how to break up a zero of multiplicity k of an Abelian differential into two zeroes of multiplicities k', k'', where $k' + k'' = k$, and $k' \geq k''$. The construction is local; it is parameterized by the two free real parameters (actually, by one complex parameter): by the small distance 2δ between the newborn zeroes, and by the direction φ of the short geodesic segment joining the two newborn zeroes. In particular, as a parameter space for this construction one can choose a punctured disk.

Now we can proceed with the second step of the construction.

Bubbling a handle into a slit. Let us slit the surface along the short geodesic segment of the length 2δ joining the newborn zeroes P', P'' and let us identify the endpoints of the slit. The resulting surface has two boundary components joined together at the point $P' = P''$. By construction the boundary components are geodesics in the flat metric determined by ω; they have the same length 2δ. Take a small flat cylinder with a waist curve of length 2δ and paste it into our surface. The surface M_{g+1}^2 is constructed. The flat structure on M_{g+1}^2 together with the choice of the horizontal direction uniquely determine an Abelian differential $\hat{\omega}$ on M_{g+1}^2. By construction the resulting Abelian differential $\hat{\omega}$ belongs to the stratum $\mathcal{H}^{num}(k_1, \dots, k_i + 2, \dots, k_n)$, where $\omega \in \mathcal{H}^{num}(k_1, \dots, k_i, \dots, k_n)$, and k_i is the multiplicity of the zero P_i (the case $k_i = 0$ is not excluded; in this case P_i is just a marked point). The Abelian differential $\hat{\omega}$ is obtained from ω by "bubbling a small handle" at the zero P_i (see Fig. 3).

This surgery is parameterized by the following list of parameters:

— Discrete parameter k', where $k_i/2 \leq k' \leq k_i$. This parameter indicates the number of sectors between the distinguished pair of sectors. For the resulting Abelian differential $\hat{\omega}$ there are $2k' + 2$ sectors on the one side and $2k'' + 2$ on the other side; see also Fig. 3, where m denotes $(k'' + 1)$;

— Pair of free real parameters δ and φ responsible for the breaking up a zero; in the resulting construction they represent the length of the waist curve of the cylinder and direction in which goes the corresponding closed geodesic. This pair of real parameters can be seen as one complex parameter: the period of the Abelian differential $\hat{\omega}$ along the waist curve of the new cylinder;

— Finally, we have two more free real parameters representing the height of the cylinder, and the twist which we used pasting it into the surface. They can be organized in a complex parameter representing the period of $\hat{\omega}$ along the cycle following the new handle.

Let us describe now the properties of this surgery.

Lemma 9. *Consider Abelian differentials $\hat{\omega}_1, \hat{\omega}_2 \in \mathcal{H}^{num}(k_1, \dots, k_i + 2, \dots, k_n)$ obtained by "bubbling a handle" at the same zero P_i of an Abelian differential $\omega \in \mathcal{H}^{num}(k_1, \dots, k_i, \dots, k_n)$. If the angle $2\pi m$ between the sectors corresponding to the handle (see Fig. 3) is the same for $\hat{\omega}_1$ and for $\hat{\omega}_2$, then there exist a continuous path in $\mathcal{H}^{num}(k_1, \dots, k_j + 2, \dots, k_n)$ joining $\hat{\omega}_1$ with $\hat{\omega}_2$; in particular, $\hat{\omega}_1, \hat{\omega}_2$ belong to the same connected component of the stratum $\mathcal{H}^{num}(k_1, \dots, k_i + 2, \dots, k_n)$.*

Proof. Fixing the discrete parameter k' (which is equal to $m - 1$ if $m > k_i/2 - 1$, and to $k_i - m + 1$ otherwise) we describe "bubbling a handle" at the zero P_i of ω by two pairs of continuous parameters as above. Note that the space of parameters δ, φ describing breaking up the zero into two is a punctured disk; the space of parameters parameterizing the cylinder (the height of the cylinder and the twist used to paste it into the surface) is also a punctured disk. Thus the total space of parameters is a direct product of two punctured disks, which is obviously path-connected. □

Lemma 10. *Let an Abelian differential $\hat{\omega} \in \mathcal{H}^{num}(k_1, \dots, k_i + 2, \dots, k_n)$ be obtained from an Abelian differential $\omega \in \mathcal{H}^{num}(k_1, \dots, k_i, \dots, k_n)$ by "bubbling a handle" at some zero P_i. Any path $\rho : [0; 1] \to \mathcal{H}^{num}(k_1, \dots, k_i, \dots, k_n)$ which starts at ω can be lifted to a path $\hat{\rho} : [0; 1] \to \mathcal{H}^{num}(k_1, \dots, k_i + 2, \dots, k_n)$ starting at $\hat{\omega}$ by continuous "bubbling a handle" along ρ.*

Proof. Note that we can bubble a small handle into *any* Abelian differential of the given stratum. This implies that we can choose an appropriate subset in the stratum $\mathcal{H}^{num}(k_1, \dots, k_i + 2, \dots, k_n)$ which would have the structure of a (singular) fiber bundle over the stratum $\mathcal{H}^{num}(k_1, \dots, k_i, \dots, k_n)$. The regular fiber is a direct product of two disks punctured at the centers; a singular fiber is a quotient of the direct product of two disks punctured at the centers by a finite group of symmetry. Thus the lifting described in the lemma is the particular case of lifting of a path from the base to the total space of a fiber bundle. □

Remark 5. In other words Lemma 10 means the following. Let an Abelian differential $\hat{\omega}$ on a Riemann surface of genus $g + 1$ be constructed from an Abelian differential ω on a surface of genus g by "bubbling a small handle". Morally, we can temporarily "forget" (or "erase") corresponding handle; modify Abelian differential ω on a surface of genus g inside his stratum in a continuous way, and then "recall" the "forgotten" handle for the resulting Abelian differential ω' provided that the metric parameters of the handle are sufficiently small. Then Abelian differential $\hat{\omega}'$ on a surface of genus $g+1$ which we obtain as a result of this construction belongs to the same connected component as Abelian differential $\hat{\omega}$.

Suppose that we "bubble a handle" into a flat surface corresponding to an Abelian differential ω having zeroes of even multiplicities. The resulting Abelian differential $\hat{\omega}$ would have the same property. In particular, both Abelian differentials determine spin structures on corresponding surfaces. The following lemma compares the parities $\varphi(\omega)$ and $\varphi(\hat{\omega})$ of these spin structures.

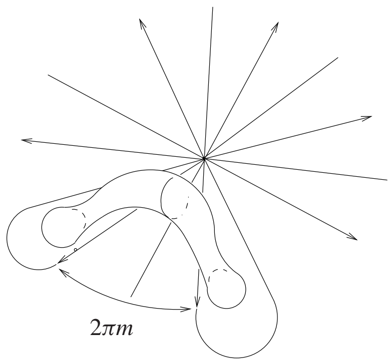

Fig. 3. Two simple separatrix loops determining a handle

Lemma 11. *Let an Abelian differential $\hat{\omega} \in \mathcal{H}^{num}(2(l_1+1), 2l_2, \ldots, 2l_n)$ on a surface of genus $g+1$ be obtained from an Abelian differential $\omega \in \mathcal{H}^{num}(2l_1, 2l_2, \ldots, 2l_n)$ on a surface of genus g by "bubbling a handle". Let $2\pi m$ be the angle of one of the two sectors complementary to the handle, see Fig. 3. The parities of the spin structures determined by ω and by $\hat{\omega}$ are related in the following way:*

$$\varphi(\hat{\omega}) - \varphi(\omega) = m + 1 (mod\, 2).$$

Proof. Choose a collection of oriented simple curves $(\alpha_i, \beta_i)_{i=1,\ldots,g}$ on the initial surface of genus g representing a symplectic basis in the first homology group. Deforming, if necessary, the paths inside their isotopy classes we can make them stay away from some small neighborhood $U(P_1)$ of the zero P_1 under consideration. We can assume that the surgery ("bubbling a handle") was performed inside this small neighborhood U. In particular, the surgery does not affect the initial collection of paths.

The initial basis of cycles can be completed to a symplectic basis on the surface of genus $g+1$ obtained after "bubbling a handle" by adding two additional curves on the handle created in the surgery.

One of these new curves, which we denote by α_{g+1}, can be represented by a waist curve of the cylinder which we pasted in; moreover, we can choose this waist curve to be a closed geodesic. By Lemma 2 we get $ind_{\alpha_{g+1}} = 0$.

The second curve, β_{g+1}, can be chosen as follows. Suppose for simplicity that the handle was bubbled with the slit made in the horizontal direction, and with a trivial twist, i.e., the leaf of the vertical foliation emitted from zero P_1 into the new handle returns back to P_1. (Changing the twist we stay in the same connected component of the stratum $\mathcal{H}^{num}(2(l_1+1), 2l_2, \dots, 2l_n)$; in particular, we do not change the parity of the spin structure defined by the corresponding $\hat{\omega}$.) Take a circle centered at P_1 in the initial surface. Making the radius of the circle sufficiently small we get the circle inside the neighborhood U; in particular, it does not intersect any of the initial curves α_i, β_i, where $i \leq g$. Choose an arc of this circle joining two distinguished sectors (see Fig. 3). Then the endpoints of the arc might be joint by a piece of leaf of the vertical foliation along the new cylinder (recall, that the twist of the handle is by assumption trivial). By construction the resulting closed path β_{g+1} is smooth; it does not intersect any of the initial curves; and $[\alpha_{g+1}] \cdot [\beta_{g+1}] = 1$. Hence we got the desired symplectic basis of cycles.

Direct calculation gives $ind_{\beta_{g+1}} = m$, since the tangent vector to the path β_{g+1} turns by the angle $2\pi m$ along the arc, and does not turn at all along the vertical segment. Now everything is ready to compare the parities of the spin structures of ω and $\hat{\omega}$; here we use formula 4

$$\begin{aligned}\varphi(\omega') &= \sum_{i=1}^{g+1}(ind_{\alpha_i}+1)\cdot(ind_{\beta_i}+1) = \\ &= \sum_{i=1}^{g}(ind_{\alpha_i}+1)\cdot(ind_{\beta_i}+1) + (ind_{\alpha_{g+1}}+1)\cdot(ind_{\beta_{g+1}}+1) = \\ &= \varphi(\omega) + (0+1)\cdot(m+1).\end{aligned}$$

□

Consider now "bubbling a handle" in the particular case when the horizontal foliation of the initial Abelian differential ω has only closed leaves. If at the intermediate stage of "bubbling a handle" we break up the zero in the horizontal direction, then the horizontal foliation of the resulting Abelian differential $\hat{\omega}$ obtained after "bubbling a handle" also has only closed leaves. It would be convenient for us to reformulate the lemmas above in this particular case in terms of the diagrams of the Abelian differentials ω and $\hat{\omega}$.

