

Symmetries of WDVV equations

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Abstract

We say that a function $F(\tau)$ obeys WDVV equations, if for a given invertible symmetric matrix $\eta^{\alpha\beta}$ and all $\tau \in \mathcal{T} \subset \mathbb{R}^n$, the expressions $c_{\beta}^{\alpha\gamma}(\tau) = \eta^{\alpha\lambda} c_{\lambda\beta\gamma}(\tau) = \eta^{\alpha\lambda} \partial_{\lambda} \partial_{\beta} \partial_{\gamma} F$ can be considered as structure constants of commutative associative algebra; the matrix $\eta_{\alpha\beta}$ inverse to $\eta^{\alpha\beta}$ determines an invariant scalar product on this algebra. A function $x^{\alpha}(z, \tau)$ obeying $\partial_{\alpha} \partial_{\beta} x^{\gamma}(z, \tau) = z^{-1} c_{\alpha}^{\varepsilon\beta} \partial_{\varepsilon} x^{\gamma}(z, \tau)$ is called a calibration of a solution of WDVV equations. We show that there exists an infinite-dimensional group acting on the space of calibrated solutions of WDVV equations (in different form such a group was constructed in [A. Givental, math.AG/0305409]). We describe the action of Lie algebra of this group.

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1. Introduction

Two-dimensional topological quantum field theory can be identified with a solution of WDVV equations, or, in geometric terms, with a Frobenius manifold. In such a theory the algebra of observables \mathcal{H} depends on parameters τ_1, \dots, τ_n (on a point of a manifold \mathcal{T}). Here $n = \dim \mathcal{H}$. Two-point correlation functions

$$\eta_{\alpha\beta} = \langle e_{\alpha} e_{\beta} \rangle \tag{1.1}$$

determine nondegenerate bilinear inner product $(\ , \)$ on \mathcal{H} . Three-point correlation functions

$$c_{\alpha\beta\gamma} = \langle e_{\alpha} e_{\beta} e_{\gamma} \rangle \tag{1.2}$$

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determine structure constants

$$c_{\beta}^{\alpha}{}_{\gamma}(\tau) = \eta^{\alpha\lambda} c_{\lambda\beta\gamma}(\tau) \tag{1.3}$$

of the algebra $\mathcal{H}(\tau)$, which is commutative and associative (in the above formulas (e_1, \dots, e_n) is a basis of vector space \mathcal{H} of observables; we assume that the vector space \mathcal{H} and the basis (e_1, \dots, e_n) do not depend on τ , but the correlation function $\langle e_{\alpha} e_{\beta} e_{\gamma} \rangle$ depends on τ).

One assumes that $\eta_{\alpha\beta}$ does not depend on τ and the expression $\partial_{\alpha} c_{\beta\gamma\lambda}$ is symmetric with respect to indices $\alpha, \beta, \gamma, \lambda$ (we use the notation ∂_{α} for the derivative $\partial/\partial\tau^{\alpha}$). This means that there exists a function $F(\tau_1, \dots, \tau_n)$, the free energy, obeying

$$c_{\alpha\beta\gamma}(\tau) = \partial_{\alpha\beta\gamma} F(\tau). \tag{1.4}$$

The function F satisfies the WDVV equations

$$\sum_{\delta, \gamma=1}^n \frac{\partial^3 F(\tau)}{\partial \tau^{\alpha} \partial \tau^{\beta} \partial \tau^{\delta}} \eta^{\delta\gamma} \frac{\partial^3 F(\tau)}{\partial \tau^{\gamma} \partial \tau^{\omega} \partial \tau^{\rho}} = \sum_{\delta, \gamma=1}^n \frac{\partial^3 F(\tau)}{\partial \tau^{\alpha} \partial \tau^{\omega} \partial \tau^{\delta}} \eta^{\delta\gamma} \frac{\partial^3 F(\tau)}{\partial \tau^{\gamma} \partial \tau^{\beta} \partial \tau^{\rho}}, \tag{1.5}$$

reflecting associativity of the algebra with structure constants $c_{\beta}^{\alpha}{}_{\gamma}(\tau) = \eta^{\alpha\lambda} c_{\lambda\beta\gamma}(\tau)$. Usually one requires that the first element e_1 of the basis of \mathcal{H} is the unit element and $\eta_{\alpha\beta} = c_{1\alpha\beta}$, then

$$\frac{\partial^3 F}{\partial \tau^{\alpha} \partial \tau^{\beta} \partial \tau^1} = \eta_{\alpha\beta}. \tag{1.6}$$

It would be convenient for us to impose the condition (1.6) only at the end of our consideration.

In the above formulas we assumed that the vector space \mathcal{H} of observables does not depend on the parameters τ_1, \dots, τ_n . This assumption is justified by a remark that starting with one of the points of the manifold \mathcal{T} one can obtain all other points of this manifold using perturbation theory (every observable can be considered as a perturbation). This construction can be used to specify a natural identification of all tangent spaces to this manifold. These tangent spaces can be considered as Frobenius algebras (associative commutative algebras with nondegenerate inner product). The geometry arising on \mathcal{T} was analyzed in [1]. Using the terminology of [1] one can say that \mathcal{T} is equipped with a structure of Frobenius manifold.

Let us define operators $\nabla_{\alpha}(z)$ on the space of \mathcal{H} -valued functions on \mathcal{T} by the formula

$$\nabla_{\alpha} = \partial_{\alpha} - z^{-1} \hat{c}_{\alpha}. \tag{1.7}$$

Here z is a complex parameter; if $\varphi = \varphi^{\gamma} e_{\gamma}$, then $\hat{c}_{\alpha} \varphi = c_{\alpha}^{\beta}{}_{\gamma} \varphi^{\gamma} e_{\beta}$. It is easy to check that operators ∇_{α} corresponding to $c_{\alpha\beta\gamma}, \eta_{\alpha\beta}$ obeying WDVV equations satisfy

$$[\nabla_{\alpha}(z), \nabla_{\beta}(z)] = 0, \tag{1.8a}$$

$$(\nabla_{\alpha}(z)\varphi, \psi) + (\varphi, \nabla_{\alpha}(-z)\psi) = \partial_{\alpha}(\varphi, \psi). \tag{1.8b}$$

Conversely, if (1.8a) and (1.8b) are satisfied, then $c_{\alpha\beta\gamma}$ and $\eta_{\alpha\beta}$ correspond to a solution of WDVV equations.

