PERTURBATION THEORY FOR LYAPUNOV EXPONENTS OF A TORAL MAP: EXTENSION OF A RESULT OF SHUB AND WILKINSON.

by David Ruelle*.

Abstract. Starting from a hyperbolic toral automorphism times a rotation of the circle, we obtain, for a small volume preserving perturbation, an exact and rigorous second order perturbation expansion of the Lyapunov exponents.

Keywords: Lyapunov exponent, toral automorphism, hyperbolicity.

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We consider volume preserving perturbations F of a diffeomorphism $F_0 = (\Phi, J)$ of $\mathbf{T}^{m+1} = \mathbf{T}^m \times \mathbf{T}$, where Φ is a hyperbolic automorphism of $\mathbf{T}^m$, and J is a translation of $\mathbf{T}$. Writing $F = F_0 + aF'$, we shall show that the Lyapunov exponents for (F, volume) can be expanded to second order in a (Theorem 1). In particular, the central Lyapunov exponent λ^c of (F, volume), to second order in a, is generally $\neq 0$ (Corolary 11). For a special family of perturbations one obtains particularly simple formulae, first noted by Shub and Wilkinson [17]. We recover their result in Theorem 12. We deviate from [17] mostly in that we don't have differentiability of λ^c, only a second order expansion around $a = 0$. The ideas used here are largely those in Shub and Wilkinson [17], and can be appreciated in the background provided by Hirsh, Pugh and Shub [9], Burns and Wilkinson [5], Ruelle and Wilkinson [16], Niţică and Török [12], Pugh, Shub and Wilkinson [14]. Among older regularity results let us mention Katok, Knieper and Weiss [11], Flaminio [9], Ruelle [15]. For recent work concerning Lyapunov exponents, see Bonatti, Gómez-Mont and Viana [3], Avila and Bochi [2]. Closely related to the subject of the present paper are the references [4] and [6].

After completing the writing of this paper, the author received a preprint by D. Dolgopyat [7], which develops similar ideas in a more general setting, but without the specific formulas we obtain here.

1. Theorem.

Let Φ be a hyperbolic automorphism of $\mathbf{T}^m$, and $J : y \mapsto y + \alpha \pmod 1$ a translation of $\mathbf{T}$. Define $F_0 = (\Phi, J)$, and let $F = F_0 + aF'(+\text{higher order in } a)$ be a C^2 perturbation of F_0, volume preserving to first order in a. (We take $F' : \mathbf{T}^{m+1} \mapsto \mathbf{R}^{m+1}$ and $F_0\xi + aF'(\xi)$ has to be understood $\pmod 1$ in each component). Let $\lambda_1 < \lambda_2 < \ldots$ be the Lyapunov exponents of (F_0, volume) and $m_1, m_2, \ldots$ their multiplicities (the exponent $= 0$ occurs with multiplicity 1). Also let $\lambda_a^{(1)} \leq \lambda_a^{(2)} \leq \ldots$ be the Lyapunov exponents of (F, volume) repeated according to multiplicity. Then we have the second order expansion

$$\sum_{\ell=m_1+\ldots+m_{r-1}+1}^{m_1+\ldots+m_r} \lambda_a^{(\ell)} = m_r \lambda_r + a^2 L_r + o(a^2)$$

If $m_r = 1$, and writing $\lambda_r = \lambda_0^{(\ell)}$, we have

$$\lambda_a^{(\ell)} = \lambda_0^{(\ell)} + a^2 L^{(\ell)} + o(a^2)$$

(this applies in particular to $\lambda^c = \lambda_a^{(\ell)}$ for $\lambda_0^{(\ell)} = 0$).

An explicit expression for L_r can be obtained (see Proposition 9). We do not assume ergodicity of (F, volume), and therefore we use *integrated* Lyapunov exponents (averaged over the volume), see however Remark 15(a).

Because the perturbation $+aF'(+\text{higher order in } a)$ to F_0 gives only a quadratic contribution in the above formulas, the higher order terms do not contribute to order a^2. Since the higher order terms do not change our results, these terms will be omitted in what follows.