Example 3. Consider a flat torus and chose the horizontal direction on it in such way that the leaves of the horizontal foliation would be closed. "Bubbling a handle" in the horizontal direction at some point of the torus we get a surface of genus 2 with horizontal foliation having only closed leaves. The diagram of this foliation is presented at Fig. 1.

We call a separatrix loop *simple* if the corresponding outgoing and ingoing separatrix rays are neighbors (see Fig. 3). In terms of diagrams of separatrix loops "bubbling a handle" in a horizontal direction corresponds to adding a pair of simple separatrix loops of the same color to the initial diagram, see Example 3 above.

Note that when we "bubble a handle" at a true zero of ω (and not at a regular point as in the example above) there are several horizontal directions at a conical point. In particular, even when we fix the discrete parameter k' there are several ways to "bubble a handle" in the horizontal direction. The first lemma says that fixing the discrete parameter k' we get to the same connected component, no matter which of these horizontal direction we choose.

Lemma 12. *Rotating a pair of simple separatrix loops of the same color corresponding to the handle just "bubbled" in such way that the number of sectors between the pair of simple separatrix loops stay unchanged we obtain diagrams representing Abelian differentials from the same connected component of the corresponding stratum.*

Proof. The proof is a direct corollary of Lemma 6 and Lemma 9. □

Example 4. The realizable diagrams presented at Fig. 8 correspond to Abelian differentials from the same connected component.

Now let us reformulate Lemma 11 in the case when the leaves of horizontal foliations of ω and of $\hat{\omega}$ are closed. We assume that all zeroes of ω and $\hat{\omega}$ have even degrees, or equivalently that the corresponding separatrix diagrams have vertices only of valences $2(mod\, 4)$.

Lemma 13. *Let Abelian differentials ω on a surface of genus g and $\hat{\omega}$ on a surface of genus $g+1$ have horizontal foliations with only closed leaves. Suppose that the corresponding separatrix diagram of ω is obtained from the diagram of $\hat{\omega}$ by erasing a pair of simple separatrix loops corresponding to the same zero and bounding a pair of sectors of the same color (see Fig. 3). Let $2\pi m$ be the angle of one of the two complementary sectors. The parities of the spin structures determined by ω and by $\hat{\omega}$ are related in the following way:*

$$\varphi(\hat{\omega}) - \varphi(\omega) = m + 1 (mod\, 2).$$

Proof. Bubbling an appropriate handle in the flat surface determined by ω we obtain an Abelian differential with the same diagram as $\hat{\omega}$. It follows from Lemma 6 that all Abelian differentials corresponding to the same diagram have the same parity of the spin structure. Thus the lemma is a direct corollary of Lemma 11. □

5. Connected components of the strata

5.1. Connected components of the minimal stratum $\mathcal{H}(2g-2)$. Here we will proceed by induction in genus g and will use the fact that any connected component of the minimal stratum $\mathcal{H}(2g)$ in genus $g+1$ can be accessed from some connected component of the minimal stratum $\mathcal{H}(2g-2)$ in genus g by "bubbling a handle". Surprisingly, this statement is not trivial. Despite of many attempts we were unable to find a purely geometric proof of the lemma below; we use the arguments based on combinatorial Lemma 20 from Appendix A.3. The difficulty which one meets here is as follows. In every connected component of $\mathcal{H}(2g)$ one can easily find Abelian differentials having a cylinder filled by regular closed leaves of the horizontal foliation. Moreover, one can get examples when such cylinder is bounded by a pair of simple separatrix loops. However, we should warn the reader that many of these Abelian differentials are *not* results of "bubbling a handle", see [3] for details.

Lemma 14. *Any connected component of the minimal stratum $\mathcal{H}(2g)$ can be accessed from some connected component of the minimal stratum $\mathcal{H}(2g-2)$ by "bubbling a handle" into a flat surface corresponding to an Abelian differential $\omega \in \mathcal{H}(2g-2)$.*

Proof. Consider the extended Rauzy class $\mathfrak{R}_{ex}$ corresponding to the connected component of the stratum $\mathcal{H}(2g)$ under consideration. Choose a permutation $\pi \in \mathfrak{R}_{ex}$ as in Lemma 20 from Appendix A. Consider an Abelian differential $\hat{\omega}$ obtained as a suspension over an interval exchange transformation with the permutation π, with integer λ_i, and with $\lambda_1 = \lambda_m$. Since all λ_i are integer, the vertical foliation of $\hat{\omega}$ has only closed leaves. It is easy to see that the vertical foliation has a pair of simple separatrix loops: that is an ingoing and outgoing separatrices in each loop are neighbors, and this pair of simple loops determines a handle (cf. Fig. 3); subintervals I_1 and I_m belong to this handle.

Consider now an Abelian differential ω obtained as a suspension over an interval exchange transformation with permutation π' as in Lemma 20, and with the vector of lengths $\lambda_2, \dots, \lambda_{m-1}$, where λ_i are same as above. By the choice of the permutations π and π' we get $\omega \in \mathcal{H}(2g-2)$. Since all λ_i are integer, the vertical foliation of ω also has only closed leaves. Moreover, it is easy to check that the separatrix diagram of the vertical foliation of $\hat{\omega}$ is obtained from the separatrix diagram of the vertical foliation of ω by "bubbling a handle". The statement of the lemma now follows from Lemma 6. □

From now on till the end of this subsection we will work in terms of separatrix diagrams representing Abelian differentials from stratum $\mathcal{H}(2g-2)$.

Consider the following three diagrams depending on genus g (cf Fig. 4 for the case $g = 5$). Each diagram contains $2g-1$ separatrix loops ($g \geq 2$) corresponding to $4g-2$ separatrix rays $r_1, \dots, r_{4g-2}$. We enumerate the rays counterclockwise starting from the one pointing to the "south". Join the following ordered pairs of rays by arcs; the order of the rays in each pair defines the orientation of the separatrix loop. Join r_1 and r_{2g}; join r_{2i+1} and r_{2i}, for $i = 1, \dots, g-1$; join r_{2i-1} and r_{2i} for $i = g+1, \dots, 2g-1$. The diagrams differ only by the way we paint them.

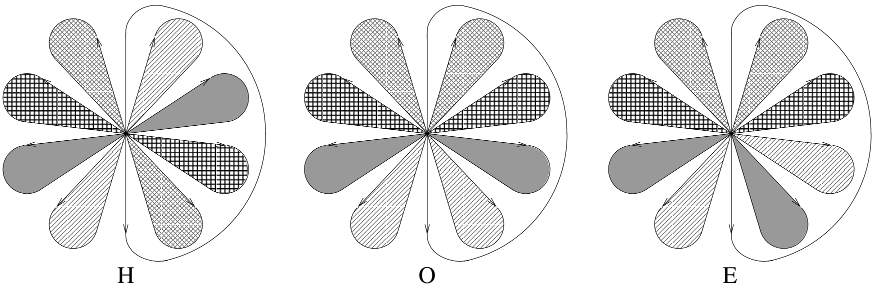

Fig. 4. The diagrams represent the following components on a surface of genus 5: H) hyperelliptic component; O) odd spin structure; E) even spin structure

— Case H. ($g \geq 2$) Paint with the same colors pairs of domains centrally symmetric to each other.

— Case O. ($g \geq 2$) Paint with the same colors pairs of domains symmetric with respect to the vertical axis.

— Case E. ($g \geq 3$) Paint with the same colors the sectors corresponding to the loops $r_3 r_2$ and $r_{4g-5} r_{4g-4}$. Paint with the same colors the sectors corresponding to the loops $r_5 r_4$ and $r_{4g-3} r_{4g-2}$. Paint with the same colors the rest pairs of domains symmetric to each other with respect to the vertical axis (cf. Fig. 4).

Lemma 15. *Every diagram of the type H, O, or E is realizable by the horizontal foliation of an Abelian differential from the stratum $\mathcal{H}(2g-2)$.*

Proof. In our case the system of linear equations on the lengths of separatrix loops (see Lemma 5) is trivial: simple separatrix loops bounding the domains of the same color have equal length. Thus it obviously has positive solutions. □

Lemma 16. *Let an Abelian differential ω have the horizontal foliation represented by one of the diagrams H, O, E.*

— If the diagram is diagram H, then $\omega \in \mathcal{H}^{hyp}(2g-2)$.

— If the diagram is diagram O, then ω has odd spin structure. If $g = 2$, then the diagram O coincides with the diagram H, and ω is hyperelliptic; for $g > 2$ it is not hyperelliptic.

— If the diagram is diagram E, then ω has even spin structure. If $g = 3$ then the diagram E coincides with the diagram H, and ω is hyperelliptic; for $g > 3$ it is not hyperelliptic.

Proof. Diagram H is always centrally symmetric ; diagram O is centrally symmetric only for $g = 2$ when it coincides with the diagram H; diagram E is centrally symmetric only for $g = 3$ when it coincides with the diagram H. Thus according to Lemma 8 these are the only cases when we get a hyperelliptic Abelian differential.

