Note added in proof.

It should be noted that the existence of the C^α hyperbolic splitting $T_K M = V^u \oplus V^s$ for f does not imply that the zero section is a hyperbolic fixed point for $u \to Tf(u \circ f^{-1})$ acting on C^α sections of $T_K M$ (I owe this remark to Liu Pei-Dong). But it is not hard to prove that the zero section is a hyperbolic fixed point for $u \to Tf(u \circ f^{-1})$ acting on $C^{\alpha'}$ sections of $T_K M$ for some $\alpha' \in (0, \alpha)$. One should thus replace α by α' in Proposition 1.2, and similarly later, but this is of no consequence for our results.

Earlier references for the smooth dependence on parameters of the conjugacy in structural stability are the following:

R. de la Llave, J.M. Marco and R. Moriyon. "Canonical perturbation theory of Anosov systems and regularity results for the Livsic cohomology equation." Ann. of Math. **123**,537-611(1986).

A. Katok, G. Knieper, M. Pollicott, and H. Weiss. "Differentiability and analyticity of topological entropy for Anosov and geodesic flows." Invent. Math. **98**,581-597(1989).

G. Contreras. "Regularity of topological and metric entropy of hyperbolic flows." Math. Z. **210**,97-111(1992).

I am indebted to Rafael de la Llave and Viviane Baladi for pointing out these references to me.

Hyperbolicity of $u \to Tf(u \circ f^{-1})$.

Let K be a compact hyperbolic set $\subset M$ for the $C^{1+\delta}$ diffeomorphism f of M. The splitting $T_K M = E^s + E^u$ is C^α for some $\alpha > 0$. We claim that the linear map $u \to Tf(u \circ f^{-1})$ is hyperbolic on the Banach space of C^β sections of $T_K M$, *provided β is sufficiently small* $(0 < \beta < \alpha)$.

We have to show that $u \to Tf(u \circ f^{-1})$ is a contraction of the space of C^β sections of E^s. We may take an adapted metric, so that $\|Tf|E^s\|_0 \leq \lambda < 1$. In the definition of the C^β norm:

$$\|\Phi\| = \max(\sup_x |\phi(x)\|, \sup_{x \neq y} \frac{|\Phi(x) - \Phi(y)|}{|x-y|^\beta})$$

we take the second sup only over pairs (x, y) such that $d(x, y) < \epsilon$, where the constant ϵ will be fixed later (small but > 0).

Write $T_x = T_x f$, $\xi = d(x, y)$. Given a section u of E^s (with C^β norm $\|u\|$) we may for each pair (x, y) with small ξ choose $v \in E^s_x$ with $|v - u_y| \leq \|u\| O(\xi^\alpha)$. We have

$$T_x u_x - T_y u_y = T_x(u_x - v) + T_x v - T_y v + T_y(v - u_y)$$

$$|T_x(u_x - v)| \leq \lambda |u_x - v| \leq \lambda |u_x - u_y| + \|u\| O(\xi^\alpha)$$

$$|T_x v - T_y v| \leq \|u\| O(\xi)$$

$$|T_y(v - u_y)| \leq \|u\| O(\xi^\alpha)$$

hence

$$|T_x u_x - T_y u_y| \leq \|u\|(\lambda \xi^\beta + O(\xi^\alpha))$$

Since $d(fx, fy) \geq C\xi$ we have

$$\frac{|T_x u_x - T_y u_y|}{d(fx, fy)^\beta} \leq \|u\| \frac{\lambda \xi^\beta + O(\xi^\alpha)}{C^\beta \xi^\beta} = \|u\|(\frac{\lambda}{C^\beta} + O(\xi^{\alpha - \beta}))$$

For small β we have $\lambda / C^\beta < 1$, and we may take ϵ such that

$$\lambda / C^\beta + O(\xi^{\alpha - \beta}) < 1 \qquad \text{if} \qquad 0 < \xi < \epsilon$$

Dedicated to Klaus Hepp
and Walter Hunziker
for their 60-th birthday.

DIFFERENTIATION OF SRB STATES.

by David Ruelle*

Abstract. Let f be a diffeomorphism of a manifold M, and ρ_f a (generalized) SRB state for f. If suppρ_f is a hyperbolic compact set we show that the map $f \mapsto \rho_f$ is differentiable in a suitable functional setup, and we compute the derivative. When suppρ_f is an attractor, the derivative is given by

$$\delta\rho_f(\Phi) = \sum_{n=0}^{\infty} \rho_f \langle \text{grad}(\Phi \circ f^n), X \rangle$$

where X is the vector field $\delta f \circ f^{-1}$. This formula can be extended to time dependent situations and also, at least formally, to nonuniformly hyperbolic situations.

The above results will find their use in the study of the Onsager reciprocity relations and the fluctuation-dissipation formula of nonequilibrium statistical mechanics.

* IHES (91440 Bures sur Yvette, France) <ruelle@ihes.fr>, and Math. Dept., Rutgers University (New Brunswick, NJ 08903, USA).

0. Introduction.

In a recent paper [7], G.Gallavotti has outlined a new proof of Onsager's reciprocity relations, based on the study of the SRB measure ρ_f for a hyperbolic dynamical system (M,f). To give a rigorous and general version of Gallavotti's argument, one has to study the dependence $f \mapsto \rho_f$, and in particular compute the derivative. In fact, one may argue that these problems are at the core of nonequilibrium statistical mechanics; they are the subject of the present paper. We do not make here the assumption of [7] that we are close to a Hamiltonian situation (where f has a smooth invariant measure); our analysis will thus be valid "far from equilibrium". In what follows we concentrate on the mathematics, and leave the application to nonequilibrium statistical mechanics for other occasions.

Let K be a mixing Axiom A attractor for the diffeomorphism f. In a suitable functional setup we shall show that the SRB state ρ_f on K depends differentiably on f. A variation δf of f corresponds to a vector field $X = \delta f \circ f^{-1}$, and we shall obtain the formula

$$\delta\rho_f(\Phi) = \sum_{k=0}^{\infty} \rho_f(\langle \mathrm{grad}(\Phi \circ f^k), X\rangle)$$

This formula is relatively easy to guess, but its proof requires some care. Instead of the Axiom A attractor case we shall in fact deal with the more general situation where K is a hyperbolic set with local product structure, and ρ_f the corresponding generalized SRB state (Sections 1, 2 and 3). In Section 4 we shall see how the definition of attractor and of SRB state can be extended to a general bounded time dependent perturbations of f. Finally, in Section 5 we shall discuss a formula for the formal derivative of the SRB state ρ_f with respect to f, without uniform hyperbolicity assumption.

The rest of this introduction is a brief summary of facts concerning hyperbolic sets. For more details see Smale [20], Shub [16], Ruelle [14], and references quoted there.

Hyperbolicity.

Let K be a compact invariant set for the diffeomorphism f of a finite-dimensional manifold M, we assume f to be of class C^r, with $r \geq 1$. We choose some Riemann metric on M. Suppose that $T_K M$ (the tangent bundle restricted to K) has a continuous Tf-invariant splitting $T_K M = V^- \oplus V^+$ and that there are constants $C \geq 1$, $\theta > 1$ such that

$$\max_{x\in K} \|(T_x f^{\mp n} | V^{\pm}(x))\| \leq C\theta^{-n} \qquad \text{for} \quad n \geq 0$$

Then K is called a hyperbolic (compact invariant) set for f. We call $V^- = V^s$ and $V^+ = V^u$ the stable and unstable subbundles respectively.

