ROZANSKY-WITTEN INVARIANTS VIA FORMAL GEOMETRY

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0. INTRODUCTION

Recently L. Rozansky and E. Witten (see [RW]) proposed a topological quantum field theory depending on a compact oriented 3-dimensional manifold M and on a compact hyperkähler manifold X. In the case $H^1(M, \mathbf{Q}) = 0$ (i.e. when M is a rational homology 3-sphere), the partition function $Z(M, X) \in \mathbf{C}$ of this field theory can be calculated using finitely many terms of the perturbation theory. As a function on M this is so called invariant of finite type, of order $2n$ where $4n$ is the dimension of X.

More generally, for every connected finite 3-valent graph Γ with $2n$ vertices, endowed with a cyclic order in the star of each vertex, Rozansky and Witten associated a function $X \mapsto Z_\Gamma(X)$ on the space of isometry classes of hyperkähler manifolds of dimension $4n$. This function is given by an integral over X of certain invariant polynomial in coefficients of the curvature tensor of the hyperkähler metric on X.

Here we propose a simple construction of RW invariants. It consists of two steps:

1) with every 3-valent graph endowed with orientations at vertices, or with every oriented rational homology 3-sphere M, we associate a cohomology class of the Lie algebra of formal Hamiltonian vector fields in an arbitrary finite-dimensional symplectic vector space. This cohomology class is stable. Stable cohomology groups under the question are called Graph Cohomology groups, because they can be calculated via certain complex constructed from finite graphs. Universal finite type invariants of links and of rational homology 3-spheres take values in certain subspace of the Graph Cohomology. Morally, this construction is a universal perturbative quantum Chern-Simons theory.

This construction is known already for a while, see [Ko1] for the general overview, [BN] for the discussion of 3-valent graphs and finite type invariants of links, and [LT] for the construction of invariants of homology 3-spheres.

2) it is known from early 70-ies that cohomology groups of Lie algebras of formal vector fields give characteristic classes of foliations (see [BR], [BH]). In the case of Hamiltonian vector fields we get characteristic classes of symplectic foliations, i.e. of foliations endowed with a symplectic structure in the transversal direction. Formally, any complex manifold can be considered as a foliation in anti-holomorphic direction. Analogously, a holomorphic symplectic manifold X gives a formal complex-valued symplectic foliation. Applying a small modification of the original construction of characteristic classes we get a homomorphism from the Lie algebra cohomology to the the cohomology of coherent sheaves $H^\bullet(X, \mathcal{O}_X)$. This construction could have been invented 25 years ago.

RW invariants can be read from these characteristic classes. Moreover, it follows directly from our description that one doesn't need the hyperkähler metric on X in the construction. One can formulate it in purely holomorphic terms (or in algebro-geometric terms if X is algebraic). An easy argument shows that numbers $Z(M, X)$ or $Z_\Gamma(X)$ are deformation invariants of hyperkähler manifolds X.

As a by-product, we obtain a construction of finite type invariants of 3-manifolds based on symplectic foliations instead of hyperkähler manifolds.

This paper is an extended version of my letters to V. Ginzburg and to E. Witten (January 1997). Recently M. Kapranov, stimulated by these letters, found a different approach to RW invariants. He noticed that cohomology classes in $H^\bullet(X, \mathcal{O}_X)$ associated with all 3-valent graphs can be written down in terms of just one class, so called Atiyah class. His construction is shorter than mine, but basically is the same. M. Kapranov wrote a beautiful and detailed exposition (see [Ka]) with many interesting deviations from the main theme. Still, I think that it is reasonable to give an account of the original geometric approach. Strictly speaking, my present paper contains no really new ideas. Nevertheless, I hope that it could help to clarify the picture.

I am grateful to M. Kapranov and to A. Schwarz for useful communications, and to E. Witten for showing me a preliminary version of the paper [RW].