The parity of the spin structure determined by an Abelian differential with a horizontal foliation of the type O or E is computed inductively using Lemma 13, with Lemma 3 serving as the base of induction. Strictly speaking, we can not apply Lemma 13 to the bubbling a handle to the torus ($g = 1$ case) because we introduced bubbling only at zeroes of positive multiplicity. It is easy to check that our arguments work as well for the bubbling at a regular point, i.e. at a "zero of multiplicity zero". The diagram for $g = 1$ case consists of one vertex and one loop. □

Proposition 2. *Any connected component of the stratum $\mathcal{H}(2g-2)$ for $g \geq 2$ contains an Abelian differential with a horizontal foliation represented by one of the diagrams H, O, E.*

Corollary 1. *For $g = 2$ the stratum $\mathcal{H}(2)$ is connected; it coincides with the hyperelliptic component. For $g = 3$ the stratum $\mathcal{H}(4)$ has exactly two connected components: the hyperelliptic one $\mathcal{H}^{hyp}(4)$, and one having odd parity of the spin structure $\mathcal{H}^{odd}(4)$. For $g \geq 4$ the stratum $\mathcal{H}(2g-2)$ has exactly three different connected components: $\mathcal{H}^{hyp}(2g-2)$, $\mathcal{H}^{odd}(2g-2)$, and $\mathcal{H}^{even}(2g-2)$.*

Proof. The corollary follows immediately from combination of Proposition 2 with Lemma 16 and Lemma 6. □

The rest part of this section is devoted to the proof of Proposition 2.

Proof of Proposition 2. The diagrams which can be obtained one from the other by reversing the arrows are equivalent in our considerations, see Remark 4. Throughout this proof we mostly do not distinguish such diagrams.

First note that for the connected component $\mathcal{H}^{hyp}(2g-2)$ the statement of the proposition is obvious: by Lemmas 15 and 16 the diagram H is realizable by the horizontal foliation of a hyperelliptic Abelian differential.

Every Riemann surface of genus $g = 2$ is hyperelliptic which implies that any Abelian differential in the stratum $\mathcal{H}(2)$ is hyperelliptic, $\mathcal{H}(2) = \mathcal{H}^{hyp}(2)$ and hence $\mathcal{H}(2)$ is connected. Thus for $g = 2$ the proposition is proved.

Assume that Proposition 2 is proved for all genera smaller than or equal to g, where $g \geq 2$. Let us prove it for genus $g + 1$. To make a step of induction we have to decrease the genus of the surface by one. In order to do this we use Lemma 14 saying that in any connected component of $\mathcal{H}(2g)$, $g \geq 2$, one can find an Abelian differential $\hat{\omega}$ obtained from some Abelian differential $\omega \in \mathcal{H}(2g-2)$ by "bubbling a handle".

"Forget" the corresponding handle (see Remark 5). By the induction assumption we can deform continuously the corresponding Abelian differential ω on a surface of genus g to fit one of the diagrams H, O, or E. Now we can "bubble the forgotten handle" along the path in the stratum $\mathcal{H}(2g-2)$, see Lemma 10. Proposition 2 now follows from the lemma below. □

Lemma 17. *Consider an Abelian differential $\hat{\omega} \in \mathcal{H}(2g)$ obtained by "bubbling a handle" at the zero of an Abelian differential $\omega \in \mathcal{H}(2g-2)$ having the horizontal foliation of one of the types H, O, E in genus g. There exist a continuous path in $\mathcal{H}(2g)$ joining $\hat{\omega}$ with an Abelian differential having the horizontal foliation of one of the types H, O, E in genus $g+1$.*

Proof. Note that by Lemma 12 we may always assume that the pair of simple separatrix loops representing the handle just "bubbled" is symmetric with respect to the vertical axis.

If the diagram obtained after "bubbling a handle" is already of one of the types H, O, E, the statement of the lemma becomes trivial.

The pair of simple separatrix loops representing the handle just "bubbled" is colored in black on all the figures. Speaking about a pair of simple separatrix loops we always mean a pair of simple separatrix loops representing a handle, and thus colored by the same color at the diagram.

The idea of the proof is the following. Our diagrams have numerous pairs of simple separatrix loops representing handles. We choose an appropriate pair of simple separatrix loops of the same color and temporarily "forget" it, see Remark 5. Using Lemma 5 we check that the modified diagram is realizable. Using the induction assumption we deform continuously the corresponding Abelian differential in genus g to one having one of the diagrams H, O, E. Then we "recall" the "forgotten" handle. By Lemma 10 the resulting diagram in genus $g+1$ represents an Abelian differential which can be obtained from the initial one $\hat{\omega}$ by a continuous deformation inside $\mathcal{H}(2g)$.

We start with the general case assuming that $g \geq 4$; we consider the small genera $g = 2, 3$ separately.

Case h) Let the initial Abelian differential $\hat{\omega}$ in genus $g+1$ correspond to the diagram of the type H with a "handle bubbled into it". If the resulting diagram is centrally symmetric, it satisfies conditions from Lemma 8 and therefore Abelian differential $\hat{\omega}$ belongs to the hyperelliptic connected component $\mathcal{H}^{hyp}(2g)$. Hence, we can join it by a continuous path with an Abelian differential corresponding to the diagram H in genus $g+1$.

Suppose now that the diagram obtained after "bubbling a handle" in the diagram H is not centrally symmetric. For the initial genus $g \geq 4$ we obtain a diagram in genus $g+1$ having a centrally symmetric pair of *simple* separatrix loops different from the pair just "bubbled". We can always choose this new pair of simple separatrix loops in such a way that the diagram obtained after "forgetting" this new pair would not be centrally symmetric. It is easy to see that the resulting diagram is realizable. Thus, by induction assumption we can deform the resulting Abelian differential inside $\mathcal{H}(2g-2)$ to one corresponding to one of the diagrams O or E. "Recall" the "forgotten" handle. The resulting diagram is obtained from one of the diagrams of the type E or O by "bubbling a handle". We have reduced this case to one of the cases o) or e).

Case o) A diagram obtained by "bubbling a handle" into the diagram of type O in genus $g \geq 4$ is either again of the type O, or it is of the type presented at Fig. 5.

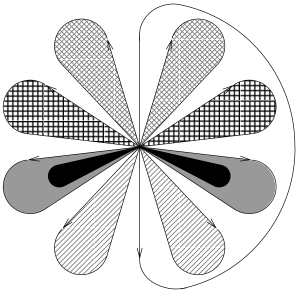

Fig. 5. Case o)

In the latter case there is a symmetric pair of simple separatrix loops next to the vertical axis. By reversing the arrows on the diagram, if necessary, we may assume that this new pair of simple separatrix loops is next to the top vertical ray (one without arrow). Let us "forget" this new pair of simple separatrix loops. The resulting diagram is obviously realizable. By Lemma 11 it represents an Abelian differential ω' having even parity of the spin structure. For initial genus $g \geq 4$ it would not be centrally-symmetric. Thus, by the induction assumption we can deform ω' by a continuous deformation inside $\mathcal{H}(2g-2)$ to an Abelian differential representing diagram E. "Recall" the "forgotten" handle. Using Lemma 12 we can assume that it is located near the marked ray. It is represented by a pair of simple separatrix loops next to the top vertical separatrix ray symmetric with respect to the vertical diameter. The diagram thus obtained is diagram E in genus $g+1$.

Case e) Consider a diagram obtained after "bubbling a handle" into diagram E in genus $g \geq 4$. If it is already of the type E, we have nothing to modify.

If the new handle was "bubbled" inside the pair of top symmetric sectors (see, for example, Fig. 6) we may turn the pair of black petals (keeping fixed the angle between them), say, placing them inside the bottom two petals (see Lemma 12). Thus we may assume that the top symmetric petals of the diagram stay unchanged upon "bubbling a handle", see, e.g. Fig. 7.

"Forgetting" the top pair of symmetric petals we obtain a diagram which is obviously realizable by an Abelian differential $\omega' \in \mathcal{H}(2g-2)$, and which is not centrally-symmetric, except only one case when $g = 4$ and black petals are inserted between the bottom pairs of petals. Thus, if we are not in this exceptional case, by the induction assumption we can deform ω' by a continuous deformation inside $\mathcal{H}(2g-2)$ to an Abelian differential representing one of the diagrams O or E. "Recalling" the "forgotten" handle we obtain an Abelian differential representing one of the diagrams O or E in genus $g+1$.

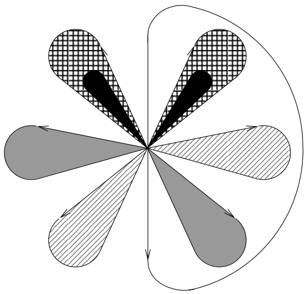

Fig. 6. A handle "bubbled" into diagram E in genus $g = 4$

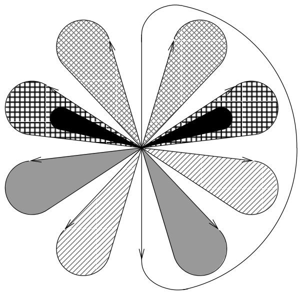

Fig. 7. Case e)

In the exceptional case when the initial genus g is equal to 4 and black petals are inserted between the bottom pairs of petals of diagram E "recall" the "forgotten" top pair of symmetric petals at the initial place. Turn the pair of black petals (keeping fixed the angle between them) by seven sectors. The black petals are again symmetric with respect to the vertical axis and we obtain a diagram of the type E in genus $g = 5$.

To complete the proof of the lemma we need to consider the small genera $g = 2, 3$.

Small genera) For initial genus $g = 2$ the diagram E does not exist, and the diagrams H and O coincide. Thus, the new diagram is obtained by "bubbling a handle" into the diagram of the type H in genus $g = 2$. The diagram obtained is either of the type O in genus $g = 3$, or it is centrally-symmetric. In the latter case by Lemma 8 the corresponding Abelian differential belongs to the hyperelliptic component. Hence, it can be joined by a continuous path to an Abelian differential corresponding to the diagram of the type H. This completes consideration of genus $g = 2$.

Consider now diagrams in genus $g = 3$. The case when a handle is "bubbled" in a diagram of the type O in genus $g = 3$ can be treated the same way as in the general case o), except that we can now obtain a diagram which is centrally symmetric, thus reducing it to the case of 'bubbling a handle" into the diagram of the type H.

