HOMOLOGICAL ALGEBRA OF MIRROR SYMMETRY

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Mirror Symmetry was discovered several years ago in string theory as a duality between families of 3-dimensional Calabi-Yau manifolds (more precisely, complex algebraic manifolds possessing holomorphic volume elements without zeroes). The name comes from the symmetry among Hodge numbers. For dual Calabi-Yau manifolds V, W of dimension n (not necessarily equal to 3) one has

$$\dim\ \mathrm{H}^p(V,\Omega^q) = \dim\ \mathrm{H}^{n-p}(W,\Omega^q)\ .$$

Physicists conjectured that conformal field theories associated with mirror varieties are equivalent. Mathematically, MS is considered now as a relation between numbers of rational curves on such a manifold and Taylor coefficients of periods of Hodge structures considered as functions on the moduli space of complex structures on a mirror manifold. Recently it has been realized that one can make predictions for numbers of curves of positive genera and also on Calabi-Yau manifolds of arbitrary dimensions.

We will not describe here the complicated history of the subject and will not mention many beautiful contsructions, examples and conjectures motivated by MS. On the contrary, we want to give an outlook of the story in general terms and propose a conceptual framework for a possible explanation of the mirror phenomenon. We will restrict ourselves to a half of MS considering it as a relation between symplectic structures on one side and complex structures on another side. Actually, we will deal only with a half of this half, ignoring the holomorphic anomaly effects (see [BCOV]) in the symplectic part (A-model) and the polarization of Hodge structures in the complex part (B-model). For an introduction to Mirror Symmetry we recommend [M] and [Y].

At the moment there are only few completely solid statements, essentially because there were no universal definition of the "number of curves" for a long time.

Comparision of symplectic and complex geometry.

We start with a recollection of well-known facts concerning symplectic and complex manifolds.

Let V be a compact smooth $2n$-dimensional manifold.

1.S. A symplectic structure on V is given by a reduction of the structure group $GL(2n, \mathbf{R})$ of the tangent bundle T_V to the subgroup $Sp\,(2n, \mathbf{R})$ satisfying certain integrability conditions (the associated 2-form ω is closed, or, equivalently the associated Poisson bracket on smooth functions satisfies the Jacobi identity).

1.C. A complex structure on V is given by a reduction of the structure group $GL(2n, \mathbf{R})$ of the tangent bundle T_V to the subgroup $GL(n, \mathbf{C})$ satisfying certain integrability conditions (the Newlander-Nirenberg theorem).

Notice that both groups $Sp\,(2n, \mathbf{R})$ and $GL(n, \mathbf{C})$ are homotopy equivalent to $U(n)$. Thus, they have the same algebra of characteristic classes generated by Chern classes $c_i \in \mathrm{H}^{2i}(BU(n), \mathbf{Z}),\ \ 1 \le i \le n$.

Basic examples of compact symplectic or complex manifolds are complex projective algebraic manifolds endowed with the pull-back of the Fubini-Studi Kähler form on the projective space.

2.S. First-order deformations of symplectic structures on V are in one-to-one correspondence with $\mathrm{H}^2(V, \mathbf{R})$. The deformation theory is unobstructed and the local moduli space of symplectic structures on V can be identified with a domain in the affine space $\mathrm{H}^2(V, \mathbf{R})$ via map $\omega \mapsto [\omega] \in \mathrm{H}^2(V, \mathbf{R})$ (J. Moser).

2.C. First-order deformations of complex structures on V near a fixed one are in one-to-one correspondence with $\mathrm{H}^1(V, T_V^{hol})$ where T_V^{hol} denotes the sheaf of holomorphic vector fields on V (Kodaira-Spencer theory). If $c_1(V) = 0$ and V admits a Kähler structure then the deformation theory of V is unobstructed and the local moduli space can be identified with a domain in the affine space $\mathrm{H}^1(V, T_V^{hol})$ (the Bogomolov-Tian-Todorov theorem).

The following two facts concern only complex manifolds.

3.C. For a complex manifold V admitting a Kähler structure there is a pure Hodge structure on the singular cohomology groups:

$$\mathrm{H}^k(V, \mathbf{Z}) \otimes \mathbf{C} \simeq \bigoplus_{p+q=k} \mathrm{H}^q(V, \Omega^p) \ \ .$$

4.C. With a complex algebraic manifold V one can associate the abelian category $Coh(V)$ of coherent sheaves on V and the triangulated category $\mathcal{D}^b(Coh(V))$ (the bounded derived category).

Our aim in this talk is to propose candidates for 4.S and 5.S in the context of symplectic geometry. The mirror symmetry should be a correspondence (partially defined and multiple valued) between symplectic and complex manifolds (both with $c_1 = 0$) identifying structures 2-4.

To get a feeling of what is going on it is instructive to look at a simplest case of the mirror symmetry, which is already highly non-trivial.

Two-dimensional tori (after R. Dijkgraaf).

Let Σ be a complex elliptic curve and $p_1, \ldots, p_{2g-2}$ are pairwise distinct points of Σ, where $g \geq 2$ is an integer. We consider holomorphic maps $\phi : C \longrightarrow \Sigma$ from compact connected smooth complex curves C to Σ, which have only one double ramification point over each point $p_i \in \Sigma$ and no other ramification points. By the Hurwitz formula the genus of C is equal to g. The set $X_g(d)$ of equivalence classes of such maps of degree $d \geq 1$ is finite, and for each $\phi : C \longrightarrow \Sigma$ its automorphism group

$$\text{Aut }(\phi) := \{f : C \to C \mid \phi \circ f = \phi\}$$

is finite. For $g \geq 2$ we introduce the generating series in one variable q as follows:

$$F_g(q) := \sum_{d \geq 1} \left(\sum_{[\phi] \in X_d(d)} \frac{1}{|\text{ Aut }(\phi)\ |} \right) q^d \in \mathbf{Q}[[q]] \ .$$

The following statement is now rigorously established due to efforts of several people (R. Dijkgraaf, M. Douglas, D. Zagier, M. Kaneko):

$$F_g \in \mathbf{Q}[E_2, E_4, E_6] \ ,$$

where E_k are the classical Eisenstein series,

$$E_k = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \left(\sum_{a|n} a^{k-1} \right) q^n \ .$$

E_k is a modular form of weight k for even $k \geq$ and E_2 is *not* a modular form. Here $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $\ldots$ are Bernoulli numbers. If one associates with E_k, $k = 2, 4, 6$, the degree k, then F_g has degree $6g - 6$.

One can regard Σ as a symplectic 2-dimensional manifold $(S^1 \times S^1, \omega)$ with the area $\int_\Sigma \omega$ equal to $-\log(q)$, $0 < q < 1$, and interpret weights q^d of ramified coverings as

$$\exp\left(-\text{area of } C \text{ with respect to the pullback of } \omega\right) \ .$$

Mirror symmetry in this example is the claim that the generating function for certain invariants of symplectic structures on $S^1 \times S^1$ is a "nice" function on the moduli space of complex structures on $S^1 \times S^1$. The two-dimensional torus is a self-dual manifold for MS.

Notice that the standard local coordinate $q = \exp(2\pi i \tau)$, $\operatorname{Im} \tau > 0$ on the moduli space of elliptic curves,

$$\text{Elliptic curve } = \mathbf{C} / \left(\mathbf{Z} + \mathbf{Z}\tau\right) \ ,$$

can be written as

$$q = \exp\left(2\pi i \frac{\int_{\gamma_1} \Omega}{\int_{\gamma_0} \Omega}\right)$$

where γ_0, γ_1 are two generators of $\mathrm{H}_1(\text{elliptic curve}, \mathbf{Z})$ and Ω is a non-zero holomorphic 1-form.