Local stable manifolds $\mathcal{V}^-(x) = \mathcal{V}^s(x)$ and unstable manifolds $\mathcal{V}^+(x) = \mathcal{V}^u(x)$ are defined by

$$\mathcal{V}^{\pm}(x) = \{y \in M : d(f^{\mp n}y, f^{\mp n}x) < R \qquad \text{for} \quad n \geq 0\}$$

The $\mathcal{V}^{\pm}(x)$ are C^r manifolds, respectively tangent to $V^{\pm}(x)$, and $x \mapsto \mathcal{V}^{\pm}(x)$ is continuous $K \to C^r$. Furthermore, there are $C' \geq 1$, $\theta' > 1$ such that if $y, z \in \mathcal{V}^{\pm}(x)$,

$$d(f^{\mp n}y, f^{\mp n}z) \leq C'\theta'^{-n}d(y,z) \qquad \text{for} \quad n \geq 0$$

Expansiveness, Hölder continuity of hyperbolic splitting, Axiom A attractors.

The map f restricted to the hyperbolic invariant set K is an *expansive homeomorphism.* This means that $d(f^k x, f^k y) < \epsilon$ for all $k \in \mathbf{Z}$, implies $x = y$.

If $r > 1$, the stable and unstable subbundles $V^{\pm}$ are Hölder continuous, *i.e.*, the sections $x \mapsto V^{\pm}(x)$ of the Grassmannian over K are C^α for some $\alpha > 0$.

We say that the compact hyperbolic f-invariant set K is *transitive* if K contains a dense orbit $(f^k a)_{k \in \mathbf{Z}}$. We say that K is an *Axiom A attractor* if K is transitive and has an open neighborhood U such that

$$\cap_{n \geq 0} f^n U = K$$

It follows that the local unstable manifolds $\mathcal{V}^u(x)$ of points of K lie in K (this is also true for the global unstable manifolds $\cup_{n=1}^{\infty} f^n \mathcal{V}^u(x)$). One can then show that the f-periodic points are dense in K. The local stable manifolds $\mathcal{V}^s(x)$ of points of K fill a neighborhood (say U) of K. Consider a continuous map $\phi : S_1 \to S_2$ along the $\mathcal{V}^s_x$ between two smooth transverse sections S_1 and S_2 (for instance two pieces of unstable manifolds). One can show that ϕ is Hölder continuous, and absolutely continuous (for the Riemann volume elements of S_1, S_2) with Hölder continuous Jacobian.

Local product structure, shadowing.

We say that the compact hyperbolic f-invariant set K has local product structure if R can be chosen in the definition of $\mathcal{V}^{\pm}(x)$ such that, for all $x, y \in K$

$$\mathcal{V}^-(x) \cap \mathcal{V}^+(y) \subset K$$

In particular, an Axiom A attractor has local product structure. For small R, we may assume that the $\mathcal{V}^{\pm}(x)$ are nearly flat, so that $\mathcal{V}^-(x) \cap \mathcal{V}^+(y)$ consists of at most one point. One can check that the map $(x, y) \mapsto [x, y]$, where $[x, y]$ is the only point in $\mathcal{V}^-(x) \cap \mathcal{V}^+(y)$, defines a product structure in a neighborhood of each point of K.

A remarkable feature of hyperbolic sets with local product structure is that δ-pseudo-orbits are well approximated by true orbits. We say that $(x_k)_{k \in [k_0, k_1]}$ is a *δ-pseudoorbit* for f if $d(fx_k, x_{k+1}) < \delta$ for every finite $k \in [k_0, k_1 - 1]$, where k_0, k_1 may be finite or $\pm\infty$. The pseudoorbit (x_k) is ϵ-shadowed by the orbit $(f^k x)$ if $d(f^k x, x_k) < \epsilon$ for all $k \in [k_0, k_1]$. Bowen has proved the following *shadowing lemma:*

Let K be a hyperbolic set with local product structure for f. For every $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudoorbit in K is ϵ-shadowed by an orbit in K.

This is a very efficient tool in the study of hyperbolic systems; it was for instance used by Bowen [3] to prove the existence of Markov partitions (first introduced by Sinai [17], [18]) in general and natural fashion. For a discussion of Markov partitions and symbolic dynamics we must however refer to the original papers.

1. Stuctural stability results.

The spaces $\mathcal{M}, \mathcal{B}, \mathcal{A}$.

From now on we take r integer> 1, and let K_0 be a hyperbolic set for f_0 of class C^r. Then, the stable and unstable subbundles $V_0^{\pm}$ are C^α for some $\alpha > 0$. The C^α maps $K_0 \to M$ form a Banach manifold $\mathcal{M}$. The maps close to the inclusion map $K \hookrightarrow M$ are described by a chart of $\mathcal{M}$ which we may take to be the open ϵ-ball B around 0 in a Banach space $\mathcal{B}$. Using the exponential map $TM \to M$, we may take for $\mathcal{B}$ the space of C^α sections of $T_{K_0}M$. Finally, we shall denote by $\mathcal{A}$ the space of C^r diffeomorphisms sufficiently close to f_0 in a fixed neighborhood U of K_0 in M.

1.1 Proposition.

Let $r \geq 2$.

(a) The map $\mathcal{A} \times \mathcal{M} \to \mathcal{M}$ defined by $(f, j) \mapsto f \circ j \circ f_0^{-1}$ is C^{r-1}.

(b) The tangent map $\mathcal{T}$ to $j \mapsto f \circ j \circ f_0^{-1}$ is given by

$$(\mathcal{T}_j \delta)(x) = (T_{j(f_0^{-1}x)} f)\delta(f_0^{-1}x)$$

where $\delta \in T_j\mathcal{M}$.

To prove (a), it will suffice to show that $(f, j) \mapsto f \circ j$ is C^{r-1}. Furthermore the problem is local, *i.e.*, it suffices to consider j and $f \circ j$ near $x_0 \in K_0$. The map $f \mapsto f \circ j$ is C^ω (in fact linear, using suitable local charts). Differentiating k times $f \circ j$ with respect to j introduces the k-th derivative of f, which is C^{r-k}, and composed with j this gives a C^α function if $r - k \geq 1$. Therefore $(f, j) \mapsto f \circ j$ is C^{r-1} as announced.

(b) follows directly from the definitions. $\square$

For the next proposition, remember that $\mathcal{A}$ is a *sufficiently small* neighborhood of f_0.

1.2 Proposition.

Let $r \geq 2$.

(a) The inclusion map $K_0 \hookrightarrow M$ is a hyperbolic fixed point of the map $\mathcal{M} \to \mathcal{M}$ defined by $j \mapsto f_0 \circ j \circ f_0^{-1}$.

(b) For $f \in \mathcal{A}$, the map $\mathcal{M} \to \mathcal{M}$ defined by $j \mapsto f \circ j \circ f_0^{-1}$ has a unique fixed point $j(f)$ close to $K_0 \hookrightarrow M$. This fixed point is hyperbolic and is a C^α homeomorphism $K_0 \to K = j(f)K_0$.

