CONVERSATIONS ON MATHEMATICS WITH A VISITOR FROM OUTER SPACE

by David Ruelle*.

Abstract. While it is probably unrewarding to try to imagine what extraterrestrial mathematics might be like, we may more reasonably try to find out what is peculiar about human mathematics. An investigation of that sort was started by J. v. Neumann in his book *The computer and the brain.* We shall discuss some of the characteristics, and especially deficiencies, of the human brain as revealed by the neurosciences and comparison with the computer. We shall argue that they explain features of human mathematics that we take for granted, but that a mathematical visitor from outer space might find very striking.

English versions of such compendia as *Encyclopedia Galactica* (Concise Edition), *Standard Galactic Dictionary of Mathematics*, and a few other basic references were made available to me by an extraterrestrial friend. I can give here only few details of how this happened, but I must explain that, after extreme initial enthusiasm, browsing through this material got me perplexed at first and then frankly distressed. So-called galactic mathematics appears to consist of huge computer programs which, run on suitable (galactic) computers, deal very efficiently with all kinds of difficult mathematical problems. As my friend Pallas explained, such programs are like large mathematical libraries, but much easier to use. Also, producing such a program is much more challenging than writing down a human type mathematical theory. It is, as she told me, like constructing a brain instead of writing a book.

My friend Pallas landed in my garden on a fine morning of May in her silent spaceship, bright and tall, and departed three months later. She had convincing and pleasant female human form, to the extent that I often doubted if she really was extraterrestrial. She told me, however, that her female guise was only for convenience. I should understand that mathematicians are sexless, and should be referred to as “it” rather than “he or she”.

Be that as it may; we had quite nice conversations on all sorts of topics ranging from poetry to religion and from music to science. To my relief we abandoned galactic mathematics after a while, and turned to human mathematics on which Pallas was writing her galactic Ph.D. thesis. She had the strangest ideas on the subject, but was well documented, and little by little convinced me. I shall now relate the results of our conversations. The

* IHES, 91440 Bures sur Yvette, France <ruelle@ihes.fr>.

basic ideas come from her, and my contribution was mostly to get this material out of her by my questions and then writing it down.

I had the greatest difficulty with her first statement, that

"...to appreciate human mathematics you have to understand how peculiar the human intellect is, compared with that of the ancient civilizations of the Galaxy."

"How on earth can you say such a thing" I answered, "and what right have you to say that the mind of a human is more peculiar than that of a slimy galactic superoctopus?"

"Why in the galaxy don't you use your brains! Think how primitive a machine your Personal Computer is, yet it completely outwits you on simple mathematical problems like deciding if a ten digit number is an exact square. You can imagine that an ancient civilization would have fixed such intellectual inadequacies by assisted coevolution, biological engineering, and so on. I shan't go into details, as they might greatly upset you."

"You mean that human civilization too will ...?"

"Yes, if human civilization survives to become ancient."

After that we argued, point by point, what made the human brain so *peculiar*. Pallas made some comparisons between humans and slimy galactic superoctopi, but these comparisons made little sense to me. So she replaced galactic beasts by human computers, with which I am more familiar. "These computers are really very stupid things", she said, "but they already give a good idea of features desirable for doing mathematics, and which are not possessed by the human brain". Here is now a list of four or five peculiarities of the human brain that we came up with, very reluctantly on my part, after many hours of discussion.

Slowness and high parallelism.

The human brain is a highly connected net of several times 10^{10} neurons. Local characteristic times are at least 1 millisecond, and the speed of propagation of the nervous influx is from 1 to 100 meters/second, so that times of the order of 100 milliseconds are easily reached. By comparison, the processor of your PC has a speed measured in millions of instructions/second. The high speed of computers permits repetitive calculations where each loop provides an updated input for the next loop. The slowness of the brain is compensated by a high parallelism of operations. An example of this parallelism occurs in the sensory pathways to the central nervous system: they preserve the spatial relations of the receptors. In the visual system this is called *retinotopy*. At a higher level, the visual system also uses parallel processing of different aspects of visual information (like color, motion, ...[3]).

The computer and the brain are both information processing systems, and functional requirements imply some structural analogies such as existence of input, output, memory. A detailed comparison shows huge differences, which were first analyzed by J. von Neumann in *The computer and the brain* [5]. This was his last book. He wrote it as his own brain was being destroyed by cancer.

Deficient memory.