Quintic 3-folds (after [COGP]).

Here we describe the first famous prediction of physicists. Let V be a non-singular hypersurface in complex projective space $\mathbf{P}^4$ given by an equation $Q(x_1,\dots,x_5)=0$ of degree 5 in 5 homogeneous variables $(x_1:\dots:x_5)$. This complex manifold carries a top degree holomorphic differential form of non-degenerate at all points (a holomorphic volume element):

$$\Omega = \frac{1}{dQ}\sum_{i=1}^{5}(-1)^i x_i dx_1\wedge\dots\wedge\widehat{dx_i}\wedge\dots\wedge dx_5 \ .$$

H. Clemens conjectured that smooth rational curves on a generic quintic 3-fold are isolated. Recently it was checked up to degree 9. It is natural to count the number N_d of rational curves on V of fixed degree d. In fact, there are singular rational curves on V of degree 5, and one has to take them into account as well. At the end of the next section we will propose an algebro-geometric formula for the "physical" number of curves on V, both smooth and singular, without assuming the validity of the Clemens conjecture.

The mirror symmetry prediction is the following. First of all, we define the virtual number of curves of degree d as

$$N_d^{virt} := \sum_{k|d}\frac{1}{k^3}N_{d/k} \ \in \ \mathbf{Q} \ .$$

The reason for this fomula is that in string theory one counts not just curves in V but maps from rational curves to V. Any map $\mathbf{P}^1\longrightarrow V$ of positive degree is the composition of a rational map $\mathbf{P}^1\longrightarrow\mathbf{P}^1$ and of an embedding $\mathbf{P}^1\hookrightarrow V$. It was argued first by Aspinwall and Morrison [AM] that the factor associated with multiple coverings of degree k should be equal to $1/k^3$.

The complete generating function for rational curves is

$$F(t) := \frac{5}{6}t^3 + \sum_{d\geq 1} N_d^{virt}\exp{(dt)} \ .$$

The first summand here represents the contribution of maps of degree 0 (i.e., maps to a point of V).

On the mirror side we consider functions

$$\psi_0(z) = \sum_{n=0}^{\infty}\frac{(5n)!}{(n!)^5}z^n$$

$$\psi_1(z) = \ \log z\cdot\psi_0(z) + 5\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}\left(\sum_{k=n+1}^{5n}\frac{1}{k}\right)z^n$$

$$\psi_2(z) = \frac{1}{2}\,(\,\log z)^2\cdot\psi_0(z)+\dots$$

$$\psi_3(z) = \frac{1}{6}\,(\,\log z)^3 \cdot \psi_0(z) + \dots$$

which are solutions of the linear differential equation

$$\left(\left(z\frac{d}{dz}\right)^4 - 5z(5z\frac{d}{dz}+1)(5z\frac{d}{dz}+2)(5z\frac{d}{dz}+3)(5z\frac{d}{dz}+4)\right)\psi(z) = 0\ .$$

More precisely,

$$\sum_{i=0}^{3}\psi_i(z)\epsilon^i + O(\epsilon^4) = \sum_{n=0}^{\infty}\frac{(1+5\epsilon)\,(2+5\epsilon)\,\dots\,(5n+5\epsilon)}{\left((1+\epsilon)\,(2+\epsilon)\,\dots\,(n+\epsilon)\right)^5}\ z^{n+\epsilon}\ .$$

Functions $\psi_i(z)$ are periods $\int_{\gamma_i}\omega$ of the Calabi-Yau manifold $W = W(z)$ which is a resolution of singularities of the following singular variety:

$$\{(x_1 : x_2 : x_3 : x_4 : x_5)|\ x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = z\,x_1x_2x_3x_4x_5\}/(\mathbf{Z}/5\mathbf{Z})^3\ .$$

Here the group $(\mathbf{Z}/5\mathbf{Z})^3$ is the group of diagonal matrices preserving the equation,

$$\{\ \mathrm{diag}(\xi_1,\dots,\xi_5)\,|\ \ \xi_i^5 = 1,\ \prod_{i=1}^{5}\xi_i = 1\,\}/\{\,\xi\,\mathrm{Id}\ |\ \xi^5 = 1\,\}$$

and γ_i are certain singular homology classes with complex coefficients. Family of varieties $W(z)$ depending on 1 parameter is mirror dual to a universal family of smooth quintic 3-folds depending on 101 parameters.

The predictions of physicists is that

$$F\left(\frac{\psi_1}{\psi_0}\right) = \frac{5}{2}\cdot\frac{\psi_1\psi_2 - \psi_0\psi_3}{\psi_0^2}\ .$$

One of miracles in this formula is that

$$\exp\left(\frac{\psi_1}{\psi_0}\right) \in \mathbf{Z}[[z]]\ .$$

Also, numbers N_d computed via the mirror prediction are positive integers.

It is interesting that the contribution of *individual* non-parametrized rational curves on 3-dimensional Calabi-Yau manifolds is connected with variations of mixed Hodge structures in a fashion analogous to the mirror symmetry predictions. Namely, according to the Aspinwall and Morrison formula [AM] we have the following generating function:

$$F(t) = \sum_{d\geq 1}\frac{\exp\,(dt)}{d^3}\ .$$

We introduce functions ψ_*:

$$\psi_0(z)=1\,,\;\;\psi_1(z)=\log z\,,\;\;\psi_2(z)=\frac{1}{2}\,(\log z)^2\,,\;\;\psi_3(z)=\mathrm{Li}_3(z)\;:=\sum_{d\geq 1}\frac{z^d}{d^3}$$

which are solutions of the linear differential equation

$$\frac{d}{dz}\left(\frac{1-z}{z}\left(z\frac{d}{dz}\right)^3\,\psi(z)\right)=0\;.$$

Functions F and ψ_* are related by the evident formula

$$F\left(\frac{\psi_1}{\psi_0}\right)=\frac{\psi_3}{\psi_0}\;\;.$$

Gromov-Witten invariants.

We describe here a not yet completely constructed theory which has potentially wider domain of applications than mirror symmetry. It is based on pioneering ideas of M. Gromov [G] on the role of $\overline{\partial}$-equations in symplectic geometry, and certain physical intuition proposed by E. Witten [W1], [W2]. There are many evidences that the following picture from [KM] is correct.

Let (V,ω) be a closed symplectic manifold, $\beta\in\mathrm{H}_2(V,\mathbf{Z})$ be a homology class and $g,n\geq 0$ be integers satisfying the inequality $2-2g-n<0$. Gromov-Witten classes

$$I_{g,n;\beta}\in\mathrm{H}_D\left(\overline{\mathcal{M}}_{g,n}(\mathbf{C})\times V^n;\mathbf{Q}\right)$$

are homology classes with rational coefficients of degree

$$D=D(g,n,\beta)=(\dim V-6)(1-g)+2n+2\int_\beta c_1(T_V)\;\;.$$

Here $\overline{\mathcal{M}}_{g,n}$ denotes the Deligne-Mumford compactification of the moduli stack of smooth connected algebraic curves of genus g with n marked points. Recall that an algebraic curve C with marked points $p_1,\dots,p_n$ is called *stable* if

(1) all singular points of C are ordinary double points,
(2) marked points p_i are pairwise distinct and smooth, $p_i\in C^{smooth}$,
(3) the group of automorphisms of $(C:p_1,\dots,p_n)$ is finite, or, equivalently, the Euler characteristic of each connected component of $C^{smooth}\setminus\{p_1,\dots,p_n\}$ is negative.

