

# The Jacobian Conjecture is stably equivalent to the Dixmier Conjecture

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## 1 Introduction

The **Jacobian Conjecture**  $JC_n$  in dimension  $n \geq 1$  asserts that *for any field  $\mathbf{k}$  of characteristic zero any polynomial endomorphism  $\phi$  of the  $n$ -dimensional affine space  $\mathbb{A}_{\mathbf{k}}^n = \text{Spec } \mathbf{k}[x_1, \dots, x_n]$  over  $\mathbf{k}$ , with Jacobian 1:*

$$\det(\partial\phi^*(x_i)/\partial x_j)_{1 \leq i, j \leq n} = 1$$

*is an automorphism.* Equivalently, one can say that  $\phi$  preserves the standard top-degree differential form  $dx_1 \wedge \dots \wedge dx_n \in \Omega^n(\mathbb{A}_{\mathbf{k}}^n)$ .

The reference due to this well known problem and related questions can be found in [5], [3].

By the Lefschetz principle it is sufficient to consider the case  $\mathbf{k} = \mathbb{C}$ . Obviously,  $JC_n$  implies  $JC_m$  if  $n > m$ . We denote by  $JC_{\infty}$  the stable Jacobian conjecture, the conjunction of conjectures  $JC_n$  for all finite  $n$ . The conjecture  $JC_n$  is obviously true in the case  $n = 1$ , and it is open for  $n \geq 2$ .

The **Dixmier Conjecture**  $DC_n$  for integer  $n \geq 1$  (see [4]) asserts that *for any field  $\mathbf{k}$  of characteristic zero any endomorphism of the  $n$ -th Weyl algebra  $A_{n, \mathbf{k}}$  over  $\mathbf{k}$  is an automorphism.*

Here  $A_{n, \mathbf{k}}$  is the associative unital algebra over  $\mathbf{k}$  with  $2n$  generators  $y_1, \dots, y_{2n}$  and relations

$$[y_i, y_j] = \omega_{ij},$$

where  $(\omega_{ij})_{1 \leq i, j \leq 2n}$  is the following standard  $2n \times 2n$  skew-symmetric matrix:

$$\omega_{ij} = \delta_{i, j+n} - \delta_{i+n, j}.$$

The algebra  $A_{n,\mathbf{k}}$  coincides with the algebra  $D(\mathbb{A}_{\mathbf{k}}^n)$  of polynomial differential operators on  $\mathbb{A}_{\mathbf{k}}^n$ . For any  $i$ ,  $1 \leq i \leq n$  element  $y_i$  acts as the multiplication operator by the variable  $x_i$ , and element  $y_{n+i}$  acts by the differentiation  $\partial/\partial x_i$ . Again, it is sufficient to consider the case  $\mathbf{k} = \mathbb{C}$ . The conjecture  $\text{DC}_n$  implies  $\text{DC}_m$  for  $n > m$ , and we can consider the stable Dixmier conjecture  $\text{DC}_\infty$ . The conjecture  $\text{DC}_n$  is open for any  $n \geq 1$ .

It is well-known that  $\text{DC}_n$  implies  $\text{JC}_n$  (in particular  $\text{DC}_\infty$  implies  $\text{JC}_\infty$ ) (see [5], [3]). The argument is very easy. Let  $\phi : \mathbb{A}_{\mathbf{k}}^n \rightarrow \mathbb{A}_{\mathbf{k}}^n$  be a counterexample to  $\text{JC}_n$ . Then  $\phi$  is a non-invertible étale map, and it induces a pullback homomorphism  $\phi_{diff}^*$  of the algebra of differential operators on  $\mathbb{A}_{\mathbf{k}}^n$ . The endomorphism  $\phi_{diff}^*$  of the Weyl algebra preserves the degree of differential operators. Restricting  $\phi_{diff}^*$  to zero order differential operators, we obtain the usual pullback  $\phi^*$  of functions on  $\mathbb{A}_{\mathbf{k}}^n$ . By our assertion it is not surjective, hence we obtain a counterexample to  $\text{DC}_n$ .