It follows from (1.8a) that there exists a function $S(\tau, z)$ defined for $z \neq 0$, which takes values in invertible $n \times n$ matrices (or, speaking in more invariant way, in the group $\text{Aut } \mathcal{H}$ of automorphisms of vector space \mathcal{H}) and obeys

$$\nabla_{\alpha}(z)S(\tau, z) = 0, \tag{1.9}$$

or, more precisely

$$\partial_\alpha S^\beta_\gamma = z^{-1} c_\alpha^\epsilon c_\gamma S^\beta_\epsilon. \tag{1.10}$$

Using (1.8b) one verifies that $S(z)$ can be chosen (nonuniquely) in such a way that the normalization conditions

$$S(z = \infty) = 1, \tag{1.11a}$$

$$S(\tau, z) S^*(\tau, -z) = 1 \tag{1.11b}$$

are satisfied. Here S^* stands for an operator adjoint to S with respect to bilinear inner product $(,)$. The choice of $S(\tau, z)$ is called calibration of Frobenius manifold (of solution of WDVV equations).

We assume that $S(z)$ is a holomorphic function of $z \in \mathbb{P}^1 \setminus \{0\}$. (Here $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.) It is clear that two solutions of (1.10) obeying normalization conditions (1.11a) and (1.11b) are related by the formula

$$\tilde{S}(\tau, z) = S(\tau, z) M(z), \tag{1.12}$$

where M is a holomorphic function on $\mathbb{P}^1 \setminus \{0\}$ obeying

$$M(\infty) = 1, \tag{1.13a}$$

$$M(z) M^*(-z) = 1. \tag{1.13b}$$

In other words, the choice of calibration is governed by the group \mathbf{B}_+ consisting of holomorphic matrix functions on $\mathbb{P}^1 \setminus \{0\}$ satisfying (1.13a) and (1.13b). Notice that the functions $S^\alpha_\beta(\tau, z)$ obey

$$\partial_\lambda S^\alpha_\beta = \partial_\beta S^\alpha_\lambda, \tag{1.14}$$

which follows from symmetry of structure constants: $c_\beta^\epsilon c_\lambda^\epsilon = c_\lambda^\epsilon c_\beta^\epsilon$. Using (1.14) one can construct a function $x^\alpha(\tau, z)$ satisfying

$$S^\alpha_\beta(\tau, z) = \partial_\beta x^\alpha(\tau, z); \tag{1.15}$$

this function is defined up to a holomorphic summand that does not depend on τ (i.e. we can take $x^\alpha(\tau, z) + \rho^\alpha(z)$ instead of x^α). For fixed z we can consider $(x^1(\tau, z), \dots, x^n(\tau, z))$ as coordinates on manifold \mathcal{T} ; these coordinates are defined up to affine transformations (a choice of basis in the space of observables and a choice of calibration result in linear transformations of these coordinates, the freedom in a choice of x^α for given S leads to a shift of them).

In what follows we will modify the notion of calibration, saying that a calibrated solution of WDVV equations is a solution of these equations together with a solution of equations

$$\partial_\beta \partial_\gamma x^\alpha = z^{-1} c_\beta^\epsilon c_\gamma^\epsilon \partial_\epsilon x^\alpha, \quad \alpha = 1, 2, \tag{1.16}$$

obeying normalization conditions

$$x^\alpha(\tau, \infty) = \tau^\alpha, \tag{1.17}$$

$$\eta_{\alpha\beta} \partial_\lambda x^\alpha(\tau, z) \partial_\mu x^\beta(\tau, -z) = \eta_{\lambda\mu}. \tag{1.18}$$

Let us consider holomorphic functions on $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, which take values in the group $GL(\mathcal{H})$ of invertible linear transformations of \mathcal{H} or, more generally, in the group $\text{Aff}(\mathcal{H})$ of invertible affine transformations of \mathcal{H} . Using bilinear inner product on \mathcal{H} we define a group \mathbf{B} (twisted

loop group) as a group of $GL(\mathcal{H})$ -valued holomorphic functions $B(z)$ on \mathbb{C}^\times obeying

$$B(z)B^*(-z) = 1. \tag{1.19}$$

Similarly, we define a group \mathbf{A} as a group of $\text{Aff}(\mathcal{H})$ -valued functions $(B(z), d(z))$, where the linear part $B(z)$ obeys (1.19). Here we write an affine transformation as a pair (B, d) , where B is a linear transformation and d is a shift:

$$x \rightarrow Bx + d. \tag{1.20}$$

Lie algebras of groups \mathbf{B} and \mathbf{A} are denoted by \mathcal{B} and \mathcal{A} correspondingly. Notice that in the modified definition a choice of calibration is governed by the subgroup \mathbf{A}_+ of the group \mathbf{A} consisting of transformations of the form (1.20) with $B \in \mathbf{B}_+$.