Just as your laptop has several memories (RAM, floppy drive, hard disk, CD-ROM) with different characteristics, the human brain has several functionally different memories. Short term memory allows us to repeat immediately a random sequence of letters or digits, but is typically limited to about seven items. This limitation is a nuisance when dialing a number that you just read in the telephone book, and it also has consequences for the way humans do mathematics. If the solution of a problem depends on having ten items of data (say) readily available, it is necessary to put these items in long term memory, (*i.e.*, somehow learn them by heart) or to create an external memory in the form of a sheet of paper on which the data are written. What corresponds to short term memory on your laptop is the random-access memory (RAM) of 8 or 16 megabytes. The difference in favor of the laptop is huge. If natural selection had favored immediate memorization of telephone numbers, we would have a much better short term memory. Our long term memory is more satisfactory, but wouldn't it be nice if your brain could store a cool 500 megabytes of error-free data in as little space as a CD-ROM (a compact disk that can hold the 46 million words of the *Encyclopedia Britannica*, and has room left for a number of other things).

Search for regularities.

A ten digit number may be hard to remember, but with 9876543210 we have no difficulty because it is just "the ten digits from zero to nine written in reverse order", and 3141592653 is also easy because it is "the first digits of pi". In the same way we find hidden regularities in telephone numbers, and if we face a wall with cracks we shall often interpret the cracks as human profiles. In general we seem to make up for inadequacies of the human mind, like poor memory, by a search for "order" or "meaning" often pushed to absurd limits. The unceasing, obsessional search for regularities is certainly fundamental to human intelligence, and in particular to the human mathematical genius.

Importance of visualization.

Evolution has given a lot of weight to the visual system, which occupies a big part of our brain. Most mathematicians are thus happy when they can make use of visual intuition in their mathematical work. For instance, it is considered a success if one can geometrize a theory, interpreting its concepts in terms of points, spaces, topology and the like. There are however non visual mathematicians. Laurent Schwartz for example boasts a complete lack of geometric intuition. While some of his colleagues have difficulty in believing him, it must be admitted that the use of geometric intuition has no logical necessity in mathematics, and is often left out of the formal presentation of the results. If one had to construct a mathematical brain, one would probably use resources more efficiently than creating a visual system. But the system is there already, it is used to great advantage by human mathematicians, and it gives a special flavor to human mathematics.

Lack of formal precision.

One wrong step in a long proof is sufficient to make the proof worthless. The ability to check mechanically that the rules are respected would thus seem to be an important mathematical asset. Our computers are good at doing this sort of mechanical work without error, but are not good at doing creative mathematics. The human brain, by contrast, is not very good at long complicated logical tasks which have to be performed without error, yet can do very difficult mathematics. Human mathematics consists in fact in talking about formal proofs, and not actually performing them. One argues quite convincingly that certain formal texts exist, and it would in fact not be impossible to write them down. But it is not done: it would be hard work, and useless because the human brain is not good at checking that a formal text is error-free. Human mathematics is a sort of dance around an unwritten formal text, which if written would be unreadable. This may not seem very promising, but human mathematics has in fact been prodigiously successful.

"I am glad to hear that human mathematics is prodigiously successful", I said. "But I thought that in discussing peculiarities of the human mind you would address the problem of consciousness, which seems so fundamental to us."

"Consciousness is a much disputed topic among humans", Pallas answered "and I didn't want to base my discussion on questionable premises. I really don't think the question is all that important, but let us discuss it if you wish."

Consciousness and attention.

Consciousness is an introspective concept, and difficult to approach scientifically. It has nevertheless attracted the attention of brain specialists (see for instance [1]). Consciousness is related to attention, which is our ability to direct intellectual resources to a specific task. (Attention is demonstrably correlated with greater blood irrigation of areas of the brain concerned with the task performed). It may be that consciousness arises naturally in a highly parallel system like the human brain, where some coordination of activities is needed to avoid chaos. Note however that many tasks are performed unconsciously. It is a natural idea that mathematical work should require conscious attention. For instance when Polya (in his book *How to solve it* [7]) describes how one should attack a mathematical problem, he uses phrases like "to understand the problem", "to have the data well in mind", which have absolutely no mathematical meaning, but reflect the importance of conscious attention. Nevertheless, unconscious mathematical work may play an essential role, as stressed by Poincaré*, once all the aspects of a problem have become familiar by conscious study. Perhaps the conscious study is needed to put all the useful information about the problem in long term memory, so that the combinatorial task of actually finding a solution can then be managed unconsciously.

"Now that you have reluctantly agreed that the human brain is a bit peculiar" Pallas said "I would like to convince you that the peculiarities we discussed have a profound impact on the human mathematical output."