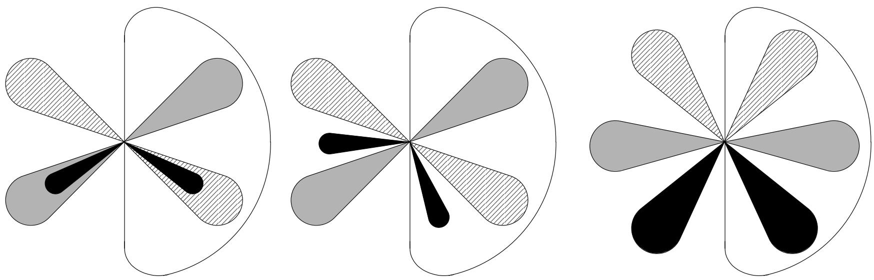

Fig. 8. Modifying diagrams by rotating a pair of petals of the same color

The diagrams H and E coincide in genus $g = 3$. "Bubbling a handle" in the diagram H in genus $g = 3$ we obtain either one of the diagrams H, E in genus $g = 4$ (up to the change of the orientation of the foliation, see Remark 4) or the diagram presented at Fig. 8 on the left.

In the latter case we can rotate the pair of black petals one sector clockwise to obtain the diagram presented at Fig. 8 in the middle; by Lemma 12 this realizable diagram represents the same connected component of the same stratum as the initial one. Using Lemma 5 one can check that erasing *any* pair of petals of the same color from the resulting diagram (the middle one on Fig. 8) one gets a realizable diagram. In particular, we may think of the pair of petals colored in light grey (the petals corresponding to North-East and South-West directions) as of the pair of petals "just bubbled". Hence, we may apply Lemma 12 to this pair of petals. Rotating two sectors clockwise this pair of petals of the same color (the ones corresponding to North-East and South-West directions) we modify the middle diagram on Fig. 8 to the diagram of the type O in genus $g = 4$ (the right one on Fig. 8).

Lemma 17 is proved. □

5.2. Stratification of $\mathcal{H}_g$ near a given stratum. Let (C, ω) be a complex curve C with an Abelian differential ω representing a point $x = [(C, \omega)] \in \mathcal{H}_g$ of the moduli space $\mathcal{H}_g$ of *all* Abelian differentials, $g \geq 2$. A germ U of the orbifold $\mathcal{H}_g$ at the point x is the quotient $\tilde{U}/\Gamma$ where $\tilde{U}$ is a germ of a complex manifold of dimension $4g - 3$ and $\Gamma = Aut(C, \omega)$ is a finite group acting on $\tilde{U}$, preserving the base point $\tilde{x} \in \tilde{U}$ corresponding to x. (For a generic point $x = [(C, \omega)] \in \mathcal{H}_g$ the group of automorphisms Γ is trivial.) Our goal here is to describe the germ $\tilde{U}$ together with the stratification of $\tilde{U}$ by multiplicities of zeroes induced from $\mathcal{H}_g$.

By definition $\tilde{U}$ is a universal analytic deformation of the pair (C, ω), i.e. it is an analytic family of pairs $(C_y, \omega_y)_{y \in \tilde{U}}$ together with an identification $i : (C_{\tilde{x}}, \omega_{\tilde{x}}) \simeq (C, \omega)$ of the distinguished element $(C_{\tilde{x}}, \omega_{\tilde{x}})$ of the family with (C, ω), such that any germ of deformations of (C, ω) is induced canonically from $\tilde{U}$.

Let $P_1, \dots, P_n$ be all zeroes of ω (enumerated in some order), and $k_1, \dots, k_n$ be their multiplicities. Let us also choose local coordinates $z_1, \dots, z_n$ near the zeroes in such way that $z_i(P_i) = 0$ and $\omega = z_i^{k_i} dz_i$. The choice of local coordinates z_i is canonical up to a transformation $z_i \mapsto \xi_i z_i$ where ξ_i is a root of unity, $\xi_i^{k_i+1} = 1$. A deformation $(C_y, \omega_y)_{y \in \tilde{U}}$ defines for each i, $1 \le i \le n$, a deformation of the germ $z_i^{k_i} dz_i$ of a holomorphic 1-form on a complex curve.

It is an easy and well-known corollary of the standard deformation theory of singularities of functions that for the germs $z^k dz$ of 1-forms there exists a universal deformation over a germ of $(k-1)$-dimensional manifold. Namely, the following proposition holds.

Proposition 3. *Let $\pi : (\mathcal{C}, c) \to (\mathcal{B}, b)$ be a map between germs of analytic spaces with based points with non-singular fibers of dimension* 1*, and let $\omega \in \Gamma(\mathcal{C}, T^*_{\mathcal{C}/\mathcal{B}})$ be a* 1*-form along the fibers of π not equal identically to zero. Let z_b be a local coordinate on $\pi^{-1}(b)$ such that $\omega|_{\pi^{-1}(b)}$ is equal to $z_b^k dz_b$ for some $k \ge 0$. Assume that $k \ge 1$. Then there exist a unique collection of $k-1$ functions $a_2, \dots a_k$ on $\mathcal{B}$ vanishing at b, and a holomorphic function z on $\mathcal{C}$ extending z_b such that*

$$\omega = \left(z^k + a_2 z^{k-2} + \cdots + a_k\right) dz.$$

Proof. Germs of 1-forms in one variable can be identified by integration with germs of functions modulo constants. Now apply the standard fact: the universal deformation of the germ z^{k+1} is given by the formula $z^{k+1} + a'_2 z^{k-1} + \cdots + a'_{k+1}$ where $(a'_i)_{i=2,\dots,k+1}$ are parameters. It remains to let $a_i := a'_i \cdot (k+1-i)/(k+1)$ for $i = 2, \dots, k$. □

We denote by $\mathcal{P}_k$ the germ in the space $\mathbb{C}^{k-1}$ endowed with coordinates $a_2, \dots, a_k$ parameterizing 1-forms $(z^k + a_2 z^{k-2} + \cdots + a_k) dz$ near $z = 0 \in \mathbb{C}$. The above proposition says that any deformation of a germ of a zero of order k of a 1-form is induced canonically from $\mathcal{P}_k$. We also get a canonical local coordinate z on deformed germs of curves. In notations of the proposition we call a point $z_{b'} \in \pi^{-1}(b')$ given by the equation $z(z_{b'}) = 0$ the *holomorphic center of masses* of zeroes (near the base point c). The reason is that for a polynomial form $\omega = (z^k + a_2 z^{k-2} + \cdots + a_k) dz$ the arithmetic mean of zeroes of ω coincides with $0 \in \mathbb{C}$. (Note that the notion of the holomorphic center of masses is *not* invariant under the $GL(2, \mathbb{R})_+$-action for the case $k \ge 3$.)

Now we are ready to construct local coordinates on the germ $\tilde{U}$ associated with a global curve C endowed with an Abelian differential ω. Using the notations introduced above, we define a map

$$\Phi : \tilde{U} \to \prod_{i=1}^{n} \mathcal{P}_{k_i} \times H^1(C, \{P_1, \dots, P_n\}; \mathbb{C}). \tag{5}$$

The components of this map have the following description. First of all, for each i, $1 \le i \le n$, we construct a canonical map $\tilde{U} \to \mathcal{P}_{k_i}$, applying Proposition 3 to a neighborhood of the point P_i. Secondly, for deformed curves (C', ω') with Abelian differentials we have *canonical* local holomorphic centers of masses $P'_1, \dots, P'_n \in C'$. We associate with (C', ω') an element in $H^1(C, \{P_1, \dots, P_n\}; \mathbb{C})$ (close to $[\omega]$) using $[\omega'] \in H^1(C', \{P'_1, \dots, P'_n\}; \mathbb{C})$ and an identification of the cohomology spaces

$$H^1(C, \{P_1, \dots, P_n\}; \mathbb{C}) \simeq H^1(C', \{P'_1, \dots, P'_n\}; \mathbb{C})$$

given by any continuous map

$$(C, \{P_1, \dots, P_n\}) \to (C', \{P'_1, \dots, P'_n\})$$

close to the identity map (in other words, using the holonomy of the Gauss–Manin connection).

An easy calculation with the tangent spaces shows that Φ is a local isomorphism. Thus, we constructed, in a sense, local coordinates in a neighborhood of any point of $\mathcal{H}_g$. The stratification of $\tilde{U}$ given by the multiplicities of zeroes is obvious. Namely, we should count the numbers of zeroes of given multiplicities in deformed polynomial Abelian differentials. Also, the transversal slice in $\tilde{U}$ to the stratum containing the base point $\tilde{x}$ is identified with the product of germs $\mathcal{P}_{k_i}$.

Using this description of the local structure of $\mathcal{H}_g$ we draw the main conclusion for our classification program:

Corollary 2. *For any stratum $\mathcal{H}(k_1, \dots, k_n)$ of $\mathcal{H}_g$ and for any connected component S of $\mathcal{H}(2g-2)$ there exists exactly one connected component S' of $\mathcal{H}(k_1, \dots, k_n)$ adjacent to S, i.e. such that S is contained in the closure $\overline{S'}$ of S' in $\mathcal{H}_g$.*

Proof. Let us prove that for any point $x = [(C, \omega)] \in \mathcal{H}(2g-2)$ of the minimal stratum $\mathcal{H}(2g-2) \subset \mathcal{H}_g$ one can find a sufficiently small neighborhood $U(x) \subset \mathcal{H}_g$ of x in $\mathcal{H}_g$ such that the intersection of $U(x)$ with $\mathcal{H}(k_1, \dots, k_n)$ is nonempty and connected. Obviously lemma follows from this statement.

Applying formula 5 to a point $x = [(C, \omega)] \in \mathcal{H}(2g-2)$ of the minimal stratum $\mathcal{H}(2g-2)$ we establish a local diffeomorphism between the germ $\tilde{U}(x)$ and $\mathcal{P}_{2g-2} \times H^1(C, \{P_1\}; \mathbb{C})$. Here $P_1 \in C$ is the single zero of order $2g-2$. Our statement now follows from the fact that the germ of any stratum in $\mathcal{P}_{2g-2}$ is nonempty and connected. □

Corollary 3. *For any parameters $(k_1, \dots, k_n)$, where $\sum k_i = 2g-2$ and $n \ge 2$, any component S of the stratum $\mathcal{H}(k_1+k_2, k_3, \dots, k_n)$ of $\mathcal{H}_g$ and for any two connected components S_1, S_2 of $\mathcal{H}(2g-2)$ to which S is adjacent, there exists a connected component S' of $\mathcal{H}(k_1, k_2, \dots, k_n)$ which is also adjacent to S_1 and S_2.*

Proof. By assumption we have $\overline{S} \supset S_1 \cup S_2$. It follows from our picture of the local behavior of the stratification that there exists a connected component S' of $\mathcal{H}(k_1, k_2, \ldots, k_n)$ adjacent to S, i.e. $\overline{S'} \supset S$. Therefore we have $\overline{S'} \supset \overline{S} \supset S_1 \cup S_2$. □

5.3. Merging zeroes and adjacency to the minimal stratum. In this section we prove

Proposition 4 (Merging zeroes). *For any given parameters $k_1, \ldots, k_n$, where $n \geq 2$ and $\sum_i k_i = 2g - 2$, the closure of any connected component of a stratum $\mathcal{H}(k_1, k_2, \ldots, k_n) \subset \mathcal{H}_g$ contains some connected component of the stratum $\mathcal{H}(k_1 + k_2, k_3, \ldots, k_n)$.*

Remark 6. Note that in general the similar statement is no longer true for the strata of quadratic differentials. Say, one of the two connected components of the stratum of meromorphic quadratic differentials having a single simple pole, and a single zero of degree 9 is adjacent to the stratum of quadratic differentials with a single zero of degree 8, and the other component — not.