Our result is an opposite implication. Namely, we prove the following

**Theorem 1** *Conjecture  $\text{JC}_{2n}$  implies  $\text{DC}_n$ .*

In particular, we obtain that the stable conjectures  $\text{JC}_\infty$  and  $\text{DC}_\infty$  are equivalent.

**Remark 1** *A. van den Essen ([5], Theorem 10,4.2) proved a weaker result: the conjecture  $\text{JC}_{2n}$  implies the invertibility of any endomorphism of  $A_{n,\mathbf{k}} = D(\mathbb{A}_{\mathbf{k}}^n)$  preserving the filtration by the degrees of differential operators.*

For the convenience of the reader, and in order to make the text self-contained, we include in the paper proofs of several known results scattered in the literature. During the preparation of this paper we have learned from K. Adjagbo about the preprint [10] where two key results concerning the Weyl algebra in finite characteristic were established (Propositions 2 and 4 from Section 4 in the present paper), see also a very recent preprint [2].

**Remark 2** *The present paper is written in the standard language of algebraic geometry. It is possible (and reasonable for some minds) to use the model-theoretic language of non-standart analysis, instead of schéme-theoretic considerations. In particular, in the proofs of several results of our paper one can use the reduction modulo an infinitely large prime.*

**Remark 3** *After this paper was written, we were told by Ken Goodearl about a paper "Endomorphisms of Weyl algebra and p-curvatures" (Osaka Journal of Mathematics Volume 42, Number 2 (June 2005)) by Yoshifumi Tsuchimoto which contains the proof of our main result. The proofs by Tsuchimoto and in the present paper are different (although there are many similarities), hence we think that it is reasonable to keep our paper on archive.*

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## 2 A reformulation of the Jacobian conjecture

For given integers  $n \geq 2, d \geq 1$  we denote by  $\text{JE}_{n,d}$  an affine scheme of finite type over  $\mathbb{Z}$  representing the following functor. For any commutative ring  $R$  the set  $\text{JE}_{n,d}(R)$  is the set of endomorphisms  $f$  of  $R$ -algebra  $R[x_1, \dots, x_n]$  such that

- $\det(\partial f(x_i)/\partial x_j)_{1 \leq i,j \leq n} = 1 \in R[x_1, \dots, x_n]$ ,
- $\deg(f(x_i)) \leq d \quad \forall i, 1 \leq i \leq n$ .

We say that  $f$  as above is an *endomorphism of degree  $\leq d$*  (and with Jacobian 1). The ring of functions  $\mathcal{O}(\text{JE}_{n,d})$  is finitely generated, its generators are coefficients  $c_{i,\alpha}$  which appear in the universal endomorphism over  $\mathcal{O}(\text{JE}_{n,d})$ :

$$f_{\text{univ}}(x_i) = \sum_{\alpha: |\alpha| \leq d} c_{i,\alpha} x^\alpha$$

Here  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$  is a multi-index,  $x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$ ,  $|\alpha| := \sum_{i=1}^n \alpha_i$ .

Similarly, for  $n \geq 2, d \geq 1, d' \geq 1$  we denote by  $\text{JA}_{n,d,d'}$  an affine scheme of finite type over  $\mathbb{Z}$  parameterizing pairs of endomorphisms  $(f, f')$  of  $n$ -dimensional affine space, with Jacobian 1, of degrees  $\leq d$  and  $\leq d'$  respectively, mutually inverse to each other:  $f \circ f' = f' \circ f = \text{Id}_{\mathbb{A}^n}$ .

We have an obvious forgetting map  $\text{pr}_{n,d,d'}^{(J)} : \text{JA}_{n,d,d'} \rightarrow \text{JE}_{n,d}$ ,  $(f, f') \mapsto f$  which is an immersion (i.e.  $\text{JA}_{n,d,d'}$  is identified with a locally closed

subscheme of  $\mathrm{JE}_{n,d}$ ). The Jacobian conjecture  $\mathrm{JC}_n$  means that for any  $d \geq 1$

$$\mathrm{JE}_{n,d} \times \mathrm{Spec} \mathbb{Q} = \bigcup_{d' \geq 1} \mathrm{pr}_{n,d,d'}^{(J)}(\mathrm{JA}_{n,d,d'} \times \mathrm{Spec} \mathbb{Q}) .$$

For given  $n, d$  the set  $X_{d'} := (\mathrm{JA}_{n,d,d'} \times \mathrm{Spec} \mathbb{Q}) \subset \mathrm{JE}_{n,d} \times \mathrm{Spec} \mathbb{Q}$  is a constructible set. Therefore, we get an infinite growing chain of constructible subsets  $X_1 \subset X_2 \subset \dots$  of the scheme of finite type  $\mathrm{JE}_{n,d} \times \mathrm{Spec} \mathbb{Q}$  over  $\mathbb{Q}$ .

