

Stability structures, motivic Donaldson-Thomas invariants and cluster transformations

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1 Introduction

1.1 Counting problems for 3-dimensional Calabi-Yau varieties

Let X be a compact complex 3-dimensional Kähler manifold such that $c_1(T_X) = 0 \in \text{Pic}(X)$ (hence by Yau theorem X admits a Calabi-Yau metric). We can associate with X several moduli spaces which have the virtual dimension zero:

- a) moduli of holomorphic curves in X with fixed genus and degree;
- b) moduli of holomorphic vector bundles on X (or, more generally, of coherent sheaves) with a fixed Chern character;
- c) moduli of special Lagrangian submanifolds¹ with a fixed homology class endowed with a $U(1)$ local system.

In order to have a well-defined virtual number of points of the moduli space one needs compactness and a perfect obstruction theory with virtual dimension zero (see [4], [67], [68]).² The compactification is known in the case a). It is given by the moduli of stable maps. The corresponding virtual numbers are Gromov-Witten invariants (GW-invariants for short). Donaldson and Thomas in [19],[68] addressed the cases b) and c). Analytical difficulties there are not completely resolved. The most understood example is the one of torsion-free sheaves of rank one with the fixed Chern character of the form $(1, 0, a, b) \in H^{ev}(X)$. The corresponding virtual numbers are called Donaldson-Thomas invariants (DT-invariants for short). One sees that the number of (discrete) parameters describing GW-invariants is equal to $1 + \dim H^2(X)$ (genus and degree) and coincides with the number of parameters describing DT-invariants. The conjecture from [47] (proved in many cases) says that GW-invariants and DT-invariants can be expressed one through another. The full putative virtual numbers in the case b) should

¹Recall that a Lagrangian submanifold $L \subset X$ is called *special* iff the restriction to L of a holomorphic volume form on X is a real volume form on L .

²The latter means that the deformation theory of a point is controlled by a differential-graded Lie algebra \mathfrak{g} such that $H^i(\mathfrak{g}) = 0$ for $i \neq 1, 2$ and $\dim H^1(\mathfrak{g}) = \dim H^2(\mathfrak{g})$.

depend on as twice as many parameters (i.e. $\dim H^{ev}(X)$). By mirror symmetry one reduces the case c) to the case b) for the dual Calabi-Yau manifold. Unlike to GW-invariants and DT-invariants these virtual numbers should depend on some choices (the Kähler structure in the case b) and the complex structure in the case c), see [68]). In particular, in the case c), for a compact $3d$ Calabi-Yau manifold X we should have an even function

$$\Omega_{SLAG} : H^3(X, \mathbf{Z}) \setminus \{0\} \rightarrow \mathbf{Q} ,$$

which depends on the complex structure on X in such a way that for any non-zero $\gamma \in H^3(X, \mathbf{Z})$ the number $\Omega_{SLAG}(\gamma)$ is a constructible function with respect to a real analytic stratification of the moduli space of complex structures. Moreover this number is integer for a generic complex structure. The invariant $\Omega_{SLAG}(\gamma)$ is the virtual number of special Lagrangian submanifolds $L \subset X$ in the class γ (or more generally, special Lagrangian submanifolds endowed with local systems of arbitrary rank).

Our aim in this paper is to describe a framework for “generalized Donaldson-Thomas invariants” and their wall-crossing formulas in the case of non-commutative compact $3d$ Calabi-Yau varieties. A choice of polarization (“complexified Kähler structure”) will be encoded into a choice of “stability condition” on \mathcal{C} . Then we define a generalized Donaldson-Thomas invariant $\Omega(\gamma)$ as the “number” of stable objects in \mathcal{C} with a fixed class γ in the K -group. Similar problem for abelian categories was addressed in the series of papers by Joyce [32][33][34] and in the recent paper of Bridgeland and Toledano Laredo [10]. Our paper can be thought of as a generalization to the case of triangulated categories (the necessity of such a generalization is motivated by both mathematical and physical applications, see e.g. [67], [21]). One of motivations for our counting formula was the microlocal formula by K. Behrend (see [2]) for the virtual number in the case of so called symmetric obstruction theory (see [4]), which is the case for objects in $3d$ Calabi-Yau categories. The above example b) corresponds to the bounded derived category $D^b(X)$ of coherent sheaves on X (more precisely to its A_∞ -enrichment). The example c) corresponds to the Fukaya category. In that case the wall-crossing formulas describe the behavior of Ω_{SLAG} . Even in the geometric situation our formalism extends beyond the case of smooth compact Calabi-Yau varieties.

1.2 Non-commutative varieties with polarization

All A_∞ -categories in this paper will be ind-constructible. This roughly means that their spaces of objects are countable inductive limits of constructible sets (for more details see Section 3). We define a *non-commutative proper algebraic variety over a base field \mathbf{k}* as an *Ext*-finite ind-constructible \mathbf{k} -linear triangulated A_∞ -category \mathcal{C} . For two objects E and F we denote by $\mathrm{Hom}^\bullet(E, F)$ the complex of morphisms and by $\mathrm{Ext}^\bullet(E, F)$ its cohomology.

Here are few examples of such categories.

Example 1 a) A_∞ -version of $D^b(X)$, the bounded derived category of the category of coherent sheaves on a smooth projective algebraic variety X/\mathbf{k} . In this case $D^b(X)$ coincides with the triangulated category $\mathrm{Perf}(X)$ of perfect complexes on X .

b) More generally, for a (not necessarily proper) smooth variety X endowed with a closed proper subset $X_0 \subset X$, the corresponding triangulated category is the full subcategory of $\mathrm{Perf}(X)$ consisting of complexes of sheaves with cohomology supported on X_0 .

c) Also for a (not necessarily proper) smooth variety X we can consider the the full subcategory of $\mathrm{Perf}(X)$ consisting of complexes of sheaves with compactly supported cohomology.

d) The A_∞ -version of the category $\mathrm{Perf}(X)$ of perfect complexes on a proper, not necessarily smooth scheme X over \mathbf{k} .

e) If A is an A_∞ -algebra with finite-dimensional cohomology then $\mathcal{C} = \mathrm{Perf}(A)$ is the category of perfect A -modules.

f) If \mathbf{k} is the field of characteristic zero and A is finitely generated in the sense of [71], (in particular it is homologically smooth, see [42]) then \mathcal{C} is the category of A -modules of finite dimension over \mathbf{k} .

g) If the category \mathcal{C} is ind-constructible and $E \in \mathrm{Ob}(\mathcal{C})$ then left and right orthogonal to the minimal triangulated subcategory generated by E are also ind-constructible (since the conditions $\mathrm{Ext}^\bullet(X, E) = 0$ and $\mathrm{Ext}^\bullet(E, X) = 0$ are “constructible”).

Let us make few comments on the list. Example a) is a particular case of examples b),c),d). Using the results of [7] we can reduce geometric examples b),d) to the algebraic example e), and also the example c) to the example f). Let us discuss a typical (and most important) example e) at the level of objects of the category. We claim that the set of isomorphism classes of objects of \mathcal{C} can be covered by an inductive limit of constructible sets.

First, replacing A by its minimal model we may assume that A is finite-dimensional. Basic examples of perfect A -modules are direct sums of shifts of A , i.e. modules of the type

$$M = A[n_1] \oplus A[n_2] \oplus \cdots \oplus A[n_r], \quad r \geq 0$$

and their “upper-triangular deformations” (a.k.a. twisted complexes). The latter are described by solutions to the Maurer-Cartan equations

$$\sum_{1 \leq l \leq r-1} m_l(\alpha, \dots, \alpha) = 0$$

where $\alpha = (a_{ij})_{i < j}$ is an upper-triangular $r \times r$ matrix with coefficients in A and $\deg a_{ij} = n_i - n_j + 1$.

This gives a closed scheme. For a point x of this scheme we have the corresponding A_∞ -module M_x over A . In order to describe all other objects of $Perf(A)$ we need to take direct summands (up to homotopy) of such modules M_x . The set of such summands is parametrized by all A_∞ -morphisms of the non-unital algebra $P = \mathbf{k} \cdot p$ with the product given by $p^2 = p$ (and with the trivial differential) to $\text{End}^\bullet(M_x)$. Every such morphism is described by a collection of linear maps $f_n : P^{\otimes n} \rightarrow \text{End}^{1-n}(M_x)$ satisfying a system of polynomial equations. Notice that if n is large then $f_n = 0$ because the corresponding negative graded components of $\text{End}^\bullet(M_x)$ are trivial. Therefore we again have a finite system of polynomial equations.

For given $N = \sum_{1 \leq j \leq r} (|n_j| + 1)$ we obtain a scheme of finite type Mod_N parametrizing some objects of $Perf(A)$. Clearly $Ob(Perf(A)) = \cup_{N \geq 1} Mod_N$. Each isomorphism class of an object appears in the union for infinitely many values of N . In order to avoid the “overcounting” we define a subscheme of finite type $Mod_N^0 \subset Mod_N$ consisting of objects not isomorphic to objects from $Mod_{N'}$ for $N' < N$. We conclude that objects of $Perf(A)$ form an ind-constructible set (more precisely, an ind-constructible stack). One can take care about morphisms in the category in a similar way. This explains the example e).

We define a *polarization* on a non-commutative proper algebraic variety over \mathbf{k} (a version of Bridgeland stability condition, see [9]) by the following data and axioms:

- an ind-constructible homomorphism $\text{cl} : K_0(\mathcal{C}) \rightarrow \Gamma$, where $\Gamma \simeq \mathbf{Z}^n$ is a free abelian group of finite rank endowed with a bilinear form³

³In physics literature Γ is called the charge lattice.

$\langle \bullet, \bullet \rangle : \Gamma \otimes \Gamma \rightarrow \mathbf{Z}$ such that for any two objects $E, F \in \text{Ob}(\mathcal{C})$ we have

$$\langle \text{cl}(E), \text{cl}(F) \rangle = \chi(E, F) := \sum_i (-1)^i \dim \text{Ext}^i(E, F) ,$$

- an additive map $Z : \Gamma \rightarrow \mathbf{C}$, called the central charge,
- a collection \mathcal{C}^{ss} of (isomorphism classes of) non-zero objects in \mathcal{C} called the semistable ones, such that $Z(E) \neq 0$ for any $E \in \mathcal{C}^{ss}$, where we write $Z(E)$ for $Z(\text{cl}(E))$,
- a choice $\text{Log } Z(E) \in \mathbf{C}$ of the logarithm of $Z(E)$ defined for any $E \in \mathcal{C}^{ss}$.

Making a connection with [9] we say that the last three items define a *stability structure (or stability condition)* on the category \mathcal{C} .

For $E \in \mathcal{C}^{ss}$ we denote by $\text{Arg}(E) \in \mathbf{R}$ the imaginary part of $\text{Log } Z(E)$.

The above data satisfy the following axioms:

- for all $E \in \mathcal{C}^{ss}$ and for all $n \in \mathbf{Z}$ we have $E[n] \in \mathcal{C}^{ss}$ and

$$\text{Arg } Z(E[n]) = \text{Arg } Z(E) + \pi n ,$$

- for all $E_1, E_2 \in \mathcal{C}^{ss}$ with $\text{Arg}(E_1) > \text{Arg}(E_2)$ we have

$$\text{Ext}_{\mathcal{C}}^{\leq 0}(E_1, E_2) = 0 ,$$

- for any object $E \in \text{Ob}(\mathcal{C})$ there exist $n \geq 0$ and a chain of morphisms $0 = E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_n = E$ (an analog of filtration) such that the corresponding “quotients” $F_i := \text{Cone}(E_{i-1} \rightarrow E_i)$, $i = 1, \dots, n$ are semistable and $\text{Arg}(F_1) > \text{Arg}(F_2) > \dots > \text{Arg}(F_n)$,
- for each $\gamma \in \Gamma \setminus \{0\}$ the set of isomorphism classes of a $\mathcal{C}_{\gamma}^{ss} \subset \text{Ob}(\mathcal{C})_{\gamma}$ consisting of semistable objects E such that $\text{cl}(E) = \gamma$ and $\text{Arg}(E)$ is fixed, is a constructible set,
- (Support Property) Pick a norm $\| \cdot \|$ on $\Gamma \otimes \mathbf{R}$, then there exists $C > 0$ such that for all $E \in \mathcal{C}^{ss}$ one has $\| E \| \leq C |Z(E)|$.

In the above definition one can allow Γ to have a torsion. In geometric examples a), d) for $\mathbf{k} = \mathbf{C}$ one can take $\Gamma = K_{top}^0(X(\mathbf{C}))$ where K_{top}^0 denotes the topological K^0 -group. Similarly, in examples b),c) one should take the K^0 -groups with appropriate supports. Another choice for Γ is the image of the algebraic K^0 -group under the Chern character. Yet another choice is $\Gamma = K_0^{num}(\mathcal{C})$, which is the quotient of the group $K_0(\mathcal{C})$ by the intersection of the left and right kernels of the Euler form $\chi(E, F)$. Finally one can pick a finite collection of ind-constructible functors $\Phi_i : \mathcal{C} \rightarrow Perf(\mathbf{k}), 1 \leq i \leq n$ and define

$$\text{cl}(E) = (\chi(\Phi_1(E)), \dots, \chi(\Phi_n(E))) \in \mathbf{Z}^n =: \Gamma ,$$

where $\chi : K_0(Perf(\mathbf{k})) \rightarrow \mathbf{Z}$ is the isomorphism of groups given by the Euler characteristic.

Remark 1 *The origin of the Support Property is geometric and can be explained in the case of the category of A-branes (the derived Fukaya category $D^b(\mathcal{F}(X))$) of a compact 3-dimensional Calabi-Yau manifold X . Let us fix a Calabi-Yau metric g_0 on X . Asymptotically, in the large volume limit (as the rescaled symplectic form approaches infinity) it gives rise to the stability condition on $D^b(\mathcal{F}(X))$ such that stable objects are special Lagrangian submanifolds, and $|Z(L)|$ is the volume of L with respect to g_0 . Then for any harmonic form η one has $|\int_L \eta| \leq C|Z(L)|$. It follows that the norm of the cohomology class of L is bounded (up to a scalar factor) by the norm of the linear functional Z .*

The Support Property implies that the set $\{Z(E) \in \mathbf{C} \mid E \in \mathcal{C}^{ss}\}$ is a discrete subset of \mathbf{C} with at most polynomially growing density at infinity. It also implies that the stability condition is locally finite in the sense of Bridgeland (see [9]). Any stability condition gives a bounded t -structure on \mathcal{C} with the corresponding heart consisting of semistable objects E with $\text{Arg}(E) \in (0, \pi]$ and their extensions.

Remark 2 *The case of the classical Mumford notion of stability with respect to an ample line bundle (and its refinement for coherent sheaves defined by Simpson) is not an example of the Bridgeland stability condition, it is rather a limiting degenerate case of it (see [1], [73] and Remark at the end of Section 2.1).*

For given \mathcal{C} and a homomorphism $\text{cl} : K_0(\mathcal{C}) \rightarrow \Gamma$ as above, let us denote by $\text{Stab}(\mathcal{C}) := \text{Stab}(\mathcal{C}, \text{cl})$ the set of stability conditions $(Z, \mathcal{C}^{ss}, (\text{Log } Z(E))_{E \in \mathcal{C}^{ss}})$. Space $\text{Stab}(\mathcal{C})$ can be endowed with a Hausdorff topology, which we discuss in detail in Section 3.4. Then we have an ind-constructible version of the following fundamental result of Bridgeland (see [9]).

Theorem 1 *The forgetting map $\text{Stab}(\mathcal{C}) \rightarrow \mathbf{C}^n \simeq \text{Hom}(\Gamma, \mathbf{C})$ given by $(Z, \mathcal{C}^{ss}, (\text{Log } Z(E))_{E \in \mathcal{C}^{ss}}) \mapsto Z$, is a local homeomorphism.*

Hence, $\text{Stab}(\mathcal{C})$ is a complex manifold, not necessarily connected. Under appropriate assumptions one can show also that the group of autoequivalences $\text{Aut}(\mathcal{C})$ acts properly and discontinuously on $\text{Stab}(\mathcal{C})$. On the quotient orbifold $\text{Stab}(\mathcal{C})/\text{Aut}(\mathcal{C})$ there is a natural non-holomorphic action of the group $GL^+(2, \mathbf{R})$ of orientation-preserving \mathbf{R} -linear automorphisms of $\mathbf{R}^2 \simeq \mathbf{C}$.

1.3 Donaldson-Thomas invariants for non-commutative 3d Calabi-Yau varieties

Recall that a non-commutative Calabi-Yau variety of dimension d (a.k.a Calabi-Yau category of dimension d) is given by an *Ext*-finite triangulated A_∞ -category \mathcal{C} which carries a functorial non-degenerate pairing

$$(\bullet, \bullet) : \text{Hom}_{\mathcal{C}}^\bullet(E, F) \otimes \text{Hom}_{\mathcal{C}}^\bullet(F, E) \rightarrow \mathbf{k}[-d]$$

(see e.g. [42], [65], [44]), such that the polylinear forms $(m_n(f_0, \dots, f_n), f_{n+1})$ defined on $\otimes_{0 \leq i \leq n+1} \text{Hom}_{\mathcal{C}}^\bullet(E_i, E_{i+1})$ by higher compositions m_n are cyclically invariant. We will discuss mainly the case $d = 3$ and assume that our non-commutative 3d Calabi-Yau variety is ind-constructible and endowed with polarization.

Under these assumptions we define *motivic Donaldson-Thomas invariants* which take values in certain Grothendieck groups of algebraic varieties (more details are given in Sections 4 and 6). Assuming some “absence of poles” conjectures, which we discuss in detail in Section 7 one can pass to the “quasi-classical limit” which corresponds to the taking of Euler characteristic of all relevant motives. In this way we obtain the putative *numerical DT-invariants* $\Omega(\gamma) \in \mathbf{Q}, \gamma \in \Gamma \setminus \{0\}$. Morally, $\Omega(\gamma)$ counts semistable objects of \mathcal{C} with a given class $\gamma \in \Gamma \setminus \{0\}$.

There is a special case when our formulas can be compared with those from [2] (see Section 7.1). Namely, let us define a *Schur object* $E \in \text{Ob}(\mathcal{C})$ as such that

$$\text{Ext}^{<0}(E, E) = 0, \text{Ext}^0(E, E) = \mathbf{k} \cdot \text{Id}_E .$$

By the Calabi-Yau property in the dimension $d = 3$ we know that the only possibly non-trivial groups $\text{Ext}^i(E, E), i = 0, 1, 2, 3$ are

$$\text{Ext}^0(E, E) \simeq \text{Ext}^3(E, E) \simeq \mathbf{k}, \quad \text{Ext}^1(E, E) \simeq (\text{Ext}^2(E, E))^* .$$

In other words the ranks are $(1, a, a, 1), a \in \mathbf{Z}_{\geq 0}$. Recall (see [41], [42]) that the deformation theory of any object $E \in \text{Ob}(\mathcal{C})$ is controlled by a differential-graded Lie algebra (DGLA for short) \mathfrak{g}_E such that $H^i(\mathfrak{g}_E) \simeq \text{Ext}^i(E, E), i \in \mathbf{Z}$. For a given Schur object E instead of \mathfrak{g}_E we can use a DGLA $\widehat{\mathfrak{g}}_E = \tau_{\leq 2}(\mathfrak{g}_E)/\tau_{\leq 0}(\mathfrak{g}_E)$ where $\tau_{\leq i}$ is the truncation functor. This makes sense since $\tau_{\leq 0}(\mathfrak{g}_E)$ is an ideal (in the homotopy sense) in \mathfrak{g}_E . The modified deformation theory gives rise to a perfect obstruction theory in the sense of [2], [4]. The corresponding moduli space is the same as the original one, although controlling DGLAs are not quasi-isomorphic. The contribution of Schur objects to $\Omega(\gamma)$ can be identified with the Behrend microlocal formula for DT-invariants. From this point of view objects of the category \mathcal{C} should be interpreted as critical points of the function (called the potential), which is obtained from the solution to the so-called classical master equation. The latter has a very natural interpretation in terms of the non-commutative formal symplectic dg-scheme defined by the A_∞ -category \mathcal{C} endowed with a Calabi-Yau structure (see [42]).

1.4 Multiplicative wall-crossing formula

The wall-crossing formulas for the numerical Donaldson-Thomas invariants do not depend on their origin and can be expressed in terms of graded Lie algebras. This is explained in Section 2. Our main application is the case of $3d$ Calabi-Yau categories. Let us recall that if \mathcal{C} is an *Ext*-finite Calabi-Yau category of the odd dimension d (e.g. $d = 3$) then the Euler form

$$\chi : K_0(\mathcal{C}) \otimes K_0(\mathcal{C}) \rightarrow \mathbf{Z}, \quad \chi(E, F) := \sum_{n \in \mathbf{Z}} (-1)^n \dim \text{Ext}^n(E, F)$$

is skew-symmetric. In this case we also assume that if \mathcal{C} is endowed with polarization, then a skew-symmetric bilinear form $\langle \bullet, \bullet \rangle : \Gamma \otimes \Gamma \rightarrow \mathbf{Z}$ is given and satisfies

$$\langle \text{cl}(E), \text{cl}(F) \rangle = \chi(E, F) \quad \forall E, F \in \text{Ob}(\mathcal{C}).$$

In general, having a free abelian group Γ of finite rank endowed with an integer-valued skew-symmetric form $\langle \bullet, \bullet \rangle$, we define a Lie algebra over \mathbf{Q} $\mathfrak{g}_\Gamma := \mathfrak{g}_{(\Gamma, \langle \bullet, \bullet \rangle)}$, with the basis $(e_\gamma)_{\gamma \in \Gamma}$ and the Lie bracket

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2}.$$

This Lie algebra is isomorphic (non-canonically) to the Lie algebra of regular functions on the algebraic Poisson torus $\text{Hom}(\Gamma, \mathbf{G}_m)$ endowed with the natural translation-invariant Poisson bracket.⁴

An additive map $Z : \Gamma \rightarrow \mathbf{C}$ is called *generic* if there are no two \mathbf{Q} -independent elements of the lattice Γ which are mapped by Z into the same straight line in $\mathbf{R}^2 = \mathbf{C}$. The set of non-generic maps is a countable union of real hypersurfaces in $\mathbf{C}^n = \text{Hom}(\Gamma, \mathbf{C})$. These hypersurfaces are called *walls*.

Let us choose such an additive map Z and an arbitrary norm $\| \bullet \|$ on the real vector space $\Gamma_{\mathbf{R}} = \Gamma \otimes \mathbf{R}$. We will keep the same notation for the \mathbf{R} -linear extension of Z to $\Gamma_{\mathbf{R}}$. Finally, assume that we are given an even map $\Omega : \Gamma \setminus \{0\} \rightarrow \mathbf{Z}$ supported on the set of $\gamma \in \Gamma$ such that $\| \gamma \| \leq C|Z(\gamma)|$ for some given constant $C > 0$.

Let $(Z_t)_{t \in [0,1]}$ be a generic piece-wise smooth path in $\mathbf{C}^n = \text{Hom}(\Gamma, \mathbf{C})$ such that Z_0 and Z_1 are generic. The wall-crossing formula calculates the function Ω_1 corresponding to Z_1 in terms of the function $\Omega = \Omega_0$ corresponding to Z_0 . This is analogous to the analytic continuation of a holomorphic function expressed in terms of its Taylor coefficients. The continuation is unique if it exists, and is not changed under a small deformation of the path with the fixed endpoints.

Let us call *strict* a sector in \mathbf{R}^2 with the vertex at the origin $(0,0)$ which is less than 180° . With a strict sector $V \subset \mathbf{R}^2$ we associate a group element A_V given by the infinite product

$$A_V := \prod_{\gamma \in Z^{-1}(V) \cap \Gamma} \exp \left(-\Omega(\gamma) \sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2} \right).$$

⁴Later we will use the multiplication as well. It is given explicitly by $e_{\gamma_1} e_{\gamma_2} = (-1)^{\langle \gamma_1, \gamma_2 \rangle} e_{\gamma_1 + \gamma_2}$.

The product takes value in a certain pro-nilpotent Lie group $G_V := G_{V,\Gamma,(\bullet,\bullet)}$, which we will describe below. The right arrow in the product sign means that the product is taken in the *clockwise* order on the set of rays $\mathbf{R}_+ \cdot Z(\gamma) \subset V \subset \mathbf{C}$. For the product in the anti-clockwise order we will use the left arrow.

Let us describe the Lie algebra $\mathfrak{g}_V = \text{Lie}(G_V)$ of the pro-nilpotent Lie group G_V . We denote by $C(V)$ a convex cone in $\Gamma_{\mathbf{R}}$ which is the convex hull of the set of points $v \in Z^{-1}(V)$ such that $\|v\| \leq C|Z(v)|$. The Lie algebra $\text{Lie}(G_V)$ is the infinite product $\prod_{\gamma \in \Gamma \cap C(V)} \mathbf{Q} \cdot e_\gamma$ equipped with the above Lie bracket.

Now we can formulate the wall-crossing formula. It says (roughly) that A_V does not change as long as no lattice point $\gamma \in \Gamma$ with $\Omega_t(\gamma) \neq 0$ crosses the boundary of the cone $Z_t^{-1}(V)$ (here Ω_t corresponds to the point $t \in [0, 1]$). By our assumptions, if $t = t_0$ corresponds to a non-generic central charge Z_{t_0} then there exists a 2-dimensional lattice $\Gamma_0 \subset \Gamma$ such that $Z_{t_0}(\Gamma_0)$ belongs to a real line $\mathbf{R}e^{i\alpha}$ for some $\alpha \in [0, \pi]$.

The wall-crossing formula describes the change of values $\Omega(\gamma)$ for $\gamma \in \Gamma_0$ and depends only on the restriction $\Omega|_{\Gamma_0}$ of Ω to the lattice Γ_0 . Values $\Omega(\gamma)$ for $\gamma \notin \Gamma_0$ do not change at $t = t_0$. Denote by $k \in \mathbf{Z}$ the value of the form $\langle \bullet, \bullet \rangle$ on a fixed basis γ_1, γ_2 of $\Gamma_0 \simeq \mathbf{Z}^2$ such that $C(V) \cap \Gamma_0 \subset \mathbf{Z}_{\geq 0} \cdot \gamma_1 \oplus \mathbf{Z}_{\geq 0} \cdot \gamma_2$. We assume that $k \neq 0$, otherwise there will be no jump in values of Ω on Γ_0 . The group elements which we are interested in can be identified with products of the following automorphisms⁵ of $\mathbf{Q}[[x, y]]$ preserving the symplectic form $k^{-1}(xy)^{-1}dx \wedge dy$:

$$T_{a,b}^{(k)} : (x, y) \mapsto (x \cdot (1 - (-1)^{kab} x^a y^b)^{-kb}, y \cdot (1 - (-1)^{kab} x^a y^b)^{ka}), a, b \geq 0, a + b \geq 1.$$

For $\gamma = a\gamma_1 + b\gamma_2$ we have

$$T_{a,b}^{(k)} = \exp \left(- \sum_{n \geq 1} \frac{e_{n\gamma}}{n^2} \right)$$

in the above notation. Any exact symplectomorphism ϕ of $\mathbf{Q}[[x, y]]$ can be

⁵Here we write an automorphism as acting on elements of the algebra of functions. The corresponding automorphism on *points* is given by the inverse formula.

decomposed uniquely into a clockwise and an anti-clockwise product:

$$\phi = \prod_{a,b}^{\rightarrow} \left(T_{a,b}^{(k)} \right)^{c_{a,b}} = \prod_{a,b}^{\leftarrow} \left(T_{a,b}^{(k)} \right)^{d_{a,b}}$$

with certain exponents $c_{a,b}, d_{a,b} \in \mathbf{Q}$. These exponents should be interpreted as the limiting values of the functions $\Omega_{t_0}^{\pm} = \lim_{t \rightarrow t_0 \pm 0} \Omega_t$ restricted to Γ_0 . The passage from the clockwise order (when the slope $a/b \in [0, +\infty] \cap \mathbb{P}^1(\mathbf{Q})$ decreases) to the anti-clockwise order (when the slope increases) gives the change of $\Omega|_{\Gamma_0}$ as we cross the wall. It will be convenient to denote $T_{a,b}^{(1)}$ simply by $T_{a,b}$. The pro-nilpotent group generated by transformations $T_{a,b}^{(k)}$ coincides with the one generated by transformations $T_{a,|k|b}$.

The compatibility of the wall-crossing formula with the *integrality* of the numbers $\Omega(\gamma)$ is not obvious but follows from:

Conjecture 1 *If for $k > 0$ one decomposes the product $T_{1,0} \cdot T_{0,k}$ in the opposite order:*

$$T_{1,0} \cdot T_{0,k} = \prod_{a/b \text{ increases}} (T_{a,kb})^{d(a,b,k)},$$

then $d(a,b,k) \in \mathbf{Z}$ for all a, b, k .

An equivalent form of this conjecture says that if one decomposes $T_{1,0}^k \cdot T_{0,1}^k$ in the opposite order then all exponents will belong to $k\mathbf{Z}$.

Here are decompositions for $k = 1, 2$

$$T_{1,0} \cdot T_{0,1} = T_{0,1} \cdot T_{1,1} \cdot T_{1,0} ,$$

$$T_{1,0}^{(2)} \cdot T_{0,1}^{(2)} = T_{0,1}^{(2)} \cdot T_{1,2}^{(2)} \cdot T_{2,3}^{(2)} \cdots (T_{1,1}^{(2)})^{-2} \cdots T_{3,2}^{(2)} \cdot T_{2,1}^{(2)} \cdot T_{1,0}^{(2)} ,$$

or equivalently

$$T_{1,0} \cdot T_{0,2} = T_{0,2} \cdot T_{1,4} \cdot T_{2,6} \cdots T_{1,2}^{-2} \cdots T_{3,4} \cdot T_{2,2} \cdot T_{1,0} .$$

Greg Moore and Frederik Denef pointed out that the factors in the last formula correspond to the BPS spectrum of $N = 2, d = 4$ super Yang-Mills model studied by Seiberg and Witten in [63]. A “physical” explanation of our formulas in this context was given in [24], see also our Section 2.7.

For $k \geq 3$ or $k \leq -1$ the decomposition of $T_{1,0}^{(k)} \cdot T_{0,1}^{(k)}$ is not yet known completely. Computer experiments give a conjectural formula for the diagonal term with the slope $a/b = 1$. The corresponding symplectomorphism is given by the map

$$(x, y) \mapsto (x \cdot F_k(xy)^{-k}, y \cdot F_k(xy)^k),$$

where the series $F_k = F_k(t) \in 1 + t\mathbf{Z}[[t]]$ is an algebraic hypergeometric series given for $k \geq 3$ by the formulas:

$$\sum_{n=0}^{\infty} \binom{(k-1)^2 n + k - 1}{n} \frac{t^n}{(k-2)n + 1} = \exp \left(\sum_{n=1}^{\infty} \binom{(k-1)^2 n}{n} \frac{k}{(k-1)^2} \frac{t^n}{n} \right).$$

The function F_k satisfies the equation

$$F_k(t) (1 - tF_k^{k-2}(t))^k - 1 = 0 .$$

Remark 3 *The above example for $k = 1$ is compatible with the expected behavior of Donaldson-Thomas invariants when we have two spherical objects $E_1, E_2 \in \mathcal{C}$ (sphericity means that $\text{Ext}^\bullet(E_i, E_i) = H^\bullet(S^3)$) such that*

$$\text{Ext}^1(E_2, E_1) = \mathbf{k}, \quad \text{Ext}^n(E_2, E_1) = 0 \text{ for } n \neq 1 .$$

In this case on the one side of the wall we have two semistable objects E_1, E_2 , and on the other side we have three semistable objects E_1, E_2, E_{12} where E_{12} is the extension of E_2 by E_1 . In the case of the derived of the Fukaya category the objects $E_i, i = 1, 2$ can correspond to embedded special Lagrangian spheres intersecting transversally at one point. Then E_{12} corresponds to their Lagrangian connected sum.

The automorphisms $T_{a,b}$ are a special case of the more general ones. Namely, we can consider the following rational automorphisms of \mathfrak{g}_Γ (considered as a Poisson algebra):

$$T_\gamma : e_\mu \mapsto (1 - e_\gamma)^{\langle \gamma, \mu \rangle} e_\mu, \quad \gamma, \mu \in \Gamma .$$

The group element A_V in the above notation has the form

$$A_V = \prod_{\gamma \in Z^{-1}(V) \cap \Gamma} \overrightarrow{T_\gamma^{\Omega(\gamma)}}$$

and acts on a completion of \mathfrak{g}_Γ . It is easy to quantize this Poisson algebra. The corresponding algebra (quantum torus) is additively generated by quantum generators $\hat{e}_\gamma, \gamma \in \Gamma$ subject to the relations

$$\hat{e}_\gamma \hat{e}_\mu = q^{\frac{1}{2}\langle \gamma, \mu \rangle} \hat{e}_{\gamma+\mu} ,$$

where q is a parameter (with the classical limit $q^{\frac{1}{2}} \rightarrow -1$). Then one has formulas similar to the above for the “quantum” analogs of automorphisms $T_\gamma, \gamma \in \Gamma$ (see Sections 6.4 and 7.1).

For general $k \geq 2$ the decomposition of the product $T_{1,0} \cdot T_{0,k}$ as in Conjecture 1, describes numerical DT-invariants of the Calabi-Yau category associated with the Kronecker quiver consisting of two vertices and k parallel arrows (see Section 8 for a general theory). Recent paper [58] gives an explicit formula for $d(a, b, k)$ in terms of the Euler characteristic of the framed moduli space of semistable representations of the quiver. Moreover, a weak form of the integrality Conjecture 1 is proved in [58].

1.5 Some analogies and speculations

The above formulas for symplectomorphisms are partially motivated by [40], Section 10, where similar formulas appeared in a different problem. Both formulas involve Hamiltonian vector fields associated with the dilogarithm function. The problem discussed in [40] was the reconstruction of the rigid analytic K3 surface from its skeleton, which is a sphere S^2 equipped with an integral affine structure, singular at a finite set of points. The group which is very similar to the pro-nilpotent group G_V was introduced in the loc. cit. where we assigned symplectomorphisms to edges of a certain tree in S^2 . That tree should be thought of as an analog of the walls in the space of stability structures. Edges of the tree (we called them “lines” in [40]) correspond to pseudo-holomorphic discs with the boundary on the Lagrangian toric fibers of the dual K3 surface. When we approach the “large complex structure limit” cusp in the moduli space of K3 surfaces, the discs degenerate into gradient lines of some smooth functions on S^2 , thus defining edges of the tree. Hence the reconstruction problem for K3 surfaces (and for higher-dimensional Calabi-Yau manifolds, see [27], [28]) is governed by the counting of rational curves in the mirror dual Calabi-Yau manifold. This observation suggests by analogy the questions below.

1) First, let us note that we may assume that the bilinear form $\langle \bullet, \bullet \rangle$ is non-degenerate on Γ by replacing Γ by a “larger” lattice (e.g. by $\Gamma \oplus \Gamma^\vee$,

where $\Gamma^\vee = \text{Hom}(\Gamma, \mathbf{Z})$ is the dual lattice, see Section 2.6). Then the Lie algebra \mathfrak{g}_Γ will be realized as the Lie algebra of exact Hamiltonian vector fields on the algebraic symplectic torus $\text{Hom}(\Gamma, \mathbf{G}_m)$. The collection of formal symplectomorphisms A_V defined above give rise to a rigid analytic space \mathcal{X}^{an} over any non-archimedean field, similarly to [40]. This space carries an analytic symplectic form and describes “the behavior at infinity” of a (possibly non-algebraic) formal smooth symplectic scheme over \mathbf{Z} . String Theory suggests that there exists an actual complex symplectic manifold \mathcal{M} (vector or hyper multiplet moduli space) admitting a (partial) compactification $\overline{\mathcal{M}}$ and such that

$$\mathcal{X}^{an}(\mathbf{C}((t))) = \overline{\mathcal{M}}(\mathbf{C}[[t]]) \setminus (\mathcal{M}(\mathbf{C}[[t]]) \cup (\overline{\mathcal{M}} \setminus \mathcal{M})(\mathbf{C}[[t]])) ,$$

i.e. it is the space of formal paths hitting the compactifying divisor but not belonging to it). In the case of the Fukaya category of a complex 3d Calabi-Yau manifold X the space \mathcal{M} looks “at infinity” as a deformation of a complex symplectic manifold \mathcal{M}^{cl} where $\dim \mathcal{M} = \dim \mathcal{M}^{cl} = \dim H^3(X)$. The latter is the total space of the bundle $\mathcal{M}^{cl} \rightarrow \mathcal{M}_X$, where \mathcal{M}_X is the moduli space of complex structures on X . The fiber of the bundle is isomorphic to the space

$$(H^{3,0}(X) \setminus \{0\}) \times (H^3(X, \mathbf{C})/H^{3,0}(X) \oplus H^{2,1}(X) \oplus H^3(X, \mathbf{Z}))$$

parametrizing pairs (*holomorphic volume element, point of the intermediate Jacobian*).⁶ Furthermore, as we discuss in Section 7.2, we expect that there is a complex integrable system associated with an arbitrary homologically smooth 3d Calabi-Yau category and the fiber being the “Deligne cohomology” of the category.

2) Is it true that the counting of the invariants $\Omega(\gamma)$ for \mathcal{C} is equivalent to the counting of (some) holomorphic discs “near infinity” in $\overline{\mathcal{M}}$? Is it possible to construct an A_∞ -category associated with those discs and to prove that it is a 3-dimensional Calabi-Yau category?

3) The study of the dependence of BPS states on a point of the moduli space of vector and hyper multiplets given in [18] and [13] suggests that \mathcal{M} is hyperkähler and the invariants $\Omega(\gamma)$ for \mathcal{C} (counting of objects of \mathcal{C}) can be interpreted as the counting of some “quaternion curves” in \mathcal{M} . Punctures

⁶In a very interesting paper [24] a construction of the hyperkähler structure on \mathcal{M} was suggested by means of our wall-crossing formulas.

“at infinity” of those curves can be interpreted as $4d$ black holes. It would be nice to think about the problem of counting such maps as a “quaternionic analog” of the counting of rational Gromov-Witten invariants. Hopefully (by the analogy with the “Gromov-Witten story”) one can define an appropriate A_∞ -category (“quaternionic Fukaya category”) and prove that it is a 3-dimensional Calabi-Yau category. This would relate our invariants $\Omega(\gamma)$ with “quaternionic” Gromov-Witten invariants.

4) Geometry similar to the one discussed in this paper also appears in the theory of moduli spaces of holomorphic abelian differentials (see e.g. [78]). The moduli space of abelian differentials is a complex manifold, divided by real “walls” of codimension one into pieces glued from convex cones. It also carries a natural non-holomorphic action of the group $GL^+(2, \mathbf{R})$. There is an analog of the central charge Z in the story. It is given by the integral of an abelian differential over a path between marked points in a complex curve. This makes plausible the idea that the moduli space of abelian differentials associated with a complex curve with marked points, is isomorphic to the moduli space of stability structures on the (properly defined) Fukaya category of this curve.

5) We expect that our wall-crossing formulas are related to those in the Donaldson theory of $4d$ manifolds with $b_2^+ = 1$ (cf. e.g. recent paper [52]) as well as with Borcherds hyperbolic Kac-Moody algebras and multiplicative automorphic forms. The formulas from [12] also look very similar.

1.6 About the content of the paper

In Section 2 we work out in detail the approach to the invariants $\Omega(\gamma)$ and the wall-crossing formula sketched in the Introduction in the framework of graded Lie algebras. It is based on the notion of stability data which admits two equivalent descriptions: in terms of a collection of elements $a(\gamma)$ of a graded Lie algebra $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ and in terms of a collection of group elements A_V which satisfy the “Factorization Property”. The latter says that $A_{V_1} A_{V_2} = A_V$ for any strict sector V and its decomposition into two sectors V_1, V_2 (in the clockwise order) by a ray emanating from the vertex. We define the topology on the space of stability data. It immediately leads to the wall-crossing formula. Then we discuss a special case when the lattice carry an integer-valued skew-symmetric bilinear form. The skew-symmetric form on the lattice Γ gives rise to a Poisson structure on the torus $\text{Hom}(\Gamma, \mathbf{G}_m)$ of its characters. Then we introduce a *double symplectic* torus, which cor-

responds to the lattice $\Gamma \oplus \Gamma^\vee$. This allows us to construct an embedding of the pro-nilpotent groups G_V (see Section 1.4) into the group of formal symplectomorphisms of the double torus. We show how the “numerical DT-invariants” $\Omega(\gamma)$ arise from a collection of elements A_V which satisfy the Factorization Property $A_{V_1}A_{V_2} = A_V$ for any strict sector V . We introduce the notion of the “wall of second kind” such that (in the categorical framework) crossing such a wall corresponds to a change of the t -structure. Then the multiplicative wall-crossing formula is equivalent to the triviality of the monodromy of a “non-linear connection” on the space of numerical stability data. Also we discuss the relationship with the works of Joyce, and Bridgeland and Toledano-Laredo by introducing (under certain conditions) a connection with irregular singularities on \mathbf{C} . In Section 2.7. we explain how stability data arise from complex integrable systems. We illustrate our consideration by an example of Seiberg-Witten curve. Arising geometry is the same as in the “string junction” interpretation of Seiberg-Witten model (see e.g. [53]). The last section is devoted to stability data on $\mathfrak{gl}(n, \mathbf{Q})$. It is related to the study of vacua in $N = 2$ supersymmetric Quantum Field Theories (see [11]).

Section 3 is devoted to some basics on ind-constructible categories, including the definition of the topology on the space of stability structures. Also we discuss the notion of the potential of an object of Calabi-Yau category and the categorical version of the wall-crossing formula. The way it is formulated is close intuitively to the physics considerations: we look how the “motive” of the moduli space of semistable objects changes when some of exact triangles become unstable.

Section 4 is devoted to motivic functions and motivic Milnor fiber. We start by recalling basics on motivic functions and motivic integration, including their equivariant versions (motivic stack functions, see also [35]). Then we discuss the notion of motivic Milnor fiber introduced by Denef and Loeser as well as its l -adic incarnation. Rough idea is to use the motivic Milnor fiber of the potential of the $3d$ Calabi-Yau category in order to define invariants of the ind-constructible set of semistable objects. The technical question arises: there might be two quadrics with the same rank and determinant but different Chow motives. In order to resolve this difficulty we introduce certain equivalence relation on motivic functions, so that in the quotient such quadrics are the same. Also, we discuss an important integral identity which will play the key role in Section 6.

Section 5 is devoted to an additional structure, which we call orientation

data. It is a super line bundle on the space of objects of our category. Roughly, it is a square root of the super line bundle of cohomology. Although the numerical DT-invariants do not depend on the orientation data, the motivic DT-invariants introduced in Section 6 depend on it in an essential way.

Section 6 is devoted to the definition of motivic DT-invariants. First we define the *motivic Hall algebra* of an ind-constructible triangulated A_∞ -category and prove its associativity. It generalizes the derived Hall algebra introduced by Toën in [70]. We define the motivic version A_V^{Hall} of the element A_V as an invertible element of the completed motivic Hall algebra associated with the sector V . The elements A_V^{Hall} satisfy the Factorization Property. Basic idea behind the Factorization Property (and hence the multiplicative wall-crossing formula) is that the infinite product in the latter corresponds to the integration over the space of *all* objects of the category \mathcal{C}_V generated by extensions of semistable objects with the central charge in V . The latter can be easily controlled when we cross the wall.

Motivic DT-invariants appear as elements of a certain quantum torus with the coefficient ring given by the equivalence classes of motivic functions. Basic fact is the theorem which says that in the case of $3d$ Calabi-Yau category there is a homomorphism of the motivic Hall algebra into the motivic quantum torus defined in terms of the motivic Milnor fiber of the potential. In many cases the images of the elements A_V^{Hall} can be computed explicitly in terms of the motivic version of the quantum dilogarithm function. The images of A_V^{Hall} are denoted by A_V^{mot} . This collection (one element for every strict sector V) is called the motivic DT-invariant. The collection of these elements satisfy the Factorization Property. Replacing motives by their Serre polynomials, we obtain q -analogs of Donaldson-Thomas invariants, denoted by $A_{V,q}$. We discuss their properties as well as the “quasi-classical limit” A_V as $q^{1/2} \rightarrow -1$. We formulate the conjectures about the existence of the limit (absence of poles conjecture) and integrality property of the limits (integrality conjecture). The latter are related to the Conjecture 1 from Section 1.4. These conjectures are discussed in detail in Section 7, where we present various arguments and computations in their favor. Presumably, the technique developed by D. Joyce can lead to the proof of our conjectures. The numerical DT-invariants $\Omega : \Gamma \setminus \{0\} \rightarrow \mathbf{Z}$ are defined as coefficients in the decomposition of symplectomorphism A_V into the product of powers $T_\gamma^{\Omega(\gamma)}$ in the clockwise order.

In Section 8 we consider in detail the case of 3-dimensional Calabi-Yau category endowed with a finite collection of spherical generators satisfying some extra property (cluster collection). Such categories correspond to quivers with potentials (Theorem 9). Applying general considerations from the previous sections we formulate some results about quivers and mutations. They are almost obvious in the categorical framework, but seem to be new in the framework of quivers. Finally we explain that cluster transformations appear naturally as birational symplectomorphisms of the double torus in the case when crossing of the wall of second kind corresponds to a mutation at a vertex of the quiver (equivalently, to a mutation at the corresponding spherical object of the Calabi-Yau category).

Several parts of the theory presented here have to be developed in more detail. This concerns ind-constructible categories and motivic stack functions. Also, we present only a sketch of the proof of the l -adic version of the main identity in Section 4.4, leaving aside few technical details (which are not difficult to restore), and the definition of the orientation data for cluster categories in Section 8.2 is left as a conjecture (although there is no doubt that it should be true).

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2 Stability conditions for graded Lie algebras

2.1 Stability data

Let us fix a free abelian group Γ of finite rank, and a graded Lie algebra $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ over \mathbf{Q} .⁷

Definition 1 *Stability data on \mathfrak{g} is a pair $\sigma = (Z, a)$ such that:*

1) $Z : \Gamma \rightarrow \mathbf{R}^2 \simeq \mathbf{C}$ is a homomorphism of abelian groups called the central charge;

⁷In examples \mathfrak{g} is a R -linear Lie algebra, where R is a commutative unital \mathbf{Q} -algebra.

2) $a = (a(\gamma))_{\gamma \in \Gamma \setminus \{0\}}$ is a collection of elements $a(\gamma) \in \mathfrak{g}_\gamma$, satisfying the following

Support Property:

Pick a norm $\|\bullet\|$ on the vector space $\Gamma_{\mathbf{R}} = \Gamma \otimes_{\mathbf{Z}} \mathbf{R}$. Then there exists $C > 0$ such that for any $\gamma \in \text{Supp } a$ (i.e. $a(\gamma) \neq 0$) one has

$$\|\gamma\| \leq C|Z(\gamma)|.$$

Obviously the Support Property does not depend on the choice of the norm. We will denote the set of all stability data on \mathfrak{g} by $\text{Stab}(\mathfrak{g})$. Later we will equip this set with a Hausdorff topology.

The Support Property is equivalent to the following condition (which we will also call the Support Property):

There exists a quadratic form Q on $\Gamma_{\mathbf{R}}$ such that

- 1) $Q|_{\text{Ker } Z} < 0$;
- 2) $\text{Supp } a \subset \{\gamma \in \Gamma \setminus \{0\} \mid Q(\gamma) \geq 0\}$,

where we use the same notation Z for the natural extension of Z to $\Gamma_{\mathbf{R}}$.

Indeed, we may assume that the norm $\|\bullet\|$ is the Euclidean norm in a chosen basis and take $Q(\gamma) = -\|\gamma\|^2 + C_1|Z(\gamma)|^2$ for sufficiently large positive constant C_1 . Generically Q has signature $(2, n-2)$, where $n = \text{rk } \Gamma$. In degenerate cases Q can have signature $(1, n-1)$ or $(0, n)$.

For a given quadratic form Q on $\Gamma_{\mathbf{R}}$ we denote by $\text{Stab}_Q(\mathfrak{g}) \subset \text{Stab}(\mathfrak{g})$ the set of stability data satisfying the above conditions 1) and 2). Obviously $\text{Stab}(\mathfrak{g}) = \cup_Q \text{Stab}_Q(\mathfrak{g})$, where the union is taken over all quadratic forms Q .

Remark 4 *In the case of a 3-dimensional Calabi-Yau manifold X there is a natural candidate for the quadratic form Q of the signature $(2, n-2)$ needed to formulate the Support Property. Namely, identifying $H^3(X, \mathbf{R})$ with $H^{3,0}(X, \mathbf{C}) \oplus H^{2,1}(X, \mathbf{C})$ we can equip $H^3(X, \mathbf{R})$ with the complex structure. Furthermore, the natural symplectic form coming from the Hodge structure gives rise to a pseudo-hermitian form on $H^3(X, \mathbf{R})$ of the signature $(2, n-2)$, where $n = \dim_{\mathbf{R}} H^3(X, \mathbf{R})$. One can ask whether this form is positive on every special Lagrangian submanifold of X . If this is true, then the Support Property gives rise to a bound on the support of the function Ω discussed in Section 1.4.*

Support Property implies the following estimate for the number of points in the $\text{Supp } a$ with the central charge inside of the disc of radius R :

$$\#(Z(\text{Supp } a) \cap \{z \in \mathbf{C} \mid |z| \leq R\}) = O(R^n),$$

where $R \rightarrow \infty$ and $n = \text{rk} \Gamma$, Therefore the set $Z(\text{Supp } a)$ is discrete in \mathbf{C} and does not contain zero.

Remark 5 *It seem reasonable to consider “limiting cases” of stability data when the Support Property is not satisfied. Then the numbers $\text{Re } Z$ and $\text{Im } Z$ are allowed to take values in arbitrary totally ordered fields, e.g. $\mathbf{R}((t))$ (here t is a formal parameter such that $t > 0$ and $t < x$ for any $x \in \mathbf{R}_{>0}$). Some of our considerations below make sense in this situation. In the framework of stability conditions on triangulated categories such structures appeared in [1], [73].*

2.2 Reformulation of the stability data

In what follows we will consider various cones in $\Gamma_{\mathbf{R}}$ and in \mathbf{R}^2 i.e. subsets, which are closed under addition and multiplication by a positive real number. We assume that the vertex of the cone (i.e. the zero of the vector space) does not belong to the cone. We will call a cone *strict* if it is non-empty and does not contain a straight line. In particular, all strict cones on the plane (we will call them strict sectors) are sectors, which are smaller than 180 degrees (not necessarily closed or open). We allow the sector to be degenerate (which means that it is a ray with the vertex at the origin). We orient the plane (and hence all sectors) in the clockwise direction. We write $l_1 \leq l_2$ if the rays l_1, l_2 bound a strict closed sector and l_1 precedes l_2 in the clockwise order (we allow $l_1 = l_2$).

Let us fix a quadratic form Q on $\Gamma_{\mathbf{R}}$. We are going to describe below another set of data and will show that it is naturally isomorphic to the set $\text{Stab}_Q(\mathfrak{g})$. Let \mathcal{S} be the set of strict sectors in \mathbf{R}^2 possibly degenerate (rays).