We start the proof with the following technical result (see the similar result in [10] for the principal stratum of quadratic differentials).

Lemma 18. *Every connected component of every stratum contains an Abelian differential whose horizontal foliation has only closed leaves and the corresponding diagram has only one cylinder.*

Proof. First of all, let us choose an Abelian differential in a given component of a stratum whose *vertical* foliation has only closed leaves, see Lemma 4. Deforming slightly this differential preserving the structure of the vertical measured foliation we can assume that the horizontal foliation is uniquely ergodic, in particular, minimal. This follows immediately from results of H. Masur, see [11], and W.A. Veech, see [17].

Let us pick a point P on our surface which is not connected by a leaf of the horizontal foliation to a critical point. The leaf of the horizontal foliation starting at P is everywhere dense. For any $\epsilon > 0$ we can now follow the leaf until it returns for the first time to the distance ϵ from the point P. For sufficiently small ϵ if we connect the endpoints of our piece of horizontal leaf by the geodesic interval of length ϵ, and then perturb slightly the obtained closed curve, we obtain a smooth closed curve γ transversal to the vertical foliation (see [20] for details).

If we choose curve γ long enough, it will intersect all cylinders of the vertical foliation and, moreover, all vertical saddle connections. Our goal now is to modify the flat structure on our surface preserving the vertical measured foliation and making the horizontal foliation satisfy the properties in the lemma.

Let us remove all vertical saddle connections and curve γ from our surface; let us forget the horizontal foliation. We obtain a finite collection of curvilinear 4-gons endowed with vertical measured foliations. Now let us construct on 4-gons new horizontal measured foliations, in such way that the horizontal parts of the boundary of every 4-gon would be leaves of the new foliation. Namely, we say that the vertical length of each 4-gon is equal to one, and if a critical point is situated on the vertical part of the boundary of a 4-gon, it should be exactly in the middle. The last condition can be always fulfilled because by our assumption we cannot have more than one critical point on any vertical part of the boundary of any 4-gon.

We endow each 4-gon with the canonical flat metric (and with two measured foliations) and then glue the rectangles thus obtained together. On the new surface both the vertical and the horizontal foliations have only closed leaves. The diagram corresponding to the vertical foliation of the new surface coincides with the diagram for the initial Abelian differential, thus we land to the same connected component. Finally, the horizontal foliation of the new surface has only one cylinder, the union of two strips of width $1/2$ on the left and on the right of the closed line γ. □

Remark 7. Actually, we have proved a stronger statement: *Abelian differentials satisfying conditions of Lemma 18 are dense in every connected component of any stratum.*

Here is the proof of this statement. We did not change the vertical foliation at all, so we do not deform the closed 1-form ω_v. We modify the horizontal foliation in two steps: at the first step we perturb the closed 1-form ω_h to some ω'_h to make the horizontal foliation uniquely ergodic. At the second step we replace ω'_h by a closed 1-form ω''_h, whose cohomology class, by construction, is Poincaré-dual to the cycle $[\gamma]$; the final horizontal foliation is the kernel foliation of ω''_h. Note that it follows from ergodicity of the intermediate foliation corresponding to ω'_h and from definition of ω''_h that choosing the curve γ sufficiently long we get

$$\int_\rho \omega'_h \approx \frac{1}{|\gamma|} \cdot (\text{number of intersections of } \gamma \text{ with } \rho) \approx \frac{1}{|\gamma|} \int_\rho \omega''_h$$

for any path ρ transversal to ω'_h (here we assume that the total area of the surface measured in the flat metric is normalized to one). By construction the integral of $\frac{1}{|\gamma|} \cdot \omega''_h$ along a piece of leaf of ω' is close to zero (provided this piece of leaf is much shorter than γ). Thus, choosing γ sufficiently long we can make ω'_h and $\frac{1}{|\gamma|} \cdot \omega''_h$ arbitrarily close. Hence, the resulting Abelian differential determined by a pair of closed 1-forms $(\omega_v, \frac{1}{|\gamma|} \cdot \omega''_h)$ is close to the initial Abelian differential corresponding to (ω_v, ω_h).

Now we are ready to prove Proposition 4.

Proof of Proposition 4. Given a connected component S' of a stratum $\mathcal{H}(k_1, \ldots, k_n)$, choose an Abelian differential ω in this component as in Lemma 18. Consider the diagram of its horizontal foliation. Consider saddle connections of this diagram. It is easy to see that *any* choice of strictly positive lengths of these saddle connections gives a solution of the system of linear equations (as in Lemma 5).

Since the union of all nonsingular leaves of the horizontal foliation forms a single cylinder, the underlying graph of the diagram is connected. In particular, every saddle is connected to at least one *other* saddle by a separatrix.

Consider a diagram obtained by shrinking this saddle connection to a point. The diagram is obviously realizable; it represents an Abelian differential from the stratum $\mathcal{H}(k_1+k_{j_1}, k_2, k_3, \dots, \widehat{k_{j_1}}, \dots, k_n)$. (For the moment we cannot control the index j_1; we just know that $j_1 \neq 1$.) We may shrink the saddle connection continuously without changing other parameters of the diagram. Thus we get a continuous path with interior in the chosen component of $\mathcal{H}(k_1, \dots, k_n)$ and with one of the endpoints belonging to $\mathcal{H}(k_1 + k_{j_1}, k_2, k_3, \dots, \widehat{k_{j_1}}, \dots, k_n)$. We have proved that the connected component S' is adjacent to some connected component of a stratum with a smaller number of zeroes. Repeating inductively this procedure we conclude that S' is adjacent to some connected component $S_1 \in \mathcal{H}(2g-2)$ of the minimal stratum.

By Corollary 2 there exist a connected component $S \subset \mathcal{H}(k_1 + k_2, k_3, \dots, k_n)$ adjacent to S_1. By Corollary 3 there exist a connected component $S'' \subset \mathcal{H}(k_1, \dots, k_n)$ adjacent to S and to S_1. Since both connected components $S', S'' \in \mathcal{H}(k_1, \dots, k_n)$ are adjacent to $S_1 \in \mathcal{H}(2g-2)$ Corollary 2 implies that $S' = S''$. By construction S'' is adjacent to the stratum $\mathcal{H}(k_1 + k_2, k_3, \dots, k_n)$ which completes the proof of Proposition 4. □

It would be convenient to formulate an intermediate result of our proof as a separate corollary.

Corollary 4. *The closure of any component of any stratum $\mathcal{H}(k_1, \dots, k_n)$ contains a connected component of the stratum $\mathcal{H}(2g-2)$, where $2g-2 = k_1 + \dots + k_n$.*

5.4. Connected components of general strata. First note that any stratum $\mathcal{H}(k_1, \dots, k_n)$ is nonempty for any collection of positive integers k_i, such that the sum of all k_i is even, see [12]. Another way to see this is to perturb Abelian differentials from $\mathcal{H}(2g-2)$ (see Corollary 2) which we have constructed directly, see Lemma 15.

Moreover, for any positive integers $l_1, \dots, l_n$, $l_1 + \dots + l_n = g-1$, perturbing an Abelian differential $\omega \in \mathcal{H}(2g-2)$ we obtain an Abelian differential $\omega' \in \mathcal{H}(2l_1, \dots, 2l_n)$, having the same parity of the spin structure as the initial Abelian differential $\omega \in \mathcal{H}(2g-2)$, as follows from invariance of the parity of the spin-structure under continuous deformations, see [1], [14]. Thus, using our direct construction of hyperelliptic components $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$, and perturbing Abelian differentials from the connected components $\mathcal{H}^{odd}(2g-2)$, $\mathcal{H}^{even}(2g-2)$ we can get all the components listed in Theorems 1 and 2. To complete the proofs of these theorems we have to prove that all the components listed in the theorems are connected, and that there are no other components.

Lemma 19. *Any stratum $\mathcal{H}(k_1, \dots, k_n)$ has at most three connected components.*

Proof. The statement follows immediately from combination of Corollary 1, Corollary 2, and Corollary 4. □

A component of $\mathcal{H}(2g-2)$ uniquely determines the embodying component of $\mathcal{H}(k_1, \dots, k_n)$, but the embodying component may contain in the closure two, or even three components of $\mathcal{H}(2g-2)$, see Propositions 5 and 6 below.

Proposition 5. *For any genus $g \geq 3$ and any k, $1 \leq k < g-1$, there is a continuous path $\gamma : [0; 1] \to \mathcal{H}_g$ such that $\gamma(]0, 1[) \subset \mathcal{H}(k, 2g-k-2)$, endpoint $\gamma(1)$ belongs to the hyperelliptic component of $\mathcal{H}(2g-2)$, and endpoint $\gamma(0)$ belongs to one of two nonhyperelliptic components of $\mathcal{H}(2g-2)$.*

Proof. The path is presented at Fig. 9. Every diagram is easily seen to be realizable. Note that we may preserve the heights and the widths (measured in our flat metric) of all cylinders along the path; we just change the identification of the boundary components. This implies the continuity of the path.