0.1. *Notations*

Let $\mathbf{g}$ be a Lie algebra over $\mathbf{R}$, $\mathbf{k} \subset \mathbf{g}$ be a finite-dimensional Lie subalgebra of $\mathbf{g}$ and K be a Lie group (not necessarily connected) with the Lie algebra $\mathbf{k}$. We assume also that an action of K by automorphisms of $\mathbf{g}$ is given, such that the induced action of $\mathbf{k} = Lie(K)$ is the adjoint action. Let V be a $(\mathbf{g}, K)$-module. Relative cohomology group $H^\bullet(\mathbf{g}, K; V)$ is defined as the cohomology group of the complex of K-invariant skew-symmetric polylinear maps from $\mathbf{g}$ to V vanishing if one of arguments belongs to $\mathbf{k}$:

$$C^i(\mathbf{g}, K; V) := \left(Hom(\wedge^i(\mathbf{g}/\mathbf{k}), V)\right)^K, \quad i \geq 0\,.$$

This complex is a subcomplex of the standard cochain complex of $\mathbf{g}$ with coefficients in V. The differential in $C^\bullet(\mathbf{g}, K; V)$ is induced from the standard differential in $C^\bullet(\mathbf{g}; V)$.

For the case of trivial coefficients the cochain complex $C^\bullet(\mathbf{g}; \mathbf{R})$ can be interpreted as the complex of (left) G-invariant differential forms on G, where G is any Lie group with the Lie algebra equal to $\mathbf{g}$. Analogously, the relative cochain complex $C^\bullet(\mathbf{g}, K; \mathbf{R})$ can be identified with the complex of G-invariant differential forms on G/K.

For $n \geq 0$ we denote by Ham_{2n} the Lie algebra of formal Hamiltonian vector fields in the standard symplectic vector space $\mathbf{R}^{2n}$. This Lie algebra is endowed with the topology of the inverse limit of finite-dimensional vector spaces. Elements of Ham_{2n} are in one-to-one correspondence with formal Hamiltonians modulo constants:

$$Ham_{2n} \simeq \mathbf{R}[[p_1, \ldots, p_n, q_1, \ldots, q_n]]/\mathbf{R}\ \ .$$

We denote by Ham^0_{2n} the subalgebra of Ham_{2n} consisting of formal vector fields vanishing at zero:

$$Ham_{2n}/Ham^0_{2n} \simeq \mathbf{R}^{2n}\ \ .$$

Lie algebra Ham^0_{2n} contains subalgebra $sp(2n, \mathbf{R})$ consisting of linear Hamiltonian vector fields. The Lie group $Sp(2n, \mathbf{R})$ acts on Ham^0_{2n}. Thus, we can define cohomology groups with coefficients in the trivial one-dimensional module

$$H^i_{2n} := H^i_{cont}(Ham^0_{2n}, Sp(2n, \mathbf{R}); \mathbf{R})\ \ .$$

Here the subscript *cont* means that we consider only continuous cochains, i.e. polylinear functionals depending on finitely many terms in the Taylor expansions at zero (Gelfand-Fuks cohomology).

1. FROM GRAPHS TO COHOMOLOGY

Let Γ be a 3-valent graph with $2N$ vertices. We associate with it an element I_Γ in H^{2N}_{2n} for any n.

Lie algebra Ham^0_{2n} is a semi-direct product of $sp(2n, \mathbf{R})$ and of the subalgebra Ham^1_{2n} consisting of Hamiltonians $H \in \mathbf{R}[[p_i, q_i]]$ such that the Taylor series of H starts at terms of order at least 3. Thus, the relative cochains $C^i_{cont}(Ham^0_{2n}, Sp(2n, \mathbf{R}); \mathbf{R})$ can be identified with $Sp(2n, \mathbf{R})$-invariant cochains of Ham^1_{2n} with trivial coefficients.

The group $Sp(2n, \mathbf{R})$ acts semi-simply on Ham^1_{2n}, and on its cochain complex. This implies that we have an isomorphism

$$H^i_{2n} \simeq \left(H^i_{cont}(Ham^1_{2n}; \mathbf{R})\right)^{Sp(2n,\mathbf{R})} \quad .$$

The first cohomology group of Ham^1_{2n} (i.e. the co-abelianization) is non-trivial, it contains $Sym^3(\mathbf{R}^{2n})$. The corresponding 1-cochain associates to a formal Hamiltonian H its third Taylor coefficient. Using cup-products we construct cohomology classes in higher degrees:

$$\wedge^{2N}(Sym^3(\mathbf{R}^{2n})) \longrightarrow H^{2N}_{cont}(Ham^1_{2n}; \mathbf{R}) \ .$$

The map from above is evidently $Sp(2n, \mathbf{R})$-equivariant. Now, any 3-valent graph Γ with $2N$ vertices (the number of vertices of Γ is necessarily even) gives, up to a sign, an invariant tensor in $\wedge^{2N}(Sym^3(\mathbf{R}^{2n}))$. We use graph as a scheme for contracting indices. Applying isomorphisms as above we get relative cohomology class I_Γ.