Let us take a calibrated solution $(F(\tau), x^\alpha(\tau, z))$ of WDVV equations and an element $(b(z), d(z)) \in \mathcal{A}$. We define

$$\delta F(\tau) = -\frac{1}{2\pi i} \int_\Gamma \left[\frac{b_{\alpha\beta}(\zeta)x^\alpha(\tau, \zeta)}{2} + d_\beta(\zeta) \right] x^\beta(\tau, -\zeta) d\zeta, \tag{1.21}$$

$$\delta x^\alpha(\tau, z) = \frac{1}{2\pi i} \int_\Gamma \frac{\eta^{\rho\epsilon} [b_{\lambda\sigma}(\zeta)x^\lambda(\tau, \zeta) + d_\sigma(\zeta)] \partial_\epsilon x^\sigma(\tau, -\zeta)}{\zeta - z} \partial_\rho x^\alpha(\tau, z) d\zeta. \tag{1.22}$$

Here Γ is a circle with the center at $z = 0$; in (1.22) we assume that the radius of this circle is less than $|z|$. We will prove that $(F + \delta F, x + \delta x)$ is a calibrated solution of WDVV equations. In other words, the Lie algebra \mathcal{A} (or, more precisely, its extension) acts on the space of calibrated solutions of WDVV equations.

We did not include the existence of unit element in our axioms of two-dimensional TQFT. If the existence of unit is required, we consider transformations given by elements $(b(z), d(z)) \in \mathcal{A}$, where $d(z)$ is related to $b(z)$ in the following way

$$d_\alpha(z) = zb_{1\alpha}(z). \tag{1.23}$$

It follows from (1.6) that we can impose a normalization condition for $x^\alpha(\tau, z)$

$$\partial_1 x^\alpha(\tau, z) = z^{-1} x^\alpha(\tau, z) + \delta^\alpha_1. \tag{1.24}$$

Conversely, differentiating (1.24) and comparing with (1.6) we obtain that (1.24) implies $e_1 = 1$. The relation (1.23) is compatible with the normalization condition (1.24). More precisely, if we define the variation of x^α by means of (1.22) where $d(z)$ is related to $b(z)$ as in (1.23) the new x^α again obeys the same normalization condition; hence e_1 is the unit element also after variation.

In this way we obtain an action of Lie algebra \mathcal{B} on the space of calibrated Frobenius manifolds with unit elements. This action corresponds to the action of twisted loop group constructed by Givental in [2]. Notice that Givental’s construction provides an action of twisted loop group on the space of genus 0 TQFT coupled to gravity. If the Frobenius manifold is constructed by means of semi-infinite variation of Hodge structures the existence of the action of the twisted loop group follows from results of Barannikov [3]. In semi-simple case a different construction of the action of the same group was given by van de Leur (see [4]). Givental presented strong evidence that for all genera the same group acts on the space of TQFT’s coupled to gravity, at least at the level of perturbation theory. Recently one of us (M.K., in preparation) defined the action of twisted loop group also for open–closed theory confirming a conjecture of another of us (A. Schwarz).

2. Geometry of Frobenius manifolds

We have seen that coordinates $x^\alpha(\tau, z)$ are defined up to affine transformations. This means that we obtained a family of affine structures T_z on manifold \mathcal{T} depending on parameter $z \in \mathbb{P}^1 \setminus \{0\}$. We can consider the direct product of $\mathbb{P}^1 \times \mathcal{T}$ as a holomorphic bundle over \mathbb{P}^1 ; all fibers of this bundle except the fiber over $z = 0$ are affine spaces.

It is useful to give an invariant description of the above structure. To do this we notice that our construction of affine structures T_z can be regarded as a particular case of general construction of affine structure by means of torsion-free flat connection $\Gamma_{\alpha\beta}^\gamma$ (the connection is torsion-free if $\Gamma_{\alpha\beta}^\gamma = \Gamma_{\beta\alpha}^\gamma$; it is flat if covariant derivatives constructed by means of Christoffel symbols $\Gamma_{\alpha\beta}^\gamma$ commute). We can say that $z^{-1}c_\alpha^\beta{}_\gamma$ determines a family of torsion-free flat connections on \mathcal{T} that has a pole of order 1 at $z = 0$. Conversely, let us consider a holomorphic bundle \mathcal{E} over \mathbb{P}^1 with the fiber over $0 \in \mathbb{P}^1$ identified with \mathcal{T} . Let us assume that all fibers over points $z \in \mathbb{P}^1 \setminus \{0\}$ are equipped with affine structure and that the affine structure is defined by means of torsion-free flat connection $\Gamma_{\alpha\beta}^\gamma(z)$ having a pole of order 1 at $z = 0$. If the bundle at hand is holomorphically trivial we can identify its total space with $\mathbb{P}^1 \times \mathcal{T}$; this identification gives a coordinate system on the total space where $\Gamma_{\alpha\beta}^\gamma$ is linear with respect to $1/z$ (we use the fact that every holomorphic function of $z \in \mathbb{P}^1 \setminus \{0\}$ having first-order pole at $z = 0$ is linear with respect to $1/z$).

Due to (1.18) one can construct a nondegenerate bilinear pairing between tangent spaces to affine spaces T_z and T_{-z} where $z \in \mathbb{P}^1 \setminus \{0\}$ (notice that tangent spaces at different points of affine space are identified). More precisely, we can rewrite (1.18) as

$$\eta_{\alpha\beta} \delta x^\alpha(\tau, z) \delta x^\beta(\tau, -z) = \eta_{\mu\nu} \delta \tau^\mu \delta \tau^\nu.$$

This equation shows that the metric $\eta_{\alpha\beta}$ induces covariantly constant pairing between tangent spaces to affine spaces T_z and T_{-z} ; for fixed τ and z tending to zero this pairing has a limit (in our coordinate system it does not depend on z). The statement about existence of limit remains correct if τ is not fixed, but depends analytically on z in a neighborhood of $z = 0$. To analyze the case when $\tau = \tau(z)$ we should prove that the expression

$$\eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial \tau^\lambda} \Big|_{(\tau=\tau(z), z)} \frac{\partial x^\beta}{\partial \tau^\mu} \Big|_{(\tau=\tau(-z), -z)} \tag{2.1}$$

has a finite limit as z tends to zero. To use the relation (1.18) we decompose

$$\frac{\partial x^\alpha}{\partial \tau^\lambda} \Big|_{(\tau, z)}$$

into Taylor series at the point (τ_0, z) ; we apply this decomposition to the case $\tau = \tau(z)$, $\tau_0 = \tau(-z)$. One can check that for z tending to zero the leading term can be written in the form

$$\frac{\partial x^\alpha}{\partial \tau^\lambda} \Big|_{(\tau, z)} = \frac{\partial x^\alpha}{\partial \tau^\lambda} \Big|_{(\tau_0, z)} + \sum_{n>0} \frac{1}{n!} z^{-n} (\hat{C}^n)_\lambda^\sigma \frac{\partial x^\alpha}{\partial \tau^\sigma}. \tag{2.2}$$