The bottom diagram is centrally symmetric and obeys the conditions of Lemma 8. Thus it corresponds to an Abelian differential from $\mathcal{H}^{hyp}(2g-2)$. By assumption of the proposition $k > 0$ and $g-k-1 > 0$. Thus the top diagram is not centrally symmetric (see Fig. 9). Hence it corresponds to a nonhyperelliptic component. □

Proposition 6. *For any genus $g \geq 4$ and any k, $1 \leq k \leq g/2$, there is a continuous path $\gamma : [0; 1] \to \mathcal{H}_g$ such that $\gamma(]0, 1[) \subset \mathcal{H}(2k-1, 2(g-k)-1)$, one of the endpoints lies in the component $\mathcal{H}^{even}(2g-2)$ and another endpoint lies in the component $\mathcal{H}^{odd}(2g-2)$.*

Proof. The path is presented at Fig. 10. Again it is easy to see that all the diagrams are realizable, and that we may preserve the heights and the widths of all cylinders along the path. It is easy to see that neither top nor bottom diagram is centrally symmetric, thus they represent nonhyperelliptic components of $\mathcal{H}(2g-2)$.

Let us prove that the parities of the spin structure corresponding to the top and the bottom diagrams are different. Constructing the diagrams we may "bubble" the handle painted in black at the very last step. Thus we may "erase" corresponding pair of simple loops both on the top and on the bottom diagram. After erasing this pair of simple loops we obtain the same diagram on top and at the bottom. Let φ_0 be the parity of the spin structure corresponding to the diagram thus obtained. By Lemma 13 the parity of the spin structure corresponding to the initial top diagram equals φ_0, while the parity of the spin structure corresponding to the initial bottom diagram equals $\varphi_0 + 1$. □

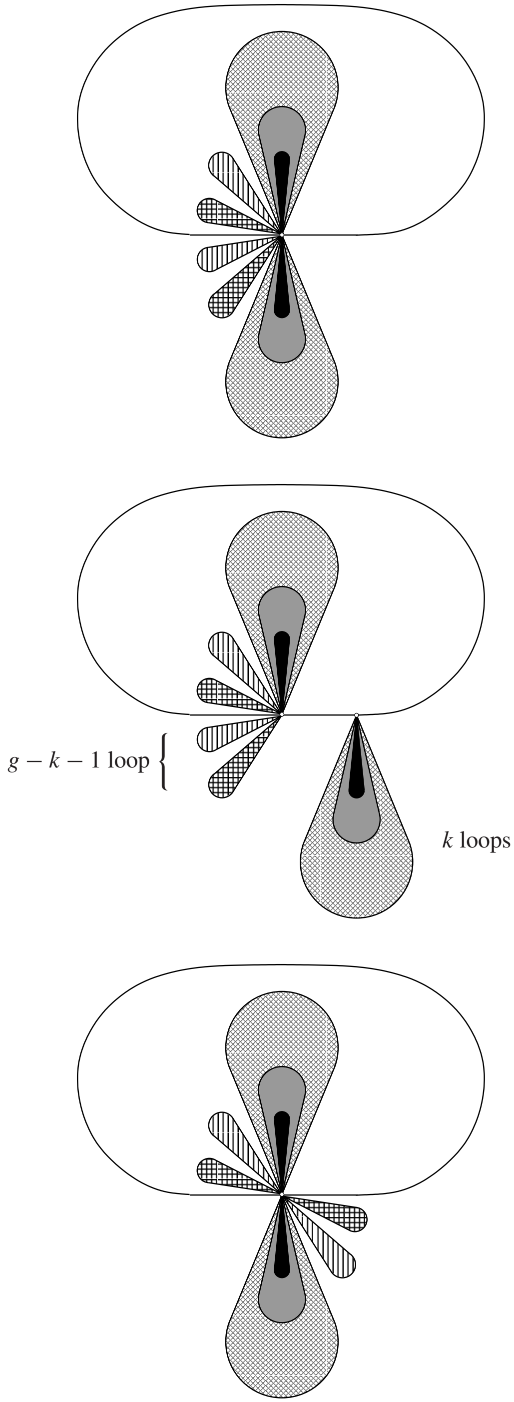

Fig. 9. A path in $\mathcal{H}(k, 2g-2-k)$ joining hyperelliptic and nonhyperelliptic components of $\mathcal{H}(2g-2)$

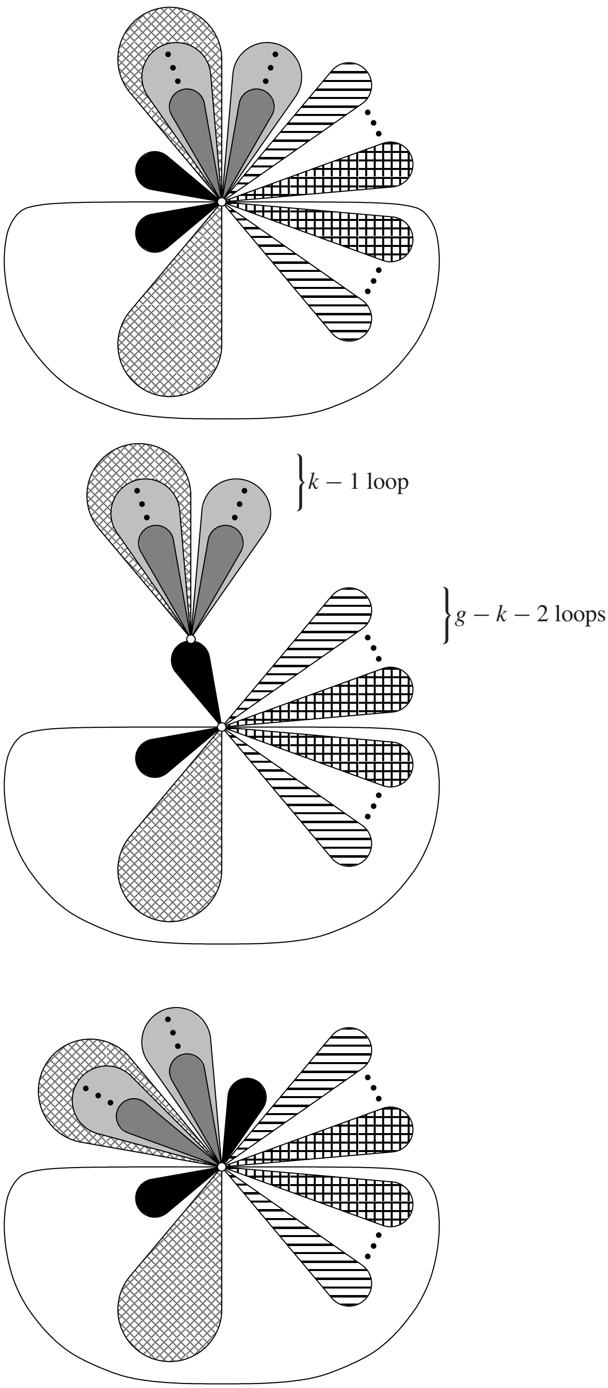

Fig. 10. A path in $\mathcal{H}(2k-1, 2(g-k)-1)$ joining $\mathcal{H}^{even}(2k-2)$ and $\mathcal{H}^{odd}(2k-2)$

Now we are ready to finish the proof of the classification Theorems 1 and 2. Recall that we already have a surjection from the set of connected components of the minimal stratum $\mathcal{H}(2g-2)$ to the set of connected components of any other stratum, and also two invariants (to be or not to be hyperelliptic, or to have even or odd spin structure) separating connecting components.

Let us start with the case $g \geq 4$. For stratum $\mathcal{H}(2g-2)$ we have achieved already the classification with three components, see Corollary 1. Similarly, we treat the case of the stratum $\mathcal{H}(2l, 2l)$, $l \geq 2$. For all the other strata of the form $\mathcal{H}(2l_1, \ldots, 2l_n)$ we will have only two connected components distinguished by the parity of spin structure. The reason is that the component adjacent to the hyperelliptic component in $\mathcal{H}(2g-2)$ is also adjacent to a nonhyperelliptic component, as follows from Proposition 5 and Corollary 3.

For strata $\mathcal{H}(2k-1, 2k-1)$ with $k \geq 2$ we will have only two components distinguished now by hyperellipticity, now we use Proposition 6 and Corollary 3.

For any other stratum we will have at least one of multiplicities which is odd and not equal to $g-1$. In this case Propositions 5 and 6 together with Corollary 3 finish the job, showing that we have only one connected component. Thus, we proved Theorem 1.

In the case $g = 3$ the upper bound of the number of connected components is 2. We treat cases $\mathcal{H}(4)$ and $\mathcal{H}(2, 2)$ as above and get two components. In all other cases we will have at least one of multiplicities equal to 1 and here we apply Proposition 5. In the case $g = 2$ the upper bound is already equal to 1 and we conclude that all strata are connected. □

Appendix A. Rauzy classes and zippered rectangles

A.1. Interval exchange transformations. In this section we recall the notions of *interval exchange transformation*, of *Rauzy class*, see [15], and the construction of a complex curve endowed with an Abelian differential by means of *"zippered rectangles"*, see [17].

Consider an interval $I \subset \mathbb{R}$, and cut it into m subintervals of lengths $\lambda_1, \ldots, \lambda_m$. Now glue the subintervals together in another order, according to some permutation $\pi \in \mathfrak{S}_m$ and preserving the orientation. We again obtain an interval I of the same length, and hence we get a mapping $T : I \to I$, which is called *interval exchange transformation*. Our mapping is a piecewise isometry, and it preserves the orientation and Lebesgue measure. It is singular at the points of cuts, unless two consecutive intervals separated by a point of cut are mapped to consecutive intervals in the image.

Remark 8. Note, that actually there are two ways to glue the subintervals "according to permutation π". We may send the k-th interval to the place $\pi(k)$, or we may have the intervals in the image appear in the order $\pi(1), \ldots, \pi(m)$. We use the first way; under this choice the second way corresponds to permutation π^{-1}.