For general discussion of Graph Cohomology and its relation with cohomology of Lie algebras of formal vector fields we refer the reader to [Ko2].

2. CHARACTERISTIC CLASSES OF FLAT BUNDLES

Before going further we remind a general construction of characteristic classes of flat bundles. In Section 3 we will apply it to foliations.

Let X be a smooth manifold, G be a finite-dimensional Lie group with the Lie algebra $\mathbf{g}$. Let $E \longrightarrow X$ be a principal G-bundle endowed with a flat connection ∇.

Assume that E is trivial as a topological G-bundle. Let us choose a smooth trivialization of E. Then the connection ∇ is given by a 1-form A on X with values in $\mathbf{g}$, satisfying the Maurer-Cartan equation

$$dA + \frac{1}{2}[A, A] = 0 \ .$$

We can consider A as a linear map from $\mathbf{g}^* = C^1(\mathbf{g}; \mathbf{R})$ to $\Omega^1(X)$. Let us extend it to the map from the whole cochain complex of $\mathbf{g}$

$$\bigoplus_i \wedge^i(\mathbf{g}^*) = \bigoplus_i C^i(\mathbf{g}; \mathbf{R})$$

to $\bigoplus_i \Omega^i(X)$ using cup-products on the cochain complex and on differential forms. The Maurer-Cartan equation guarantees that this map is a morphism of complexes. Thus, we have a map of cohomology groups:

$$H^i(\mathbf{g};\mathbf{R}) \longrightarrow H^i(X,\mathbf{R}) \ .$$

This map does not change if we choose another trivialization of E in the same homotopy class of trivializations. The proof is immediate because in such a situation we have a flat connection in the trivialized G-bundle over the product $X \times [0,1]$.

Another way to describe the same construction is to use the natural G-invariant $\mathbf{g}$-valued 1-form A_E on E satisfying the Maurer-Cartan equation. It gives a homomorphism $H^i(\mathbf{g},\mathbf{R}) \longrightarrow H^i(E,\mathbf{R})$. A trivialization of E gives a section $s : X \longrightarrow E$ of E. Then we can restrict cohomology classes form E to $s(X)$.

For a semisimple group G and for primitive classes in $H^i(\mathbf{g};\mathbf{R})$ we get odd-dimensional characteristic classes for flat connections in topologically trivial bundles, essentially Chern-Simons secondary characteristic classes. For connected nilpotent groups G we always get characteristic classes because these groups are contractible and all G-bundles are topologically trivial.

Suppose now that we fixed a Lie subgroup K of G such that the bundle E is topologically equivalent to a bundle induced from a K-bundle. In other words, the bundle with fiber G/K associated with E, has a continuous section. Analogously to the previous construction, we can define a map

$$C^i(\mathbf{g},K;\mathbf{R}) \longrightarrow \Omega^i(X) \ .$$

The induced map on cohomology

$$H^i(\mathbf{g},K;\mathbf{R}) \longrightarrow H^i(X,\mathbf{R})$$

depends only on the homotopy class of an identification of E with induced bundles (i.e. of the section of the associated G/K-bundle).

If the inclusion $K \subset G$ is homotopy equivalence then there is unique homotopy class of identifications. For example, flat connections in complex n-dimensional vector bundles have characteristic classes (Chern-Simons classes)

$$cs_i \in H^{2i+1}(gl(n,\mathbf{C}),U(n);\mathbf{R}) \longrightarrow H^{2i+1}(X,\mathbf{R}), \ \ 0 \le i \le n-1 \ .$$

The next generalization consists in consideration of an infinite-dimensional group G, for example a projective limit of finite-dimensional groups. Also, one can consider "Lie groups" which are formal manifolds in some directions. The algebra of functions on such a "group" is the algebra of formal power series in several variables with coefficients in usual smooth functions on a Lie group. In algebraic geometry one can consider for these purposes a mixture of pro- and ind- schemes. Any pair $(\mathbf{g},K)$ (as in the definition of relative cohomology, see subsection 0.1) produces a partially formal Lie group.