Let us assume  $\mathrm{JC}_n$  and fix an integer  $d \geq 1$ . Then  $\cup_{d' \geq 1} X_{d'} = X$  where  $X := \mathrm{JE}_{n,d} \times \mathrm{Spec} \mathbb{Q}$ . Then it follows from the standard properties of constructible sets (see [6], Corrolaire 1.9.8, Chapitre IV) that there exists an integer  $d'$  such that  $X_{d'} = X$ . Alternatively, one can use a result of O. Gabber (see [3], Theorem 1.2) which says that for an automorphism  $f$  of  $\mathbf{k}[x_1, \dots, x_n]$  of degree  $\leq d$  in the above sense ( $\mathbf{k}$  is a field of any characteristic), the inverse map has the degree  $\leq d^{n-1}$ . Hence one can a priori set  $d' = d^{n-1}$ . Anyhow, the Jacobian conjecture can be rephrased as the equality  $\mathrm{JA}_{n,d,d'}(\mathbb{C}) = \mathrm{JE}_{n,d}(\mathbb{C})$ .

The following statement is obvious.

**Lemma 1** *Let  $\phi : A \rightarrow B$  be an immersion of schemes of finite type over  $\mathbb{Z}$ . Then  $\phi$  induces a bijection between  $A(\mathbb{C})$  and  $B(\mathbb{C})$  if and only if there exists a finite set of primes  $S$  such that  $\phi$  induces a bijection between  $A(\mathbf{k})$  and  $B(\mathbf{k})$  for any field  $\mathbf{k}$  with  $\mathrm{char} \mathbf{k} \notin S \cup \{0\}$ .*

We apply it to the projection  $\mathrm{pr}_{n,d,d'}^{(J)}$ . The conclusion is that the Jacobian conjecture  $\mathrm{JC}_n$  is equivalent to the following

**Conjecture 1** *( $\mathrm{JC}_n$  in finite characteristic) For any  $d \geq 1$  there exists  $d' \geq 1$  and a finite set of primes  $S$  such that for any field  $\mathbf{k}$  with  $\mathrm{char} \mathbf{k} \notin S \cup \{0\}$  and any polynomial map  $\phi : \mathbb{A}_{\mathbf{k}}^n \rightarrow \mathbb{A}_{\mathbf{k}}^n$  of degree  $\leq d$  with Jacobian 1, the inverse map exists and has degree  $\leq d'$ .*

The equivalence of  $\mathrm{JC}_n$  and the above conjecture in finite characteristic was first established by K. Adjamagbo in [1].

## 3 More about Weyl algebras

### 3.1 Weyl algebras over an arbitrary base

One can define an algebra  $A_{n,R}$  for arbitrary commutative ring  $R$  exactly in the same way as for fields of characteristic zero. This algebra is free as a  $R$ -

module. It has a canonical basis consisting of monomials  $(y_1^{\alpha_1} \cdots y_{2n}^{\alpha_{2n}})_{(\alpha_1, \dots, \alpha_{2n}) \in \mathbb{Z}_{\geq 0}^{2n}}$ . Although for any  $R$  the algebra  $A_{n,R}$  maps to the algebra  $D(\mathbb{A}_R^n)$  of differential operators acting on  $R[x_1, \dots, x_n]$ , these two algebras are not isomorphic in general. For example, if  $R$  is an algebra over  $\mathbb{Z}/p\mathbb{Z}$  for some prime  $p$ , then the operator  $(d/dx_1)^p$  is zero.