Given an interval exchange transformation T corresponding to a pair (λ, π), $\lambda \in \mathbb{R}_+^m$, $\pi \in \mathfrak{S}_m$, set $\beta_0 = 0$, $\beta_i = \sum_{j=1}^{i} \lambda_j$, and $I_i = [\beta_{i-1}, \beta_i[$. Define skew-symmetric $m \times m$-matrix $\Omega(\pi)$ as follows:

$$\Omega_{ij}(\pi) = \begin{cases} 1 \text{ if } i < j \text{ and } \pi(i) > \pi(j) \\ -1 \text{ if } i > j \text{ and } \pi(i) < \pi(j) \\ 0 \text{ otherwise.} \end{cases} \tag{6}$$

Consider a translation vector $\delta = \Omega(\pi) \cdot \lambda$. Our interval exchange transformation T is defined as follows:

$$T(x) = x + \delta_i, \qquad \text{for } x \in I_i,\ 1 \le i \le m.$$

A.2. Extended Rauzy classes. Consider an Abelian differential $\omega \in \mathcal{H}(k_1, ..., k_n)$ on a surface of genus $g \geq 2$. Consider corresponding vertical (or horizontal) measured foliation on the Riemann surface. For generic ω every nonsingular leaf of the foliation is dense on the surface. Take an interval I transversal to the foliation. Our foliation is oriented, so it defines the Poincaré map (the first return map) $I \to I$. It is easy to see that the map T is an interval exchange transformation. The number of intervals under exchange is $2g + n - 1$, $2g + n$, or $2g + n + 1$ depending on the choice of I. (Morally, one has to place the endpoints of the transversal interval on the critical leaves of the foliation to obtain the minimal possible number of subintervals.) In particular the choice of transversal interval I determines some permutation π. Consider the set $\mathfrak{R}_{ex}$ of all possible permutations $\pi \in \mathfrak{S}_{2g+n-1}$ which can be obtained by choosing different transversal intervals I. It was proved by W.A. Veech in [17] that the set $\mathfrak{R}_{ex}$ does not depend on the choice of a generic representative ω in any connected component of $\mathcal{H}(k_1, \ldots, k_n)$. The set $\mathfrak{R}_{ex}$ is called *extended Rauzy class*, see [15], [17].

Conversely, given an interval exchange transformation $T : I \to I$ one can construct a complex curve C_g and an Abelian differential ω on it, such that the Poincaré map induced by the vertical foliation on the appropriate embedded subinterval would give the initial interval exchange transformation. Though the choice of the pair (C_g, ω) is not unique, topology of (C_g, ω) (genus, degrees $k_1, \ldots, k_n$ of zeroes of ω, and even the connected component of $\mathcal{H}(k_1, \ldots, k_n)$) are uniquely determined by the permutation π. We review the construction of *suspension over an interval exchange transformation* in Appendix A.4, more details can be found in [11] or in [17].

In the section below we present a direct combinatorial definition of the extended Rauzy class, see [15], [17].

A.3. Combinatorics of Rauzy classes. Note, that if for some $k < m$ we have $\pi\{1, \dots, k\} = \{1, \dots, k\}$, then the corresponding interval exchange transformation T decomposes into two interval exchange transformations. We consider only the class $\mathfrak{S}_m^0$ of *irreducible* permutations — those which have no invariant subsets of the form $\{1, \dots, k\}$, where $1 \le k < m$.

Permutation π is called *degenerate* if it obeys one of the following conditions (see 3.1–3.3 in [11] or equivalent conditions 5.1–5.5 in [17]):

for some $1 \le j < m$,

$$\begin{aligned} \pi(j) &= m \\ \pi(j+1) &= 1 \\ \pi(1) &= \pi(m) + 1 \end{aligned}$$

for some $1 \le j < m$,

$$\begin{aligned} \pi(j+1) &= 1 \\ \pi(1) &= \pi(j) + 1 \end{aligned}$$

for some $1 \le j < m$,

$$\begin{aligned} \pi(j+1) &= \pi(m) + 1 \\ \pi(j) &= m. \end{aligned}$$

Otherwise permutation π is called *nondegenerate*.

We denote by $\tau_k \in \mathfrak{S}_m$, $1 \le k < m$ the following permutation:

$$\begin{aligned} \tau_k \quad &= (1, 2, \dots, k, k+2, \dots, m, k+1) \quad 1 \le k < m-1 \\ \tau_{m-1} &= (1, 2, \dots, m) = id. \end{aligned}$$

Permutation τ_k cyclically moves one step forward all the elements occurring after the element k.

Consider two maps $a, b : \mathfrak{S}_m^0 \to \mathfrak{S}_m^0$ on the set of irreducible permutations (see [15]):

$$\begin{aligned} a(\pi) &= \pi \cdot \tau^{-1}_{\pi^{-1}(m)} \\ b(\pi) &= \tau_{\pi(m)} \cdot \pi \end{aligned}$$

where one should consider product of permutations as composition of operators — from right to left. Say, $b(2, 3, 1) = (1, 3, 2) \cdot (2, 3, 1) = (3, 2, 1)$. We may consider permutation as a pair of orderings of a finite set: a "domain" ordering and an "image" ordering. Operator b corresponds to the modification of the image ordering by cyclically moving one step forward those letters occurring after the image of the last letter in the domain, i.e., after the letter m. Operation a corresponds to the modification of the ordering of the domain by cyclically moving one step forward those letters occurring after one going to the last place, i.e., after $\pi^{-1}(m)$.

Note, that

$$(a(\pi))^{-1} = b(\pi^{-1}).$$

In components the maps a, b are represented as follows:

$$a(\pi)(j) = \begin{cases} \pi(j) & j \le \pi^{-1}(m) \\ \pi(m) & j = \pi^{-1}(m) + 1 \\ \pi(j-1) & \text{other } j \end{cases}$$

$$b(\pi)(j) = \begin{cases} \pi(j) & \pi(j) \le \pi(m) \\ \pi(j) + 1 & \pi(m) < \pi(j) < m \\ \pi(m) + 1 & \pi(j) = m. \end{cases}$$

Definition 6. *The Rauzy class $\mathfrak{R}(\pi)$ of an irreducible permutation π is the subset of those permutations in $\mathfrak{S}_m^0$ which can be obtained from π by some composition of maps a and b.*

Consider the permutation $\pi_0 = (m, m-1, \dots, 2, 1)$, and the map

$$\mathrm{Ad}_{\pi_0} : \pi \mapsto \pi_0^{-1} \pi \pi_0 = \pi_0 \pi \pi_0.$$

Note that the map Ad_{π_0} maps an irreducible permutation to an irreducible one.

Definition 7. *The extended Rauzy class $\mathfrak{R}_{ex}(\pi)$ of an irreducible permutation π is the subset of permutations which can be obtained from π by some composition of the maps a, b, and* Ad_{π_0}.

Remark 9. A Rauzy class $\mathfrak{R}(\pi)$ (extended Rauzy class $\mathfrak{R}_{ex}(\pi)$) of a nondegenerate permutation π contains only nondegenerate permutations.

Theorem (W.A. Veech, [17]). *The extended Rauzy classes of nondegenerate permutations are in the one-to-one correspondence with the connected components of the strata in the moduli spaces of Abelian differentials.*

Using classification of the strata obtained in the current paper, article [21] presents an explicit construction of a representative of any extended Rauzy class.

Lemma (G. Rauzy, [15]). *Any Rauzy class $\mathfrak{R}$ contains at least one permutation π with the property*

$$\pi(m) = 1 \qquad \pi(1) = m.$$

For the convenience of the reader we give a sketch of the proof. We want to fulfill constraints $\pi(m) = 1$ and $\pi^{-1}(m) = 1$. Suppose that it is not the case. Let us compare numbers $\pi(m)$ and $\pi^{-1}(m)$. If the smallest of them is greater than 1, then applying one of operations a or b several times one can make another number strictly smaller. If the smallest among $\pi(m)$ and $\pi^{-1}(m)$ is equal to 1, then applying one of operations a or b several times one can make *both* numbers $\pi(m)$ and $\pi^{-1}(m)$ equal to 1. □

We need the following modification of this lemma.

Lemma 20. *Any extended Rauzy class $\mathfrak{R}_{ex}$ of nondegenerate permutations contains at least one permutation π with the following two properties*

$$\pi(m) = 1 \qquad \pi(1) = m.$$

The permutation

$$\pi' := \begin{pmatrix} \pi(2)\,, \pi(3)\,, \cdots, \pi(m-2)\,, \pi(m-1) \\ 2 \quad, \quad 3 \quad, \cdots, \quad m-2 \quad, \quad m-1 \end{pmatrix}$$

obtained as a restriction of π to the ordered set $\{2, 3, \dots, m-1\}$ is irreducible.

Proof. Consider a permutation π as in the previous lemma. Suppose that the restriction π' of π to the ordered subset $\{2, 3, \dots, m-1\}$ is reducible. Choose the maximal integer $a < m-1$ such that π' leaves the set $\{2, \dots, a\}$ invariant. In other words chose the rightmost position where we can break permutation π' into two nonempty permutations.

Consider the following ordered subsets:

$$\begin{aligned} A &:= \{2, \dots, a\} \\ B_1 &:= \{a+1, \dots, \pi(m-1)-1\} \\ B_2 &:= \{\pi(m-1), \dots, m-1\} \end{aligned}$$

where B_1 is an empty set when $\pi(m-1) = a+1$. Replace the initial permutation π by the following one contained in the same extended Rauzy class:

$$\begin{pmatrix} m \;\; 1 & & \pi(A) & & \pi(B_1) & & & \pi(B_2) & & \\ & B_2 & & 1 & & A & | & & B_1 & m \end{pmatrix}. \tag{7}$$

This permutation is obtained from permutation π by composition of the following two operations. We first make modification from the right by cyclically moving one step forward the elements of the top line occurring after the letter m. Then we make modification from the left by cyclically moving $card(B_2)$ steps forward the elements of the bottom line occurring before letter m.

After reenumeration of the elements in the standard order we see that in this standard enumeration our new permutation π_2 again has the property $\pi_2(1) = m$ and $\pi_2(m) = 1$. Restriction π_2' of this new permutation to the subset $\{2, \dots, m-1\}$ may be again reducible. We are going to prove that the restricted permutation may split only to the right of the marked place. In other words we are going to prove that if the subset $\{2, \dots, a_2\}$ is invariant under π_2' then $a_2 \ge a + \text{Card } B_2 > a$.