We denote by $\widehat{\text{Stab}}_Q(\mathfrak{g})$ the set of pairs (Z, A) such that:

a) $Z : \Gamma \rightarrow \mathbf{R}^2$ is an additive map such that $Q|_{\text{Ker } Z} < 0$;

b) $A = (A_V)_{V \in \mathcal{S}}$ is a collections of elements $A_V \in G_{V,Z,Q}$, where $G_{V,Z,Q}$ is a pro-nilpotent group with the pro-nilpotent graded Lie algebra

$$\mathfrak{g}_{V,Z,Q} = \prod_{\gamma \in \Gamma \cap C(V,Z,Q)} \mathfrak{g}_{\gamma} ,$$

where $C(V, Z, Q)$ is the convex cone generated by the set

$$S(V, Z, Q) = \{x \in \Gamma_{\mathbf{R}} \setminus \{0\} \mid Z(x) \in V, Q(x) \geq 0\} .$$

The above definition makes sense because the cone $C(V, Z, Q)$ is strict, as one can easily see by elementary linear algebra. Hence for a triangle Δ which is cut from V by a straight line, any $\gamma \in Z^{-1}(\Delta)$ can be represented as a sum of other elements of $\Gamma \cap C(V, Z, Q)$ in finitely many ways. Furthermore, the triangle Δ defines an ideal $J_\Delta \subset \mathfrak{g}_{V,Z,Q}$ consisting of elements $y = (y_\gamma) \in \mathfrak{g}_{V,Z,Q}$ such that for every component y_γ the corresponding γ does not belong to the convex hull of $Z^{-1}(\Delta)$. Then the quotient $\mathfrak{g}_\Delta := \mathfrak{g}_{V,Z,Q}/J_\Delta$ is a nilpotent Lie algebra, and $\mathfrak{g}_{V,Z,Q} = \varprojlim_{\Delta \subset V} \mathfrak{g}_\Delta$.

Let $G_\Delta = \exp(\mathfrak{g}_\Delta)$ be the nilpotent group corresponding to the Lie algebra \mathfrak{g}_Δ . Then $G_{V,Z,Q} = \varprojlim_{\Delta} G_\Delta$ is a pro-nilpotent group. If $V = V_1 \sqcup V_2$ (in the clockwise order) then there are natural embeddings $G_{V_i,Z,Q} \rightarrow G_{V,Z,Q}$, $i = 1, 2$.

We impose the following axiom on the set of pairs (Z, A) :

Factorization Property:

The element A_V is given by the product $A_V = A_{V_1} A_{V_2}$ where the equality is understood in $G_{V,Z,Q}$.

We remark that if $Q_1 \leq Q$ and both forms Q, Q_1 are negative on $\text{Ker } Z$ then $G_{V,Z,Q_1} \subset G_{V,Z,Q}$ for any $V \in \mathcal{S}$. We say that the $(Z, A) \in \widehat{Stab}_Q(\mathfrak{g})$ and $(Z', A') \in \widehat{Stab}_{Q'}(\mathfrak{g})$ are equivalent if $Z = Z' := Z$ and there exists Q_0 such that $Q \leq Q_0, Q' \leq Q_0, Q_0|_{\text{Ker } Z} < 0$ and moreover for any $V \in \mathcal{S}$ we have $A_V = A'_V$ as elements of the group G_{V,Z,Q_0} .

Theorem 2 1) *For a fixed Q there is a natural bijection between sets $\widehat{Stab}_Q(\mathfrak{g})$ and $Stab_Q(\mathfrak{g})$.*

2) *Any two elements of $\widehat{Stab}_Q(\mathfrak{g})$ and $\widehat{Stab}_{Q'}(\mathfrak{g})$ are equivalent if and only if they define the same element in $Stab(\mathfrak{g})$.*

Proof. Suppose that we are given a pair $(Z, A) \in \widehat{Stab}_Q(\mathfrak{g})$. In order to construct the corresponding element in $Stab_Q(\mathfrak{g})$ we take the same Z as a homomorphism $\Gamma \rightarrow \mathbf{R}^2$. What is left is to construct a collection $a(\gamma) \in \mathfrak{g}_\gamma$. We define it such as follows.

a) If $Z(\gamma) = 0$ then we set $a(\gamma) = 0$.

b) Suppose $Z(\gamma) \neq 0$. Let us consider the ray $l = \mathbf{R}_{>0} Z(\gamma)$. Then we have an element $\log(A_l) \in \mathfrak{g}_{l,Z,Q} \subset \prod_{\gamma \in \Gamma} \mathfrak{g}_\gamma$. We denote by $a(\gamma)$ the component of $\log(A_l)$ which belongs to \mathfrak{g}_γ . This assignment gives rise to stability data $(Z, a) \in Stab_Q(\mathfrak{g})$. In order to show that it is injective, we observe that

the Factorization Property implies that $A_V = \prod_{l \subset V}^{\rightarrow} A_l$, where the product is taken in the clockwise order over the set of all rays l which belong to V . Indeed, let us consider the image of A_V in G_Δ . Then only finitely many rays contribute to the product $\prod_{l \subset V}^{\rightarrow} A_l$, and the product formula follows from the Factorization Property. Since $G_{V,Z,Q} = \varinjlim_{\Delta} G_\Delta$ the desired equality holds.

Conversely, if we have stability data $(Z, a) \in \text{Stab}_Q(\mathfrak{g})$, then we construct a pair (Z, A) taking the same Z and Q , and for any ray l we set

$$A_l = \exp \left(\sum_{\gamma \in \Gamma \cap C(l, Z, Q)} a(\gamma) \right).$$

Notice that $A_l = 1$ if there are no elements γ such that $Z(\gamma) \in l$. We define A_V for any $V \in a$ using the Factorization Property, i.e. $A_V = \prod_{l \subset V}^{\rightarrow} A_l$. This proves part 1) of the theorem. Part 2) follows immediately from definitions. The theorem is proved. ■

Remark 6 *We will use the same name “stability data” for either of the set of data which appear in the above theorem and will denote either set by $\text{Stab}(\mathfrak{g})$.*

Remark 7 *Let $\mathbf{R}^2 \setminus \{(0, 0)\} = \sqcup_{1 \leq i \leq n} V_i$, where $V_i, 1 \leq i \leq n$ are strict (semiclosed) sectors. Then the stability data with a given central charge Z are uniquely determined by an arbitrary collection of elements $A_{V_i} \in G_{V_i, Z, Q}$ for some quadratic form Q .*

There exists a generalization of stability data suitable for motivic Hall algebras. Namely, let us assume that the Lie algebra \mathfrak{g} carries an automorphism η such that $\eta(\mathfrak{g}_\gamma) = \mathfrak{g}_{-\gamma}$ for any $\gamma \in \Gamma$.

Definition 2 *Symmetric stability data for (\mathfrak{g}, η) is a pair (Z, \hat{a}) where $Z : \Gamma \rightarrow \mathbf{C}$ is an additive map and \hat{a} is a map $(\gamma, \varphi) \mapsto \hat{a}(\gamma, \varphi) \in \mathfrak{g}_\gamma$ where $\varphi \in \mathbf{R}$, $\gamma \in \Gamma$ is such that $Z(\gamma) \in \mathbf{R}_{>0} e^{i\varphi}$ and*

$$\hat{a}(\gamma, \varphi + \pi) = \eta(\hat{a}(\gamma, \varphi)).$$

All the considerations about stability data admit a straightforward generalization to the symmetric case. We will use them without further comments.

Remark 8 Let H_Γ be a Γ -graded unital associative algebra considered as a graded Lie algebra. Then the pro-nilpotent groups $G_{V,Z,Q}$ discussed above are the groups of invertible elements of the form $f = 1 + \dots$ in the pro-nilpotent associative algebras which are completions of H_Γ .

Remark 9 Decomposition $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ and the Lie algebras $\mathfrak{g}_{V,Z,Q}$ are similar to the root decomposition and nilpotent subalgebras in Kac-Moody Lie algebras. The involution $\gamma \mapsto -\gamma$ is similar to the ‘‘Cartan involution’’. These analogies deserve further study, since Donaldson-Thomas invariants (more precisely, counting functions for BPS states) appear in physics as a kind of character formulas (see e.g. [18], formula (2.7)). In particular our multiplicative wall-crossing formulas in the case of wall of second kind should be related to automorphic forms of Borcherds (see [8]). The motivic Hall algebra defined below in Section 6 could be thought of as the motivic version of the algebra of BPS states (see [29]).

2.3 Topology and the wall-crossing formula

Here we are going to introduce a Hausdorff topology on the set of stability data in such a way that the forgetting map

$$Stab(\mathfrak{g}) \rightarrow \text{Hom}(\Gamma, \mathbf{C}) \simeq \mathbf{C}^n, (Z, a) \mapsto Z$$

will be a local homeomorphism. In particular $Stab(\mathfrak{g})$ carries a structure of a complex manifold (in general with an uncountable number of components, each of which is paracompact). In order to define the topology we define the notion of a continuous family of points in $Stab(\mathfrak{g})$.

Let X be a topological space, $x_0 \in X$ be a point, and $(Z_x, a_x) \in Stab(\mathfrak{g})$ be a family parametrized by X .

Definition 3 We say that the family is continuous at x_0 if the following conditions are satisfied:

- a) The map $X \rightarrow \text{Hom}(\Gamma, \mathbf{C}), x \mapsto Z_x$ is continuous at $x = x_0$.
- b) Let us choose a quadratic form Q_0 such that $(Z_{x_0}, a_{x_0}) \in Stab_{Q_0}(\mathfrak{g})$. Then there exists an open neighborhood U_0 of x_0 such that $(Z_x, a_x) \in Stab_{Q_0}(\mathfrak{g})$ for all $x \in U_0$.
- c) For any closed strict sector V such that $Z(\text{Supp } a_{x_0}) \cap \partial V = \emptyset$ the map

$$x \mapsto \log A_{V,x,Q_x} \in \mathfrak{g}_{V,Z_x,Q_x} \subset \prod_{\gamma \in \Gamma} \mathfrak{g}_\gamma$$

is continuous at $x = x_0$. Here we endow the vector space $\prod_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ with the product topology of discrete sets, and A_{V,x,Q_x} is the group element associated with (Z_x, a_x) , sector V and a quadratic form Q_x such that $(Z_x, a_x) \in \text{Stab}_{Q_x}(\mathfrak{g})$.

Remark 10 Part c) of the Definition 3 means that for any $\gamma \in \Gamma \setminus \{0\}$ the component of $\log A_{V,x,Q_x}$ belonging to \mathfrak{g}_γ is locally constant as a function of x in a neighborhood of x_0 .

The element $\log(A_{V,x,Q_x}) \in \prod_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ does not depend on Q_x , e.g. we can take $Q_x := Q_0$ for x close to x_0 . The continuity means informally that for any closed triangle $\Delta \subset \mathbf{R}^2$ with one vertex at the origin, the projection of $\log A_{V,Z_x,Q_x}$ into the vector space $\oplus_{\gamma \in \Delta} \mathfrak{g}_\gamma$ does not depend on $x \in X$ as long as there is no element $\gamma \in \text{Supp } a_x$ such that $Z(\gamma)$ crosses the boundary $\partial\Delta$.

It is easy to see that the above definition gives rise to a topology on $\text{Stab}(\mathfrak{g})$.

Proposition 1 *This topology is Hausdorff.*

Proof. Let (Z, a) and (Z', a') be two limits of a sequence (Z_n, a_n) as $n \rightarrow \infty$. We have to prove that $(Z, a) = (Z', a')$. It is clear that $Z = Z'$ since $\text{Hom}(\Gamma, \mathbf{C})$ is Hausdorff. Let us now choose quadratic forms Q and Q' which are compatible with a and a' respectively in the sense of Definition 3. Then there exists a quadratic form Q_0 such that Q_0 is negative on $\text{Ker } Z = \text{Ker } Z'$ and also $Q \leq Q_0, Q' \leq Q_0$. Then for all sufficiently large n the form Q_0 is compatible with a_n .

For a generic sector $V \subset \mathbf{R}^2$ its boundary rays do not intersect $Z(\Gamma)$. By part c) of the Definition 3 we have: $A_{V,Z,Q_0} = A'_{V,Z',Q_0}$ since the product $\prod_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ is Hausdorff. Since any ray in \mathbf{R}^2 with the vertex at the origin can be obtained as an intersection of generic sectors then we conclude that $a = a'$. The Proposition is proved. ■

Let us fix an element $Z_0 \in \text{Hom}(\Gamma, \mathbf{C})$ and a quadratic form Q_0 compatible with Z_0 (i.e. negative on its kernel). We denote by U_{Q_0,Z_0} the connected component containing Z_0 in the domain $\{Z \in \text{Hom}(\Gamma, \mathbf{C}) \mid (Q_0)|_{\text{Ker } Z} < 0\}$. In what follows we will frequently use the following elementary observation.

Proposition 2 *If Q is a quadratic form on a finite-dimensional vector space $\Gamma_{\mathbf{R}}$ and $Z : \Gamma_{\mathbf{R}} \rightarrow \mathbf{C}$ is an \mathbf{R} -linear map such that $Q|_{\text{Ker } Z} < 0$ then the intersection $\{x \in \Gamma_{\mathbf{R}} \mid Q(x) \geq 0\} \cap Z^{-1}(l)$ is a convex cone (possibly empty) for any ray $l \subset \mathbf{C}$ with the vertex at the origin.*

Let $\gamma_1, \gamma_2 \in \Gamma \setminus \{0\}$ be two \mathbf{Q} -linearly independent elements such that $Q_0(\gamma_i) \geq 0, Q_0(\gamma_1 + \gamma_2) \geq 0, i = 1, 2$. We introduce the set

$$\mathcal{W}_{\gamma_1, \gamma_2}^{Q_0} = \{Z \in U_{Q_0, Z_0} \mid \mathbf{R}_{>0}Z(\gamma_1) = \mathbf{R}_{>0}Z(\gamma_2)\} .$$

In this way we obtain a countable collection of hypersurfaces $\mathcal{W}_{\gamma_1, \gamma_2}^{Q_0} \subset U_{Q_0, Z_0}$ called the *walls corresponding to Q_0, γ_1, γ_2* . We denote their union by $\mathcal{W}_1 := \mathcal{W}_1^{Q_0}$ and sometimes call it *the wall of first kind* (physicists call it the wall of marginal stability).

Let us consider a continuous path $Z_t, 0 \leq t \leq 1$ in U_{Q_0, Z_0} which intersects each of these walls for finitely many values of $t \in [0, 1]$. Suppose that we have a continuous lifting path (Z_t, a_t) of this path such that Q_0 is compatible with each a_t for all $0 \leq t \leq 1$. Then for any $\gamma \in \Gamma \setminus \{0\}$ such that $Q_0(\gamma) \geq 0$ the element $a_t(\gamma)$ does not change as long as t satisfies the condition

$$Z_t(\gamma) \notin \cup_{\gamma_1, \gamma_2 \in \Gamma \setminus \{0\}, \gamma_1 + \gamma_2 = \gamma} \mathcal{W}_{\gamma_1, \gamma_2}^{Q_0} .$$

If this condition is not satisfied we say that t is a discontinuity point for γ . For a given γ there are finitely many discontinuity points.

Notice that for each $t \in [0, 1]$ there exist limits

$$a_t^\pm(\gamma) = \lim_{\varepsilon \rightarrow 0, \varepsilon > 0} a_{t \pm \varepsilon}(\gamma)$$

(for $t = 0$ or $t = 1$ only one of the limits is well-defined). Then the continuity of the lifted path (Z_t, a_t) is equivalent to the following *wall-crossing formula* which holds for *any* $t \in [0, 1]$ and arbitrary $\gamma \in \Gamma \setminus \{0\}$:

$$\begin{aligned} & \overrightarrow{\prod}_{\mu \in \Gamma^{prim}, Z_t(\mu) \in l_{\gamma, t}} \exp \left(\sum_{n \geq 1} a_t^-(n\mu) \right) = \\ & = \exp \left(\sum_{\mu \in \Gamma^{prim}, Z_t(\mu) \in l_{\gamma, t}, n \geq 1} a_t(n\mu) \right) = \overrightarrow{\prod}_{\mu \in \Gamma^{prim}, Z_t(\mu) \in l_{\gamma, t}} \exp \left(\sum_{n \geq 1} a_t^+(n\mu) \right), \end{aligned}$$

where $l_{\gamma, t} = \mathbf{R}_{>0}Z_t(\gamma)$, and $\Gamma^{prim} \subset \Gamma$ is the set of primitive vectors. The first and the last products are taken in the clockwise order of $\text{Arg}(Z_{t-\varepsilon})$ and $\text{Arg}(Z_{t+\varepsilon})$ respectively, where $\varepsilon > 0$ is sufficiently small. Moreover, for each γ we have $a_t^-(\gamma) = a_t^+(\gamma) = a_t(\gamma)$ unless there exist non-zero γ_1, γ_2 such that $\gamma = \gamma_1 + \gamma_2$ and $Z_t \in \mathcal{W}_{\gamma_1, \gamma_2}^{Q_0}$.

Remark 11 *Informally speaking, the wall-crossing formula says that for a very small sector V containing the ray $l_{\gamma,t}$ the corresponding element A_V , considered as a function of time, is locally constant in a neighborhood of t .*

For each $\gamma \in \Gamma \setminus \{0\}$ the wall-crossing formula allows us to calculate $a_1(\gamma)$ in terms of $a_0(\gamma')$ for a finite collection of elements $\gamma' \in \Gamma \setminus \{0\}$. Morally it is an inductive procedure on the ordered set of discontinuity points $t_i \in [0, 1]$. The only thing we need to check is that for each $\gamma \in \Gamma \setminus \{0\}$ the computation involves finitely many elements of Γ . For that we need some preparation. First we introduce a partial order on the set $S_{Q_0} = (\Gamma \cap Q_0^{-1}(\mathbf{R}_{\geq 0})) \times [0, 1]$ generated by the following relations:

- a) $(\gamma, t) \geq (\gamma, t')$ if $t \geq t'$;
- b) if $\gamma = \sum_{1 \leq i \leq m} \gamma_i$, $Q_0(\gamma_i) \geq 0$, $Z_t(\gamma_i) \in \mathbf{R}_{>0} Z_t(\gamma)$, $1 \leq i \leq m$, $m \geq 2$, where not all γ_i belong to $\mathbf{Q} \cdot \gamma$, then $(\gamma, t) \geq (\gamma_i, t)$ for all $1 \leq i \leq m$.

Lemma 1 *For any $(\gamma, t) \in S_{Q_0}$ the set $(\gamma', t') \in S_{Q_0}$ such that $(\gamma', t') \leq (\gamma, t)$ is a finite union of sets of the form $\{\gamma_\alpha\} \times [0, t_\alpha]$.*

The Lemma immediately implies the desired result.

Corollary 1 *The element $a_t(\gamma)$ is a finite Lie expression of the elements $a_0(\gamma_\alpha)$.*

Proof of the Lemma. Let us assume the contrary. Then we have an infinite sequence $t_1 > t_2 > t_3 > \dots$ such that

$$(\gamma_1, t_1) \geq_a (\gamma_2, t_2) \geq_b (\gamma_3, t_3) \geq_a (\gamma_4, t_4) \geq_b \dots,$$

where the subscript $a)$ or $b)$ denotes the two different possibilities for the partial order defined above. Let $t_\infty = \lim_{n \rightarrow \infty} t_n$. It is easy to see that there exists a Euclidean norm $\|\bullet\|$ on $\Gamma_{\mathbf{R}}$ such that for any $v_1, v_2 \in \Gamma_{\mathbf{R}}$ satisfying the properties $Q_0(v_i) \geq 0$, $Z_{t_\infty}(v_1) \in \mathbf{R}_{>0} Z_{t_\infty}(v_2)$ we have the inequalities $\|v_i\| < \|v_1 + v_2\|$ for $i = 1, 2$.

Moreover the same property holds if we replace the map Z_{t_∞} by an additive map Z which is close to it. Then we conclude that

$$\|\gamma_{2n}\| > \|\gamma_{2n+1}\| = \|\gamma_{2n+2}\|$$

for all sufficiently large n . This contradicts to the fact the lattice Γ is discrete in $\Gamma_{\mathbf{R}}$. The Lemma is proved. ■

The previous discussion allows us to lift a generic path $Z_t, 0 \leq t \leq 1$ as above to a unique continuous path $(Z_t, a_t) \in \text{Stab}(\mathfrak{g}), 0 \leq t \leq 1$ which starts at a given point $(Z_0, a_0) \in \text{Stab}(\mathfrak{g})$. In other words, we have the notion of a *parallel transport* along a generic path. This observation is a part of the following more general statement.

Theorem 3 *For given quadratic form Q_0 and $(Z_0, a_0) \in \text{Stab}_{Q_0}(\mathfrak{g})$ there exists a unique continuous map $\phi : U_{Z_0, Q_0} \rightarrow \text{Stab}(\mathfrak{g})$ such that it is a section of the natural projection $\text{Stab}(\mathfrak{g}) \rightarrow U_{Z_0, Q_0}$, and $\phi(Z_0) = (Z_0, a_0)$.*

Proof. We have already proved the existence of a lifted path (Z_t, a_t) for a generic path Z_t provided the beginning point Z_0 is fixed. What is left to prove that the endpoint (a_1, Z_1) does not depend on a choice of the generic path Z_t . We are going to sketch the proof leaving the details for the reader.

Let us consider an infinitesimally small loop around the intersection point Z of two or more walls. We would like to prove that the monodromy of the parallel transport along the loop is trivial. There are two possibilities:

- a) there are two different sublattices $\Gamma_1, \Gamma_2 \subset \Gamma$ of ranks ≥ 2 such that $Z(\Gamma_i), i = 1, 2$ belong to two different lines in the plane \mathbf{R}^2 ;
- b) there exists a sublattice $\Gamma_3 \subset \Gamma$ such that $\text{rk } \Gamma_3 \geq 3$ and $Z(\Gamma_3)$ belongs to a line in \mathbf{R}^2 .

In the case a) the corresponding Lie subalgebras of the completion of \mathfrak{g}_Γ are graded by non-intersecting subsets of Γ . Hence the corresponding wall-crossing transformations commute.

In the case b) let us choose a decomposition $\mathbf{R}^2 \setminus \{(0, 0)\} = \sqcup_{1 \leq i \leq 4} V_i$, where $V_i, 1 \leq i \leq 4$ are strict sectors such that $\mathbf{R} \cdot Z(\Gamma_3) \subset V_1 \sqcup V_3 \sqcup \{(0, 0)\}$. When we move around the infinitesimally small loop the element $a(\gamma)$ can change only for $\gamma \in \Gamma_3$. Hence we can replace Γ by Γ_3 in all computations. The wall-crossing formula implies that the elements $A_{V_i}, 1 \leq i \leq 4$ do not change along the loop (moreover, by our assumption we have $A_{V_2} = A_{V_4} = 1$). By Remark 7 from Section 2.2 we conclude that the stability data with the central charge Z' which is close to Z are uniquely determined by Z' and the collection of elements $A_{V_i}, 1 \leq i \leq 4$. Hence the monodromy around the loop is trivial.

Finally one has to check the the *global* monodromy around a loop in U_{Z_0, Q_0} is trivial. It follows from the fact that the fundamental group $\pi_1(U_{Z_0, Q_0})$ is generated by the loop $Z \mapsto Ze^{2\pi it}, t \in [0, 1]$. But the monodromy around this loop is trivial for generic Z , because the loop does not intersect the walls. ■

We can write the wall-crossing formula in the way similar to the one from the Introduction. In the case of generic path we have at a discontinuity point $t_0 \in [0, 1]$ a two-dimensional lattice $\Gamma_0 \simeq \mathbf{Z}^2$ which is projected by Z_{t_0} into a real line in \mathbf{R}^2 . We choose an isomorphism $\Gamma_0 \simeq \mathbf{Z}^2$ in such a way that $Q_0^{-1}(\mathbf{R}_{\geq 0}) \cap (\Gamma_0 \setminus \{0\})$ is contained in $\mathbf{Z}_{>0}^2 \cup \mathbf{Z}_{<0}^2$. Also we assume that the orientation of $\Gamma_0 \otimes \mathbf{R}$ defined by Z_t agrees with the one on $\mathbf{Z}_{>0}^2$ for $t = t_0 - \varepsilon$ and is opposite to it for $t = t_0 + \varepsilon$, where $\varepsilon > 0$ is sufficiently small.

Then if $\gamma = (m, n) \in \mathbf{Z}_{>0}^2$, and $a_{t_0}^\pm(\gamma) := a^\pm(m, n)$, we can write the the wall-crossing formula in the following way:

$$\prod_{(m,n)=1}^{\rightarrow} \exp\left(\sum_{k \geq 1} a^-(km, kn)\right) = \prod_{(m,n)=1}^{\leftarrow} \exp\left(\sum_{k \geq 1} a^+(km, kn)\right),$$

where in the LHS we take the product over all coprime m, n in the increasing order of $m/n \in \mathbf{Q}$, while in the RHS we take the product over all coprime m, n in the decreasing order. Both products are equal to $\exp(\sum_{m,n>0} a_{t_0}(m, n))$.

2.4 Crossing the wall of second kind

Here we will interpret the parallel transport in a different way, introducing a wall of another kind. We use the notation from the previous section. In particular, we fix the quadratic form Q_0 and the connected component U of the set $\{Z \in \text{Hom}(\Gamma, \mathbf{C}) \mid (Q_0)|_{\text{Ker } Z} < 0\}$.

For a given primitive $\gamma \in \Gamma \setminus \{0\}$ we introduce the set $\mathcal{W}_\gamma^{Q_0} = \{Z \in U \mid Z(\gamma) \in \mathbf{R}_{>0}\}$. It is a hypersurface in U . We call it a *wall of second kind* associated with γ . We call the union $\cup_\gamma \mathcal{W}_\gamma^{Q_0}$ the *wall of second kind* and denote it by \mathcal{W}_2 .

Definition 4 *We say that a path $\sigma = (Z_t)_{0 \leq t \leq 1} \subset U$ is short if the convex cone C_σ which is the convex hull of $(\cup_{0 \leq t \leq 1} Z_t^{-1}(\mathbf{R}_{>0})) \cap \{Q_0 \geq 0\}$ is strict.*

With a short path we associate a pro-nilpotent group G_{C_σ} with the Lie algebra $\mathfrak{g}_{C_\sigma} = \prod_{\gamma \in C_\sigma \cap \Gamma} \mathfrak{g}_\gamma$.

The following result is obvious.

Proposition 3 *For a generic short path $\sigma = (Z_t)_{0 \leq t \leq 1}$ there exists no more than countable set $t_i \in [0, 1]$ and corresponding primitive $\gamma_i \in \Gamma \setminus \{0\}$ such that $Z_{t_i} \in \mathcal{W}_{\gamma_i}^{Q_0}$. For each i we have: $\text{rk } Z_{t_i}^{-1}(\mathbf{R}) \cap \Gamma = 1$.*

Let us recall the continuous lifting map $\phi : U \rightarrow \text{Stab}(\mathfrak{g})$ from the previous section. In the notation of the previous Proposition we define for any t_i a group element

$$A_{t_i} = \exp \left(\varepsilon_i \sum_{n \geq 1} a_{t_i}(n\gamma_i) \right) \in G_{C_\sigma} ,$$

where $\varepsilon_i = \pm 1$ depending on the direction in which the path $Z_t(\gamma_i)$ crosses $\mathbf{R}_{>0}$ for t sufficiently close to t_i .

Theorem 4 *For any short loop the monodromy $\prod_{t_i}^{\rightarrow} A_{t_i}$ is equal to the identity (here the product is taken in the increasing order of the elements t_i).*

Proof. Here we also present a sketch of the proof. Similarly to the proof of the Theorem 3 we consider the case of infinitesimally small loop σ around a point Z such that $\text{rk } \Gamma_2 = 2$ where $\Gamma_2 := Z^{-1}(\mathbf{R}) \cap \Gamma$ (i.e. Z is a point where two, and hence infinitely many, walls of second kind intersect). Since σ is infinitesimally small we can replace Γ by Γ_2 . Then we have the space $\text{Hom}(\Gamma_2, \mathbf{C}) \simeq \mathbf{R}^4$ which contains a countable collection of walls consisting of those $Z : \Gamma_2 \rightarrow \mathbf{C}$ for which there exists $\gamma \in \Gamma_2 \setminus \{0\}$ such that $Q_0(\gamma) \geq 0$ and $Z(\gamma) \in \mathbf{R}$. All the hypersurfaces contain $\mathbf{R}^2 = \text{Hom}(\Gamma_2, \mathbf{R}) \subset \text{Hom}(\Gamma_2, \mathbf{C})$. Factorizing by this subspace \mathbf{R}^2 we obtain a collection of lines with rational slopes in the union of two opposite strict sectors $S \cup (-S) \subset \mathbf{R}^2 = \text{Hom}(\Gamma_2, i\mathbf{R})$.

We have to prove that the product over a loop surrounding $(0, 0)$ is the identity element. But it is easy to see that the product over the rays belonging to each of the sectors is equal to the left (resp. right) hand side of the wall-crossing formula. ■

Let us now introduce a set $\mathcal{X}_1 \subset \Gamma \times U$ which consists of pairs (γ, Z) such that $\gamma \in \Gamma \setminus \{0\}$ is a non-zero element, $Q_0(\gamma) \geq 0$, $Z(\gamma) \in \mathbf{R}_{>0}$ and $Z^{-1}(\mathbf{R}_{>0}) \cap \Gamma = (\mathbf{Q}_{>0} \cdot \gamma) \cap \Gamma$.

Proposition 4 *The set of continuous sections $\psi : U \rightarrow \text{Stab}(\mathfrak{g})$ such that $\psi(Z)$ is compatible with Q_0 for any $Z \in U$ is in one-to-one correspondence with functions $\tilde{a} : \mathcal{X}_1 \rightarrow \mathfrak{g}$ such that $\tilde{a}(\gamma, Z) \in \mathfrak{g}_\gamma$ satisfying the property that for any small loop σ the monodromy defined in the previous theorem is equal to the identity.*

Proof. The bijection is given by the formula $\tilde{a}(\gamma, Z) = a_{\psi(Z)}(\gamma)$. By the previous theorem the corresponding monodromy is trivial. Conversely, the

triviality of the monodromy is equivalent to the wall-crossing formula in the special case when a 2-dimensional sublattice of Γ is mapped by Z into the line $\mathbf{R} \subset \mathbf{C}$. The general case of an arbitrary line can be reduced to this one by a rotation $Z \mapsto Ze^{2\pi it}$ (it does not change the values $a(\gamma)$ because we do not cross the wall of first kind). ■

Let us also introduce a set $\mathcal{X}_2 \subset \Gamma \times U$ which consists of such pairs (γ, Z) that $Q_0(\gamma) \geq 0, Z(\gamma) > 0$ and there are no non-zero \mathbf{Q} -independent elements $\gamma_1, \gamma_2 \in \Gamma$ with the property $\gamma = \gamma_1 + \gamma_2, Q_0(\gamma_i) \geq 0, Z(\gamma_i) > 0, i = 1, 2$. Since \mathcal{X}_2 is a locally-closed hypersurface in $U \times \text{Hom}(\Gamma, \mathbf{C})$ it has finitely many connected components. Obviously, we have $\mathcal{X}_1 \subset \mathcal{X}_2$.

It follows from the wall-crossing formula that for a continuous section $\psi : U \rightarrow \text{Stab}(\mathfrak{g})$ the restriction of the function a to \mathcal{X}_2 is locally-constant and uniquely determines the section ψ . Therefore, the values of the restriction $a|_{\pi_0(\mathcal{X}_2)}$ provides a countable coordinate system (satisfying a countable system of equations) on the set of continuous sections $\{\psi : U \rightarrow \text{Stab}(\mathfrak{g})\}$ as above. It can be compared with another countable coordinate system (with no constraints) given the value $\psi(Z_0)$ for $Z_0 \in U$. The latter coordinate system is not very convenient since one has to choose a generic Z_0 .

2.5 Invariants $\Omega(\gamma)$ and the dilogarithm

Let Γ be a free abelian group of finite rank n as before, endowed with a skew-symmetric integer-valued bilinear form $\langle \bullet, \bullet \rangle : \Gamma \times \Gamma \rightarrow \mathbf{Z}$. Recall the Lie algebra $\mathfrak{g}_\Gamma = \mathfrak{g}_{\Gamma, \langle \bullet, \bullet \rangle} = \bigoplus_{\gamma \in \Gamma} \mathbf{Q} \cdot e_\gamma$ with the Lie bracket

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2} .$$

Let us introduce a commutative associative product on \mathfrak{g}_Γ by the formula

$$e_{\gamma_1} e_{\gamma_2} = (-1)^{\langle \gamma_1, \gamma_2 \rangle} e_{\gamma_1 + \gamma_2} .$$

We denote by $\mathbb{T}_\Gamma := \mathbb{T}_{\Gamma, \langle \bullet, \bullet \rangle}$ the spectrum of this commutative algebra. It is easy to see that \mathbb{T}_Γ is a torsor over the algebraic torus $\text{Hom}(\Gamma, \mathbf{G}_m)$. Moreover \mathbb{T}_Γ is an algebraic Poisson manifold with the Poisson bracket

$$\{a, b\} := [a, b] .$$

The Poisson structure on \mathbb{T}_Γ is invariant with respect to the action of $\text{Hom}(\Gamma, \mathbf{G}_m)$.

We can specify the results of the previous sections to the Lie algebra \mathfrak{g}_Γ . For stability data (Z, a) we can write uniquely (by the Möbius inversion formula)

$$a(\gamma) = - \sum_{n \geq 1, \frac{1}{n}\gamma \in \Gamma \setminus \{0\}} \frac{\Omega(\gamma/n)}{n^2} e_\gamma ,$$

where $\Omega : \Gamma \setminus \{0\} \rightarrow \mathbf{Q}$ is a function. Then we have

$$\exp \left(\sum_{n \geq 1} a(n\gamma) \right) = \exp \left(- \sum_{n \geq 1} \Omega(n\gamma) \sum_{k \geq 1} \frac{e_{kn\gamma}}{k^2} \right) := \exp \left(- \sum_{n \geq 1} \Omega(n\gamma) \text{Li}_2(e_{n\gamma}) \right) ,$$

where $\text{Li}_2(t) = \sum_{k \geq 1} \frac{t^k}{k^2}$ is the dilogarithm function.

The Lie algebra \mathfrak{g}_Γ acts on \mathbb{T}_Γ by Hamiltonian vector fields. Let us denote by T_γ the formal Poisson automorphism

$$T_\gamma = \exp(\{-\text{Li}_2(e_\gamma), \bullet\}) , \quad T_\gamma(e_\mu) = (1 - e_\gamma)^{\langle \gamma, \mu \rangle} e_\mu$$

considered as an automorphism of algebra of functions.

More precisely for any strict convex cone $C \subset \Gamma_{\mathbf{R}}$ containing γ the element T_γ acts on the formal scheme $\text{Spf}(\prod_{\mu \in \Gamma \cap C} \mathbf{Q}e_\mu)$. Moreover T_γ is the Taylor expansion of a birational automorphism of \mathbb{T}_Γ .

Finally, in the case when Γ comes from a $3d$ Calabi-Yau category the numbers $\Omega(\gamma)$ are (conjecturally) integers for $(\gamma, Z) \in \mathcal{X}_2$ in notation of Section 2.4. They provide generalization of DT-invariants (BPS degeneracies in physics language).

2.6 Symplectic double torus

If the skew-symmetric bilinear form on Γ is degenerate, then the action of \mathfrak{g}_Γ on \mathbb{T}_Γ is not exact. In order to remedy the problem we can embed $(\Gamma, \langle \bullet, \bullet \rangle)$ into a larger symplectic lattice. A possible choice is $\Gamma \oplus \Gamma^\vee$, where $\Gamma^\vee = \text{Hom}(\Gamma, \mathbf{Z})$. The corresponding non-degenerate bilinear form is

$$\langle (\gamma_1, \nu_1), (\gamma_2, \nu_2) \rangle = \langle \gamma_1, \gamma_2 \rangle + \nu_2(\gamma_1) - \nu_1(\gamma_2) .$$

Let us choose a basis $e_i, 1 \leq i \leq n = \text{rk} \Gamma$ of Γ . It gives rise to the coordinates $y_i, 1 \leq i \leq n$ on \mathbb{T}_Γ . The Poisson structure on \mathbb{T}_Γ can be written as

$$\{y_i, y_j\} = b_{ij} y_i y_j ,$$

where $b_{ij} = \langle e_i, e_j \rangle$.

Let us also introduce additional coordinates $x_j, 1 \leq j \leq \text{rk } \Gamma^\vee$ in such a way that $(y_i, x_j), 1 \leq i, j \leq n$ will be the coordinates on the double torus $D(\mathbb{T}_\Gamma)$ with the Poisson brackets

$$\{x_i, x_j\} = 0, \{y_i, x_j\} = \delta_{ij} y_i x_j .$$

There is a projection

$$\pi : D(\mathbb{T}_\Gamma) \rightarrow \mathbb{T}_\Gamma, \pi((y_i)_{1 \leq i \leq n}, (x_j)_{1 \leq j \leq n}) = (y_i)_{1 \leq i \leq n} .$$

Notice that π is a Poisson morphism of the symplectic manifold $D(\mathbb{T}_\Gamma)$ onto the Poisson manifold \mathbb{T}_Γ .

Let $C \subset \Gamma_{\mathbf{R}}$ be a closed convex strict cone. Let us choose a closed convex strict cone $C_1 \subset (\Gamma \oplus \Gamma^\vee) \otimes \mathbf{R}$ which contains $C \oplus \{0\}$. With the cone C_1 we associate the Poisson algebra $\mathbf{Q}[[C_1]]$ consisting of series $\sum_{\gamma, \delta \in C_1 \cap (\Gamma \oplus \Gamma^\vee)} c_{\gamma, \delta} y^\gamma x^\delta$. The pro-nilpotent group $G_C = \exp(\prod_{\gamma \in C \cap \Gamma} \mathfrak{g}_\gamma)$ acts by Poisson automorphisms of $\mathbf{Q}[[C_1]]$.

Let us consider a closed algebraic submanifold $N \subset D(\mathbb{T}_\Gamma)$ defined by the equations

$$y_i \prod_j x_j^{b_{ij}} = -1, \quad 1 \leq i \leq n .$$

Lemma 2 *The image of the group G_C preserves the corresponding completion of N .*

Proof. It suffices to check that the image of the Lie algebra \mathfrak{g}_C preserves the equations of N . Notice that this image belongs to the Lie algebra of Hamiltonian vector fields on $D(\mathbb{T}_\Gamma)$ generated by $\{y^\gamma, \bullet\}$, where $\gamma \in \Gamma$ and $y^\gamma = y_1^{\gamma_1} \dots y_n^{\gamma_n}$. Taking logarithms we see that

$$\{\log(y^\gamma), \log(y_i) + \sum_j b_{ij} \log(x_j)\} = \sum_j \gamma_j b_{ji} + \sum_j b_{ij} \gamma_j = 0 .$$

This concludes the proof. ■

Remark 12 *It is clear that the action of the image of G_C also commutes with the map π . Moreover the image of G_C in the group of exact symplectomorphisms of the completion of \mathbb{T}_Γ corresponding to C can be characterized by the property that it preserves the completion of N and commutes with π .*

Let us finally make a remark about a possible non-archimedean geometry interpretation of our construction. Let us choose a complete non-archimedean field K with the residue field of characteristic zero. Extending scalars we can think of the algebraic variety $D(\mathbb{T}_\Gamma)$ as of variety over K . We denote by $D(\mathbb{T}_\Gamma)^{an}$ the corresponding non-archimedean K -analytic space in the sense of Berkovich (see [40] for the explanation of the relevance of Berkovich approach to the large complex structure limit of Calabi-Yau varieties). Then the group G_C acts on the analytic subset of $D(\mathbb{T}_\Gamma)^{an}$ given by inequalities $\{|e_\gamma| < 1, \gamma \in C \setminus \{0\}\}$. Here we interpret e_γ as a Laurent monomial on $D(\mathbb{T}_\Gamma)$.

The symplectic double torus together with submanifold N will be used again only in Section 8.

2.7 Complex integrable systems and stability data

In this section we explain how complex integrable systems (with some additional structures) give rise to stability data in the graded Lie algebra \mathfrak{g}_Γ associated with a symplectic lattice. In particular, Seiberg-Witten differential can be interpreted as the central charge for a complex integrable system, while the BPS degeneracies are interpreted via our “numerical” Donaldson-Thomas invariants as the number of certain gradient trees on the base of a complex integrable system.

Recall that a *complex integrable system* is a holomorphic map $\pi : X \rightarrow B$ where $(X, \omega_X^{2,0})$ is a holomorphic symplectic manifold, $\dim X = 2 \dim B$, and the generic fiber of π is a Lagrangian submanifold, which is a polarized abelian variety. We assume (in order to simplify the exposition) that the polarization is principal. The fibration π is non-singular outside of a closed subvariety $B^{sing} \subset B$ of codimension at least one. It follows that on the open subset $B^{sm} := B \setminus B^{sing}$ we have a local system Γ of symplectic lattices with the fiber over $b \in B^{sm}$ equal to $\Gamma_b := H_1(X_b, \mathbf{Z})$, $X_b = \pi^{-1}(b)$ (the symplectic structure on Γ_b is given by the polarization).

Furthermore, the set B^{sm} is locally (near each point $b \in B^{sm}$) embedded as a holomorphic Lagrangian subvariety into an affine symplectic space parallel to $H_1(X_b, \mathbf{C})$. Namely, let us choose a symplectic basis $\gamma_i \in \Gamma_b, 1 \leq i \leq 2n$. Then we have a collection of holomorphic closed 1-forms $\alpha_i = \int_{\gamma_i} \omega_X^{2,0}, 1 \leq i \leq 2n$ in a neighborhood of b . There exists (well-defined locally up to an additive constant) holomorphic functions $z_i, 1 \leq i \leq 2n$ such that $\alpha_i = dz_i, 1 \leq i \leq 2n$. They define an embedding of a neighborhood of b into \mathbf{C}^{2n} . The collection of 1-forms α_i gives rise to an ele-

ment $\delta \in H^1(B^{sm}, \Gamma^\vee \otimes \mathbf{C})$. We assume that $\delta = 0$. This assumption is equivalent to an existence of a section $Z \in \Gamma(B^{sm}, \Gamma \otimes \mathcal{O}_{B^{sm}})$ such that $\alpha_i = Z(\gamma_i), 1 \leq i \leq 2n$.

Definition 5 We call Z the central charge of the integrable system.

Hence, for every point $b \in B^{sm}$ we have a symplectic lattice Γ_b endowed with an additive map $Z_b : \Gamma_b \rightarrow \mathbf{C}$. Our goal will be to define a continuous family of stability data on graded Lie algebras \mathfrak{g}_{Γ_b} with central charges Z_b .

First, we show an example of section Z .

Example 2 (Seiberg-Witten curve)

Let $B = \mathbf{C}$ be a complex line endowed with a complex coordinate u . We denote by $X^0 = T^*(\mathbf{C} \setminus \{0\})$ the cotangent bundle to the punctured line. We endow it with the coordinates $(x, y), y \neq 0$ and the symplectic form

$$\omega^{2,0} = dx \wedge \frac{dy}{y}.$$

There is a projection $\pi^0 : X^0 \rightarrow B$ given by

$$\pi(x, y) = \frac{1}{2}\left(x^2 - y - \frac{c}{y}\right),$$

where c is a fixed constant. Fibers of π^0 are punctured elliptic curves

$$y + \frac{c}{y} = x^2 - 2u.$$

We denote by X the compactification of X^0 obtained by the compactifications of the fibers. We denote by $\pi : X \rightarrow B$ the corresponding projection. Then $Z_u \in H^1(\pi^{-1}(u), \mathbf{C})$ is represented by a meromorphic 1-form $\lambda_{SW} = \frac{xdy}{y}$ (Seiberg-Witten form). The form λ_{SW} has zero residues, hence it defines an element of $H^1(\pi^{-1}(u), \mathbf{C})$ for any $u \in B^{sm}$, where $B^{sm} = B \setminus \{b_-, b_+\}$ consists of points where the fiber of π is a non-degenerate elliptic curve.

The dense open set $B^{sm} \subset B$ carries a Kähler form

$$\omega_B^{1,1} = \text{Im} \left(\sum_{1 \leq i \leq n} \alpha_i \wedge \overline{\alpha_{n+i}} \right).$$

We denote by g_B the corresponding Kähler metric.

For any $t \in \mathbf{C}^*$ we define an integral affine structure on C^∞ -manifold B^{sm} given by a collection of closed 1-forms $\text{Re}(t\alpha_i), 1 \leq i \leq 2n$. For any simply-connected open subset $U \subset B^{sm}$ and a covariantly constant section $\gamma \in \Gamma(B^{sm}, \Gamma)$ we have a closed 1-form

$$\alpha_{\gamma,t} = \text{Re} \left(t \int_{\gamma} \omega_X^{2,0} \right) = d \text{Re}(tZ(\gamma)) ,$$

and the corresponding gradient vector field $v_{\gamma,t} = g_B^{-1}(\alpha_{\gamma,t})$. Notice that this vector field is a constant field with integral direction in the integral affine structure associated with closed 1-forms $\text{Im}(t\alpha_i), 1 \leq i \leq 2n$.

Similarly to [40] we can construct infinite oriented trees lying in B such that its external vertices belong to B^{sing} , and edges are *positively oriented* trajectories of vector fields $v_{\gamma,t}$. All internal vertices have valency at least 3, and every such vertex should be thought of as a splitting point: a trajectory of the vector field $v_{\gamma,t}$ is split at a vertex into several trajectories of vector fields $v_{\gamma_1,t}, \dots, v_{\gamma_k,t}$ such that $\gamma = \gamma_1 + \dots + \gamma_k$.

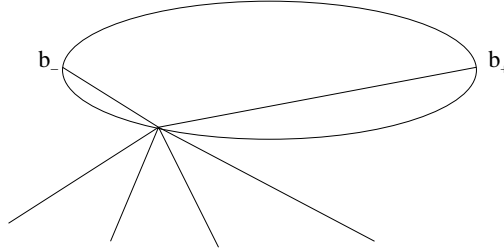
The restriction of the function Z to a tree gives rise to a \mathbf{C} -valued function such that on the trajectory of vector field $v_{\gamma,t}$ it is equal to the restriction of $Z(\gamma)$ to this trajectory. We assume that this function approach to zero as long as we approach an external vertex of the tree (which belongs to B^{sing}). It is easy to see that $tZ(\gamma)$ is a positive number at any other point of the tree (hence it defines a length function). We expect that for any point $b \in B^{sm}$ and $\gamma \in \Gamma_b$ there exist finitely many such trees which pass the point b in the direction of γ (we can think of b as a root of the tree, hence we can say above that we consider oriented trees such that all external vertices except of the root belong to B^{sing}). Here we choose an affine structure with $t \in \mathbf{R}_{>0}(Z(\gamma)_b^{-1})$. Probably the number of such trees for fixed b, γ is finite, since their lengths should be bounded.⁸

For a fixed $t \in \mathbf{C}^*$ the union W_t of all trees as above is in fact a countable union of real hypersurfaces in B^{sm} . They are analogs of the walls of second kind. The set W_t depends on $\text{Arg } t$ only. The union $\cup_{\theta \in [0, 2\pi i)} W_{te^{i\theta}}$ swap the whole space B^{sm} . Let us denote by $W^{(1)}$ the union over all $t \in \mathbf{C}^*/\mathbf{R}_{>0}$ of the sets of internal vertices of all trees in $W^{(1)}$ (splitting points of the gradient trajectories). This is an analog of the wall of first kind.

⁸In [40] we modified the gradient fields near B^{sing} in order to guarantee the convergence of infinite products in the adic topology. It seems that we were too cautious, and the convergence holds without any modification.

In [40] we suggested a procedure of assigning rational multiplicities to edges of trees (see also [27],[28]). This leads to the following picture. Consider the total space $tot(\mathbf{\Gamma})$ of the local system $\mathbf{\Gamma}$. It follows from above assumptions and considerations that we have a locally constant function $\Omega : tot(\mathbf{\Gamma}) \rightarrow \mathbf{Q}$ which jumps at the subset consisting of the lifts of the wall $W^{(1)}$ to $tot(\mathbf{\Gamma})$. Then for a fixed $b \in B^{sm}$ the pair (Z, Ω) defines stability data on the graded Lie algebra \mathfrak{g}_{Γ_b} of the group of formal symplectomorphisms of the symplectic torus \mathbb{T}_{Γ_b} . In this way we obtain a local embedding $B^{sm} \hookrightarrow Stab(\mathfrak{g}_{\Gamma_b})$.

In the above example of Seiberg-Witten curve, the wall $W^{(1)}$ is an oval-shaped curve which contains two singular points $b_{\pm} \in B^{sing}$. A typical W_t looks such as follows.



The wall-crossing formula coincides with the one for $T_{a,b}^{(2)}$ (see Introduction).

Remark 13 1) *We expect that the above considerations are valid for a large class of complex integrable systems, e.g. Hitchin system.*

2) *In the case when we have a 3d complex compact Calabi-Yau manifold X , the moduli space \mathcal{M}_X of complex structures on X is locally embedded into the projective space $\mathbf{P}(H^3(X, \mathbf{C}))$ as a base of a Lagrangian cone $\mathcal{L}_X \subset H^3(X, \mathbf{C})$. It carries a Kähler metric (Weil-Petersson metric). We can repeat the above considerations given for integrable systems, replacing the gradient flows by the attractor flow (see e.g. [13]). The above case of integrable systems is obtained in the limit, when the cone becomes “very sharp”.*

2.8 Relation with the works of Joyce, and of Bridgeland and Toledano-Laredo

Let \mathfrak{g}, Γ be as in Section 2.1. We assume that the ground field is \mathbf{C} . Suppose that $C \subset \Gamma_{\mathbf{R}}$ is a strict convex cone. We are interested in such stability data

(Z, a) that $\text{Supp } a \subset C \cup (-C)$. We define D as an open subset of $\text{Hom}(\Gamma, \mathbf{C})$ which consists of additive maps such that $C \cap \Gamma$ is mapped into the upper-half plane $\mathcal{H}_+ = \{z \in \mathbf{C} \mid \text{Im}(z) > 0\}$. We interpret D as an open subset of $\text{Stab}(\mathfrak{g})$. Every $\alpha \in C \cap \Gamma$ gives rise to an invertible function (coordinate) $z_\alpha \in \mathcal{O}(D)^\times$ such that $z_\alpha(Z, a) = Z(\alpha)$.

Recall the pro-nilpotent Lie algebra $\mathfrak{g}_C = \prod_{\gamma \in C \cap \Gamma} \mathfrak{g}_\gamma$ and the corresponding pro-nilpotent group G_C .