The arithmetic genus of stable curve C is defined by the formula

$$g_a(C):=\;\dim\mathrm{H}^1(C,\mathcal{O})=\chi(C^{smooth})/2-1\;\;.$$

The stack $\overline{\mathcal{M}}_{g,n}$ is the moduli stack of stable marked curves of arithmetic genus g with n marked points. The associated coarse moduli space $\overline{\mathcal{M}}_{g,n}(\mathbf{C})$ is a compact complex orbifold.

One expects that $I_{g,n;\beta}$ is invariant under continuous deformations of the symplectic structure on V.

Analogously, we expect that the Gromov-Witten invariants can be defined for non-singular projective algebraic varieties over arbitrary fields and they take values in the Chow groups with rational coefficients instead of the singular homology groups.

Intuitively, the geometrical meaning of Gromov-Witten classes in the symplectic case can be described as follows. Let us choose an almost-complex structure on V compatible in the evident way with the symplectic form ω. Notice that the space of almost-complex structures compatible with the fixed ω is contractible. Denote by $X_{g,n}(V,\beta)$ the space of equivalence classes of $(C; x_1, \dots, x_n; \phi)$ where C is a smooth complex curve of genus g with pairwise distinct marked points x_i, and $\phi : C \to V$ is a pseudo-holomorphic map (i.e., a solution of the Cauchy-Riemann equation $\overline{\partial}\phi = 0$) such that the image of the fundamental class of C equal to β. There is a natural map from $X_{g,n}(V,\beta)$ to $\mathcal{M}_{g,n}(\mathbf{C}) \times V^n$ associating with $(C; x_*; \phi)$ the equivalence class of $(C; x_*)$ and the sequence of points $(\phi(x_1), \dots, \phi(x_n))$. One can show easily that the dimension of the space $X_{g,n}(V,\beta)$ at each point is bigger than or equal to $D(g,n,\beta)$. Also, under appropriate genericity assumptions, $X_{g,n}(V,\beta)$ is *smooth* of dimension $D(g,n,\beta)$. We want to define $I_{g,n;\beta}$ as the image of the fundamental class of a compactification $\overline{X_{g,n}(V,\beta)}$. The problem here is to find a correct compactification and to define the "fundamental class" if there are components of dimensions bigger than expected. Also one has to prove that classes $I_{g,n;\beta}$ do not depend on the choice of an almost-complex structure.

There are now two approaches to the rigorous construction of Gromov-Witten classes. First one is due to Y. Ruan and G. Tian [RT] and it suffices for the genus zero case. This construction works only for so called semi-positive manifolds (including Fano and Calabi-Yau manifolds), but it gives classes with integral coefficients. The idea of this construction is to perturb generically $\overline{\partial}$-equations and check that there are no strata of dimension larger than $D(0,n,\beta)$ in Gromov's compactification of the space of pseudo-holomorphic curves. In fact, Ruan and Tian define not GW-classes but the number of maps from a fixed complex curve to V satisfying general incidence conditions (counted with signs). Using algebraic results on the structure of $\mathrm{H}^*(\overline{\mathcal{M}}_{0,n})$ it is possible to reconstruct whole genus-zero part of Gromov-Witten classes (see [KM]). Another construction [K1] is based on a new compactification of the moduli space of maps and should work, presumably, for all genera, for all symplectic manifolds and also for all non-singular projective varieties over arbitrary fields. At least, one can produce now purely algebro-geometric definitions of genus-zero Gromov-Witten invariants in the case of complete intersections in projective spaces. Its advantage is that it will not use any general position argument, and its weak point is the lack of control on integrality of arising classes.

As an example we give a definition of "numbers of rational curves" on a quintic 3-fold. Denote by $\overline{\mathcal{M}}_{0,0}(\mathbf{P}^4, d)$ the moduli stack of equivalence classes of maps $\phi : C \longrightarrow \mathbf{P}^4$ where C is a connected curve of arithmetic genus zero and having only odrinary double points as singular points (i.e., C is a tree of rational curves) such that each irreducible component of C mapping to a point has at least 3 singular points. The parameter d, $d \geq 1$, denotes the degree of the image of the fundamental class $[C]$ in $\mathrm{H}_2(\mathbf{P}^4, \mathbf{Z}) \simeq \mathbf{Z}$. It is proven in [K1] that $\overline{\mathcal{M}}_{0,0}(\mathbf{P}^4, d)$ is a smooth proper algebraic stack of finite type. The set of its complex points is a compact complex orbifold of dimension $5d+1$.

We define a vector bundle $\mathcal{E}_d$ of rank $5d+1$ over $\overline{\mathcal{M}}_{0,0}(\mathbf{P}^4, d)$. The fiber of $\mathcal{E}_d$ at $\phi : C \longrightarrow \mathbf{P}^4$ is equal to $\mathrm{H}^0(C, \phi^*\mathcal{O}(5))$. Notice that if a quintic 3-fold V is given by an equation Q of degree 5 in 5 variables, $Q \in \Gamma(\mathbf{P}^4, \mathcal{O}(5))$, then there is an associated section $\widetilde{Q}$ of $\mathcal{E}_d$ whose zeroes are exactly maps into V. In general, there are connected components of the set of zeroes of $\widetilde{Q}$ of positive dimensions arising from multiple covering maps to rational curves in V. Nevertheless, we define the "virtual" number of curves by the formula

$$N_d^{virt} := \int_{\overline{\mathcal{M}}_{0,0}(\mathbf{P}^4,d)} c_{5d+1}(\mathcal{E}_d) \ .$$

Here the integral is understood in the orbifold sense. Thus, the numbers $N_d^{virt} \in \mathbf{Q}$ in general are not integers. We are sure that our formula will give the same numbers as physicists predict. This formula was checked up to degree 4. Also we obtained in [K1] a closed expression for the generating function for the numbers N_d^{virt}. Hence the mirror prediction in the quintic case is reduced to an explicit identity.

There is an extension of the definition above to the case of complete intersections in toric varieties and for counting of higher genus curves in flag varieties.

Axioms.

The system of axioms formulated in [KM] is a formalization of what physicists call 2-dimensional topological field theory coupled with gravity (see [W2]). We reproduce here only one of axioms from [KM] which is the basic one. Other axioms encode more evident properties of Gromov-Witten classes, like the invariance under permutation of indices, etc.

It will be convenient to associate with the class $I_{g,n;\beta}$ a linear map

$$J_{g,n;\beta} : (\mathrm{H}^*(V, \mathbf{Q}))^{\otimes n} \longrightarrow \mathrm{H}^*(\overline{\mathcal{M}}_{g,n}, \mathbf{Q})$$

using the Künneth formula and the Poincaré duality. A *splitting* axiom describes the restriction of Gromov-Witten classes to boundary divisors of $\overline{\mathcal{M}}_{g,n}$. Namely, for $g_1, g_2 \geq 0$ and $n_1, n_2 \geq 0$ such that

$$g_1 + g_2 = g, \ \ n_1 + n_2 = n + 2, \ \ 2 - 2g_i - n_i < 0 \text{ for } i = 1, 2$$

there exists a natural inclusion $i_{g_*,n_*} : \overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_2,n_2} \hookrightarrow \overline{\mathcal{M}}_{g,n}$

$$(C_1; x_1, \dots, x_{n_1}) \times (C_2; y_1, \dots, y_{n_2}) \longmapsto (C_1 \bigcup_{x_1 = y_1} C_2; x_2, \dots, x_{n_1}, y_2, \dots, y_{n_2}) \ .$$