2. Normal hyperbolicity.

As in [17], we invoque the theory of normal hyperbolicity of [10]. We start from the fact that F_0 is normally hyperbolic to the smooth fibration of $\mathbf{T}^{m+1}$ by circles $\{x\} \times \mathbf{T}$. Taking some $k \geq 2$ we apply [10] Theorems (7.1), (7.2). Thus we obtain a C^1 neighborhood U of F_0 in the C^k diffeomorphisms of $\mathbf{T}^{m+1}$ such that, for $F \in U$, there is an equivariant fibration $\pi : \mathbf{T}^{m+1} \to \mathbf{T}^m$ with

$$\pi F = \Phi \pi$$

The fibers $\pi^{-1}\{x\}$ are C^k circles forming a continuous fibration of $\mathbf{T}^{m+1}$ (this fibration is in general not smooth). Furthermore there is a TF-invariant continuous splitting of $T\mathbf{T}^{m+1}$ into three subbundles:

$$T\mathbf{T}^{m+1} = E^s + E^u + E^c$$

such that E^c is 1-dimensional tangent to the circles $\pi^{-1}\{x\}$, E^s is m^s-dimensional contracting and E^u is m^u-dimensional expanding for TF.

If $\lambda_r < 0$ (and F is in a suitable C^1-small neighborhood U of F_0), we can introduce a continuous vector subbundle E^r of $T\mathbf{T}^{m+1}$ which consists of vectors contracting under TF^n faster than $(\lambda_r + \epsilon)^n$ where $\epsilon > 0$ and $\lambda_r + \epsilon < \lambda_{r+1}$. In fact E^r is a hyperbolic (attracting) fixed point for the action induced by TF^{-1} on the bundle of $m_1 + \ldots + m_r$ dimensional linear subspaces of $T\mathbf{T}^{m+1}$ (over F^{-1} acting on T^{m+1}).

If $\lambda_r > 0$, replacement of F by F^{-1} similarly yields a continuous subbundle $\bar{E}^r$ of $m_r + \ldots$ dimensional subspaces.

3. Proposition.

Assume that F is of class C^k, $k \geq 2$, and that F is C^k close to F_0. The bundles E^r, $\bar{E}^r$, when restricted to a circle $\pi^{-1}\{x\}$ are of class C^{k-1}, continuously in x.

If $\mathcal{G}$ denotes the (Grassmannian) manifold of $m_1+\ldots+m_r$ dimensional linear subspaces of $\mathbf{R}^{m+1}$, we may identify the bundle of $m_1 + \ldots + m_r$ dimensional linear subspaces of $T\mathbf{T}^{m+1}$ with $\mathbf{T}^{m+1} \times \mathcal{G}$. We denote by $\mathcal{E} \in \mathcal{G}$ the spectral subspace of the matrix defining Φ corresponding to the smallest $m_1+\ldots+m_r$ eigenvalues (in absolute value, and repeated according to multiplicity).

If $\mathcal{F}_0$ is the action defined by TF_0 on $T\mathbf{T}^{m+1} \times \mathcal{G}$, the circles $\{x\} \times \mathbf{T} \times \{\mathcal{E}\}$ form an $\mathcal{F}_0$ invariant fibration of $\mathbf{T}^{m+1} \times \{\mathcal{E}\}$, to which $\mathcal{F}_0$ is normally hyperbolic. If F is C^k close to F_0, the corresponding C^{k-1} action $\mathcal{F}$ is normally hyberbolic to a pertubed fibration where $\{x\} \times \mathbf{T} \times \{\mathcal{E}\}$ is replaced by $E^r|\pi^{-1}\{x\}$. According to [10] Theorem 7.4, Corollary (8.3) and the following Remark 2, we find that the C^{k-1} circle $E^r|\pi^{-1}\{x\} \subset \mathbf{T}^{m+1} \times \mathcal{G}$ depends continuously on $x \in \mathbf{T}^{m+1}$. Similarly for $\bar{E}$. $\square$

Note that in [17], the C^r section theorem is used in a similar situation, giving estimates uniform in x. However, continuity in x (not just uniformity) will be essential for us in what follows.