In the paper [32] by D. Joyce the following system of differential equations for a collection of holomorphic functions $(f_\alpha)_{\alpha \in C \cap \Gamma}$, $f_\alpha \in \mathcal{O}(D) \otimes \mathfrak{g}_\alpha$ was considered:

$$\forall \alpha \in C \cap \Gamma \quad df_\alpha = -\frac{1}{2} \sum_{\beta+\gamma=\alpha} [f_\beta, f_\gamma] d \log \frac{z_\beta}{z_\gamma} .$$

It follows that if (f_α) satisfies the above system of equations then the differential 1-form

$$\omega = \sum_{\alpha} f_\alpha d \log z_\alpha \in \Omega^1(D) \widehat{\otimes} \mathfrak{g}_C := \prod_{\alpha \in C \cap \Gamma} (\Omega^1(\mathfrak{g}) \otimes \mathfrak{g}_\alpha)$$

gives rise to the flat connection, since

$$d\omega + \frac{1}{2}[\omega, \omega] = 0 .$$

Moreover, setting $F = \sum_{\alpha} f_\alpha$ we observe that

$$dF + [\omega, F] = 0 ,$$

i.e. F is a flat section of this connection in the adjoint representation.

One can check by induction that there exists a unique solution to the above system of differential equations (modulo constants for each function f_α). This means that the set of solutions is isomorphic to \mathfrak{g}_C (non-canonically).

For any $n \geq 0$ and pairwise different numbers $x_i \in \mathbf{C} \setminus \{0, 1\}$, $i = 1, \dots, n$ we introduce the following function (multilogarithm) which is holomorphic when all x_i lie outside of the interval $[0, 1]$:

$$L_n(x_1, \dots, x_n) := v.p. \int_{0 < t_1 < t_2 < \dots < t_n < 1} \prod_{1 \leq i \leq n} \frac{dt_i}{t_i - x_i} ,$$

where *v.p.* means “principal value”. Then $L_0 = 1, L_1(x) = \log(1 - 1/x)$ where we define the branch of the logarithm by taking the cut along the ray $(-\infty, 0)$.

For a given collection (f_α) as above, Joyce defined a collection of functions \mathcal{E}_α on D with values in the completed universal enveloping algebra $U(\mathfrak{g}_C)$:

$$\mathcal{E}_\alpha = \sum_{n \geq 1} \sum_{\alpha_1 + \dots + \alpha_n = \alpha} f_{\alpha_1} \dots f_{\alpha_n} I_n(z_{\alpha_1}, z_{\alpha_2}, \dots, z_{\alpha_n}),$$

where for $z_1, \dots, z_n \in \mathbf{C}$ such that $0 < \text{Im } z_1 < \text{Im } z_2 < \dots < \text{Im } z_n$ we set⁹

$$\begin{aligned} I_n(z_1, z_2, \dots, z_n) &:= \\ &= 2\pi i (-1)^{n-1} L_{n-1} \left(\frac{z_1}{z_1 + \dots + z_n}, \frac{z_1 + z_2}{z_1 + \dots + z_n}, \dots, \frac{z_1 + z_2 + \dots + z_{n-1}}{z_1 + \dots + z_n} \right). \end{aligned}$$

One can show that in fact $\mathcal{E}_\alpha \in \mathfrak{g}_C$, and it is a locally constant along strata of the stratification defined by the walls $z_\beta/z_\gamma \in \mathbf{R}$ where $\alpha = \beta + \gamma$ with $\beta, \gamma \in C \cap \Gamma$ and β is not parallel to γ .

For a solution (f_α) of the above system of differential equations we define a differential 1-form on $D \times \mathbf{C}^*$ such that

$$\widehat{\omega} := \sum_{\alpha} f_{\alpha} e^{v z_{\alpha}} d \log(v z_{\alpha}),$$

where v is the standard coordinate on \mathbf{C}^* .

Then one checks that

$$d\widehat{\omega} + \frac{1}{2}[\widehat{\omega}, \widehat{\omega}] = 0.$$

Let $M_{(f_\alpha)} \in G_C$ be the monodromy of the corresponding flat connection computed along a closed loop in the complex v -plane, which starts at $+i\infty$ and goes in the anti-clockwise direction around $v = 0$. The flatness implies that the monodromy does not depend on the point of D .

On the other hand let us consider the element $N_{(f_\alpha)} \in G_C$ defined as

$$\prod_{l \subset \mathcal{H}_+}^{\rightarrow} \exp \left(\sum_{\alpha \in C \cap \Gamma, z_\alpha \in l} \mathcal{E}_\alpha \right),$$

where the product is taken over all rays $l \subset \mathcal{H}_+$ with the vertex at the origin.

⁹This formula was proposed in [10] as the inversion of the Joyce formula which expressed f_α 's in terms of \mathcal{E}_α 's.

Conjecture 2 *We have $M_{(f_\alpha)} = N_{(f_\alpha)}$.*

The conjecture implies that the elements $a_\alpha := \mathcal{E}_\alpha$ satisfy the wall-crossing formula. The element $M_{(f_\alpha)}$ is equal (in our notation) to the element A_V , where V is a strict sector in \mathcal{H}_+ containing $Z(C)$. We will discuss below a sequence of identities which imply the conjecture. But we need to introduce certain functions first.

Let $\varphi : (0, 1) \rightarrow \mathbf{C} \setminus \{0\}$ be the infinite contour which starts and ends at $+i\infty$, goes in the anti-clockwise direction and surrounds the point $0 \in \mathbf{C}$. With the contour φ we associate the following function on $(\mathcal{H}_+)^n$, $n \geq 1$:

$$K_n(z_1, \dots, z_n) := \int_{0 < t_1 < \dots < t_n < 1} \exp\left(\sum_{1 \leq i \leq n} \varphi(t_i) z_i\right) \prod_{1 \leq i \leq n} \varphi'(t_i)/\varphi(t_i) dt_i .$$

Notice that this function can be written as a Chen iterated integral

$$K_n(z_1, \dots, z_n) = \int_{\varphi} \omega_1 \circ \omega_2 \circ \dots \circ \omega_n ,$$

where $\omega_i = e^{vz_i} dv/v$, $1 \leq i \leq n$, $v \in \mathbf{C} \setminus \{0\}$.

Let us fix $n \geq 1$ and a collection of complex numbers $z_i \in \mathcal{H}_+$, $1 \leq i \leq n$. We call a sequence $0 = i_0 < i_1 < \dots < i_{k-1} < i_k$ *admissible* if

$$\text{Arg}(z_1 + \dots + z_{i_1}) \geq \text{Arg}(z_{i_1+1} + \dots + z_{i_2}) \geq \dots \geq \text{Arg}(z_{i_{k-1}+1} + \dots + z_{i_k}) .$$

For a fixed admissible sequence we have a partition $k = l_1 + \dots + l_m$ where l_1, l_2, \dots, l_m are the numbers of consecutive equalities in the above sequence of inequalities for the arguments. Let $\Omega_{k, l_1, \dots, l_m}(z_1, \dots, z_n)$ be the set of all admissible sequences $0 = i_0 < i_1 < \dots < i_{k-1} < i_k$ with the given partition $k = l_1 + \dots + l_m$. Under these assumptions and notation one can see that the previous Conjecture 2 is equivalent to

Conjecture 3 *We have*

$$K_n(z_1, \dots, z_n) = \sum_{\Omega_{k, l_1, \dots, l_m}(z_1, \dots, z_n)} \prod_{1 \leq j \leq m} \frac{1}{l_j!} I_{i_1}(z_1, \dots, z_{i_1}) \cdot I_{i_2 - i_1}(z_{i_1+1}, \dots, z_{i_2}) \cdot \dots \\ \cdot I_{i_k - i_{k-1}}(z_{i_{k-1}+1}, \dots, z_n) .$$

Indeed, for $z_i = z_{\alpha_i}$, $i = 1, \dots, n$ the l.h.s. of the formula is the contribution of the term $f_{\alpha_1} \dots f_{\alpha_n}$ in the expansion of $M_{(f_\alpha)}$. Similarly, the r.h.s. is the contribution of the same term in $N_{(f_\alpha)}$.

Here we give a proof of the above conjecture in the special case:

Proposition 5 *If $0 < \text{Arg } z_1 < \dots < \text{Arg } z_n < \pi$ then*

$$K_n(z_1, \dots, z_n) = I_n(z_1, z_2, \dots, z_n) .$$

Proof. For $n = 1$ both sides are equal to $2\pi i$. For $n \geq 2$ we proceed by induction. First one checks directly that

$$dK_n(z_1, \dots, z_n) = - \sum_{i=1}^{n-1} d \log \left(\frac{z_{i+1}}{z_i} \right) K_{n-1}(z_1, \dots, z_i + z_{i+1}, \dots, z_n) .$$

The same formula holds if we replace K_n, K_{n-1} by I_n, I_{n-1} respectively. Thus we see by induction that $K_n - I_n = \text{const}_n$. We want to prove that $\text{const}_n = 0$. In order to do that we take $z_j = z_j(\varepsilon)$, $1 \leq j \leq n$, such as follows:

$$z_1(\varepsilon) = i + \frac{1}{\varepsilon}, \quad z_n(\varepsilon) = i - \frac{1}{\varepsilon}, \quad z_k(\varepsilon) = i - k, \quad 2 \leq k \leq n - 1 .$$

Here $i = \sqrt{-1}$. Then

$$0 < \text{Arg } z_1(\varepsilon) < \text{Arg } z_2(\varepsilon) < \dots < \text{Arg } z_n(\varepsilon) < \pi$$

and $|\sum_{1 \leq j \leq k} z_j(\varepsilon)| \rightarrow \infty$ as $\varepsilon \rightarrow 0$ for $k = 1, \dots, n - 1$, and moreover $|\sum_{1 \leq j \leq n} z_j(\varepsilon)|$ is a constant function of ε . Therefore,

$$I_n(z_1(\varepsilon), \dots, z_n(\varepsilon)) \rightarrow 0$$

as $\varepsilon \rightarrow 0$, since all the arguments of the function L_{n-1} in the definition of I_n approach infinity.

Hence in order to finish the proof it suffices to show that

$$K_n(z_1(\varepsilon), \dots, z_n(\varepsilon)) \rightarrow 0$$

as $\varepsilon \rightarrow 0$. Here is the sketch of the proof.¹⁰ Notice that

$$\int_{+i\infty}^{v_2} e^{v_1 z_1(\varepsilon)} dv_1 / v_1 = \frac{1}{z_1(\varepsilon)} e^{v_1 z_1(\varepsilon)} / v_2 + r_1(\varepsilon) ,$$

¹⁰We thank Andrei Okounkov for the idea of the proof.

where $r_1(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Repeating we obtain that

$$K_n(z_1(\varepsilon), \dots, z_n(\varepsilon)) = \frac{1}{z_1(\varepsilon)} \frac{1}{z_1(\varepsilon) + z_2(\varepsilon)} \cdots \frac{1}{z_1(\varepsilon) + z_2(\varepsilon) + \cdots + z_{n-1}(\varepsilon)} \times \\ \times \int_{\varphi} e^{v_n(z_1(\varepsilon) + \cdots + z_n(\varepsilon))} dv_n / v_n + r_n(\varepsilon) ,$$

where the integral is taken over the contour φ described before, and $r_n(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. It follows from our choice of numbers $z_j(\varepsilon), 1 \leq j \leq n$ that $K_n(z_1(\varepsilon), \dots, z_n(\varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$. ■

One can hope that the technique developed in [32] helps in proving the general case.

A relationship between Joyce formulas and iterated integrals is discussed in [10] in a slightly different form. In that paper the elements $N_{(f_\alpha)}$ are interpreted as Stokes multipliers for a different system of differential equations on \mathbf{C} (with coordinate t) with values in the Lie algebra which is an extension of $\mathfrak{g}_{\mathbf{C}}$ by the abelian Lie algebra $\text{Hom}(\Gamma, \mathbf{C})$ (an analog of the Cartan subalgebra). It has irregular singularity at the origin given by $\frac{Z}{t^2}$, where Z is the central charge of the stability structure. In fact the connection from [10] reduces to our connection after the change of variables $v = 1/t$ and the conjugation by $\exp(-vZ)$.

2.9 Stability data on $\mathfrak{gl}(n)$

Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbf{Q})$ be the Lie algebra of the general linear group. We consider it as a Γ -graded Lie algebra $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$, where

$$\Gamma = \{(k_1, \dots, k_n) \mid k_i \in \mathbf{Z}, \sum_{1 \leq i \leq n} k_i = 0\}$$

is the root lattice. We endow \mathfrak{g} with the Cartan involution η . Algebra \mathfrak{g} has the standard basis $E_{ij} \in \mathfrak{g}_{\gamma_{ij}}$ consisting of matrices with the single non-zero entry at the place (i, j) equal to 1. Then $\eta(E_{ij}) = -E_{ji}$. In what follows we are going to consider symmetric (with respect to η) stability data on \mathfrak{g} .

We notice that

$$\text{Hom}(\Gamma, \mathbf{C}) \simeq \mathbf{C}^n / \mathbf{C} \cdot (1, \dots, 1) .$$

We define a subspace $\text{Hom}^\circ(\Gamma, \mathbf{C}) \subset \text{Hom}(\Gamma, \mathbf{C})$ consisting (up to a shift by the multiples of the vector $(1, \dots, 1)$) of vectors (z_1, \dots, z_n) such that $z_i \neq z_j$

if $i \neq j$. Similarly we define a subspace $\text{Hom}^{\circ\circ}(\Gamma, \mathbf{C}) \subset \text{Hom}(\Gamma, \mathbf{C})$ consisting (up to the same shift) of such (z_1, \dots, z_n) that there is no z_i, z_j, z_k belonging to the same real line as long as $i \neq j \neq k$. Obviously there is an inclusion $\text{Hom}^{\circ\circ}(\Gamma, \mathbf{C}) \subset \text{Hom}^{\circ}(\Gamma, \mathbf{C})$.

For $Z \in \text{Hom}(\Gamma, \mathbf{C})$ we have $Z(\gamma_{ij}) = z_i - z_j$. If $Z \in \text{Hom}^{\circ\circ}(\Gamma, \mathbf{C})$ then symmetric stability data with such Z is the same as a skew-symmetric matrix (a_{ij}) with rational entries determined from the equality $a(\gamma_{ij}) = a_{ij}E_{ij}$. Every continuous path in $\text{Hom}^{\circ}(\Gamma, \mathbf{C})$ admits a unique lifting to $\text{Stab}(\mathfrak{g})$ as long as we fix the lifting of the initial point. The matrix (a_{ij}) changes when we cross walls in $\text{Hom}^{\circ}(\Gamma, \mathbf{C}) \setminus \text{Hom}^{\circ\circ}(\Gamma, \mathbf{C})$. A typical wall-crossing corresponds to the case when in the above notation the point z_j crosses a straight segment joining z_i and $z_k, i \neq j \neq k$. In this case the only change in the matrix (a_{ij}) is of the form:

$$a_{ik} \mapsto a_{ik} + a_{ij}a_{jk} .$$

This follows from the multiplicative wall-crossing formula which is of the form:

$$\begin{aligned} \exp(a_{ij}E_{ij}) \exp(a_{ik}E_{ik}) \exp(a_{jk}E_{jk}) &= \\ &= \exp(a_{jk}E_{jk}) \exp((a_{ik} + a_{ij}a_{jk})E_{ik}) \exp(a_{ij}E_{ij}) . \end{aligned}$$

Same wall-crossing formulas appeared in [11] in the study of the change of the number of solitons in $N = 2$ two-dimensional supersymmetric QFT. In [11] the numbers a_{ij} were integers, and the wall-crossing preserved integrality. In our considerations, for any $Z \in \text{Hom}^{\circ\circ}(\Gamma, \mathbf{C})$ the fundamental group $\pi_1(\text{Hom}^{\circ}(\Gamma, \mathbf{C}), Z)$ acts on the space of skew-symmetric matrices by polynomial transformations with integer coefficients. It can be identified with the well-known actions of the pure braid group on the space of upper-triangular matrices in the theory of Gabrielov bases of isolated singularities and in the theory of triangulated categories endowed with exceptional collections. Furthermore, the matrices $\exp(a_{ij}E_{ij}) = 1 + a_{ij}E_{ij}$ can be interpreted as Stokes matrices of a certain connection in a neighborhood of $0 \in \mathbf{C}$, which has irregular singularities (tt^* -connection from [11], see also [24]). This observation should be compared with the results about the irregular connection from the previous section.

3 Ind-constructible categories and stability structures

3.1 Ind-constructible categories

Here we introduce an ind-constructible version of the notion of a (triangulated) A_∞ -category. Let \mathbf{k} be a field, $\bar{\mathbf{k}}$ be its algebraic closure. By a variety over \mathbf{k} (not necessarily irreducible) we mean a reduced separated scheme of finite type over \mathbf{k} . Recall the following definition.

Definition 6 *Let S be a variety over \mathbf{k} . A subset $X \subset S(\bar{\mathbf{k}})$ is called constructible over \mathbf{k} if it belongs to the Boolean algebra generated by $\bar{\mathbf{k}}$ -points of open (equivalently closed) subschemes of S .*

Equivalently, a constructible set is the union of a finite collection of $\bar{\mathbf{k}}$ -points of disjoint locally closed subvarieties $(S_i \subset S)_{i \in I}$.

For any field extension $\mathbf{k} \subset \mathbf{k}' \subset \bar{\mathbf{k}}$ we define the set of \mathbf{k}' -points $X(\mathbf{k}')$ of the constructible set X as $(X \cap S(\mathbf{k}')) \subset S(\bar{\mathbf{k}})$. In particular, $X(\bar{\mathbf{k}}) = X$.

We define the category $\mathcal{CON}_{\mathbf{k}}$ of constructible sets over \mathbf{k} as a category with objects (X, S) , where X and S as above. The set of morphisms $\text{Hom}_{\mathcal{CON}_{\mathbf{k}}}((X_1, S_1), (X_2, S_2))$ is defined to be the set of maps $f : X_1 \rightarrow X_2$ such that there exists a decomposition of X_1 into the finite disjoint union of $\bar{\mathbf{k}}$ -points of varieties $(S_i \subset S_1)_{i \in I}$ such that the restriction of f to each $S_i(\bar{\mathbf{k}})$ is a morphism of schemes $S_i \rightarrow S_2$. We see that there is a natural faithful functor from $\mathcal{CON}_{\mathbf{k}}$ to the category of sets equipped with the action of $\text{Aut}(\bar{\mathbf{k}}/\mathbf{k})$.

Definition 7 *An ind-constructible set over \mathbf{k} is given by a chain of embeddings of constructible sets $X := (X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \dots)$ over \mathbf{k} . A morphism of ind-constructible sets is defined as a map $g : \cup_i X_i(\bar{\mathbf{k}}) \rightarrow \cup_i Y_i(\bar{\mathbf{k}})$ such that for any i there exists $n(i)$ such that $g|_{X_i(\bar{\mathbf{k}})} : X_i(\bar{\mathbf{k}}) \rightarrow Y_{n(i)}(\bar{\mathbf{k}})$ comes from a constructible map.*

Ind-constructible sets form a full subcategory $\mathcal{IC}_{\mathbf{k}}$ of the category of ind-objects in $\mathcal{CON}_{\mathbf{k}}$.

Remark 14 *Equivalently, we can consider a countable collection $Z_i = X_i \setminus X_{i-1}$ of non-intersecting constructible sets. Then a morphism $\sqcup_{i \in I} Z_i \rightarrow \sqcup_{j \in J} Z'_j$ is given by a collection of constructible maps $f_i : Z_i \rightarrow \sqcup_{j \in J_i} Z'_j$, where each J_i is a finite set.*

The category of constructible (or ind-constructible) sets has fibered products. There is a notion of constructible (or ind-constructible) vector bundle (i.e. the one with the fibers which are affine spaces of various finite dimensions).

Definition 8 *An ind-constructible A_∞ -category over \mathbf{k} is defined by the following data:*

1) *An ind-constructible set $\mathcal{M} = \text{Ob}(\mathcal{C}) = \sqcup_{i \in I} X_i$ over \mathbf{k} , called the set of objects.*

2) *A collection of ind-constructible vector bundles*

$$\mathcal{HOM}^n \rightarrow \mathcal{M} \times \mathcal{M}, n \in \mathbf{Z}$$

called the bundles of morphisms of degree n . The restriction $\mathcal{HOM}^n \rightarrow X_i \times X_j$ is a finite-dimensional constructible vector bundle for any $n \in \mathbf{Z}, i, j \in I$, and the restriction $\mathcal{HOM}^n \rightarrow X_i \times X_j$ is a zero bundle for $n \leq C(i, j)$, where $C(i, j)$ is some constant.

3) *For any $n \geq 1, l_1, \dots, l_n \in \mathbf{Z}$, ind-constructible morphisms of ind-constructible bundles*

$$m_n : p_{1,2}^* \mathcal{HOM}^{l_1} \otimes \dots \otimes p_{n,n+1}^* \mathcal{HOM}^{l_n} \rightarrow p_{1,n+1}^* \mathcal{HOM}^{l_1 + \dots + l_n + 2 - n},$$

where $p_{i,i+1}$ denote natural projections of \mathcal{M}^{n+1} to \mathcal{M}^2 . These morphisms are called higher composition maps.

The above data are required to satisfy the following axioms A1)-A3):

A1) *Higher associativity property for $m_n, n \geq 1$ in the sense of A_∞ -categories. We leave for the reader to write down the corresponding well-known identities (see [37],[42]).*

This axiom implies that we have a small $\bar{\mathbf{k}}$ -linear non-unital A_∞ -category $\mathcal{C}(\bar{\mathbf{k}})$ with the set of objects $\mathcal{M}(\bar{\mathbf{k}})$ and morphisms $\mathcal{HOM}^\bullet(\bar{\mathbf{k}})$.

A2) *(weak unit) There is a constructible section s of the ind-constructible bundle $\mathcal{HOM}_{\text{Diag}}^0 \rightarrow \mathcal{M}$ such that the image of s belongs to the kernel of m_1 and gives rise to the identity morphisms in \mathbf{Z} -graded $\bar{\mathbf{k}}$ -linear category $H^\bullet(\mathcal{C}(\bar{\mathbf{k}}))$.*

Alternatively, instead of A2) one can use the axiomatics of A_∞ -categories with strict units (see [42], [46]).

An ind-constructible A_∞ -category \mathcal{C} gives rise to a collection of ind-constructible bundles over $\text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{C})$ given by

$$\mathcal{E}\mathcal{X}\mathcal{T}^i := H^i(\mathcal{HOM}^\bullet), i \in \mathbf{Z}$$

with the fiber over a pair of objects (E, F) equal to

$$\mathrm{Ext}^i(E, F) := H^i(\mathcal{HOM}_{E,F}^\bullet) .$$

The cohomology groups are taken with respect to the differential m_1 .

A3) (local regularity) There exists a countable collection of schemes $(S_i)_{i \in I}$ of finite type over \mathbf{k} , a collection of algebraic \mathbf{k} -vector bundles $\mathrm{HOM}_i^n, n \in \mathbf{Z}$ over $S_i \times S_i$ for all i , and ind-constructible identifications

$$\sqcup_i S_i(\bar{\mathbf{k}}) \simeq \mathcal{M}, \mathrm{HOM}_i^n \simeq \mathcal{HOM}_{|S_i \times S_i}^n, n \in \mathbf{Z}$$

such that all higher compositions $m_n, n \geq 2$, considered for objects from S_i for any given $i \in I$, become morphisms of algebraic vector bundles.

We will often call ind-constructible A_∞ -categories simply by ind-constructible categories. The basic example of an ind-constructible category is the category $\mathrm{Perf}(A)$ of perfect A -modules where A an A_∞ -algebra over \mathbf{k} with finite-dimensional cohomology (see the discussion after the Example 1 in Section 1.2 of Introduction).

We define a *functor* between two ind-constructible categories mimicking the usual definition of an A_∞ -functor.

A functor $\Phi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is called an *equivalence* if Φ is a full embedding, i.e. it induces an isomorphism

$$\mathrm{Ext}^\bullet(E, F) \simeq \mathrm{Ext}^\bullet(\Phi(E), \Phi(F)) \quad \forall E, F \in \mathrm{Ob}(\mathcal{C}_1)(\bar{\mathbf{k}})$$

and moreover, there exists an ind-constructible *over $\bar{\mathbf{k}}$* map

$$s : \mathrm{Ob}(\mathcal{C}_2)(\bar{\mathbf{k}}) \rightarrow \mathrm{Ob}(\mathcal{C}_1)(\bar{\mathbf{k}})$$

such that for any object $E \in \mathrm{Ob}(\mathcal{C}_2)(\bar{\mathbf{k}})$ we have $E \simeq \Phi(s(E))$.

Using the notions of a functor and of an equivalence we can define the property of an ind-constructible weakly unital A_∞ -category \mathcal{C} to be triangulated. For example, the property to have exact triangles can be formulated as follows. Consider a finite A_∞ -category \mathcal{C}_3 consisting of 3 objects E_1, E_2, E_3 with non-trivial morphism spaces

$$\mathrm{Hom}^0(E_i, E_i) = \mathbf{k} \cdot \mathrm{id}_{E_i}, \mathrm{Hom}^0(E_1, E_2) \simeq \mathrm{Hom}^0(E_2, E_3) \simeq \mathrm{Hom}^1(E_3, E_1) \simeq \mathbf{k}$$

equivalent to the full subcategory of the category of representations of the quiver A_2 consisting of modules of dimensions $(0, 1), (1, 1), (1, 0)$. Let $\mathcal{C}_2 \subset \mathcal{C}_3$ be the full subcategory consisting of first two objects.

It is easy to see directly from the definitions that for any ind-constructible category \mathcal{C} there are natural ind-constructible categories $\text{Fun}(\mathcal{C}_i, \mathcal{C})$, $i = 2, 3$ whose objects over $\bar{\mathbf{k}}$ are the usual A_∞ -categories of functors from $\mathcal{C}_i(\bar{\mathbf{k}})$ to $\mathcal{C}(\bar{\mathbf{k}})$ as defined e.g. in [41],[42] and [37]. There exists a natural restriction functor

$$r_{32} : \text{Fun}(\mathcal{C}_3, \mathcal{C}) \rightarrow \text{Fun}(\mathcal{C}_2, \mathcal{C}).$$

Similarly to the setting of usual A_∞ -categories, the ind-constructible version of the axiom of exact triangles says that r_{32} is an equivalence. In the same manner one can define other properties of triangulated A_∞ -categories (i.e. the existence of shift functors, finite sums, see [42], [65]) in the ind-constructible setting.

In Sections 5,6 we will use a simplified notation $\text{Cone}(f)$ for a cone of morphism f in $\mathcal{C}(\bar{\mathbf{k}})$ “pretending” that cones are functorial. The precise prescription is to take an object in $\text{Fun}(\mathcal{C}_2, \mathcal{C})$ corresponding to f , find an isomorphic object in $\text{Fun}(\mathcal{C}_3, \mathcal{C})$, and then take the image in $\mathcal{C}(\bar{\mathbf{k}})$ of the object E_3 . All this can be properly formulated using the language of constructible stacks, see 3.2 and 4.2.

Let us call an ind-constructible A_∞ -category *minimal on the diagonal* if the restriction of m_1 to the diagonal $\Delta \subset \mathcal{M} \times \mathcal{M}$ is trivial. One can show that any ind-constructible A_∞ -category is equivalent to a one which is minimal on the diagonal.

Remark 15 *Typically in practice one has a decomposition $\text{Ob}(\mathcal{C}) = \sqcup_{i \in I} X_i$ where X_i are schemes, not just constructible sets. Moreover, for any $E \in X_i(\bar{\mathbf{k}})$ there is a natural map $T_E X_i \rightarrow \text{Ext}^1(E, E)$. The reason for this is the fact that the deformation theory of the object E should be controlled by the DGLA $\text{Ext}^\bullet(E, E)$. We did not include the above property into the list of axioms since it does not play any role in our constructions.*

3.2 Stack of objects

In this section we assume that the ground field \mathbf{k} is perfect, i.e. the $\bar{\mathbf{k}}$ is a Galois field over \mathbf{k} . Our goal in this section is to explain how to associate with an ind-constructible A_∞ -category \mathcal{C} over \mathbf{k} an ordinary \mathbf{k} -linear A_∞ -category $\mathcal{C}(\mathbf{k})$, in such a way that ind-constructible equivalences will induce the usual equivalences. For any field extension $\mathbf{k}' \supset \mathbf{k}$ (e.g. for $\mathbf{k}' = \mathbf{k}$) one can define A_∞ -category $\mathcal{C}^{\text{naive}}(\mathbf{k}')$ to be the small \mathbf{k}' -linear category with the set of objects given by $(\text{Ob}(\mathcal{C}))(\mathbf{k}')$ and obvious morphisms and compositions. This

is not a satisfactory notion because in the definition of the equivalence we demand only the surjectivity on isomorphism classes of objects over $\bar{\mathbf{k}}$. The naive category $\mathcal{C}^{naive}(\mathbf{k})$ will be a full subcategory of the “correct” category $\mathcal{C}(\mathbf{k})$. One should read carefully brackets, as in our notation

$$Ob(\mathcal{C}(\mathbf{k})) \neq (Ob(\mathcal{C}))(\mathbf{k}) =: Ob(\mathcal{C}^{naive}(\mathbf{k})),$$

contrary to the case of $\bar{\mathbf{k}}$ where we have

$$Ob(\mathcal{C}(\bar{\mathbf{k}})) = (Ob(\mathcal{C}))(\bar{\mathbf{k}}).$$

We will see also the “set of isomorphism classes of objects” in \mathcal{C} should be better understood as an ind-constructible stack¹¹.

Let $\mathbf{k}' \subset \bar{\mathbf{k}}$ be a finite Galois extension of \mathbf{k} and consider an element $E' \in (Ob(\mathcal{C}))(\mathbf{k}') \subset Ob(\mathcal{C}(\bar{\mathbf{k}}))$ such that $\sigma(E')$ is isomorphic to E' for all $\sigma \in \text{Gal}(\mathbf{k}'/\mathbf{k})$. We would like to define the descent data for such E' , which should be data necessary to define an object in (not yet defined) \mathbf{k} -linear category $\mathcal{C}(\mathbf{k})$ which becomes isomorphic to E' after the extension of scalars from \mathbf{k} to \mathbf{k}' .

First, for a finite non-empty collection of objects $(E_i)_{i \in I}$ of any A_∞ -category \mathcal{C}' linear over a field \mathbf{k}' (not necessarily a perfect one) we define an *identification data* for objects of this collection to be an A_∞ -functor Φ from the A_∞ -category $\mathcal{C}_{I, \mathbf{k}'}$ describing I copies of the same object:

$$Ob(\mathcal{C}_I) = I, \text{Hom}^\bullet(i, j) = \text{Hom}^0(i, j) \simeq \mathbf{k}'$$

to \mathcal{C}' . In plain terms, to give such a functor is to give a closed morphism of degree 0 for any pair of objects E_i, E_j (representing the identity id_{E_i} in $H^\bullet(\mathcal{C}(\bar{\mathbf{k}}))$ for $i = j$), a homotopy for any triple of objects, homotopy between homotopies for any quadruple of objects, etc. Thus, we in a sense identify all the objects of the collection $(E_i)_{i \in I}$ and hence can treat it as a new object (canonically isomorphic to all $(E_i)_{i \in I}$), without choosing any specific element $i \in I$.

Returning to the case of $E' \in (Ob(\mathcal{C}))(\mathbf{k}')$, we define the descent data as the identification of the collection of objects $(\sigma(E'))_{\sigma \in \text{Gal}(\mathbf{k}'/\mathbf{k})}$ of the category $\mathcal{C}^{naive}(\mathbf{k}')$ equivariant with respect to the action of $\text{Gal}(\mathbf{k}'/\mathbf{k})$ acting both on the collection and on the coefficients in the identification.

¹¹Even in the case when $\mathbf{k} = \bar{\mathbf{k}}$ it is important to keep track on automorphisms groups of objects (and not only on the set of isomorphism classes), e.g. for the correct definition of the motivic Hall algebra in 6.1.

We define the set $Ob(\mathcal{C}(\mathbf{k}))$ of objects of $\mathcal{C}(\mathbf{k})$ to be the inductive limit over finite Galois extensions \mathbf{k}'/\mathbf{k} of descent data as above. Also one can define morphisms and higher compositions. We leave the following Proposition without a proof.

Proposition 6 *There is a natural structure of a \mathbf{k} -linear A_∞ -category on $\mathcal{C}(\mathbf{k})$ containing $\mathcal{C}^{naive}(\mathbf{k})$ as a full subcategory. Any equivalence $\Phi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ in ind-constructible sense induces an equivalence $\mathcal{C}_1(\mathbf{k}) \rightarrow \mathcal{C}_2(\mathbf{k})$. If \mathcal{C} is triangulated in ind-constructible sense then $\mathcal{C}(\mathbf{k})$ is also triangulated.*

If E is an object of $\mathcal{C}(\mathbf{k})$ then any other object E' of $\mathcal{C}(\mathbf{k})$ which is isomorphic to E after the extension of scalars to $\mathcal{C}(\bar{\mathbf{k}})$ is in fact isomorphic to E in $\mathcal{C}(\mathbf{k})$ (in other words, there are no non-trivial twisted forms). The reason is that (as follows directly from definitions) the set of such “ \mathbf{k} -forms” of E is classified by $H^1(\text{Gal}(\bar{\mathbf{k}}/\mathbf{k}), \mathcal{G}_E)$, where \mathcal{G}_E is a simplicial group associated with the A_∞ -algebra $\text{End}_{\mathcal{C}(\mathbf{k})}^\bullet(E, E)$. There is a spectral sequence which converges to this set and has the second term $E_2 = (E_2^{pq})$ given by

$$\begin{aligned} & H^1(\text{Gal}(\bar{\mathbf{k}}/\mathbf{k}), (\text{Ext}_{\mathcal{C}(\bar{\mathbf{k}})}^0(E, E)^\times) , \\ & H^2(\text{Gal}(\bar{\mathbf{k}}/\mathbf{k}), \text{Ext}_{\mathcal{C}(\bar{\mathbf{k}})}^{-1}(E, E)) , \\ & H^3(\text{Gal}(\bar{\mathbf{k}}/\mathbf{k}), \text{Ext}_{\mathcal{C}(\bar{\mathbf{k}})}^{-2}(E, E)), \dots \end{aligned}$$

We observe that all Galois cohomology groups with coefficients in $\text{Ext}_{\mathcal{C}(\bar{\mathbf{k}})}^{<0}(E, E)$ are trivial (since $\text{Ext}_{\mathcal{C}(\bar{\mathbf{k}})}^i(E, E), i < 0$ are just sums of copies of the additive group $\mathbf{G}_a(\bar{\mathbf{k}})$). Also the set $H^1(\text{Gal}(\bar{\mathbf{k}}/\mathbf{k}), (\text{Ext}_{\mathcal{C}(\bar{\mathbf{k}})}^0(E, E)^\times)$ is the one-element set, because for any finite-dimensional algebra A over \mathbf{k} we have $H^1(\text{Gal}(\bar{\mathbf{k}}/\mathbf{k}), A^\times) = 0$ (a version of Hilbert 90 theorem, see also section 2.1 in [35]). One can deduce from the above spectral sequence an important corollary: the set of isomorphism classes of objects of $\mathcal{C}(\mathbf{k})$ is in a natural bijection with the set of isomorphism classes of the usual descent data in category $H^\bullet(\mathcal{C}(\bar{\mathbf{k}}))$ endowed with the strict action of $\text{Gal}(\bar{\mathbf{k}}/\mathbf{k})$.

Finally, we will explain how to associate an ind-constructible stack to an ind-constructible category \mathcal{C} over \mathbf{k} . First of all, we can always assume that \mathcal{C} satisfies the following axiom

A4) There exists a decomposition $\mathcal{M} = \sqcup_{i \in I} X_i$ into the countable disjoint union of constructible sets over \mathbf{k} such that any two isomorphic objects of $H^\bullet(\mathcal{C}(\bar{\mathbf{k}}))$ belong to the same part $X_i(\bar{\mathbf{k}})$ for some $i \in I$.

Indeed, if we choose any decomposition $\mathcal{M} = \sqcup_{i \in I} X'_i$ into disjoint union of constructible sets over \mathbf{k} and identify I with the set of natural number $\{1, 2, \dots\}$, then we can shrink $X'_i(\bar{\mathbf{k}})$ to the subset consisting of objects which are not isomorphic to objects from $\cup_{j < i} X'_j(\bar{\mathbf{k}})$.

By axioms A3), A4) we may assume that $Ob(\mathcal{C})(\bar{\mathbf{k}})$ is decomposed into the union of $\bar{\mathbf{k}}$ -points of schemes $(S_i)_{i \in I}$ (as in Axiom A3)) such that any isomorphism class of $\mathcal{C}(\bar{\mathbf{k}})$ belongs only to one of the sets $S_i(\bar{\mathbf{k}})$. Let us call it a locally regular subdivision of $Ob(\mathcal{C})$. Moreover, we can assume that all S_i are smooth and equidimensional, and such that for any $i \in I$ there exists $\delta(i) \in \mathbf{Z}_{\geq 0}$ such that for any $E \in S_i(\bar{\mathbf{k}})$ the subset of objects in $S_i(\bar{\mathbf{k}})$ isomorphic to E has dimension $\delta(i)$. This can be achieved by subdividing each S_i into smaller pieces, and by removing some unnecessary pieces consisting of objects which belong to other pieces. Then taking a generic slice of codimension $\delta(i)$ (and thus shrinking \mathcal{C} to an equivalent subcategory), and taking further subdivisions, one may assume that we have a locally regular subdivision of $Ob(\mathcal{C})$ such that any isomorphism class of objects in $S_i(\bar{\mathbf{k}})$ is finite. Moreover, we may assume that the cardinality c_i of all isomorphism classes in $S_i(\bar{\mathbf{k}})$ depend only on i , and also the dimension d_i of the algebra $\text{Ext}^0(E, E)$ for $E \in S_i(\bar{\mathbf{k}})$ also depends only on i .

For any given $i \in I$ let us consider the constructible set Z_i over \mathbf{k} parametrizing isomorphism classes of objects in $S_i(\bar{\mathbf{k}})$. There is a natural constructible (over \mathbf{k}) bundle of finite-dimensional unital associative algebras \mathcal{A} , with the fiber \mathcal{A}_x over any full collection $x = (E_1, \dots, E_{c_i})$ (up to permutation) of different isomorphic objects equal to

$$\oplus_{1 \leq j_1, j_2 \leq c_i} \text{Ext}^0(E_{j_1}, E_{j_2}).$$

The above algebra is Morita equivalent to $\text{Ext}^0(E_j, E_j)$ for every $j \leq c_i$, and in fact is isomorphic to the matrix algebra

$$\mathcal{A}_x \simeq \text{Mat}(c_i \times c_i, \text{Ext}^0(E_j, E_j)) \quad \forall j \leq c_i.$$

Informally speaking, the “stack” of objects from S_i is the stack of projective modules M over algebra \mathcal{A}_x for some $x \in Z_i(\bar{\mathbf{k}})$ which are isomorphic after Morita equivalence to a free module of rank one over $\text{Ext}^0(E_j, E_j)$ where E_j is some representative of the equivalence class x , i.e. M is isomorphic to the standard module $(\text{Ext}^0(E_j, E_j))^{\oplus c_i}$ over the matrix algebra for every $j \leq c_i$. We see that M has dimension

$$N_i := c_i d_i$$

over $\bar{\mathbf{k}}$. This leads to the following construction. Define a constructible set Y_i over \mathbf{k} to be the set of pairs (x, f) where $x \in Z_i(\bar{\mathbf{k}})$ is a point and f is a homomorphism of A_x to the algebra of matrices $\text{Mat}(N_i \times N_i, \bar{\mathbf{k}})$ such that the resulting structure of \mathcal{A}_x -module on $\bar{\mathbf{k}}^{N_i}$ belongs to the isomorphism class of projective \mathcal{A}_x -modules discussed above. The group $GL(N_i, \bar{\mathbf{k}})$ acts naturally on Y_i by changing the basis in the standard coordinate space $\bar{\mathbf{k}}^{N_i}$. The quotient set is naturally identified with $Z_i(\bar{\mathbf{k}})$, and the stabilizer of every point is isomorphic to $\text{Ext}^0(E_j, E_j)^\times$ in the above notation.

The essential element of the presented construction is that everything is equivariant with respect to the action of $\text{Gal}(\bar{\mathbf{k}}/\mathbf{k})$. Hence, we come to the conclusion that one associates (making many choices) with an ind-constructible category \mathcal{C} over \mathbf{k} a countable collection of varieties $(Y_i)_{i \in I}$ (we can assume that Y_i are not just constructible sets but varieties after making further subdivisions) endowed with algebraic actions of affine algebraic groups $GL(N_i)$ such that the groupoid of isomorphism classes of $\mathcal{C}(\bar{\mathbf{k}})$ is naturally equivalent to the groupoid of the disjoint union of sets $Y_i(\bar{\mathbf{k}})$ endowed with $GL(N_i, \bar{\mathbf{k}})$ -actions. If we replace \mathcal{C} by an equivalent ind-constructible category, or make different choices in the construction, we obtain an equivalent in an obvious sense “ind-constructible stack”. We will discuss ind-constructible stacks later, in Section 4.2.

Moreover, using the fact that the first Galois cohomology with coefficients in $GL(N_i)$ vanish, one can see that the same is true for $\mathcal{C}(\mathbf{k})$ (and replace $Y_i(\bar{\mathbf{k}})$, $GL(N_i, \bar{\mathbf{k}})$ -actions by $Y_i(\mathbf{k})$ and $GL(N_i, \mathbf{k})$ -action $\forall i \in I$). In general, for any field \mathbf{k}' , $\mathbf{k} \subset \mathbf{k}' \subset \bar{\mathbf{k}}$ one can define the descent data for \mathbf{k}' and a \mathbf{k}' -linear A_∞ -category $\mathcal{C}(\mathbf{k}')$ (which is triangulated if \mathcal{C} is triangulated in the ind-constructible sense). The groupoid of isomorphism classes of objects of $H^\bullet(\mathcal{C}(\mathbf{k}'))$ is equivalent to the groupoid of the disjoint union of sets $Y_i(\mathbf{k}')$ endowed with $GL(N_i, \mathbf{k}')$ -actions. More generally, one can define the category $\mathcal{C}(\mathbf{k}')$ for *any* field extension $\mathbf{k}' \supset \mathbf{k}$, not necessarily an algebraic one. In the case $\mathbf{k}' = \bar{\mathbf{k}}$ we get a non-fatal crash of notations, because the A_∞ -category $\mathcal{C}(\bar{\mathbf{k}})$ in last sense is equivalent to the previously defined $\mathcal{C}(\bar{\mathbf{k}})$.

In what follows, we will assume for convenience that $Ob(\mathcal{C})$ for an ind-constructible category \mathcal{C} is described by schemes Y_i with $GL(N_i)$ -actions. In particular, for any extension $\mathbf{k}' \supset \mathbf{k}$ we will have a bijection

$$\text{Iso}(\mathcal{C}(\mathbf{k}')) \simeq \sqcup_{i \in I} Y_i(\mathbf{k}') / GL(N_i, \mathbf{k}')$$

between the set of isomorphism classes in $\mathcal{C}(\mathbf{k}')$ and the set of orbits.

Remark 16 *In fact, objects of an A_∞ -category form not a stack but a higher stack, i.e. one should speak about isomorphisms between isomorphisms etc. Passing to the level of ordinary stacks we make a truncation. Presumably, for a proper treatment of ind-constructible categories and problems like non-functoriality of cones, one should introduce higher constructible stacks. Looking on the guiding example of identification data for finite non-empty collections, one can guess an appropriate notion of a higher constructible stack. Namely, it should be a simplicial constructible set X_\bullet which satisfies a constructible version of the Kan property (i.e. there exists a constructible lifting from horns to simplices) and such that*

1) *for any $k \geq 2$ the constructible map $(\partial_0, \dots, \partial_k) : X_k \rightarrow (X_{k-1})^{k+1}$ is a constructible vector bundle over its image (i.e. there exists a constructible identification of non-empty fibers of this map with vector spaces).*

2) $\exists k_0$ *such that $\forall k \geq k_0$ the above map is an inclusion.*

The reason for the first property is that in the case of identification on each step (except first two) we have to solve linear equations. The second property comes from the property $\mathcal{HOM}_{|X_i \times X_i}^n = 0$ for $n \ll 0$ in our axiomatics of ind-constructible categories.

3.3 Ind-constructible Calabi-Yau categories and potentials

Let \mathbf{k} be a field of characteristic zero. Recall that a Calabi-Yau category of dimension d is a weakly unital \mathbf{k} -linear triangulated A_∞ -category \mathcal{C} (see [41], [42], [65]), such that for any two objects E, F the \mathbf{Z} -graded vector space $\mathrm{Hom}^\bullet(E, F) = \bigoplus_{n \in \mathbf{Z}} \mathrm{Hom}^n(E, F)$ is finite-dimensional (hence the space $\mathrm{Ext}^\bullet(E, F)$ is also finite-dimensional) and moreover:

1) We are given a non-degenerate pairing

$$(\bullet, \bullet) : \mathrm{Hom}^\bullet(E, F) \otimes \mathrm{Hom}^\bullet(F, E) \rightarrow \mathbf{k}[-d] ,$$

which is symmetric with respect to interchange of objects E and F ;

2) For any $N \geq 2$ and a sequence of objects E_1, E_2, \dots, E_N we are given a polylinear $\mathbf{Z}/N\mathbf{Z}$ -invariant map

$$W_N : \bigotimes_{1 \leq i \leq N} (\mathrm{Hom}^\bullet(E_i, E_{i+1})[1]) \rightarrow \mathbf{k}[3 - d] ,$$

where $[1]$ means the shift in the category of \mathbf{Z} -graded vector spaces, and we set $E_{N+1} = E_1$;

3) We have:

$$W_N(a_1, \dots, a_N) = (m_{N-1}(a_1, \dots, a_{N-1}), a_N) ,$$

where $m_n : \otimes_{1 \leq i \leq n} \text{Hom}^\bullet(E_i, E_{i+1}) \rightarrow \text{Hom}^\bullet(E_1, E_{n+1})[2-n]$ are higher composition maps.

The collection $(W_N)_{N \geq 2}$ is called the *potential of \mathcal{C}* . If $d = 3$ then for any object $E \in \text{Ob}(\mathcal{C})$ we define a formal series W_E^{tot} at $0 \in \text{Hom}^\bullet(E, E)[1]$ by the formula:

$$W_E^{tot}(\alpha) = \sum_{n \geq 2} \frac{W_n(\alpha, \dots, \alpha)}{n} .$$

We call W_E^{tot} the *total (or full) potential of the object E* . We call the *potential of E* the restriction of W_E^{tot} to the subspace $\text{Hom}^1(E, E)$. We will denote it by W_E .

The notion of a Calabi-Yau category admits a natural generalization to the ind-constructible case (the pairing is required to be a morphism of constructible vector bundles). It follows from the Axiom A3) that there exists a decomposition of $\text{Ob}(\mathcal{C}) \simeq \sqcup S_i$ into the disjoint union of schemes such that all Taylor components W_N of the potential are symmetrizations of regular sections of cyclic powers of algebraic vector bundles on schemes S_i . Therefore we can treat the family of potentials $W_{\mathcal{C}} = (W_E)_{E \in \text{Ob}(\mathcal{C})}$ as a function, which is regular with respect to the variable E and formal in the direction $\alpha \in \text{Hom}^1(E, E)$ (or $\alpha \in \text{Ext}^1(E, E)$ if our category is minimal on the diagonal). Also the potential W_E considered as a function of $E \in S_j$ becomes a section of the pro-algebraic vector bundle $\prod_{n \geq 2} \text{Sym}^n(\mathcal{HOM}_{|_{\text{Diag}(S_j) \subset S_j \times S_j}}^1)^*$, where *Diag* denotes here the diagonal embedding.

Proposition 7 *In the case of 3d Calabi-Yau category \mathcal{C} consisting of one object E the potential W_E admits (after a formal change of coordinates) a splitting:*

$$W_E = W_E^{min} \oplus Q_E \oplus N_E ,$$

where W_E^{min} is the potential of the minimal model \mathcal{C}^{min} (i.e. it is a formal series on $\text{Ext}^1(E, E)$), the quadratic form Q_E is defined on the vector space $\text{Hom}^1(E, E)/\text{Ker}(m_1 : \text{Hom}^1(E, E) \rightarrow \text{Hom}^2(E, E))$ by the formula $Q_E(\alpha, \alpha) = \frac{m_2(\alpha, \alpha)}{2}$, and N_E is the zero function on the image of the map $m_1 : \text{Hom}^0(E, E) \rightarrow \text{Hom}^1(E, E)$. In the above splitting formula we use the notation $(f \oplus g)(x, y) = f(x) + g(y)$ for the direct sum of formal functions f and g .

The above Proposition follows from the minimal model theorem for Calabi-Yau algebras (i.e. Calabi-Yau categories with only one object). In its formulation below we are going to use the language of formal non-commutative geometry from [42]. We assume that the ground field has characteristic zero.

Theorem 5 *a) Let (X, x_0, ω, d_X) be a $\mathbf{Z}/2\mathbf{Z}$ -graded non-commutative formal pointed manifold (X, x_0) endowed with an odd symplectic form ω and homological vector field d_X which preserves ω and vanishes at x_0 . Then it is isomorphic to the product*

$$(X', x'_0, \omega', d_{X'}) \times (X'', x''_0, \omega'', d_{X''}) ,$$

where $(X', x'_0, \omega', d_{X'})$ is minimal in the sense that $(\text{Lie}_{d_{X'}})|_{T_{x'_0}X'} = 0$ (i.e. $d_{X'}$ vanishes quadratically at x'_0), and the second factor satisfies the following property: there exists a finite-dimensional super vector space V endowed with an even non-degenerate quadratic form Q_V such that $(X'', x''_0, \omega'', d_{X''})$ is isomorphic to the non-commutative formal pointed manifold associated with $V \oplus \Pi V^*$ (here Π is the change of parity functor) endowed with a constant symplectic form ω_V coming from the natural pairing between V and ΠV^* , and homological vector field d_V is the Hamiltonian vector field associated with the pull-back of Q_V under the natural projection $V \oplus \Pi V^* \rightarrow V$.

b) In the \mathbf{Z} -graded case when X corresponds to a 3d Calabi-Yau algebra (i.e. ω has degree -1) a similar statement holds. In this case V is \mathbf{Z} -graded vector space, Q_V has degree 0, and the tangent space $T_{x''_0}X''$ isomorphic to $V \oplus V^[-1]$.*

Proof. One can prove part a) similarly to the usual minimal model theorem for A_∞ -algebras or L_∞ -algebras (it is induction by the order of the Taylor expansion, see e.g. [41]). Part b) is a \mathbf{G}_m -equivariant version of part a). ■

The Proposition follows from part b) of the Theorem, since we have a decomposition $\text{Hom}^1(E, E) \simeq \text{Ext}^1(E, E) \oplus V^0 \oplus (V^1)^*$ where $V^i, i \in \mathbf{Z}$ are the graded components of V . The restriction of W_E^{tot} to $\text{Hom}^1(E, E)$ is the direct sum of W_E^{min} , the restriction of Q_V to V^0 (we identify Q_V with Q_E) and the zero function on $(V^1)^*$.