Here \hat{C} stands for the matrix

$$C_\rho^\nu = (\tau^\mu - \tau_0^\mu) c_{\rho\mu}^\nu(\tau_0).$$

Applying (2.2) and (1.18) we obtain that (2.1) has a limit as z tends to zero; it converges to

$$(\exp \tilde{C})_\lambda^\sigma \eta_{\sigma\mu} \tag{2.3}$$

where

$$\tilde{C}_\rho^v = 2 \frac{d\tau^\mu}{dz} \Big|_{z=0} c_{\rho\mu}^v(\tau_0).$$

In the derivation of (2.2) one should use the formula

$$\frac{\partial^2 x^\alpha}{\partial \tau^\beta \partial \tau^\gamma} = z^{-1} c_{\beta\gamma}^\epsilon \frac{\partial x^\alpha}{\partial \tau^\epsilon} \tag{2.4}$$

and similar formulas expressing higher partial derivatives of x^α in terms of the first derivatives. (One obtains such formulas differentiating (2.4).) We assume that z tends to zero and keep only the leading term; then there is no necessity to take derivatives of $c_{\beta\gamma}^\epsilon$ differentiating (2.4). We obtain that in our approximation

$$\frac{\partial^{k+1} x^\alpha}{\partial \tau^\beta \partial \tau^{\gamma_1} \dots \partial \tau^{\gamma_k}} = z^{-k} c_{\gamma_k \rho_{k-1}}^{\rho_k} c_{\gamma_{k-1} \rho_{k-2}}^{\rho_{k-1}} \dots c_{\gamma_1 \beta}^{\rho_1} \frac{\partial x^\alpha}{\partial \tau^{\rho_1}}. \tag{2.5}$$

It is easy to check that (2.2) follows from (2.5).

Let us describe now geometric data that permit us to construct a calibrated solution of WDVV. We start with holomorphic bundle over \mathbb{P}^1 and a family of torsion-free flat connections $\Gamma_{\alpha\beta}^\gamma(z, \tau)$ on the fibers T_z of this bundle that depend holomorphically on $z \in \mathbb{P}^1 \setminus \{0\}$ and have a pole of order 1 at $z = 0$. We assume that the bundle is holomorphically trivial; trivialization permits us to represent the connections in the form $\Gamma_{\alpha\beta}^\gamma(z, \tau) = z^{-1} c_{\alpha\gamma}^\beta(\tau)$ in corresponding coordinate system.

Let us suppose in addition that we have a covariantly constant nondegenerate bilinear pairing between tangent spaces to fibers over z and over $-z$ for $z \neq 0$ that depends holomorphically on z . (As we noticed affine structure on T_z permits us to identify tangent spaces at all points of T_z .) This pairing gives a flat metric on T_∞ . We assume that the pairing between tangent spaces to T_z and T_{-z} has a limit if the points of T_z and T_{-z} tend to a point of T_0 staying on a holomorphic section of our bundle over a neighborhood of $z = 0$. (It is sufficient to assume that the pairing is bounded, because a holomorphic function bounded in a neighborhood of a point has a removable singularity at this point.)

Writing the bilinear pairing in the coordinate system coming from a trivialization of the bundle we obtain that the pairing can be described by means of matrix $\eta_{\alpha\beta}(\tau)$ that does not depend on z . (Independence of z follows from the fact that a bounded holomorphic function of z is a constant.) We conclude from our assumption about the fiber T_∞ that the matrix $\eta_{\alpha\beta}(\tau)$ specifies a nondegenerate flat metric on fibers.

It follows from above remarks that $c_{\alpha\gamma}^\beta$ and $\eta_{\alpha\beta}$ obey (1.8a) and (1.8b), hence they specify a solution to WDVV equation; the choice of trivialization of affine bundle over $\mathbb{P}^1 \setminus \{0\}$ corresponds to the choice of calibration.

3. Symmetries of WDVV equation

Let us consider a solution to WDVV equations and its calibration $x^\alpha(\tau, z)$. As we have seen, the functions $x^\alpha(\tau, z)$ determine an affine structure on fibers of the trivial bundle over \mathbb{P}^1 except the fiber over $z = 0$. One can obtain \mathbb{P}^1 pasting together two open disks $|z| < \infty$ and $|z| > 0$. We can construct a new holomorphic bundle over \mathbb{P}^1 twisting the direct product $\mathbb{P}^1 \times \mathcal{T}$ over \mathbb{C}^\times , which is the common region of these two disks D_0 and D_∞ . The total space of the new bundle is obtained by means of identification of $\mathbb{C}^\times \times \mathcal{T} \subset D_0 \times \mathcal{T}$ with $\mathbb{C}^\times \times \mathcal{T} \subset D_\infty \times \mathcal{T}$ by the

formula $(z, \tau) \sim (z, f_z(\tau))$. Here $f_z: T_z \rightarrow T_z$ is an analytic automorphism of T_z (of the fiber over $z \in \mathbb{C}^\times$) that depends analytically of $z \in \mathbb{C}^\times$. The fiber T_z is equipped with affine structure. We will assume that $f_z: T_z \rightarrow T_z$ is an affine transformation; then the fibers of the new bundle over all points $z \in \mathbb{P}^1 \setminus \{0\}$ also can be considered as affine spaces.