The following diagram should be commutative:

$$\begin{array}{ccc}
\mathrm{H}^*(V)^{\otimes n} & \xrightarrow{\simeq} & \mathrm{H}^*(V)^{\otimes(n_1-1)} \otimes \mathrm{H}^*(V)^{\otimes(n_2-1)} \\
{\scriptstyle J_{g,n;\beta}}\downarrow & & \downarrow{\scriptstyle \otimes\Delta} \\
\mathrm{H}^*(\overline{\mathcal{M}}_{g,n}) & & \mathrm{H}^*(V)^{\otimes n} \otimes \mathrm{H}^*(V)^{\otimes n_2} \\
{\scriptstyle (i_{g_*,n_*})^*}\downarrow & & \downarrow{\scriptstyle \sum_{\beta_1+\beta_2=\beta} J_{g_1,n_1;\beta_1}\otimes J_{g_2,n_2;\beta_2}} \\
\mathrm{H}^*(\overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_2,n_2}) & \xrightarrow[\text{Künneth}]{\simeq} & \mathrm{H}^*(\overline{\mathcal{M}}_{g_1,n_1}) \otimes \mathrm{H}^*(\overline{\mathcal{M}}_{g_2,n_2})
\end{array}$$

Here all cohomology are taken with coefficients in $\mathbf{Q}$ and Δ denotes the Poincaré dual to the fundamental class of the diagonal $V \subset V \times V$. The geometric meaning of this axiom is clear: a map ϕ of the glued curve C from the image of i_{g_*,n_*} is the same as two maps ϕ_1, ϕ_2 from C_1 and C_2 with $\phi_1(x_1) = \phi_2(y_1)$.

The splitting axiom in the case $g_1 = g_2 = 0$ was checked by Y. Ruan and G. Tian for semi-positive manifolds and by me for complete intersections using the stable map approach.

Associativity equation.

For a compact symplectic manifold (V, ω) denote by $\mathcal{H} := \oplus_k \mathrm{H}^k(V, \mathbf{C})$ the total cohomology space of V considered as a $\mathbf{Z}$-graded vector space (super vector space) and also as a complex supermanifold. It means that the underlying topological space of $\mathcal{H}$ is $\mathrm{H}^{even}(V, \mathbf{C})$ and functions $\mathcal{O}(\mathcal{H})$ on $\mathcal{H}$ are holomorphic functions on $\mathrm{H}^{even}(V, \mathbf{C})$ with values in the exterior algebra generated by $\left(\mathrm{H}^{odd}(V, \mathbf{C})\right)^*$.

Using Gromov-Witten classes for genus zero we define the following function (pre-potential) on $\mathcal{H}$:

$$\Phi(\gamma) := \sum_{\beta \in \mathrm{H}_2(V,\mathbf{Z})} e^{-\int_\beta \omega} \sum_{n \geq 3} \frac{1}{n!} \int_{I_{0,n;\beta}} 1_{\overline{\mathcal{M}}_{0,n}} \otimes \gamma \otimes \cdots \otimes \gamma \,.$$

Here γ denotes an even element of $\mathcal{H} \otimes \Lambda$ where Λ is an arbitrary auxilary supercommutative algebra (as usual in the theory of supermanifolds). The element $1_{\overline{\mathcal{M}}_{0,n}}$ is the identity in the cohomology ring of $\overline{\mathcal{M}}_{0,n}$.

Conjecture. *The series Φ is absolutely convergent in a neighbourhood $\mathcal{U}$ of 0 in $\mathcal{H}$, if the cohomology class $[\omega] \in H^2(V, \mathbf{R})$ is sufficiently positive.*

Without assuming the validity of this conjecture we can work not over the field $\mathbf{C}$ but over the field of fractions of the semigroup ring $\mathbf{Q}[B]$ where B is the semigroup generated by classes β such that $\int_\beta \omega' \geq 0$ for all symplectic froms ω' close to ω. Other homology classes are excluded because they cannot be represented by pseudo-holomorphic curves.

The function Φ in its definition domain $\mathcal{U}$ satisfies a system of non-linear differential equations of the third order (due to R. Dijkgraaf, E. Verlinde, H. Verlinde, and E. Witten, see [W2]). Let us choose a basis x_i of the space $\mathcal{H}$ and denote by x^i the corresponding coordinate system on $\mathcal{H}$. Denote by (g_{ij}) the matrix of the Poincaré pairing, $g_{ij} := \int_V x_i \wedge x_j$, and by (g^{ij}) the inverse matrix. For all i, j, k, l, we have (modulo appropriate sign corrections for odd-degree classes):

$$\sum_{m,m'} \frac{\partial^3 \Phi}{\partial x^i \partial x^j \partial x^m} g^{mm'} \frac{\partial^3 \Phi}{\partial x^k \partial x^l \partial x^{m'}} = \sum_{m,m'} \frac{\partial^3 \Phi}{\partial x^i \partial x^k \partial x^m} g^{mm'} \frac{\partial^3 \Phi}{\partial x^j \partial x^l \partial x^{m'}}$$

This equation can be reformulated as the condition of associativity of the algebra given by the structure constants $A_{ij}^k := \sum_{k'} g^{kk'} \partial_{ijk'} \Phi$. In invariant terms it means that Φ defines a supercommutative associative multiplication on the tangent bundle to $\mathcal{H}$ (the quantum cohomology ring).

The associativity equation follows from the splitting axiom and from a certain linear relation among components of the compactification divisor of $\overline{\mathcal{M}}_{0,n}$. Denote by D_S for $S \subset \{1, \dots, n\}$, $2 \le \#S \le n-2$, the divisor in $\overline{\mathcal{M}}_{0,n}$ which is the closure of the moduli of stable curves $(C; p_1, \dots, p_n)$ consisting of two irreducible components C_1, C_2 such that $p_i \in C_1$ for $i \in S$ and $p_i \in C_2$ for $i \notin S$.

Lemma. *We have the following identity in* $H^2(\overline{\mathcal{M}}_{0,n}, \mathbf{Z})$

$$\sum_{\substack{S:1,2 \in S \\ 3,4 \notin S}} [D_S] = \sum_{\substack{S:1,3 \in S \\ 2,4 \notin S}} [D_S] \ .$$

Both sides in the equality above are pullbacks under the forgetful map $\overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,4}$ of points $D_{\{1,2\}}$, $D_{\{1,3\}} \in \overline{\mathcal{M}}_{0,4} \simeq \mathbf{P}^1$. It is clear that any two points on $\mathbf{P}^1$ are rationally equivalent as divisors. □

Conversely, one can show using the splitting axiom that one can reconstruct the whole system of genus zero GW-classes starting from Φ. The equation of the associativity is a necessary and sufficient condition for the existence of such reconstruction (see [KM]).

The associativity equation was studied by B.Dubrovin [D]. He discovered that it is a completely integrable system in many cases (but not for CY manifolds). For example, for $V \simeq \mathbf{P}^2$ the associativity equation is equivalent to the Painlevé VI equation. It means that via a simple recursion formula we can compute the number of rational curves of degree d in the projective plane passing through generic $3d-1$ points.

Notice that by dimensional reasons, the associativity equation is an empty condition for 3-dimensional Calabi-Yau manifolds, because the virtual dimension of the space of rational curves is zero, curves do not intersect each other and the degeneration argument is unapplicable.

Let us introduce a connection on the tangent bundle $T_{\mathcal{U}}$ by the formula $\nabla = \nabla_{0|\mathcal{U}} + A$, where ∇_0 is the standard connection of the affine space $\mathcal{H}$. The associativity equation implies the flatness of ∇.