(c) The map $f \mapsto j(f)$ is $C^{r-1} : \mathcal{A} \to \mathcal{M}$, and the tangent map $\delta f \mapsto \delta j$ is given by

$$\delta j = (1 - \mathcal{T}_{j(f)})^{-1}(\delta f \circ f^{-1} \circ j(f))$$

Clearly $K_0 \hookrightarrow M$ is a fixed point of $j \mapsto f_0 \circ j \circ f_0^{-1}$. The corresponding tangent map is $\mathcal{T}_0 : \mathcal{B} \to \mathcal{B}$ given by

$$(\mathcal{T}_0 \delta)(x) = (T_{f_0^{-1}x} f_0)\delta(f_0^{-1}x)$$

(see Proposition 1.(b)). We have to show that this is a hyperbolic linear map, *viz.*, its spectrum is disjoint from the unit circle. Here we use the fact that the splitting of $T_{K_0}M$ into stable and unstable subbundles is C^α, giving a decomposition $\mathcal{B} = \mathcal{B}^s \oplus \mathcal{B}^u$ such that $\mathcal{T}_0|\mathcal{B}^s$ and $\mathcal{T}_0^{-1}|\mathcal{B}^u$ have spectral radius < 1. This proves (a).

Using Proposition 1(a), Proposition 2(a), and the implicit function theorem, we see that $j \mapsto f \circ j \circ f_0^{-1}$ has a unique fixed point $j(f)$ close to $K_0 \hookrightarrow M$. By continuity, this fixed point is hyperbolic (*i.e.*, $\mathcal{T}_{j(f)}$ is a hyperbolic linear map). By expansiveness of f_0 on K_0, $j(f)$ cannot collapse different orbits, and is thus injective. This proves (b).

[We have here followed Hirsch and Pugh [8] in establishing the persistence of the hyperbolic set K].

The implicit function theorem also yields that $f \mapsto j(f)$ is C^{r-1}, and by differentiating $j \circ f_0 = f \circ j$ we get

$$\delta j \circ f_0 = \delta f \circ j + Tf \circ \delta j$$

hence

$$(1 - \mathcal{T}_{j(f)})\delta j = \delta f \circ j \circ f_0^{-1} = \delta f \circ f^{-1} \circ j$$

hence

$$\delta j = (1 - \mathcal{T}_{j(f)})^{-1}(\delta f \circ f^{-1} \circ j(f))$$

proving (c). ∎

1.3 Proposition.

Let $r \geq 3$. We denote by $\pi : \widetilde{M} \to M$ the Grassmannian of TM, and let $\tilde{f} : \widetilde{M} \to \widetilde{M}$ be induced by Tf. Also let $\widetilde{\mathcal{M}}$ denote the Banach manifold of C^β maps: $K_0 \to \widetilde{M}$, for some suitably small $\beta > 0$ (we take $\beta \leq \alpha$).

(a) The map $\mathcal{A} \times \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ defined by $(f, \tilde{\jmath}) \mapsto \tilde{f} \circ \tilde{\jmath} \circ f_0^{-1}$ is C^{r-2}.

(b) The canonical lifting $K_0 \to V_0^u$ is a hyperbolic fixed point of the map $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ defined by $\tilde{\jmath} \mapsto \tilde{f}_0 \circ \tilde{\jmath} \circ f_0^{-1}$.

(c) For $f \in \mathcal{A}$, the map $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ defined by $\tilde{\jmath} \mapsto \tilde{f} \circ \tilde{\jmath} \circ f_0^{-1}$ has a unique fixed point $\tilde{\jmath}(f)$ close to $K_0 \to V_0^u$. Furthermore $\pi \circ \tilde{\jmath}(f) = j(f)$, $\tilde{\jmath}(f)x = V^u(j(f)x)$, and $f \mapsto \tilde{\jmath}(f)$ is C^{r-2}: $\mathcal{A} \to \widetilde{\mathcal{M}}$.

(a) is proved like Proposition 1.1(a), taking into account the fact that $\tilde{f}$ is of class C^{r-1}.

From the hyperbolic splitting $T_{K_0}M = V_0^s \oplus V_0^u$ (for Tf), one also obtains a hyperbolic splitting $T_{V_0^u}\widetilde{M} = \widetilde{V}_0^s \oplus \widetilde{V}_0^u$ (for $T\tilde{f}$). In fact

$$\widetilde{V}_0^s = (T\pi | T_{V_0^u}\widetilde{M})^{-1} V_0^s$$

and

$$\widetilde{\mathcal{V}}_0^u = \{\xi : \pi\xi \in \mathcal{V}_0^u \text{ and } \xi \text{ is the tangent space to } \mathcal{V}_0^u \text{ at } \pi\xi\}$$

Note that $x \mapsto \widetilde{\mathcal{V}}_0^u(x)$ is continuous because $x \mapsto \mathcal{V}_0^u(x)$ is continuous $K \to C^r$. Therefore, the splitting $\widetilde{V}_0^s \oplus \widetilde{V}_0^u$ is again C^β for some $\beta > 0$, and (b) follows.

Using (a), (b), and the implicit function theorem, we see that $\tilde{\jmath} \mapsto \tilde{f} \circ \tilde{\jmath} \circ f_0^{-1}$ has a unique fixed point $\tilde{\jmath}(f)$ close to $K_0 \to V_0^u$. Since $\pi \circ \tilde{f} = f \circ \pi$, we have

$$\pi \circ \tilde{\jmath}(f) = \pi \circ \tilde{f} \circ \tilde{\jmath}(f) \circ f_0^{-1} = f \circ \pi \circ \tilde{\jmath}(f) \circ f_0^{-1}$$

which shows that $\pi \circ \tilde{\jmath}(f) = j(f)$. Since $\widetilde{K} = \tilde{\jmath}(f)K_0$ is $\tilde{f}$-invariant and close to V_0^u, we have $\widetilde{K} = V^u$, *i.e.*, $\tilde{\jmath}(f)x = V^u(j(f)x)$. Finally, the implicit function theorem also shows that $f \mapsto \tilde{\jmath}(f)$ is C^{r-2}: $\mathcal{A} \to \widetilde{\mathcal{M}}$, concluding the proof of (c). $\square$

2. Generalized SRB measures: smooth dependence on f.

We assume from now on that K_0 has local product structure, and that $f_0|K_0$ is mixing (for instance f_0 satisfies Smale's Axiom A, and K_0 is a mixing basic set). Then also $K = K_f = j(f)K_0$ has local product stucture for f, and $f|K$ is mixing.

If $f \in \mathcal{A}$, the (generalized) SRB measure* with respect to f on K is the unique equilibrium state for $-\log J_f^u$, *i.e.*, the unique f-invariant probability measure $\rho = \rho_f$ on K making

$$h_f(\rho) - \rho(\log J_f^u) \tag{1}$$

maximum. Here $h_f(\rho)$ is the *entropy* of ρ, and J_f^u is the *unstable Jacobian* [therefore, $\rho(\log J_f^u)$ is the sum of the positive Lyapunov exponents for ρ]. We do not make the usual assumption that K is an attractor**. The maximum of (1) is $P(\log J_f^u) \leq 0$ [the value 0 is obtained if and only if K is an attractor, see [5]].