If f_z is sufficiently close to identity the new holomorphic bundle is holomorphically trivial (this follows from the remark that holomorphically trivial vector bundles form an open subset of the space of holomorphic vector bundles over \mathbb{P}^1). As we know, there exists a pairing between tangent spaces to the fibers T_z and T_{-z} . To guarantee the existence of similar pairing for the new bundle we should impose some conditions on affine maps $f_z: T_z \rightarrow T_z$. Namely, if

$$f_z = (B(z), d(z)) \tag{3.1}$$

we should require

$$B(z)B(-z)^* = 1. \tag{3.2}$$

Notice that we did not change the pairing over the inner disk, therefore the behavior of the pairing as z tends to zero does not change. We imposed the condition the pairing is bounded on every holomorphic section over a neighborhood of $z = 0$; this condition is fulfilled for the new bundle. Hence the new bundle also specifies a calibrated solution to WDVV equations.

The group of matrix functions $B(z)$ obeying (3.2) is a (version of) twisted loop group of [2]. The group of affine transformations of the form (3.1) with the linear part obeying (3.2) was denoted by \mathbf{A} in Section 1 and the corresponding Lie algebra was denoted by \mathcal{A} . In what follows it will be convenient to consider f_z as an operator on the vector space of observables \mathcal{H} .

It follows from the above consideration that the Lie algebra \mathcal{A} acts on the space of calibrated solutions of WDVV equations. Our goal is to calculate this action more explicitly.

Elements of \mathcal{A} can be considered as pairs $(b(z), d(z))$, where $b(z)$ is a holomorphic function on \mathbb{C}^\times with values in linear maps $\mathcal{H} \rightarrow \mathcal{H}$ obeying $b(z) + b(-z)^* = 0$ and $d(z)$ is a holomorphic \mathcal{H} -valued function on \mathbb{C}^\times . To calculate the action of $(b(z), d(z))$ we should trivialize the twisted bundle. This means that for every $\tau \in \mathcal{T} = T_\infty$, we should find a holomorphic section of the twisted bundle taking the value τ at $z = \infty$. We will write this holomorphic section as a pair of holomorphic sections of direct product $D_0 \times \mathcal{T}$ and $D_\infty \times \mathcal{T}$ with appropriate gluing conditions on \mathbb{C}^\times . Representing sections of $D_i \times \mathcal{T}$ as \mathcal{T} -valued functions we can write down the gluing condition in the form

$$(\delta^\mu_\lambda + b_\lambda^\mu(z))x^\lambda(\kappa(z), z) + d^\mu(z) = x^\mu(\alpha(z), z), \tag{3.3}$$

where $\alpha^\rho(z)$ and $\kappa^\rho(z)$ close to τ^ρ can be expressed as

$$\begin{aligned} \alpha^\rho(z) &= \tau^\rho + a^\rho(z), & \text{inner disk } D_0, \\ \kappa^\rho(z) &= \tau^\rho + k^\rho(z), & \text{outer disk } D_\infty. \end{aligned} \tag{3.4}$$

Here we are working with infinitesimal group transformation, corresponding to Lie algebra element $(b(\zeta), d(\zeta))$. We require $k(\tau, \infty) = 0$ to get a section containing $\tau \in \mathcal{T} = T_\infty$.

From (3.3) we have

$$\varphi^\rho(\zeta) \equiv a^\rho(\zeta) - k^\rho(\zeta) = (S^{-1}(\zeta))^\rho_\mu [b_\lambda^\mu(\zeta)x^\lambda(\zeta) + d^\mu(\zeta)]. \tag{3.5}$$

Knowing (3.5), we can express a and k in terms of Cauchy integral

$$\Phi^\rho(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi^\rho(\zeta)}{\zeta - z} d\zeta, \tag{3.6}$$

where Γ is any circle with the center at the origin. This integral represents a function that is holomorphic everywhere except for Γ , on which it has a jump φ^ρ , and then tends to zero at infinity. Using these facts we obtain

$$a^\rho(z) = \Phi^\rho(z) \tag{3.7}$$

inside Γ ,

$$k^\rho(z) = \Phi^\rho(z) \tag{3.8}$$

outside Γ and

$$k^\rho(z) \rightarrow 0 \quad \text{as } z \rightarrow \infty. \tag{3.9}$$

The expression (3.8) for k^ρ leads immediately to the expression (1.22) for δx^α .

To calculate the variation of structure constants c we notice that we do not change the connection over the inner disk; the only change in c comes from the change of coordinates. Using the standard rules for the variation of connection one can obtain that the structure constants behave like a tensor by the change of coordinates; this fact was used in [3]. (Recall, that structure constants describe the behavior of connection at the point $z = 0$ where the connection has a pole.) The change of coordinate τ at $z = 0$ is governed by the formula

$$a^\rho(z=0) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi^\rho(\zeta)}{\zeta} d\zeta = \frac{1}{2\pi i} \int_\Gamma \frac{\eta^{\rho\epsilon} [b_{\lambda\sigma}(\zeta)x^\lambda(\zeta) + d_\sigma(\zeta)] \partial_\epsilon x^\sigma(-\zeta)}{\zeta} d\zeta. \tag{3.10}$$