Corollary 2 *The minimal model potential W_E^{min} does not depend on a choice of minimal model for $\text{End}^\bullet(E)$, if considered up to a formal non-linear automorphism of the bundle \mathcal{HOM}^1 restricted to the diagonal $\text{Diag}(S_j) \subset S_j \times S_j$.*

Proof. Change of the minimal model is a \mathbf{Z} -graded change of coordinates. It preserves the topological ideal generated by all coordinates of non-zero degrees. ■

We remark that there is a notion of Calabi-Yau category valid over a field \mathbf{k} of arbitrary characteristic. In the case of a category with one object E let us denote by A the A_∞ -algebra $\mathrm{Hom}^\bullet(E, E)$. We assume that $\mathrm{Ext}^\bullet(E, E) = H^\bullet(A)$ is finite-dimensional. Then a Calabi-Yau structure of dimension d on A is given by a functional Tr of degree $-d$ on the cyclic homology $HC_\bullet(A)$ such that the induced functional on $H^\bullet(A)/[H^\bullet(A), H^\bullet(A)]$ gives rise to a non-degenerate bilinear form $(a, b) \mapsto Tr(ab)$, where $a, b \in H^\bullet(A)$.

In the case of positive characteristic the notion of the potential does not exist in the conventional sense. This can be seen in the example $A = \mathbf{F}_3\langle\xi\rangle/(\xi^4)$, $\deg \xi = +1$. The potential should have the form $W(\xi) = \xi^3/3 + \dots$ which does not make sense over the field \mathbf{F}_3 .

In general it seems that although the potential does not exist, its differential is well-defined as a closed 1-form.

Remark 17 *In the case of characteristic zero the cyclic homology $HC_\bullet(A)$ can be identified with the cohomology of the complex $\bigoplus_{n \geq 1} \mathrm{Cycl}^n(A[1])$ of cyclically invariant tensors (see [42]). Therefore the potential W becomes a functional of degree $3 - d$ on the latter complex, vanishing on the image of the differential. Hence it defines a class $[W]$ in $(HC_\bullet(A))^*$. The latter space is a $\mathbf{k}[[u]]$ -module, where u is a variable, $\deg u = +2$ (see loc. cit.). The class $[W]$ is related to the functional Tr discussed above by the formula $[W] = uTr$. In the case of a Calabi-Yau algebra of dimension $d = 2k + 1$ it is natural to introduce a cyclic functional W_k with the corresponding class $[W_k] = u^k Tr$. It can be thought of as a higher-dimensional Chern-Simons action. In particular, it defines a formal power series W_k^0 of degree zero such that it vanishes with the first k derivatives on the formal scheme of solutions to the Maurer-Cartan equation.*

3.4 Topology on the space of stability structures

Let \mathcal{C} be an ind-constructible weakly unital A_∞ -category over a field \mathbf{k} of arbitrary characteristic. Let $\mathrm{cl} : \mathrm{Ob}(\mathcal{C}) \rightarrow \Gamma \simeq \mathbf{Z}^n$ be a map of ind-constructible sets (where Γ is considered as a countable set of points) such that the induced map $\mathrm{Ob}(\mathcal{C})(\bar{\mathbf{k}}) \rightarrow \Gamma$ factorizes through a group homomorphism $\mathrm{cl}_{\bar{\mathbf{k}}} : K_0(\mathcal{C}(\bar{\mathbf{k}})) \rightarrow \Gamma$. It is easy to see that for any field extension

$\mathbf{k}' \supset \mathbf{k}$ we obtain a homomorphism $\text{cl}_{\overline{\mathbf{k}'}} : K_0(\mathcal{C}(\overline{\mathbf{k}'})) \rightarrow \Gamma$.

In the case when \mathcal{C} is a Calabi-Yau category we require that Γ is endowed with an integer-valued bilinear form $\langle \bullet, \bullet \rangle$ and the homomorphism $\text{cl}_{\overline{\mathbf{k}}}$ is compatible with $\langle \bullet, \bullet \rangle$ and the Euler form on $K_0(\mathcal{C}(\overline{\mathbf{k}}))$.

For ind-constructible triangulated A_∞ -categories the notion of stability structure admits the following version.

Definition 9 *A constructible stability structure on (\mathcal{C}, cl) is given by the following data (cf. Introduction, Section 1.2):*

- an ind-constructible subset

$$\mathcal{C}^{ss} \subset \text{Ob}(\mathcal{C})$$

consisting of objects called semistable, and satisfying the condition that with each object it contains all isomorphic objects,

- an additive map $Z : \Gamma \rightarrow \mathbf{C}$ called central charge, such that $Z(E) := Z(\text{cl}(E)) \neq 0$ if $E \in \mathcal{C}^{ss}$,
- a choice of the branch of logarithm $\text{Log } Z(E) \in \mathbf{C}$ for any $E \in \mathcal{C}^{ss}$ which is constructible as a function of E .

These data are required to satisfy the corresponding axioms from Section 1.2 for the category $\mathcal{C}(\overline{\mathbf{k}})$.

In particular

- *the set of $E \in \mathcal{C}^{ss}(\overline{\mathbf{k}}) \subset \text{Ob}(\mathcal{C})(\overline{\mathbf{k}})$ with the fixed $\text{cl}(E) \in \Gamma \setminus \{0\}$ and fixed $\text{Log } Z(E)$ is a constructible set.*

Before we proceed with the topology let us make a comparison with the ‘‘Lie-algebraic’’ story of Section 2. First, we observe that the set \mathcal{C}^{ss} can be thought of as an analog of the collection of elements $(a(\gamma))_{\gamma \in \Gamma \setminus \{0\}}$ from Section 2.1. Then we give the following definition of another data and axioms which is equivalent to the one given above and can be thought of as an analog of the collection of the group elements A_V .

Definition 10 *A constructible stability structure on (\mathcal{C}, cl) is given by the following data:*

- an additive map $Z : \Gamma \rightarrow \mathbf{C}$,

- for any bounded connected set $I \subset \mathbf{R}$ an ind-constructible subset $\mathcal{P}(I) \subset \text{Ob}(\mathcal{C})(\overline{\mathbf{k}})$ which contains which every object all isomorphic objects.

These data are required to satisfy the following axioms¹²:

- the zero object of the category $\mathcal{C}(\overline{\mathbf{k}})$ belongs to all $\mathcal{P}(I)$,
- $\cup_{n \in \mathbf{Z}_{>0}} \mathcal{P}([-n, n]) = \text{Ob}(\mathcal{C})(\overline{\mathbf{k}})$,
- if $I_1 < I_2$ in the sense that every element of I_1 is strictly less than any element of I_2 then for any $E_1 \in \mathcal{P}(I_1)$ and $E_2 \in \mathcal{P}(I_2)$ one has $\text{Ext}^{\leq 0}(E_2, E_1) = 0$,
- $\mathcal{P}(I + 1) = \mathcal{P}(I)[1]$ where $[1]$ is the shift functor in $\mathcal{C}(\overline{\mathbf{k}})$,
- (Extension Property) If $I = I_1 \sqcup I_2$ and $I_1 < I_2$ in the above sense then the ind-constructible set $\mathcal{P}(I)$ is isomorphic to the ind-constructible subset of such objects $E \in \text{Ob}(\mathcal{C})(\overline{\mathbf{k}})$ which are extensions $E_2 \rightarrow E \rightarrow E_1$ with $E_m \in \mathcal{P}(I_m)$, $m = 1, 2$,
- if I is an interval of the length strictly less than one, $E \in \mathcal{P}(I)$, $E \neq 0$, then $Z(\text{cl}(E))$ belongs to the strict sector

$$V_I = \{z = re^{\pi i \varphi} \in \mathbf{C}^* | r > 0, \varphi \in I\},$$

- there is a non-degenerate quadratic form Q on $\Gamma_{\mathbf{R}}$ such that $Q|_{\text{Ker } Z} < 0$, and for any interval I of the length strictly less than 1 the set

$$\{\text{cl}(E) \in \Gamma | E \in \mathcal{P}(I)\} \subset \Gamma$$

belongs to the convex cone $C(V_I, Z, Q)$ defined in Section 2.2,

- let I be an interval of the length strictly less than 1, and $\gamma \in \Gamma$. Then the set $\{E \in \mathcal{P}(I) | \text{cl}(E) = \gamma\}$ is constructible.

The equivalence of Definitions 9,10 can be proved similarly to the proof of Theorem 2.

With this equivalent description of a constructible stability condition we observe that the collection of sets $\mathcal{P}(I)$ considered for all intervals I with

¹²One should read expressions $\text{Ob}(\mathcal{C})$, $\mathcal{P}(I)$ etc. in this list of axioms as sets of $\overline{\mathbf{k}}$ -points.

the length less than 1 are analogous to the collection of elements A_V where $V = V_I$ (see Section 2.2) and the Extension Property is analogous to the Factorization Property.

One has the following result.

Proposition 8 *For any constructible stability structure on \mathcal{C} and any field extension $\mathbf{k} \subset \mathbf{k}'$ the category $\mathcal{C}(\mathbf{k}')$ carries a locally finite stability structure in the sense of [9] with the central charge given by $Z \circ \text{cl}_{\mathbf{k}'}$ and the collection of additive subcategories $\mathcal{P}(I)(\mathbf{k}')$, where I runs through the set of bounded connected subsets of \mathbf{R} as above.*

Proof. The proof is straightforward. Local finiteness in the sense of [9] follows from our (stronger) assumption on the quadratic form Q . ■

Let us denote by $\text{Stab}(\mathcal{C}, \text{cl})$ the set of constructible stability structures on \mathcal{C} with a fixed class map cl . Our goal is to introduce a topology on $\text{Stab}(\mathcal{C}, \text{cl})$.

Let $\Delta \subset \mathbf{C}$ be a triangle with one vertex at the origin. We choose a branch of the function $z \mapsto \text{Log } z$ for $z \in \Delta$. We denote the corresponding argument function by $\text{Arg}(z)$. We denote by $\mathcal{C}_{\Delta, \text{Log}}$ an A_∞ -subcategory of \mathcal{C} generated by the zero object $\mathbf{0}$, semistable objects E with $Z(E) \in \Delta$, $\text{Arg}(E) \in \text{Arg}(\Delta)$ as well as extensions J of such objects satisfying the condition $Z(J) \in \Delta$. We allow the case $\Delta = V$ where V is a sector, in which case we will use the notation $\mathcal{C}_{V, \text{Log}}$. It is easy to see that $\mathcal{C}_{\Delta, \text{Log}}$ is an ind-constructible category. Notice that in the language of ind-constructible sets $\mathcal{P}(I)$ we have $\text{Ob}(\mathcal{C}_{V_I, \text{Log}}) = \mathcal{P}(I)$ for some choice of the branch Log . The condition of genericity of the sector V_I corresponding to a closed interval $I = [a, b]$ of the length less than 1 corresponds to the following genericity condition of the set $\mathcal{P}(I)$: both $\mathcal{P}(\{a\})$ and $\mathcal{P}(\{b\})$ are zero categories (equivalently $\mathcal{P}([a, b]) = \mathcal{P}((a, b))$).

Let us fix a non-degenerate quadratic form Q on $\Gamma_{\mathbf{R}}$ such that $Q|_{\text{Ker } Z} < 0$ and $Q(\text{cl}(E)) > 0$ for any $E \in \mathcal{C}^{ss}$. We introduce the topology on $\text{Stab}(\mathcal{C}, \text{cl})$ in the following way. Let us consider a family $\sigma_x = (Z_x, \mathcal{C}_x^{ss}, \dots)$, $x \in X$ of stability structures in a neighborhood of $x_0 \in X$. Then for every point x , a generic closed interval $I = [a, b]$ of the length less than 1 we have the full category $\mathcal{P}(I)_x \subset \text{Ob}(\mathcal{C})$. For a given $\gamma \in \Gamma \setminus \{0\}$ we denote by $\mathcal{P}(I)_{x, \gamma}$ the constructible subset of objects $E \in \mathcal{P}(I)_x$ such that $\text{cl}(E) = \gamma$.

We say that a family $\sigma_x = (Z_x, \mathcal{C}_x^{ss}, \dots)$, $x \in X$ of stability structures parametrized by a topological space X is *continuous at a given stability structure* $\sigma_{x_0} = (Z_0, \mathcal{C}_0^{ss}, \dots)$ if:

- a) The map $x \rightarrow Z_x$ is continuous at $x = x_0$.
- b) There exists a neighborhood U of x_0 such that for any $E \in \mathcal{C}_x^{ss}, x \in U$ we have $Q(\text{cl}(E)) \geq 0$.
- c) For any generic closed interval I of the length strictly less than 1 the constructible set $\mathcal{P}(I)_{x,\gamma}$ is locally constant near x_0 (cf. Definition 3c)).

In this way we obtain a Hausdorff topology on $Stab(\mathcal{C}, \text{cl})$. We can define a parallel transport along a path σ_t in the space $Stab(\mathcal{C}, \text{cl})$ similarly to the case of stability structures in graded Lie algebras discussed in Section 2. Each time when we cross the wall of first kind we use the above property c) in order to “recalculate” the set of semistable objects. In order to do this we use the following property: $E \in \text{Ob}(\mathcal{C}_{\Delta, \text{Log}})$ is semistable iff there is no extension $E_2 \rightarrow E \rightarrow E_1$ where $E_i, i = 1, 2$ are non-zero objects of $\mathcal{C}_{\Delta, \text{Log}}$ such that $\text{Arg}(E_2) > \text{Arg}(E_1)$. These considerations also ensure that the Theorem 1 from Introduction holds (i.e. the natural projection of the space of stability conditions to the space of central charges is a local homeomorphism).

4 Motivic functions and motivic Milnor fiber

4.1 Recollection on motivic functions

Recall (see [14]) that for any constructible set X over \mathbf{k} one can define an abelian group $Mot(X)$ of motivic functions as the group generated by symbols $[\pi : S \rightarrow X] := [S \rightarrow X]$ where π is a morphism of constructible sets, subject to the relations

$$[(S_1 \sqcup S_2) \rightarrow X] = [S_1 \rightarrow X] + [S_2 \rightarrow X] .$$

For any constructible morphism $f : X \rightarrow Y$ we have two homomorphisms of groups:

- 1) $f_! : Mot(X) \rightarrow Mot(Y)$, such that $[\pi : S \rightarrow X] \mapsto [f \circ \pi : S \rightarrow Y]$;
- 2) $f^* : Mot(Y) \rightarrow Mot(X)$, such that $[S' \rightarrow Y] \mapsto [S' \times_Y X \rightarrow X]$.

Moreover, $Mot(X)$ is a commutative ring via the fiber product operation. We denote by $\mathbb{L} \in Mot(\text{Spec}(\mathbf{k}))$ the element $[\mathbf{A}_{\mathbf{k}}^1] := [\mathbf{A}_{\mathbf{k}}^1 \rightarrow \text{Spec}(\mathbf{k})]$. It is customary to add its formal inverse \mathbb{L}^{-1} to the ring $Mot(\text{Spec}(\mathbf{k}))$ (or more generally to the ring $Mot(X)$ which is a $Mot(\text{Spec}(\mathbf{k}))$ -algebra).

There are several “realizations” of the theory of motivic functions which we are going to recall below.

(i) There is a homomorphism of rings

$$\chi : Mot(X) \rightarrow Constr(X, \mathbf{Z}) ,$$

where $Constr(X, \mathbf{Z})$ is the ring of integer-valued constructible functions on X endowed with the pointwise multiplication. More precisely, the element $[\pi : Y \rightarrow X]$ is mapped into $\chi(\pi)$, where $\chi(\pi)(x) = \chi(\pi^{-1}(x))$, which is the Euler characteristic of the fiber $\pi^{-1}(x)$.

(ii) Let now X be a scheme of finite type over a field \mathbf{k} , and $l \neq char \mathbf{k}$ be a prime number. There is a homomorphism of rings

$$Mot(X) \rightarrow K_0(D_{\text{constr}}^b(X, \mathbf{Q}_l)) ,$$

where $D_{\text{constr}}^b(X, \mathbf{Q}_l)$ is the bounded derived category of étale l -adic sheaves on X with constructible cohomology, such that

$$[\pi : S \rightarrow X] \mapsto \pi_!(\mathbf{Q}_l) ,$$

which is the direct image of the constant sheaf \mathbf{Q}_l . Notice that $D_{\text{constr}}^b(X, \mathbf{Q}_l)$ is a tensor category, hence Grothendieck group K_0 is naturally a ring. The homomorphisms $f_!$ and f^* discussed above correspond to the functors $f_!$ (direct image with compact support) and f^* (pullback), which we will denote by the same symbols. We will also use the notation $\int_X \phi := f_!(\phi)$ for the canonical map $f : X \rightarrow Spec(\mathbf{k})$.

(iii) In the special case $X = Spec(\mathbf{k})$ the above homomorphism becomes a map

$$[S] \mapsto \sum_i (-1)^i [H_c^i(S \times_{Spec(\mathbf{k})} Spec(\bar{\mathbf{k}}), \mathbf{Q}_l)] \in K_0(\text{Gal}(\bar{\mathbf{k}}/\mathbf{k}) - mod_{\mathbf{Q}_l}) ,$$

where $\text{Gal}(\bar{\mathbf{k}}/\mathbf{k}) - mod_{\mathbf{Q}_l}$ is the tensor category of finite-dimensional continuous l -adic representations of the Galois group $\text{Gal}(\bar{\mathbf{k}}/\mathbf{k})$, and we take the étale cohomology of S with compact support.

(iv) If $\mathbf{k} = \mathbf{F}_q$ is a finite field then for any $n \geq 1$ we have a homomorphism

$$Mot(X) \rightarrow \mathbf{Z}^{X(\mathbf{F}_{q^n})}$$

given by

$$[\pi : Y \rightarrow X] \mapsto (x \mapsto \#\{y \in X(\mathbf{F}_{q^n}) \mid \pi(y) = x\}) .$$

Operations $f^!, f_*$ correspond to pullbacks and pushforwards of functions on finite sets.

(v) If $\mathbf{k} \subset \mathbf{C}$ then the category of l -adic constructible sheaves on a scheme of finite type X can be replaced in the above considerations by the Saito's category of mixed Hodge modules (see [61]).

(vi) In the case $X = \text{Spec}(\mathbf{k})$ one has two additional homomorphisms:

a) Serre polynomial

$$\text{Mot}(\text{Spec}(\mathbf{k})) \rightarrow \mathbf{Z}[q^{1/2}]$$

such that

$$[Y] \mapsto \sum_i (-1)^i \sum_{w \in \mathbf{Z}_{\geq 0}} \dim H_c^{i,w}(Y) q^{w/2},$$

where $H_c^{i,w}(Y)$ is the weight w component in the i -th Weil cohomology group with compact support.

b) If $\text{char } \mathbf{k} = 0$ then we have the Hodge polynomial

$$\text{Mot}(\text{Spec}(\mathbf{k})) \rightarrow \mathbf{Z}[z_1, z_2]$$

such that

$$[Y] \mapsto \sum_{i \geq 0} (-1)^i \sum_{p, q \geq 0} \dim \text{Gr}_F^p(\text{Gr}_{p+q}^W H_{DR,c}^i(Y)) z_1^p z_2^q,$$

where Gr_{\bullet}^W and Gr_F^{\bullet} denote the graded components with respect to the weight and Hodge filtrations, and $H_{DR,c}^i$ denotes the de Rham cohomology with compact support.

Clearly the Hodge polynomial determines the Serre polynomial via the homomorphism $\mathbf{Z}[z_1, z_2] \rightarrow \mathbf{Z}[q^{1/2}]$ such that $z_i \mapsto q^{1/2}, i = 1, 2$.

4.2 Motivic functions in the equivariant setting

Here we give a short exposition of the generalization of the theory of motivic functions in the equivariant setting (essentially due to Joyce [35], here we use slightly different terms).

Let X be a constructible set over a field \mathbf{k} and G be an affine algebraic group acting on X , in the sense that $G(\overline{\mathbf{k}})$ acts on $X(\overline{\mathbf{k}})$ and there exists a G -variety S over \mathbf{k} with a constructible equivariant identification $X(\overline{\mathbf{k}}) \simeq S(\overline{\mathbf{k}})$.

We define the group $\text{Mot}^G(X)$ of G -equivariant motivic functions as abelian group generated by G -equivariant constructible maps $[Y \rightarrow X]$ modulo the relations

- $[(Y_1 \sqcup Y_2) \rightarrow X] = [Y_1 \rightarrow X] + [Y_2 \rightarrow X]$,
- $[Y_2 \rightarrow X] = [(Y_1 \times \mathbf{A}_{\mathbf{k}}^d) \rightarrow X]$ if $Y_2 \rightarrow Y_1$ is a G -equivariant constructible vector bundle of rank d .

This group form a commutative ring via the fiber product, and a morphism of constructible sets with group actions induces a pullback homomorphism of corresponding rings. There is no natural operation of a pushforward for equivariant motivic functions, and for that one has to enlarge ring of functions.

Consider the following 2-category of constructible stacks. First, its objects are pairs (X, G) as above¹³, and the objects of the category of 1-morphisms $\text{Hom}((X_1, G_1), (X_2, G_2))$ are pairs (Z, f) where Z is a $G_1 \times G_2$ -constructible set such that $\{e\} \times G_2$ acts freely on Z in such a way that we have the induced G_1 -equivariant isomorphism $Z/G_2 \simeq X_1$, and $f : Z \rightarrow X_2$ is a $G_1 \times G_2$ -equivariant map (G_1 acts trivially on X_2). An element of $\text{Hom}((X_1, G_1), (X_2, G_2))$ defines a map of sets $X_1(\bar{\mathbf{k}})/G_1(\bar{\mathbf{k}}) \rightarrow X_2(\bar{\mathbf{k}})/G_2(\bar{\mathbf{k}})$. Furthermore, elements of $\text{Hom}((X_1, G_1), (X_2, G_2))$ form naturally objects of a groupoid, so we obtain a 2-category $\mathcal{SCON}_{\mathbf{k}}$ of *constructible stacks over \mathbf{k}* . The 2-category of constructible stacks carries a direct sum operation induced by the disjoint union of stacks

$$(X_1, G_1) \sqcup (X_2, G_2) = ((X_1 \times G_2 \sqcup X_2 \times G_1), G_1 \times G_2),$$

as well as the product induced by the Cartesian product

$$(X_1, G_1) \times (X_2, G_2) = (X_1 \times X_2, G_1 \times G_2).$$

The abelian group of stack motivic function $\text{Mot}_{st}((X, G))$ is generated by the group of isomorphism classes of 1-morphisms of stacks $[(Y, H) \rightarrow (X, G)]$ with the fixed target (X, G) , subject to the relations

- $[(Y_1, G_1) \sqcup (Y_2, G_2) \rightarrow (X, G)] = [(Y_1, G_1) \rightarrow (X, G)] + [(Y_2, G_2) \rightarrow (X, G)]$
- $[(Y_2, G_1) \rightarrow (X, G)] = [(Y_1 \times \mathbf{A}_{\mathbf{k}}^d, G_1) \rightarrow (X, G)]$ if $Y_2 \rightarrow Y_1$ is a G_1 -equivariant constructible vector bundle of rank d .

¹³Strictly speaking, we should denote such stacks as triples (X, G, α) where α is the action of G on X .

The ring $Mot^G(X)$ maps to $Mot_{st}((X, G))$. Notice that every isomorphism class $[(Y, H) \rightarrow (X, G)]$ corresponds to an ordinary morphism of constructible sets acted by algebraic groups. Indeed, in the notation of the definition of 1-morphism of stacks with $(X_1, G_1) := (Y, H)$ and $(X_2, G_2) := (X, G)$, we can replace the source (Y, H) by an equivalent stack $(Z, G_1 \times G_2)$ and get an ordinary morphism $(Z, G_1 \times G_2) \rightarrow (X, G)$ of constructible sets acted by algebraic groups. One can define the pullback, the pushforward and the product of elements of $Mot_{st}((X, G))$.

Finally, for a constructible stack $\mathcal{S} = (X, G)$ we define its class in the ring $K_0(Var_{\mathbf{k}})[[\mathbb{L}]^{-1}, ([GL(n)]^{-1})_{n \geq 1}]$ as

$$[\mathcal{S}] = \frac{[(X \times GL(n))/G]}{[GL(n)]},$$

where we have chosen an embedding $G \rightarrow GL(n)$ for some $n \geq 1$, and $(X \times GL(n))/G$ is the ordinary quotient by the diagonal free action (thus in the RHS we have the quotient of motives of ordinary varieties). The result does not depend on the choice of embedding (see [3], Lemma 2.3). Then we define the integral $\int_{\mathcal{S}} : Mot_{st}(\mathcal{S}) \rightarrow K_0(Var_{\mathbf{k}})[[\mathbb{L}]^{-1}, ([GL(n)]^{-1})_{n \geq 1}]$ as $\int_{\mathcal{S}}[\mathcal{S}' \rightarrow \mathcal{S}] = [\mathcal{S}']$.

If \mathbf{k} is finite, one can associate with every constructible stack $\mathcal{S} = (X, G)$ a finite set $\mathcal{S}(\mathbf{k})$, the set of orbits of $GL(n, \mathbf{k})$ acting on $((X \times GL(n))/G)(\mathbf{k})$. There is a homomorphism of the algebra of stack motivic functions to the algebra of \mathbf{Q} -valued functions on $\mathcal{S}(\mathbf{k})$. The identity function

$$\mathbf{1}_{(X, G)} := [(X, G) \rightarrow (X, G)]$$

represented by the identity map, when interpreted as a measure (for pushforwards) maps to the “stack counting measure” on $\mathcal{S}(\mathbf{k})$ which is equal to $\#(GL(n, \mathbf{k}))^{-1}$ times the direct image of the ordinary counting measure¹⁴ on $((X \times GL(n))/G)(\mathbf{k})$. Its density with respect to the ordinary counting measure on $\mathcal{S}(\mathbf{k})$ is given by the inverse to the order of the stabilizer.

Our construction in Section 3.2 can be rephrased as a construction of an ind-constructible stack of objects. Hence we can speak about stack motivic functions on an ind-constructible category.

¹⁴For every finite set S the counting measure on S has weight 1 for each element $s \in S$.

4.3 Motivic Milnor fiber

Let M be a complex manifold, $x_0 \in M$. Recall, that for a germ f of an analytic function at x_0 such that $f(x_0) = 0$ one can define its *Milnor fiber* $MF_{x_0}(f)$, which is a locally trivial C^∞ -bundle over S^1 of manifolds with the boundary (defined only up to a diffeomorphism):

$$\{z \in M \mid \text{dist}(z, x_0) \leq \varepsilon_1, |f(z)| = \varepsilon_2\} \rightarrow S^1 = \mathbf{R}/2\pi\mathbf{Z} ,$$

where $z \mapsto \text{Arg } f(z)$. Here dist is any smooth metric on M near x_0 , and there exists a constant $C = C(f, \text{dist})$ and a positive integer $N = N(f)$ such that for $0 < \varepsilon_1 \leq C$ and $0 < \varepsilon_2 < \varepsilon_1^N$ the C^∞ type of the bundle is the same for all $\varepsilon_1, \varepsilon_2, \text{dist}$.

In particular, taking the cohomology of the fibers we obtain a well-defined local system on S^1 .

There are several algebro-geometric versions of this construction (theories of nearby cycles). They produce analogs of local systems on S^1 , for example l -adic representations of the group $\text{Gal}(\mathbf{k}((t))^{\text{sep}}/\mathbf{k}((t)))$ where $l \neq \text{char } \mathbf{k}$.

There is a convenient model of the Milnor fiber in non-archimedean geometry. In order to describe it we note that the field $K = \mathbf{k}((t))$ is a non-archimedean field endowed with the (standard) valuation, and with the norm given by $|a| = c^{\text{val}(a)}$ for a given constant $c \in (0, 1)$. Let

$$f \in \mathbf{k}[[x_1, \dots, x_n]], f(0) = 0$$

be a formal series considered as an element of $K[[x_1, \dots, x_n]]$. Clearly it is convergent in the non-archimedean sense in the domain $U \subset (\mathbf{A}_{\mathbf{k}}^n)^{\text{an}}$ defined by inequalities $|x_i| < 1, 1 \leq i \leq n$. The non-archimedean analog of the Milnor fiber is given (at the level of points) by the fibration

$$\begin{aligned} \{x = (x_1, \dots, x_n) \in U \mid \max_i |x_i| \leq \varepsilon_1, 0 < |f(x)| \leq \varepsilon_2\} \rightarrow \\ \rightarrow \{w \in (\mathbf{A}_{\mathbf{k}}^1)^{\text{an}} \mid 0 < |w| \leq \varepsilon_2\} , \end{aligned}$$

where $\varepsilon_1, \varepsilon_2$ are positive numbers as above (cf. [54]).

Ideally, we would like to have the following picture. Let $V \rightarrow X$ be a vector bundle over a scheme of finite type X/\mathbf{k} , and

$$f \in \prod_{n \geq 1} \Gamma(X, \text{Sym}^n(V^*))$$

be a function on the formal completion of zero section of $\text{tot}(V)$ vanishing on X . We would like to associate with such data a motivic Milnor fiber

$$MF(f) \in \text{Mot}(X \times \mathbf{G}_m) .$$

Here the factor \mathbf{G}_m replaces the circle S^1 in the analytic picture. Moreover, the motivic function $MF(f)$ should be “unramified” in \mathbf{G}_m -direction (i.e. it should correspond to a \mathbf{G}_m -invariant stratification of $X \times \mathbf{G}_m$). In the case $\mathbf{k} = \mathbf{C}$, assuming that f is convergent near zero section, the value $MF_x(f)$ at a point $(x, \epsilon) \in X(\mathbf{C}) \times \mathbf{C}^*$ should be thought of as a representative of the alternating sum

$$\sum_i (-1)^i [H^i(f^{-1}(\epsilon) \cap B_{0,x})]$$

where $\epsilon \neq 0$ is a sufficiently small complex number and $B_{0,x}$ is a small open ball around 0 in the fiber V_x . Notice that here we use the usual cohomology and not the one with compact support.

Also, we can consider the case when X is a constructible set and $V \rightarrow X$ is a constructible vector bundle. We say that $f \in \prod_{n \geq 1} \Gamma(X, \text{Sym}^n(V^*))$ is constructible if (X, V, f) is constructibly isomorphic to an algebraic family of formal functions over a scheme of finite type as above.

This goal was achieved by Denef and Loeser (see [14]) in the case $\text{char } \mathbf{k} = 0$ by using motivic integration and resolution of singularities. In this case the group $\text{Mot}(X \times \mathbf{G}_m)$ is replaced by $\text{Mot}^\mu(X)$, where $\mu = \varprojlim_n \mu_n$ and μ acts trivially on X . Here μ_n is the group of n -th roots of 1 in \mathbf{k} . We will always assume that μ -action is “good” in the sense that μ acts via a finite quotient μ_n and every its orbit is contained in an affine open subscheme. Notice that there is a homomorphism of groups

$$\text{Mot}^\mu(X) \rightarrow \text{Mot}(X \times \mathbf{G}_m), \quad [\pi : Y \rightarrow X] \mapsto [\pi_1 : (Y \times \mathbf{G}_m)/\mu_n \rightarrow X \times \mathbf{G}_m] ,$$

where μ acts on Y via its quotient μ_n and $\pi_1(y, t) = \pi(y)t^n$.

As we work with constructible sets, it is sufficient to define the motivic Milnor fiber not for a family, but for an individual formal germ of a function. Let M be a smooth formal scheme over \mathbf{k} with closed point x_0 and f be a function on M vanishing at x_0 (e.g. M could be the formal completion at 0 of a fiber of vector bundle $V \rightarrow X$ in the above notation). We assume that f is not identically equal to zero near x_0 , otherwise the Milnor fiber would be empty. Let us choose a simple normal crossing resolution of singularities $\pi : M' \rightarrow M$ of the hypersurface in M given by the equation $f = 0$ with

exceptional divisors $D_j, j \in J$. The explicit formula for the motivic Milnor fiber from [14] looks such as follows¹⁵.

$$MF_{x_0}(f) = \sum_{I \subset J, I \neq \emptyset} (1 - \mathbb{L})^{\#I-1} [\widetilde{D_I^0} \cap \pi^{-1}(x_0)] \in Mot^\mu(Spec(\mathbf{k})) ,$$

where $D_I = \bigcap_{j \in I} D_j$, D_I^0 is the complement in D_I to the union of all other exceptional divisors, and $\widetilde{D_I^0} \rightarrow D_I^0$ is a certain Galois cover with the Galois group μ_{m_I} , where m_I is the g.c.d. of the multiplicities of all divisors $D_i, i \in I$ (see [14] for the details). Informally speaking, the fiber of the cover $\widetilde{D_I^0} \rightarrow D_I^0$ is the set of connected components of a non-zero level set of function $f \circ \pi$ near a point of D_I^0 .

The space $Mot^\mu(X)$ carries an associative product introduced by Looijenga (see [45]) which is different from the one defined above. It is essential for the motivic Thom-Sebastiani theorem which will be discussed later. Let us sketch a construction of this product.

First, let us introduce the commutative ring $Mot(X \times \mathbf{A}_k^1)_{conv}$ which coincides as an abelian group with $Mot(X \times \mathbf{A}_k^1)$ but carries the ‘‘convolution product’’

$$[f_1 : S_1 \rightarrow X \times \mathbf{A}_k^1] \circ [f_2 : S_2 \rightarrow X \times \mathbf{A}_k^1] = [f_1 \oplus f_2 : S_1 \times_X S_2 \rightarrow X \times \mathbf{A}_k^1],$$

where $(f_1 \oplus f_2)(s_1, s_2) = (pr_X(f_1(s_1)), pr_{\mathbf{A}_k^1}(f_1(s_1)) + pr_{\mathbf{A}_k^1}(f_2(s_2)))$. The ring $Mot(X \times \mathbf{A}_k^1)_{conv}$ contains the ideal

$$\mathbf{I} := pr_X^*(Mot(X)) .$$

By definition we have an epimorphism of abelian groups $Mot(X) \rightarrow \mathbf{I}$. Let

$$i : X \rightarrow X \times \mathbf{A}_k^1, x \mapsto (x, 0), \quad j : X \times (\mathbf{A}_k^1 \setminus \{0\}) \rightarrow X \times \mathbf{A}_k^1$$

be natural embeddings. They give rise to an isomorphism of abelian groups

$$i^* \oplus j^* : Mot(X \times \mathbf{A}_k^1) \simeq Mot(X) \oplus Mot(X \times (\mathbf{A}_k^1 \setminus \{0\})) .$$

Since $i^* \circ pr_X^* = id_{Mot(X)}$ we see that the restriction of pr_X^* to \mathbf{I} gives an isomorphism of abelian groups

$$\mathbf{I} \simeq Mot(X) ,$$

¹⁵In [14] it was assumed that f is a regular function on a smooth scheme, but the formula and all the arguments work in the formal setting as well.

and j^* induces the isomorphism of groups

$$\text{Mot}(X \times \mathbf{A}_{\mathbf{k}}^1)/\mathbf{I} \simeq \text{Mot}(X \times (\mathbf{A}_{\mathbf{k}}^1 \setminus \{0\})) .$$

Using the latter isomorphism we transfer the convolution product and endow $\text{Mot}(X \times (\mathbf{A}_{\mathbf{k}}^1 \setminus \{0\}))$ with an associative product which we will call *exotic*.

Recall that we have a homomorphism of groups

$$\text{Mot}^\mu(X) \rightarrow \text{Mot}(X \times \mathbf{G}_m) = \text{Mot}(X \times (\mathbf{A}_{\mathbf{k}}^1 \setminus \{0\})) .$$

One can check that the image of $\text{Mot}^\mu(X)$ is closed under the exotic product. Intuitively, the image consists of isotrivial families of varieties over $X \times (\mathbf{A}_{\mathbf{k}}^1 \setminus \{0\})$ equipped with a flat connection which has finite (i.e. belonging to some μ_n) monodromy. The complicated formula from [45] coincides with the induced product on $\text{Mot}^\mu(X)$. In what follows we will use the notation

$$\mathcal{M}^\mu(X) := (\text{Mot}^\mu(X), \text{exotic product}) .$$

Let $V \rightarrow X, V' \rightarrow Y$ be two constructible vector bundles endowed with constructible families f, g of formal power series. We denote by $f \oplus g$ the sum of pullbacks of f and g to the constructible vector bundle

$$\text{pr}_X^*V \oplus \text{pr}_Y^*V' \rightarrow X \times Y .$$

Then we have the following motivic version of Thom-Sebastiani theorem.

Theorem 6 ([15]) *One has*

$$(1 - MF(f \oplus g)) = \text{pr}_X^*(1 - MF(f)) \cdot \text{pr}_Y^*(1 - MF(g)) \in \mathcal{M}^\mu(X \times Y) .$$

One can make similar constructions in the equivariant setting. Let X/\mathbf{k} be a constructible set endowed with the good action of an affine algebraic group G . We endow X also with the trivial μ -action. Then, similarly to the above, we can equip $\text{Mot}^{G \times \mu}(X)$ with the exotic product (by considering G -equivariant families over X in the previous considerations). We will denote the resulting ring by $\mathcal{M}^{G, \mu}(X)$. Using the canonical resolution of singularities (see e.g. [5]) one can define the equivariant motivic Milnor fiber in the case of equivariant families of functions, and state the corresponding version of Thom-Sebastiani theorem.

In the case of arbitrary \mathbf{k} there is an l -adic version of the above results. More precisely, the theory of Milnor fiber is replaced by the theory of nearby

cycles (see [64]), with the convolution defined by Laumon [43]. The Thom-Sebastiani theorem was proved in this case by Pierre Deligne and probably by Lei Fu (both unpublished).

There is an analog of the Hodge polynomial in the story (see [14], 3.1.3). Let us assume that $\mathbf{k} = \mathbf{C}$ for simplicity. Then we have a homomorphism of rings

$$\mathcal{M}^\mu(\text{Spec}(\mathbf{k})) \rightarrow \left\{ \sum_{\alpha, \beta \in \mathbf{Q}, \alpha + \beta \in \mathbf{Z}} c_{\alpha, \beta} z_1^\alpha z_2^\beta \mid c_{\alpha, \beta} \in \mathbf{Z} \right\} .$$

Namely, for a smooth projective μ_n -scheme Y we set

$$\begin{aligned} [Y] \mapsto & \sum_{p, q \geq 0, p, q \in \mathbf{Z}} (-1)^{p+q} \dim H^{p, q, 0}(Y) z_1^p z_2^q + \\ & + \sum_{p, q \geq 0, p, q \in \mathbf{Z}} \sum_{1 \leq i \leq n-1} (-1)^{p+q} \dim H^{p, q, i}(Y) z_1^{p+i/n} z_2^{q+1-i/n} , \end{aligned}$$

where $H^{p, q, i}(Y)$ is the subspace of the cohomology $H^{p, q}(Y)$, where an element $\xi \in \mu_n$ acts by multiplication by ξ^i . The appearance of rational exponents was first time observed in the Hodge spectrum of a complex isolated singularity. Taking $z_1 = z_2 = q^{1/2}$ we obtain the corresponding Serre polynomial.

4.4 An integral identity

In this section we are going to discuss the identity which will be crucial in the proof of the main theorem of Section 6.

Let \mathbf{k} be a field of characteristic zero, and V_1, V_2, V_3 be finite-dimensional \mathbf{k} -vector spaces.

Conjecture 4 *Let W be a formal series on the product $V_1 \times V_2 \times V_3$ of three vector spaces, depending in a constructible way on finitely many extra parameters, such that $W(0, 0, 0) = 0$ and W has degree zero with respect to the diagonal action of the multiplicative group \mathbf{G}_m with the weights $(1, -1, 0)$. We denote by \widehat{W} the \mathbf{G}_m -equivariant extension of W to the formal neighborhood \widehat{V}_1 of $V_1 \times \{0\} \times \{0\} \subset V_1 \times V_2 \times V_3$. Then we have the following formula (where we denote the direct image by the integral):*

$$\int_{v_1 \in V_1} (1 - MF_{(v_1, 0, 0)}(\widehat{W})) = \mathbb{L}^{\dim V_1} (1 - MF_{(0, 0, 0)}(W|_{(0, 0, V_3)})) ,$$

where in the RHS we consider the motivic Milnor fiber at $(0, 0, 0)$ of the restriction of W to the subspace $(0, 0, V_3)$.

Using the obvious equality $\int_{V_1} 1 = \mathbb{L}^{\dim V_1}$ we can rewrite the identity as

$$\int_{v_1 \in V_1} MF_{(v_1, 0, 0)}(\widehat{W}) = \mathbb{L}^{\dim V_1} \cdot MF_{(0, 0, 0)}(W|_{(0, 0, V_3)}) .$$

Let us discuss the l -adic version of the Conjecture. For simplicity we assume that the vector spaces do not depend on extra parameters. Then we have a morphism of formal schemes $\pi : \widehat{V}_1 \rightarrow \text{Spf}(\mathbf{k}[[w]])$ such that $w \mapsto \widehat{W}$ as well as an embedding $i_{V_1} : V_1 \rightarrow \widehat{W}^{-1}(0)$. For any morphism $\pi : X \rightarrow \text{Spf}(\mathbf{k}[[w]])$ of formal schemes we denote by $\mathbf{R}\psi_\pi$ the functor of nearby cycles. It acts from the bounded derived category of l -adic constructible sheaves on X to the bounded derived category of l -adic constructible sheaves on $X_0 = \pi^{-1}(0)$ endowed with the action of the inertia group (hence they can be informally thought of as l -adic constructible sheaves on $X_0 \times_{\text{Spec}(\mathbf{k})} \mathbf{k}((w))$).

Proposition 9 *The complex $\mathbf{R}\Gamma_c(i_{V_1}^* \mathbf{R}\psi_\pi(\mathbf{Q}_l))$ is isomorphic (as a complex of $\text{Gal}(\overline{\mathbf{k}((w))}/\mathbf{k}((w)))$ -modules) to the complex $\mathbf{R}\Gamma_c(V_1, \mathbf{Q}_l) \otimes j^* \mathbf{R}\psi_{\widehat{\pi}}(\mathbf{Q}_l)$, where $j : \text{Spec}(\mathbf{k}) \rightarrow V_3, j(0) = 0$ is the natural embedding and $\widehat{\pi}$ is the morphism of the formal completion of $0 \in \{0\} \times \{0\} \times V_3$ to $\text{Spf}(\mathbf{k}[[w]])$ given by the restriction $W|_{\{0\} \times \{0\} \times V_3}$.*

Proof. We will give a sketch of the proof based on the non-archimedean model for the Milnor fiber described in Section 4.3.

Let us consider the $\mathbf{k}((t))$ -analytic space (in the sense of Berkovich) associated with the scheme $(V_1 \times V_2 \times V_3) \times_{\text{Spec}(\mathbf{k})} \text{Spec}(\mathbf{k}((t)))$. Let us choose sufficiently small positive numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ (we will specify them later) and define an analytic subspace $Y = Y_{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4}$ by the following inequalities:

$$|v_1| \leq 1 + \varepsilon_1, \quad |v_2|, |v_3| \leq \varepsilon_2, \quad \varepsilon_4 \leq |W(v_1, v_2, v_3)| \leq \varepsilon_3 .$$

Notice that the series $W(v_1, v_2, v_3)$ is convergent on Y because of homogeneity property. We introduce another analytic space $Y' = Y'_{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4} \subset Y_{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4}$ by changing the inequality $|v_1| \leq 1 + \varepsilon_1$ to the equality $|v_1| = 1 + \varepsilon_1$ (all other inequalities remain unchanged). There is a natural projection $pr_{Y \rightarrow A}$ (resp. its restriction $pr_{Y' \rightarrow A}$) of Y (resp. of Y') to the annulus $A = \{w \mid \varepsilon_4 \leq |w| \leq \varepsilon_3\}$. Let us now consider the complex

$$\text{Cone}((pr_{Y \rightarrow A})_*(\mathbf{Q}_l) \rightarrow (pr_{Y' \rightarrow A})_*(\mathbf{Q}_l))[-1] .$$

This is a *lisse* sheaf on the annulus, i.e. a continuous l -adic representation of the fundamental group of the $\mathbf{k}((t))$ -analytic space A . There is a tautological embedding $\mathbf{k}((w)) \rightarrow \mathcal{O}^{an}(A)$. It induces the homomorphism of profinite groups $\pi_1(A) \rightarrow \text{Gal}(\overline{\mathbf{k}((w))}/\mathbf{k}((w)))$. Then one can show that the complex

$$\begin{aligned} \text{Cone}((pr_{Y \rightarrow A})_*(\mathbf{Q}_l) \rightarrow (pr_{Y' \rightarrow A})_*(\mathbf{Q}_l))[-1] &\simeq \\ &\simeq (pr_{Y \rightarrow A})_*(\text{Cone}(\mathbf{Q}_l \rightarrow (i_{Y' \hookrightarrow Y})_*\mathbf{Q}_l)) \end{aligned}$$

on A is quasi-isomorphic to the pull-back of the complex $\mathbf{R}\Gamma_c(i_{V_1}^* \mathbf{R}\psi_\pi(\mathbf{Q}_l))$ of $\text{Gal}(\overline{\mathbf{k}((w))}/\mathbf{k}((w)))$ -modules.

We decompose the space Y into a disjoint union $Y_0 \sqcup Y_1$ where for Y_0 we have $|v_1||v_2| = 0$ while for Y_1 we have $|v_1||v_2| \neq 0$. Similarly, we have a decomposition $Y' = Y'_0 \sqcup Y'_1$. We claim that the complex

$$(pr_{Y_0 \rightarrow A})_*(\text{Cone}(\mathbf{Q}_l \rightarrow (i_{Y'_0 \hookrightarrow Y_0})_*\mathbf{Q}_l))$$

is quasi-isomorphic to the pull-back of the complex $\mathbf{R}\Gamma_c(V_1, \mathbf{Q}_l) \otimes j^* \mathbf{R}\psi_{\bar{\pi}}(\mathbf{Q}_l)$ of $\text{Gal}(\overline{\mathbf{k}((w))}/\mathbf{k}((w)))$ -modules. Notice that $W_{|Y_0}$ depends on v_3 only. Furthermore, Y_0 and Y'_0 can be decomposed as the products

$$Y_0 = Y_3 \times Z_0, \quad Y'_0 = Y_3 \times Z'_0,$$

where

$$\begin{aligned} Z_0 &:= \{(v_1, v_2) \in V_1^{an} \times V_2^{an} \mid v_1 = 0 \text{ or } v_2 = 0, |v_1| \leq 1 + \varepsilon_1, |v_2| \leq \varepsilon_2\}, \\ Z'_0 &:= \{(v_1, v_2) \in V_1^{an} \times V_2^{an} \mid v_2 = 0, |v_1| = 1 + \varepsilon_1\}, \\ Y_3 &:= \{v_3 \in V_3^{an} \mid |v_3| \leq \varepsilon_2, \varepsilon_4 \leq |W(0, 0, v_3)| \leq \varepsilon_3\}. \end{aligned}$$

Here we denote by $V_i^{an}, i = 1, 2, 3$ the $\mathbf{k}((t))$ -analytic space associated with the scheme $V_i \times_{\text{Spec}(\mathbf{k})} \text{Spec}(\mathbf{k}((t))), i = 1, 2, 3$. Notice that Z_0 is the bouquet of two (non-archimedean) balls. It follows that the inclusion of the ball

$$Z''_0 = \{(v_1, 0) \mid |v_1| \leq 1 + \varepsilon_1\}$$

into Z_0 induces isomorphisms of the Berkovich étale cohomology groups of the l -adic sheaves on the analytic spaces. Therefore the cohomology groups of the pair (Z_0, Z'_0) coincide with the cohomology groups of the pair (Z''_0, Z'_0) . The latter are equal to $\mathbf{R}\Gamma_c(V_1^{an}, \mathbf{Q}_l)$ (which corresponds to the image of $\mathbb{L}^{\dim V_1}$ in $K_0(\text{Gal}(\overline{\mathbf{k}((t))}/\mathbf{k}((t))) - \text{mod})$).

We have an obvious morphism of complexes of sheaves on Y :

$$f : Cone(\mathbf{Q}_l \rightarrow (i_{Y' \hookrightarrow Y})_* \mathbf{Q}_l) \rightarrow Cone((i_{Y_0 \hookrightarrow Y})_* \mathbf{Q}_l \rightarrow (i_{Y' \hookrightarrow Y})_* \mathbf{Q}_l) .$$

In order to prove the Proposition we have to prove that the $(pr_{Y \rightarrow A})_* f$ is a quasi-isomorphism, i.e. $(pr_{Y \rightarrow A})_*(Cone(f))$ is zero. The compactness of spaces Y, Y', Y_0, Y'_0 implies that

$$(pr_{Y \rightarrow A})_*(Cone(f)) \simeq (pr_{Y_1 \rightarrow A})_!(Cone(\mathbf{Q}_l \rightarrow (i_{Y'_1 \hookrightarrow Y_1})_* \mathbf{Q}_l)) .$$

The (partially defined) actions of the group \mathbf{G}_m on Y_1 and Y'_1 are free, and the value of W does not change under the action. More precisely, one can define easily analytic “spaces of orbits” $\tilde{Y}_1 \supset \tilde{Y}'_1$ of \mathbf{G}_m acting on Y_1 and Y'_1 respectively. The projections

$$Y_1 \rightarrow \tilde{Y}_1, \quad Y'_1 \rightarrow \tilde{Y}'_1$$

are proper maps, and the map W factors through them. Hence it is enough to check that

$$(pr_{Y_1 \rightarrow \tilde{Y}_1})_*(Cone(\mathbf{Q}_l \rightarrow (i_{Y'_1 \hookrightarrow Y_1})_* \mathbf{Q}_l)) \simeq 0 .$$

This follows from the fact that every orbit in Y_1 is a closed annulus, its intersection with Y'_1 is a circle, and the inclusion of a circle into an annulus induces an isomorphism of étale cohomology groups.

This concludes the sketch of the proof. ■

Remark 18 1) In the proof we did not specify the values of $\varepsilon_i, i = 1, 2, 3, 4$. We can take $\varepsilon_4 = O(\varepsilon_3)$ (e.g. take $\varepsilon_4 = \frac{1}{2}\varepsilon_3$), $\varepsilon_2 = o(1)$ and $\varepsilon_3 = O(\varepsilon_2^N), \varepsilon_1 = O(\varepsilon_3^M)$ for some integers $N, M > 0$.

2) In the proof we used the comparison of the cohomology of the sheaf of nearby cycles with the étale cohomology of subvarieties of $\mathbf{k}((t))$ -analytic spaces (see [54]).

We strongly believe that the analog of the Proposition holds at the level of motivic rings in the case $char \mathbf{k} = 0$.

4.5 Equivalence relation on motivic functions

We start with a motivation for this section. There are many examples of pairs of constructible sets (or even varieties) X_1, X_2 over a field \mathbf{k} such that their classes $[X_1]$ and $[X_2]$ in $Mot(Spec(\mathbf{k}))$ are different (or at least not obviously coincide), but X_1 and X_2 coincide in each realization described in Section 4.1 (i)-(vi) (e.g. when X_1, X_2 are isogeneous abelian varieties). In particular we will be interested in the case when $X_l, l = 1, 2$ are affine quadrics given by the equations $\sum_{1 \leq i \leq n} a_{i,l} x_i^2 = 1$, such that they have equal determinants: $\prod_{1 \leq i \leq n} a_{i,1} = \prod_{1 \leq i \leq n} a_{i,2} \in \mathbf{k}^\times$.