We say that an endomorphism  $f$  of the algebra  $A_{n,R}$  has degree  $\leq d$  if the image  $f(y_i)$  of any generator  $y_i \in A_{n,R}$ ,  $i = 1, \dots, 2n$  is a linear combination with coefficients in  $R$  of the monomials of degree  $\leq d$ . In a manner completely parallel to the previous section, we can define schemes of finite type  $\text{DE}_{n,d}$ ,  $\text{DA}_{n,d,d'}$ , and the projection  $\text{pr}_{n,d,d'}^{(D)}$ . Also, we can make a reformulation of the Dixmier conjecture in the same way as for the Jacobian conjecture.

### 3.2 The Weyl algebra in finite characteristic as an Azumaya algebra

It is a classical fact that in finite characteristic the algebra  $A_{n,R}$  has a big center, and it is moreover an Azumaya algebra of its center (see [8]).

We will use the following slightly non-standard definition of an Azumaya algebra (see e.g. [7], Proposition 2.1, Chapter IV):

**Definition 1** *For a commutative ring  $R$  an Azumaya algebra over  $R$  of rank  $N \geq 1$  is an associative unital algebra  $A$  over  $R$  which is a finitely generated  $R$ -module and such that there exists a finitely generated faithfully flat extension  $R' \subset R$  of  $R$  such that the pullback algebra  $A' := A \otimes_R R'$  is isomorphic to the matrix algebra*

$$\text{Mat}(N \otimes N, R') = \text{Mat}(N \times N, \mathbb{Z}) \otimes R'$$

as an algebra over  $R'$ .

It follows by descent that the center of an Azumaya algebra over  $R$  coincides with  $R$ . Also, an Azumaya algebra  $A$  considered as a  $R$ -module is a finitely generated projective module, in other words, a vector bundle over  $\text{Spec } R$ . This bundle has rank  $N^2$ , its fibers are associative algebras, and the fiber over any point of  $\text{Spec } R$  over an algebraically closed field  $\mathbf{k}$  is isomorphic to the matrix algebra  $\text{Mat}(N \times N, \mathbf{k})$ .

**Proposition 1** *For any commutative algebra  $R$  over  $\mathbb{Z}/p\mathbb{Z}$  where  $p$  is a prime, the algebra  $A_{n,R}$  is an Azumaya algebra of rank  $p^n$  over  $R[x_1, \dots, x_{2n}]$ . The central element of  $A_{n,R}$  corresponding to variable  $x_i$  is  $y_i^p$ .*

**Proof:** Let us introduce a faithfully flat extension  $R' := R[\xi_1, \dots, \xi_{2n}]$  of  $R[x_1, \dots, x_{2n}]$ , where the inclusion of  $R[x_1, \dots, x_{2n}]$  into  $R'$  is given by

$$x_i \mapsto \xi_i^p, \quad i \in \{1, \dots, 2n\}.$$

We claim that the algebra over  $R'$

$$A' := A_{n,R} \otimes_{R[x_1, \dots, x_{2n}]} R[\xi_1, \dots, \xi_{2n}]$$

is isomorphic to the matrix algebra of rank  $p^n$  over  $R'$ . Namely, the algebra  $A'$  considered as an algebra over  $R' = R[\xi_1, \dots, \xi_{2n}]$ , has generators  $y_i$ ,  $i \in \{1, \dots, 2n\}$  and defining relations

$$[y_i, y_j] = \omega_{ij}, \quad y_i^p = \xi_i^p.$$

Introduce a new set of generators  $y'_i \in A'$ ,  $i \in \{1, \dots, 2n\}$  by the formula

$$y'_i := y_i - \xi_i.$$

These generators have defining relations

$$[y'_i, y'_j] = \omega_{ij}, \quad (y'_i)^p = 0.$$

Hence, we see that the algebra  $A'$  over  $R'$  is isomorphic to the tensor product over  $\mathbb{Z}/p\mathbb{Z}$  of  $R'$  and a finite-dimensional algebra over  $\mathbb{Z}/p\mathbb{Z}$  given by the generators  $(y'_i)_{1 \leq i \leq 2n}$  and the relations as above. The last algebra is the tensor product of  $n$  copies of its version in the case  $n = 1$ . The statement of the proposition now follows from the following