Let $\bar{\jmath}(f) : K \to K_0$ be the inverse of $j(f)$ considered as a map $K_0 \to K$, and define $\mu_f = \bar{\jmath}(f)^*\rho_f$. Then, μ_f is the unique equilibrium state with respect to f_0 on K_0 for $-\log J_f^u \circ j(f)$. [This follows from $j(f) \circ f_0 = f \circ j(f)$].

2.1 Proposition.

Let $r \geq 3$. We assume that K has local product structure with respect to f, and that $f|K$ is mixing.

(a) The map $f \mapsto J_f^u \circ j(f)$ is C^{r-2}: $\mathcal{A} \to C^\beta(K_0)$.

(b) The map $f \mapsto \mu_f|C^\beta(K_0)$ is C^{r-2}: $\mathcal{A} \to C^\beta(K_0)^*$.

Let u be the dimension of the unstable subspaces. We note that $J_f^u \circ j(f)$ is the norm of $(Tf)^{\wedge u}$ evaluated at $\tilde{\jmath}(f)$, and that $f \mapsto Tf$ is C^ω: $\mathcal{A} \to C^{r-1}$. Since, by Proposition 1.3(c), $f \mapsto \tilde{\jmath}(f)$ is C^{r-2}: $\mathcal{A} \to \widetilde{\mathcal{M}}$, we see that $f \mapsto J_f^u \circ j(f)$ is C^{r-2}: $\mathcal{A} \to C^\beta(K_0)$, proving (a).

We shall now use the fact that, if I is the set of f_0-invariant probability measures on K_0, then the *pressure*

$$A \mapsto P(A) = \max_{\mu \in I}[h_{f_0}(\mu) + \mu(A)]$$

---

* SRB mesures were introduced by Sinai [19] for Anosov diffeomorphisms and extended to Axiom A attractors for diffeomorphisms (Ruelle [12]) and flows (Bowen and Ruelle [5]). For the general situation where uniform hyperbolicity is not required see Ledrappier and Young [10]. In this Section and the next we consider another generalization where we assume uniform hyperbolicity, but not attractivity. The uniqueness of ρ maximizing (1) is because $\log J_f^u$ is Hölder continuous, and $f|K$ mixing (see Bowen [4], or Ruelle [13]).

** When K is not an attractor, ρ_f serves to describe diffusion away from K under f. This is the content of Proposition 3.1 in Ruelle [15]. See also Bowen and Ruelle [5], Young [21], Lopes and Markarian [11] (for a special case: open billiard described by a Cantor set), Eckmann and Ruelle [6] Section IV E. The work by Kaplan, Yorke, Kantz, Grassberger, Gaspard, and Nicolis should also be mentioned here.

is a C^ω function on $C^\beta(K_0)$. Furthermore, the derivative of P at A (which is an element of the dual $C^\beta(K_0)^*$) is the restriction to $C^\beta(K_0)$ of the equilibrium state μ^A for A. [For these results, see [13]]. Therefore the map $A \mapsto \mu^A | C^\beta(K_0)$ is C^ω: $C^\beta(K_0) \to C^\beta(K_0)^*$. Applying this to $A = -\log J_f^u \circ j(f)$, and $\mu^A = \mu_f$, we see (using (a)) that $f \mapsto \mu_f | C^\beta(K_0)$ is C^{r-2}: $\mathcal{A} \to C^\beta(K_0)^*$, proving (b). $\square$

2.2 Proposition.

Let $r \geq 3$. The map $f \mapsto \rho_f | C^{r-1}(M)$ (where ρ_f is the SRB state for f) is C^{r-2}: $\mathcal{A} \to C^{r-1}(M)^*$.

We use the fact that $\rho_f = j(f)^* \mu_f$, so that

$$\rho_f | C^{r-1}(M) = \ell(f)^*(\mu_f | C^\beta(K_0))$$

where the bounded operator $\ell(f) : C^{r-1}(M) \to C^\beta(K_0)$ is defined by $\ell(f)\Phi = \Phi \circ j(f)$ and $\ell(f)^*$ is its adjoint. Differentiation of μ_f proceeds according to Proposition 2.1(b). The function $\ell : \mathcal{A} \to \mathcal{L}(C^{r-1}(M), C^\beta(K_0))$ is $r-2$ times continuously differentiable (as seen by direct computation because if $\Phi \in C^{r-1}$, its first $r-2$ derivatives are still C^1, which by composition with a C^β function gives a C^β function). The same holds therefore for

$$\ell^* : \mathcal{A} \to \mathcal{L}(C^\beta(K_0)^*, C^{r-1}(M)^*)$$

We may now differentiate $\ell(f)^*(\mu_f | C^\beta(K_0))$, and we find that the derivatives up to order $r-2$ are in $C^{r-1}(M)^*$. $\square$

2.3 Remark.

One can probably improve Proposition 2.2 to the statement that $f \mapsto \rho_f | C^{r-2+\epsilon}(M)$ is C^{r-2}: $\mathcal{A} \to C^{r-2+\epsilon}(M)^*$ when $\epsilon > 0$.

3. Generalized SRB measures: differentiation with respect to f.

For $r \geq 3$, we have just seen that $f \mapsto \rho_f = j(f)^* \mu_f$ is C^1: $\mathcal{A} \to C^2(M)^*$. We may thus differentiate this map, or equivalently compute the tangent map $\delta\rho_f(\Phi)$ to

$$f \mapsto \rho_f(\Phi) = \mu_f(\Phi \circ j(f))$$

for $\Phi \in C^2(M)$. The linear functional $\delta f \mapsto \delta\rho_f(\Phi)$ corresponds to a linear functional $X \mapsto \delta\rho_f(\Phi)$, where $X = \delta f \circ f^{-1}$ is a C^{r-1} vector field on M. We shall evaluate $X \mapsto \delta\rho_f(\Phi)$ in two steps.

First step: computing $(\delta\mu_f)(\Phi \circ j(f))$.

By assumption we have the hyperbolic splitting $T_K M = V^s \oplus V^u$ for Tf over K. Let $F = F(f)$ be a section (not necessarily continuous) of $(V^u)^{\wedge u}$, such that $\|F_x\| = 1$ for all $x \in K$. (We use the norm defined from the Riemann metric; since $(V^u)^{\wedge u}$ is 1-dimensional, F_x is unique up to a factor ± 1). We have

$$(T_x f)^{\wedge u} F_x = \lambda(x) F_{fx}$$

$$|\lambda(x)| = J_f^u(x) \tag{2}$$

Let now $V^{s\perp} \subset T^*M$ be the subbundle orthogonal to V^s. There is a unique section $F^* = F^*(f)$ of the 1-dimensional bundle $(V^{s\perp})^{\wedge u}$ such that $\langle F_x^*, F_x \rangle = 1$ for all $x \in K$. We have

$$(T_x^* f)^{\wedge u} F_{fx}^* = \lambda(x) F_x^*$$

and

$$\lambda(x) = \langle F_{fx}^*, (T_x f)^{\wedge u} F_x \rangle$$

Remember that $f \mapsto x = j(f)x_0$, and $F_x(f)$, $F_x^*(f)$ depend differentiably on f. We may thus estimate δJ_f^u in terms of δf by straightforward first order calculus. [The fact that $j(f) : K_0 \to K$ is in general not smooth plays no role here]. It is convenient to embed M isometrically in $\mathbf{R}^N$ with the Euclidean metric (for suitably large N). Then $x + T_x M$ may be viewed as an affine subspace of $\mathbf{R}^N$, and a local chart of M is provided by orthogonal projection on $x + T_x M$. Let $|x - y| < \epsilon/10$. In an ϵ-neighborhood of x, the manifolds M, $x + T_x M$, and $y + T_y M$ are $O(\epsilon^2)$-close, and the projections $M \to x + T_x M$, or $y + T_y M$ preserve distances up to order ϵ^2. This means that for first order calculations we may consider M as a piece of Euclidean space near x (or similarly near fx), and identify $T_x M$ with $T_y M$.