Variations of Hodge structures.

Suppose that $c_1(V) = 0$, and V carries at least one integrable complex structure compatible with ω such that $\mathrm{H}^{2,0}(V) = 0$. For any such complex structure we have a Hodge decomposition $\oplus \mathrm{H}^k(V, \mathbf{C}) = \oplus \mathrm{H}^{p,q}$. We expect that all cycles $I_{g,n;\beta}$ are Hodge cycles of (complex) dimension equal to $(n + \dim_{\mathbf{C}} V - 3)$. It follows that the restriction of ∇ to the convergence domain of the series Φ in the second cohomology group:

$$\mathcal{U}^{cl} := \mathcal{U} \cap \mathrm{H}^2(V, \mathbf{C}) \subset \mathrm{H}^2(V, \mathbf{C}) = \mathrm{H}^{1,1}$$

maps $\mathrm{H}^{p,q} \otimes \mathcal{O}(\mathcal{U}^{cl})$ to $\mathrm{H}^{p+1,q+1} \otimes \Omega^1(\mathcal{U}^{cl})$. We call $\mathcal{U}^{cl}$ the classical moduli space because it is locally isomorphic to a complexification of the moduli space of symplectic structures on V.

We introduce filtrations $\oplus_{p \leq p_0} \mathrm{H}^{p,q}$ on trivial bundles over $\mathcal{U}^{cl}$ with fibers equal to $\oplus_{p-q \text{ is fixed}} \mathrm{H}^{p,q}$. Hence we have flat connections and filtrations on holomorphic vector bundles over a complex manifold satisfying the Griffiths transversality conditions. We call such data a *complex* variation of pure Hodge structures. One can prove by using formal arguments with Hodge-Tate groups that the equivalence classes of such complex variations of pure Hodge structures do not change under deformations of the complex structure on V.

For general symplectic manifolds V with $c_1(V) = 0$ we can consider just the two trivial vector bundles H^{ev} and H^{odd} on $\mathcal{U}^{cl} := \mathcal{U} \cap \mathrm{H}^2(V, \mathbf{C}) \subset \mathcal{H}$ endowed with the flat connection induced from ∇ and the filtration by subbundles $\oplus_{k \leq k_0} \mathrm{H}^k$.

Algebro-geometric complex variations of pure Hodge structures are defined as subquotients of variations of pure Hodge structures on cohomology groups of complex projective algebraic manifolds depending algebraically on parameters.

Mirror Conjecture. *Complex variations of pure Hodge structures constructed using Gromov-Witten invariants of symplectic manifold V as above are locally equivalent to algebro-geometric variations.*

In almost all known examples such variations of Hodge structures should be locally equivalent to variations of Hodge structures on total cohomology bundles of a mirror family of complex manifolds with $c_1 = 0$. Exceptions come mostly from CY-manifolds V such that $\dim \mathrm{H}^1(V, T_V) = 0$, i.e. rigid manifolds. In this case dual manifolds with rotated Hodge diamond could not exist, because $\dim \mathrm{H}^1(W, \Omega^1_W) \neq 0$ for Kähler manifolds. Physicists proposed as candidates certain substructures of Hodge structures on cohomology groups of Fano varieties (=algebraic manifolds with an ample anti-canonical bundle). Also, calculations of numbers of curves on projective spaces suggest that in general there exists some relation between the pre-potential of *non* Calabi-Yau manifolds and algebro-geometric variations of Hodge structures.

In the case of a quintic V in $\mathbf{P}^4$ the function Φ is the sum of two terms: the contribution of maps to points of V and the contribution of rational curves in V (and their multiple covers). We introduce coordinates t^i, $i = 0, 1, 2, 3$ in one-dimensional spaces $\mathrm{H}^{i,i}(V)$ and odd coordinates ξ^j, η^j, $j = 1, \ldots, 102$ in

$\mathrm{H}^3(V, \mathbf{C})$. In these coordinates we have (modulo adding a polynomial of degree 2)

$$\Phi(t^i, \xi^j, \eta^j) = \frac{5}{6} \sum_{i+j+k=3} t^i t^j t^k + t^0 \sum_j \xi^j \eta^j + \sum_{d \geq 1} N_d^{virt} \exp\left(dt^1\right) .$$

One can deduce an example from [COGP] from this formula.

The flat coordinates x^i on the moduli space of complex structures on dual manifolds are equal to the ratios of periods $\left(\int_{\gamma_i} \Omega\right) / \left(\int_{\gamma_0} \Omega\right)$ where Ω is a holomorphic volume element on the mirror manifold W and γ_i are elements of $H_*(W, \mathbf{C})$ locally constant with respect to the Gauss-Manin connection on the homology bundle.

There exists a generalization of the mirror correspondence to higher genera. First of all, the dimension formula for degrees of Gromov-Witten classes shows that one can expect a non-negative dimension for the space of genus g curves for Calabi-Yau varieties V only in the following cases:

(1) $g = 0$ and an arbitrary dimension $n := \dim V$ (this is what we have described right now),
(2) $g = 1$ and arbitrary n,
(3) $g \geq 2$ and $n \leq 3$.

The Harvard group of physicists in the remarkable paper [BCOV] proposed a procedure ("quantum Kodaira-Spencer theory") giving numbers of curves for cases $g = 1$ or $n = 3$. It relates GW-invariants with certain structures on the moduli of dual varieties which are more complicated than just variations of Hodge structures and are not understood mathematically yet. The example of R. Dijkgraaf (elliptic curves) is a 1-dimensional version of this theory.

In the rest of this talk we give an outline of a program relating Mirror Symmetry to general structures of Homological Algebra. The central ingredient here is a fundamental construction of K. Fukaya based on ideas of S. Donaldson, A. Floer and G. Segal.

Extended moduli spaces.

When we restrict the flat bundle $T_{\mathcal{U}}$ to the subspace $\mathcal{U}^{cl} = \mathrm{H}^2(V, \mathbf{C})$, much of information will be lost. It seems very reasonable to extend the moduli space of symplectic structures to the whole domain $\mathcal{U}$ in $\mathcal{H}$ in which the potential Φ is defined. Hence the tangent space to the extended moduli space at classical points $\mathcal{U}^{cl}$ should be equal to $\mathcal{H} = \oplus \mathrm{H}^k$.

Now we want to construct an extended moduli space $\mathcal{M}$ for a complex Calabi-Yau W containing the ordinary moduli space of complex structures on W. The natural candidate for the tangent bundle to $\mathcal{M}$ at classical points $\mathcal{M}^{cl} := \mathrm{Moduli}\,(W)$ should be equal to the direct sum $\oplus \mathrm{H}^p(W, \wedge^q T_W)$. The problem of constructing $\mathcal{M}$ was already dicussed by E. Witten (see [W3]).

We anticipate that $\oplus \mathrm{H}^p(W, \bigwedge^q T_W)$ can be interpreted as the total Hochschild cohomology of the sheaf $\mathcal{O}_W$ of algebras of holomorphic functions on W.

For an algebra A/k over a field its Hochschild cohomology $\mathrm{HH}^*(A) = \mathrm{H}^*(A, A)$ is defined as $\mathrm{Ext}^*_{A-mod-A}(A, A)$. The second Hochschild cohomology $\mathrm{HH}^2(A)$ classifies infinitesimal deformations of A. Notice that each A-bimodule M defines a functor from the category of A-modules into itself:

$$M\otimes_A : \ \ A - mod \ \to \ A - mod\,, \quad N \mapsto M \otimes_A N$$

and A corresponds to the identity functor Id_{A-mod}.

