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Integrality of instanton numbers and p-adic B-model

Maxim Kontsevich [a], Albert Schwarz [b,*], Vadim Vologodsky [c]

[a] IHES, 91440 Bures-sur-Yvette, France
[b] Department of Mathematics, University of California, Davis, Davis, CA 95616, USA
[c] Department of Mathematics, University of Chicago, 5734 S. University Avenue, Chicago, IL 60637, USA

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Abstract

We study integrality of instanton numbers (genus zero Gopakumar–Vafa invariants) for quintic and other Calabi–Yau manifolds. We start with the analysis of the case when the moduli space of complex structures is one-dimensional; later we show that our methods can be used to prove integrality in general case. We give an expression of instanton numbers in terms of Frobenius map on p-adic cohomology; the proof of integrality is based on this expression.

1. Introduction

The basic example of mirror symmetry was constructed in [1]. In this example one starts with holomorphic curves on the quintic $\mathcal{A}$ given by the equation

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 + \psi x_1 x_2 x_3 x_4 x_5 = 0$$

in projective space. (In other words, one considers A-model on this quintic.) Mirror symmetry relates this A-model to the B-model on $\mathcal{B}$ (on the quintic factorized with respect to the finite symmetry group $(\mathbb{Z}_5)^3$). Instanton numbers are defined mathematically in terms of Gromov–Witten invariants, i.e., by means of integration over the moduli space of curves. The moduli space is an orbifold, therefore it is not clear that this construction gives integer numbers. The mirror conjecture proved by Givental [2] permits us to express the instanton numbers in terms of solutions of Picard–Fuchs equations on mirror quintic $\mathcal{B}$; however, integrality is not clear from this expression. Gopakumar and Vafa [3] introduced BPS invariants that are integer numbers by definition; it should be possible to prove that instanton numbers can be considered as a particular case of GV-invariants. However, such a proof is unknown; moreover, there exists no rigorous definition of GV-invariants.

The goal of present Letter is to prove the integrality of instanton numbers. However, we will be able to check only a weaker statement: the instanton numbers become integral after multiplication by some fixed number. We work in the framework of B-model definition. In Section 2 we consider the case when the moduli space of deformations of complex structure on a Calabi–Yau threefold is one-dimensional. The proof can be generalized to the case when the moduli space is multidimensional (Section 3). The considerations of the Letter are not rigorous. To make the Letter accessible to physicists we have hidden mathematical difficulties in the exposition below. The paper [5] will contain a rigorous mathematical proof of the results of present Letter.

We will use freely the well-known mathematical results about sigma-models on Calabi–Yau threefolds; see, for example, [6] or [7]. We will follow the notations of [7].

The proof of integrality of instanton numbers is based on an important statement that these numbers can be expressed in terms of arithmetic geometry. May be, the fact that physical quantities can be studied in terms of number theory is more significant than the proof itself.

Instanton numbers we consider can be identified with genus 0 Gopakumar–Vafa (GV) invariants. GV-invariants can

* Corresponding author.
E-mail addresses: maxim@ihes.fr (M. Kontsevich), schwarz@math.ucdavis.edu (A. Schwarz), volgdsky@math.uchicago.edu (V. Vologodsky).

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be expressed in terms of Gromov–Witten invariants, but their integrality is not clear from this expression. One can give a condition of integrality of GV-invariants in terms of Frobenius map generalizing Lemma 2 of present Letter. It seems that GV-invariants also can be expressed in terms of p-adic B-model.

The relation between topological sigma-models and number theory was anticipated long ago. The existence of such a relation is strongly supported by the fact that Picard–Fuchs equations that play important role in B-model appear also in Dwork's theory of zeta-functions of manifolds over finite fields.

Calabi–Yau manifolds over finite fields and arithmetic analog of mirror conjecture where considered in very interesting papers by Candelas, de la Ossa and Rodriguez-Villegas [8], see also [9]. We go in different direction: our main goal is to obtain the information about sigma-models over complex numbers using methods of number theory.