Calculating the Jacobian matrix at $z = 0$

$$\begin{aligned} L^\mu_\nu &= \frac{\partial \tau^\mu}{\partial \alpha^\nu} \\ &= \delta^\mu_\nu - \frac{1}{2\pi i} \int_\Gamma \frac{\eta^{\mu\epsilon} b_{\lambda\sigma}(\zeta)}{\zeta} [\partial_\nu x^\lambda(\zeta) \partial_\epsilon x^\sigma(-\zeta) + x^\lambda(\zeta) \partial_\nu \partial_\epsilon x^\sigma(-\zeta)] d\zeta \\ &\quad - \frac{1}{2\pi i} \int_\Gamma \frac{\eta^{\mu\epsilon} d_\sigma(\zeta)}{\zeta} \partial_\nu \partial_\epsilon x^\sigma(-\zeta) d\zeta, \end{aligned} \tag{3.11}$$

we obtain the variation of structure constants

$$\begin{aligned} \delta c_\alpha^\beta{}_\gamma(\tau) &= (L^{-1})^\mu_\alpha L^\beta_\nu (L^{-1})^\xi_\gamma c_{\mu\nu}{}^\xi(\tau) - c_\alpha^\beta{}_\gamma(\tau) - \partial_\epsilon (c_\alpha^\beta{}_\gamma) \delta \tau^\epsilon \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{\eta^{\mu\epsilon} d_\sigma(\zeta)}{\zeta} [\partial_\alpha (c_\epsilon^\beta{}_\gamma) \partial_\mu x^\sigma(-\zeta) + c_{\mu\nu}{}^\beta{}_\gamma \partial_\alpha \partial_\epsilon x^\sigma(-\zeta)] d\zeta \\ &\quad + \frac{1}{2\pi i} \int_\Gamma \frac{b_{\sigma\lambda}(\zeta)}{\zeta} [\eta^{\mu\epsilon} c_{\mu\nu}{}^\beta{}_\gamma \partial_\epsilon x^\lambda(-\zeta) \partial_\alpha x^\sigma(\zeta) + \eta^{\beta\epsilon} c_\alpha{}^\nu{}_\gamma \partial_\nu x^\lambda(-\zeta) \partial_\epsilon x^\sigma(\zeta) \\ &\quad + \eta^{\xi\epsilon} c_\alpha^\beta{}_\xi \partial_\epsilon x^\lambda(-\zeta) \partial_\gamma x^\sigma(\zeta) + \eta^{\mu\epsilon} \partial_\alpha (c_\epsilon^\beta{}_\gamma) \partial_\mu x^\lambda(-\zeta) x^\sigma(\zeta) \\ &\quad + \eta^{\mu\epsilon} c_{\mu\nu}{}^\beta{}_\gamma \partial_\alpha \partial_\epsilon x^\lambda(-\zeta) x^\sigma(\zeta)] d\zeta. \end{aligned} \tag{3.12}$$

Taking into account (1.4), we see that (3.12) leads to the expression (1.21) for δF .

Let us assume now that the first element of the basis of \mathcal{H} is the unit element and $\eta_{\alpha\beta} = c_{1\alpha\beta}$. We would like to require that these assumptions are fulfilled also for the deformed solution of

WDVV equations. Then we should have

$$\delta(\delta^\beta_\alpha) = \delta c_{\alpha\beta}{}^1(\tau) = \frac{1}{2\pi i} \int_\Gamma \frac{\eta^{\beta\epsilon} [d_\sigma(\zeta) - \zeta b_{1\sigma}(\zeta)]}{\zeta} \partial_\alpha S^\sigma_\epsilon(-\zeta) d\zeta = 0. \tag{3.13}$$

This is true if we have the relation (1.23) between the functions $d(z)$ and $b(z)$. We have seen in the introduction how to change the expressions for δF and δx^α if we require the existence of unit element. We came again to the same prescription.

The monomials

$$b_{\alpha\beta,m} z^{-m}$$

with coefficients obeying

$$b_{\alpha\beta,m} = (-1)^{m+1} b_{\beta\alpha,m}, \tag{3.14}$$

can be considered as (topological) generators of the Lie algebra **B**. Let us write down corresponding variations of F and of coefficients h_n that appear in the decomposition of $x^\alpha(\tau, z)$ in a series $x^\alpha(\tau, z) = \sum_{n=0}^\infty h^{\alpha,n} z^{-n}$. It follows from (1.21), (1.22) and (1.23) that

$$\begin{aligned} \delta F(\tau) = & b_{\alpha 1,2} \tau^\alpha + b_{\alpha 1,1} h^{\alpha,1} - \frac{1}{2} b_{\alpha\beta,1} \tau^\alpha \tau^\beta + b_{\alpha 1,0} h^{\alpha,2} + b_{\alpha\beta,0} \tau^\alpha h^{\beta,1} \\ & + \sum_{m < 0} \left\{ b_{\alpha 1,m} h^{\alpha,-m+2} + b_{\alpha\beta,m} \left[(-1)^m \tau^\alpha h^{\beta,-m+1} \right. \right. \\ & \left. \left. + \frac{1}{2} \sum_{n=m}^{-1} (-1)^{m+n} h^{\alpha,-n} h^{\beta,-m+n+1} \right] \right\}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} \delta x^\alpha(\tau, z) = & \eta^{\rho\epsilon} \sum_{k \geq 0} \partial_\rho h^{\alpha,k} z^{-k} \sum_{i \geq 0} \sum_{l=0}^i \partial_\epsilon h^{\sigma,l} (-1)^l \left(b_{1\sigma,-i+1} + \sum_{m \leq -i} b_{\lambda\sigma,m} h^{\lambda,-m-i} \right) z^{i-l} \\ & - \sum_{m=-\infty}^\infty \eta^{\alpha\sigma} [z b_{1\sigma,m} + b_{\lambda\sigma,m} x^\lambda(\tau, z)] z^{-m}, \end{aligned} \tag{3.16}$$

$$\begin{aligned} \delta h^{\alpha,n} = & \eta^{\rho\epsilon} \sum_{0 \leq l \leq i \leq \infty} \partial_\rho h^{\alpha,i+n-l} \partial_\epsilon h^{\sigma,l} (-1)^l b_{1\sigma,-i+1} \\ & + \eta^{\rho\epsilon} \sum_{0 \leq l \leq i \leq -m \leq \infty} \partial_\rho h^{\alpha,i+n-l} \partial_\epsilon h^{\sigma,l} (-1)^l b_{\lambda\sigma,m} h^{\lambda,-m-i} \\ & - \eta^{\alpha\sigma} \left[b_{1\sigma,n+1} + \sum_{m \leq n} b_{\lambda\sigma,m} h^{\lambda,n-m} \right]. \end{aligned} \tag{3.17}$$