An important example is the "Lie group" $FDiff(\mathbf{R}^n)$ (formal diffeomorphisms of $\mathbf{R}^n$). Its Lie algebra is the Lie algebra W_n of formal vector fields in $\mathbf{R}^n$. The underlying topological space of $FDiff(\mathbf{R}^n)$ is the group all automorphisms of $\mathbf{R}$-algebra

$\mathbf{R}[[x_1,\ldots,x_n]]$ (i.e. the group of formal diffeomorphisms of $\mathbf{R}^n$ fixing 0). Functions on this group are formal power series on n-dimensional space

$$\mathbf{R}^n \simeq \mathbf{R}\langle\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\rangle$$

with coefficients in smooth functions on spaces of jets of a finite order of diffeomorphisms of $\mathbf{R}^n$ fixing 0.

3. FOLIATIONS AND FLAT BUNDLES

If X is a smooth manifold and $\mathcal{F}$ is a foliation of codimension n on X then we construct a linear map

$$H^i_{cont}(W_n, O(n,\mathbf{R});\mathbf{R}) \longrightarrow H^i(X,\mathbf{R}) \ .$$

Namely, we have a principal bundle over X with the structure group equal to the group of formal diffeomorphisms of $\mathbf{R}^n$ fixing 0. The fiber of this bundle at each point $x\in X$ consists of identifications of formal neighborhoods of x in the space of leaves $U/\mathcal{F}$, where U is sufficiently small neighborhood of x in X, with the formal neighborhood of 0 in $\mathbf{R}^n$. This bundle carries a natural flat connection only along $\mathcal{F}$. We can consider the associate principal $FDiff(\mathbf{R}^n)$-bundle. This bundle carries a natural flat connection along all directions in X.

As a topological group $FDiff(\mathbf{R}^n)$ is the same as $Aut(\mathbf{R}[[x_1,\ldots,x_n]])$ (because formal coordinates do not change the topology). The group $Aut(\mathbf{R}[[x_1,\ldots,x_n]])$ is homotopy equivalent to $GL(n,\mathbf{R})$, being a pro-nilpotent extension of it. The group $GL(n,\mathbf{R})$ is homotopy equivalent to its subgroup $O(n,\mathbf{R})$. Thus, we get characteristic classes as above.

What we explained above is the standard construction of characteristic classes of foliations phrased in somewhat new terms.

4. APPLICATION TO HAMILTONIAN VECTOR FIELDS

Let us suppose now that the foliation $\mathcal{F}$ is endowed with the transversal symplectic structure and has codimension $2n$ instead of n as above. To have such a structure is the same as to have a closed degenerate 2-form ω on X of constant rank $2n$. The foliation $\mathcal{F}$ is given by the kernel of ω.

4.1. *Remark*

As a side remark, we want to notice that the natural source of degenerate 2-forms is the variational principle. On the set of solutions of Euler-Lagrange equations one has naturally a closed 2-form, which could be degenerate in some cases. The standard point of view is opposite to this. People usually consider Poisson manifolds, i.e. bivector fields satisfying the Jacobi identity as a degeneration of symplectic geometry. In general, Poisson manifolds describe a limiting behavior of quantum mechanics.

For a foliation $\mathcal{F}$ one can introduce the de Rham complex along $\mathcal{F}$:

$$\Omega^i_{\mathcal{F}}(X) := \Gamma(X, \wedge^i(T^*_{\mathcal{F}}))$$

where $T_{\mathcal{F}}$ denote the tangent bundle to $\mathcal{F}$. This complex is a quotient complex of the de Rham complex of X. Cohomology $H^i_{\mathcal{F}}(X)$ of the complex $\Omega^\bullet_{\mathcal{F}}(X)$ is quite a wild object, non-computable in simple terms in general.