**Lemma 2** *For any prime number  $p$  the algebra  $A$  over  $\mathbb{Z}/p\mathbb{Z}$  with two generators  $y_1, y_2$  and relations*

$$[y_1, y_2] = 1, \quad y_1^p = y_2^p = 0$$

*is isomorphic to  $\text{Mat}(p \times p, \mathbb{Z}/p\mathbb{Z})$ .*

**Proof:** Consider the finite ring  $B := \mathbb{Z}/p\mathbb{Z}[x]/(x^p) = \mathbb{Z}[x]/(x^p, p)$ . It is isomorphic to  $(\mathbb{Z}/p\mathbb{Z})^p$  as an abelian group (and as  $\mathbb{Z}/p\mathbb{Z}$ -module). Differential operators  $Y_1, Y_2$  acting on  $B$  and given by the formulas

$$Y_1(f) = df/dx, \quad Y_2 f = xf, \quad f \in B$$

are well-defined, and satisfy the relations  $[Y_1, Y_2] = 1$ ,  $Y_1^p = Y_2^p = 0$ . Hence we obtain a homomorphism  $A \rightarrow \text{End}_{\mathbb{Z}/p\mathbb{Z}\text{-mod}}(B)$ . A direct calculation shows that this is an isomorphism.  $\square$

## 4 The proof of the implication $\text{JC}_{2n} \rightarrow \text{DC}_n$

Let us assume that the conjecture  $\text{JC}_{2n}$  (phrased in the form of Conjecture 1) is true, our goal is to prove  $\text{DC}_n$ .

Let  $f : A_{n,\mathbb{C}} \rightarrow A_{n,\mathbb{C}}$  be an endomorphism of degree  $\leq d$ . We have to prove that  $f$  is invertible.

Denote by  $R$  the subring of  $\mathbb{C}$  generated by the coefficients of elements  $f(y_i) \in A_{n,\mathbb{C}}$ ,  $i \in \{1, \dots, 2n\}$  in the standard basis of  $A_{n,\mathbb{C}}$ . The ring  $R$  is a finitely generated integral domain. Moreover, we may assume that for any prime  $p$  the ring  $R/pR$  is either zero or an integral domain, in particular it has no non-zero nilpotents. In order to achieve this property it is enough to extend  $R$  by adding inverses to finitely many primes.

For any prime  $p$  the endomorphism  $f$  induces an endomorphism

$$f_p : A_{n,R/pR} \rightarrow A_{n,R/pR}$$

of an Azumaya algebra of rank  $p^n$  over

$$C_p := R/pR[x_1, \dots, x_{2n}] = \text{Center}(A_{n,R/pR}) .$$

The following result was proved first by Y. Tsuchimoto [10], it follows also from a more general recent result from [2].

**Proposition 2** *The endomorphism  $f_p$  maps  $C_p$  to itself.*

**Proof:** Denote by  $\mathbf{k}$  an algebraically closed field of characteristic  $p$ . For any  $\mathbf{k}$ -point  $v$  of  $\text{Spec } C_p$  the fiber  $A_v$  is an algebra over  $\mathbf{k}$  isomorphic to  $\text{Mat}(p^n \times p^n, \mathbf{k})$ .

**Lemma 3** *An element  $a \in A_{n,R/pR}$  belongs to  $C_p$  if and only if for any  $\mathbf{k}$ -point  $v$  of  $\text{Spec } C_p$  where  $\mathbf{k}$  is an algebraically closed field, the image of  $a$  in  $A_v$  is central, i.e. it is a scalar matrix.*

**Proof:** One direction is obvious, i.e. if  $a$  is central then its image in  $A_v$  is central. Conversely, if  $a \in A_{n,R/pR}$  is not central then there exists  $b \in A_{n,R/pR}$  such that  $[a, b] \neq 0$ . For any non-zero section  $s$  of the vector bundle  $A_{n,R/pR}/C_p$  there exists a  $\mathbf{k}$ -point at which this section does not vanish, because the algebra  $C_p = R/pR[x_1, \dots, x_{2n}]$  has no non-zero nilpotents by our assumption that  $R/pR$  has no non-zero nilpotents. We apply this

argument to the section  $s = [a, b]$  and conclude that the image of  $a$  in  $A_v$  is not central for some  $v$ .  $\square$