Notice that p-adic methods were used in [13] to prove integrality of mirror map for quintic and in [4] to prove integrality in general case. The idea to use the Frobenius map on p-adic cohomology to prove some integrality statements related to mirror symmetry appeared in [15].

In arithmetic geometry one can consider Hodge structure on cohomology; this means that one can define the main notions of B-model theory in p-adic framework [11,12]. Our computations are based on the fact that in the situation we consider one can obtain the information about the conventional sigma-model from analysis of its p-adic analog. This is a nontrivial mathematical fact; however, in this Letter we will skip the justification of this statement referring to [5].

2. Integrality of instanton numbers: the simplest case

Instead of working with A-model we consider mirror B-model.

Our starting point is the well-known formula relating the Yukawa coupling Y in canonical coordinates (normalized Yukawa coupling) to instanton numbers n_k:

$$Y(q) = \text{const} + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d}.$$

(This formula is valid in the case when the moduli space of complex structures is one-dimensional; for the quintic $\text{const} = 5$.)

Lemma 1. *Let us assume that*

$$\sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d} = \sum_{k=1}^{\infty} m_k q^k. \tag{1}$$

If the numbers n_k are integers then for every prime number p the difference $m_{kp} - m_k$ is divisible by $p^{3(\alpha+1)}$ where α is defined as the number of factors equal to p in the prime decomposition of k. Conversely, if

$$p^{3(\alpha+1)} \big| m_{kp} - m_k \tag{2}$$

for every prime p and every k, the numbers n_k are integers.

To prove the statement we notice that the following expression for m_k in terms of n_k can be derived from (1):

$$m_k = \sum_{d|k} n_d d^3. \tag{3}$$

Let us suppose that $k = p^\alpha r$ where r is not divisible by p. Then

$$m_{kp} - m_k = \sum_{s|r} n_{p^{\alpha+1}s} \left(p^{\alpha+1}s\right)^3. \tag{4}$$

(We are summing over all divisors of kp that are not divisors of k, i.e., over all $p^{\alpha+1}s$, where $s|r$.)

We see immediately that $m_{kp} - m_k$ is divisible by $p^{3(\alpha+1)}$.

To derive integrality of n_k from this property one can use the Moebius inversion formula

$$n_k k^3 = \sum_{d|k} \mu(d) m_{\frac{k}{d}}, \tag{5}$$

where $\mu(d)$ stands for Moebius function. Recall, that Moebius function can be defined by means of the following properties $\mu(ab) = \mu(a) \cdot \mu(b)$ if a and b are relatively prime, $\mu(p) = -1$ if p is a prime number, $\mu(p^\alpha) = 0$ if $\alpha > 1$. Again we represent k as $p^\alpha r$ where p does not divide r and a divisor d of k as $p^\beta s$ where $s|r$ and $\beta \leqslant \alpha$. Taking into account that $\mu(p^\beta s) = 0$ if $\beta > 1$ we obtain

$$\begin{aligned} n_k k^3 &= \sum_{s|r} \mu(s) m_{\frac{k}{s}} + \sum \mu(sp) m_{\frac{k}{sp}} \\ &= \sum_{s|r} \mu(s) (m_{\frac{k}{s}} - m_{\frac{k}{sp}}). \end{aligned} \tag{6}$$

It follows from our assumption that the left-hand side of (5) is divisible by $p^{3\alpha}$, hence n_k does not contain p in the denominator (in other words, n_k can be considered as an integer p-adic number). In the above calculation we assumed that $\alpha \geqslant 1$; the case $\alpha = 0$ is trivial. If the condition (2) is satisfied for every prime p we obtain that the numbers n_k are integers.

Lemma 2. *The numbers n_k defined in terms of $Y(q)$ by the formula (1) are integers if and only if for every prime p there exists such a series $\psi(q) = \sum s_k q^k$ having p-adic integer coefficients that*

$$Y(q) - Y(q^p) = \delta^3 \psi(q). \tag{7}$$

Here δ stands for the logarithmic derivative $q\frac{d}{dq}$.