In view of the above considerations we may write, to first order in δf,

$$\delta\lambda(x) = \lambda(x)[\phi(x) - \phi(fx)] + \langle F_{fx}^*, [\delta(T_x f)^{\wedge u}] F_x \rangle$$

where

$$\phi(x) = \langle F_x^*, \delta F_x \rangle = -\langle \delta F_x^*, F_x \rangle$$

Note that the arbitrary ± 1 factor encountered earlier disappears in the definition of $\phi(x)$, and that $\phi(\cdot)$ is a continuous function.

We have

$$\delta(T_x f) = T_x(\delta f) = [T_{fx}(\delta f \circ f^{-1})](T_x f)$$

hence

$$\delta(T_x f)^{\wedge u} = [(1 + T_{fx}(\delta f \circ f^{-1}))^{\wedge u} - 1](T_x f)^{\wedge u}$$

hence

$$\delta\lambda(x) - \lambda(x)[\phi(x) - \phi(fx)]$$

$$= \lambda(x)\langle F^*_{fx}, [(1 + T_{fx}(\delta f \circ f^{-1}))^{\wedge u} - 1]F_{fx}\rangle = \lambda(x)[\mathrm{div}^u X](fx) \tag{3}$$

where $\mathrm{div}^u X$ is the *divergence of* $X = \delta f \circ f^{-1}$ *in the unstable direction* defined as follows. The orthogonal projection $M \to x + T_x M$ replaces the vector field X by a function $X' : x + T_x M \to T_x M$. Restriction of X' to $x + V^u(x)$, and projection parallel to $V^s(x)$ gives a function $X'' : x + V^u(x) \to V^u(x)$. Using an orthonormal basis of $V^u(x)$, we let $\xi_1, \ldots, \xi_u$ be the corresponding coordinates in $x + V^u(x)$, and $X''_1, \ldots, X''_u$ the corresponding components of X''. It is now readily checked that (3) holds if we write

$$\mathrm{div}^u X = \sum_{i=1}^{u} \frac{\partial}{\partial \xi_i} X_i$$

[Note that with our choice of coordinates, the metric tensor may be considered as constant near x; otherwise the expression for div^u would be more complicated].

From (2), and (3) we obtain

$$\delta[-\log J^u_f \circ j(f)]x_0 = -\frac{\delta\lambda(x)}{\lambda(x)}$$

$$= [-\mathrm{div}^u X](fj(f)x_0) + \phi(fj(f)x_0) - \phi(j(f)x_0)$$

or

$$\delta[-\log J^u_f \circ j(f)] = [-\mathrm{div}^u X] \circ j(f) \circ f_0 + \text{coboundary}$$

where the coboundary term $\psi \circ f_0 - \psi$ does not change the equilibrium state.

Write $\Psi = [-\mathrm{div}^u X] \circ j(f)$ so that $\Psi \in C^\beta(K_0)$. Taking also $\Phi \in C^\beta(K_0)$, we have

$$(\delta\mu_f)(\Phi) = \sum_{k \in \mathbf{Z}} [\mu_f((\Phi \circ f_0^k).\, \Psi) - \mu_f(\Phi).\, \mu_f(\Psi)]$$

[See [13] Chapter 5, Exercise 5, and use a Markov partition to apply this result to the present situation]. Finally (with $\Phi \in C^2(M)$)

$$(\delta\mu_f)(\Phi \circ j(f)) = \sum_{k \in \mathbf{Z}} [\rho_f((\Phi \circ f^k).\, (-\mathrm{div}^u X)) - \rho(\Phi).\, \rho_f(-\mathrm{div}^u X)]$$

Second step: computing $\mu_f(\delta(\Phi\circ j(f)))$.

Using Proposition 1.2(c) we have

$$\delta(\Phi\circ j(f))x_0=\langle T_{j(f)x_0}\Phi,\delta j(f)x_0\rangle=\langle T_{j(f)x_0}\Phi,(1-\mathcal{T}_{j(f)})^{-1}(\delta f\circ f^{-1}\circ j(f))x_0\rangle$$

where

$$(\mathcal{T}_{j(f)}(Y\circ j(f))x_0=(T_{j(f_0^{-1}x_0)}f)(Y\circ j(f)\circ f_0^{-1})x_0$$

Write again $x=j(f)x_0$, $X=\delta f\circ f^{-1}$, and let $X(x)=X^s(x)+X^u(x)$ with $X^s(x)\in V^s(x)$, $X^u(x)\in V^u(x)$. We have then

$$(\mathcal{T}^k_{j(f)}(Y\circ j(f))x_0=(T_{f^{-k}x}f^k)(Y\circ f^{-k})x$$

and

$$\delta(\Phi\circ j(f))x_0$$

$$=\langle T_x\Phi,\sum_{n=0}^{\infty}\mathcal{T}^n_{j(f)}(X^s\circ j(f))x_0\rangle-\langle T_x\Phi,\sum_{n=1}^{\infty}\mathcal{T}^{-n}_{j(f)}(X^u\circ j(f))x_0\rangle$$

$$=\langle T_x\Phi,\sum_{n=0}^{\infty}(T_{f^{-n}x}f^n)X^s(f^{-n}x)\rangle-\langle T_x\Phi,\sum_{n=1}^{\infty}(T_{f^nx}f^{-n})X^u(f^nx)\rangle$$

$$=\sum_{n=0}^{\infty}\langle T_{f^{-n}x}(\Phi\circ f^n),X^s(f^{-n}x)\rangle-\sum_{n=1}^{\infty}\langle T_{f^nx}(\Phi\circ f^{-n}),X^u(f^nx)\rangle$$

Using the f_0-invariance of μ_f, and writing $\operatorname{grad}\Phi$ for the element of T^*_xM defined by $T_x\Phi$ we have thus

$$\mu_f(\delta(\Phi\circ j(f)))$$

$$=\int\mu_f(dx_0)[\sum_{n=0}^{\infty}\langle T_{j(f)x_0}(\Phi\circ f^n),X^s(j(f)x_0)\rangle-\sum_{n=1}^{\infty}\langle T_{j(f)x_0}(\Phi\circ f^{-n}),X^u(j(f)x_0)\rangle]$$

$$=\rho_f[\sum_{n=0}^{\infty}\langle\operatorname{grad}(\Phi\circ f^n),X^s\rangle-\sum_{n=1}^{\infty}\langle\operatorname{grad}(\Phi\circ f^{-n}),X^u\rangle]$$

3.1 Theorem.

Let K be a compact invariant set for the C^3 diffeomorphism f of M. We assume that K is hyperbolic with local product structure and that $f|K$ is mixing. We denote by ρ_f the generalized SRB state on K.