As in Section 3, we have a principal bundle over X with the structure group

$$Aut(\mathbf{R}[[p_i, q_i]], \sum dp_i \wedge dq_i) \ .$$

This bundle carries a natural flat connection along $\mathcal{F}$. The structure group is homotopy equivalent to the subgroup $Sp(2n, \mathbf{R})$, and also to a smaller subgroup $U(n)$. Hence, we have natural maps

$$H^i_{2n} = H^i_{cont}(Ham^0_{2n}, Sp(2n, \mathbf{R}); \mathbf{R}) \longrightarrow H^i_{cont}(Ham^0_{2n}, U(n); \mathbf{R}) \longrightarrow H^i_{\mathcal{F}}(X) \ .$$

Foliation F has a natural transversal volume element vol represented by the differential form $\omega^n/n!$ where ω is the closed 2-form defining $\mathcal{F}$. Multiplication by this volume element gives a morphism of complexes and of cohomology spaces:

$$vol\wedge : \Omega^i_{\mathcal{F}}(X) \longrightarrow \Omega^{i+2n}(X), \quad H^i_{\mathcal{F}}(X) \longrightarrow H^{i+2n}(X, \mathbf{R}) \ .$$

Thus, we get characteristic classes with values in the de Rham cohomology of X in degrees shifted by $2n$. Alternatively, there is an analogous map for Lie algebra cohomology:

$$vol\wedge : H^i_{2n} \longrightarrow H^{i+2n}_{cont}(Ham_{2n}, Sp(2n, \mathbf{R}); \mathbf{R}) \ .$$

5. GROUPS OF SYMPLECTOMORPHISMS

Let (Y, ω) be a compact symplectic $2n$-dimensional manifold. Let us denote by $Sympl(Y)$ (or $Sympl(Y, \omega)$) the group of symplectomorphisms of Y, and by $sympl(Y)$ the Lie algebra of Hamiltonian vector fields on Y. A version of the construction of characteristic classes of foliations gives the following homomorphism:

$$H^i_{2n} = H^i_{cont}(Ham^0_{2n}, Sp(2n, \mathbf{R}); \mathbf{R}) \longrightarrow H^i_{cont}(Ham^0_{2n}, U(n); \mathbf{R}) \longrightarrow H^i(Sympl(Y)^\delta, \mathbf{R}) \ .$$

Here the upper index δ means that we consider $Sympl(Y)$ as a discrete group.

This homomorphism takes values in the group of characteristic classes of flat non-linear symplectic bundles. Let $E \longrightarrow B$ be a smooth bundle with a flat connection and a covariantly constant symplectic form on fibers. We assume that fibers are isomorphic as symplectic manifolds to Y. Such a structure is given by a homomorphism (up to a conjugacy)

$$\pi_1(B) \longrightarrow Sympl(Y)^\delta$$

The total space E carries a symplectic foliation. By constructions described above in Section 4 we get classes with values in $H^{i+2n}(E,\mathbf{R})$. Fibers of the bundle $E\longrightarrow B$ are compact and naturally oriented. Thus, we can integrate cohomology classes along fibers landing at the space $H^i(B,\mathbf{R})$.

Analogously, one can construct homomorphisms for the Lie algebra cohomology

$$H^i_{2n}=H^i_{cont}(Ham^0_{2n},Sp(2n,\mathbf{R});\mathbf{R})\longrightarrow H^i_{cont}(Ham^0_{2n},U(n);\mathbf{R})\longrightarrow H^i(sympl(Y),\mathbf{R}) \ .$$

We refer the reader to [F]. In general, it seems that there is a natural homomorphism from $H^i_{cont}(Ham^0_{2n},U(n);\mathbf{R})$ to the Van Est cohomology of $Sympl(Y)$ (the cohomology of a subcomplex of continuous cochains of the group $Sympl(Y)$).

5.1. *Example of a flat symplectic bundle*

There is a natural series of finite-dimensional bundles with flat symplectic connections. The base is the moduli space of complex curves of genus $g\geq 2$, the fiber is the moduli space of irreducible unitary local systems (or, generally, flat connections with compact structure groups) on a surface of genus g. There is a standard symplectic form on the moduli space of flat connections which is defined purely topologically. Of course, sometimes such moduli spaces are non-compact, or singular after the compactification. Modulo these technical difficulties, this example gives a series of homomorphisms, labeled by compact Lie groups, from the Graph Cohomology to the cohomology of moduli spaces of curves. In [Ko2] we constructed another homomorphism which maps the Graph Cohomology to the *homology* groups of moduli spaces of curves.