It is easy to check that this lemma is a reformulation of Lemma 1. The kth coefficient of the decomposition of $Y(q) - Y(q^p)$ into q-series is equal to

$$m_k - m_{\frac{k}{p}} = m_{p^\alpha s} - m_{p^{\alpha-1}s}$$

if $k = p^\alpha s$ and $\alpha \geqslant 1$, and to m_k if k is not divisible by p. From the other side, the coefficients of $\delta^3\psi(q)$ are equal to $k^3 s_k$.

Notice, that Lemma 2 can be formulated in terms of the Frobenius map φ. This map transforms q into q^p; corresponding map φ^* on functions of variable q transforms $f(q)$ into

$f(q^p)$. The formula (7) can be rewritten in the form

$$Y - \varphi^* Y = \delta^3 \psi.$$

The above statements show that it is natural to apply p-adic methods attempting to prove integrality of instanton numbers.

Recall that B-model is formulated in terms of Hodge filtration on the middle-dimensional cohomology of Calabi–Yau manifold; one should consider deformation of complex structure of Calabi–Yau manifold and corresponding variations of Hodge filtration. Analogous problems can be considered in p-adic setting.

If we work with mirror quintic $\mathcal{B}$ (and more general if the moduli space $\mathcal{M}$ of complex structures on a Calabi–Yau threefold is one-dimensional) then one can find a local coordinate q in a neighborhood of maximally unipotent boundary point of $\mathcal{M}$ (canonical coordinate) and a basis $e^0(q)$, $e^1(q)$, $e_1(q)$, $e_0(q)$ in three-dimensional cohomology that satisfy the following conditions:

(1) The basis e^0, e^1, e_1, e_0 is symplectic: $\langle e_0, e^0\rangle = -1$, $\langle e_1, e^1\rangle = 1$, all other inner products vanish.

(2) Gauss–Manin connection acts in the following way:

$$\nabla_\delta e^0 = 0, \tag{8}$$

$$\nabla_\delta e^1 = e^0, \tag{9}$$

$$\nabla_\delta e_1 = Y(q) e^1, \tag{10}$$

$$\nabla_\delta e_0 = e_1. \tag{11}$$

Here δ stands for logarithmic derivative $q\frac{d}{dq}$ and ∇_δ for corresponding Gauss–Manin covariant derivative

(3)

$$e^0 \in \mathcal{F}^0 \cap \mathcal{W}_0,$$

$$e^1 \in \mathcal{F}^1 \cap \mathcal{W}_2,$$

$$e_1 \in \mathcal{F}^2 \cap \mathcal{W}_4,$$

$$e_0 \in \mathcal{F}^3 \cap \mathcal{W}_6.$$

Here $\mathcal{F}^p$ stands for the Hodge filtration and $\mathcal{W}_k$ for the weight filtration (the covariantly constant filtration associated with monodromy around maximally unipotent boundary point).

(4) $Y(q) = \text{const} + \sum m_k q^k$ is a q-series with integer coefficients m_k.

We assume that $q = 0$ corresponds to maximally unipotent boundary point of the space $\mathcal{M}$ and that we are working in a neighborhood of this point.

The conditions we imposed specify the canonical coordinate q and the vectors of the basis only up to a constant factor. One can fix the canonical coordinate and the vectors $e^0(q)$, $e^1(q)$, $e_1(q)$, $e_0(q)$ up to a sign requiring that the vectors $e^0(0)$, $e^1(0)$ form a $\mathbb{Z}$-basis of W_2. (Recall, that the bundle of cohomology groups can be extended to the point $q = 0$; one denotes by W_k the weight filtration on the fiber over this point. One can talk about $\mathbb{Z}$-basis because the fiber over $q = 0$ is equipped by an integral structure that depends on the choice of coordinate on the moduli space.)

All of the statements above are well known; see, for example, [7], Chapters 5 and 6.