Analogously, we define the Hochschild cohomology of the structure sheaf $\mathcal{O}_W$ of a scheme W over k (or of an analytic space) as the global Ext-functor

$$\mathrm{HH}^*(\mathcal{O}_W) := \ \mathrm{Ext}^*_{W\times W}\,(\delta_*(\mathcal{O}_W), \delta_*(\mathcal{O}_W))$$

where $\delta : W \hookrightarrow W \times W$ is the diagonal embedding. Another definition of the Hochschild cohomology for algebraic varieties (in fact, equivalent to the ours) was proposed by M. Gerstenhaber and S. D. Schack [GS]. The following fact proven in hidden form in [GS] seems to be new in algebraic geometry:

Theorem. *For smooth (and not necessarily compact) variety W over a field of characteristic zero there is a canonical isomorphism*

$$HH^n(\mathcal{O}_W) \simeq \bigoplus_{k+l=n} H^k(W, \bigwedge^l T_W) \ .$$

For smooth W the second Hochschild cohomology $\mathrm{HH}^2(\mathcal{O}_W)$ splits into the direct sum of ordinary first-order deformations $\mathrm{H}^1(W, T_W)$, non-commutative deformations $\mathrm{H}^0(W, \bigwedge^2 T_W)$ of the sheaf $\mathcal{O}_W$ of associative algebras (global Poisson brackets on W), and a little bit more misterious piece $\mathrm{H}^2(W, \mathcal{O}_W)$. This third part can be interpreted as locally trivial first-order deformations of the sheaf of abelian categories of $\mathcal{O}_W$-modules.

In the next section we will propose an interpretation of the total Hochschild cohomology as the tangent space to "extended moduli space" $\mathcal{M}$ containing the classical moduli space $\mathcal{M}^{cl}$ as a part.

A_∞-algebras and categories.

A_∞-algebras were introduced by J.Stasheff in 1963 (see [S]). Let $A = \oplus A^k$ be a $\mathbf{Z}$-graded vector space. The structure of A_∞-algebra on A is an infinite sequence of linear maps $m_k : A^{\otimes k} \to A, \ k \geq 1, \ \deg m_k = 2 - k$ satisfying the (higher) associativity conditions:

(1) $m_1^2 = 0$, (we can consider m_1 as a differential and (A, m_1) as a complex),
(2) $m_1(m_2(a \otimes b) = m_2(m_1(a) \otimes b) \pm m_2(a \otimes m_1(b))$, ($m_2$ is a morphism of complexes),
(3) $m_3(m_1(a)\otimes b\otimes c) \pm m_3(a\otimes m_1(b)\otimes c) \pm m_3(a\otimes b\otimes m_1(c)) \pm m_1(m_3(a\otimes b\otimes c) =$

$= m_2(m_2(a \otimes b) \otimes c) - m_2(a \otimes m_2(b \otimes c))$, ($m_2$ is associative up to homotopy),

(4) and so on...

In one sentence one can define the A_∞-algebra structure on A as a co-derivation in the graded sense $d, \ d^2 = 0$ of degree 1 on the co-free co-associative algebra without a co-unit co-generated by the $\mathbf{Z}$-graded vector space $A[1], \quad A[1]^k := A^{k+1}$.

A morphism of A_∞-algebras (from A to B) is an infinite collection of linear maps $A^{\otimes k} \to B\,, \ k \geq 1$ satisfying some equations analogous to the defining equations for individual A_∞-algebras. In terms of co-free co-algebras it is the same as a differential graded homomorphism. A homotopy equivalence of A_∞-algebras is a morphism whose linear part induces an isomorphism of cohomology groups with respect to the differential m_1.

In general, A_∞-algebras are closely related to differential graded algebras. Namely, a dg-algebra is the same as an A_∞-algebra with $m_3 = m_4 = \cdots = 0$. Conversely, for an A_∞-algebra A one can construct using the bar-construction a differential graded algebra B homotopy equivalent to A.

An additive category over a field k is a category C with finite direct sums such that all sets of morphisms $\mathrm{Hom}_C(X,Y)$ are endowed with structure of vector spaces over k and where the composition of morphisms is a bilinear map. In a sense, one can approximate additive categories by algebras of endomorphisms of their objects. Analogously, one can define differential graded category as an additive category with the structure of complexes on $\mathrm{Hom}_C(X,Y)$ such that the composition is a morphism of complexes.

An A_∞-category C is a collection of objects and $\mathbf{Z}$-graded spaces of morphisms $\mathrm{Hom}_C(X,Y)$ for each two objects endowed with higher compositions of morphisms satisfying relations parallel to the defining relations of A_∞-algebras. We require the existence of identity morphisms $Id_X \in \mathrm{Hom}_C(X,X)$ which obey the usual properties of identity for composition m_2 and vanish under substitution in other (higher) compositions. We can also require the existence of finite direct sums in C in an obvious sense. Notice that C is not a category in general, because the composition of morphisms is not associative. Nevertheless one can construct an additive category $H(C)$ from C with the same class of objects by defining new $\mathbf{Z}$-graded spaces of morphisms as

$$\mathrm{Hom}_{H(C)}(X,Y) := \frac{\mathrm{Ker}\ (m_1 : \mathrm{Hom}^0_C(X,Y) \to \mathrm{Hom}^1_C(X,Y)\,)}{\mathrm{Im}\ (m_1 : \mathrm{Hom}^{-1}_C(X,Y) \to \mathrm{Hom}^0_C(X,Y)\,)} \quad .$$

There exists a generalization of Hochschild cohomology to the case of A_∞-algebras. The meaning of $\mathrm{HH}^*(A)$ for $* > 0$ is the space of equivalence classes of first-order deformations of A_∞-structure on A over $\mathbf{Z}$-graded bases. We hope that there exists an appropriate version of Hochschild cohomology for some good class of A_∞-categories as well.

One can show under some mild assumptions that if the formal $\mathbf{Z}$-graded moduli space $\mathcal{M}$ of A_∞-categories is smooth than there exists the canonical structure of an associative commutative algebra on the tangent bundle to $\mathcal{M}$. In the case of a an A_∞-category consisting just of one object X with morphisms $\mathrm{Hom}_C(X,X)$

consisting of an associative algebra A in degreee 0 the product in Hochschild cohomology (i.e. in the tangent space to $\mathcal{M}$)

$$\mathrm{HH}^*(A) := \mathrm{Ext}^*_{A-mod-A}(A, A)$$

coincides with the usual Yoneda composition of *Ext*-groups.

Triangulated categories.

One of fundamental tools in homological algebra is the triangulated category $\mathcal{D}(C)$ associated with an abelian category C satisfying certain conditions (J.-L. Verdier, see [V]). A triangulated category is an additive category endowed with a shift functor and a class of so-called exact triangles, obeying a complicated list of axioms. For example, for C equal to the category of A-modules, where A is an associative algebra, the category $\mathcal{D}(C)$ is equivalent to the category whose objects are complexes of free A-modules and morphisms are equal to homotopy classes of differential graded morphisms of degree 0:

$$\mathrm{Hom}_{D(C)}(X, Y) := \mathrm{H}^0\Big(\bigoplus_k \prod_j \mathrm{Hom}_C(X^j, Y^{j+k})\Big) \ .$$

The bounded derived category $\mathcal{D}^b(C)$ is the full subcategory of $\mathcal{D}(C)$ consisting of complexes of A-modules with non-vanishing cohomology groups only in finitely many degrees.