"Some of our social scientists make similar statements, to the effect that scientific theories are shaped by social forces, and that what is true today may be false tomorrow when the political power has changed hands. A mathematical text would be a narrative like any other narrative, and literary criticism could reveal its true social content: racist, male chauvinistic, and the like. We, human mathematicians, have a very different view of the nature of our art. We believe in absolute truth: 137 is a prime number, and no social event will change this fact. We use the name *Teichmüller space* to denote some mathematical object, even though Teichmüller was a Nazi, and most of us abhor the Nazi ideology. It is our pride that that we have access to a world of ideas were political ideologies do not count. Would you now replace social relativism by another kind of relativism where truth would not be the same for humans and for slimy galactic superoctopusses? Would you. . . "

* See "L'invention mathématique", which is Chapter 3 of [6].

Pallas interrupted me there: “No! no! no!” she said. “Logical truth is absolute. It is not determined by social circumstances or by the particular structure of the mind of a galactic species with mathematical abilities. But mathematical style depends enormously on the structure of the mind which produces it. I leave you till tomorrow to find an example of that yourself, and explain it to me.”

Of course the mathematical discussions which I have been relating were spread over many days, and we did other things than talking about mathematics. In June-July in particular, we had splendid weather and took long walks through fields and forest. “Coming from an advanced civilization” she said “it is a bit of a shock to be exposed to all these creepy-crawlies that you take for granted: mosquitos, spiders, slugs, and the ever-present flies eating from your food and shitting on it. Yet I have come to love the sort of nonchalant absurdity of your primitive culture, its art, and its archaic science. I am even starting to enjoy being a female human...”

“That is called a girl, or a woman.”

“I know, but I don’t like this sexist language.”

Anyway, the next day I presented my homework to Pallas, a double homework actually.

Greek geometry.

A contemporary mathematician leafing through Euclid will find absolutely nontrivial theorems even if they are well-known to it (following the advice of Pallas I am not saying “to him or to her”). Greek geometry is early but in a sense completely modern mathematics. It does however show more clearly than later mathematics two peculiarities of the human brain that produced it:

(1) it uses the human visual system, in fact geometry is directly derived from visual experience and intuition,

(2) it uses an external memory in the form of a drawing formed of lines and circles, with points labeled by letters*.

Combining these two tricks permits elaborate logical constructions which the Greeks rightly considered as prodigious intellectual feats. Hilbert’s version of Euclidean geometry without the help of (1) and (2) shows how hard the subject really is.

Parsimony.

Instead of asking how the characteristics of the human brain influence human mathematics, one could ask what kind of humans are likely to do good mathematics. A productive artist might be insane or a heavy drug abuser, but a working mathematician must be relatively normal. Some level of paranoia is however acceptable, and not uncommon. But the most widespread feature is an obsessive disposition associated with the characteristic traits of *order*, *parsimony*, and *stubbornness*. Freud has interpreted those traits as leftover from the so-called anal-sadistic stage of the child’s libido evolution. Whatever the interpretation, it is clear why order and stubbornness should be assets for a scientist in general. But parsimony is particularly meaningful for human mathematics. We know indeed that

* It seems that the drawings of Greek geometry have not been much studied (historians of science study *texts*). See however [4]; I am indebted to Karine Chemla for this reference and for an enlightening conversation on the subject.

mathematical proofs tend to be long (this is related to Gödel's theorem), and that the human ability to check the correctness of formal texts is limited. One has thus to cut proofs into segments which make sense (*i.e.* make sense for the easily tired attention of a human mathematician). The compromise between long proofs and short attention span calls for an ungenerous, parsimonious attitude. Many mathematicians, of course, have grown up to be generous in social relations where systematic parsimony would be crippling.

Pallas had remained silent during the presentation of my homework, doodling as it seemed on a large sheet of paper. As I looked at it I saw that she had drawn a kind of octopus, a rather noble-looking creature with a high forehead and intelligent eyes.

"This is a slimy galactic superoctopus" she said. "Notice that each arm trifurcates: it has twenty-four fingers, an important number."

I reflected that the slimy galactic superoctopodes must have discovered arithmetics by counting their fingers and toes, exactly like humans.

"But let us return to your homework. I am glad that you recognized that Greek geometry is human mathematics at its most human, and I would say at its most beautiful. Concerning the psychological setup of mathematicians, I did not know Freud's ideas on the anal-sadistic stage, and I have no personal insight into the matter, but I shall mention your remark in my PhD thesis. What you say about parsimony goes to the heart of trying to characterize human mathematics, but does not solve the problem. And the problem, as far as I am concerned remains unsolved."