Let  $a \in C_p \subset A_{n,R/pR}$  be a central element. We want to prove that  $f(a)$  is central. Assume the opposite. Then by the above lemma there exists a homomorphism  $\rho : A_{n,R/pR} \rightarrow \text{Mat}(p^n \times p^n, \mathbf{k})$  such that  $\rho(f(a))$  is not a scalar matrix. Let us denote by  $V_0 \simeq \mathbf{k}^{p^n}$  the module over  $A_{n,R/pR}$  associated to the homomorphism  $f \circ \rho$ . Our assumption mean that  $V_0$  considered as a module over  $\mathbf{k} \otimes C_p$  is *not* isomorphic to the sum of  $p^n$  copies of the simple module  $M_v \simeq \mathbf{k}$  associated with any  $\mathbf{k}$ -point  $v$  of  $\text{Spec } C_p$ . The support of the module  $V_0$  is a non-empty finite subscheme of  $\text{Spec } C_p$  defined over  $\mathbf{k}$ , hence there exists a  $\mathbf{k}$ -point  $v$  in it support. Moreover, for any such point  $v$  the tensor product  $V := V_0 \otimes_{C_p \otimes \mathbf{k}} M_v$  is a vector space over  $\mathbf{k}$  such that  $0 < \dim V < \dim V_0$ . The algebra  $A_{n,R/pR}$  maps to the algebra of endomorphisms of  $C_p$ -module  $V_0$ , hence it maps to the algebra of  $\mathbf{k}$ -linear endomorphisms of  $V$ . In this representation of  $A_{n,R/pR}$  the center  $C_p$  acts by scalars, by the nature of the construction.

Therefore, we obtain a homomorphism  $C_p \rightarrow \mathbf{k}$ , i.e. a  $\mathbf{k}$ -point  $v$  of  $\text{Spec } C_p$ , and a homomorphism of  $\mathbf{k}$ -algebras

$$A_v \simeq \text{Mat}(p^n \times p^n, \mathbf{k}) \rightarrow \text{Mat}(M \times M, \mathbf{k}), \quad 0 < M < p^n,$$

here  $M := \dim V$ . This is impossible because  $\text{Mat}(p^n \times p^n, \mathbf{k})$  is simple and  $0 < \dim_{\mathbf{k}} \text{Mat}(M \times M, \mathbf{k}) < \dim_{\mathbf{k}} \text{Mat}(p^n \times p^n, \mathbf{k})$ . We obtain a contradiction. The Proposition is proven.  $\square$

Denote by  $f_p^{centr}$  the endomorphism of  $C_p$  induced by  $f$ . Our next goal is to prove that  $f_p^{centr}$  **preserves certain  $R/pR$ -linear Poisson bracket** on  $C_p$ .

Namely, we define an operation  $\{, \} : C_p \otimes_{R/pR} C_p \rightarrow C_p$  by the formula

$$\{a, b\} = \frac{[\tilde{a}, \tilde{b}]}{p} \pmod{pA_{n,R}} \in A_{n,R/pR} = A_{n,R}/pA_{n,R}$$

where  $\tilde{a}, \tilde{b} \in A_{n,R}$  are arbitrary lifts of the elements  $a, b \in C_p \subset A_{n,R/pR}$ . First of all, it is easy to see that the commutator  $[\tilde{a}, \tilde{b}]$  vanishes modulo  $p$ , hence the division by  $p$  makes sense. It is uniquely defined because  $R$  and hence  $A_{n,R}$  both have no torsion. A straightforward check shows that  $\{a, b\}$  defined as above does not depend on the choice of the lifts  $\tilde{a}, \tilde{b}$ , and it belongs to the center  $C_p$ . Moreover, the commutator  $\{, \}$  on  $C_p$  is a  $R/pR$ -linear, skew-symmetric operation satisfying the Jacobi identity (hence  $C_p$  becomes a Lie