(a) The derivative of $f\mapsto\rho_f$ is given by

$$\delta f\mapsto\delta\rho_f=\delta^{(1)}\rho_f+\delta^{(2)}\rho_f$$

and, for $\Phi \in C^2(M)$,

$$\delta^{(1)}\rho_f(\Phi) = \sum_{k=-\infty}^{\infty} [\rho_f((\Phi \circ f^k)(-\mathrm{div}^u X^u) - \rho_f(\Phi)\rho_f(-\mathrm{div}^u X^u)]$$

$$\delta^{(2)}\rho_f(\Phi) = \sum_{k=0}^{\infty} \rho_f\langle \mathrm{grad}(\Phi \circ f^n), X^s\rangle - \sum_{k=1}^{\infty} \rho_f\langle \mathrm{grad}(\Phi \circ f^{-n}), X^u\rangle$$

where X^s, X^u *are the components of the vector field* $X = \delta f \circ f^{-1}$ *along the stable and unstable subbundles of the hyperbolic decomposition* $T_K M = V^s \oplus V^u$.

(b) If K *is an attractor, we have* $\rho_f(\mathrm{div}^u Y) = 0$ *for any smooth vector field* Y, *and therefore*

$$\delta\rho_f(\Phi) = \sum_{n=0}^{\infty} \rho_f\langle \mathrm{grad}(\Phi \circ f^n), X\rangle$$

$$= \sum_{n=0}^{\infty} \rho_f[\langle(\mathrm{grad}\Phi) \circ f^n, (Tf^n)X^s\rangle - (\Phi \circ f^n)\mathrm{div}^u X^u]$$

The proof of (a) has been given above. For (b) we use a Markov partition and a disintegration of ρ_f into measures carried by pieces of unstable manifolds. By a change of variable $x \mapsto y = f^N x$ for N large, and use of Gauss's formula we see that $\rho_f(\mathrm{div}^u Y)$ reduces to boundary terms, and since these cancel pairwise $\rho_f(\mathrm{div}^u Y) = 0$. Therefore $\rho_f(\mathrm{div}^u X^u) = 0$ and

$$\rho_f[(\Phi \circ f^k)(-\mathrm{div}^u X^u)] = \rho_f\langle \mathrm{grad}(\Phi \circ f^k), X^u\rangle$$

so that

$$\delta\rho_f(\Phi) = \sum_{n=0}^{\infty} \rho_f\langle \mathrm{grad}(\Phi \circ f^n), X^s + X^u\rangle$$

as announced. ∎

3.2 Remarks.

(a) In the attractor case the formula for $\delta\rho_f(\Phi)$ contains a term

$$\sum_{n=0}^{\infty} \rho\langle \mathrm{grad}\Phi, ((Tf^n)X^s) \circ f^{-n}\rangle$$

which converges exponentially because Tf is a contraction on V^s, and a term

$$\sum_{n=0}^{\infty} \rho[\Phi\cdot ((\mathrm{div}^u X^u) \circ f^{-n})]$$

which converges exponentially because of the exponential decay of correlations for the Gibbs state ρ.

(b) Let m be a probability measure absolutely continuous with respect to Riemann volume on M, and with support in the basin of the attractor K. Then $f^{*n}m$ has the weak limit ρ_f when $n\to\infty$. We may write

$$\delta[(f^{*n}m)(\Phi)] = \delta m(\Phi\circ f^n) = \int m(dx)\,\delta\Phi(f^n x)$$

$$= \int m(dx)\langle(\mathrm{grad}\Phi)(f^n x),\delta f^n x\rangle$$

$$= \int m(dx)\langle(\mathrm{grad}\Phi)(f^n x),\sum_{k=0}^{n-1}(Tf^k)\delta f(f^{n-k-1}x)\rangle$$

$$= \sum_{k=0}^{n-1}\int((f^{n-k})^*m)(dy)\langle(\mathrm{grad}\Phi)(f^k y),(Tf^k)\delta f(f^{-1}y)\rangle$$

$$= \sum_{k=0}^{n-1}\int((f^{n-k})^*m)(dy)\langle(\mathrm{grad}(\Phi\circ f^k))(y),X(y)\rangle$$

When $n\to\infty$ we obtain formally

$$\delta\rho_f(\Phi) = \sum_{k=0}^{\infty}\rho_f\langle\mathrm{grad}(\Phi\circ f^k),X\rangle$$

as asserted in the theorem.

4. Bounded time dependent perturbations.

Let $\mathcal{B}_\infty \subset \mathcal{B}^{\mathbf{Z}}$ be the Banach space of sequences $(X_k)_{k\in\mathbf{Z}}$ such that

$$\|(X_k)\|_\infty = \sup_k \|X_k\| < \infty$$

Then, with the notation of Section 1, $B^{\mathbf{Z}} \subset \mathcal{B}_\infty$ ($B^{\mathbf{Z}}$ contains the open ϵ-ball of $\mathcal{B}_\infty$). Note that $0 \in B^{\mathbf{Z}}$ corresponds to $(K \hookrightarrow M)^{\mathbf{Z}}$ and is a fixed point of the map

$$(j_k)_{k\in\mathbf{Z}} \to (f \circ j_{k-1} \circ f^{-1})_{k\in\mathbf{Z}}$$

This map is differentiable, and its derivative at 0 is a hyperbolic linear operator in $\mathcal{B}_\infty$. Therefore if $\mathbf{f} = (f_k) \in \mathcal{A}^{\mathbf{Z}}$, the map

$$(j_k)_{k\in\mathbf{Z}} \to (f_k \circ j_{k-1} \circ f^{-1})_{k\in\mathbf{Z}}$$

has a unique fixed point $\mathbf{j} \in B^{\mathbf{Z}}$, yielding a diagram

$$\begin{array}{ccccccccc}
\cdots & \to & K_{k-1} & \stackrel{f_k}{\to} & K_k & \stackrel{f_{k+1}}{\to} & K_{k+1} & \to & \cdots \\
 & & \uparrow j_{k-1} & & \uparrow j_k & & \uparrow j_{k+1} & & \\
\cdots & \to & K & \stackrel{f}{\to} & K & \stackrel{f}{\to} & K & \to & \cdots
\end{array}$$

where the vertical arrows are the components j_k of $\mathbf{j}$ and $K_k = j_k K$. The diagram is commutative because $j_k = f_k \circ j_{k-1} \circ f^{-1}$. Using the expansiveness of f on K, one checks that the j_k are homeomorphisms. The diagram expresses structural stability at the level of bounded time dependent perturbations of a hyperbolic dynamical system.

Because the j_k are close to the identity, and the f_k close to f, one can define (un)stable bundles $V_k^\pm$ with the obvious properties, and (un)stable manifolds $\mathcal{V}_k^\pm(x)$, such that $j_k^{-1}\mathcal{V}_k^\pm(j_k x)$ coincides with $\mathcal{V}^\pm(x)$ in a sufficiently small neighborhood of x. The proofs of these facts go along standard lines, and we do not give them here. We shall now outline how SRB states can be defined in the present situation where there is no time stationarity. The proofs will only be sketched.