6. COMPLEX GEOMETRY AND FOLIATIONS

Let X be a complex manifold of dimension N. We denote by $\widetilde{X}$ the underlying smooth manifold of dimension $2N$. It is well-known that the almost-complex structure of X can be considered as a vector subbundle $T^{0,1}$ of the complexified tangent bundle $T_{\widetilde{X}}\otimes\mathbf{C}$. The integrability of almost complex structures is equivalent to the formal integrability of $T^{0,1}$. Thus, we get formally a "complex foliation" on $\widetilde{X}$.

There is still a better point of view. In order to describe it we introduce an auxiliary "complex manifold" $\widetilde{X}_{\mathbf{C}}$. The underlying topological space of this manifold $\widetilde{X}_{\mathbf{C}}$ is $\widetilde{X}$. The sheaf of functions on $\widetilde{X}_{\mathbf{C}}$ is the sheaf of complex-valued smooth functions on $\widetilde{X}$ considered as an algebra over $\mathbf{C}$. We look at this sheaf as at a completion of the sheaf of holomorphic functions on complex manifold $X\times\overline{X}$ defined in small neighborhoods of the closed subset $X_{diag}:=\{(x,x)|x\in X\}$ of $X\times\overline{X}$. From this picture it is clear that $\widetilde{X}_{\mathbf{C}}$ has formally the structure of the product of two manifolds, and carries two transversal foliations. We would like to forget about one of them and leave the other. Thus, holomorphic functions on X are functions constant along $\overline{\partial}$-foliation. Also, the de Rham complex along $\overline{\partial}$-foliation is nothing but the standard Dolbeault complex of X. Absence of higher cohomology groups for "coherent sheaves" on $\widetilde{X}_{\mathbf{C}}$ can be viewed as a Stein property. In algebraic geometry we would call such spaces affine schemes.

Let us return to symplectic geometry. If X carries a holomorphic symplectic form ω, then on $\widetilde{X}_{\mathbf{C}}$ we have a holomorphic symplectic foliation. Now we can apply the same construction as in Section 4 and get a map

$$H^i_{2n} \otimes \mathbf{C} := H^i_{cont}(Ham^0_{2n} \otimes \mathbf{C}, Sp(2n, \mathbf{C}); \mathbf{C}) \longrightarrow H^i_{\overline{\partial}}(\widetilde{X}) = H^i(X, \mathcal{O}) \ .$$

It is almost evident from our description that this construction is complex-analytic, i.e. if (X, ω) holomorphically depends on parameters then corresponding classes also depend holomorphically. Moreover, the construction can be phrased in the language of algebraic geometry, see [Ka].

A small extension of this construction involves cohomology with non-trivial coefficients. For example, we have natural maps

$$H^i_{cont}(Ham^0_{2n} \otimes \mathbf{C}, Sp(2n, \mathbf{C}); \wedge^j(\mathbf{C}^{2n})) \longrightarrow H^i(X, \Omega^j) \ .$$

Corresponding graph complexes are associated with graphs with free legs. Part of these graph cohomology spaces corresponding to 3-valent graphs appears as universal Vassiliev invariants of knots, see [BN]. Definitely there are other non-trivial cohomology classes corresponding to graphs of higher valency, as follows from simple estimates of Euler characteristics of Graph Complexes.

The construction of M. Kapranov of characteristic classes associated with 3-valent graphs can be phrased as follows. The Atiyah class α_T, introduced in [Ka], is the image in $H^1(X, Sym^3 T_X)$ of a natural class (see Section 1) in

$$H^1_{cont}(Ham^0_{2n} \otimes \mathbf{C}, Sp(2n, \mathbf{C}); Sym^3(\mathbf{C}^{2n})) \ .$$

Together with the symplectic form ω, which is an element of $H^0(X, \wedge^2(T^*_X))$ (or, of $H^0_{cont}(Ham^0_{2n} \otimes \mathbf{C}, Sp(2n, \mathbf{C}); \wedge^2(\mathbf{C}^{2n}))$), one can construct characteristic classes contracting indices in the tensor product of copies of α_T and of ω.