We considered quintic as a Calabi–Yau complex threefold. However, it is possible to consider it and $\mathcal{M}$ over $\mathbb{Z}$ or over the ring $\mathbb{Z}_p$ of integer p-adic numbers and to study its cohomology over $\mathbb{Z}_p$. It is well known that the Hodge filtration and weight filtration on cohomology can be defined also in this case [10–12]. It is natural to assume that in p-adic setting all of the statements (1)–(4) remain valid. This can be proven under certain conditions, however, the proof is not simple. A rigorous proof for general Calabi–Yau threefolds is given in [5]. (For quintic one can derive these statements from known integrality of mirror map [13] and integrality of one of periods.) Notice, that the Yukawa coupling in p-adic situation remains the same, but the coefficients m_k are considered as p-adic integers.

It is important to emphasize that in our consideration we should assume that the manifold at hand remains nonsingular after reduction with respect to prime number p. This requirement can be violated for finite number of primes. (For example, the mirror quintic $\mathcal{B}$ becomes singular after reduction mod 5.)

In the p-adic theory there exists an additional symmetry: the so-called Frobenius map. (In the situation we need the Frobenius map was analyzed in [10,12].[1]) Namely, the map $q \to q^p$ of the moduli space of quintics into itself can be lifted to a homomorphism Fr of cohomology groups of corresponding quintics. Here q is considered as a formal parameter or as a p-adic integer; cohomology are taken with coefficients in $\mathbb{Z}_p$. We will express instanton numbers in terms of this map, assuming that $p > 3$.

Notice first of all that the Frobenius map Fr preserves the weight filtration $\mathcal{W}_k$. It does not preserve the Hodge filtration $\mathcal{F}^p$, but it has the following property:

$$\operatorname{Fr} \mathcal{F}^s \subset p^s \mathcal{F}^0. \tag{12}$$

Notice that to prove (12) one should assume that $p > 3$ (the proof is based on the inequality $s < p$).

The Frobenius map is compatible with symplectic structure on 3-dimensional cohomology; more precisely,

$$\langle \operatorname{Fr} a, \operatorname{Fr} b\rangle = p^3 \operatorname{Fr}\langle a, b\rangle, \tag{13}$$

where $\langle a, b\rangle$ stands for the inner product of cohomology classes.

It is compatible with Gauss–Manin connection ∇; namely,

$$\nabla_\delta \operatorname{Fr} a = p \operatorname{Fr} \nabla_\delta a. \tag{14}$$

The matrix of Frobenius map is triangular; this follows from the fact that Fr preserves the weight filtration. Using (14) one can check that the diagonal elements ϵ_i of this matrix obey $\delta\epsilon_i = 0$; hence they do not depend on q. In the same way one can prove that two neighboring diagonal elements are equal up to a factor of p; in other words $\operatorname{Fr} e^0 = \epsilon e^0$ and all other diagonal elements have the form $p^i \epsilon$. From (13) we conclude that

[1] A transparent explanation of the origin of Frobenius map based on the ideas of supergeometry will be given in [14]. We are using the Frobenius map in canonical coordinates, but one can define this map in any coordinate system.

$\epsilon = \pm 1$. In what follows we assume that $\epsilon = 1$; the modifications necessary in the case $\epsilon = -1$ are obvious.

Taking into account (12) and (13) we can write

$$\mathrm{Fr}\, e^0 = e^0, \tag{15}$$
$$\mathrm{Fr}\, e^1 = pe^1 + pm_{12}e^0, \tag{16}$$
$$\mathrm{Fr}\, e_1 = p^2 e_1 + p^2 m_{23} e^1 + p^2 m_{13} e^0, \tag{17}$$
$$\mathrm{Fr}\, e_0 = p^3 e_0 + p^3 m_{34} e_1 + p^3 m_{24} e^1 + p^3 m_{14} e^0, \tag{18}$$

where $m_{ij} \in \mathbb{Z}_p[q]$ are q-series with integer p-adic coefficients. (The powers of p in RHS come from (12).) Using (13) we can obtain also that