The shift functors at the level of objects just shifts degree of complexes: $X \longrightarrow X[n],\ \ X[1]^k = X^{k+n}$ and $(X[n])[m] = X[n+m],\ \ X[0] = X$.

We will not describe Verdier's axiomatics of exact triangles here because it does not look completely satisfactory, although it was generally adopted and widely used. Certain improvement of axioms was proposed by A. Bondal and M. Kapranov in [BK]. The main ingredient in their definition is the notion of a twisted complex in a differential graded category.

We can extend the construction of [BK] to the case of A_∞-categories. We assume that an A_∞-category C is endowed with shift functors such that

$$\mathrm{Hom}_C(X[i], Y[j]) = \mathrm{Hom}_C(X, Y)\,[j-i] \ .$$

By definition, a (one-sided) *twisted complex* is a family $(X^{(i)})_{i\in\mathbf{Z}}$ of objects of an A_∞-category C such that $X^{(i)} = 0$ for almost all i together with a collection of morphisms $d_{ij} \in \mathrm{Hom}_C(X^{(i)}, X^{(j)})^{j-i}$ for $i < j$ obeying a generalization of the Maurer-Cartan equation:

$$\text{for fixed } \ i,\, j \qquad \sum_{\substack{k;i_0,\dots,i_k \\ i_0=i, i_k=j}} m_k(f_{i_0,i_1}, \dots, f_{i_{k-1},i_k}) = 0 \ .$$

We define the $\mathbf{Z}$-graded space of morphisms between twisted complexes X and Y as

$$\bigoplus_{k,j} \mathrm{Hom}_C(X^{(j)}, Y^{(j+k)})[-k] \ .$$

Using higher compositions in C one can define the structure of an A_∞-category on $\{$twisted complexes of $C\}$. Any higher composition of morphisms of twisted complexes is defined as the sum over all possible products which one can imagine.

One can check without difficulties that the derived category

$$\mathcal{D}^b(C) := H(\text{twisted complexes of } C)$$

satisfies Verdier axioms for triangulated category.

Fukaya's A_∞-category.

In this section we describe a remarkable contsruction of Kenji Fukaya [F] with few minor modifications.

Let V be a closed symplectic manifold with $c_1(T_V) = 0$.

Denote by LV the space of pairs (x, L) where x is a point of V and L is a Lagrangian subspace in T_xV. The space LV is fibered over V with fibers equal to Lagrangian Grassmanians. Thus the fundamental group of the fibers is isomorphic to $\mathbf{Z}$.

The condition on V posed above guarantees that there exists a $\mathbf{Z}$-covering $\widetilde{LV}$ of LV inducing a universal cover of each fiber. Let us fix $\widetilde{LV}$.

Objects of Fukaya's category $F(V)$ are Lagrangian submanifolds $\mathcal{L} \subset V$ endowed with a continuous lift of the evident map $\mathcal{L} \to LV$ to a map $\mathcal{L} \to \widetilde{LV}$. In fact, it is only a first approximation to right objects, see remarks after the defintion. For subvarieties $\mathcal{L}_1, \mathcal{L}_2$ intersecting each other transversally at a point $x \in V$ and endowed with lifts to $\widetilde{LV}$, we can define the Maslov index $\mu_x(\mathcal{L}_1, \mathcal{L}_2) \in \mathbf{Z}$. Notice that

$$\mu_x(\mathcal{L}_1, \mathcal{L}_2) + \mu_x(\mathcal{L}_2, \mathcal{L}_1) = n := \frac{1}{2}\mathrm{dim}(V) \ .$$

K. Fukaya defines the space of morphisms $Mor_F(\mathcal{L}_1, \mathcal{L}_2)$ only if $\mathcal{L}_1, \mathcal{L}_2$ intersect transversally:

$$\mathrm{Hom}_F(\mathcal{L}_1, \mathcal{L}_2) := \mathbf{C}^{\mathcal{L}_1 \cap \mathcal{L}_2}$$

with $\mathbf{Z}$-grading coming from the Maslov index.

The differential in $\mathrm{Hom}_F(\mathcal{L}_1, \mathcal{L}_2)$ is a version of Floer's differential. Its matrix coefficient associated with two intersection points $p_1, p_2 \in \mathcal{L}_1 \cap \mathcal{L}_2$ is defined as

$$\sum_{\phi : D^2 \longrightarrow V} \pm \exp(-\text{area of } D^2)$$

where $\phi : D^2 \longrightarrow V$ is a pseudo-holomorphic map from the standard disc $D^2 := \{z \mid \ |z| \le 1\} \in \mathbf{C}$ to V such that $\phi(-1) = p_1,\ \phi(+1) = p_2$ and

$$\phi(\exp(i\alpha)) \in \mathcal{L}_1 \text{ for } 0 < \alpha < \pi, \ \ \phi(\exp(i\alpha)) \in \mathcal{L}_2 \text{ for } \pi < \alpha < 2\pi \ .$$

More precisely, we consider *equivalence classes* of maps ϕ modulo the action of the group of holomorphic automorphisms of D^2 stabilizing points 1 and -1:

$$\mathbf{R}_+^\times \subset PSL(2, \mathbf{R}) = \ \mathrm{Aut}\ (D^2) \ .$$

The area of D^2 with respect to the pullback of ω depends only on the homotopy type of $\phi \in \pi_2(V, \mathcal{L}_1 \cup \mathcal{L}_2)$. One expects that for sufficiently large ω the infinite series is absolutely convergent.

The sign $\pm$ in the definition of Floer differential comes from a natural orientation of the space of pseudo-holomorphic maps. One expects that there will be finitely many such maps for a generic almost-complex structure on V if $\mu_{p_2} - \mu_{p_1} = 1$. Presumably, one can develop a general technique of stable maps for surfaces with boundaries or extend Ruan-Tian's methods.

Analogously, one can define higher order compositions using zero-dimensional components of spaces of equivalence classes modulo $PSL(2, \mathbf{R})$-action of maps ϕ from the standard disc D^2 to V with the boundary $\phi(\partial D^2)$ sitting in a union of Lagrangian subvarieties. More precisely, if $\mathcal{L}_1, \ldots, \mathcal{L}_{k+1}$ are Lagrangian submanifolds intersecting each other transversally and $p_j \in \mathcal{L}_j \cap \mathcal{L}_{j+1},\ j = 1, \ldots, k$ are chosen intersection points, then we define the composition of corresponding base elements in spaces of morphisms as

$$m_k(p_1, \ldots, p_k) := \sum_{\substack{\phi : D^2 \longrightarrow V,\, q \in \mathcal{L}_1 \cap \mathcal{L}_{k+1} \\ 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_k < \alpha_{k+1} = 2\pi}} \pm \exp\left(-\int_{D^2} \phi^* \omega\right) q \in \mathrm{Hom}_F(\mathcal{L}_1, \mathcal{L}_{k+1})$$

where $\phi(\exp(i\alpha)) \in \mathcal{L}_j$ for $\alpha_{j-1} < \alpha < \alpha_j$ and $\phi(\exp(i\alpha_j)) = p_j$ for $j = 1, \ldots, k+1$; $p_{k+1} := q$. Again, we expect that there exist only finitely many equivalence classes modulo the action of $\mathrm{Aff}(\mathbf{R}) = \mathrm{Stab}_{1 \in D^2} \subset PSL(2, \mathbf{R})$ of such maps in each homotopy class if $\mu_q = \mu_{p_1} + \cdots + \mu_{p_k} + 2 - k$ and the infinite series in the definition of m_k converges absolutely.