7. HYPERKÄHLER MANIFOLDS

If X is a compact hyperkähler manifold then we have have 3 complex structures I, J, K on X. Let us pick one of them, say I. Complex manifold X_I carries a holomorphic symplectic form ω_I. We can construct numerical invariants multiplying characteristic classes of (X_I, ω_I) in $H^i(X_I, \Omega^j)$ by appropriate powers of the holomorphic symplectic form and of the cohomology class of the Kähler form, and then integrating over X. In [Ka] the reader can find arguments showing that we get the same formulas as in the paper of Rozansky and Witten.

For 3-valent graphs the number which we get is invariant under deformations preserving the cohomology class of the Kähler form. The argument is is that 1) $Z_\Gamma(X, \omega)$ it is a holomorphic function on the moduli space of complex symplectic manifolds with fixed polarization, depending only on the symplectic form modulo the multiplication by a constant scalar, 2) by twistor construction one can produce a lot of rational in these moduli spaces, 3) holomorphic functions on $\mathbf{C}P^1$ are constant.

The moduli space of complex structures on a compact hyperkähler manifold is a locally symmetric Hermitean spaces of non-compact type. Thus, hyperkähler manifolds should have degenerations to (possibly) simpler objects. Eventually, one expects that one can get a combinatorial objects at the limit, something like toric varieties. These combinatorial objects should produce weight systems for Vassiliev invariants.

8. SUPERSYMMETRIC FORMULATION

We have seen two sources of linear functionals on Graph Cohomology (and invariants of knots and rational homology 3-spheres): symplectic foliations and complex symplectic manifolds. Another (standard) construction of invariants uses a finite-dimensional Lie algebra $\mathbf{g}$ endowed with an invariant non-degenerate scalar product, see [BN]. A bit more general construction (see [Ko1]) involves homotopy Lie algebras with scalar products.

In this section we demonstrate that all these constructions are special cases of one universal construction.

The main notion here is the notion of a differential $\mathbf{Z}/2\mathbf{Z}$-graded supermanifold, or of a Q-manifold in short (the terminology is borrowed from [AKSZ], the letter Q comes from the standard notation for the generator of BRST symmetry in mathematical physics). By definition, a Q-manifold is a super manifold endowed with the action of super Lie group $\mathbf{R}^{0|1}$. In other terms, the Q-structure is given by an odd vector field $Q \in \Pi\Gamma(T_X)$ satisfying the equation $[Q, Q] = 0$. One defines complex Q-manifolds analogously, (with possible versions like infinite-dimensional manifolds, or partially formal manifolds, like our spaces $\widetilde{X}_{\mathbf{C}}$ as in Section 6).

Basic examples of Q-manifolds are

1) $X =$ an ordinary manifold with $Q = 0$,

2)$X = \Pi TY = Spec(\bigoplus_i \Omega^i(Y))$, the odd tangent space to an ordinary manifold Y. Vector field Q is the de Rham differential,

3) $X = \Pi T_{\mathcal{F}} Y = Spec(\bigoplus_i \Omega^i_{\mathcal{F}}(Y))$, an extension of the previous example to the case of foliated manifold $(Y, \mathcal{F})$,

4) $X = \Pi T^{0,1} \widetilde{Y}_{\mathbf{C}} = Spec(\bigoplus_i \Omega^{0|i}(Y))$ for complex manifold Y, with Q equal to the Dolbeault differential $\overline{\partial}$,

5) $X = \Pi\mathbf{g} = Spec(\bigoplus_i \wedge^i(\mathbf{g})^*)$, where g is a Lie algebra, with the Q equal to the standard differential in the cochain complex of $\mathbf{g}$ with trivial coefficients,

6) a tautological extension of the previous example to homotopy Lie algebras. Homotopy Lie algebras are defined as formal Q-manifolds such that the vector field Q vanishes at the origin.

For Q-manifold X we define $\mathbf{Z}/2\mathbf{Z}$-graded cohomology group $H^\bullet_Q(X)$ as the cohomology $Ker(Q)/Im(Q)$ of the differential Q on the super vector space $\mathcal{O}(X)$ of global functions on X for C^∞, Stein , affine,... spaces X (and sheaf hypercohomology for the non-affine case). This cohomology group is equal to the space of functions in example 1), to the de Rham cohomology in example 2), to the de Rham cohomology for foliations in example 3), to the Dolbeault cohomology in example 4), and to the Lie algebra cohomology in example 5).