Here we propose a modification of the notion of motivic function which is a version of the Grothendieck's approach to the theory of pure motives with numerical equivalence. Let us explain it in the case of $X = Spec(\mathbf{k})$, where \mathbf{k} is a field of characteristic zero (which we will assume throughout this section).

We start with the symmetric monoidal \mathbf{Q} -linear category $\mathcal{M}^{eff}(\mathbf{k})$. Its objects are smooth projective varieties over \mathbf{k} and

$$\mathrm{Hom}_{\mathcal{M}^{eff}(\mathbf{k})}(Y_1, Y_2) = \mathbf{Q} \otimes \mathrm{Im}(Z^{\dim Y_2}(Y_1 \times Y_2) \rightarrow H_{DR}^{2 \dim Y_2}(Y_1 \times Y_2)) ,$$

where $Z^n(X)$ denotes as usual the space of algebraic cycles in X of codimension n , and we take the image of the natural map into the algebraic de Rham cohomology. Then $\mathrm{Hom}_{\mathcal{M}^{eff}(\mathbf{k})}(Y_1, Y_2)$ is a finite-dimensional \mathbf{Q} -vector space. Instead of de Rham cohomology we can use Betti cohomology (for an embedding $\mathbf{k} \hookrightarrow \mathbf{C}$) or l -adic cohomology. Comparison theorems imply that the image of the group of cycles in the cohomology does not depend on a cohomology theory.

Composition of morphisms is given by the usual composition of correspondences, and the tensor product is given by the Cartesian product of varieties. We extend the category $\mathcal{M}^{eff}(\mathbf{k})$ by adding formally finite sums (then it becomes an additive category), and finally taking the Karoubian envelope. The K_0 -ring of the resulting category contains the element $\mathbb{L} = [\mathbf{P}_k^1] - [Spec(\mathbf{k})]$. Adding formally the inverse \mathbb{L}^{-1} we obtain the ring which we denote by $Mot_{coh}(Spec(\mathbf{k}))$. It is an easy corollary of Bittner theorem (see [6]) that the natural map $Mot(Spec(\mathbf{k})) \rightarrow Mot_{coh}(Spec(\mathbf{k}))$ which assigns to a smooth projective variety its class in $Mot_{coh}(Spec(\mathbf{k}))$ is a homomorphism of rings.

The above considerations can be generalized to the case of motives over constructible sets.

Definition 11 Let X be a constructible set over a field \mathbf{k} , $\text{char } \mathbf{k} = 0$. A constructible family of smooth projective varieties over X is represented by a pair consisting of a smooth projective morphism $h : Y \rightarrow X_0$ of schemes of finite type over \mathbf{k} and a constructible isomorphism $j : X_0 \xrightarrow{\text{constr}} X$. Two such representations

$$h : Y \rightarrow X_0, j : X_0 \xrightarrow{\text{constr}} X, \quad h' : Y' \rightarrow X'_0, j' : X'_0 \xrightarrow{\text{constr}} X$$

are identified if we are given constructible isomorphisms $f : Y \xrightarrow{\text{constr}} Y'$, $g : X_0 \xrightarrow{\text{constr}} X'_0$ such that $h' \circ f = g \circ h$, $j' \circ g = j$, and for any point $x \in X(\bar{\mathbf{k}})$ the induced constructible isomorphism between smooth projective varieties $(j \circ h)^{-1}(x)$ and $(j' \circ h')^{-1}(x)$ is an isomorphism of schemes.

For a constructible family of smooth projective varieties over X and a point $x \in X$ there is a well-defined smooth projective variety Y_x over the residue field $\mathbf{k}(x)$ called *the fiber over x* . Moreover, one can define the notion of constructible family of algebraic cycles of the fixed dimension. We say that such a family is homologically equivalent to zero if for any $x \in X$ the corresponding cycles in Y_x map to zero in $H_{DR}^\bullet(Y_x)$. Also, having two constructible families of smooth projective varieties over X one easily defines their product, which is again a constructible family of smooth projective varieties over X . All that allows us to generalize our constructions from the case $X = \text{Spec}(\mathbf{k})$ to the general case. In this way we obtain the ring $\text{Mot}_{\text{coh}}(X)$ as well as the natural homomorphism of rings $\text{Mot}(X) \rightarrow \text{Mot}_{\text{coh}}(X)$.

Definition 12 We say that two elements of $\text{Mot}(X)$ are (cohomologically) equivalent if their images in $\text{Mot}_{\text{coh}}(X)$ coincide.

The set of equivalence classes (in fact the ring) will be denoted by $\overline{\text{Mot}}(X)$. It is isomorphic to the image of $\text{Mot}(X)$ in $\text{Mot}_{\text{coh}}(X)$. In particular, the above-mentioned quadrics define the same element in $\overline{\text{Mot}}(\text{Spec}(\mathbf{k}))$.

Let now X be a constructible set over \mathbf{k} , endowed with an action of an affine algebraic group G . We define an equivalence relation on $\text{Mot}^G(X)$ in the following way. First we choose an embedding $G \hookrightarrow GL(N)$. We say that $f, g \in \text{Mot}^G(X)$ are equivalent if their pull-backs to $\text{Mot}((X \times GL(N))/G)$ have the same image in $\overline{\text{Mot}}(X \times GL(N))/G$. Using the fact that all $GL(N)$ -torsors over a constructible set are trivial it is easy to show that this equivalence relation does not depend on the embedding $G \hookrightarrow GL(N)$.

The ring of equivalence classes is denoted by $\overline{Mot}^G(X)$.¹⁶ Similarly one defines the ring $\overline{Mot}^{G \times \mu}(X)$, where $\mu = \varprojlim_n \mu_n$. The exotic product descends to $\overline{Mot}^{G \times \mu}(X)$. Hence we obtain the ring $\overline{\mathcal{M}}^{G, \mu}(X)$ of equivalence classes as well as homomorphism of rings $\mathcal{M}^{G, \mu}(X) \rightarrow \overline{\mathcal{M}}^{G, \mu}(X)$.

4.6 Numerical realization of motivic functions

This section is not used in further consideration, its goal is only to show that the abstractly defined ring $\overline{\mathcal{M}}^{G, \mu}(X)$ can be realized as certain ring of functions with numerical values.

Let Z be a scheme of finite type over a finite field $\mathbf{k} \simeq \mathbf{F}_q$ endowed with an action of the group μ_n of roots of 1 such that $n < p = \text{char } \mathbf{F}_q$. We choose a prime $l \neq p$. Let us define Y as the quotient $(Z \times (\mathbf{A}_{\mathbf{k}}^1 \setminus \{0\})) / \mu_n$ with respect to the diagonal action of μ_n . Then we have a morphism

$$\pi : Y \rightarrow \mathbf{A}_{\mathbf{k}}^1 \setminus \{0\}, \quad (z, t) \mapsto t^n \quad \forall (z, t) \in (Z \times \mathbf{A}_{\mathbf{k}}^1 \setminus \{0\})(\overline{\mathbf{F}}_q).$$

Let $j : \mathbf{A}_{\mathbf{k}}^1 \setminus \{0\} \rightarrow \mathbf{A}_{\mathbf{k}}^1$ be the natural embedding. We define the number

$$N_Z = \text{Tr}_{Fr}(\mathcal{F}(j_* \pi_!(\mathbf{Q}_{l, Y}))|_{s=1}) \in \overline{\mathbf{Q}}_l,$$

which is the trace of the Frobenius Fr of the fiber at $s = 1$ of the Fourier transform of the l -adic sheaf $j_* \pi_!(\mathbf{Q}_{l, Y})$, where $\mathbf{Q}_{l, Y}$ denotes the constant sheaf \mathbf{Q}_l on Y . In fact the number N_Z can be considered as an element of the cyclotomic field $\mathbf{Q}(\eta_p)$, where η_p is a primitive p -th root of 1:

$$1 + \eta_p + \cdots + \eta_p^{p-1} = 0.$$

We have a canonical non-trivial character $\chi : \mathbf{F}_q \rightarrow \mathbf{Q}(\eta_p)^*$ given by the composition of the trace $\text{Tr}_{\mathbf{F}_q \rightarrow \mathbf{F}_p} : \mathbf{F}_q \rightarrow \mathbf{F}_p$ with the additive character

$$\chi_p : \mathbf{F}_p \rightarrow \mathbf{Q}(\eta_p)^*, \quad m \bmod p \mapsto \eta_p^m.$$

Then

$$N_Z = \sum_{s \in (\mathbf{A}_{\mathbf{k}}^1 \setminus \{0\})(\mathbf{F}_q)} \#(\pi^{-1}(s)(\mathbf{F}_q)) \chi(s).$$

Notice that the last formula makes sense for constructible Z as well.

Let now X be a constructible set over a field \mathbf{k} , $\text{char } \mathbf{k} = 0$, endowed with an action of an affine algebraic group G .

¹⁶In this way we have circumvented the problem of defining the category of G -equivariant motivic sheaves.

Definition 13 We call a model for (X, G) the following choices:

- a finitely generated subring $R \subset \mathbf{k}$,
- a scheme $\mathcal{X} \rightarrow \text{Spec}(R)$ of finite type,
- an affine group scheme $\mathcal{G} \rightarrow \text{Spec}(R)$ together with an embedding $\mathcal{G} \hookrightarrow \text{GL}(N)_R$,
- an action of \mathcal{G} on \mathcal{X} ,
- a constructible identification over \mathbf{k} of $\mathcal{X} \times_{\text{Spec}(R)} \text{Spec}(\mathbf{k})$ with X , as well as an isomorphism of groups $\mathcal{G} \times_{\text{Spec}(R)} \text{Spec}(\mathbf{k}) \simeq G$ over k , compatible with the actions.

Such a model always exists, and models form a filtered system. With a given model for (X, G) we associate a commutative unital ring

$$K(\mathcal{X}) = \varinjlim_{\{\text{open } U \subset (\mathcal{X} \times \text{GL}(n))/\mathcal{G}\}} \prod_{\{\text{closed } x \in U\}} \mathbf{Q}(\eta_{\text{char } \mathbf{k}(x)}) ,$$

where $\mathbf{k}(x)$ is the residue field of x (which is a finite field).

Suppose that we are given a model for (X, G) and let $f \in \mathcal{M}^{G, \mu}(X)$. As always we assume that the μ -action on X is good and factors through the action of some μ_n .

Definition 14 A model for f compatible with the model $(R, \mathcal{X}, \mathcal{G})$ for (X, G) consists of the following data:

- a finite set J , numbers $N_j, d_j, n_j \in \mathbf{Z}$, where $j \in J$, such that all numbers N_j are positive and invertible in $\mathcal{O}(\mathcal{X})$,
- $\mathcal{G} \times \mu_{N_j}$ -equivariant morphisms of constructible sets $\mathcal{Y}_j \rightarrow \mathcal{X}$ given for each $j \in J$, where $\mathcal{Y}_j \rightarrow \mathcal{X}$ are $\mathcal{G} \times \mu_{N_j}$ -schemes of finite type, and we endow \mathcal{X} with the trivial action of the group μ_{N_j} ,

These data are required to satisfy the condition that

$$f = \sum_{j \in J} n_j [\mathcal{Y}_j \times_{\text{Spec}(R)} \text{Spec}(\mathbf{k}) \rightarrow X] \cdot \mathbb{L}^{d_j} .$$

Models for f always exist. Moreover, for any finite collection $(f_i)_{i \in I}$ of elements of $\mathcal{M}^{G,\mu}(X)$ there exists a model for (X, G) with compatible models for $(f_i)_{i \in I}$.

Having a model for f we can associate with it an element $f_{num} \in K(\mathcal{X})$ in the following way. Let $x \in (\mathcal{X} \times GL(N)_R)/\mathcal{G}$ be a closed point. We can apply the considerations of the beginning of this section to the fiber $Z_{j,x}$ over the point x , of the map $(\mathcal{Y}_j \times GL(N)_R)/\mathcal{G} \rightarrow (\mathcal{X} \times GL(N)_R)/\mathcal{G}$, where \mathcal{Y}_j is the scheme from the definition of the model for f . Then for each j we obtain an element $N_{Z_{j,x}} \in \mathbf{Q}(\eta_p)$, $p = \text{char } \mathbf{k}(x)$. Finally, we set

$$f_{num}(x) := \sum_{j \in J} n_j q_x^{d_j} N_{Z_{j,x}} ,$$

where $q_x := \#\mathbf{k}(x)$. Hence we realize f as a function with values in numbers.

5 Orientation data on odd Calabi-Yau categories

Considerations of this section are reminiscent of those in Quantum Field Theory when one tries to define determinants for the Gaussian integral in a free theory.

5.1 Remarks on the motivic Milnor fiber of a quadratic form

Although the theory of motivic Milnor fiber was defined over a field of characteristic zero, an essential part of considerations below has meaning over an arbitrary field \mathbf{k} , $\text{char } \mathbf{k} \neq 2$ if we replace the notion of motivic Milnor fiber by its l -adic version.

Let now V be a \mathbf{k} -vector space endowed with a non-degenerate quadratic form Q . We define an element

$$I(Q) = (1 - MF_0(Q)) \mathbb{L}^{-\frac{1}{2} \dim V} \in \mathcal{M}^\mu(\text{Spec}(\mathbf{k}))[\mathbb{L}^{\pm 1/2}] ,$$

where $\mathbb{L}^{1/2}$ is a formal symbol which satisfies the relation $(\mathbb{L}^{1/2})^2 = \mathbb{L}$, and Q is interpreted as a function on V . Then the motivic Thom-Sebastiani theorem implies that

$$I(Q_1 \oplus Q_2) = I(Q_1)I(Q_2) .$$

Also we have $I(Q) = 1$, if Q is a split form: $Q = \sum_{1 \leq i \leq n} x_i y_i$ for $V = \mathbf{k}^{2n}$. Therefore, we have a homomorphism of groups

$$I : Witt(\mathbf{k}) \rightarrow (\mathcal{M}^\mu(Spec(\mathbf{k}))[\mathbb{L}^{\pm 1/2}])^\times ,$$

where $Witt(\mathbf{k})$ is the Witt group of the field \mathbf{k} . We can think of it as a multiplicative character. Let us denote by $J_2(\mathbf{k}) := J_2(Spec(\mathbf{k}))$ the quotient of the group $\mathbf{Z} \times \mathbf{k}^\times / (\mathbf{k}^\times)^2$ by the subgroup generated by the element $(2, -1)$. There is an obvious homomorphism $Witt(\mathbf{k}) \rightarrow J_2(\mathbf{k})$ given for a quadratic form Q by

$$[Q] \mapsto (\text{rk } Q, \det(Q) \bmod (\mathbf{k}^\times)^2) .$$

Notice that all ‘‘motivic realizations’’ of $I(Q)$ in the sense of Section 4.1 depend only on the image of $[Q]$ in $J_2(\mathbf{k})$. This is similar to the classical formula (for $\mathbf{k} = \mathbf{R}$ and a positive definite form Q)

$$\int_V \exp(-Q(x)) dx = (2\pi)^{-\frac{1}{2} \dim V} (\det(Q))^{-1/2}$$

in the sense that the answer depends on $\dim V$ and $\det(Q)$ only. In particular, the homomorphism of rings

$$\mathcal{M}^\mu(Spec(\mathbf{k})) \rightarrow K_0(D_{\text{constr}}^b(\mathbf{G}_m, \mathbf{Q}_l))$$

(see Section 4.1, (ii), (iii) and Section 4.2) induces (by combining with the above character) a homomorphism of abelian groups

$$Witt(\mathbf{k}) \rightarrow (K_0(D_{\text{constr}}^b(\mathbf{G}_m, \mathbf{Q}_l))[\mathbb{L}^{\pm 1/2}])^\times .$$

It is easy to see that it factors through the homomorphism $Witt(\mathbf{k}) \rightarrow J_2(\mathbf{k})$. For example the element of K_0 -group corresponding to the pair (n, a) , $n \in 2\mathbf{Z} + 1, a \in \mathbf{k}^\times$ is represented by $\mathbb{L}^{-1/2}[F]$ where F is a local system on \mathbf{G}_m associated with the double cover of \mathbf{G}_m given by $y \mapsto y^2 a, y \in \mathbf{G}_m(\bar{\mathbf{k}})$.

Question 1 *The above considerations give rise to the following question. Let us consider the family of quadratic forms $Q_{a_1, a_2}(x, y) = a_1 x_1^2 + a_2 x_2^2$ where $a_1, a_2 \in \mathbf{k}^\times$. Is it true that $[Q_{a_1, a_2}] = [Q_{a'_1, a'_2}]$ in $K_0(Var_{\mathbf{k}})$ as long as $a_1 a_2 = a'_1 a'_2$?*

We expect that the answer to the Question is negative, and this is the main reason for introducing the equivalence relation for the motivic functions in Section 4.5.

The above considerations can be generalized to arbitrary constructible (or ind-constructible) sets. Namely, let X be a constructible set over \mathbf{k} . We define the group $J_2(X)$ as the quotient of the group

$$\text{Constr}(X, \mathbf{Z}) \times (\text{Constr}(X, \mathbf{G}_m) / \text{Constr}(X, \mathbf{G}_m)^2)$$

by the subgroup consisting of the elements $(2f, (-1)^f)$, $f \in \text{Constr}(X, \mathbf{Z})$, where we denote by $\text{Constr}(X, Y)$ the set of constructible maps $X \rightarrow Y$. To a constructible vector bundle $V \rightarrow X$ endowed with a non-degenerate quadratic form $Q = (Q_x)_{x \in X}$ we associate the element

$$I(V, Q) := (1 - MF_0(Q)) \mathbb{L}^{-\frac{1}{2} \dim V} \in (\mathcal{M}^\mu(X) [\mathbb{L}^{\pm 1/2}])^\times .$$

Here we treat each Q_x as a formal power series on the fiber V_x and $\dim V \in \text{Constr}(X, \mathbf{Z})$. One can show that this correspondence gives rise to a homomorphism of groups

$$\bar{I} : J_2(X) \rightarrow (\overline{\mathcal{M}}^\mu(X) [\mathbb{L}^{\pm 1/2}])^\times .$$

This fact has a simple “numerical” counterpart: for the case $\mathbf{k} \simeq \mathbf{F}_q$, two affine quadrics given by equations $Q_1(x) = 0, Q_2(x) = 0$ have the same number of points if Q_1, Q_2 are two non-degenerate quadratic forms of equal rank and determinant.

Let us consider a symmetric monoidal category $sPic_2(X)$ consisting of constructible super line bundles $L \rightarrow X$ endowed with an isomorphism $L^{\otimes 2} \simeq \mathbf{1}_X$, where $\mathbf{1}_X$ is a trivial even line bundle on X . It is easy to see that the group $J_2(X)$ is the group of isomorphism classes of objects of $sPic_2(X)$. If $V \rightarrow X$ is a constructible super vector bundle, $V = V^{even} \oplus \Pi V^{odd}$ then there is a well-defined super line bundle (called super determinant bundle) $\text{sdet}(V) \rightarrow X$ given by

$$\text{sdet}(V) := \Pi^{\dim V^{even} - \dim V^{odd}} (\wedge^{\text{top}} V^{even} \otimes (\wedge^{\text{top}} V^{odd})^*) ,$$

where Π is the parity change functor.

Recall canonical isomorphisms:

1. $\text{sdet}(V^*) \simeq \text{sdet}(\Pi(V)) \simeq (\text{sdet}(V))^*$,
2. $\text{sdet}(\bigoplus_{i \in I} V_i) \simeq \bigotimes_{i \in I} \text{sdet}(V_i)$,
3. if V carries an odd differential d then $\text{sdet}(V) \simeq \text{sdet}(H^\bullet(V, d))$,

4. for an exact triangle of complexes $(V_1^\bullet, d_1) \rightarrow (V_2^\bullet, d_2) \rightarrow (V_3^\bullet, d_3)$ there is a canonical isomorphism

$$\mathrm{sdet}(V_2^\bullet) \simeq \mathrm{sdet}(V_1^\bullet) \otimes \mathrm{sdet}(V_3^\bullet) .$$

In case if V carries a non-degenerate quadratic form $Q = (Q_x)_{x \in X}$ we have a canonical isomorphism $\mathrm{sdet}(V) \simeq (\mathrm{sdet}(V))^*$. Therefore in this case we have a well-defined object of $sPic_2(X)$. Its class in the group $J_2(X)$ is represented by the pair

$$(\dim V, \{\det(Q_x)\}_{x \in X} \bmod (\mathrm{Constr}(X, \mathbf{G}_m))^2) .$$

The above considerations can be generalized to the case when X is acted by an algebraic group G . Then one replaces the category $sPic_2(X)$ by the category $sPic_2(X, G)$ of G -equivariant constructible super line bundles L endowed with a G -equivariant isomorphism $L^{\otimes 2} \simeq \mathbf{1}_X$. The group of isomorphism classes of $sPic_2(X, G)$ we denote by $J_2(X, G)$. In what follows we will often omit the word “equivariant” in the considerations involving the category $sPic_2(X, G)$.

Remark 19 *Let us make an additional assumption that $\sqrt{-1} \in \mathbf{k}$. In this case the quadratic form $x^2 + y^2 = (x + \sqrt{-1}y)(x - \sqrt{-1}y)$ is split. Then $MF_0(x^2 + y^2) = 1 - \mathbb{L}$ and we can consider the element $\mathbb{L}^{1/2} := 1 - MF_0(x^2)$ which enjoys the property $(\mathbb{L}^{1/2})^2 = \mathbb{L}$. Furthermore, the group $J_2(X)$ can be canonically identified with the product*

$$\mathrm{Constr}(X, \mathbf{Z}/2\mathbf{Z}) \times \mathrm{Constr}(X, \mathbf{G}_m) / (\mathrm{Constr}(X, \mathbf{G}_m))^2 .$$

Therefore the isomorphism classes of objects of $sPic_2(X)$ can be identified with pairs (constructible μ_2 -function, constructible μ_2 -torsor).

5.2 Orientation data

Let \mathcal{C} be an ind-constructible \mathbf{k} -linear 3-dimensional Calabi-Yau category.¹⁷ Then we have a natural ind-constructible super line bundle \mathcal{D} over $Ob(\mathcal{C})$ with the fiber over E given by $\mathcal{D}_E = \mathrm{sdet}(\mathrm{Ext}^\bullet(E, E))$. It follows that on

¹⁷There is a notion of $\mathbf{Z}/2\mathbf{Z}$ -graded odd or even Calabi-Yau category, see [42]. Some considerations of this section can be generalized to $\mathbf{Z}/2\mathbf{Z}$ -graded case.

the ind-constructible stack of exact triangles $E_1 \rightarrow E_2 \rightarrow E_3$ we have an isomorphism of the pull-backed line bundles which fiberwise reads as

$$\mathcal{D}_{E_2} \otimes \mathcal{D}_{E_1}^{-1} \otimes \mathcal{D}_{E_3}^{-1} \simeq (\text{sdet}(\text{Ext}^\bullet(E_1, E_3)))^{\otimes 2} .$$

Let us explain this isomorphism. The multiplicativity of superdeterminants on exact triangles gives rise to a canonical isomorphism

$$\begin{aligned} \text{sdet}(\text{Ext}^\bullet(E_2, E_2)) &\simeq \text{sdet}(\text{Ext}^\bullet(E_1, E_1)) \otimes \text{sdet}(\text{Ext}^\bullet(E_1, E_3)) \otimes \\ &\otimes \text{sdet}(\text{Ext}^\bullet(E_3, E_1)) \otimes \text{sdet}(\text{Ext}^\bullet(E_3, E_3)) . \end{aligned}$$

By the Calabi-Yau property we have

$$\text{sdet}(\text{Ext}^\bullet(E_3, E_1)) \simeq \text{sdet}(\Pi(\text{Ext}^\bullet(E_3, E_1)))^* \simeq \text{sdet}(\text{Ext}^\bullet(E_1, E_3))$$

which implies the desired formula. When $Ob(\mathcal{C}) = \sqcup_{i \in I} Y_i$ is a decomposition into the union of $GL(N_i)$ -invariant constructible sets as at the end of 3.2, then the restriction $\mathcal{D}|_{Y_i}$ is a $GL(N_i)$ -equivariant super line bundle and the above isomorphisms are also equivariant.

Definition 15 *Orientation data on \mathcal{C} consists of a choice of an ind-constructible super line bundle $\sqrt{\mathcal{D}}$ on $Ob(\mathcal{C})$ such that its restriction to each $Y_i, i \in I$ is $GL(N_i)$ -equivariant, endowed on each X_i with $GL(N_i)$ -equivariant isomorphisms $(\sqrt{\mathcal{D}})^{\otimes 2} \simeq \mathcal{D}$ and such that for the natural pull-backs to the ind-constructible stack of exact triangles $E_1 \rightarrow E_2 \rightarrow E_3$ we are given equivariant isomorphisms:*

$$\sqrt{\mathcal{D}}_{E_2} \otimes (\sqrt{\mathcal{D}}_{E_1})^{-1} \otimes (\sqrt{\mathcal{D}}_{E_3})^{-1} \simeq \text{sdet}(\text{Ext}^\bullet(E_1, E_3))$$

such that the induced equivariant isomorphism

$$\mathcal{D}_{E_2} \otimes \mathcal{D}_{E_1}^{-1} \otimes \mathcal{D}_{E_3}^{-1} \simeq (\text{sdet}(\text{Ext}^\bullet(E_1, E_3)))^{\otimes 2}$$

coincides with the one which we have a priori.

We define the group $J_2(\mathcal{C}) := \prod_{i \in I} J_2(Y_i, GL(N_i))$. We have a canonical equivariant super line bundle $\mathcal{D}_{\leq 1}$ whose fiber at $E \in Ob(\mathcal{C})$ is

$$\mathcal{D}_{\leq 1, E} := \text{sdet}(\tau_{\leq 1}(\text{Ext}^\bullet(E, E))) ,$$

where $\tau_{\leq i}, i \in \mathbf{Z}$ denotes the standard truncation functor. It is easy to see that we have an equivariant isomorphism $\mathcal{D}_{\leq 1}^{\otimes 2} \simeq \mathcal{D}$. Then on the space of exact triangles $E_1 \rightarrow E_2 \rightarrow E_3$ we have an equivariant isomorphism of super line bundles fiberwise given by

$$(\mathcal{D}_{\leq 1, E_2} \otimes \mathcal{D}_{\leq 1, E_1}^{-1} \otimes \mathcal{D}_{\leq 1, E_3}^{-1})^{\otimes 2} \simeq (\text{sdet}(\text{Ext}^\bullet(E_1, E_3)))^{\otimes 2}.$$

Let now $\text{Fun}(\mathcal{C}_3, \mathcal{C})$ be the ind-constructible category of A_∞ -functors from the the category \mathcal{C}_3 considered in 3.1. Its objects can be thought of as exact triangles

$$E_1 \rightarrow E_2 \rightarrow E_3 = \text{Cone}(E_1 \rightarrow E_2)$$

in \mathcal{C} . There are three functors $\text{Funct}(\mathcal{C}_3, \mathcal{C}) \rightarrow \mathcal{C}$ which associate to an exact triangle $E_1 \rightarrow E_2 \rightarrow E_3$ the objects E_1, E_2, E_3 respectively. These functors induce three homomorphisms

$$\phi_i : J_2(\mathcal{C}) \rightarrow J_2(\text{Funct}(\mathcal{C}_3, \mathcal{C})), i = 1, 2, 3 .$$

The super line bundle L with the fiber

$$L_{E_1 \rightarrow E_2 \rightarrow E_3} = (\mathcal{D}_{\leq 1, E_2} \otimes \mathcal{D}_{\leq 1, E_1}^{-1} \otimes \mathcal{D}_{\leq 1, E_3}^{-1}) \otimes (\text{sdet}(\text{Ext}^\bullet(E_1, E_3)))^{-1}$$

defines an element $l \in J_2(\text{Funct}(\mathcal{A}_2, \mathcal{C}))$, since $L^{\otimes 2} \simeq \mathbf{1}_{\text{Funct}(\mathcal{C}_3, \mathcal{C})}$. Then a choice of orientation data on \mathcal{C} is equivalent to a choice of $h \in J_2(\mathcal{C})$ such that $-\phi_1(h) + \phi_2(h) - \phi_3(h) = l$. Indeed a choice of orientation data gives rise to a super line bundle $\sqrt{\mathcal{D}}$ such that $\sqrt{\mathcal{D}}^{\otimes 2} \simeq \mathcal{D}_{\leq 1}^{\otimes 2}$. Therefore the quotient $h = \sqrt{\mathcal{D}} \otimes \mathcal{D}_{\leq 1}^{-1}$ defines an element in $J_2(\mathcal{C})$, and the condition for the tensor squares of the super line bundles $\sqrt{\mathcal{D}}_{E_i}, i = 1, 2, 3$ on the space of exact triangles is equivalent to the equation $-\phi_1(h) + \phi_2(h) - \phi_3(h) = l$.

Remark 20 *All the above considerations admit a straightforward generalization to the case of Calabi-Yau category of arbitrary odd dimension d . In the case $d = 1 \pmod{4}$ we have canonical orientation data given by*

$$\sqrt{\mathcal{D}}_\varepsilon := \text{sdet}(\tau_{\leq \frac{d-1}{2}}(\text{Ext}^\bullet(E, E))) .$$

This is due to the observation that in the explicit description of the analog of the obstruction element l defined above in terms of a super vector bundle endowed with a symmetric bilinear form, the super vector bundle turns out to be purely odd, hence the bilinear form is split. It follows that the obstruction element is trivial. In the case $d = 3$ considered in this paper the obstruction does not have to vanish.

5.3 Orientation data from a splitting of bifunctors

Let \mathcal{C} be a triangulated ind-constructive category over a field \mathbf{k} . We will assume that all functors, bifunctors etc. respect this structure. In this section we are going to discuss a special framework in which orientation data is easy to construct.

Let $F : \mathcal{C} \times \mathcal{C}^{op} \rightarrow Perf(Spec(\mathbf{k}))$ be a biadditive bifunctor and d be an integer. We define the *dual bifunctor of degree d* as a bifunctor $F^\vee = F^{\vee, d}$ given by

$$F^\vee(E_2, E_1) := F(E_1, E_2)^*[-d] .$$

Clearly $F \mapsto F^\vee$ is an involution.

Definition 16 *A self-duality structure on F of degree d is an isomorphism $F \rightarrow F^\vee$ of bifunctors such that for any two objects E_1, E_2 the induced non-degenerate pairing*

$$F(E_1, E_2) \otimes F(E_1, E_2) \rightarrow \mathbf{k}[-d]$$

is symmetric on the level of cohomology $H^\bullet(F(E_1, E_2))$. If F is endowed with a self-duality structure of degree d then we call it self-dual.

For a Calabi-Yau category of dimension d the bifunctor $(E_1, E_2) \mapsto \text{Hom}^\bullet(E_1, E_2)$ is self-dual.

For any self-dual bifunctor F of odd degree d we can repeat considerations of Section 5.2. Namely, we define an ind-constructible super line bundle \mathcal{D}^F with the fiber $\mathcal{D}_E^F := \text{sdet}(F(E, E))$. Then for any exact triangle $E_1 \rightarrow E_2 \rightarrow E_3$ we have a canonical isomorphism

$$\sqrt{\mathcal{D}_{E_2}^F} \otimes (\sqrt{\mathcal{D}_{E_1}^F})^{-1} \otimes (\sqrt{\mathcal{D}_{E_3}^F})^{-1} \simeq \text{sdet}(F(E_1, E_3))^{\otimes 2}.$$

Then one can ask the same question: *is there an ind-constructible super line bundle $\sqrt{\mathcal{D}^F}$ which is compatible with the above isomorphism in the sense of Definition 15?*

The answer is positive for any bifunctor of the form $F \simeq H \oplus H^\vee$, with the obvious self-duality structure. In this case we set

$$\sqrt{\mathcal{D}_E^F} := \text{sdet}(H(E, E)) .$$

More generally we can use an \mathbf{A}^1 -homotopy in this special case. More precisely, suppose we are given a bifunctor¹⁸ $G : \mathcal{C} \times \mathcal{C}^{op} \rightarrow Perf(\mathbf{A}_k^1)$. It can be thought of as a family $G_t : \mathcal{C} \times \mathcal{C}^{op} \rightarrow Perf(Spec(\mathbf{k}))$ of bifunctors, parametrized by $t \in \mathbf{A}^1(\mathbf{k})$, namely $G_t = i_t^* \circ G$, where $i_t : Spec(\mathbf{k}) \rightarrow \mathbf{A}^1$ is the embedding corresponding to t . Since the category $Perf(\mathbf{A}_k^1)$ has an obvious duality functor (taking dual to a complex of vector bundles) then the definition of self-duality structure extends naturally to families. Suppose that we have a family of self-dual bifunctors $G_t, t \in \mathbf{A}^1(\mathbf{k})$ such that $G_0 \simeq \text{Hom}^\bullet(\bullet, \bullet)$ and $G_1 \simeq H \oplus H^\vee$ for some bifunctor H , and the isomorphisms preserve the self-duality structures. Then we have a canonical orientation data on \mathcal{C} , since any super line bundle over \mathbf{A}_k^1 is trivial and all fibers are canonically isomorphic.

6 Motivic Donaldson-Thomas invariants

6.1 Motivic Hall algebra and stability data

In this section the field \mathbf{k} can have arbitrary characteristic.

Let \mathcal{C} be an ind-constructible triangulated A_∞ -category over a field \mathbf{k} . We are going to describe a motivic generalization of the derived Hall algebras from [70].

As usual, we have a constructible countable decomposition $Ob(\mathcal{C}) = \sqcup_{i \in I} Y_i$ with group $GL(N_i)$ acting on Y_i . Let us consider a $Mot(Spec(\mathbf{k}))$ -module $\oplus_i Mot_{st}(Y_i, GL(N_i))$ (see section 4.2) and extend it by adding negative powers $\mathbb{L}^n, n < 0$ of the motive of the affine line \mathbb{L} . We denote the resulting $Mot(Spec(\mathbf{k}))$ -module by $H(\mathcal{C})$. We understand elements of $H(\mathcal{C})$ as *measures* (and not as functions), because in the definition of the product we will use the pushforward maps.

We would like to make $H(\mathcal{C})$ into an associative algebra, called the *motivic Hall algebra*. We need some preparations for that. First we observe that if $[\pi_i : Z_i \rightarrow Ob(\mathcal{C})], i = 1, 2$ are two elements¹⁹ of $H(\mathcal{C})$ then one has a

¹⁸In fact we would like to say that G is “ind-constructible” in some sense. A sufficient, but not necessary condition would be the existence of an ind-constructible functor G' from $\mathcal{C} \times \mathcal{C}^{op}$ to $Perf(\mathbf{P}_k^1)$ such that G is isomorphic to the composition of G' and the restriction functor $Perf(\mathbf{P}_k^1) \rightarrow Perf(\mathbf{A}_k^1)$.

¹⁹Here we consider for simplicity the case when the groups acting on Z_1, Z_2 are trivial, the generalization to the case of non-trivial groups is straightforward.

constructible set $tot((\pi_1 \times \pi_2)^*(\mathcal{E}\mathcal{X}\mathcal{T}^1))$ which is the total space of the pull-back of the ind-constructible bundle $\mathcal{E}\mathcal{X}\mathcal{T}^1$ over $Ob(\mathcal{C}) \times Ob(\mathcal{C})$. Then the map $Cone$ (see Section 3.1) after the shift [1] maps the total space to $Ob(\mathcal{C})$.

For any $N \in \mathbf{Z}$ we introduce the ‘‘truncated’’ Euler characteristic

$$(E, F)_{\leq N} := \sum_{i \leq N} (-1)^i \dim \text{Ext}^i(E, F) .$$

In the future we will use the notation $(E, F)_i$ for $\dim \text{Ext}^i(E, F)$, hence $(E, F)_{\leq N} = \sum_{i \leq N} (-1)^i (E, F)_i$.

With the pair $[\pi_i : Y_i \rightarrow X_i], i = 1, 2$ as above we can associate a collection of constructible sets

$$W_n = \{(y_1, y_2, \alpha) \mid y_i \in Y_i, \alpha \in \text{Ext}^1(\pi_2(y_2), \pi_1(y_1)), (\pi_2(y_2), \pi_1(y_1))_{\leq 0} = n\} ,$$

where $n \in \mathbf{Z}$ is arbitrary. Clearly

$$[tot((\pi_1 \times \pi_2)^*(\mathcal{E}\mathcal{X}\mathcal{T}^1)) \rightarrow Ob(\mathcal{C})] = \sum_{n \in \mathbf{Z}} [W_n \rightarrow Ob(\mathcal{C})] .$$

We define the product

$$[Y_1 \rightarrow Ob(\mathcal{C})] \cdot [Y_2 \rightarrow Ob(\mathcal{C})] = \sum_{n \in \mathbf{Z}} [W_n \rightarrow Ob(\mathcal{C})] \mathbb{L}^{-n} ,$$

where the map $W_n \rightarrow Ob(\mathcal{C})$ is given by the formula

$$(y_1, y_2, \alpha) \mapsto Cone(\alpha : \pi_2(y_2)[-1] \rightarrow \pi_1(y_1)) .$$

Proposition 10 *The above formula makes $H(\mathcal{C})$ into an associative algebra.*

Proof. We are going to prove the result for the ‘‘delta functions’’

$$\nu_E = [pt \rightarrow Ob(\mathcal{C})], pt \mapsto E ,$$

where E is an object of $\mathcal{C}(\bar{\mathbf{k}})$. The case of equivariant families is similar. In other words, we would like to prove that

$$(\nu_{E_1} \cdot \nu_{E_2}) \cdot \nu_{E_3} = \nu_{E_1} \cdot (\nu_{E_2} \cdot \nu_{E_3}) .$$

Replacing the category by its minimal model we may replace in all considerations Hom^\bullet by Ext^\bullet . Let us also remark that an element $\alpha \in \text{Ext}^1(E, F)$

defines an extension E_α which we can interpret as a deformation of the object $E \oplus F$ (the trivial extension). Therefore for any object G the group $\text{Ext}^\bullet(G, E_\alpha)$ is equal to the cohomology of the complex $(\text{Ext}^\bullet(G, E \oplus F), d_\alpha)$, where d_α is the operator of multiplication (up to a sign) by α .

Notice that

$$\nu_{E_1} \cdot \nu_{E_2} = \mathbb{L}^{-(E_2, E_1) \leq 0} [\text{Ext}^1(E_2, E_1) \rightarrow \text{Ob}(\mathcal{C})] := \mathbb{L}^{-(E_2, E_1) \leq 0} \int_{\alpha \in \text{Ext}^1(E_2, E_1)} \nu_{E_\alpha} ,$$

where E_α is the object corresponding to the extension α , i.e.

$$E_\alpha = \text{Cone}(\alpha : E_2[-1] \rightarrow E_1) .$$

It follows that

$$(\nu_{E_1} \cdot \nu_{E_2}) \cdot \nu_{E_3} = \mathbb{L}^{-(E_2, E_1) \leq 0} \int_{\alpha \in \text{Ext}^1(E_2, E_1), \beta \in \text{Ext}^1(E_3, E_\alpha)} \mathbb{L}^{-(E_3, E_\alpha) \leq 0} \nu_{E_\beta} .$$

We observe that

$$(E_3, E_\alpha)_{\leq 0} = (E_3, E_2 \oplus E_1)_{\leq 0} - l_\alpha = (E_3, E_2)_{\leq 0} + (E_3, E_1)_{\leq 0} - l_\alpha ,$$

where the ‘‘error term’’ $l_\alpha \geq 0$ can be computed in terms of the linear map d_α . Therefore one can write

$$(\nu_{E_1} \cdot \nu_{E_2}) \cdot \nu_{E_3} = \mathbb{L}^{-(E_2, E_1) \leq 0 - (E_3, E_1) \leq 0 - (E_3, E_2) \leq 0} \int_{\alpha \in \text{Ext}^1(E_2, E_1), \beta \in \text{Ext}^1(E_3, E_\alpha)} \mathbb{L}^{l_\alpha} \nu_{E_\beta} .$$

One can write a similar expression for $\nu_{E_1} \cdot (\nu_{E_2} \cdot \nu_{E_3})$. In this case the ‘‘error term’’ will be denoted by r_α instead of l_α .

Notice that the differential

$$d_\alpha : \text{Ext}^0(E_3, E_2) \oplus \text{Ext}^0(E_3, E_1) \rightarrow \text{Ext}^1(E_3, E_1) \oplus \text{Ext}^1(E_3, E_2)$$

satisfies the property that the only non-trivial component is the map $\alpha_R : \text{Ext}^0(E_3, E_2) \rightarrow \text{Ext}^1(E_3, E_1)$. Here we denote by α_R the linear operator of multiplication by $\alpha \in \text{Ext}^1(E_2, E_1)$ from the right. We will use the same convention for the linear operator $\alpha_R : \text{Ext}^1(E_3, E_2) \rightarrow \text{Ext}^2(E_3, E_1)$. Hence we see that

$$\dim \text{Ext}^1(E_3, E_\alpha) = \dim \text{Ker} (\alpha_R : \text{Ext}^1(E_3, E_2) \rightarrow \text{Ext}^2(E_3, E_1)) +$$

$$+ \dim \text{Coker} (\alpha_R : \text{Ext}^0(E_3, E_2) \rightarrow \text{Ext}^1(E_3, E_1)) .$$

Let us now consider the constructible set

$$X_{1,2,3} = \{(\alpha, \gamma, \delta) \in \text{Ext}^1(E_2, E_1) \oplus \text{Ext}^1(E_3, E_2) \oplus \text{Ext}^1(E_3, E_1) \mid \alpha \circ \gamma = 0\} .$$

Notice that a triple $(\alpha, \gamma, \delta) \in X_{1,2,3}$ defines the deformation of the object $E_1 \oplus E_2 \oplus E_3$ preserving the filtration

$$E_1 \subset E_1 \oplus E_2 \subset E_1 \oplus E_2 \oplus E_3 .$$

More precisely, the triple gives rise to a twisted complex, which is defined by the corresponding to (α, γ, δ) solution to the Maurer-Cartan equation (strictly upper-triangular matrix acting on $E_1 \oplus E_2 \oplus E_3$). The latter observation means that there is an ind-constructible map $X_{1,2,3} \rightarrow \text{Ob}(\mathcal{C})$ which assigns to a point (α, γ, δ) the corresponding twisted complex.

Let us now fix $\alpha \in \text{Ext}^1(E_2, E_1)$ and consider the ind-constructible subset $X_{1,2,3}^\alpha \subset X_{1,2,3}$ which consists of the triples with fixed α . There is a natural projection $(\alpha, \gamma, \delta) \mapsto (\gamma, \delta)$, which gives rise to the constructible map $X_{1,2,3}^\alpha \rightarrow \text{Ext}^1(E_3, E_\alpha)$. This is a constructible affine bundle with the fibers isomorphic to $\text{Im}(\alpha_R : \text{Ext}^0(E_3, E_2) \rightarrow \text{Ext}^1(E_3, E_1))$. Also, one can see directly that the dimension of the latter space is

$$l_\alpha = (E_3, E_2 \oplus E_1)_{\leq 0} - (E_3, E_\alpha)_{\leq 0} .$$

Hence we have the following identity in $H(\mathcal{C})$:

$$\int_{\alpha \in \text{Ext}^1(E_2, E_1), \beta \in \text{Ext}^1(E_3, E_\alpha)} \mathbb{L}^{l_\alpha} \nu_{E_\beta} = [X_{1,2,3} \rightarrow \text{Ob}(\mathcal{C})] .$$

Therefore,

$$(\nu_{E_1} \cdot \nu_{E_2}) \cdot \nu_{E_3} = \mathbb{L}^{-(E_2, E_1)_{\leq 0} - (E_3, E_1)_{\leq 0} - (E_3, E_2)_{\leq 0}} \cdot [X_{1,2,3} \rightarrow \text{Ob}(\mathcal{C})] .$$

Similar considerations show that

$$\nu_{E_1} \cdot (\nu_{E_2} \cdot \nu_{E_3}) = \mathbb{L}^{-(E_2, E_1)_{\leq 0} - (E_3, E_1)_{\leq 0} - (E_3, E_2)_{\leq 0}} \cdot [X_{1,2,3} \rightarrow \text{Ob}(\mathcal{C})] .$$

This proves the associativity of the product in $H(\mathcal{C})$. ■

For a constructible stability structure on \mathcal{C} with an ind-constructible class map $\text{cl} : K_0(\mathcal{C}) \rightarrow \Gamma$, a central charge $Z : \Gamma \rightarrow \mathbf{C}$, a strict sector $V \subset \mathbf{R}^2$

and a branch Log of the logarithm function on V we have the category $\mathcal{C}_V := \mathcal{C}_{V, \text{Log}}$ defined in Section 3.4. Hence we have the *completion*

$$\widehat{H}(\mathcal{C}_V) := \prod_{\gamma \in (\Gamma \cap \mathcal{C}(V, Z, Q)) \cup \{0\}} H(\mathcal{C}_V \cap \text{cl}^{-1}(\gamma)) .$$

Then we have an invertible element $A_V^{\text{Hall}} \in \widehat{H}(\mathcal{C}_V)$ such that

$$A_V^{\text{Hall}} := 1 + \cdots = \sum_{i \in I} \mathbf{1}_{(\text{Ob}(\mathcal{C}_V) \cap Y_i, GL(N_i))} ,$$

where $\mathbf{1}_S$ is the identity function (see 4.2) but interpreted as a counting measure²⁰. In short, element A_V^{Hall} is given by the counting measure restricted to \mathcal{C}_V . The summand 1 comes from zero object.

Proposition 11 *Elements A_V^{Hall} satisfy the Factorization Property:*

$$A_V^{\text{Hall}} = A_{V_1}^{\text{Hall}} \cdot A_{V_2}^{\text{Hall}}$$

for a strict sector $V = V_1 \sqcup V_2$ (decomposition in the clockwise order).

Proof. The proof follows from the following observations:

1) For any $E_i \in \text{Ob}(\mathcal{C}_{V_i}(\overline{\mathbf{k}}))$, $i = 1, 2$ one has $(E_2, E_1)_{\leq 0} = \dim \text{Ext}^0(E_2, E_1)$ because $\text{Ext}^i(E_2, E_1) = 0$ for $i < 0$.

2) The set $\{[E] \in \text{Iso}(\mathcal{C}_V(\overline{\mathbf{k}}))\}$ is in one-to-one correspondence with the set of isomorphism classes of the triples (E_1, E_2, α) such that $E_i \in \text{Ob}(\mathcal{C}_{V_i}(\overline{\mathbf{k}}))$, $i = 1, 2$ and $\alpha \in \text{Ext}^1(E_2, E_1)$ (the map between the sets assigns to the triple the extension E_α).

3) The automorphism group of the triple (E_1, E_2, α) is the stabilizer of α for the natural action of the group $\text{Aut}(E_2) \times \text{Aut}(E_1)$ on the vector space $\text{Ext}^1(E_2, E_1)$.

4) There is an exact sequence of groups

$$1 \rightarrow \text{Ext}^0(E_2, E_1) \rightarrow \text{Aut}(E_\alpha) \rightarrow \text{Aut}(E_1, E_2, \alpha) \rightarrow 1$$

In order to apply these observations one uses the fact that an object $E \in \mathcal{C}_V(\overline{\mathbf{k}})$ contains a unique subobject $E_1 \in \text{Ob}(\mathcal{C}_{V_1}(\overline{\mathbf{k}}))$ such that the quotient object E_2 belongs to $\mathcal{C}_{V_2}(\overline{\mathbf{k}})$, and then the factor $\mathbb{L}^{-(E_2, E_1)_{\leq 0}}$ cancels the ratio between the stabilizer of α and the automorphism group of the extension E_α . ■

²⁰The same is true if one uses the language of higher stacks because for any $E \in \text{Ob}(\mathcal{C}(V))(\overline{\mathbf{k}})$ one has $\text{Ext}^{<0}(E, E) = 0$.

Corollary 3 *Let us endow $H(\mathcal{C})$ with an automorphism η given by the shift functor [1]. Then the collection (A_V^{Hall}) gives rise to a symmetric stability data on $H(\mathcal{C})$ considered as a graded Lie algebra (see Definition 2 and Remark 8 in Section 2.2). Moreover we obtain a local homeomorphism $\text{Stab}(\mathcal{C}, \text{cl}) \rightarrow \text{Stab}(H(\mathcal{C}))$.*

The above considerations can be illustrated in the case of finite fields. Namely, let us assume that \mathcal{C} is a triangulated category over a finite field \mathbf{F}_q . We define the Hall algebra $H(\mathcal{C})$ as an associative unital algebra over \mathbf{Q} , which is a \mathbf{Q} -vector space spanned by isomorphism classes $[E]$ of objects $E \in \text{Ob}(\mathcal{C})$. The multiplication is given by the formula

$$[E] \cdot [F] = q^{-(F,E) \leq 0} \sum_{\alpha \in \text{Ext}^1(F,E)} [E_\alpha],$$

where E_α is the extension corresponding to $\alpha \in \text{Ext}^1(F, E)$.

We define a stability condition on \mathcal{C} in the same way as in the Introduction (or Section 3.4) without imposing any constructibility condition (since we do not assume that our category is ind-constructible). Inside of the set $\text{Stab}(\mathcal{C})$ of stability conditions on \mathcal{C} we consider a subset $\text{Stab}^0(\mathcal{C})$ consisting of such stability conditions that the set $\{E \in \mathcal{C}_\gamma^{ss} \mid \text{Arg}(E) = \varphi\}$ is finite for any $\gamma \in \Gamma, \varphi \in [0, 2\pi)$. This property is analogous to the one in the ind-constructible setting which says that \mathcal{C}_γ^{ss} is a constructible set. Then for any strict sector V and a choice of the branch Log we have an element $A_V^{\text{Hall}} \in \widehat{H}(\mathcal{C}_V)$ given by

$$A_V^{\text{Hall}} = 1 + \dots = \sum_{[E] \in \text{Iso}(\mathcal{C}_V)} \frac{[E]}{\#\text{Aut}(E)}.$$

Similarly to the motivic case the collection of elements (A_V^{Hall}) satisfies the Factorization property. Hence it defines a stability data on the space $H(\mathcal{C})$ considered as a graded Lie algebra.

The relationship of our version of Hall algebra to the Toën derived Hall algebra from [70] is described in the following proposition.

Proposition 12 *There is a homomorphism of rings $H(\mathcal{C}) \rightarrow H_{\text{To}}(\mathcal{C})$, where $H_{\text{To}}(\mathcal{C})$ is the derived Hall algebra over \mathbf{F}_q defined by Toën in [70] (see also [76]), such that*

$$[\pi : Y \rightarrow \text{Ob}(\mathcal{C})] \mapsto \sum_{y \in Y(\mathbf{F}_q)} [\pi(y)] \# \text{Aut}(y)(\mathbf{F}_q) q^{(y,y) < 0}.$$

Moreover, for any strict sector V the above homomorphism admits a natural extension to the completed Hall algebras such that the element $A_V^{\text{Hall}} \in \widehat{H}(\mathcal{C}_V)$ is mapped to the element of the completed Hall algebra $\widehat{H}_{\text{To}}(\mathcal{C}_V)$ given by $\sum_{[x] \in \text{Iso}(\mathcal{C}_V)} [x]$.