In particular, for $m = 1$ we obtain

$$\begin{aligned} \delta F(\tau) = & b_{\alpha 1,1} h^{\alpha,1} - \frac{1}{2} b_{\alpha\beta,1} \tau^\alpha \tau^\beta, \\ \delta c_{\alpha\beta\gamma}(\tau) = & \eta^{\lambda\epsilon} b_{\lambda 1,1} \partial_\alpha c_{\epsilon\beta\gamma}, \end{aligned} \tag{3.18}$$

$$\begin{aligned} \delta\tau^\alpha &= \delta h^{\alpha,0} = 0, \\ \delta h^{\alpha,n} &= \eta^{\rho\sigma} b_{1\sigma,1} \partial_\rho h^{\alpha,n} - \eta^{\alpha\sigma} b_{\lambda\sigma,1} h^{\lambda,n-1}, \quad n \geq 1. \end{aligned} \tag{3.19}$$

For $m = 2$

$$\begin{aligned} \delta F(\tau) &= b_{\alpha 1,2} \tau^\alpha, \\ \delta c_{\alpha\beta\gamma}(\tau) &= 0, \end{aligned} \tag{3.20}$$

$$\begin{aligned} \delta\tau^\alpha &= \delta h^{\alpha,0} = 0, \\ \delta h^{\alpha,1} &= -\eta^{\alpha\sigma} b_{1\sigma,2}, \\ \delta h^{\alpha,n} &= -\eta^{\alpha\sigma} b_{\lambda\sigma,2} h^{\lambda,n-2}, \quad n \geq 2. \end{aligned} \tag{3.21}$$

For $m > 2$ we obtain

$$\begin{aligned} \delta F(\tau) &= 0, \\ \delta c_{\alpha\beta\gamma}(\tau) &= 0, \end{aligned} \tag{3.22}$$

$$\begin{aligned} \delta h^{\alpha,n} &= 0, \quad 0 \leq n \leq m, \\ \delta h^{\alpha,m-1} &= -\eta^{\alpha\sigma} b_{1\sigma,m}, \\ \delta h^{\alpha,n} &= -\eta^{\alpha\sigma} b_{\lambda\sigma,m} h^{\lambda,n-m}, \quad n \geq m. \end{aligned} \tag{3.23}$$

4. Comparison with Givental’s approach

Givental’s construction of twisted loop group acting on the space of Frobenius manifolds is based on another geometric interpretation of the notion of Frobenius structure. For every calibrated solution of WDVV equation one can construct a functional \mathcal{F} that depends on infinite number of parameters q^0, \dots, q^n, \dots , where $q^k \in \mathcal{H}$. (The functional \mathcal{F} can be interpreted as free energy depending on primary fields and descendants. The construction of \mathcal{F} in terms of free energy F , defined on “small phase space” \mathcal{H} , and calibration is given in [1,5].) More precisely, one should consider \mathcal{F} as a formal power series with respect to its arguments; it is important to emphasize that parameters q^k used by Givental are obtained from standard parameters $\tau^0, \dots, \tau^n, \dots$ in the expression of free energy by means of a shift in τ^1 . Notice, that in Givental’s paper it is assumed that there exists a unit element and $\eta_{\alpha\beta} = c_{1\alpha\beta}$. (This is a standard assumption relaxed in present paper mostly to simplify the exposition.)

Let us introduce a space of \mathcal{H} -valued Laurent polynomials $a(z)$ equipped with symplectic form

$$\begin{aligned} \omega(a, b) &= \text{res}(a(z), b(-z)) = (2\pi i)^{-1} \int_\Gamma (a(z), b(-z)) dz \\ &= (2\pi i)^{-1} \int_\Gamma a^\alpha(z) \eta_{\alpha\beta} b^\beta(-z) dz. \end{aligned} \tag{4.1}$$

This space can be identified with T^*L where L is a Lagrangian subspace consisting of polynomials $q^0 + q^1 z + q^2 z^2 + \dots$. The functional \mathcal{F} can be considered as a function on L ; we construct a Lagrangian submanifold \mathcal{L} corresponding to \mathcal{F} by means of standard formula

$$p_i = \partial\mathcal{F}/\partial q^i. \tag{4.2}$$

One can prove [2], that \mathcal{L} is a Lagrangian cone with the vertex at the origin and that every tangent space $T_x\mathcal{L}$ to \mathcal{L} is tangent to \mathcal{L} exactly along $zT_x\mathcal{L}$. (More precisely, if K is a tangent space to \mathcal{L} then its intersection with \mathcal{L} consists of points of the form zk where $k \in K$.) Conversely, every Lagrangian cone obeying these conditions corresponds to calibrated solution of WDVV equations. Givental defines twisted loop group as a group of matrix Laurent series satisfying

$$M(z)M^*(-z) = 1. \tag{4.3}$$

It is easy to check that elements of this group can be interpreted as linear symplectic transformations commuting with multiplication by z . Lagrangian cones corresponding to calibrated solutions of WDVV equations are defined in terms of symplectic geometry and multiplication by z . This means that an element of twisted loop group transforms such a cone into another cone of the same kind and a calibrated solution of WDVV equations into another calibrated solution. (This is a rigorous statement if one works at the level of Lie algebras; to define the action of an element of twisted loop group one should be more precise with the definition of free energy in terms of formal series.) Using the construction of \mathcal{F} in terms of F and calibration one can calculate the action of Lie algebra of twisted loop group on calibrated solutions of WDVV equations; one gets formulas that agree with (3.15), (3.16). One should emphasize, however, that Givental works with formal series instead of functions.