algebra), and for any  $a \in C_p$  the operator  $\{a, \cdot\} : C_p \rightarrow C_p$  is a  $R/pR$ -linear derivation of  $C_p$ , i.e. the bracket satisfies the Leibniz rule

$$\{a, bb'\} = \{a, b\}b' + \{a, b'\}b .$$

**Lemma 4** *The above defined canonical Poisson bracket on  $C_p \simeq R/p[x_1, \dots, x_{2n}]$  is given by the standard formula*

$$\{a, b\} = \sum_{i=1}^n \left( \frac{\partial a}{\partial x_i} \frac{\partial b}{\partial x_{n+i}} - \frac{\partial b}{\partial x_i} \frac{\partial a}{\partial x_{n+i}} \right) .$$

**Proof:** By the Leibniz rule it follows that it suffices to calculate the bracket  $\{x_i, x_j\}$  for any two generators of  $C_p$ . The calculation reduces to the case  $n = 1$ . It is convenient to calculate first the commutator in the algebra  $A_{1, \mathbb{Z}}$  and then make the reduction modulo  $p$ :

$$\frac{1}{p} [(d/dx)^p, x^p] = \frac{1}{p} \sum_{i=0}^{p-1} \frac{(p!)^2}{(i!)^2 (p-i)!} x^i (d/dx)^i = -1 \pmod{p}$$

Then the statement of the lemma follows immediately.  $\square$

The next lemma follows directly from the definition of the bracket:

**Lemma 5** *The homomorphism  $f_p^{centr} : C_p \rightarrow C_p$  preserves the canonical Poisson bracket.*

It is well-known in symplectic geometry that a non-degenerate Poisson structure on a  $C^\infty$  manifold  $X$  is essentially the same as a symplectic structure, i.e. a non-degenerate closed 2-form. The same is true in the algebraic context, in characteristic  $> 2$ . Namely, a Poisson bracket gives a section

$$\alpha \in \Gamma(\text{Spec } C_p, \wedge^2 T_{\text{Spec } C_p / \text{Spec } R/pR})$$

of the wedge square of the tangent bundle, defined by the formula

$$\{f, g\} = \langle df \wedge dg, \alpha \rangle \in C_p, \quad \forall f, g \in C_p .$$

This section can be interpreted as an operator from the cotangent bundle to the tangent bundle. This operator is invertible in our case, the inverse operator can be interpreted as a 2-form

$$\omega := \alpha^{-1} = \frac{1}{2} \sum_{1 \leq i, j \leq 2n} \omega_{ij} dx_i \wedge dx_j = \sum_{i=1}^n dx_i \wedge dx_{n+i} .$$

**Lemma 6** For  $p > n$  the endomorphism  $f_p^{centr}$  of  $C_p = R/pR[x_1, \dots, x_{2n}]$  preserves the top-degree form  $dx_1 \wedge \dots \wedge dx_{2n} \in \Omega^{2n}(B/(R/pR))$ .

**Proof:** It follows from the previous lemma that  $f_{centr}$  preserves the symplectic 2-form  $\omega$ . The volume form from above is equal to  $\pm\omega^n/n!$  for  $p > n$ .  $\square$

The next result implies that the degree of  $f_p^{centr}$  is  $\leq d$ .

**Proposition 3** For any field  $\mathbf{k}$  of characteristic  $p$  and any  $\mathbf{k}$ -point  $v$  of  $\text{Spec } R$ , the degree of  $f_v$  (as an endomorphism of the Weyl algebra  $A_{n,\mathbf{k}}$ ) is equal to the degree of  $f_v^{centr}$  (as an endomorphism of the polynomial algebra) where  $f_v^{centr}$  is the endomorphism of  $\text{Center}(A_{n,\mathbf{k}}) \simeq \mathbf{k}[x_1, \dots, x_{2n}]$  induced from  $f_p^{centr}$ .