Proof. Straightforward. ■

In fact, in the Toën version of the Hall algebra the factorization property

$$A_V^{\text{Hall}} = A_{V_1}^{\text{Hall}} A_{V_2}^{\text{Hall}}$$

is essentially trivial. The reason is that the structure constants in $\widehat{H}_{\text{To}}(\mathcal{C}_V)$ for the elements of the basis corresponding to objects in a heart of a t -structure are the usual one, i.e. they count the number of 2-step filtrations of a given object with given isomorphism classes of the associate graded factors. The factorization property means that any object in \mathcal{C}_V has a unique subobject in \mathcal{C}_{V_1} with quotient in \mathcal{C}_{V_2} .

Remark 21 *One can try to go even further in an attempt to “categorify” the motivic Hall algebra. Here one has to assume that objects of \mathcal{C} form not just an ind-constructible stack, but a higher stack of locally finite type in the sense of Toën and Vezzosi (see [72]). The corresponding category will be the monoidal category of motivic sheaves on $\text{Ob}(\mathcal{C})$. The motivic Hall algebra is the K_0 -ring of this category. In the case of the non-commutative variety endowed with polarization one can define (for any strict sector V) the subcategory \mathcal{F}_V of “motivic sheaves with central charges in V ”. Nevertheless, the Factorization Property could fail since the object A_V^{Hall} can be non-isomorphic to the object $A_{V_1}^{\text{Hall}} \otimes A_{V_2}^{\text{Hall}}$ (but their images in K_0 coincide).*

Finally, we explain how to rephrase the factorization property in terms of t -structures, without the use of stability conditions. Here we understand a t -structure α on a small triangulated category \mathcal{C} as a pair of strictly full subcategories (i.e. a pair of sets of equivalence classes of objects)

$$\mathcal{C}_{-, \alpha}, \mathcal{C}_{+, \alpha} \subseteq \mathcal{C}$$

such that for any $E_- \in \mathcal{C}_{-, \alpha}$, $E_+ \in \mathcal{C}_{+, \alpha}$ we have $\text{Ext}^{\leq 0}(E_-, E_+) = 0$, and any object $E \in \mathcal{C}$ can be represented (uniquely) as an extension

$$\tau_{-, \alpha}(E) \rightarrow E \rightarrow \tau_{+, \alpha}(E), \quad \tau_{\pm, \alpha}(E) \in \mathcal{C}_{\pm, \alpha} .$$

Any stability condition on \mathcal{C} defines two t -structures α_l, α_r such that $\mathcal{C}_{-, \alpha_l}$ (resp. $\mathcal{C}_{-, \alpha_r}$) consists of extensions of semistable objects E with $\text{Arg}(E) \geq 0$ (resp. with $\text{Arg}(E) > 0$). These two t -structures do not change under the action of the group

$$\left\{ \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \mid a_{11}, a_{22} > 0 \right\} \subset \widetilde{GL}_+(2, \mathbf{R})$$

of transformations preserving the upper half-plane. In particular, we see that while a connected component in the $Stab(\mathcal{C})$ is a real $2n$ -dimensional manifold for $n := \text{rank}(\Gamma)$, the set of corresponding t -structures is at most $(n - 1)$ -dimensional.

Introduce an order on the set of t -structures by

$$\alpha_1 \leq \alpha_2 \iff \mathcal{C}_{-, \alpha_1} \subseteq \mathcal{C}_{-, \alpha_2} \iff \mathcal{C}_{+, \alpha_1} \supseteq \mathcal{C}_{+, \alpha_2} .$$

The shift functor acts on t -structures, and $\alpha[1] \leq \alpha$ for any t -structure α .

Let now \mathcal{C} be an ind-constructible category endowed with an ind-constructible homomorphism $cl : K_0(\mathcal{C}(\bar{\mathbf{k}})) \rightarrow \Gamma$ and α_1, α_2 are two ind-constructible t -structures. We say

$$\alpha_1 \leq_{constr} \alpha_2$$

iff

- $\alpha_1 \leq \alpha_2 \leq \alpha_1[-1]$,
- $\forall \gamma \in \Gamma \quad \mathcal{C}_{+, \alpha_1} \cap \mathcal{C}_{-, \alpha_2} \cap cl^{-1}(\gamma)$ is constructible,
- the cone generated by $\{\gamma \in \Gamma \mid \mathcal{C}_{+, \alpha_1} \cap \mathcal{C}_{-, \alpha_2} \cap cl^{-1}(\gamma) \neq 0\}$ is strict.

If $\alpha_1 \leq_{constr} \alpha_2$ then we define an element A_{α_1, α_2} of an appropriately completed Hall algebra as the sum of the “counting measure” over the objects in $\mathcal{C}_{+, \alpha_1} \cap \mathcal{C}_{-, \alpha_2}$. Obviously, elements A_V (for an open, or a closed, or a semi-open strict sector V) are of the form A_{α_1, α_2} for appropriate t -structures α_1, α_2 . The factorization property generalizes to

$$A_{\alpha_1, \alpha_3} = A_{\alpha_1, \alpha_2} \cdot A_{\alpha_2, \alpha_3}$$

if $\alpha_1 \leq_{constr} \alpha_2, \alpha_2 \leq_{constr} \alpha_3, \alpha_1 \leq_{constr} \alpha_3$.

Notice that in the case of stability conditions the element A_V is preserved under the action of a subgroup of $\widetilde{GL}_+(2, \mathbf{R})$ conjugated to the group of positive diagonal matrices. This action on $Stab(\mathcal{C})/Aut(\mathcal{C})$ has a good chance to be ergodic, as indicates a similar example with the moduli spaces of curves with abelian differentials (see a review [78]).

6.2 Motivic weights and stability data on motivic quantum tori

Let \mathcal{C} be a 3-dimensional ind-constructible Calabi-Yau category over a field of characteristic zero (see Section 3.3). In this section we are going to define motivic Donaldson-Thomas invariants associated with a constructible stability condition and an orientation data on \mathcal{C} .

Step 1.

Let us define the ring

$$D^\mu = \mathcal{M}^\mu(\text{Spec}(\mathbf{k}))[\mathbb{L}^{-1}, \mathbb{L}^{1/2}, ([GL(n)]^{-1})_{n \geq 1}] ,$$

where the ring $\mathcal{M}^\mu(\text{Spec}(\mathbf{k}))$ was defined in 4.3, and $\mathbb{L} = [\mathbf{A}_k^1]$ is the motive of the affine line. The element $\mathbb{L}^{1/2}$ is a formal symbol satisfying the equation $(\mathbb{L}^{1/2})^2 = \mathbb{L}$. Instead of inverting motives

$$[GL(n)] = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \dots (\mathbb{L}^n - \mathbb{L}^{n-1})$$

of all general linear groups we can invert motives of all projective spaces

$$[\mathbf{P}^n] = \frac{\mathbb{L}^{n+1} - 1}{\mathbb{L} - 1} .$$

We also will consider the ring \overline{D}^μ of equivalence classes of functions from D^μ by the equivalence relation defined in Section 4.5. The ring \overline{D}^μ will play the role of the universal coefficient ring where motivic Donaldson-Thomas invariants take value.

Step 2.

We define an algebra $\overline{\mathcal{M}}^\mu(\text{Ob}(\mathcal{C}))$ associated with \mathcal{C} which will contain certain canonical element called the motivic weight. First, we define

$$\mathcal{M}^\mu(\text{Ob}(\mathcal{C})) := \prod_i \mathcal{M}^{GL(N_i), \mu}(Y_i)[\mathbb{L}^{-1}, \mathbb{L}^{1/2}] ,$$

where $(Y_i, GL(N_i))_{i \in I}$ is a decomposition of the stack of objects of \mathcal{C} as at the end of 3.2. Algebra $\overline{\mathcal{M}}^\mu(\text{Ob}(\mathcal{C}))$ is obtained from it by passing to the equivalence classes in the sense of Section 4.5.

For any $GL(N_i)$ -invariant constructible set $Z \subset Y_i$ for some $i \in I$, we have a $\mathcal{M}^\mu(\text{Spec}(\mathbf{k}))[\mathbb{L}^{-1}, \mathbb{L}^{1/2}]$ -linear map

$$\int_Z : \mathcal{M}^\mu(\text{Ob}(\mathcal{C})) \rightarrow D^\mu$$

which is the μ -equivariant version of integral over stack $(Z, GL(N_i))$ (see 4.2) of the restriction to Z . Explicitly, if $f|_{Y_i}$ is represented by a $\mu \times GL(N_i)$ -equivariant map $X \rightarrow Y_i$ then

$$\int_Z f = [X \times_{Y_i} Z]/[GL(N_i)] \in D^\mu$$

where $[X \times_{Y_i} Z]$ is interpreted a constructible set with μ -action. By additivity we extend the integral to the case when Z is a finite union of $GL(N_i)$ -invariant constructible set $Z_i \subset Y_i$ for different $i \in I$.

Step 3.

Now we are going to define the motivic weight. Recall that for any $E \in \text{Ob}(\mathcal{C})(\bar{\mathbf{k}})$ we have defined the potential W_E^{min} which is a formal power series in $\alpha \in \text{Ext}^1(E, E)$ which starts with cubic terms. We denote by

$$MF(E) := MF_0(W_E^{min})$$

the motivic Milnor fiber of W_E^{min} at $0 \in \text{Ext}^1(E, E)$. Then the assignment $E \mapsto MF(E)$ can be interpreted as the value of some function $MF \in \mathcal{M}^\mu(\text{Ob}(\mathcal{C}))$.

Let us choose an orientation data $\sqrt{\mathcal{D}}$ for \mathcal{C} . Recall that in Section 5.2 we defined the element $h \in J_2(\mathcal{C})$ represented by the equivariant super line bundle $\sqrt{\mathcal{D}} \otimes \mathcal{D}_{\leq 1}^{-1}$ with trivialized tensor square. For a representative of h given by a pair (V, Q) we have $\bar{I}(h) = (1 - MF_0(Q))\mathbb{L}^{-\frac{1}{2}\text{rk}Q}$. Let us choose such a representative.

Definition 17 *The motivic weight $w \in \mathcal{M}^\mu(\text{Ob}(\mathcal{C}))$ is the function defined on objects by the formula*

$$w(E) = \mathbb{L}^{\frac{1}{2}\sum_{i \leq 1} (-1)^i \dim \text{Ext}^i(E, E)} (1 - MF(E))(1 - MF_0(Q_E))\mathbb{L}^{-\frac{1}{2}\text{rk}Q_E} .$$

The image $\bar{w} \in \overline{\mathcal{M}^\mu}(\text{Ob}(\mathcal{C}))$ does not depend on the choice of a representative of h and is equal to

$$\bar{w}(E) = \mathbb{L}^{\frac{1}{2}\sum_{i \leq 1} (-1)^i \dim \text{Ext}^i(E, E)} (1 - MF(E))\bar{I}(h(E)) ,$$

where $h(E)$ is the value of the obstruction h at the point E .

Step 4.

Let us now fix the following data:

- a triple $(\Gamma, \langle \bullet, \bullet \rangle, Q)$ consisting of a free abelian group Γ of finite rank endowed with a skew-symmetric bilinear form $\langle \bullet, \bullet \rangle : \Gamma \otimes \Gamma \rightarrow \mathbf{Z}$, and a quadratic form Q on $\Gamma_{\mathbf{R}} = \Gamma \otimes \mathbf{R}$;
- an ind-constructible, $\text{Gal}(\bar{\mathbf{k}}/\mathbf{k})$ -equivariant homomorphism

$$\text{cl}_{\bar{\mathbf{k}}} : K_0(\mathcal{C}(\bar{\mathbf{k}})) \rightarrow \Gamma$$

compatible with the Euler form and the skew-symmetric bilinear form;

- a constructible stability structure $\sigma \in \text{Stab}(\mathcal{C}, \text{cl})$ compatible with the quadratic form Q in the sense that $Q|_{\text{Ker } Z} < 0$ and $Q(\text{cl}_{\bar{\mathbf{k}}}(E)) \geq 0$ for $E \in \mathcal{C}^{ss}(\bar{\mathbf{k}})$.

In the next section we are going to define a homomorphism from the motivic Hall algebra to the associative unital algebra called motivic quantum torus. The latter is defined in the following way.

For any commutative unital ring C which contains an invertible symbol $\mathbb{L}^{1/2}$ we introduce a C -linear associative algebra

$$\mathcal{R}_{\Gamma, C} := \bigoplus_{\gamma \in \Gamma} C \cdot \hat{e}_{\gamma}$$

where the generators $\hat{e}_{\gamma}, \gamma \in \Gamma$ satisfy the relations

$$\hat{e}_{\gamma_1} \hat{e}_{\gamma_2} = \mathbb{L}^{\frac{1}{2} \langle \gamma_1, \gamma_2 \rangle} \hat{e}_{\gamma_1 + \gamma_2}, \quad \hat{e}_0 = 1.$$

We will call it the *quantum torus* associated with Γ and C .

For any strict sector $V \subset \mathbf{R}^2$ we define

$$\mathcal{R}_{V, C} := \prod_{\gamma \in \Gamma \cap C_0(V, Z, Q)} C \cdot \hat{e}_{\gamma}$$

and call it the quantum torus associated with V . Here we introduce a notation which will be used later:

$$C_0(V, Z, Q) := C(V, Z, Q) \cup \{0\}$$

where the cone $C(V, Z, Q)$ was defined in 2.2. Algebra $\mathcal{R}_{V, C}$ is the natural completion of the subalgebra $\mathcal{R}_{V, C} \cap \mathcal{R}_{\Gamma, C} \subset \mathcal{R}_{\Gamma, C}$.

Let us choose as C the ring D^{μ} . We denote $\mathcal{R}_{\Gamma} := \mathcal{R}_{\Gamma, D^{\mu}}$ the corresponding quantum torus and call it the *motivic quantum torus* associated with Γ . Similarly, we have motivic quantum tori R_V associated with strict sectors V .

Step 5.

We define an element $A_V^{mot} \in \mathcal{R}_V := \mathcal{R}_{V, \overline{\mathcal{D}}^\mu}$ in the following way. First, we fix a branch of the function $\text{Log } z$, where $z \in V$ (the result will not depend on the choice of the branch). Recall the category $\mathcal{C}_{V, \text{Log}} \subset \mathcal{C}$ (see Section 3.4). It follows from our assumptions that for any $\gamma \in \Gamma$ the set $\mathcal{C}_{V, \gamma} = \{E \in \text{Ob}(\mathcal{C}_{V, \text{Log}}) \mid \text{cl}(E) = \gamma\}$ is constructible.

Finally, we define the desired element

$$A_V^{mot} = \sum_{\gamma} \left(\int_{\mathcal{C}_{V, \gamma}} w \right) \cdot \hat{e}_{\gamma} .$$

The element A_V^{mot} in fact depends only on \overline{w} .

Informally, one can write

$$A_V^{mot} = \sum_{E \in \text{Iso}(\mathcal{C}_{V, \text{Log}})} \frac{w(E)}{[\text{Aut}(E)]} \hat{e}_{\text{cl}(E)} = 1 + \dots \in \mathcal{R}_V ,$$

where $\text{Iso}(\mathcal{C}_{V, \text{Log}})$ denotes the set of isomorphism classes of objects of the category $\mathcal{C}_{V, \text{Log}}$.

Theorem 7 *Assuming the integral identity, the collection of elements (A_V^{mot}) satisfies the Factorization Property: if a strict sector V is decomposed into a disjoint union $V = V_1 \sqcup V_2$ (in the clockwise order) then*

$$A_V^{mot} = A_{V_1}^{mot} A_{V_2}^{mot} .$$

Moreover we have a local homeomorphism $\text{Stab}(\mathcal{C}) \rightarrow \text{Stab}(\mathcal{R}_{\Gamma, \overline{\mathcal{D}}^\mu})$.

This theorem follows immediately from the statement of Proposition 11 (see 6.1) about the elements A_V^{Hall} , and the Theorem 8 from the next section.

6.3 From motivic Hall algebra to motivic quantum torus

Assume that \mathcal{C} is an ind-constructible $3d$ Calabi-Yau category endowed with polarization and orientation data $\sqrt{\mathcal{D}}$. The Hall algebra of \mathcal{C} is graded by the corresponding lattice Γ : $H(\mathcal{C}) = \bigoplus_{\gamma \in \Gamma} H(\mathcal{C})_{\gamma}$. Main result of this section is the following theorem.

Theorem 8 *The map $\Phi : H(\mathcal{C}) \rightarrow \mathcal{R}_\Gamma$ given by the formula*

$$\Phi(\nu) = (\nu, w)\hat{e}_\gamma, \quad \nu \in H(\mathcal{C})_\gamma$$

is a homomorphism of Γ -graded \mathbf{Q} -algebras. Here w is the motivic weight and (\bullet, \bullet) is the pairing between motivic measures and functions.

In other words, the homomorphism $H(\mathcal{C}) \rightarrow \mathcal{R}_\Gamma$ can be written as

$$\begin{aligned} & [\pi : Y \rightarrow \text{Ob}(\mathcal{C})] \mapsto \\ & \mapsto \int_Y (1 - MF(\pi(y))) (1 - MF_0(Q_{\pi(y)})) \mathbb{L}^{-\frac{1}{2} \text{rk } Q_{\pi(y)}} \mathbb{L}^{\frac{1}{2}(\pi(y), \pi(y)) \leq 1} \hat{e}_{\text{cl}(\pi(y))} , \end{aligned}$$

where \int_Y is understood as the direct image functor (see Section 4.2).

The natural extension of the above homomorphism to the completion of $\widehat{H}(\mathcal{C}_V)$ maps the element A_V^{Hall} to the element A_V^{mot} defined in Section 6.2.

Proof. For simplicity we will present the proof of the Theorem for

$$\nu_E := [\delta_E : pt \rightarrow \text{Ob}(\mathcal{C})] ,$$

where $\delta_E(pt) = E \in \text{Ob}(\mathcal{C}(\mathbf{k}))$ is the “delta-function”. The general proof for equivariant constructible families is similar. We will also assume that our category is minimal on the diagonal.

The proof will consists of several steps.

Step 1.

We have:

$$\nu_{E_1} \cdot \nu_{E_2} = \mathbb{L}^{-(E_2, E_1) \leq 0} [\pi_{21} : \text{Ext}^1(E_2, E_1) \rightarrow \text{Ob}(\mathcal{C})] ,$$

the map π_{21} is the restriction of the cone map²¹

$$\text{tot}(\text{Ker}(m_1 : \mathcal{HOM}^0 \rightarrow \mathcal{HOM}^1)) \rightarrow \text{Ob}(\mathcal{C})$$

to the fiber over the point $(E_2[-1], E_1)$. Under this map the element $\alpha \in \text{Ext}^1(E_2, E_1)$ is mapped to the object

$$E_\alpha = \text{Cone}(\alpha : E_2[-1] \rightarrow E_1) .$$

²¹Recall that we pretend that such a map exists. In fact, it is defined only as a 1-morphism of stacks.

Let us denote by γ_i the class $\text{cl}(E_i) \in \Gamma, i = 1, 2$. Then we have:

$$\Phi(\nu_{E_i}) = \mathbb{L}^{\frac{1}{2}\langle E_i, E_i \rangle \leq 1} (1 - MF_0(W_{E_i}^{min})) \bar{I}(h(E_i)) \hat{e}_{\gamma_i}, \quad i = 1, 2,$$

where $h(E_i)$ is the value at E_i of the element $h \in J_2(\mathcal{C})$ (i.e. the image of the restriction map to $J_2(\mathcal{C})$) given by the super line bundle $\sqrt{\mathcal{D}} \otimes \mathcal{D}_{\leq 1}^{-1}$ with trivialized tensor square.

We have:

$$\begin{aligned} \Phi(\nu_{E_1})\Phi(\nu_{E_2}) &= \mathbb{L}^{\frac{1}{2}((E_1, E_1)_{\leq 1} + (E_2, E_2)_{\leq 1})} \times \\ &\times \mathbb{L}^{\frac{1}{2}((E_1, E_2)_{\leq 1} - (E_2, E_1)_{\leq 1})} (1 - MF_0(W_{E_1}^{min} \oplus W_{E_2}^{min})) \bar{I}(h(E_1)) \bar{I}(h(E_2)) \hat{e}_{\gamma_1 + \gamma_2}. \end{aligned}$$

In order to obtain this formula we used the Calabi-Yau property, which implies that

$$\begin{aligned} \langle \gamma_1, \gamma_2 \rangle &= \sum_{j \in \mathbf{Z}} (-1)^j (E_1, E_2)_j = (E_1, E_2)_{\leq 1} + (E_1, E_2)_{\geq 2} = \\ &= (E_1, E_2)_{\leq 1} - (E_2, E_1)_{\leq 1}, \end{aligned}$$

where we employ the notation $(E_1, E_2)_{\geq m} = \sum_{j \geq m} (-1)^j (E_1, E_2)_j$. Also we used the motivic Thom-Sebastiani theorem for the Milnor fibers and the product formula for the basis elements in the motivic quantum torus \mathcal{R}_Γ .

On the other hand, we can apply Φ to the product $\nu_{E_1} \cdot \nu_{E_2}$ and obtain:

$$\begin{aligned} \Phi(\nu_{E_1} \cdot \nu_{E_2}) &= \\ &= \mathbb{L}^{-(E_2, E_1)_{\leq 0}} \int_{\alpha \in \text{Ext}^1(E_2, E_1)} \mathbb{L}^{\frac{1}{2}\langle E_\alpha, E_\alpha \rangle \leq 1} (1 - MF_0(W_{E_\alpha}^{min})) \bar{I}(h(E_\alpha)) \hat{e}_{\gamma_1 + \gamma_2}. \end{aligned}$$

Using the identity $(E_2, E_1)_{\leq 0} = (E_2, E_1)_{\leq 1} + (E_2, E_1)_1$ (and also recall that $(E_2, E_1)_1 = \dim \text{Ext}^1(E_2, E_1)$) and observing that

$$(E_1 \oplus E_2, E_1 \oplus E_2)_{\leq 1} = (E_1, E_1)_{\leq 1} + (E_2, E_2)_{\leq 1} + (E_1, E_2)_{\leq 1} + (E_2, E_1)_{\leq 1}$$

we arrive to the following equality which is equivalent to

$$\Phi(\nu_{E_1} \cdot \nu_{E_2}) = \Phi(\nu_{E_1})\Phi(\nu_{E_2})$$

and hence implies the Theorem:

$$\begin{aligned} &\mathbb{L}^{(E_2, E_1)_1} (1 - MF_0(W_{E_1}^{min} \oplus W_{E_2}^{min})) \bar{I}(h(E_1)) \bar{I}(h(E_2)) = \\ &= \int_{\alpha \in \text{Ext}^1(E_2, E_1)} \mathbb{L}^{\frac{1}{2}((E_\alpha, E_\alpha)_{\leq 1} - (E_1 \oplus E_2, E_1 \oplus E_2)_{\leq 1})} (1 - MF_0(W_{E_\alpha}^{min})) \bar{I}(h(E_\alpha)). \end{aligned}$$

Step 2.

Now we would like to express the difference

$$(E_\alpha, E_\alpha)_{\leq 1} - (E_1 \oplus E_2, E_1 \oplus E_2)_{\leq 1}$$

as the rank of a certain linear operator. Recall that the object E_α can be thought of as a deformation of the object $E_0 := E_1 \oplus E_2$. Therefore, there is a spectral sequence which starts at $\text{Ext}^\bullet(E_1 \oplus E_2, E_1 \oplus E_2)$ and converges to $\text{Ext}^\bullet(E_\alpha, E_\alpha)$. Using the A_∞ -structure one can make it very explicit. Namely, let us denote by $d_\alpha : \text{Ext}^\bullet(E_1 \oplus E_2, E_1 \oplus E_2) \rightarrow \text{Ext}^\bullet(E_1 \oplus E_2, E_1 \oplus E_2)$ the differential of degree +1 given by the formula

$$d_\alpha = m_2(\alpha, \bullet) + m_2(\bullet, \alpha) + m_3(\alpha, \bullet, \alpha) .$$

Then the graded vector space $\text{Ext}^\bullet(E_\alpha, E_\alpha)$ is isomorphic to the cohomology of d_α (cf. e.g. [42], Remark 10.1.5).

It is clear that for any cohomological complex (C^\bullet, d) of finite-dimensional vector spaces we have the equality

$$\sum_{i \leq 1} (-1)^i \dim H^i(C) - \sum_{i \leq 1} (-1)^i \dim C^i = \text{rk } d^{(1)},$$

where $d^{(1)} : C^1 \rightarrow C^2$ is the component of d . Applying this observation to our complex we obtain that

$$(E_\alpha, E_\alpha)_{\leq 1} - (E_1 \oplus E_2, E_1 \oplus E_2)_{\leq 1} = \text{rk } d_\alpha^{(1)} .$$

Step 3.

Let us introduce a $\bar{\mathbf{k}}$ -vector space

$$\begin{aligned} \mathcal{M}_{E_1, E_2} &= \text{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2) = \\ &= \text{Ext}^1(E_1, E_1) \oplus \text{Ext}^1(E_2, E_1) \oplus \text{Ext}^1(E_1, E_2) \oplus \text{Ext}^1(E_2, E_2) . \end{aligned}$$

It can be interpreted as the tangent space to the moduli space of formal deformations of the object $E_1 \oplus E_2$. We choose coordinates (x, α, β, y) on this space in such a way that x denotes the coordinates on $\text{Ext}^1(E_1, E_1)$, α denotes the coordinates on $\text{Ext}^1(E_2, E_1)$, β denotes the coordinates on $\text{Ext}^1(E_1, E_2)$ and y denotes the coordinates on $\text{Ext}^1(E_2, E_2)$. Then the point $(0, \alpha, 0, 0)$ corresponds (by abuse of notation) to the isomorphism class $\alpha \in \text{Ext}^1(E_2, E_1)$ of an exact triangle $E_1 \rightarrow E_\alpha \rightarrow E_2$. Later we are going to use

the integral identity from Section 4.4 applying it to the formal neighborhood of the subspace consisting of the points $(0, \alpha, 0, 0)$. In order to do that we will relate the potential of the object E_α with a certain formal function on \mathcal{M}_{E_1, E_2} .

We may assume that the full subcategory $\mathcal{C}(E_1, E_2)$ consisting of the pair of objects E_1, E_2 is minimal. As in the case of one object the potential of \mathcal{C} induces a formal power series $W_{E_1, E_2} = W(x, \alpha, \beta, y)$ on \mathcal{M}_{E_1, E_2} . It is defined as the abelianization of a series $\sum_{n \geq 3} W_n/n$ in cyclic paths in the quiver Q_{E_1, E_2} with the vertices E_1 and E_2 and $(E_i, E_j)_1$ edges between vertices E_i and E_j for $i, j \in \{1, 2\}$. Since any cyclic path has the same number of edges in the direction $E_1 \rightarrow E_2$ as in the direction $E_2 \rightarrow E_1$ we conclude that the potential W_{E_1, E_2} is \mathbf{G}_m -invariants with respect to the \mathbf{G}_m -action on the graded vector space \mathcal{M}_{E_1, E_2} with the weights $wt x = wt y = 0$ and $wt \alpha = -wt \beta = 1$. The potential W_{E_1, E_2} is obtained from the potential $W_{E_1 \oplus E_2}$ by a formal change of variables.

It follows from \mathbf{G}_m -invariance of W_{E_1, E_2} that it belongs to $\bar{\mathbf{k}}[\alpha][[x, \beta, \gamma]]$. Therefore it defines a function on the formal neighborhood of the affine subspace $\{(0, \alpha, 0, 0)\} \subset \mathcal{M}_{E_1, E_2}$. In particular, for any $\alpha \in \text{Ext}^1(E_2, E_1)$ we obtain a formal power series $W_{E_1, E_2, \alpha}$ on \mathcal{M}_{E_1, E_2} which is the Taylor expansion of W_{E_1, E_2} at the point $(0, \alpha, 0, 0)$. Similarly to the Proposition 7 from Section 3.3 the series $W_{E_1, E_2, \alpha}$ becomes (after a formal change of coordinates) a direct sum $W_{E_\alpha}^{min} \oplus Q_{E_\alpha} \oplus N_{E_\alpha}$, where Q_{E_α} is a non-degenerate quadratic form, N_{E_α} is the zero function on a vector subspace, and $W_{E_\alpha}^{min}$ does not contain terms of degree less than 3 in its Taylor expansion. By the motivic Thom-Sebastiani theorem we have

$$(1 - MF_0(W_{E_1, E_2, \alpha})) = (1 - MF(E_\alpha))(1 - MF_0(Q_{E_\alpha})) .$$

Let us consider the quadratic form $((W_{E_1, E_2})''|_{(0, \alpha, 0, 0)})$ on \mathcal{M}_{E_1, E_2} , where $(W_{E_1, E_2})''$ denotes the second derivative of the potential with respect to the affine coordinates. It follows from the above discussion that this quadratic form is equal to the direct sum of Q_{E_α} and the zero quadratic form.

It is easy to check that $((W_{E_1, E_2})''|_{(0, \alpha, 0, 0)})(v)$ is equal to $(d_\alpha^{(1)}v, v)$ for any $v \in \mathcal{M}_{E_1, E_2}$. Hence Q_{E_α} can be identified with the quadratic form on $\text{Im}(d_\alpha^{(1)})$ given by $(u, (d_\alpha^{(1)})^{-1}u)$.

Step 4.

Recall (see Section 5.2) that for any exact triangle $E_1 \rightarrow E_\alpha \rightarrow E_2$ we have a super line bundle L with a canonically trivialized square:

$$L_{E_1 \rightarrow E_\alpha \rightarrow E_2} = (\mathcal{D}_{\leq 1, E_\alpha} \otimes \mathcal{D}_{\leq 1, E_1}^{-1} \otimes \mathcal{D}_{\leq 1, E_2}^{-1}) \otimes (\text{sdet}(\text{Ext}^\bullet(E_1, E_2)))^{-1}.$$

For a split triangle $E_\alpha \simeq E_1 \oplus E_2$ (i.e. $\alpha = 0$) this line bundle is canonically trivialized since

- by definition, for $E_\alpha \simeq E_1 \oplus E_2$ we have

$$\begin{aligned} \mathcal{D}_{\leq 1, E_\alpha} &\simeq \mathcal{D}_{\leq 1, E_1} \otimes \mathcal{D}_{\leq 1, E_2} \otimes \\ &\otimes \text{sdet}(\text{Ext}^{\leq 1}(E_1, E_2)) \otimes \text{sdet}(\text{Ext}^{\leq 1}(E_2, E_1)) , \end{aligned}$$

- by the Calabi-Yau property we have

$$\text{sdet}(\text{Ext}^{\leq 1}(E_2, E_1)) \simeq \text{sdet}(\text{Ext}^{\geq 2}(E_1, E_2)) .$$

Therefore, for any exact triangle $E_1 \rightarrow E_\alpha \rightarrow E_2$ we have an isomorphism

$$L_{E_1 \rightarrow E_\alpha \rightarrow E_2} \simeq \mathcal{D}_{\leq 1, E_\alpha} \otimes \mathcal{D}_{\leq 1, E_1 \oplus E_2}^{-1} .$$

On the other hand, considerations similar to those in Step 2 give rise to a canonical isomorphism

$$\mathcal{D}_{\leq 1, E_\alpha} \otimes \mathcal{D}_{\leq 1, E_1 \oplus E_2}^{-1} \simeq \text{sdet}(\text{Im}(d_\alpha^{(1)})) .$$

One can see that the trivialization $(\text{sdet}(\text{Im}(d_\alpha^{(1)})))^{\otimes 2} \simeq \mathbf{1}$ comes exactly from the non-degenerate quadratic form Q_{E_α} . Therefore, for an arbitrary exact triangle $E_1 \rightarrow E_\alpha \rightarrow E_2$ we have an isomorphism of super lines compatible with the trivializations of squares:

$$L_{E_1 \rightarrow E_\alpha \rightarrow E_2} \simeq \text{sdet}(\text{Im}(d_\alpha^{(1)})) .$$

This implies that

$$\bar{I}(Q_{E_\alpha}) = \bar{I}(l(E_1 \rightarrow E_\alpha \rightarrow E_2)) ,$$

where $l \in J_2(\text{Funct}(\mathcal{C}_3, \mathcal{C}))$ was defined in Section 5.2.

Step 5.

Let us apply the integral identity from Section 4.4 to the potential W_{E_1, E_2} . We put

$$V_1 := \text{Ext}^1(E_2, E_1), V_2 := \text{Ext}^1(E_1, E_2), V_3 := \text{Ext}^1(E_1, E_1) \oplus \text{Ext}^1(E_2, E_2) .$$

We have:

$$\begin{aligned} & \int_{\alpha \in \text{Ext}^1(E_2, E_1)} (1 - MF_{(0, \alpha, 0, 0)}(W_{E_1, E_2})) = \\ & = \mathbb{L}^{(E_2, E_1)1} (1 - MF_0((W_{E_1, E_2})|_{\text{Ext}^1(E_1, E_1) \oplus \text{Ext}^1(E_2, E_2)})) . \end{aligned}$$

On the other hand the LHS of the integral identity is equal to

$$\begin{aligned} & \int_{\alpha \in \text{Ext}^1(E_2, E_1)} (1 - MF_0(Q_{E_\alpha}))(1 - MF(W_{E_\alpha}^{min})) = \\ & = \int_{\alpha \in \text{Ext}^1(E_2, E_1)} \mathbb{L}^{\frac{1}{2} \text{rk } Q_{E_\alpha}} \bar{I}(Q_{E_\alpha})(1 - MF(W_{E_\alpha}^{min})) . \end{aligned}$$

Recall that $\text{rk } Q_{E_\alpha} = \text{rk}(d_\alpha^{(1)}) = (E_\alpha, E_\alpha)_{\leq 1} - (E_1 \oplus E_2, E_1 \oplus E_2)_{\leq 1}$ by Steps 2 and 3. Then the integral identity becomes the following equality:

$$\begin{aligned} & \int_{\alpha \in \text{Ext}^1(E_2, E_1)} \mathbb{L}^{\frac{1}{2}((E_\alpha, E_\alpha)_{\leq 1} - (E_1 \oplus E_2, E_1 \oplus E_2)_{\leq 1})} \bar{I}(Q_{E_\alpha})(1 - MF(W_{E_\alpha}^{min})) = \\ & = \mathbb{L}^{(E_2, E_1)1} (1 - MF_0(W_{E_1}^{min} \oplus W_{E_2}^{min})) . \end{aligned}$$

Comparing this formula with the one we wanted to prove on Step 1 we see that they coincide if

$$\bar{I}(Q_{E_\alpha}) = \frac{\bar{I}(h(E_\alpha))}{\bar{I}(h(E_1))\bar{I}(h(E_2))} .$$

Now using Step 4 we observe that this cocycle condition is equivalent to the main property of the orientation data on exact triangles. This concludes the proof of the Theorem. ■

Definition 18 *Let \mathcal{C} be an ind-constructible 3-dimensional Calabi-Yau category endowed with polarization, $\sigma \in \text{Stab}(\mathcal{C}, \text{cl})$. We call the collection of elements $(A_V^{mot} \in \mathcal{R}_V)$ of the completed motivic quantum tori (\mathcal{R}_V) (for all strict sectors $V \subset \mathbf{R}^2$) the motivic Donaldson-Thomas invariant of \mathcal{C} .*

Let us consider the following unital \mathbf{Q} -subalgebra of $\mathbf{Q}(q^{1/2})$:

$$D_q := \mathbf{Z}[q^{1/2}, q^{-1/2}, ((q^n - 1)^{-1})_{n \geq 1}] .$$

There is a homomorphism of rings $\overline{D}^\mu \rightarrow D_q$ given by the twisted Serre polynomial. Namely, it maps $\mathbb{L}^{1/2} \mapsto q^{1/2}$, and on \mathcal{M}^μ it is the composition

of the Serre polynomial with the involution $q^{1/2} \mapsto -q^{1/2}$. Therefore, we have a homomorphism of algebras $\mathcal{R}_\Gamma \rightarrow \mathcal{R}_{\Gamma,q}$, where $\mathcal{R}_{\Gamma,q}$ is a D_q -algebra generated by $\hat{e}_\gamma, \gamma \in \Gamma$, subject to the relations

$$\hat{e}_\gamma \hat{e}_\mu = q^{\frac{1}{2}\langle \gamma, \mu \rangle} \hat{e}_{\gamma+\mu}, \quad \hat{e}_0 = 1.$$

Similarly to the motivic case, we have the algebra $\mathcal{R}_{V,q}$ associated with any strict sector V .

The motivic DT-invariants give rise to stability data on the graded Lie algebra associated with $\mathcal{R}_{\Gamma,q}$. We will denote by $A_{V,q} \in \mathcal{R}_{V,q}$ the element corresponding to A_V^{mot} .

6.4 Examples

1) Assume that a 3-dimensional Calabi-Yau category \mathcal{C} is generated by one spherical object E defined over \mathbf{k} . Therefore $R := \text{Ext}^\bullet(E, E) \simeq H^\bullet(S^3, \mathbf{k})$. In this case we take $\Gamma = K_0(\mathcal{C}(\overline{\mathbf{k}})) \simeq \mathbf{Z} \cdot \text{cl}_{\overline{\mathbf{k}}}(E)$, and the skew-symmetric form on Γ is trivial. In order to choose an orientation data, let us fix a basis $r_0 = 1, r_3$ in the algebra R (the subscript indicates the degree). Consider R as a bimodule over itself and denote this bimodule by M . The corresponding to $1, r_3$ bimodule basis will be denoted by $1_M, r_{3,M}$. Then we have a family $M_t, 0 \leq t \leq 1$ of R -bimodule structures on M such that

$$1_M \cdot_t r_3 = r_3 \cdot_t 1_M = (1-t)r_{3,M}.$$

Hence $M_0 = M$ and $M_1 \simeq N \oplus N^\vee$ in notation of Section 5.3. The latter gives a decomposition of the bifunctor Hom^\bullet . The above family of bimodules define a homotopy which can be used for definition of an orientation data as in Section 5.2.

For any $z \in \mathbf{C}, \text{Im } z > 0$ we have a stability condition σ_z such that $E \in \mathcal{C}^{ss}$, $Z(E) := Z(\text{cl}_{\overline{\mathbf{k}}}(E)) = z$, $\text{Arg}(E) = \text{Arg}(z) \in (0, \pi)$. For a strict sector V such that $\text{Arg}(V) \subset (0, \pi)$ we have the category \mathcal{C}_V which is either trivial (if $z \notin V$) or consists of objects $0, E, E \oplus E, \dots$ (if $z \in V$). Then $A_V^{mot} = 1$ in the first case and

$$A_V^{mot} = \sum_{n \geq 0} \frac{\mathbb{L}^{n^2/2}}{[GL(n)]} \hat{e}_{\gamma_1}^n,$$

in the second case. Here $\gamma_1 := \text{cl}_{\overline{\mathbf{k}}}(E)$ is the generator of Γ .

Let us comment on the answer. In this case $\text{Ext}^1(nE, nE) = 0$, where we set $nE = E^{\oplus n}$, $n \geq 1$. Therefore $W_{nE} = 0$ which implies that $MF(W_{nE}) = 0$. The numerator is

$$\mathbb{L}^{n^2/2} = \mathbb{L}^{\frac{1}{2} \dim \text{Ext}^0(nE, nE)} = \mathbb{L}^{\frac{1}{2} \sum_{i \leq 1} (-1)^i \dim \text{Ext}^i(nE, nE)},$$

since $\text{Ext}^{\neq 0}(nE, nE) = 0$.

Let us consider the “quantum dilogarithm” series

$$\mathbf{E}(q^{1/2}, x) = \sum_{n \geq 0} \frac{q^{n^2/2}}{(q^n - 1) \dots (q^n - q^{n-1})} x^n \in \mathbf{Q}(q^{1/2})[[x]].$$

Since $[GL(n)] = (\mathbb{L}^n - 1) \dots (\mathbb{L}^n - \mathbb{L}^{n-1})$, we conclude that

$$A_V^{mot} = \mathbf{E}(\mathbb{L}^{1/2}, \hat{e}_{\gamma_1}).$$

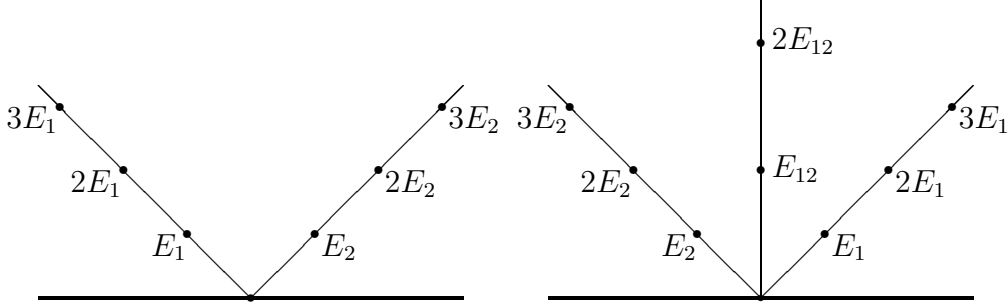
In order to simplify the notation we will denote $\mathbf{E}(q^{1/2}, x)$ simply by $\mathbf{E}(x)$. In Section 7.1 we will discuss the quasi-classical limit, and will associate numerical Donaldson-Thomas invariants $\Omega(\gamma) \in \mathbf{Q}$ for any $\gamma \in \Gamma$ for given stability structure $\sigma \in \text{Stab}(\mathcal{C}, \text{cl})$. In our basic example we have (for any σ)

$$\Omega(\pm\gamma_1) = 1, \quad \Omega(n\gamma_1) = 0 \text{ for } n \neq \pm 1.$$

2) Assume that \mathcal{C} is generated by two spherical objects E_1, E_2 defined over \mathbf{k} such that $\dim \text{Ext}^i(E_2, E_1) = 0$ if $i \neq 1$ and $\dim \text{Ext}^1(E_2, E_1) = 1$. Notice that the unique (up to isomorphism) non-trivial extension E_{12} appears in the exact triangle $E_1 \rightarrow E_{12} \rightarrow E_2$ and it is a spherical object.

For any $z_1, z_2 \in \mathbf{C}$, $\text{Im } z_i > 0, i = 1, 2$ there is a unique stability condition σ_{z_1, z_2} such that $Z(E_i) := Z(\text{cl}_{\overline{\mathbf{k}}}(E_i)) = z_i, i = 1, 2$, and the category $\mathcal{C}_V(\overline{\mathbf{k}})$ in the case $z_1, z_2 \in V, \text{Arg}(V) \subset (0, \pi)$ consists of subsequent extensions of the copies of E_1 and E_2 .

If $\text{Arg}(z_1) > \text{Arg}(z_2)$ then the only σ_{z_1, z_2} -semistable objects are (up to shifts) $E_1, 2E_1, \dots, E_2, 2E_2, \dots$, where we use the notation nE for $E^{\oplus n}$, as before. If $\text{Arg}(z_2) > \text{Arg}(z_1)$ then we have three groups of σ_{z_1, z_2} -semistable objects: $nE_1, nE_2, nE_{12}, n \geq 1$.



The wall-crossing formula implies the following well-known identity (see [22]) in the algebra $D_q\langle\langle x_1, x_2 \rangle\rangle / (x_1x_2 - qx_2x_1)$:

$$\mathbf{E}(x_1)\mathbf{E}(x_2) = \mathbf{E}(x_2)\mathbf{E}(x_{12})\mathbf{E}(x_1) ,$$

where $x_{12} = q^{-1/2}x_1x_2 = q^{1/2}x_2x_1$ and x_i corresponds to $\hat{e}_{\text{cl}_{\mathbb{K}}(E_i)}$, $i = 1, 2, 12$. Namely, both sides of the above identity are equal to $A_{V_{big,q}}$ for any sector V_{big} in the upper half-plane containing z_1, z_2 . The LHS and the RHS of the identity come from the decompositions

$$A_{V_{big,q}} = A_{V_1,q}A_{V_2,q}, \quad A_{V_{big,q}} = A_{V_2,q}A_{V_{12,q}}A_{V_1,q},$$

where V_i , $i = 1, 2, 12$ are some narrow sectors containing z_i .

Remark 22 *The function $\mathbf{E}(x)$ satisfies also the identity*

$$\mathbf{E}(x_2)\mathbf{E}(x_1) = \mathbf{E}(x_1 + x_2)$$

for x_1, x_2 obeying the relations $x_1x_2 = qx_2x_1$ as above. This follows from the formula

$$\mathbf{E}(x) = \exp_q \left(\frac{q^{1/2}}{q-1} x \right)$$

where $\exp_q(x)$ is the usual q -exponent

$$\exp_q(x) := \sum_{n \geq 0} \frac{x^n}{[n]_q!}, \quad [n]_q! := \prod_{j=1}^n [j]_q, \quad [j]_q := 1 + q + \dots + q^{j-1} .$$

The exponential property of $\mathbf{E}(x)$ seems to play no role in our considerations.

If we denote by $\gamma_i \in \Gamma \simeq \mathbf{Z}^2$ the classes $\text{cl}(E_i)$, $i = 1, 2$, then the only non-trivial numerical Donaldson-Thomas invariants are

$$\Omega(\pm\gamma_1) = \Omega(\pm\gamma_2) = 1$$

in the case $\text{Arg}(z_1) > \text{Arg}(z_2)$, and

$$\Omega(\pm\gamma_1) = \Omega(\pm\gamma_2) = \Omega(\pm(\gamma_1 + \gamma_2)) = 1$$

in the case $\text{Arg}(z_1) < \text{Arg}(z_2)$.

6.5 D0-D6 BPS bound states: an example related to the MacMahon function

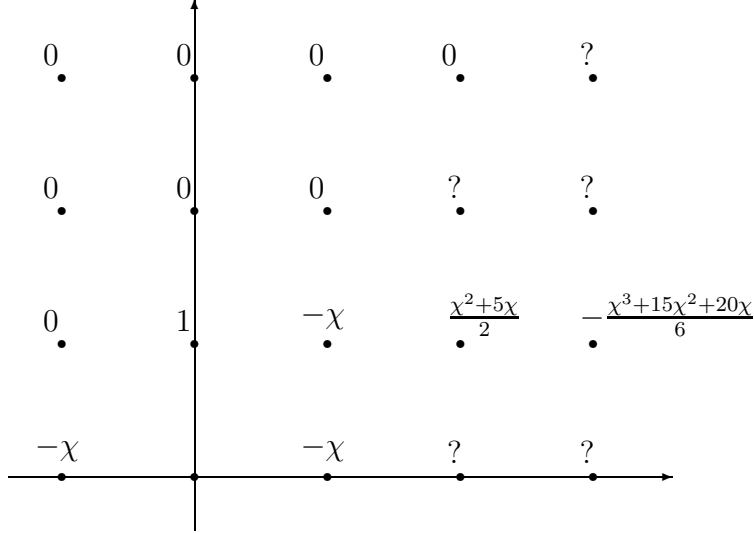
Let X be a compact $3d$ Calabi-Yau manifold over \mathbf{k} , such that $H^1(X, \mathcal{O}_X) = 0$. We denote by $\mathcal{C}^{(0,6)}$ the ind-constructible triangulated category generated by the structure sheaf \mathcal{O}_X and torsion sheaves $\mathcal{O}_x, x \in X$.²² This category has a t -structure with the heart consisting of coherent sheaves on X which are trivial vector bundles outside of a finite set. Then \mathcal{O}_X is the only spherical object in $\mathcal{C}^{(0,6)}$. We choose $\Gamma = \mathbf{Z}\gamma_1 \oplus \mathbf{Z}\gamma_2$, which is the image of $K_0(\mathcal{C}^{(0,6)})$ under the Chern class in the quotient of the Chow group by the numerical equivalence, where $\gamma_1 = \text{cl}_{\overline{\mathbf{k}}}(\mathcal{O}_x)$ for any point $x \in X$, and $\gamma_2 = \text{cl}_{\overline{\mathbf{k}}}(\mathcal{O}_X)$. We are going to consider a stability condition $\sigma = (Z, (\mathcal{C}^{(0,6)})^{ss}, \dots)$ on $\mathcal{C}^{(0,6)}$ with the above t -structure and such that

$$z_1 := Z(\gamma_1) = -1, \quad z_2 = Z(\gamma_2) = i = \sqrt{-1}.$$

Then σ -semistable objects in $\mathcal{C}^{(0,6)}$ will be either pure torsion sheaves supported at finitely many points or torsion-free sheaves.

This corresponds to the following picture for $\Omega(\gamma)$.

²²This category is related to the counting of D0-D6 BPS bound states, compare with [13], formula (6.1).



Let us comment on the last figure.

a) The vertical line corresponds to the subcategory generated by the spherical object \mathcal{O}_X , for which we know $\Omega(\gamma)$. Namely, $\Omega(\gamma_2) = 1$ and $\Omega(n\gamma_2) = 0, n \geq 2$.

b) Horizontal line $z_2 - nz_1, n \geq 0$ corresponds to sheaves of ideals of 0-dimensional subschemes. Then:

$$\sum_{n \geq 0} \Omega(\gamma_2 - n\gamma_1)t^n = M(-t)^{\chi(X)},$$

where $\chi(X)$ is the Euler characteristic of X and

$$M(x) := \prod_{n \geq 1} (1 - x^n)^{-n} \in \mathbf{Z}[[x]]$$

is the MacMahon function (see [47], [4] about this identity).

c) The torsion sheaves $\mathcal{O}_x, x \in X$ are Schur objects in $\mathcal{C}^{(0,6)}$. Their moduli space is canonically identified with X . By Behrend's formula (see [2]) their contribution to the virtual fundamental number of objects is

$$\Omega(\gamma_1) = (-1)^{\dim X} \chi(X) = -\chi(X).$$

d) The numbers marked by “?” correspond to (possibly non-Schur) objects. Notice that there are no semistable objects with the class $n\gamma_2 - m\gamma_1$ with $0 < m < n$. They correspond to the sector filled by 0's.

Let us now choose a path $\sigma_{z_1(\tau), z_2(\tau)}$ in the space of the above stability structures such that

$$z_1(\tau) = -\exp(i\tau), \quad z_2(\tau) = i,$$

where $\tau \in [0, \pi/2 + \varepsilon]$, $i = \sqrt{-1}$ and $\varepsilon > 0$ is sufficiently small. The heart of the t -structure for $\tau > 0$ consists of complexes of sheaves E such that there exists an exact triangle $E_1 \rightarrow E \rightarrow E_2[-1]$, where E_1 is a torsion-free sheaf and E_2 is a torsion sheaf (indeed, the new t -structure is obtained from the initial one by the standard tilting procedure). This heart coincides with the category $\mathcal{C}_V^{(0,6)}$ for any $\tau \in (0, \pi/2 + \varepsilon]$ where

$$V = \{z \in \mathbf{C}^* \mid 0 \leq \text{Arg}(z) \leq \pi/2 + \varepsilon\}.$$

Object $\mathcal{O}_X \in \mathcal{C}_V^{(0,6)}$ can not be represented as a non-trivial extension in $\mathcal{C}_V^{(0,6)}$, hence it is semistable for any $\tau \in [0, \pi/2 + \varepsilon]$.