$$-m_{34} + m_{12} = 0, \qquad -m_{23}m_{34} + m_{24} + m_{13} = 0.$$

Applying Fr to (9) and using (14) we obtain

$$pe^0 + p\delta m_{12} e^0 = pe^0. \tag{19}$$

This means that m_{12} is a constant. One can prove [5] that $m_{12} = 0$. We see that

$$m_{34} = m_{12} = 0, \qquad m_{24} + m_{13} = 0. \tag{20}$$

Similarly, from (10) we obtain

$$Y(q) - Y\big(q^p\big) = \delta m_{23}, \qquad m_{23} + \delta m_{13} = 0 \tag{21}$$

hence

$$Y\big(q^p\big) - Y(q) = \delta^2 m_{13}. \tag{22}$$

From (11) we see that

$$m_{34}Y + \delta m_{24} = m_{23}, \qquad \delta m_{34} = 0,$$
$$m_{24} + \delta m_{14} = m_{13}. \tag{23}$$

We know that $m_{34} = 0$, hence $m_{23} = \delta m_{24}$. Using the last equation in (23) and (20) we conclude that

$$2m_{13} = \delta m_{14}. \tag{24}$$

Combining this equation with (22) we obtain

Lemma 3.

$$Y\big(q^p\big) - Y(q) = \frac{1}{2}\delta^3 m_{14}. \tag{25}$$

Theorem. *Instanton numbers are p-adic integers if $p > 3$.*

This statement follows immediately from Lemmas 2 and 3.

Eq. (25) together with (5) leads to the representation of instanton numbers in terms of Frobenius map:

$$n_{p^a r} = \sum_{d|r} \mu\left(\frac{r}{d}\right) M_{p^a d} \frac{d^3}{r^3}, \tag{26}$$

where we assume that r is not divisible by p and use the notation M_k for the coefficients of power series expansion of $-\frac{1}{2}m_{14}$:

$$-\frac{1}{2}m_{14} = \sum M_k q^k.$$

Conversely, one can express the Frobenius map in terms of instanton numbers using Lemma 3 and (20)–(24). (Notice, that we are talking about Frobenius map in canonical coordinates.) One should notice that we assumed that diagonal entries of the matrix of the Frobenius map are positive; the assumption that they are negative leads to the change of sign of all entries of this matrix. One should mention also that our considerations do not fix the value of m_{14} at the point $q = 0$; it seems, however, that one can prove that this value is equal to zero.

3. Integrality of instanton numbers: general case

In the considerations of Section 2 we restricted ourselves to the case of quintic or, more generally, to the case when the moduli space of complex structures is one-dimensional.

Let us analyze the case when the dimension of moduli space $\mathcal{M}$ of complex structures is equal to $r > 1$.

Under certain conditions one can find a basis in three-dimensional cohomology consisting of vectors $e_0 \in I^{3,3}$, $e_1, \ldots, e_r \in I^{2,2}$, $e^1, \ldots, e^r \in I^{1,1}$, $e^0 \in I^{0,0}$ where $I^{p,p}$ stands for $\mathcal{F}^p \cap \mathcal{W}_{2p}$. In appropriate coordinate system $q_1, \ldots, q_r$ on $\mathcal{M}$ (in canonical coordinates) Gauss–Manin connection acts in the following way:

$$\nabla_{\delta_i} e^0 = 0,$$
$$\nabla_{\delta_i} e^k = \delta_{ik} e_0, \quad k = 1, \ldots, r,$$
$$\nabla_{\delta_i} e_j = \sum_k Y_{ijk}(q) e^k, \quad j = 1, \ldots, r,$$
$$\nabla_{\delta_i} e_o = e_i.$$

Here ∇_{δ_j} denotes the covariant derivative that corresponds to the logarithmic derivative $\delta_j = q_j \partial/\partial q_j$.