Since the initial permutation π is nondegenerate we have $\pi(m-1) \neq m-1$ (to see this let $j = m-1$ in the second condition on degenerate permutations at the beginning of this section). Thus card $B_2 > 1$. Hence the letter 1 cannot be the second letter in the bottom line of (7). Thus, if the splitting occurs, the leftmost invariant subset contains more than one element. Looking at the top line of (7) we see that this means that the leftmost invariant subset must contain at least one element of $\pi(A)$. Looking at the bottom line we see that the leftmost invariant subset contains at least one element of B_2. Note that the set A considered as an unordered set was chosen to be invariant under the permutation π. Thus $\pi(A)$ does not intersect with B_2. Hence the splitting may occur only to the right of the word $\pi(A)$ in the top line. Thus the leftmost invariant subset must contain all the elements of the unordered set $\pi(A) = A$. Thus the splitting may mapped only to the right of the marked position at the bottom line.

Repeating inductively this procedure we finally obtain an irreducible restricted permutation. □

A.4. Zippered rectangles (after W.A. Veech). Having an interval exchange transformation $T : I \to I$ one can "suspend" a smooth closed complex curve C_g and an Abelian differential ω over T. Here we present the idea of the "suspension"; one can find all the details in the original paper of W.A. Veech [17].

Place the interval I horizontally in the plane $\mathbb{R}^2 = \mathbb{C}$. Place a rectangle R_i over each subinterval $I_i \subset I$; the rectangle R_i has the width $\lambda_i = |I_i|$ and some altitude h_i. Later on we shall pose some restrictions on the altitudes. Glue the top horizontal side of rectangle R_i to the interval $T(I_i)$ at the base. There are still no identifications between the vertical sides of the rectangles, so we get a Riemann surface with several "holes"; each boundary component is a union of the vertical sides of the rectangles (see Figs. 11, 12). Now start "zipping" the holes (see Fig. 11). If the altitudes h_i of the rectangles, and the altitudes a_i till which we "zipper" the rectangles obey some linear equations

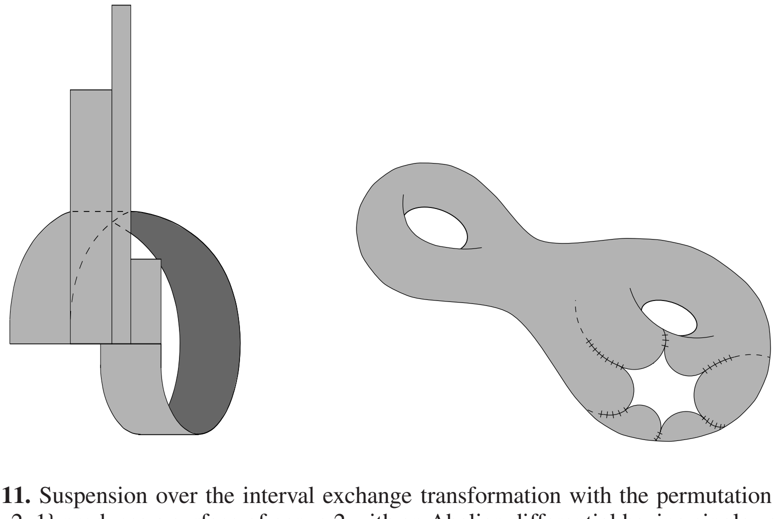

Fig. 11. Suspension over the interval exchange transformation with the permutation $\pi = \{4, 3, 2, 1\}$ produces a surface of genus 2 with an Abelian differential having single zero of order 2

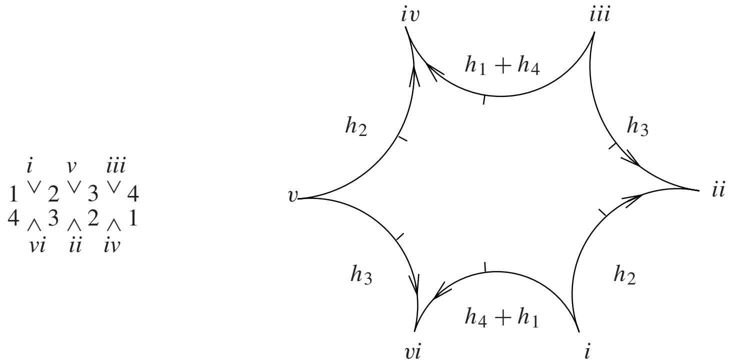

Fig. 12. The lengths of the sides of the hole for a suspension over an interval exchange transformation with the permutation $\pi = \{4, 3, 2, 1\}$

and inequalities (see [17]), then we manage to eliminate all the holes. The Riemann surface thus constructed has natural flat structure with cone-type singularities; the complex structure, coming from the initial complex structure on the plane $\mathbb{C} = \mathbb{R}^2$, extends to the conical points. The Abelian differential ω is locally represented as dz, where z is the standard coordinate in $\mathbb{C}$.

As we already mentioned the altitudes h_i, and a_i obey some linear relations (cf. Fig. 12); it is proved in [17] that the family of solutions is always nonempty. This family has dimension $m = 2g + k - 1 = \dim H^1(C_g, \{\text{zeroes of } \omega\})$, which coincides with the number m of subintervals under exchange, $\pi \in \mathfrak{S}_m$.

Appendix B. Abelian differentials on hyperelliptic curves

Let ω be an Abelian differential on a hyperelliptic complex curve such that all zeroes of ω are of even degrees. Let the canonical divisor $K(\omega)$ of ω be

$$
\begin{aligned}
K(\omega) = {} & 2(k_1 P_{i_1} + \cdots + k_p P_{i_p}) \\
& + 2\left(l_1(P_1^+ + P_1^-) + \cdots + l_q(P_q^+ + P_q^-)\right)
\end{aligned}
\tag{8}
$$

where

$$\sum_{i=1}^{p} k_i + 2\sum_{j=1}^{q} l_j = g - 1.$$

By P_{i_n} we denote the points which are invariant under hyperelliptic involution; by $P_j^{\pm}$ we denote the pairs of points symmetrical to each other under hyperelliptic involution. We assume that all the indicated points are distinct.

Proposition 7. *The parity of the spin structure defined by an Abelian differential on a hyperelliptic curve is given by the following formula:*

$$\varphi(\omega) \equiv \dim \left| \frac{1}{2} K(\omega) \right| + 1 (mod\, 2) = \sum_{i=1}^{p} \left[\frac{k_i}{2} \right] + \sum_{j=1}^{q} l_q + 1 (mod\, 2).$$

Proof. In our case a base of solutions of the linear system $\frac{1}{2}K(\omega)$ can be constructed explicitly. □

Corollary 5. *Parity of the spin structure determined by an Abelian differential from the hyperelliptic component* $\mathcal{H}^{hyp}(2g-2)$ *equals*

$$\varphi(\mathcal{H}^{hyp}(2g-2)) \equiv \left[\frac{g+1}{2} \right] (mod\, 2).$$

Parity of the spin structure of the hyperelliptic component $\mathcal{H}^{hyp}(g-1, g-1)$*, for odd genera* g *equals*

$$\varphi \left(\mathcal{H}^{hyp}(g-1, g-1) \right) \equiv \left(\frac{g+1}{2} \right) (mod\, 2) \quad for \;\; odd \;\; g.$$

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References

1. Atiyah, M.: Riemann surfaces and spin structures. Ann. Sci. Éc. Norm. Supér., IV. Sér. **4**, 47–62 (1971)
2. Calabi, E.: An intrinsic characterization of harmonic 1-forms. Global Analysis, Papers in Honor of K. Kodaira, D.C. Spencer and S. Iyanaga (eds.), 101–117 (1969)
3. Eskin, A., Masur, H., Zorich, A.: Moduli spaces of Abelian differentials: the principal boundary, counting problems and the Siegel–Veech constants, 88 pp. Submitted to Publ. Math., Inst. Hautes Étud. Sci.; electronic version in arXiv:math.DS/0202134
4. Hubbard, J., Masur, H.: Quadratic differentials and foliations, Acta Math. **142**, 221–274 (1979)
5. Johnson, D.: Spin structures and quadratic forms on surfaces. J. Lond. Math. Soc., II. Ser. **22**, 365–373 (1980)
6. Katok, A.B.: Invariant measures of flows on oriented surfaces, Soviet Math. Dokl. **14**, 1104–1108 (1973)
7. Kontsevich, M., Zorich, A.: Lyapunov exponents and Hodge theory. Preprint IHES M/97/13, 1–16 electronic version: hep-th/9701164
8. Lanneau, E.: Hyperelliptic components of the moduli spaces of quadratic differentials with prescribed singularities. To appear in Comment. Math. Helv.; electronic version in arXiv:math.GT/0210099
9. Maier, A.G.: Trajectories on closed orientable surfaces. Sb. Math. **12**, 71–84 (1943) (in Russian)

10. Masur, H.: The Jenkins–Strebel differentials with one cylinder are dense. Comment. Math. Helv. **54**, 179–184 (1979)
11. Masur, H.: Interval exchange transformations and measured foliations. Ann. Math. **115**, 169–200 (1982)
12. Masur, H., Smillie, J.: Quadratic differentials with prescribed singularities and pseudo-Anosov diffeomorphisms. Comment. Math. Helv. **68**, 289–307 (1993)
13. Milnor, J.: Remarks concerning spin manifolds. Differential and Combinatorial Topology (in Honor of Marston Morse). Princeton 1965
14. Mumford, D.: Theta-characteristics of an algebraic curve. Ann. Sci. Éc. Norm. Supér., IV. Sér. **2**, 181–191 (1971)
15. Rauzy, G.: Echanges d'intervalles et transformations induites. Acta Arith. **34**, 315–328 (1979)
16. Strebel, K.: Quadratic differentials. Springer 1984
17. Veech, W.A.: Gauss measures for transformations on the space of interval exchange maps. Ann. Math. **115**, 201–242 (1982)
18. Veech, W.A.: The Teichmüller geodesic flow. Ann. Math. **124**, 441–530 (1986)
19. Veech, W.A.: Moduli spaces of quadratic differentials. J. Anal. Math. **55**, 117–171 (1990)
20. Zorich, A.: How do the leaves of a closed 1-form wind around a surface. In: Pseudo-periodic Topology. AMS Translations, Ser. 2, **197**, 135–178. Providence, RI: Am. Math. Soc. 1999
21. Zorich, A.: Explicit construction of a representative of any extended Rauzy class. To appear