Let us now consider the case $\tau \in (\pi/2, \pi/2 + \varepsilon]$. Then object \mathcal{O}_X has the minimal argument among all non-trivial objects in $\mathcal{C}_V^{(0,6)}$. Therefore, all other indecomposable semistable objects E are strictly on the left of \mathcal{O}_X , and we have

$$\text{Ext}^0(E, \mathcal{O}_X) = 0.$$

Taking the long exact sequence of Ext -groups to the object \mathcal{O}_X one easily shows that in the decomposition $E_1 \rightarrow E \rightarrow E_2[-1]$ we have $E_1 = 0$. Hence in this new heart $\mathcal{C}_V^{(0,6)}$ the left orthogonal to \mathcal{O}_X consists of objects $F[-1]$, where F is a torsion sheaf. We conclude that for the stability condition with $\tau \in (\pi/2, \pi/2 + \varepsilon]$ the only semistable objects have classes which belong to $\mathbf{Z}_{\neq 0}\gamma_2 \sqcup \mathbf{Z}_{\neq 0}\gamma_1$. Therefore, all DT-invariants $\Omega_\tau(\gamma)$ for $\sigma_{z_1(\tau), z_2(\tau)}$ with $\tau > \pi/2$ are completely determined by the numbers $a_n = \Omega_\tau(-n\gamma_1)$, $n \geq 1$ (and known invariants $\Omega_\tau(m\gamma_2) = \delta_{m,1}$, $m \geq 1$). Then the wall-crossing formula determines all the invariants $\Omega(\gamma) = \Omega_0(\gamma)$ for the initial stability condition $\sigma_{z_1(0), z_2(0)}$ in terms of the numbers a_n , $n \geq 1$.

The wall-crossing formula implies that the following identity:

$$\prod_{n \geq 1} T_{-n\gamma_1}^{a_n} T_{\gamma_2} = \prod_{m \geq 1, n \geq 0} \overrightarrow{T_{-n\gamma_1 + m\gamma_2}^{\Omega(-n\gamma_1 + m\gamma_2)}} \prod_{n \geq 1} T_{-n\gamma_1}^{a_n}.$$

Using the known result for special values $\Omega(\gamma_2 - n\gamma_1)$, $n \geq 1$ (in terms of the MacMahon function), one can deduce that all the numbers $a_n = \Omega(n\gamma_1)$

for $n \geq 1$ are equal to $-\chi(X)$. We don't know how to prove this identity directly. We see that invariants $\Omega_\tau(\gamma)$ for $\tau > \pi/2$ have a much simpler form than $\Omega(\gamma) = \Omega_0(\gamma)$. Moreover, it is now possible (in principle) to work out a formula for $\Omega(-n\gamma_1 + m\gamma_2)$ for any given $m \geq 2$.

Remark 23 *One can try to generalize the above considerations to the case of D0-D2-D6 bound states. Mathematically this means that we consider the triangulated category generated by the sheaf \mathcal{O}_X and sheaves with at most 1-dimensional support (cf. [47]). A problem arises here, since for the natural t -structure there is no central charge which gives a stability condition on the category. Presumably, in this case one can use the limit stability conditions (see [1], [73]).*

Remark 24 *Let X be a 3d complex Calabi-Yau manifold, $C \simeq \mathbf{P}^1 \subset X$ a rational curve with normal bundle isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and \mathcal{C} be an ind-constructible A_∞ -version of the category $\text{Perf}_C(X)$ of perfect complexes supported on C . Then $\Gamma := K_0(\mathcal{C}) \simeq \mathbf{Z}^2$ carries a trivial skew-symmetric (Euler) form. The lattice Γ is generated by $\text{cl}(\mathcal{O}_{pt})$ and $\text{cl}(\mathcal{O}_C)$. It follows that there are no wall-crossings in this case, and hence our invariants $\Omega(\gamma)$ do not change under continuous deformations of a stability condition. In order to use this idea for computations one can choose two stability conditions by specifying the corresponding t -structures and central charges:*

a) *choose the t -structure with the heart consisting of coherent sheaves on X supported on C and the central charge Z such that*

$$Z(\text{cl}(\mathcal{O}_{pt})) \in \mathbf{R}_{<0}, \quad \text{Im } Z(\text{cl}(\mathcal{O}_C)) > 0 ;$$

b) *choose the t -structure given by the category of finite-dimensional representations of the quiver with two vertices and two double arrows in each direction and the potential*

$$W = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1 .$$

Then calculations from [66] give the following formulas for the invariants $\Omega(\gamma)$:

$$\begin{aligned} \Omega(n \text{cl}(\mathcal{O}_{pt})) &= -2, n \neq 0 ; \\ \Omega(n \text{cl}(\mathcal{O}_{pt}) \pm \text{cl}(\mathcal{O}_C)) &= 1, n \in \mathbf{Z} . \end{aligned}$$

In all other cases $\Omega(\gamma) = 0$.

For recent generalization of [66] see [49],[50],[51].

7 Quasi-classical limit and integrality conjecture

7.1 Quasi-classical limit, numerical DT-invariants

The elements $A_{V,q} \in \mathcal{R}_{V,q}$ corresponding to A_V^{mot} are series in $\hat{e}_\gamma, \gamma \in \Gamma$ with coefficients which are rational functions in $q^{1/2}$. They can have poles as $q^n = 1$ for some $n \geq 1$. Hence it is not clear how to take the quasi-classical limit as $q^{1/2} \rightarrow -1$ (this corresponds to the taking of Euler characteristic of the corresponding motives).

Let us assume that the skew-symmetric form on Γ is non-degenerate (otherwise we can replace Γ by the symplectic lattice $\Gamma \oplus \Gamma^\vee$). The element $A_{V,q}$ defines an automorphism of an appropriate completion of $\mathcal{R}_{\Gamma,q}$. More precisely, it acts by the conjugation $x \mapsto A_{V,q} x A_{V,q}^{-1}$ on the subring

$$\prod_{\gamma \in C_0(V) \cap \Gamma} D_q \hat{e}_\gamma$$

where $C_0(V) = C_0(V, Z, Q)$ is the union of 0 with the convex hull $C(V, Z, Q)$ of the set $Z^{-1}(V) \cap \{Q \geq 0\}$ (see Section 2).

Let us recall the example of the category generated by two spherical objects from Section 6.4. We will use notation for sectors V_1, V_2, V_{big} introduced there. One has, for quantum variables $x_1 x_2 = q x_2 x_1$ and $A_{V_1,q} = \mathbf{E}(x_1)$:

$$\begin{aligned} x_1 &\mapsto \mathbf{E}(x_1) x_1 \mathbf{E}(x_1)^{-1} = x_1 ; \\ x_2 &\mapsto \mathbf{E}(x_1) x_2 \mathbf{E}(x_1)^{-1} = x_2 (1 + q^{1/2} x_1) . \end{aligned}$$

This follows from the formula $f(x_1) x_2 = x_2 f(q x_1)$, where $f(x)$ is an arbitrary series as well from the formula

$$\mathbf{E}(x) = \prod_{n \geq 0} (1 + q^{(2n+1)/2} x)^{-1} ,$$

which is valid for $0 < q < 1$. The latter formula implies the needed identity in $\mathbf{Q}(q^{1/2})[[x]]$:

$$\mathbf{E}(q x) = (1 + q^{1/2} x) \mathbf{E}(x) .$$

A similar formula holds for the conjugation by $A_{V_2,q}$. We remark that in this example the conjugation by $A_{V,q}$ for V_1, V_2 or V_{big} preserves the subring

$\prod_{\gamma \in C_0(V) \cap \Gamma} \mathbf{Z}[q^{\pm 1/2}] \hat{e}_\gamma$. In particular, one can make a specialization at

$$q^{1/2} = -1.$$

Remark 25 Recall that at the end of Section 6.3 we defined a homomorphism

$$\mathcal{R}_\Gamma \rightarrow \mathcal{R}_{\Gamma, q}$$

as the composition of Serre polynomial with the involution $q^{1/2} \mapsto -q^{-1/2}$. In particular, the specialization $q^{1/2} = -1$ is well-defined on the subring of series in generators \hat{e}_γ with coefficients in $\mathcal{M}^\mu(\text{Spec}(\mathbf{k}))[\mathbb{L}^{-1/2}]$ (see also Section 7.3), and it corresponds to the usual Euler characteristic. We use the twisting $q^{1/2} \mapsto -q^{-1/2}$ in order to avoid a lot of minus signs in formulas.

The “integer” quantum torus

$$\bigoplus_{\gamma \in C_0(V) \cap \Gamma} \mathbf{Z}[q^{\pm 1/2}] \hat{e}_\gamma \subset \mathcal{R}_{\Gamma, q}$$

has the quasi-classical limit²³ which is the Poisson algebra with basis $e_\gamma, \gamma \in C_0(V) \cap \Gamma$ with the product and Poisson bracket given by

$$e_\gamma e_\mu = (-1)^{\langle \gamma, \mu \rangle} e_{\gamma+\mu}, \quad \{e_\gamma, e_\mu\} = (-1)^{\langle \gamma, \mu \rangle} \langle \gamma, \mu \rangle e_{\gamma+\mu}.$$

The Poisson bracket is the limit of a normalized bracket:

$$\begin{aligned} [\hat{e}_\gamma, \hat{e}_\mu] &= (q^{1/2 \langle \gamma, \mu \rangle} - q^{-1/2 \langle \gamma, \mu \rangle}) \hat{e}_{\gamma+\mu}, \\ \lim_{q^{1/2} \rightarrow -1} (q-1)^{-1} \cdot (q^{1/2 \langle \gamma, \mu \rangle} - q^{-1/2 \langle \gamma, \mu \rangle}) &= (-1)^{\langle \gamma, \mu \rangle} \langle \gamma, \mu \rangle. \end{aligned}$$

One can write informally

$$e_\gamma = \lim_{q^{1/2} \rightarrow -1} \frac{\hat{e}_\gamma}{q-1}.$$

Conjecture 5 For any 3d Calabi-Yau category with polarization and any strict sector V the automorphism $x \mapsto A_{V, q} x A_{V, q}^{-1}$ preserves the subring

$$\prod_{\gamma \in C_0(V) \cap \Gamma} D_q^+ \hat{e}_\gamma,$$

where $D_q^+ := \mathbf{Z}[q^{\pm 1/2}]$.

²³There is another quasi-classical limit $q^{1/2} \rightarrow +1$ which we do not consider here.

Later we will present arguments in favor of this conjecture as well as a stronger version. Assuming the Conjecture we can define “numerical” DT-invariants of a $3d$ Calabi-Yau category with polarization in the following way. Consider the quasi-classical limit (i.e. specialization at $q^{1/2} = -1$) of the automorphism $x \mapsto A_{V,q} x A_{V,q}^{-1}$. We will present (see Section 7.4. and Conjecture 10) an explicit conjectural formula for this “quasi-classical limit” which does not depend on the orientation data. The quasi-classical limit gives rise to a formal symplectomorphism of the torus \mathbb{T}_Γ and therefore induces the stability data on the graded Lie algebra \mathfrak{g}_Γ (see Section 2.5). Alternatively, we can define such data as

$$a(\gamma) := \lim_{q^{1/2} \rightarrow -1} (q - 1) a(\gamma)_q$$

in the obvious notation. For a generic central charge Z the symplectomorphism can be written as

$$A_V = \prod_{Z(\gamma) \in V}^{\rightarrow} T_\gamma^{\Omega(\gamma)},$$

where

$$T_\gamma(e_\mu) = (1 - e_\gamma)^{\langle \gamma, \mu \rangle} e_\mu$$

and $\Omega(\gamma) \in \mathbf{Q}$ (see Section 2.5). In the above example of the Calabi-Yau category generated by one spherical object E we have $\Omega(n \operatorname{cl}(E)) = 1$ if $n \neq 1$ and $\Omega(n \operatorname{cl}(E)) = 0$ otherwise.

Conjecture 6 *For a generic central charge Z all numbers $\Omega(\gamma), \gamma \in \Gamma \setminus \{0\}$ are integers.*

The collection $(\Omega(\gamma))_{\gamma \in \Gamma}$ seems to be the correct mathematical definition of the counting of BPS states in String Theory.

Finally, we make a comment about the relationship with the work of Kai Behrend (see [2]). Recall that he defined a \mathbf{Z} -valued invariant of a critical point x of a function f on X which is equal to

$$(-1)^{\dim X} (1 - \chi(MF_x(f))),$$

where χ denotes the Euler characteristic. By Thom-Sebastiani theorem this number does not change if we add to f a function with a quadratic singularity at x (stable equivalence).

Let M be a scheme with perfect obstruction theory (see [4]). Thus M is locally represented as a scheme of critical points of a function f on a manifold X . Then the above invariant gives rise to a \mathbf{Z} -valued constructible function B on M . The global invariant is

$$\int_M B d\chi := \sum_{n \in \mathbf{Z}} n \chi(B^{-1}(n)),$$

where χ denotes the Euler characteristic. Behrend proved that for a *proper* M the invariant $\int_M B d\chi$ coincides with the degree of the virtual fundamental class $[M]^{virt} \in H_0(M)$ given by $\int_{[M]^{virt}} 1$.

Now let us assume that $M \subset \mathcal{C}^{ss}$ consists of Schur objects E (see Section 1.3), such that $\text{cl}(E) = \gamma \in \Gamma$ is a fixed primitive class. Let us look at the contribution of M to the motivic DT-invariant $a(\gamma)_{mot}$. By definition it is equal to

$$\int_M \frac{\mathbb{L}^{\frac{1}{2}(1 - \dim \text{Ext}^1(E, E))}}{\mathbb{L} - 1} (1 - MF(E))(1 - MF_0(Q_E)) \mathbb{L}^{-\frac{1}{2} \text{rk } Q_E} \hat{e}_\gamma.$$

Mapping it to the quantum torus and taking the quasi-classical limit $q^{1/2} \rightarrow -1$, and taking into account the relation $-a(\gamma) = \Omega(\gamma)$ for primitive $\gamma \in \Gamma$ (see Section 2.5), we obtain that Behrend's formula implies that the contribution of M to the value $\Omega(\gamma)$ is equal to $\int_{[M]^{virt}} 1$.

7.2 Deformation invariance and intermediate Jacobian

We also expect the following (not very precise) conjecture to be true as well.

Conjecture 7 *The collection $(\Omega(\gamma))_{\gamma \in \Gamma}$ is invariant with respect to the “polarization preserving” deformations of \mathcal{C} , in the case when \mathcal{C} is homologically smooth in the sense of [42].*

The motivation for the last Conjecture is the deformation invariance of the virtual fundamental class in the “classical” Donaldson-Thomas theory. Recall that homologically smooth *Ext*-finite categories can be thought as non-commutative analogs of smooth proper schemes. Hence, we can expect that the moduli stacks of semistable objects in such categories are also proper in some sense. Therefore, we can also expect that the degree of the virtual fundamental class is invariant under deformations.

Also, we expect the following generalization of our theory in the case when $\mathbf{k} = \mathbf{C}$ and the $3d$ Calabi-Yau category is homologically smooth (see [42]).

1) First, we recall that even without imposing the Calabi-Yau condition one expects that a triangulated compact homologically smooth A_∞ -category \mathcal{C} (possibly $\mathbf{Z}/2\mathbf{Z}$ -graded) admits (conjecturally) a non-commutative pure Hodge structure (see [39], [36], [42] about motivations, definitions as well as some conjectures and applications of this notion). In particular, periodic cyclic homology groups $HP_{even}(\mathcal{C})$ (resp. $HP_{odd}(\mathcal{C})$) carry descending Hodge filtrations

$$\begin{aligned} HP_{even}(\mathcal{C}) \cdots \supset F_{even}^i \supset F_{even}^{i-1} \supset \dots, i \in \mathbf{Z} \\ HP_{odd}(\mathcal{C}) \cdots \supset F_{odd}^i \supset F_{odd}^{i-1} \supset \dots, i \in \mathbf{Z} + \frac{1}{2}. \end{aligned}$$

In $3d$ Calabi-Yau case we assume that the smallest non-trivial term of the filtration F_{odd}^\bullet is $F^{-3/2}$, $\dim F^{-3/2} = 1$. Moreover, in general, it is expected that there are lattices $K_{top}^{even}(\mathcal{C})$ and $K_{top}^{odd}(\mathcal{C})$ which belong to the corresponding periodic cyclic homology groups (they represent the non-commutative version of the image of the topological K -theory in the de Rham cohomology).

2) If \mathcal{C} is homologically smooth Calabi-Yau category then it is easy to see that (assuming the degeneration of the Hodge-to-de Rham conjecture, see [42]) the moduli space \mathcal{M} of formal deformations of \mathcal{C} is smooth of dimension $\dim \mathcal{M} = \frac{1}{2} \dim HP_{odd}(\mathcal{C})$ (this is a corollary of the formality of the little disc operad as well as the fact that the action of the Connes differential is represented by the rotation of the circle, which is homotopically trivial under the assumption). It is expected that the global moduli space also exists. Notice that the Calabi-Yau structure on \mathcal{C} induces a symplectic structure on the vector space $HP_{odd}(\mathcal{C})$ and in the $3d$ case the moduli space \mathcal{M} is locally embedded into $HP_{odd}(\mathcal{C})$ as a Lagrangian cone.

3) We expect that for an arbitrary triangulated compact homologically smooth A_∞ -category \mathcal{C} one has a non-commutative version of the Deligne cohomology $H_D(\mathcal{C})$ which fits into a short exact sequence

$$0 \rightarrow HP_{odd}(\mathcal{C}) / (F_{odd}^{1/2} + K_{odd}^{top}(\mathcal{C})) \rightarrow H_D(\mathcal{C}) \rightarrow F_{even}^0 \cap K_{even}^{top}(\mathcal{C}) \rightarrow 0.$$

Morally, $H_D(\mathcal{C})$ should be thought as zero cohomology group of the homotopy colimit of the following diagram of cohomology theories:

$$\begin{array}{ccc} & HC_{\bullet}^{-}(\mathcal{C}) & \\ & \downarrow & \\ K_{\bullet}^{top}(\mathcal{C}) & \longrightarrow & HP_{\bullet}(\mathcal{C}) \end{array}$$

where $HC_{\bullet}^{-}(\mathcal{C})$ is the negative cyclic homology.

Any object of \mathcal{C} should have its characteristic class in $H_D(\mathcal{C})$. More precisely, there should be a homomorphism of groups $ch_D : K_0(\mathcal{C}) \rightarrow H_D(\mathcal{C})$ (in the case of Calabi-Yau manifold it is related to holomorphic Chern-Simons functional). The reason for this is that every object $E \in Ob(\mathcal{C})$ has natural characteristic classes in $K_0^{top}(\mathcal{C})$ and in $HC_0^{-}(\mathcal{C})$ whose images in $HP_0(\mathcal{C})$ coincide with each other. The total space \mathcal{M}^{tot} of the fibration $\mathcal{M}^{tot} \rightarrow \mathcal{M}$ with the fiber $H_D(\mathcal{C})$ over the point $[\mathcal{C}] \in \mathcal{M}$ should be a holomorphic symplectic manifold (cf. [20]). Moreover, any fiber of this fibration (i.e. the group $H_D(\mathcal{C})$ for given $[\mathcal{C}]$) is a countable union of complex Lagrangian tori. By analogy with the commutative case we expect that the locus $\mathcal{L} \subset \mathcal{M}^{tot}$ consisting of values of ch_D is a countable union of Lagrangian subvarieties. Every such subvariety can be either a finite ramified covering of \mathcal{M} or a fibration over a proper subvariety of \mathcal{M} with the fibers which are abelian varieties.

4) For generic $[\mathcal{C}] \in \mathcal{M}$ one can use the triple $(K_0(\mathcal{C}), H_D(\mathcal{C}), ch_D)$ instead of the triple $(K_0(\mathcal{C}), \Gamma, cl)$. Analogs of our motivic Donaldson-Thomas invariants $A_V^{mot} \in \mathcal{R}_V$ will be formal countable sums of points in $H_D(\mathcal{C})$ with “weights” which are elements of the motivic ring \overline{D}^{μ} . The pushforward map from $H_D(\mathcal{C})$ to $\Gamma = F_{even}^0 \cap K_0^{top}(\mathcal{C})$ gives the numerical DT-invariants. The continuity of motivic DT-invariants means that after taking the quasi-classical limit the weights become integer-valued functions on the set of those irreducible components of \mathcal{L} which are finite ramified coverings on \mathcal{M} .

These considerations lead to the following

Question 2 *Is there a natural extension of the numerical DT-invariants to those components of \mathcal{L} which project to a proper subvariety of \mathcal{M} ?*

Remark 26 *Let us notice the similarity of the above considerations with those in the theory of Gromov-Witten invariants. Suppose X is a 3d complex*

compact Calabi-Yau manifold with $H^1(X, \mathbf{Z}) = 0$. Then we have an exact sequence

$$0 \rightarrow H_{DR}^3(X)/(F^2 H_{DR}^3(X) + H^3(X, \mathbf{Z})) \rightarrow H_D^4(X) \rightarrow H^4(X, \mathbf{Z}) \rightarrow 0 ,$$

where $H_D^4(X) = \mathbb{H}^4(X, \mathbf{Z} \rightarrow \mathcal{O}_X \rightarrow \Omega_X^1)$ is the Deligne cohomology. Then any curve $C \subset X$ defines the class $[C] \in H_D^4(X)$. For a generic complex structure on X the class is constant in any smooth connected family of curves. Moreover, a stable map to X defines a class in $H_D^4(X)$. Then we have exactly the same picture with holomorphic symplectic fibration $\mathcal{M}^{tot} \rightarrow \mathcal{M}$ with the Lagrangian fibers, as we discussed above. Similarly to the case of DT-invariants the GW-invariants appear as infinite linear combinations of points in $H_D^4(X)$, but this time with rational coefficients. We expect that the well-known relationship “GW=DT” (see [47]) should be a statement about the equality of the above-discussed counting functions (assuming positive answer to the above question).

7.3 Absence of poles in the series A_V^{Hall}

Here we are going to discuss a stronger version of the Conjecture 5.

Conjecture 8 Let $\overline{D}^+ := \overline{\mathcal{M}^\mu}(\text{Spec}(\mathbf{k}))[\mathbb{L}^{-1/2}]$ be the ring of equivalence classes of motivic functions. Then the automorphism of the motivic quantum torus given by $x \mapsto A_V^{\text{mot}} x (A_V^{\text{mot}})^{-1}$ preserves the subring $\prod_{\gamma \in C(V) \cap \Gamma} \overline{D}^+ \hat{e}_\gamma$ for all strict sectors $V \subset \mathbf{R}^2$.

It is enough to check the conjecture for all $x = \hat{e}_\gamma, \gamma \in \Gamma$. Moreover, because of Factorization Property it is enough to consider the case when $V = l$ is a ray. In the latter case we can split the infinite product into those corresponding to different arithmetic progression, hence reducing the conjecture to the case when $Z(\Gamma) \cap l = \mathbf{Z}_{>0} \cdot \gamma_0$ for some non-zero $\gamma_0 \in \Gamma$. Then we have

$$A_l^{\text{mot}} = A_l^{\text{mot}}(\hat{e}_{\gamma_0}) = 1 + \sum_{n \geq 1} c_n \hat{e}_{\gamma_0}^n \in \overline{D}^+[[\hat{e}_{\gamma_0}]] .$$

Using the commutation relations in the motivic quantum torus we have:

$$A_l^{\text{mot}}(\hat{e}_{\gamma_0}) \hat{e}_\gamma (A_l^{\text{mot}}(\hat{e}_{\gamma_0}))^{-1} = \hat{e}_\gamma A_l^{\text{mot}}(\mathbb{L}^{\langle \gamma_0, \gamma \rangle} \hat{e}_{\gamma_0}) A_l^{\text{mot}}(\hat{e}_{\gamma_0})^{-1} .$$

Since for any series $f(t) = 1 + \dots$ we have

$$\frac{f(\mathbb{L}^n t)}{f(t)} = \frac{f(\mathbb{L}^n t)}{f(\mathbb{L}^{n-1} t)} \cdots \frac{f(\mathbb{L} t)}{f(t)},$$

in order to prove the conjecture it suffices to check that

$$A_l^{mot}(\mathbb{L}\hat{e}_{\gamma_0})A_l^{mot}(\hat{e}_{\gamma_0})^{-1} \in \overline{D^+}[[\hat{e}_{\gamma_0}]].$$

Since in that case we are dealing with objects whose central charges belong to the ray l , we can restrict ourselves to the subcategory \mathcal{C}_l . The latter can be thought of as a heart of the t -structure of an ind-constructible 3d Calabi-Yau category with *vanishing* Euler form. More precisely, $\mathcal{C}_l(\overline{\mathbf{k}})$ is an abelian artinian category with $\text{Hom}_{\mathcal{C}_l(\overline{\mathbf{k}})}(E, F) := \text{Ext}_{\mathcal{C}(\overline{\mathbf{k}})}^0(E, F)$. Then $K_0(\mathcal{C}_l(\overline{\mathbf{k}})) \simeq \oplus_{E \neq 0} \mathbf{Z} \cdot [E]$, where the sum runs over the set of non-zero simple objects of $\mathcal{C}_l(\overline{\mathbf{k}})$.

Next, we can reduce the conjecture to a special case when $\text{cl}_{\mathbf{k}}(E) = \gamma_0$ is a fixed class for all simple objects E of $\mathcal{C}_l(\overline{\mathbf{k}})$. Indeed, let us consider an ind-constructible homomorphism $\text{cl}'_{\mathbf{k}} : K_0(\mathcal{C}_l(\overline{\mathbf{k}})) \rightarrow \Gamma' := \mathbf{Z} \oplus \mathbf{Z}$ such that $\text{cl}'_{\mathbf{k}}(E) = (1, 0)$ if $\text{cl}_{\mathbf{k}}(E) = \gamma_0$ and $\text{cl}'_{\mathbf{k}}(E) = (0, 1)$ if $\text{cl}_{\mathbf{k}}(E) \in \{2\gamma_0, 3\gamma_0, \dots\}$ for a simple object E . Let choose two complex numbers z_1, z_2 in such a way that $0 < \text{Arg}(z_1) < \text{Arg}(z_2) < \pi$ and define a central charge $Z' : \Gamma' \rightarrow \mathbf{C}$ by the formula $Z'((1, 0)) = z_1, Z'((0, 1)) = z_2$. In this way we obtain a new stability structure on the triangulated envelope of $\mathcal{C}_l(\overline{\mathbf{k}})$ with the same heart. In particular, the element A_l^{mot} will be decomposed into an infinite product:

$$A_l^{mot} = \overrightarrow{\prod} A_{l'}^{mot}$$

of series $A_{l'}^{mot}$ corresponding to abelian categories $\mathcal{C}_{l'}(\overline{\mathbf{k}})$ for the new stability structure. One of these categories will be the subcategory generated by simple objects E such that $\text{cl}'_{\mathbf{k}}(E) = \gamma_0$. Let us call such category *pure of class* γ_0 . All other categories $\mathcal{C}_{l'}(\overline{\mathbf{k}})$ do not contain objects with the class γ_0 . Repeating the procedure we reduce the conjecture to the case of pure category of the class $m\gamma_0$ for some $m \geq 1$. Similarly to the above arguments we can reduce it further to the case $m = 1$. In this case the conjecture follows from the one below which concerns Hall algebras of categories which are no longer required to carry a Calabi-Yau structure.

In order to formulate this new conjecture we are going to use the following set-up. Let $(\mathcal{C}, \mathcal{A})$ be a pair consisting of an ind-constructible triangulated A_∞ -category over a ground field \mathbf{k} and $\mathcal{A} \subset \text{Ob}(\mathcal{C})$ be an ind-constructible subset such that $\mathcal{A}(\overline{\mathbf{k}})$ is the heart of a bounded t -structure in $\mathcal{C}(\overline{\mathbf{k}})$. We assume that simple objects of the abelian category $\mathcal{A}(\overline{\mathbf{k}})$ form a constructible subset of $\text{Ob}(\mathcal{C})(\overline{\mathbf{k}})$ and every object in $\mathcal{A}(\overline{\mathbf{k}})$ is a finite extension of simple ones. These data are equivalent to a special kind of an ind-constructible category with a stability structure. Namely, let us take $\Gamma := \mathbf{Z}$ and define $\text{cl}_{\overline{\mathbf{k}}}(E) = 1$ for every simple object of $\mathcal{A}(\overline{\mathbf{k}})$. It follows that $\text{cl}_{\overline{\mathbf{k}}}(F) = \text{length}(F)$ for any object of $\mathcal{A}(\overline{\mathbf{k}})$. Furthermore, we choose a complex number z_0 in the upper-half plane and define a central charge $Z : \Gamma \rightarrow \mathbf{C}$ by the formula $Z(1) = z_0$. Then $\mathcal{A} = \mathcal{C}_l$ for $l = \mathbf{R}_{>0} \cdot z_0$. Therefore the element A_l^{Hall} defined for this stability structure can be thought of as a series in one variable:

$$A_l^{\text{Hall}}(t) = 1 + \sum_{n \geq 1} c_n t^n.$$

Let us define a subalgebra $H^+(\mathcal{C}) \subset H(\mathcal{C})$ to be the set of linear combination of elements of the form

$$\mathbb{L}^n \cdot [Z \rightarrow \text{Ob}(\mathcal{C})]$$

where $n \in \mathbf{Z}$ and $Z \rightarrow \text{Ob}(\mathcal{C})$ is a 1-morphism of ind-constructible stacks (see Section 4.2) with Z being an ordinary constructible set endowed with trivial action of the trivial group. The multiplication law in $H(\mathcal{C})$ preserves such class of elements.

Conjecture 9 *The element $F_l(t) := A_l^{\text{Hall}}(\mathbb{L}t)A_l^{\text{Hall}}(t)^{-1}$ belongs to the completed Hall algebra $\widehat{H}^+(\mathcal{C})$ (i.e. we do not invert motives $[GL(n)]$, $n \geq 1$ of the general linear groups).*

Below we discuss two special cases in which the above conjecture holds. But first we present a similar motivating statement in the case of finite fields. Let R be finitely generated algebra over a finite field \mathbf{F}_q , and $R\text{-mod}^f$ denotes the category of finite-dimensional (over \mathbf{F}_q) left R -modules. We define the Hall algebra $H(R\text{-mod}^f)$ as a unital associative algebra over the ring $\mathbf{Z}[\frac{1}{q}]$ generated by the isomorphism classes $[M]$ of objects of $R\text{-mod}^f$ with the

multiplication

$$[E] \cdot [F] = q^{-\dim \text{Hom}(F,E)} \sum_{\alpha \in \text{Ext}^1(F,E)} [E_\alpha] ,$$

where, as before, E_α denotes an extension with the class α .

Proposition 13 *Let*

$$A(t) := \sum_{[M] \in \text{Iso}(R\text{-mod}^f)} \frac{[M]}{\# \text{Aut}(M)} t^{\dim M} .$$

Then

$$F(t) := A(qt)A(t)^{-1} \in H(R - \text{mod}^f)[[t]] .$$

Moreover,

$$F(t) = \sum_{I \subset R, I=RI, \dim R/I < \infty} [R/I] t^{\dim R/I} .$$

Hence the quotient $F(t)$ does not have denominator $(q^n - 1)$, $n \geq 1$ and can be represented in terms of the “non-commutative Hilbert scheme” of left ideals in R of finite codimension.

Proof. Let us make use of the basis of “renormalized” elements

$$[\widehat{E}] := \frac{[E]}{\# \text{Aut}(E)}$$

in the \mathbf{Q} -algebra $H(R - \text{mod}^f) \otimes \mathbf{Q}$. Then the product can be rewritten in a more familiar form:

$$[\widehat{E}] \cdot [\widehat{F}] = \sum_{[G]} c_{[\widehat{E}], [\widehat{F}]}^{[\widehat{G}]} [\widehat{G}] ,$$

where the structure constant $c_{[\widehat{E}], [\widehat{F}]}^{[\widehat{G}]} \in \mathbf{Z}$ denotes the number of subobjects in G isomorphic to E and such that the quotient is isomorphic to F . In these notation we have:

$$A(t) = \sum_{[M]} [\widehat{M}] t^{\dim M} .$$

Since first statement of the Proposition follows from the second one, we are going to show the latter. In the new notation it becomes:

$$\begin{aligned} \sum_{I \subset R, I=RI, \dim R/I < \infty} [\widehat{R/I}] \# \text{Aut}(R/I) t^{\dim R/I} \cdot \sum_{[M]} [\widehat{M}] t^{\dim M} &= \\ &= \sum_{[N]} [\widehat{N}] q^{\dim N} t^{\dim N} . \end{aligned}$$

Let us fix an object N , and consider the coefficient of the term $[\widehat{N}]t^{\dim N}$. In the RHS it is equal to $q^{\dim N}$. It is easy to see that the corresponding coefficient in the LHS is of the form

$$\begin{aligned} \sum_{I \subset R, I=RI, \dim R/I < \infty} \sum_{N' \subset N, N' \simeq R/I} \# \text{Aut}(R/I) &= \\ = \sum_{I \subset R, I=RI, \dim R/I < \infty} \sum_{R/I \hookrightarrow N} 1 &= \#N = q^{\dim N} . \end{aligned}$$

Notice that in the last sum we consider all possible embeddings of R/I to N and every summand corresponds to a choice of a cyclic vector in a cyclic submodule in N . This proves the Proposition. ■

The above Proposition suggest to interpret our category as a category of modules and then apply similar arguments which reduce the sum (or even the motivic integral) to the sum over all cyclic submodules. It is useful to keep this in mind when considering two examples in the next subsection.

Remark 27 *The subalgebra $H^+(\mathcal{C})$ of the Hall algebra has the advantage that one can apply the Euler characteristic χ to its elements fiberwise over $\text{Ob}(\mathcal{C})$, and get a constructible \mathbf{Z} -valued function (with constructible support) on the ind-constructible set $\text{Iso}(\mathcal{C})$ of isomorphism classes of objects of $\mathcal{C}(\overline{\mathbf{k}})$. The multiplication in $H^+(\mathcal{C})$ descends to a multiplication on the abelian group of such functions. It is easy to see that this multiplication is commutative, and one has*

$$\nu_E \times \nu_F = \nu_{E \oplus F}$$

where ν_E etc. are delta-functions (see Section 6.1). This follows from the fact that for any non-zero $\alpha \in \text{Ext}^1(F, E)$ all objects $E_{t\alpha}$ are isomorphic to each other for $t \in \overline{\mathbf{k}}^\times$, and the Euler characteristic of \mathbf{G}_m is zero.

7.4 Reduction to the case of category of modules

Here we present two special cases when the conjecture holds.

1) Assume that the abelian category $\mathcal{A}(\bar{\mathbf{k}})$ contains only one²⁴ (up to an isomorphism) simple object $E \neq 0$, and this object is defined over the field \mathbf{k} . Hence $\text{Ext}_{\mathcal{C}(\mathbf{k})}^0(E, E) \simeq \mathbf{k}$. We also assume that A_∞ -algebra $\text{Hom}^\bullet(E, E)$ is minimal, i.e. $m_1 = 0$, and hence $\text{Hom}^\bullet(E, E) = \text{Ext}^\bullet(E, E)$.

Proposition 14 *The category $\mathcal{A}(\mathbf{k})$ is equivalent to the category $B\text{-mod}^{f,cont}$ of continuous finite-dimensional representations of a finitely generated topological algebra B .*

Proof. There is a general way to construct the algebra B from the A_∞ -structure. Let x_1, \dots, x_m be a basis in the vector space $(\text{Ext}^1(E, E))^*$. Then the higher composition maps $m_n : \text{Ext}^1(E, E)^{\otimes n} \rightarrow \text{Ext}^2(E, E)$, $n \geq 2$ define a linear map

$$\sum_{n \geq 2} m_n : (\text{Ext}^2(E, E))^* \rightarrow \mathbf{k}\langle\langle x_1, \dots, x_m \rangle\rangle = \prod_{n \geq 0} ((\text{Ext}^1(E, E))^*)^{\otimes n}.$$

We define a topological algebra $B_E := B$ as the quotient of $\mathbf{k}\langle\langle x_1, \dots, x_m \rangle\rangle$ by the closure of the 2-sided ideal generated by the image of $\sum_{n \geq 2} m_n$.

Next we observe that any object M of $\mathcal{A}(\mathbf{k})$ is a finite extension of objects isomorphic to E . Hence, it can be thought of as deformation of an object $mE := E \oplus E \oplus \dots \oplus E$ (m summands) preserving the filtration

$$E \subset E \oplus E \subset \dots \subset mE,$$

where $m = \text{length}(M)$. Every such a deformation is given by a solution to the Maurer-Cartan equation

$$\sum_{n \geq 2} m_n(\alpha, \dots, \alpha) = 0,$$

where $\alpha = (\alpha_{ij})$ is an upper-triangular matrix with coefficients from $\text{Ext}^1(E, E)$. It is easy to see that such a solution gives rise to a representation of the algebra B in the upper-triangular matrices of finite size. Furthermore one checks that this correspondence provides an equivalence of categories

$$F : \mathcal{A}(\mathbf{k}) \simeq B\text{-mod}^{f,cont}.$$

²⁴The arguments below extend immediately to the case of finitely many such objects.

This proves the Proposition. ■

Notice that

$$length(M) = \dim F(M)$$

for any object M .

Using the framework of finite-dimensional continuous representations we can modify the arguments from the proof of the Proposition 13 to the case of motivic functions instead of finite fields and obtain the formula

$$A^{\text{Hall}}(\mathbb{L}t)A^{\text{Hall}}(t)^{-1} = \sum_{n \geq 0} [\pi : \text{Hilb}_n(B) \rightarrow \text{Ob}(\mathcal{C})]t^n,$$

where $\text{Hilb}_n(B)$ is the scheme of closed left ideals in B of codimension n (non-commutative analog of Hilbert scheme) and $\pi(I) = B/I$ for any such ideal.

2) Let us assume that $\mathbf{k} = \mathbf{F}_q$ and \mathcal{A} is an abelian \mathbf{k} -linear category such that every object has finitely many subobjects. We define the map $\text{cl} : K_0(\mathcal{A}) \rightarrow \mathbf{Z}$ such that $\text{cl}([E]) = n$ if E is simple object and $\text{End}(E) \simeq \mathbf{F}_{q^n}$.

Proposition 15 *Assume that \mathcal{A} is a heart of a t -structure of a triangulated Ext-finite \mathbf{F}_q -linear A_∞ -category \mathcal{C} . Let us consider the series*

$$A(t) := \sum_{[M] \in \text{Iso}(\mathcal{A})} \frac{[M]}{\#\text{Aut}(M)} t^{\text{cl}(M)}.$$

Then we claim that

$$F(t) := A(qt)A(t)^{-1} = \sum_{[M], M \text{ is cyclic}} c_M [M] t^{\text{cl}(M)},$$

where $c_M \in \mathbf{Z}[\frac{1}{q}]$, and the notion of a cyclic object is introduced below.

We are going to reduce the proof to the case of modules over an algebra. Moreover we will give an explicit formula for the coefficients c_M . In order to do that we need the following categorical definition of a cyclic object.

Definition 19 *We say that an object N in an artinian abelian category is cyclic if there is no epimorphism $N \rightarrow E \oplus E$ where $E \neq 0$ is simple.*

In the category of finite-dimensional modules over an associative algebra (over any field), cyclic objects are the same as cyclic modules.

Any object $M \in \mathcal{A}$ admits a decomposition $M = \bigoplus_{\alpha} M_{\alpha}$ into a direct sum of indecomposables. For each indecomposable summand M_{α} we have a decomposition $M_{\alpha}^{ss} = \bigoplus_i E_{\alpha,i}$ of its maximal semisimple factor M_{α}^{ss} (called the cosocle of M_{α}) into a direct sum of simple objects $E_{\alpha,i}$.

Let us assume that M is a cyclic object. It is equivalent to the condition that all simple factors $E_{\alpha,i}$ are pairwise different. Notice that

$$\text{End}(M^{ss}) = \bigoplus_{\alpha,i} \text{End}(E_{\alpha,i}) \simeq \bigoplus_{\alpha,i} \mathbf{F}_q^{m_{\alpha,i}}$$

where $m_{\alpha,i} = \text{cl}(E_{\alpha,i}) \in \mathbf{Z}_{>0}$. Also, it follows from the cyclicity of M that

$$\text{End}(M)^{ss} \simeq \bigoplus_{\alpha} \mathbf{F}_{q^{n_{\alpha}}}$$

for some positive integers n_{α} . It follows from the definition that $m_{\alpha,i}$ is divisible by n_{α} for any pair (α, i) . Observe that in the above notation

$$\# \text{Aut}(M) = q^r \prod_{\alpha} (q^{n_{\alpha}} - 1),$$

where r is the dimension over \mathbf{F}_q of the radical of $\text{End}(M)$. Now we claim that in the above Proposition 15

$$c_M = \frac{q^{\text{cl}(M)} \cdot \prod_{\alpha,i} \frac{q^{m_{\alpha,i}} - 1}{q^{m_{\alpha,i}}}}{q^r \cdot \prod_{\alpha} (q^{n_{\alpha}} - 1)}.$$

The property $n_{\alpha} | m_{\alpha,i}$ implies that $c_M \in \mathbf{Z}[\frac{1}{q}]$. We are going to prove the Proposition together with the above formula for c_M .

Proof. We may assume that \mathcal{A} is generated by finitely many simple objects (but they can be defined over different finite extensions of \mathbf{F}_q). First, we claim that \mathcal{A} is equivalent to the category $B\text{-mod}^{f,cont}$ of finite-dimensional continuous representations of a topological algebra B , similarly to the previous example. More precisely, let $N = \bigoplus_i E_i$ be the direct sum of all simple objects E_i , and set $C := \text{End}(N)$. Then C is a semisimple associative unital \mathbf{F}_q -algebra, which is isomorphic to $\bigoplus_i \mathbf{F}_{q^{\text{cl}(E_i)}}$. Let us consider $\text{Ext}^1(N, N)$ as a C -bimodule and take

$$G := \text{Hom}_{C \otimes C^{op}\text{-mod}}(\text{Ext}^1(N, N), C \otimes C^{op})$$

to be the dual bimodule. The topological free algebra

$$\prod_{n \geq 0} G \otimes_C \otimes_C G \otimes_C \cdots \otimes_C G \quad (n \text{ tensor factors})$$

contains a closed two-sided ideal generated by the image of the map $\sum_{n \geq 2} m_n$ (here we use the “ A_∞ -origin” of our abelian category). We denote by B the quotient of the free algebra by this ideal. Then B can be thought of as a completed path algebra of the quiver defined by simple objects E_i with the arrows which correspond to a basis of $(\text{Ext}^1(E_i, E_j))^*$. Similarly to the previously considered example, we have an equivalence of categories $\Psi : \mathcal{A} \simeq B - \text{mod}^{f, \text{cont}}$. Under this equivalence simple object E_i maps to the direct summand $\mathbf{F}_{q^{\text{cl}(E_i)}}$ of C , hence $\dim \Psi(E_i) = \text{cl}(E_i)$. It follows that for any object M we have $\dim \Psi(M) = \text{cl}(M)$.

Lemma 3 *Module $M \in B - \text{mod}^{f, \text{cont}}$ is cyclic iff M^{ss} is cyclic. Moreover $v \in M$ is a generator iff its projection $\bar{v} \in M^{ss}$ is a generator.*

Proof of lemma. The first statement follows directly from the definition of a cyclic object. In order to prove the second statement assume that $\bar{v} \in M^{ss}$ is a generator. We want to prove that the quotient $M/Bv = 0$. If this is not the case then we have an epimorphism $M/Bv \rightarrow E_{i_0}$ to a simple module E_{i_0} . It follows that we have an epimorphism $M^{ss} \rightarrow E_{i_0}$ such that $\bar{v} \mapsto 0$. This contradicts to the assumption that $\bar{v} \in M^{ss}$ is a generator. The lemma is proved. ■

In order to finish the proof of the Proposition, it is enough to check that the coefficient c_M given by a product formula on the previous page, is equal to the number of isomorphism classes of generators $v \in M$ up to an automorphism of M . In order to do that we observe that the product $\prod_{\alpha, i} (q^{m_{\alpha, i}} - 1)$ from the formula for c_M is in fact equal to the number of generators of M^{ss} . Furthermore, the factor $\frac{q^{\text{cl}(M)}}{\prod_{\alpha, i} q^{m_{\alpha, i}}}$ is equal to the number of liftings of a generator of M^{ss} to a generator of M (this number is the number of elements in the kernel of the projection $M \rightarrow M^{ss}$). Finally, we recall that $q^r \prod_{\alpha} (q^{n_{\alpha}} - 1) = \# \text{Aut}(M)$. Applying the above lemma we finish the proof of the Proposition. ■

Remark 28 *It looks plausible that the Proposition holds without the assumption that \mathcal{A} is a t -structure of an A_∞ -category.*

We do not know the “motivic” analog of the above Proposition. In that case one should replace \mathcal{A} by an ind-constructible abelian category over any field. There is a notion of semisimple and cyclic modules, it is preserved under field extensions²⁵. It looks natural to expect that an analog of the quotient $\frac{\prod_{\alpha,i}(q^{m_{\alpha,i}}-1)}{\prod_{\alpha}(q^{n_{\alpha}}-1)}$ is the motive $\underline{\text{Aut}}(M^{ss})/\underline{\text{Aut}}'(M)$ where $\underline{\text{Aut}}(M^{ss})$ is the affine group scheme of automorphisms of M^{ss} and $\underline{\text{Aut}}'(M)$ is the image of the scheme of automorphisms of M in $\underline{\text{Aut}}(M^{ss})$. Both groups schemes are algebraic tori. Although the motivic version is not absolutely clear, we can write down the “numerical” version, which is the result of the quasi-classical limit $q^{1/2} \rightarrow -1$ (equivalently, this is the result of taking the Euler characteristic of the corresponding motives).

It follows from the Proposition that in the quasi-classical limit only those terms in the formula for c_M are non-zero for which $\underline{\text{Aut}}(M^{ss}) = \underline{\text{Aut}}'(M)$. Let us call such objects *special cyclic*. A cyclic object is special cyclic iff under the extension of scalars to $\bar{\mathbf{k}}$ the cocycles of all indecomposable summands (i.e. objects M_{α}^{ss} in our notation) are simple.

In the case of finite-dimensional modules over an associative algebra A , a cyclic object (or module) M is special iff the scheme of left ideals $I \subset A$ such that $M \simeq A/I$ has Euler characteristic 1. For non-special cyclic objects the corresponding Euler characteristic vanishes.

Let us return to our considerations in the case of $3d$ ind-constructible Calabi-Yau category over a field \mathbf{k} of characteristic zero. We reduced the main conjecture to the case of a single ray, hence $\mathcal{A}(\bar{\mathbf{k}})$ is the heart of a t -structure of \mathcal{C}_l . In this case isomorphism classes of special cyclic objects M with the fixed class $\text{cl}(M)$ form a constructible set \mathcal{SC}_n . Thus, we arrive to the following formula

$$\begin{aligned} \chi_{\Phi}(F_l(t)) &= \chi_{\Phi}(A_l^{\text{mot}}(\mathbb{L}t)A_l^{\text{mot}}(t)^{-1}) = \\ &= \sum_{n \geq 0} t^n \int_{\mathcal{SC}_n} (-1)^{(M,M) \leq 1} (1 - \chi(MF(M))) d\chi, \end{aligned}$$

where $\int_V f d\chi = \sum_{n \in \mathbf{Z}} n \chi(f^{-1}(n))$ denotes the “integral over Euler characteristic” χ of the map $f : V \rightarrow \mathbf{Z}$, and χ_{Φ} is the composition of the homomorphism Φ from the motivic Hall algebra to the motivic quantum torus (restricted to subalgebra $H^+(\mathcal{C}) \subset H(\mathcal{C})$), and of the Euler characteristic morphism acting on coefficients as $\chi : \bar{D}^+ \rightarrow \mathbf{Z}$.

We remark that the RHS does not depend on the orientation data.

²⁵Notice that notions of simple or indecomposable objects are *not* preserved under the field extension.

Conjecture 10 *In case if the category \mathcal{C} is not endowed with orientation data the above procedure gives rise to well-defined stability data on the graded Lie algebra \mathfrak{g}_Γ of Poisson automorphisms of the algebraic Poisson torus $\text{Hom}(\Gamma, \mathbf{G}_m)$ as well as a continuous local homeomorphism $\text{Stab}(\mathcal{C}, \text{cl}) \rightarrow \text{Stab}(\mathfrak{g}_\Gamma)$.*

7.5 Evidence for the integrality conjecture

In this section we present arguments in favor of the integrality of the “numerical” DT-invariants $\Omega(\gamma)$. Recall that if E is an object of a \mathbf{k} -linear triangulated category, then we say that E is a Schur object if

$$\text{Ext}^{<0}(E, E) = 0, \quad \text{Ext}^0(E, E) \simeq \mathbf{k} \cdot \text{id}_E .$$

Let us assume now that \mathcal{C} is an ind-constructible $3d$ Calabi-Yau category generated by a Schur object $E \in \mathcal{C}(\mathbf{k})$ in the sense that the category $\mathcal{C}(\overline{\mathbf{k}})$ consists of finite extensions of the shifts $E[i], i \in \mathbf{Z}$. In this case $K_0(\mathcal{C}(\overline{\mathbf{k}})) \simeq \mathbf{Z} \cdot \text{cl}_{\overline{\mathbf{k}}}(E)$. We take $\Gamma = K_0(\mathcal{C}(\overline{\mathbf{k}}))$ and the trivial skew-symmetric form on Γ . For any $z \in \mathbf{C}, \text{Im } z > 0$ our category carries an obvious stability condition σ_z such that $Z(E) := Z(\text{cl}_{\overline{\mathbf{k}}}(E)) = z$, $\text{Arg}(E) = \text{Arg}(z) \in (0, \pi)$. All objects $F \in \mathcal{C}^{ss}(\overline{\mathbf{k}})$ with $\text{Arg}(F) = \text{Arg}(E)$ are n -fold extensions of copies of E for some $n \geq 1$. We denote by l the ray $\mathbf{R}_{>0} \cdot z$.