We continued our discussion on other days, during the long walks that we made. I wanted her to help me guess at the future of human mathematics, which she was quite reluctant to do, "not knowing" as she said "if mankind will still exist in a few decades". She preferred to analyze the forces which would drive mathematics. Here again I shall put in my own words the tentative conclusions that we reached after hours of conversation.

Towards formalized mathematics.

Formalization is one of the great dreams of mathematicians. But they have been content with mathematics that could *in principle* be formalized, so that the correctness of the formalized text could *in principle* be checked mechanically. The progress of computers should lead in due time to the possibility of translating human mathematics into formal language so that proofs can be checked mechanically. Such an enterprise appears much less difficult than producing interesting original mathematics by computer. Computer formalization, however, may bring surprises, like any mathematical endeavor. Here, interestingly, the principle of parsimony for the length and simplicity of proofs could be largely relaxed: checking by computer that a formal text is correct would be extremely fast, and wasting a factor of ten in the length of the proof would not change things much.

Towards intuitive mathematics.

We know that theorems that are simple to state may have very long proofs, and we are seeing more and more examples of this, particularly in algebra (beginning with Feit and Thompson [2], later followed by the classification of finite simple groups). It is thus hard to exclude that some very interesting results are inherently off limits to the unaided human brain, and might only become accessible when sufficiently intelligent computers take over. As long, however, as humans use their own brains to do mathematics, some areas will be privileged. Our visual intuition of space and our intuition of time make for instance the theory of dynamical systems particularly attractive, and this has indeed been a flourishing domain of research in recent times. The human brain is also in a favorable position in branches of mathematics related to physics, not just for reasons of intuition, but because the physical world itself proposes a wealth of facts expecting theorization.

Towards natural mathematics.
Here are two facts:

$$3^2 + 4^2 = 5^2 \qquad (a) \qquad , \qquad 3^3 + 4^3 + 5^3 = 6^3 \qquad (b)$$

Many things can be said about (a), for instance it "means" that if the sides of a right triangle are 3 and 4, the hypotenuse is 5. It is much less clear what (b) means. In fact, it can be argued that many properties of integers occur, in some precise sense, at random. In apparent contradiction with this randomness, we see mathematicians put much effort to organize what they know in neat natural structures. It seems indeed natural to use compact sets, groups, or functors because it makes human mathematics efficient in its use by human mathematicians. But how much of what we consider natural follows from the specific structure of the human mind? How much is in some sense universal? My galactic friend was not very helpful in finding an answer to these questions which she found "a bit imprecisely formulated".

In the discussions with Pallas which I have just reported, on the role of formalization, intuition and naturalness, she made rather less provocative remarks than in our earlier discussions. I was almost disappointed.

"Have you then rounded up your study of human mathematics with what we have just discussed?"

"Not at all. I have a new project for after my thesis. When I have looked into it some more, I want to discuss it with you."

"What is it about?"

"The creative role of error and intellectual confusion in human mathematics."

"Oh, boy!"

As I got up one fine morning of August, I did not see Pallas. Looking through the window, I found that her spaceship had gone. There was a note on the breakfast table:

To my favorite human mathematician.*

They have scheduled my thesis for very soon, and I must leave in a hurry. Also, I am a bit homesick, and I want to get away for a while from your crazy world and all its flies. As you must have guessed, I am really a slimy galactic superoctopus, and at times a bit tired of my female human guise. What I want now is to relax in a little pool of clean warm water, and make bubbles pink and blue (as we say in our language). But as soon as I have passed my thesis, I shall apply for a travel grant, and be back.

So long,

Pallas.

---

* In fact I am a mathematical physicist, as I told her several times

References.

[1] F. Crick. *The astonishing hypothesis: the scientific search for the soul.* Touchstone, New York, 1994.

[2] W. Feit and J.G. Thompson. "Solvability of groups of odd order." Pacific J. Math. **13**,755-1029(1963).

[3] E.R. Kandel, J.H. Schwartz, and Th.M. Jessel. *Essentials of neural science and behavior.* Appleton and Lange, East Norwalk, CT, 1955.

[4] R. Netz. *The shaping of deduction in Greek mathematics. A study in cognitive history.* Cambridge U. P., 1999.

[5] J. von Neumann. *The computer and the brain.* Yale U. P., New Haven, 1958.

[6] H. Poincaré. *Science et méthode.* Ernest Flammarion, Paris, 1908.

[7] G. Polya. *How to solve it.* 2-nd ed. Princeton U. P., Princeton, 1957.