PREDICTABLE, CHAOTIC AND HISTORICAL DYNAMICS.

by David Ruelle

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Dedicated to Floris Takens on the occasion of his 60-th birthday.

When Floris Takens and myself proposed that hydrodynamic turbulence is described by what we called *strange attractors*, we expected neither the opposition that this idea would at first encounter, nor the importance it would later achieve in the development of chaos theory. At the occasion of this joint work [1] (and later) I noted the ability of Floris to combine the focussed rigorous thinking appropriate for mathematical work, and the openmindedness necessary to study physical applications. Some openmindedness will also be required from the reader of the present note, because I raise some questions about smooth dynamical systems, which I am quite unable to attack in a proper mathematical way. But, by necessity, one must ask questions before they can be answered, and smooth dynamics is a particularly daunting subject in view of the many questions which one has absolutely no idea how to answer.

A question of great interest for physical applications is that of the long time behavior of smooth dynamical systems. To fix ideas, let f be a diffeomorphism of a compact manifold M, and let $x \in M$. What can one say about $f^n x$ for $n \to \infty$? To make the problem somewhat manageable we have to put restrictions on f and x. A natural restriction is that we may assume $x \notin N$ where N is some subset of zero Lebesgue measure of M. (By Lebesgue measure we mean a measure with smooth density w.r.t. Lebesgue, in charts of M. The concept of zero measure is then independent of choices). As far as f is concerned, one is happy if one can handle some nonempty open set in some topological space of diffeomorphisms.

The problem just outlined has been heavily researched, from Hadamard to Poincaré, through Kolmogorov and the Russian school, to Smale and his friends, to Jacob Palis and the Brasilian school, in which we may include

Floris Takens. (There are many books, among which we may quote [6], [4], [3], [5], [2]). In the simplest examples, for f in some open set (Morse-Smale diffeomorphisms), $f^n x$ tends to an attracting periodic orbit for Lebesgue almost all $x \in M$. This is what we would call *predictable behavior*. In more general situations, for f in some open set (Axiom A + No Cycles), and for Lebesgue almost all $x \in M$, $f^n x$ tends for $n \to \infty$ to an attractor A (there are finitely many such attractors in M). In this situation, and again for Lebesgue almost x, the measure

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{f^k x} \qquad (*)$$

tends weekly to a limiting measure μ when $n \to \infty$. For each attractor A there is a unique μ, which is ergodic, and called SRB measure (see [7], [8], [9]).

In another situation (which is the object of KAM theory, see for instance [16], citeMo2) for f in an open set of volume preserving diffeomorphisms and x in a set of positive Lebesgue measure, $f^n x$ is confined to an m-torus, and $(*)$ tends to an ergodic measure μ on this torus, so that the dynamical system (μ, f) is quasiperiodic.

In all these examples we have recurrent behavior, described by a measure μ. If this measure has positive entropy we say that we have chaos. Computer studies have shown that chaotic behavior is quite common, but its study remains difficult in spite of the work of Pesin, Viana, L.-S. Young, and many others (see in particular [11], [12], [13], [14], [15]).

But what about nonrecurrent behavior? Is it possible that for a large set of diffeomorphisms f, and a set of positive measure of points $x \in M$, the measure $(*)$ has no limit? This absence of limit is what we want to call *historical behavior*. This means that, as the time n tends to ∞, the point $f^n x$ keeps having new ideas about what it wants to do. Can such historical, nonrecurrent behavior occur in a stable manner? (It is easy enough to concoct an example in two dimensions where $f^n x$ oscillates ever more slowly between two fixed points, so that $(*)$ does not have a limit, but this example disappears after perturbation).

It is apparently not known if historical behavior, as described above, can occur in a persistent manner. (I am indebted to Michel Herman for confirming that). Making a conjecture would involve choosing a topological space of diffeomorphisms, etc. Rather than going into such premature details let me explore the possibility that historical behavior could be persistent. Here are a few arguments in favor of this possibility.

1. As discussed earlier in the text we have a two-dimensional example of historical behavior, easily killed by perturbation. But features (like non-hyperbolic behavior) which are nongeneric in low dimension, often become generic in higher dimension. This might happen for historical behavior.

2. Computer studies of all but the simplest dynamical systems show that the limit of $(*)$ when $n \to \infty$ is often extraordinarily slow, if it takes place at all. (In this respect see for example Gr).

3. there are physical systems which are believed to have historical behavior, and which somewhat resemble smooth dynamical systems. Specifically, spin glasses (see [10]) are infinite systems of spin with historical behavior, and Markov partitions give a representation of hyperbolic systems as infinite systems of spins. (The resemblance is not close: hyperbolic systems correspond to one-dimensional chains of spins, while one would like to consider spin glasses in higher dimension. Also, the time evolutions are totally different). Note the following intuitive view of why one expects historical behavior for spin glasses. Their evolution is pictured as a random walk in a random potential. At a given time one is trapped in a valley of the potential. Eventually one crosses a barrier to another valley. As time goes on, deeper and deeper valleys are explored where one stays trapped for longer and longer. (For a rigorous study of this situation in one dimension, see [19]. Can a smooth dynamical system emulate a random walk in a random potential (and this in a persistent manner?)

We know that there are simple dynamical systems with very inventive time evolution. In particular among cellular automata (think of Conway's Game of Life, [20]). The question here is whether or not, for smooth dynamical systems, it is possible to get rid of "historical" behavior by eliminating "negligeable sets" of diffeomorphisms and of initial conditions.

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