K. Fukaya claims that the identities of A_∞-category follow from considerations analogous to the proof of the associativity equations in the case of rational curves. Also he claims that it is possible to extract an actual A_∞-category with compositions of all morphisms using an appropriate notion of a "generic" Lagrangian manifold. In particular, it is possible to restore the identity morphisms. The main idea is that two Lagrangian submanifolds obtained one from another by a Hamiltonian flow are equivalent with respect to the Floer cohomology.

There is an extension of Fukaya's category. We can consider pairs consisting of a Lagrangian submanifold $\mathcal{L}$ and a unitary local system $\mathcal{E}$ on $\mathcal{L}$ as objects of new A_∞-category. Morphism spaces will be defined as

$$\mathrm{Hom}_F((\mathcal{L}_1, \mathcal{E}_1), (\mathcal{L}_2, \mathcal{E}_2)) := \bigoplus_{p \in \mathcal{L}_1 \cap \mathcal{L}_2} \mathrm{Hom}\,(\mathcal{E}_{1|p}, \mathcal{E}_{2|p})\ .$$

In the definiton of higher composition we add a new factor to each term equal to the trace of the composition of holonomy maps along the boundary of D^2. Unitarity in this definition is an obligatory condition, otherwise the series defining higher compositions will be inevitably divergent.

It seems that there are further possible extensions of Fukaya's A_∞-category. One can consider as objects Lagrangian foliations, families of Lagrangian submanifolds parametrized by closed oriented manifolds etc.

Homological Mirror Conjecture.

We propose here a conjecture in slightly vague form which should imply the "numerical" Mirror conjecture. Let (V, ω) be a $2n$-dimensional symplectic manifold with $c_1(V) = 0$ and W be a dual n-dimensional complex algebraic manifold.

The derived category constructed from the Fukaya category $F(V)$ (or a suitably enlarged one) is equivalent to the derived category of coherent sheaves on a complex algebraic variety W.

More precisely, we expect that there is an embedding of $\mathcal{D}^b(F(V))$ as a full triangulated subcategory into $\mathcal{D}^b(Coh(W))$. We have following evidences for that.

(1) By the general philosophy, A_∞-deformations of first order of $F(V)$ should correspond to Ext-groups in a category of functors $F(V) \to F(V)$. The natural candidate for such a category is $F(V \times V)$ where the symplectic structure on $V \times V$ is $(\omega, -\omega)$. The diagonal $V_{diag} \subset V \times V$ is a Lagrangian submanifold and it corresponds to the identity functor. By a version of Floer's theorem (see [F]) there is a canonical isomorphism between the Floer cohomology $\mathrm{H}^*(\mathrm{Hom}_{F(V\times V)}(V_{diag}, V_{diag}))$ and the ordinary topological cohomology $\mathrm{H}^*(V, \mathbf{C})$. The Yoneda product on the Floer cohomology considered as Ext-groups arises from holomorphic maps from D^2 with 3 marked points on ∂D^2 to $V \times V$ with a boundary on V_{diag}. Such maps are the same as holomorphic maps to V from the 2-dimensional sphere $S^2 \simeq \mathbf{C}P^2$ with 3 marked points. Thus, it seems very reasonable to expect that we will get exactly the quantum cohomology product on $\mathrm{H}^*(V)$. We expect that the equivalence of derived categories will imply numerical predictions.

(2) Lagrangian varieties (and local systems on it) form a natural class of local boundary conditions for the A-model in topological open string theory. Also, holomorphic vector bundles form local boundary conditions for the B-model (E. Witten [W4]). Physicists beleive that the whole string theories on dual varieties are equivalent. Thus, we want to say that topological open string theory is more or less the same as a triangulated category.

(3) Both categories $\mathcal{D}^b(F(V))$ and $\mathcal{D}^b(Coh(W))$ possess a duality: a functorial isomorphism $(\mathrm{Hom}(X, Y))^* \simeq \mathrm{Hom}(Y, X[n])$. On the algebro-geometric side it is Serre duality. For Fukaya's category the definition of compositions is cyclically symmetric. The duality follows from this symmetry and from the idenitity $\mu_x(\mathcal{L}_1, \mathcal{L}_2) + \mu_x(\mathcal{L}_2, \mathcal{L}_1) = n$. We developed some time ago a theory of A_∞-algebras with duality in [K2] and proposed a combinatorial construction of cohomology classes of the moduli spaces of smooth curves $\mathcal{M}_{g,n}$ based on such algebras. This construction has a generalization to an A_∞-category with a duality. Thus, we expect that the Gromov-Witten invariants could be defined in a general purely algebraic situation. We still do not know what is missed in algebraic structures and how to define classes with values in $H^*(\overline{\mathcal{M}}_{g,n}, \mathbf{C})$.

A mirror complex manifold W is usually not unique. For example, one cannot distinguish the B-models on dual abelian varieties A, A'. It is compatible with our picture because the derived categories of coherent sheaves on A and A' are equivalent via the Fourier-Mukai transform. Also, the B-models on birationally equivalent Calabi-Yau manifolds W, W' are beleived to be isomorphic. In all known examples the Hodge structures on total cohomology depend only on a birational type. Thus, we expect that the derived categories of coherent sheaves on W and on W' are equivalent.

Our conjecture, if it is true, will unveil the mystery of Mirror Symmetry. The numerical predictions mean that two elements of an uncountable set (formal power series with integral coefficients) coincide. Our homological conjecture is equivalent to the coincidence in a *countable* set (connected components of the "moduli space of triangulated categories", whatever it means).

In the last section we show what our program looks like in the simplest case of Mirror Symmetry.

Two-dimensional tori: a return.

Let Σ be the standard flat two-dimensional torus $S^1 \times S^1$ endowed with a symplectic form ω proportional to the standard volume element. Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ be three simple closed geodesics from pairwise different homology classes and

$$p_1 \in \mathcal{L}_1 \cap \mathcal{L}_2, \;\; p_2 \in \mathcal{L}_2 \cap \mathcal{L}_3, \;\; p_3 = q \in \mathcal{L}_1 \cap \mathcal{L}_3$$

be three intersection points. We will compute now the tensor coefficient of the composition m_2 corresponding to the base vectors p_1, p_2, p_3. Each map ϕ from D^2 to Σ can be lifted to a map $\tilde{\phi}$ from D^2 to $\mathbf{R}^2$ = the universal covering space of Σ. The preimages of circles $\mathcal{L}_i$ on $\mathbf{R}^2$ form three families of parallel straight lines. Thus the images of lifted maps $\tilde{\phi}$ are triangles with sides on these lines. It is easy to see that the equivalence classes of triangles modulo the action of $\mathbf{Z}^2 = \pi_1(\Sigma)$ are labeled by terms of an arithmetic progression (the lengths of sides of triangles sitting on the pullback of $\mathcal{L}_1$). The areas of triangles are proportional to the squares of elements of this progression. The tensor element of composition m_2 can be written naturally as

$$\sum_{n \in \mathbf{Z}} \exp\left(-(an+b)^2\right)$$

for some real parameters $a \neq 0$ and b, which is a value of the classical $\theta-$function. The associativity equation is equivalent to the standard bilinear identity for θ-functions. It is well-known that θ-functions form natural bases of spaces of global sections of line bundles over elliptic curves.

It seems very plausible that the triangulated category constructed from the Fukaya category $F(\Sigma)$ enlarged using unitary local systems is equivalent to the bounded derived category of coherent sheaves on the elliptic curve with the real modular parameter $\tau := \exp\left(-\text{ area of } (\Sigma)\right)$.

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