We are working in the neighborhood of maximally unipotent boundary point $q = 0$. We assume that the B-model at hand can be obtained as a mirror of A-model. Then one can say that the so-called integrality conjecture of [6] (see also [7], Section 5.2.2) is satisfied; this is sufficient to derive the above representation for Gauss–Manin connection. As in the case $r = 1$ the conditions we imposed leave some freedom in the choice of canonical coordinates and of the basis. We will require that the vectors $e^0(0), e^1(0), \ldots, e^r(0)$ constitute a $\mathbb{Z}$-basis of W_2.

The Yukawa couplings $Y_{ijk}(q)$ can be considered as power series with respect to canonical coordinates $q_1, \ldots, q_r$; these series have integral coefficients. (The integrality of these coefficients will not be used in the proof; it can be derived from the integrality of Frobenius map and the formula (27).)

Again one can prove [5] that the Gauss–Manin connection has the same form in p-adic situation; the Yukawa couplings should be considered as elements of $\mathbb{Q}_p[q_1, \ldots, q_r]$ in this case.

The Frobenius map has the form

$$\mathrm{Fr}\, e^0 = e^0,$$
$$\mathrm{Fr}\, e^k = pe^k + p(m_{12})^k e^0,$$
$$\mathrm{Fr}\, e_j = p^2 e_j + p^2 (m_{23})_{jk} e^k + p^2 (m_{13})_j e^0,$$
$$\mathrm{Fr}\, e_0 = p^3 e_0 + p^3 (m_{34})^j e_j + p^3 (m_{24})_k e^k + p^3 m_{14} e^0,$$

where $(m_{34})^j + (m_{12})^j = 0$, $(m_{23})_{jk}(m_{34})^k + (m_{13})_j + (m_{24})_j + 0$.

We can repeat the considerations of the case $r = 1$ to obtain

$$(m_{34})^j = (m_{12})^j = 0,$$
$$Y_{ijk}(q^p) - Y_{ijk}(q) = \delta_i (m_{23})_{jk} = \delta_i \delta_j (m_{13})_k,$$
$$2(m_{13})_j = \delta_j m_{14}.$$

We come to a conclusion that

$$Y_{ijk}(q^p) - Y_{ijk}(q) = \delta_i \delta_j \delta_k \left(\frac{1}{2} m_{14}\right). \tag{27}$$

This equation permits us to prove integrality of instanton numbers in the case at hand.

Recall that the Yukawa couplings can be represented in the form

$$\begin{aligned} Y_{ijk}(q) &= \mathrm{const} + \sum_s n_s \frac{q^s}{1-q^s} s_i s_j s_k \\ &= \mathrm{const} + \sum_{s, d|s} \frac{q^s}{d^3} n_{\frac{s}{d}} s_i s_j s_k. \end{aligned} \tag{28}$$

Here s is a multiindex: $s = (s_1, \ldots, s_r)$ and $q^s = q_1^{s_1} \cdots q_r^{s_r}$. In the second sum d runs over all positive integers dividing the integer vector s.

The numbers n_s can be identified with instanton numbers of the mirror A-model. The formula (28) remains correct in p-adic setting if we consider n_s as p-adic numbers.

Comparing the above formula with (27) we see that

$$\sum_{d|s} \frac{n_{\frac{p^a s}{d}}}{d^3} = M_{p^a s}, \tag{29}$$

where M_r are coefficients of the power expansion of $-\frac{1}{2} m_{14}$ and the vector s in (29) is not divisible by p. Using Moebius inversion formula we obtain from (29) an expression of instanton numbers n_s in terms of M_r; this expression has the form

$$n_{p^a r} = \sum_{d|r} d^{-3} \mu(d) M_{\frac{p^a r}{d}}, \tag{30}$$

where r is an integer vector that is not divisible by p and the sum runs over all positive divisors of r.

It is clear from this formula that instanton numbers are p-adic integers for $p > 3$. (We use the fact that matrix elements of Frobenius map, in particular, m_{14} are power series having p-adic integers as coefficients.) Notice, that the appearance of negative power of d in (30) does not contradict p-adic integrality, because d is not divisible by p.

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