In the previous section we obtained a formula for $\chi_\Phi(F_l(t))$ in terms of the integral over Euler characteristic over the moduli space of special cyclic objects of $\mathcal{C}_l(\overline{\mathbf{k}})$. We are going to make it more explicit further, by using the potential of E . Let us recall (see Section 3.3) that with the object E we associate a collection of cyclically invariant polylinear maps

$$W_N : (\text{Ext}^1(E, E)^{\otimes N})^{\mathbf{Z}/N\mathbf{Z}} \rightarrow \mathbf{k}, N \geq 3 ,$$

$$a_1 \otimes \cdots \otimes a_N \mapsto W_N(a_1, \dots, a_N) = (m_{N-1}(a_1, \dots, a_{N-1}), a_N) .$$

Let us choose a basis x^1, \dots, x^m in $\text{Ext}^1(E, E)$. Then to any $n \geq 0$ and collection of matrices $X_1, \dots, X_m \in \text{Mat}(n \times n, \overline{\mathbf{k}})$ we associate the number

$$W_N^{(n)}(X_1, \dots, X_m) = \frac{1}{N} \sum_{1 \leq i_1, \dots, i_N \leq m} W_N(x^{i_1}, \dots, x^{i_N}) \text{Tr}(X_{i_1} \dots X_{i_N}) .$$

Thus we have a polynomial on $\overline{\mathbf{k}}^{mn^2}$. The series

$$W^{(n)} = \sum_{N \geq 3} W_N^{(n)}$$

is a formal function on the formal neighborhood of the reduced closed subscheme $Nilp_{m,n} \subset \mathbf{A}^{mn^2}$ whose $\bar{\mathbf{k}}$ -points are collections of matrices

$$(X_1, \dots, X_m) \in Mat(n \times n, \bar{\mathbf{k}})$$

which satisfy the property that there exists a basis in which all $(X_i)_{i=1, \dots, m}$ are strictly upper triangular. Equivalently, $\text{Tr}(X_{i_1} \dots X_{i_r}) = 0$ for any sequence of indices $i_\bullet \in \{1, \dots, m\}$ with $r \geq (n+1)$. This property ensures that $W_N^{(n)}$ is well-defined in a formal neighborhood of $Nilp_{m,n}$. Then

$$\chi_\Phi(F_l(t)) = 1 + \sum_{n \geq 1} t^n \int_{Nilp_{m,n}^{SC}/PGL(n)} (-1)^{n^2(1-m)} (1 - \chi(MF_{(X_1, \dots, X_m)}(W^{(n)}))) d\chi,$$

where $Nilp_{m,n}^{SC}$ for $n \geq 1$ is a subscheme of $Nilp_{m,n}$ whose $\bar{\mathbf{k}}$ -points consists of those collections (X_1, \dots, X_m) for which

$$\text{codim} \left(\sum_i \text{Im}(X_i) \right) = 1.$$

Let us comment on this formula. First we remark that it is sufficient to integrate over the set $Crit(W^{(n)})$ of critical points of $W^{(n)}$, since for all non-critical points $\chi(MF_{(X_1, \dots, X_m)}(W^{(n)})) = 1$. Repeating the arguments of the previous section we obtain that $\mathcal{C}(\bar{\mathbf{k}})$ is equivalent to the category $B_W - \text{mod}^{f, \text{cont}}$ of continuous finite-dimensional representations over $\bar{\mathbf{k}}$ of the topological \mathbf{k} -algebra $B_W = \mathbf{k}\langle\langle x_1, \dots, x_m \rangle\rangle / \overline{(\partial_{x_i} W)}$, $1 \leq i \leq m$, where x_i , $1 \leq i \leq m$ are the coordinates corresponding to the chosen basis x^i , $1 \leq i \leq m$, and $\overline{(\partial_{x_i} W)}$ denotes the closure of the 2-sided ideal generated by the cyclic derivatives of the cyclic potential $W = \sum_{N \geq 3} N^{-1} W_N$. Indeed, it is straightforward to see that a point $(X_1, \dots, X_m) \in Nilp_{m,n}(\bar{\mathbf{k}})$ gives rise to a continuous n -dimensional representation of W if and only if it belongs to $Crit(W^{(n)})$. In terms of the category $\mathcal{C}(\bar{\mathbf{k}})$ these points correspond to n -fold extensions of the Schur object E by itself. Special cyclic modules correspond to critical points belonging to $Nilp_{m,n}^{SC} \subset Nilp_{m,n}$.

Considering an object M of length n as an upper-triangular deformation of the “free” object $nE = E \oplus \dots \oplus E$ (n -times) we see that

$$(M, M)_{\leq 1} = (nE, nE)_{\leq 1} + r, \quad r := \dim \text{Im}(W^{(n)})''_{(X_1, \dots, X_m)}.$$

Then

$$1 - MF_{(X_1, \dots, X_m)}(W^{(n)}) = (1 - MF(E))(1 - MF_0(Q_E)),$$

where Q_E is a quadratic form and $\text{rk } Q_E = r$. Indeed, $W^{(n)}$ coincides with the potential W_{nE} of the object nE under the isomorphism $\text{Ext}^1(nE, nE) \simeq \overline{\mathbf{k}}^{mn^2}$. Thus we see that

$$(-1)^{(M,M)_{\leq 1}} = (-1)^{n^2(1-m)+r}.$$

Since $\chi(1 - MF_0(Q_E)) = (-1)^{\text{rk } Q_E}$ we obtain the desired formula for $\chi_{\Phi}(F_l(t))$. Alternatively, in the integral one can replace the quotient $\text{Nilp}_{m,n}^{SC}/\text{PGL}(n)$ by $\text{Nilp}_{m,n}^{cycl}/\text{GL}^{(1)}(n)$. Here $\text{Nilp}_{m,n}^{cycl} \subset \text{Nilp}_{m,n}$ consists of collection of matrices such that

$$\overline{\mathbf{k}}\langle\langle X_1, \dots, X_m \rangle\rangle v_1 = \overline{\mathbf{k}}^n$$

where $v_1 := (1, 0, \dots, 0)$ is the first base vector, and the group $\text{GL}^{(1)}(n) \subset \text{GL}(n)$ is the stabilizer of v_1 . Notice that $\text{GL}^{(1)}(n)$ acts freely on $\text{Nilp}_{m,n}^{cycl}$. The reason is that the contribution of non-special cyclic objects vanishes as follows from the vanishing of the Euler characteristic of corresponding schemes of modules with chosen cyclic generators.

Conjecture 11 *We have:*

$$\chi_{\Phi}(F_l(t)) = \prod_{n \geq 1} (1 - t^n)^{n \Omega(n)},$$

where all $\Omega(n) = \Omega(n \text{cl}_{\overline{\mathbf{k}}}(E))$ are integer numbers (see Section 1.4).

Let us illustrate the conjecture in few examples.

1) Let $m = 0$ (i.e. the case of just one spherical object). Then $W = 0$ and $B_W = \mathbf{k}$. The only non-trivial cyclic representation have dimension one, hence $\chi(F_l(t)) = 1 - t$. Then we have

$$\Omega(1) = 1, \quad \Omega(n) = 0 \text{ for } n > 1.$$

2) Let $m = 1, W^{(n)}(X_1) = \text{Tr}(X_1^d)$ for $d = 3, 4, \dots$. Then $B_W = \mathbf{k}[x_1]/(x_1^{d-1})$. There is a unique isomorphism class of cyclic B_W -modules in an dimension $n = 0, 1, 2, \dots, d-1$. One can show directly that

$$\chi(F_l(t)) = (1 - t)^{d-1}, \quad \Omega(1) = d - 1, \quad \Omega(n) = 0 \text{ for } n > 1.$$

3) Let $m \geq 1$ be arbitrary and $W = 0$. In this case

$$\chi(F_l(t)) = 1 + \sum_{n \geq 1} (-1)^{n^2(1-m)} \chi(\text{Nilp}_{m,n}^{cycl}/\text{GL}^{(1)}(n)) t^n.$$

Euler characteristic $\chi(\text{Nilp}_{m,n}^{\text{cycl}}/GL^{(1)}(n))$ coincides with the Euler characteristic of the non-commutative Hilbert scheme $H_{n,1}^{(m)}$ from [59]. The latter parametrizes left ideals of codimension n in the free algebra $\overline{\mathbf{k}}\langle x_1, \dots, x_m \rangle$. The reason why we can disregard all non-nilpotent collections (X_1, \dots, X_m) of matrices is that the latter carries a free action of the group \mathbf{G}_m , such that $X_i \mapsto \lambda X_i, 1 \leq i \leq m$ where $\lambda \in \mathbf{G}_m(\overline{\mathbf{k}})$. Hence the corresponding Euler characteristic is trivial. Then using explicit formulas from [59] we obtain

$$G_{(m)}(t) := \chi(F_l(t)) = \sum_{n \geq 0} \frac{(-1)^{n(1-m)}}{(m-1)n+1} \binom{mn}{n} t^n.$$

Notice that this series can be written as

$$\exp \left(\sum_{n \geq 1} \frac{(-1)^{n(1-m)}}{mn} \binom{mn}{n} t^n \right).$$

For $m = 1$ we have $\Omega(1) = -1, \Omega(n) = 0, n \geq 2$. In general

$$\Omega(n) = \frac{1}{mn^2} \sum_{d|n} \mu(n/d) \binom{md}{d} (-1)^{(m-1)d+1},$$

where $\mu(k)$ is the Möbius function (for $m = 2$ see the entry A131868 in the online Encyclopedia of integer sequences).

Remark 29 *One can check that the generating function $G = G_{(m)}$ is algebraic: it satisfies the equation²⁶*

$$G(t) + t(-1)^m (G(t))^m - 1 = 0.$$

An interesting question arises: which algebraic functions admit multiplicative factorization of the form $\prod_{n \geq 1} (1 - t^n)^{n \Omega(n)}$, where all $\Omega(n)$ are integer numbers?

8 Donaldson-Thomas invariants and cluster transformations

8.1 Spherical collections and mutations

Let \mathcal{C} be a 3-dimensional ind-constructible Calabi-Yau category over a field \mathbf{k} of characteristic zero. Assume that it is endowed with a finite collection of

²⁶Compare with the algebraic series in the Introduction, section 1.4.

spherical generators $\mathcal{E} = \{E_i\}_{i \in I}$ of \mathcal{C} defined over \mathbf{k} . Then $\text{Ext}_{\mathcal{C}(\mathbf{k})}^\bullet(E_i, E_i)$ is isomorphic to $H^\bullet(S^3, \mathbf{k})$, $i \in I$. The matrix of the Euler form (taken with the *minus* sign)

$$a_{ij} := -\chi(\text{Ext}_{\mathcal{C}(\mathbf{k})}^\bullet(E_i, E_j))$$

is integer and skew-symmetric. In fact, the ind-constructible category \mathcal{C} can be canonically reconstructed from the (plain, i.e. not ind-constructible) \mathbf{k} -linear Calabi-Yau A_∞ -category $\mathcal{C}(\mathbf{k})$, or even from its full subcategory consisting of the collection \mathcal{E} . In what follows we will omit the subscript $\mathcal{C}(\mathbf{k})$ in the notation for Ext^\bullet -spaces.

Definition 20 *The collection \mathcal{E} is called cluster if for any $i \neq j$ the graded space $\bigoplus_{m \in \mathbf{Z}} \text{Ext}^m(E_i, E_j)$ is either zero, or it is concentrated in one of two degrees $m = 1$ or $m = 2$ only.*

We will assume that our collection is cluster. In that case $K_0(\mathcal{C}(\overline{\mathbf{k}})) \simeq \mathbf{Z}^I$ with the basis formed by the isomorphism classes $[E_i], i \in I$.

With the cluster collection \mathcal{E} we associate a quiver Q such that Q does not have oriented cycles of lengths 1 and 2, and $a_{ij} \geq 0$ is the number of arrows from i to j (we identify the set of arrows from i to j with a basis in $\text{Ext}^1(E_i, E_j)$). Then the potential for the object $E = \bigoplus_{i \in I} E_i$ gives rise to the potential $W = W_Q$ of the quiver Q , i.e. the restriction of the potential to $\bigoplus_{i, j \in I} \text{Ext}^1(E_i, E_j)$. The latter is an infinite linear combination of cyclic words (see [17], [77] where the potential appears abstractly without the relation with Calabi-Yau categories). Any such linear combination is called a potential of Q . In our case the potential is automatically minimal, i.e. all words have length at least 3. The group of continuous automorphisms of the completed path algebra of Q preserving the projectors pr_i , $i \in I$, acts on the set of potentials of Q . We call it the gauge action. Let us state the following general result.

Theorem 9 *Let \mathcal{C} be a 3-dimensional \mathbf{k} -linear Calabi-Yau category generated by a finite collection $\mathcal{E} = \{E_i\}_{i \in I}$ of generators satisfying the condition that*

- $\text{Ext}^0(E_i, E_i) = \mathbf{k} \text{id}_{E_i}$,
- $\text{Ext}^0(E_i, E_j) = 0$ for any $i \neq j$,
- $\text{Ext}^{<0}(E_i, E_j) = 0$, for any i, j .

The equivalence classes of such categories with respect to A_∞ -transformations preserving the Calabi-Yau structure and the collection \mathcal{E} , are in one-to-one correspondence with the gauge equivalence classes of pairs (Q, W) where Q is a finite oriented quiver (possibly with cycles of length 1 or 2) and W is a minimal potential of Q (i.e. its Taylor decomposition starts with terms of degree at least 3).

The case of cluster collections corresponds to quivers without oriented cycles of length 1 and 2.

Proof. We will present the proof of the Theorem in the case of the category with single object E (i.e. $A = \text{Hom}(E, E)$ is a 3d Calabi-Yau algebra). The general case can be proved in a similar way.

Let Q be a quiver with one vertex and $|J|$ loops, where J is a finite set. We assume that Q is endowed with the potential W_0 . We would like to construct a 3d Calabi-Yau category with a single object E such that the number of loops in Q is equal to $\text{Ext}^1(E, E)$ and the restriction of the potential of the category to $\text{Ext}^1(E, E)$ coincides with the given W_0 . Our considerations proceed such as follows. Assuming that such a category exists we will find an explicit formula for the potential on $A = \text{Hom}(E, E)$. Then we simply take this explicit formula as the definition. If the desired category is constructed then we can consider the graded vector space $\text{Ext}^\bullet(E, E)[1]$ which decomposes as

$$\text{Ext}^0(E, E)[1] \oplus \text{Ext}^1(E, E) \oplus \text{Ext}^2(E, E)[-1] \oplus \text{Ext}^3(E, E)[-2] .$$

The first and the last summand are isomorphic to $\mathbf{k}[1]$ and $\mathbf{k}[-2]$ respectively, and the middle two summands are dual two each other. We introduce graded coordinates on $\text{Ext}^\bullet(E, E)[1]$ and denote them such as follows:

- a) the coordinate α of degree +1 on $\text{Ext}^0(E, E)[1]$;
- b) the coordinate a of degree -2 on $\text{Ext}^3(E, E)[-2]$;
- c) the coordinates $x_i, i \in J$ of degree 0 on $\text{Ext}^1(E, E)$;
- d) the coordinates $\xi_i, i \in J$ of degree -1 on $\text{Ext}^2(E, E)[-1]$.

The Calabi-Yau structure on A gives rise to the minimal potential $W = W(\alpha, x_i, \xi_i, a)$, which is a series in cyclic words on the space $\text{Ext}^\bullet(E, E)[1]$. If it arises from the pair (Q, W_0) , then the restriction $W(0, x_i, 0, 0)$ must coincide with $W_0 = W_0(x_i)$. Furthermore, A defines a non-commutative formal pointed graded manifold endowed with a symplectic structure (see [42]). The potential W satisfies the ‘‘classical BV equation’’ $\{W, W\} = 0$, where $\{\bullet, \bullet\}$ denotes the corresponding Poisson bracket.

With these preliminary considerations we see what problem should be solved. We need to construct an extension of W_0 to the formal series W of degree 0 in cyclic words on the graded vector space $\mathbf{k}[1] \oplus \mathbf{k}^J \oplus \mathbf{k}^J[-1] \oplus \mathbf{k}[-2]$, satisfying the classical BV-equation with respect to the Poisson bracket

$$\{W, W\} = \sum_i \partial W / \partial x_i \partial W / \partial \xi_i + \partial W / \partial \alpha \partial W / \partial a .$$

Here is the construction. Let us start with the potential

$$W_{can} = \alpha^2 a + \sum_{i \in J} (\alpha x_i \xi_i - \alpha \xi_i x_i) .$$

This potential makes the above graded vector space into a $3d$ Calabi-Yau algebra with associative product and the unit. The multiplication vanishes on the graded components

$$\text{Ext}^1(E, E) \otimes \text{Ext}^1(E, E) \rightarrow \text{Ext}^2(E, E)$$

and is a non-degenerate bilinear form on components

$$\text{Ext}^1(E, E) \otimes \text{Ext}^2(E, E) \rightarrow \text{Ext}^3(E, E) \simeq \mathbf{k} .$$

Now we see that starting with an arbitrary minimal potential W_0 on $\text{Ext}^1(E, E)$ we can lift it to the minimal potential on $\text{Ext}^\bullet(E, E)$ by setting

$$\widehat{W} := W_{can} + W_0 .$$

We claim that $\{\widehat{W}, \widehat{W}\} = 0$. Indeed, we have $\{W_{can}, W_{can}\} = \{W_0, W_0\} = 0$. Moreover,

$$\{W_{can}, W_0\} = \alpha \sum_{i \in J} [x_i, \partial W_0 / \partial x_i] = 0$$

(we use here the well-known identity $\sum_{i \in J} [x_i, \partial W_0 / \partial x_i] = 0$).

Next we need to check compatibility of the above construction with the gauge group action. Let G_0 be the subgroup of the grading preserving automorphisms of the group of continuous automorphisms of the algebra of formal series $\mathbf{k}\langle\langle \alpha, x_i, \xi_i, a \rangle\rangle, i \in J$. Let $\mathcal{J} \subset \mathbf{k}\langle\langle \alpha, x_i, \xi_i, a \rangle\rangle$ be a closed 2-sided ideal generated by α, a and ξ_i for $i \in J$. Since every generator of \mathcal{J} has non-zero degree we conclude that the group G_0 preserves \mathcal{J} (it can be deduced from the fact that it transforms generators into series of non-zero degrees). Therefore we obtain a homomorphism of groups $G_0 \rightarrow \text{Aut}(\mathbf{k}\langle\langle x_i \rangle\rangle), i \in J$. The

restriction of the potential W to $\text{Ext}^1(E, E)$ defines a surjection from the set of A_∞ -equivalence classes of $3d$ Calabi-Yau algebras to the gauge equivalence classes of (Q, W_0) , where Q is a quiver with one vertex endowed with the minimal potential W_0 . Such algebras can be thought of as deformations of the “ansatz”, which is a $3d$ Calabi-Yau algebra A_{can} corresponding to the potential W_{can} .

Finally we are going to show that the above surjection is in fact a bijection. The latter will follow from the equivalence of the corresponding deformation theories. The deformation theory of the Calabi-Yau algebra A_{can} is controlled by a DGLA $\mathfrak{g}_{A_{can}} = \bigoplus_{n \in \mathbf{Z}} \mathfrak{g}_{A_{can}}^n$, which is a DG Lie subalgebra of the DG Lie algebra

$$\widehat{\mathfrak{g}_{A_{can}}} = \left(\prod_{n \geq 1} \text{Cycl}^n((A[1])^*) \right) [-1]$$

of all cyclic series in the variables $\alpha, x_i, \xi_i, a, i \in J$ (the Lie bracket is given by the Poisson bracket and the differential is given by $\{W_{can}, \bullet\}$). Namely, the component of $\widehat{\mathfrak{g}_{A_{can}}}$ of degree N consists of all cyclic series which contain at least $2 + N$ letters $\alpha, x_i, \xi_i, a, i \in J$. We will call the degree defined in terms of these letter a *cyclic degree* in order to distinguish it from the *cohomological degree* of complexes. Notice that the set of A_∞ -equivalence classes of minimal $3d$ Calabi-Yau algebras can be identified with the set of gauge equivalence classes of solutions $\gamma \in \mathfrak{g}_{A_{can}}^1$ to the Maurer-Cartan equation

$$d\gamma + \frac{1}{2}[\gamma, \gamma] = 0 .$$

Similarly, the set of gauge equivalence classes of minimal potentials on $A^1 = \text{Ext}^1(E, E)$ can be identified with the set of gauge equivalence classes of solutions to the Maurer-Cartan equation in the DGLA $\mathfrak{h} = \mathfrak{h}^0 \oplus \mathfrak{h}^1$, where

$$\mathfrak{h}^0 = \prod_{n \geq 1} ((A^1)^*)^{\otimes n} \otimes A^1 , \quad \mathfrak{h}^1 = \prod_{n \geq 3} \text{Cycl}^n((A^1)^*) .$$

Here we identify \mathfrak{h}^0 with the Lie algebra of continuous derivations of the topological algebra $\mathbf{k}\langle\langle x_1, \dots, x_n \rangle\rangle$ preserving the augmentation ideal (x_1, \dots, x_n) , and we identify \mathfrak{h}^1 the \mathfrak{h}^0 -module of minimal cyclic potentials on A^1 .

The above construction of the “lifting” $\widehat{W} = W_0 + W_{can}$ can be interpreted as a homomorphism of DGLAs $\psi : \mathfrak{h} \rightarrow \widehat{\mathfrak{g}_{A_{can}}}$. Namely, \mathfrak{h}^0 is identified (after the shift [1]) with the space of such cyclic series in $x_i, \xi_i, i \in J$ which contain

exactly one of the variables ξ_i and at least one of the variables x_j for some $i, j \in J$. Similarly \mathfrak{h}^1 is identified with the space of cyclic series in $x_i, i \in J$ which has terms of degree at least 3.

We claim that ψ induces an epimorphism (previous considerations ensure that it is a monomorphism) of cohomology groups in degree 1, and for both DGLAs \mathfrak{h} and $\mathfrak{g}_{A_{can}}$ there is no cohomology in degree greater or equal than 2. This would imply the desired surjectivity of ψ .

Notice that the differential $\{W_{can}, \bullet\}$ preserves the difference between cyclic and cohomological degree. It follows that the complex $\mathfrak{g}_{A_{can}}$ is a direct summand of the complex $\widehat{\mathfrak{g}_{A_{can}}}$. The latter is dual to the cyclic complex $CC_\bullet(A_{can})$. Let $A_{can}^+ \subset A_{can}$ be a non-unital A_∞ -subalgebra consisting of terms of positive cohomological degree. Then, one has for the cyclic homology: $HC_\bullet(A_{can}) \simeq HC_\bullet(A_{can}^+) \oplus HC_\bullet(\mathbf{k})$. In terms of the dual complex this isomorphism means the decomposition into a direct sum of the space of cyclic series in variables $x_i, \xi_i, a, i \in J$ (corresponds to $(HC_\bullet(A_{can}^+))^*$) and the space of cyclic series in the variable α of odd cyclic degree (corresponds to $(HC_\bullet(\mathbf{k}))^*$). It is easy to see that the series in the variable α do not contribute to the cohomology of $\mathfrak{g}_{A_{can}} \subset \widehat{\mathfrak{g}_{A_{can}}}$.

The cohomological degree of series in variables $x_i, \xi_i, a, i \in I$ is non-positive. Recall that we shifted the grading in Lie algebras by 1 with respect to the cohomological grading. Hence $H^{\geq 2}(\mathfrak{g}_{A_{can}}) = 0$. Also, it is immediate that $H^1(\mathfrak{g}_{A_{can}})$ is isomorphic to the space of cyclic series in the variables $x_i, i \in I$ with terms of degree at least 3. Hence $H^1(\mathfrak{g}_{A_{can}}) \simeq \mathfrak{h}^1 \simeq H^1(\mathfrak{h})$ (the latter holds since the differential on \mathfrak{h} is trivial). This concludes the proof. ■

Next, we will introduce the notion of a *mutation* on the set of cluster collections in a given category \mathcal{C} . Let us choose an element of I which we will denote by 0. We are going to write $i < 0$ if $a_{i0} > 0$, and $i > 0$ if $i \neq 0$ and $a_{i0} \leq 0$. The mutation of \mathcal{E} at the object E_0 is defined as a new spherical collection $\mathcal{E}' = (E'_i)_{i \in I}$ such that:

$$\begin{aligned} E'_i &= E_i, & i < 0, \\ E'_0 &= E_0[-1], \\ E'_i &= R_{E_0}(E_i), & i > 0. \end{aligned}$$

where $R_{E_0}(E_i) := Cone(E_0 \otimes Ext^\bullet(E_0, E_i) \rightarrow E_i)$ is the reflection functor given by the cone of the natural evaluation map. Explicitly, the object E'_i

for $i > 0$ fits in the exact triangle

$$E_i \rightarrow E'_i \rightarrow E_0 \otimes \text{Ext}^1(E_0, E_i) .$$

Notice that all objects $E'_i, i \neq 0$ belong to the abelian category generated by $E_i, i \in I$. We remark that the spherical collection \mathcal{E}' is not necessarily a cluster one.

At the level of the lattice $\Gamma := \mathbf{Z}^I$ the change of the spherical collections $\mathcal{E} \rightarrow \mathcal{E}'$ corresponds to the following relation between the basis $v_i := \text{cl}_{\overline{\mathbf{k}}}(E_i), i \in I$ and the mutated basis $v'_i = \text{cl}_{\overline{\mathbf{k}}}(E'_i), i \in I$:

$$\begin{aligned} v'_i &= v_i, i < 0, \\ v'_0 &= -v_0, \\ v'_i &= v_i - \langle v_0, v_i \rangle v_0 = v_i + a_{0i} v_0, i > 0 . \end{aligned}$$

We recall that $a_{ij} = -\langle v_i, v_j \rangle$. The mutated matrix (a'_{ij}) is given by

$$\begin{aligned} a'_{ij} &= a_{ij} + a_{i0} a_{0j} \text{ if } i < 0 < j, \\ a'_{i0} &= -a_{i0}, \\ a'_{0i} &= -a_{0i}, \\ a'_{ij} &= a_{ij}, \text{ otherwise.} \end{aligned}$$

Thus we see that the mutation at E_0 gives rise to the mutation of the matrix (a_{ij}) in the sense of cluster algebras (see [77]). Notice that at the categorical level the mutation is not an involution. The composition of the mutation at E_0 and of the mutation at $E'_0 = E_0[-1]$ is the reflection functor R_{E_0} applied to all elements of the cluster collection.

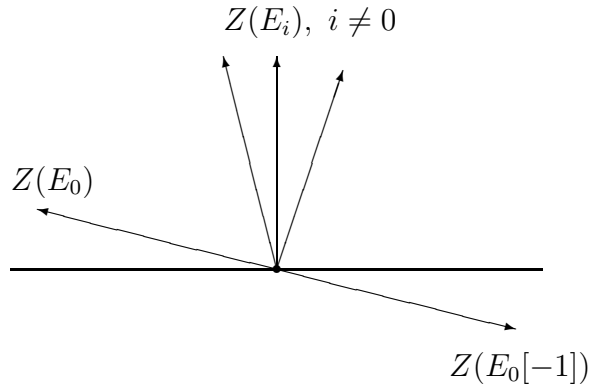
Identifying Calabi-Yau categories endowed with cluster collections with quivers with potentials we obtain the well-known notion of mutation of a quiver with potential (see [77]). Then we have the following result.

Theorem 10 *In the scheme (an infinite-dimensional affine space) of potentials \mathcal{PT} there is a countable set of algebraic hypersurfaces $X_i, i \geq 1$ invariant under the gauge group action, such that for any potential belonging to the set $\mathcal{PT} \setminus \cup_{i \geq 1} X_i$ one can make mutations indefinitely, obtaining each time a potential from $\mathcal{PT} \setminus \cup_{i \geq 1} X_i$. In particular, all corresponding quivers do not have oriented cycles of length one or two.*

Sketch of the proof. The mutated spherical collection fails to be cluster if for some $i \neq j$ we have simultaneously $\text{Ext}^1(E'_i, E'_j) \neq 0$ and $\text{Ext}^2(E'_i, E'_j) \neq 0$.

0. This property is not stable under deformations of 3-dimensional Calabi-Yau A_∞ -category, since we can add a quadratic term to the potential $W_{E'_i \oplus E'_j}$ reducing the dimension of $\text{Ext}^1(E'_i, E'_j)$ and $\text{Ext}^2(E'_i, E'_j)$. Therefore, the property that the mutated collection is also a cluster one holds on a Zariski open non-empty subset of the space of all potentials. Moreover, the mutation induces a birational identification between varieties (maybe infinite-dimensional) of gauge equivalence classes of generic potentials for quivers corresponding to skew-symmetric matrices (a_{ij}) and (a'_{ij}) . ■

Any cluster collection $\mathcal{E} = \{E_i\}_{i \in I}$ defines an open domain $U_{\mathcal{E}} \subset \text{Stab}(\mathcal{C}, \text{cl})$, where $\Gamma = K_0(\mathcal{C}(\bar{\mathbf{k}}))$, $\text{cl} = \text{id}$. Namely, for any collection $z_i \in \mathbf{C}$, $\text{Im } z_i > 0$, $i \in I$ we have a stability condition $\sigma_{(z_i)} := \sigma_{(z_i)_{i \in I}}$ with the t -structure defined by $(E_i)_{i \in I}$ and the central charge Z such that $Z(E_i) := Z(\text{cl}(E_i)) = z_i$, $i \in I$. The heart of the t -structure is an abelian category $\mathcal{A}_{\mathcal{E}}$ generated by $(E_i)_{i \in I}$, which is artinian with simple objects E_i , $i \in I$. This abelian category is equivalent to the category of continuous finite-dimensional representations of the algebra B_W where W is the potential of the path algebra of the quiver Q . If \mathcal{E}' is a cluster collection obtained from \mathcal{E} by the mutation at E_0 then the domains $U_{\mathcal{E}}$ and $U_{\mathcal{E}'}$ do not have common interior points, but have a common part of the boundary which is the wall of second kind. The common boundary corresponds to the stability structure with $Z(E_0) \in \mathbf{R}_{<0}$.



Category $\mathcal{A}_{\mathcal{E}'}$ is obtained from $\mathcal{A}_{\mathcal{E}}$ by tilting. Namely, any object M of $\mathcal{A}_{\mathcal{E}}$ admits a unique presentation as an extension

$$0 \rightarrow nE_0 \rightarrow M \rightarrow N \rightarrow 0$$

where $N \in \mathcal{B} := \{E \in \mathcal{A}_{\mathcal{E}} \mid \text{Hom}(E, E_0) = 0\}$. Similarly, any object M' of $\mathcal{A}_{\mathcal{E}'}$ admits a unique presentation as an extension

$$0 \rightarrow N \rightarrow M' \rightarrow nE_0[-1] \rightarrow 0$$

with $N \in \mathcal{B}$.

8.2 Orientation data for cluster collections

Let $\mathcal{E} = (E_i)_{i \in I}$ be a cluster collection. We set $R := R_{\mathcal{E}} = \text{Ext}^{\bullet}(E, E)$, where $E = \bigoplus_{i \in I} E_i$. Then R is an A_{∞} -algebra. We denote by $M := M_{\mathcal{E}}$ the algebra R considered as R -bimodule. Using the truncation functors $\tau_{\leq i}$ and $\tau_{\geq i}$ we define a sub-bimodule $M_{\geq 2} = \tau_{\geq 2}M$ as well as a quotient bimodule $M/M_{\geq 2}$, which is isomorphic to $M_{\leq 1} = \tau_{\leq 1}M$. Then we can deform the extension $M_{\geq 2} \rightarrow M \rightarrow M_{\leq 1}$ into the direct sum of bimodules $M_{\geq 2} \oplus M_{\leq 1}$. Moreover, one can check that there exists a deformation which consists of self-dual bimodules (i.e. they give rise to self-dual functors in the sense of Section 5.3). Thus we would like to define an orientation data using the splitting given by the bifunctor F which corresponds to the bimodule $M_{\leq 1}$, i.e. $(E_i, E_j) \mapsto \tau_{\leq 1} \text{Ext}^{\bullet}(E_i, E_j)$. Let $\mathcal{E}' = (E'_i)_{i \in I}$ be the cluster collection obtained by a mutation at $i = 0$. One can check directly that $\mathbf{Z}/2\mathbf{Z}$ -valued quadratic form defined on $K_0(\mathcal{C}(\bar{\mathbf{k}}))$ by

$$[E] = \sum_{i \in I} n_i [E_i] \mapsto \sum_{i \in I} n_i^2 - \sum_{i, j \in I, a_{ij} > 0} a_{ij} n_i n_j \pmod{2\mathbf{Z}}$$

is invariant under mutations. This means that the parity of the super line bundle $\sqrt{\mathcal{D}}_F = \text{sdet}(\tau_{\leq 1} \text{Ext}^{\bullet}(F, F))$ is preserved under mutations. This makes plausible the following conjecture.

Conjecture 12 *Bifunctors $M_{\leq 1}^{\mathcal{E}}$ and $M_{\leq 1}^{\mathcal{E}'}$ define isomorphic orientation data on \mathcal{C} .*

In order to check the conjecture one needs to find a self-dual \mathbf{A}^1 -deformation of $M_{\leq 1}^{\mathcal{E}} \oplus (M_{\leq 1}^{\mathcal{E}'})^{\vee}$ to a bifunctor of the type $N \oplus N^{\vee}$ (we identify bifunctors with bimodules here).

8.3 Quantum DT-invariants for quivers

For any $\sigma \in U_{\mathcal{E}}$ (recall domain $U_{\mathcal{E}}$ introduced at the end of Section 8.1) we have the corresponding element A_V^{Hall} , where V is any strict sector containing all $Z(E_i), i \in I$. The element A_V^{Hall} does not depend on σ . Moreover, this element depends only on the gauge equivalence class of the corresponding potential. The associated element $A_{V,q} := A_{\mathcal{E},q}$ of the quantum torus $\mathcal{R}_{V,q}$ depends (for a *generic* potential) on the matrix (a_{ij}) only.

Let us associate with our quiver Q the quantum torus $\mathcal{R}_{Q,q}$. By definition it is an associative unital algebra over the field $\mathbf{Q}(q^{1/2})$ of rational functions, with invertible generators $\hat{e}_i^{\pm 1}, i \in I$ subject to the relations

$$\hat{e}_i \hat{e}_j = q^{a_{ji}} \hat{e}_j \hat{e}_i .$$

We are going to use its double $D(\mathcal{R}_{Q,q})$, which is generated by $\mathcal{R}_{Q,q}$, new set of generators $\hat{e}_i^{\vee}, i \in I$ subject to the additional set of relations:

$$\hat{e}_i^{\vee} \hat{e}_j^{\vee} = \hat{e}_j^{\vee} \hat{e}_i^{\vee}, \quad \hat{e}_i^{\vee} \hat{e}_j = q^{-\delta_{ij}} \hat{e}_j \hat{e}_i^{\vee}, \quad i, j \in I .$$

The corresponding quasi-classical limits are Poisson tori which we will denote by \mathbb{T}_Q and $D(\mathbb{T}_Q)$ respectively.

Identifying $\mathcal{R}_{\Gamma,q}$ with $\mathcal{R}_{Q,q}$ in the obvious way we obtain an element

$$\mathbf{E}_Q = 1 + \cdots \in \widehat{\mathcal{R}_{Q,q}}$$

corresponding to $A_{\mathcal{E},q}$. We observe that \mathbf{E}_Q is a series in non-negative powers of $\hat{e}_i, i \in I$.

Conjugation with \mathbf{E}_Q gives rise to an automorphism of the quantum torus $D(\mathcal{R}_{Q,q})$. By the ‘‘absence of poles’’ conjecture it does not have poles at $q^n = 1, n \geq 1$. In particular it defines a formal symplectomorphism of the double torus $D(\mathbb{T}_Q)$ (see Section 2.6, with the notation $b_{ij} := -a_{ij}$).

8.4 Quivers and cluster transformations

The formal power series \mathbf{E}_Q in $\hat{e}_i, i \in I$ defined in the previous section satisfy a number of remarkable properties.

- 1) If $|I| = 1$ then Q is a quiver with one vertex i . We have

$$\mathbf{E}_Q = \mathbf{E}(\hat{e}_i) ,$$

where \mathbf{E} is the quantum dilogarithm function.

2) Let $I = I_1 \sqcup I_2$, and we assume that $a_{i_1, i_2} < 0$ for any $i_1 \in I_1, i_2 \in I_2$. Then we have two subquivers Q_1 and Q_2 of Q with the sets of vertices I_1 and I_2 correspondingly, and all the arrows connecting Q_1 and Q_2 go only in the direction from Q_2 to Q_1 (i.e. there is no arrows from Q_1 to Q_2).

Proposition 16 *One has:*

$$\mathbf{E}_Q = \mathbf{E}_{Q_1} \mathbf{E}_{Q_2}$$

where we embed $\mathcal{R}_{Q_j, q}$, $j = 1, 2$ into $\mathcal{R}_{Q, q}$ in the obvious way: $\hat{e}_i \mapsto \hat{e}_i$ for $i \in I_1$ or $i \in I_2$.

Proof. Consider the stability condition $\sigma \in U_{\mathcal{E}}$ on the Calabi-Yau category \mathcal{C}_Q associated with Q and a generic potential. Let $\mathcal{E} = \{E_i\}_{i \in I}$ be the corresponding cluster collection. We choose a stability condition $\sigma \in U_{\mathcal{E}}$ in such a way that $\text{Arg}(E_{i_1}) > \text{Arg}(E_{i_2})$ for $i_1 \in I_1, i_2 \in I_2$. In this case $\mathcal{C}_Q^{ss} = \mathcal{C}_{Q_1}^{ss} \sqcup \mathcal{C}_{Q_2}^{ss}$. This implies the desired identity. ■

Remark 30 *It follows from the Properties 1) and 2) that for any acyclic quiver Q the element \mathbf{E}_Q can be expressed as the product of $\mathbf{E}(\hat{e}_i)$, $i \in I$. In particular, the conjugation by \mathbf{E}_Q has a well-defined quasi-classical limit as $q^{1/2} \rightarrow -1$, which is a birational symplectomorphism of the torus $D(\mathbb{T}_Q)$.*

3) Let Q' be the quiver obtained from Q by the mutation at $0 \in I$. We denote the standard generators of the corresponding quantum tori by $(\hat{e}'_i)_{i \in I}$, $\hat{e}'_i = \hat{e}_{\text{cl}_{\mathbb{K}}(E'_i)}$ and $(\hat{e}_i)_{i \in I}$, $\hat{e}_i = \hat{e}_{\text{cl}_{\mathbb{K}}(E_i)}$ respectively. Let us introduce the elements

$$R_Q = \mathbf{E}(\hat{e}_0)^{-1} \cdot \mathbf{E}_Q, \quad R_{Q'} = \mathbf{E}_{Q'} \cdot \mathbf{E}(\hat{e}'_0)^{-1}.$$

Here R_Q is a series in variables \hat{e}_i for $i < 0$, and in (dependent) variables $\hat{e}_i, \hat{e}_i \hat{e}_0, \dots, \hat{e}_i \hat{e}_0^{a_{0i}}$ for $j > 0$. Similarly, $R_{Q'}$ is a series in variables \hat{e}'_i for $i < 0$ and $\hat{e}'_i, \hat{e}'_i \hat{e}'_0, \dots, \hat{e}'_i (\hat{e}'_0)^{a_{0i}}$ for $i > 0$.

Then $R_Q = R_{Q'}$ under the identification

$$\begin{aligned} \hat{e}'_i &= \hat{e}_i, \quad i < 0, \\ \hat{e}'_0 &= \hat{e}_0^{-1}, \\ \hat{e}'_i &= q^{-\frac{1}{2}a_{0i}} \hat{e}_i \hat{e}_0^{a_{0i}}, \quad i > 0. \end{aligned}$$

This follows from the above-discussed picture of tilting via the wall-crossing, more precisely, from the formula

$$\mathbf{E}(\hat{e}_{\text{cl}_{\overline{\mathbf{k}}}(E_0)})^{-1} A_{\mathcal{E},q} = A_{\mathcal{E}',q} \mathbf{E}(\hat{e}_{-\text{cl}_{\overline{\mathbf{k}}}(E_0)})^{-1}.$$

Element $R_Q = R_{Q'}$ corresponds to the integral over the space of objects of category \mathcal{B} in notation at the end of Section 8.1.

For the convenience of the reader we give also the formulas comparing dual coordinates on the double quantum torus:

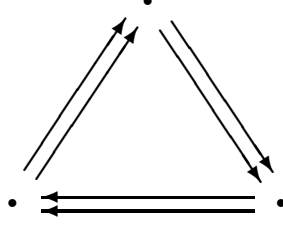
$$\begin{aligned} \hat{e}_i^{\vee'} &= \hat{e}_i^{\vee}, \quad \forall i \neq 0, \\ \hat{e}_0^{\vee'} &= (\hat{e}_0^{\vee})^{-1} \cdot \prod_{i>0} (\hat{e}_i^{\vee})^{a_{0i}}. \end{aligned}$$

Let us now consider the minimal class \mathcal{P} of oriented finite quivers which satisfies the following properties:

- a) the trivial quiver (one vertex no arrows) belongs to \mathcal{P} ;
- b) class \mathcal{P} is closed under mutations;
- c) if $Q_1, Q_2 \in \mathcal{P}$ then a quiver Q obtained from the disjoint union of Q_1 and Q_2 by inserting a finite number of arrows from Q_2 to Q_1 (without changing anything else for Q_1 and Q_2) also belongs to \mathcal{P} . We will say in this case that Q is an extension of Q_1 by Q_2 . At the level of categories this means that any object J of the category $\mathcal{A}(\mathcal{E})$ generated by $E_i \in \mathcal{E}, i \in I$ is an extension $F_1 \rightarrow J \rightarrow F_2$ where F_1 (resp. F_2) is an object of the abelian category generated by $E_i, i \in I_1$ (resp. $E_i, i \in I_2$).

This class \mathcal{P} enjoys the property that the gauge group associated with $Q \in \mathcal{P}$ when acting on the space of potentials on Q has one open orbit (this can be shown by induction), hence the corresponding 3-dimensional Calabi-Yau category is *rigid*. Moreover for any $Q \in \mathcal{P}$ the element \mathbf{E}_Q is a finite product of the elements $\mathbf{E}(f)$, where $f = \hat{e}_\gamma$ is a monomial. In particular, the conjugation with \mathbf{E}_Q has a quasi-classical limit as $q^{1/2} \rightarrow -1$, which is a birational transformation.

One of the first nontrivial examples of a quiver Q which is not in the class \mathcal{P} is the quiver Q_3 which has three vertices and two parallel arrows between any two vertices (see the Figure). This quiver is stable under mutations. The element \mathbf{E}_{Q_3} satisfies an overdetermined system of equations. The computer check shows that the conjugation with \mathbf{E}_{Q_3} has the quasi-classical limit which is not rational. It is not clear whether it admits an analytic continuation.



The mutation property of Q_3 has the following explicit corollary. Namely, there exist collections

$$c_{i,j,k}, b_{m_1,m_2,n} \in \mathbf{Q}(q^{1/2}), \quad i, j, k \in \mathbf{Z}_{\geq 0}, \quad m_1, n \geq 0, \quad -m_1 \leq m_2 \leq m_1$$

such that the following system of equations is satisfied:

$$\begin{aligned} c_{0,0,0} &= b_{0,0,0} = 1, \quad c_{i,j,k} = c_{j,k,i} = c_{k,i,j}, \\ c_{n_0,n_1,n_2} &= \sum_{l \geq 0} \varepsilon_l q^{l(n_2-n_1)} b_{n_1,n_0-l-n_1,n_2}, \\ c_{n_0,n_1,n_2} &= \sum_{l \geq 0} \varepsilon_l q^{l(n_2-n_0)} b_{n_0,n_0+l-n_1,n_2}, \end{aligned}$$

where

$$\varepsilon_l = \frac{q^{l^2/2}}{(q^l - 1) \dots (q^l - q^{l-1})}$$

are coefficients of the series \mathbf{E} . To have a solution of this system of equations is the same as to write the element

$$\mathbf{E}_Q = \sum_{i,j,k} c_{i,j,k} \hat{e}_{(i,j,k)},$$

where we identified Γ with \mathbf{Z}^3 . The above system of equations follows from the identity $R_{Q_3} = R_{Q'_3}$ since $Q_3 = Q'_3$ after the mutation. The elements $b_{m_1,m_2,n}$ are derived from $c_{i,j,k}$.

Notice that the above system of equations has a solution which is not unique. Therefore the element \mathbf{E}_Q is determined non-uniquely, but only up to a multiplication by a series of the type

$$1 + \sum_{n \geq 1} a_n \hat{e}_{1,1,1}^n, \quad a_n \in \mathbf{Q}(q^{1/2})$$

which belongs to the center of the quantum torus $\mathcal{R}_{Q_3,q}$.

Let as before $\mathcal{E} = (E_i)_{i \in I}$ be a cluster collection in \mathcal{C} such that the corresponding potential is generic. We make an additional assumption that the conjugation $\text{Ad}_{A_{\mathcal{E},q}} : x \mapsto A_{\mathcal{E},q} x A_{\mathcal{E},q}^{-1}$ is a birational transformation of the double quantum torus $\mathcal{R}_{\Gamma \oplus \Gamma^\vee, q} \simeq D(\mathcal{R}_{Q,q})$. This means that it is an automorphism of the (well-defined) skew field $K_{\Gamma \oplus \Gamma^\vee, q}$ of fractions of this quantum torus. In the equivalent language of quivers it suffices to require that $Q \in \mathcal{P}$.

Let us denote by $\Phi_{\mathcal{E}}$ the automorphism of $K_{\Gamma \oplus \Gamma^\vee, q}$ given by

$$\Phi_{\mathcal{E}}(x) = (\text{Ad}_{A_{\mathcal{E},q}}^{-1} \circ \tau)(x) ,$$

where τ is the involution induced by the antipodal involution $\gamma \mapsto -\gamma$ of $\Gamma \oplus \Gamma^\vee$.

Proposition 17 *If $\mathcal{E}' = (E'_i)_{i \in I}$ is the cluster collection obtained by the mutation at E_0 then*

$$\text{Ad}_{\mathbf{E}(\hat{e}_{\text{cl}_{\mathbf{K}}}(E_0))}^{-1} \circ \Phi_{\mathcal{E}} \circ \text{Ad}_{\mathbf{E}(\hat{e}_{\text{cl}_{\mathbf{K}}}(E_0))} = \Phi_{\mathcal{E}'} .$$

Proof. From the known identity

$$\text{Ad}_{\mathbf{E}(\hat{e}_{\text{cl}_{\mathbf{K}}}(E_0))}^{-1} \circ \text{Ad}_{A_{\mathcal{E},q}} = \text{Ad}_{A_{\mathcal{E}',q}} \circ \text{Ad}_{\mathbf{E}(\hat{e}_{\text{cl}_{\mathbf{K}}}(E_0))}^{-1}$$

we obtain the desired one by multiplying it from the right by $\tau \circ \text{Ad}_{\mathbf{E}(\hat{e}_{\text{cl}_{\mathbf{K}}}(E_0))}$.

■

Now we can state a similar result for a quiver Q which satisfies the condition that $\text{Ad}_{\mathbf{E}_Q}$ is a birational transformation of the skew field K_Q of fractions of the double quantum torus $D(\mathcal{R}_{Q,q})$. Let us define $\Phi_Q := \text{Ad}_{\mathbf{E}_Q}^{-1} \circ \tau$ where τ is the obvious involution:

$$\tau(\hat{e}_i) = \hat{e}_i^{-1}, \quad \tau(\hat{e}_i^\vee) = (\hat{e}_i^\vee)^{-1} .$$

Let Q' be the quiver obtained as a mutation of Q at the vertex $0 \in I$. Then we have the following corollary of the above Proposition.

Corollary 4 *Let us define the map $C_{Q,0} : K_{Q,q} \rightarrow K_{Q',q}$ as the composition*

$$K_{Q,q} \rightarrow K_{\Gamma \oplus \Gamma^\vee, q} \rightarrow K_{\Gamma \oplus \Gamma^\vee, q} \rightarrow K_{Q',q} ,$$

where the middle arrow is the automorphism $\text{Ad}_{\mathbf{E}(\hat{e}_{\text{cl}_{\mathbf{K}}}(E_0))}^{-1}$ while the other maps are obvious isomorphisms of skew fields. Then

$$C_{Q,0} \circ \Phi_Q = \Phi_{Q'} \circ C_{Q,0} .$$

Proof. It is just a reformulation of the previous Proposition in the language of quivers. ■

Let us compute $C_{Q,0}(\hat{e}_i)$, where $\hat{e}_i = \hat{e}_{\text{cl}_{\mathbf{k}}(E_i)}$, $i \in I$, as well as $C_{Q,0}(\hat{e}_i^\vee)$, $i \in I$. We have to compute the action of $\text{Ad}_{\mathbf{E}(\hat{e}_{\text{cl}_{\mathbf{k}}(E_0)})}^{-1}$ on these generators. Thus we obtain

$$\begin{aligned}\hat{e}_0 &\mapsto (\hat{e}'_0)^{-1}, \\ \hat{e}_i &\mapsto \hat{e}'_i \cdot \prod_{0 \leq n \leq a_{i0}-1} (1 + q^{n+1/2} (\hat{e}'_0)^{-1})^{-1}, \quad i < 0, \\ \hat{e}_i &\mapsto \hat{e}'_i \cdot \prod_{0 \leq n \leq a_{i0}-1} (1 + q^{n+1/2} \hat{e}'_0), \quad i > 0.\end{aligned}$$

Similarly we obtain that

$$\begin{aligned}\hat{e}_i^\vee &\mapsto \hat{e}_i^{\vee'}, \quad i \neq 0, \\ \hat{e}_0^\vee &\mapsto (\hat{e}_0^{\vee'})^{-1} \cdot \prod_{i>0} (\hat{e}_i^{\vee'})^{a_{0i}} \cdot (1 + q^{1/2} (\hat{e}'_0)^{-1})^{-1}.\end{aligned}$$

Under quasi-classical limit the generators \hat{e}_i , $i \in I$ go to the coordinates y_i , $i \in I$ and \hat{e}_i^\vee go to the coordinates x_i , $i \in I$ of the symplectic double torus (see Section 2.6). Then in those coordinates we obtain

$$\begin{aligned}y_i &\mapsto \frac{y'_i}{(1-1/y'_0)^{a_{i0}}}, \quad i < 0, \\ y_0 &\mapsto (y'_0)^{-1}, \\ y_i &\mapsto y'_i (1 - y'_0)^{a_{0i}}, \quad i > 0.\end{aligned}$$

For the dual coordinates we have:

$$\begin{aligned}x_i &\mapsto x'_i, \quad i \neq 0, \\ x_0 &\mapsto (x'_0)^{-1} \cdot \prod_{i>0} (x'_i)^{a_{0i}} \cdot (1 - 1/y'_0)^{-1}.\end{aligned}$$

Up to a change of sign these are cluster transformations. Namely, if we set $X_i = -y_i$, $X'_i = -y'_i$, $A_i = 1/x_i$, $i \in I$ then our formulas become formulas (17) and (18) from [23] (in the notation from loc. cit).

Remark 31 *Let us recall the variety N from Section 2.6 defined by the equations $N = \{y_i = -\prod_{j \in I} x_j^{a_{ij}}, i \in I\}$, and let N' be a similar variety defined for the transformed coordinates $x'_i, y'_i, i \in I$. One can check that the quasi-classical limit of $C_{Q,0}$ transforms N into N' . Furthermore, the quasi-classical limit of the automorphism Φ_Q preserves N .*

Remark 32 *1) Let us assume that $\text{Ad}_{A_{\varepsilon,q}}$ is birational (e.g. for $Q \in \mathcal{P}$). The above considerations show that the conjugacy class of the element Φ_Q is an*

invariant of the quiver Q under mutations. Passing to quasi-classical limit we obtain an invariant of a quiver (under mutations) which is a conjugacy class in the group of birational transformations of the classical double torus.

2) The categorical version of the above remark holds in a greater generality. Namely, let us assume that \mathcal{C} has a t -structure generated by finitely many objects. Then we can define the motivic DT-invariant $A_{\mathcal{C}}^{\text{mot}} := A_V^{\text{mot}}$ (and its quantum and semi-classical relatives) for every stability condition such that all the generators of the t -structure are stable. Here V can be any strict sector containing their central charges, so we can replace it by the upper-half plane. Then the conjugacy class of the automorphism $\Phi_{\mathcal{C}} := \text{Ad}_{A_{\mathcal{C}}}^{-1} \circ \tau$ (if it makes sense) will be independent (under appropriate conditions) of the choice of stability condition.

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