SRB states.

We first recall the definition of SRB measure in the case of a single diffeomorphism f. Suppose that K is a mixing Axiom A attractor for f, and let $m(dx) = \underline{m}(x)\,dx$ be a probability measure absolutely continuous with respect to the Riemann volume element dx, and with support in the basin of attraction of K. Then, when $n \to \infty$, $f^{*n}m$ tends to the SRB measure ρ. One way to see that the limit exists (see [12]) is to choose a Markov partition of (K, f) formed of rectangles $[S_i, U_i]$. Displacing the mass of $m(dx)$ by a bounded distance along stable manifolds, we obtain measures m_i on the pieces U_i of unstable manifolds, where m_i is absolutely continuous with respect to the Riemann volume element of U_i. The weak limit of $f^{*n}m$ remains the same if m is replaced by the sum of the m_i, and this leads to a standard transfer operator study and to the identification of the limit ρ. The SRB state ρ may be characterized in four different ways:

(i) as limit of $f^{*n}m$ where m is absolutely continuous with respect to dx,

(ii) as f-invariant measure absolutely continuous along unstable directions,

(iii) in terms of eigenfunctions of transfer operators $\mathcal{L}$ and $\mathcal{L}^*$,

(iv) by a variational principle.

In the situation of bounded time dependent perturbations as described above, we can still define SRB states as collections (ρ_k) where ρ_k *is a probability measure on* K_k *and* $f_k^*\rho_{k-1} = \rho_k$. We may take as definition the property

(i) for each k, $\rho_k = \lim_{n\to\infty} f_k^* \cdots f_{k-n}^* m$.*

To prove existence and uniqueness of the SRB states,and study their properties, we may use the maps j_k and a Markov partition into rectangles $[S_i, U_i]$ for (K, f). Note in particular that K_k is a union of sets $j_k[s, U_i]$. Choose now $s_i \in S_i$ and let $\pi_i : [S_i, U_i] \to [s_i, U_i]$ be the projection. Here is a second characterization of SRB states:

(ii) for each k, the conditional measures $\rho_{k,s,i}$ of ρ_k with respect to the partition $(j_k[s, U_i])$ are absolutely continuous with respect to the Riemann volume element on unstable manifolds. Furthermore the densities $\phi_{i,k}$ of the measures $(j_k\pi_i j_k^{-1})^*(\rho_k|j_k[S_i, U_i])$ with respect to the unstable volume element are continuous uniformly in k.*

The second condition in (ii*) could be replaced by various other uniformity properties.

We write

$$\mathcal{L}_k\phi_{k-1} = \phi_k$$

to express that the densities $\phi_{i,k}$ are obtained from the densities $\phi_{i,k-1}$ by application of a transfer operator $\mathcal{L}_k$ with coefficients constructed from unstable Jacobians. If σ_k is the collection of measures on the $j_k[s_i, U_i]$ corresponding to the unstable volume elements, and $\tilde{\phi} = (\tilde{\phi}_i)$ is arbitrary, we have

$$(\sigma_k, \mathcal{L}_k\tilde{\phi}) = (\sigma_{k-1}, \tilde{\phi})$$

i.e. $\mathcal{L}_k^*\sigma_k = \sigma_{k-1}$. Here is a third characterization of SRB states:

(iii)* $$(j_k\pi_i j_k^{-1})^*(\rho_k|j_k[S_i, U_i]) = \phi_k\sigma_k$$
where ϕ_k is (up to normalization) $\lim_{n\to\infty} \mathcal{L}_k \cdots \mathcal{L}_{k-n}1$.

The $\mathcal{L}_k$, acting on a space of Hölder continuous functions, are close to $\mathcal{L}$, and there is thus a cone C containing the "principal" eigenvector of $\mathcal{L}$, and mapped inside itself by all $\mathcal{L}_k$. From this one obtains that $\mathcal{L}_k \cdots \mathcal{L}_{k-n}1$ converges to a limit ϕ_k.

Adapting for instance the study in [12] to the time dependent situation, it is now easy to prove existence and uniqueness of SRB states, and equivalence of *(i*), (ii*), (iii*)*. Note that we have here a situation close to the study of Gibbs states and equilibrium states by Bogenschütz and Gundlach [2], Khanin and Kifer [9], Baladi [1], where however $(f_k)_{k\in\mathbf{Z}}$ is distributed according to some τ-ergodic measure $\mathbf{P}$. In that case, one obtains only $\mathbf{P}$-a.e. statements, but one gains equivalence of *(i*), (ii*), (iii*)* with a variational principle *(iv*)*.

Causality.

Note that the "attractors" K_k and the "SRB measures" ρ_k depend only on f_{k-n}, $n \geq 0$. However, the j_k, the $(j_k \pi_i j_k^{-1})^*(\rho_k | j_k[S_i, U_i])$ and the densities ϕ_k depend on all f_j (because their definitions involve projection along stable manifolds).

Differentiation of the map $\mathbf{f} \to \rho_0$.

We shall not embark in a general study of the smoothness of the map $\mathbf{f} \to \rho_0$, although such a study should be possible. What is easy is to vary a finite number of the f_k, say those with $|k| \leq N$, because ρ_{-N} then remains fixed, and we have

$$\rho_0 = f_0^* \dots f_{-N}^* \rho_{-N-1}$$

In particular,

$$\delta\rho_0(\Phi) = \delta(f_0^* \dots f_{-N}^* \rho_{-N-1})(\Phi) = \delta\rho_{-N-1}(\Phi \circ f_0 \circ \dots \circ f_{-N})$$

$$= \sum_{n=0}^{N} \rho_{-N-1}(T(\Phi \circ f_0 \circ \dots \circ f_{-n+1})\delta f_{-n} \circ f_{-n-1} \circ \dots \circ f_{-N})$$

$$= \sum_{n=0}^{N} \int \rho_{-N-1}(dx) \, \langle \mathrm{grad}_{f_{-n} \cdots f_{-N} x}(\Phi \circ f_0 \circ \dots \circ f_{-n+1}), (\delta f_{-n} \circ f_{-n}^{-1})(f_{-n} \dots f_{-N} x) \rangle$$

$$= \sum_{n=0}^{N} (f_{-n}^* \dots f_{-N}^* \rho_{-N-1}) \langle \mathrm{grad}(\Phi \circ f_0 \circ \dots \circ f_{-n+1}), X_{-n} \rangle$$

where X_k is the vector field $\delta f_k \circ f_k^{-1}$.

Finally, we have thus

$$\delta\rho_0(\Phi) = \sum_{n=0}^{\infty} \rho_{-n} \langle \mathrm{grad}(\Phi \circ f_0 \circ \dots \circ f_{-n+1}), X_{-n} \rangle$$

$$= \sum_{n=0}^{\infty} \rho_0 \langle \mathrm{grad}\Phi, (T(f_0 \circ \dots \circ f_{-n+1}) X_{-n}^s) \circ (f_0 \circ \dots \circ f_{-n+1})^{-1} \rangle$$

$$- \sum_{n=0}^{\infty} \rho_0 [\Phi \cdot ((\mathrm{div}^u X_{-n}^u) \circ (f_0 \circ \dots \circ f_{-n+1})^{-1})]$$

Note that this is formally identical with the result of theorem 3.1(b) when we replace ρ_k by ρ and f_k by f.