Let us discuss the relation between Givental’s geometric picture and the setup of the present paper. We will work with symplectic vector space \mathcal{C} of \mathcal{H} -valued holomorphic functions on \mathbb{C}^\times having a pole or a removable singularity at infinity; the symplectic form on \mathcal{C} is defined by the formula (4.1).

We start with a calibrated solution of WDVV equations and define a subset \mathcal{L} of \mathcal{C} as a set of all functions of the form $zp^\lambda \frac{\partial x(\tau, z)}{\partial \tau^\lambda}$, where $\tau \in \mathcal{T}$, $x(\tau, z) = (x^1(\tau, z), \dots, x^n(\tau, z))$ is a calibration and $p^\lambda(z)$ is a polynomial. One can say that \mathcal{L} is a union of vector spaces $\mathcal{L}_\tau = \{zp^\lambda \frac{\partial x(\tau, z)}{\partial \tau^\lambda} \mid p^\lambda \in \mathbb{C}[z]\}$ labeled by $\tau \in \mathcal{T}$. It is easy to check that \mathcal{L} is an isotropic submanifold of \mathcal{C} . Indeed, the tangent space $L_f = T_f\mathcal{L}$ to \mathcal{L} at the point $f = zp^\lambda(z) \frac{\partial x(\tau, z)}{\partial \tau^\lambda} \neq 0$ is spanned by \mathcal{L}_τ and $\partial f / \partial \tau^1, \dots, \partial f / \partial \tau^n$. Using the relation

$$\frac{\partial f}{\partial \tau^\sigma} = zp^\lambda(z) \frac{\partial^2 x(\tau, z)}{\partial \tau^\lambda \partial \tau^\sigma} = p^\lambda(z) c_{\lambda\gamma}^\sigma \frac{\partial x(\tau, z)}{\partial \tau^\gamma} \tag{4.4}$$

and (1.18) we find that the form (4.1) vanishes on all tangent spaces.

Let us suppose now that $\eta_{\alpha\beta} = c_{1\alpha\beta}$ and that the function x^α obeys the normalization condition (2.4). Introducing the notation $J^{\mu, z} = x^\mu(\tau, z) + z$, we conclude from (2.4) that the point

$$J(\tau, z) = z\partial_1 J(\tau, z) \tag{4.5}$$

belongs to \mathcal{L}_τ (it can be represented as $f = zp^\lambda(z) \frac{\partial x(\tau, z)}{\partial \tau^\lambda}$ with $p(z) = (1, 0, \dots, 0)$). Applying (4.4) we see that the tangent space $L_f = T_f\mathcal{L}$ at this point is spanned by \mathcal{L}_τ and $\partial f / \partial \tau^\sigma = \partial x / \partial \tau^\sigma$, in other words, it consists of vectors of the form $p^\lambda(z) \frac{\partial x}{\partial \tau^\lambda}$, where p^λ is an arbitrary polynomial. Using (2.4) we obtain that this tangent space is Lagrangian. The isotropic submanifold \mathcal{L} is a union of vector spaces, hence it is always a cone. The subset of \mathcal{L} consisting of points where the matrix $p^\lambda(0)c_{\lambda\gamma}^\sigma$ is nondegenerate can be regarded as Lagrangian cone. Notice that this subset can be empty if we do not assume the existence of unit element; it is nonempty if for at least one element of the algebra with structure constants $c_{\lambda\gamma}^\sigma$ the operator of multiplication by this element is invertible. (We use the fact that an isotropic subspace of T^*L is Lagrangian if its projection on L is surjective.)

Let us describe the way to obtain calibrated solutions from Lagrangian cones following the considerations applied in [2] to the case when there exists a unit element. Let us consider a Lagrangian cone $\mathcal{L} \subset T^*L$ with the vertex at the origin assuming that every tangent space $T_x\mathcal{L}$ to \mathcal{L} is tangent to \mathcal{L} exactly along $zT_x\mathcal{L}$. Let us assume that for every τ the cone \mathcal{L} contains a point $J(z, \tau) = cz + \tau + r(z, \tau)$ where $r(\infty, \tau) = 0$. Then the derivatives $\partial J/\partial\tau^\alpha$ form a basis in $\Lambda/z\Lambda$ where Λ stands for one of these tangent spaces. They can be regarded also as free generators of Λ considered as $\mathbb{C}[z]$ -module. Using the fact that $z\partial J/\partial\tau \in z\Lambda \subset \mathcal{L}$ we obtain that the second derivatives $z\partial^2 J/\partial\tau^\alpha\tau^\beta$ are in Λ and therefore can be represented as linear combinations of $\partial J/\partial\tau^\alpha$ with coefficients that are polynomials with respect to z . From the other side these second derivatives have a removable singularity at infinity, hence the coefficients do not depend on z . These coefficients $c_{\alpha\gamma\beta}$ specify a family of torsion-free flat connections by the formula (1.7). (The connections are flat because Eq. (1.9) has a nondegenerate solution $S(z, \tau) = \partial J/\partial\tau$.)

The connections we constructed together with $J(z, \tau)$ specify a calibrated solution to the WDVV equations. To finish the proof of this statement we should check (2.1). However, one can verify that (2.1) follows from (1.18). (Both of these conditions can be interpreted geometrically as compatibility of nondegenerate linear pairing between tangent spaces to affine spaces T_z and T_{-z} with the connection $\Gamma_{\alpha\beta}^\gamma(z) = z^{-1}c_{\alpha\gamma\beta}$.) We used (1.18) to prove that the cone constructed by means of a calibrated solution to WDVV equations is isotropic; one can use the same arguments in opposite direction: to show that the family of connections obtained from a Lagrangian cone obeys (1.18).

The formulas of present paper can be obtained in Givental's approach by means of reconstruction formula of [1,5]. Such a derivation is sketched in the appendix to the preprint version of the paper (hep-th/0508221).

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