Instead of flat vector bundles in usual geometry it is convenient to speak about Q-

equivariant vector bundles. For example, any flat bundle over a manifold Y produces a Q-equivariant bundle over ΠTY. Any holomorphic bundle over complex manifold X produces a Q-equivariant bundle over $\Pi T^{0,1}\widetilde{X}_{\mathbf{C}}$. Any $\mathbf{g}$-module gives a Q-equivariant bundle over $\Pi\mathbf{g}$. We define in a uniform way cohomology $H^\bullet_Q(X,E)$ with coefficients in a Q-equivariant bundle E as $Ker(Q)/Im(Q)$ in the super vector space $\Gamma(X,E)$.

Lie algebras, manifolds, foliations, complex structures, and rational homotopy types are all alike.

9. Q-FAMILIES OF SYMPLECTIC MANIFOLDS

The most general construction of characteristic classes including all previous cases is the following. Let B be a Q-manifold and $p : E \longrightarrow B$ be a Q-equivariant bundle whose fibers are symplectic supermanifolds (may be formal) of super dimension $(2n|k)$. We assume that the symplectic structure on fibers is also Q-equivariant. Let $s : B \longrightarrow E$ be a Q-equivariant section of this bundle. The formal completion of E along $s(B)$ is a Q-equivariant bundle over B of formal pointed symplectic manifolds. Repeating with appropriate modifications constructions from Section 2, we obtain a homomorpism:

$$H^i_{2n|k} \longrightarrow H^\bullet_Q(B) \ .$$

Here super vector spaces $H^i_{2n|k}$ are defined for any $n,k \geq 0$ starting with the standard symplectic super vector space $\mathbf{R}^{2n|k}$ with even coordinates $(p_1,\ldots,p_n,q_1,\ldots,q_n)$ and odd coordinates $(\xi_1,\ldots,\xi_k)$ and with the symplectic form

$$\sum_{i=1}^{n} dp_i \wedge dq_i + \sum_{j=1}^{k} d\xi_j \wedge d\xi_j \ .$$

Graph Cohomology maps to spaces $H^i_{2n|k}$, as in the purely even case.

The general situation above can be described as a family of homotopy Lie algebras with scalar products. Thus, RW invariants is a generalization of the standard construction with homotopy Lie algebras from [Ko1] to the case of families.

We repeat here this construction for the case of ordinary even Lie algebras.

If g is a Lie algebra with a non-degenerate invariant scalar product then $\Pi\mathbf{g}$ is a flat symplectic super manifold. The vector field Q from example 5) has square equal to 0 by the Jacobi identity. Also, Q preserves the symplectic structure on $\Pi\mathbf{g}$ and vanishes at 0. The corresponding Q-bundle has the base B equal to a point pt with the trivial action of Q on it. The section s maps pt to 0.

Analogous constructions can be used in other geometric situations. The advantage of symplectomorphism groups is the existence of a huge amount of stable classes with possible significance for differential topology.

10. TOPOLOGICAL QUANTUM FIELD THEORY

Up to now, 3-dimensional manifolds played a purely formal role in our exposition. We replaced them from the beginning by cohomology classes of symplectomorphism groups. In fact, in [AKSZ] a general Lagrangian was constructed, which uses as input data an oriented odd-dimensional manifold M and a symplectic manifold X. The symmetry group of this Lagrangian is the product of symplectomorphism group $Sympl(X)$ and of certain super extension of the diffeomorphism group of M. Correlators in the corresponding topological quantum field theory can be extended to cohomological correlators with values in

$$H^\bullet(BDiff(M), \mathbf{R}) \otimes H^\bullet(Sympl^\delta(X), \mathbf{R}) \ .$$

It seems that the RW Lagrangian is essentially the same as the Lagrangian in [AKSZ], applied to topological quantum field theories depending on parameters.

In the scheme presented in [AKSZ], one can repace M by an odd-dimensional complex Calabi-Yau manifold. The corollary is that finite-type invariants of rational homology 3-spheres give holomorphic invariants of 3-dimensional Calabi-Yau manifolds with holomorphic volume elements.

We are planning to discuss it in more details in a joint work with A. Schwarz.

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