5. Formal derivative of ρ_f in the general case.

We assume that the f-invariant state ρ satisfies the SRB condition, but here we do not suppose uniform hyperbolicity, (*i.e.*, suppρ need not be a hyperbolic invariant set). Thus we do not know how ρ will vary with f, but we have a good formal candidate for its derivative, *viz.*,

$$\delta\rho(\Phi) = \sum_{n=0}^{\infty} \rho\langle \mathrm{grad}(\Phi \circ f^n), X\rangle$$

where $X = \delta f \circ f^{-1}$. If there are no vanishing Lyapunov exponents, a measurable splitting $T_x M = V^s(x) \oplus V^u(x)$ is defined $\rho(dx)$-a.e., and we may write $X(x) = X^s(x) + X^u(x)$ with $X^s(x) \in V^s(x)$, $X^u(x) \in V^u(x)$. Then

$$\rho\langle \mathrm{grad}(\Phi, f^n), X\rangle = \rho\langle \mathrm{grad}(\Phi, f^n), X^s + X^u\rangle$$
$$= \rho\langle (\mathrm{grad}\Phi) \circ f^n, (Tf^n)X^s\rangle - \rho((\Phi \circ f^n)\cdot \mathrm{div}^u X^u)$$

with $\rho(\mathrm{div}^u X^u) = 0$ just as in the uniformly hyperbolic case. Formally, we have thus

$$\delta\rho(\Phi) = \sum_{n=0}^{\infty} \rho\langle (\mathrm{grad}\Phi) \circ f^n, (Tf^n)X^s\rangle - \sum_{n=0}^{\infty} \rho((\Phi \circ f^n)\cdot \mathrm{div}^u X^u)$$

The convergence of the right-hand side depends on how $(Tf^n)X^s$ and $\rho((\Phi \circ f^n)\cdot \mathrm{div}^u X^u)$ tend to 0 when $n \to \infty$.

In the time dependent case, the formula becomes

$$\delta\rho_0(\Phi) = \sum_{n=0}^{\infty} \rho_{-n}\langle \mathrm{grad}(\Phi \circ f_0 \circ \ldots \circ f_{-n+1}), X_{-n}\rangle$$

where $X_k = \delta f_k \circ f_k^{-1}$. In particular, if all f_k are equal to f and the ρ_k to ρ, we obtain

$$\delta\rho_0(\Phi) = \sum_{n=0}^{\infty} \rho\langle \mathrm{grad}(\Phi \circ f^n), X_{-n}\rangle$$

$$= \sum_{n=0}^{\infty} \rho\langle (\mathrm{grad}\Phi) \circ f^n, ((Tf^n)X^s_{-n})\rangle - \sum_{n=0}^{\infty} \rho((\Phi \circ f^n)\cdot \mathrm{div}^u X^u_{-n})$$

There are similar formulae for flows. Suppose for instance that the state ρ satisfies the SRB condition for the flow (f^t) corresponding to the vector field $\mathcal{X}$. Let X_t be a time dependent perturbation of $\mathcal{X}$, then the derivative of ρ at time 0 is given formally by

$$\delta\rho_0(\Phi) = \int_0^{\infty} dt\, \rho\langle \mathrm{grad}(\Phi \circ f^t), X_{-t}\rangle$$
$$= \int_0^{\infty} dt\, \rho\langle (\mathrm{grad}\Phi) \circ f^t, (Tf^t)X^s_{-t}\rangle$$
$$- \int_0^{\infty} dt\, \rho((\Phi \circ f^t)(\mathrm{div}^u X^u_{-t}))$$

References.

[1] V.Baladi. "Correlation spectum of quenched and annealed equilibrium states for random expanding maps." Preprint.

[2] T.Bogenschütz and V.M.Gundlach. "Ruelle's transfer operator for random subschift of finite type." Ergod. Th. Dynam. Syst. **15**,413-447(1995).

[3] R.Bowen. "Markov partitions for Axiom A diffeomorphisms." Amer. J. Math. **92**,725-747(1970).

[4] R.Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms.* Lecture Notes in Math. **470**. Springer, Berlin, 1975.

[5] R.Bowen and D.Ruelle. "The ergodic theory of Axiom A flows." Invent. Math. **29**,181-202(1975).

[6] J.-P.Eckmann and D.Ruelle. "Ergodic theory of strange attractors." Rev. Mod. Phys. **57**,617-656(1985).

[7] G.Gallavotti. "Chaotic hypothesis: Onsager reciprocity and fluctuation-dissipation theorem." J. Statist. Phys. **84**,899-926(1996).

[8] M.Hirsch and C.Pugh. "Stable manifolds and hyperbolic sets." *in*: *Global Analysis.* Proc. Symp. Pure Appl. Math. **14**,133-164(1970).

[9] K.Khanin and Y.Kifer. "Thermodynamic formalism for random transformations and statistical mechanics." Preprint.

[10] F.Ledrappier and L.S.Young. "The metric entropy of diffeomorphisms: I. Characterization of measures satisfying Pesin's formula, II. Relations between entropy, exponents and dimension." Ann. of Math. **122**,509-539,540-574(1985).

[11] A.Lopes and R.Markarian. "Open billiards: invariant and conditionally invariant probabilities on Cantor sets." SIAM J. Appl. Math. **56**,651-680(1996).

[12] D.Ruelle. "A measure associated with Axiom A attractors." Am. J. Math. **98**,619-654(1976).

[13] D.Ruelle. *Thermodynamic formalism.* Encyclopedia of Math. and its Appl., vol. 5. Addison-Wesley, Reading, Mass., 1978.

[14] D.Ruelle. *Elements of differentiable dynamics and bifurcation theory.* Academic Press, San Diego, 1989.

[15] D.Ruelle. "Positivity of entropy production in nonequilibrium statistical mechanics." J. Statist. Phys. **85**,1-25(1996).

[16] M.Shub. *Global stability of dynamical systems.* Springer, Berlin, 1987.

[17] Ya.G.Sinai. "Markov partitions and C-diffeomorphisms." Funkts. Analiz i Ego Pril. **2**, No **1**,64-89(1968). English translation: Functional Anal. Appl. **2**,61-82(1968).

[18] Ya.G.Sinai. "Construction of Markov partitions." Funkts. Analiz i Ego Pril. **2**, No **3**,70-80(1968). English translation: Functional Anal. Appl. **2**,245-253(1968).

[19] Ya.G.Sinai. "Gibbsian measures in ergodic theory." Uspehi Mat. Nauk **27**, No **4**,21-64(1972). English translation: Russian Math. Surveys **27**, No **4**,21-69(1972).

[20] S.Smale. "Differentiable dynamical systems." Bull. Amer. Math. Soc. **73**,747-817(1967).

[21] L.-S.Young. "Some large deviation results for dynamical systems." Trans. Amer. Math. Soc. **318**,525-543(1990).