**Proof:** The degree of  $f_v^{centr}$  is defined as the maximum over  $i \in \{1, \dots, 2n\}$  of the degrees of polynomials  $f_v^{centr}(x_i)$ . The degree of endomorphism  $f_v$  is defined as the maximum over  $i \in \{1, \dots, 2n\}$  of the degrees (in the sense of Bernstein filtration, by the degree of monomials in the standard basis of  $A_{n,\mathbf{k}}$ ) of elements  $f_v(y_i)$ . We claim that for each index  $i$  both degrees coincide with each other. The reason is the following. Let  $d_i$  be the degree of  $f_v(y_i)$ . We claim that the degree of  $f_v(y_i^p) = (f_v(y_i))^p$  considered as an element of  $A_{n,\mathbf{k}}$ , is equal to  $pd_i$ . It follows from the following

**Lemma 7** The degree is an additive character of the multiplicative monoid of non-zero elements in  $A_{n,\mathbf{k}}$ .

**Proof:** It follows immediately from the consideration of Bernstein filtration on  $A_{n,\mathbf{k}}$  and the remark that the product of non-zero homogeneous polynomials is a non-zero polynomial.  $\square$

The degree of  $f_v(y_i^p)$  considered as an element of  $\text{Center}(A_{n,\mathbf{k}})$  is  $1/p$  times its degree in  $A_{n,\mathbf{k}}$ , i.e. it is equal to  $d_i$ . Proposition 3 is proven.  $\square$

Now we can use finally our main assumption that the Jacobian conjecture  $\text{JC}_{2n}$  holds. Namely, by its reformulation (in form of Conjecture 1), we conclude that there exists an integer  $d' \geq 1$  and a finite set of primes  $S$  (the union of the set of excluded primes for  $\text{JC}_{2n}$  in form of Conjecture 1, and the set of primes  $\leq n$ ), such that for any algebraically closed field  $\mathbf{k}$  such that  $p = \text{char}(\mathbf{k}) \notin S \cup \{0\}$  and any  $\mathbf{k}$ -point  $v$  of  $\text{Spec } R$ , the pullback  $f_v^{centr}$  to  $v$  of  $f_p^{centr}$  is invertible and the inverse endomorphism of  $\mathbf{k}[x_1, \dots, x_{2n}]$  has the degree  $\leq d'$ .

The following result is a particular case of a more general statement proven in [2], and also follows from [10].

**Proposition 4** *For any  $v$  as above the endomorphism  $f_v$  of  $A_{n,\mathbf{k}}$  is invertible.*

**Proof:** We may assume that  $\mathbf{k}$  is algebraically closed. The endomorphism  $f_v$  of Azumaya algebra  $A_{n,\mathbf{k}}$  preserves the center and is invertible on the center. Thus, it gives a  $C_v$ -linear homomorphism  $g_v$  from one Azumaya algebra of rank  $p^n$  over  $C_v := \mathbf{k}[x_1, \dots, x_{2n}]$  (here we mean the algebra  $A_{n,\mathbf{k}}$ ), to another Azumaya algebra of rank  $p^n$  (the pullback of  $A_{n,\mathbf{k}}$  by  $f_v^{centr}$ ). Any such a homomorphism restricts to an isomorphism after the reduction to any  $\mathbf{k}$ -point of  $C_v$ , because any homomorphism of associative  $\mathbf{k}$ -algebras

$$\text{Mat}(N \times N, \mathbf{k}) \rightarrow \text{Mat}(N \times N, \mathbf{k}), \quad N := p^n$$

is an isomorphism. Therefore,  $g_v$  is an isomorphism of vector bundles.  $\square$

Finally, the degree of the inverse to  $f_v$  is  $\leq d'$ , as follows directly from Proposition 3. The conclusion is that for any point  $v$  of  $\text{Spec } R$  over an field  $\mathbf{k}$  of finite characteristic  $p \notin S$ , the corresponding point of the scheme of finite type  $\text{DE}_{n,d}$  (see Section 3.1 for the notation) belongs to the constructible set  $\text{DA}_{n,d,d'}$ . This implies (see Lemma 1) that  $f$  is invertible after the localization to zero characteristic, and the inverse endomorphism has degree  $\leq d'$ . Theorem 1 is proven.  $\square$

**Remark 4** *It is interesting that Poisson brackets appear in another situation related to polynomial automorphisms. The Poisson algebra structure was used by I. Shestakov and U. Umirbaev in their proof that the Nagata automorphism is wild (see [9]).*

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