4. Corollary.

The splitting $T\mathbf{T}^{m+1} = E^s + E^u + E^c$ *when restricted to a circle* $\pi^{-1}\{x\}$ *is of class* C^{k-1}, *continuously in* x.

It is clear that $E^c|\pi^{-1}\{x\}$ is of class C^{k-1} because it is the tangent bundle to the C^k circle $\pi^{-1}\{x\}$. As to E^s, E^u, they are special cases of E^r, $\bar{E}^r$. ∎

Notation.

Remember that $F = F_0 + aF'$, and fix F'. We shall use the notation π_a, E^r_a, ...to indicate the a-dependence of π, E^r, ...

5. Proposition.

For small $\epsilon > 0$ *there is a continuous function* $x \mapsto \gamma_x$ *from* $\mathbf{T}^m$ *to* $C^k(\mathbf{T} \times (-\epsilon, \epsilon) \to \mathbf{T}^m)$ *such that* $\gamma_x(y, 0) = 0$ *and* $\pi_a^{-1}\{x\} = \{(x + \gamma_x(y, a), y) : y \in \mathbf{T}\}$.

To see this define $\tilde{F} : \mathbf{T}^{m+1} \times (-\epsilon, \epsilon) \to \mathbf{T}^{m+1} \times (-\epsilon, \epsilon)$ by $\tilde{F}(\xi, a) = ((F_0 + aF')(\xi), a)$ and observe that $\tilde{F}$ is normally hyperbolic to the 2-dimensional manifolds

$$\cup_{a\in(-\epsilon,\epsilon)}(\pi_a^{-1}\{x\}, a)$$

and these are thus C^k 2-dimensional submanifolds of $\mathbf{T}^{m+1} \times (-\epsilon, \epsilon)$. ∎

We may in the same manner replace $\pi_a^{-1}\{x\}$ by $\cup_{a\in(-\epsilon,\epsilon)}(\pi_a^{-1}\{x\}, a)$ in Proposition 3 and Corollary 4. Writing E_a for E^r_a, $\bar{E}^r_a$, E^s_a, E^u_a, E^c_a, we obtain that $(\cdot, a) \mapsto E_a(\cdot)$, when restricted from $\mathbf{T}^{m+1} \times (-\epsilon, \epsilon)$ to $\cup_{a\in(-\epsilon,\epsilon)}(\pi_a^{-1}\{x\}, a)$ is of class C^{k-1}. We rephrase this as follows:

6. Proposition.

The map

$$x \mapsto \{(y, a) \mapsto E_a(x + \gamma_x(y, a), y)\}$$

where E_a *stands for* E^r_a, $\bar{E}^r_a$, E^s_a, E^u_a, E^c_a, *is continuous* $\mathbf{T}^m \to \mathrm{C}^{k-1}(\mathbf{T} \times (-\epsilon, \epsilon) \to$ Grassmannian of $\mathbf{R}^{m+1})$ *where we have used the identification* $T\mathbf{T}^{m+1} = \mathbf{T}^{m+1} \times \mathbf{R}^{m+1}$. ∎

Notation.

From now on we write E_a for E^r_a, $\bar{E}^r_a$, E^s_a, E^u_a, E^c_a. When $a = 0$, E_0 is a trivial subbundle of $T\mathbf{T}^{m+1} = \mathbf{T}^{m+1} \times \mathbf{R}^{m+1}$, and we shall write $E_0 = \mathbf{T}^{m+1} \times \mathcal{E}$, denoting thus by $\mathcal{E}$ a spectral subspace of the matrix on $\mathbf{R}^{m+1}$ defining $(\Phi, 1)$. We denote by $\mathcal{E}^\perp$ the complementary spectral subspace.

Taking $k = 2$ we have then:

7. Corollary.

There are linear maps $G(x, y), R(x, y, a) : \mathcal{E} \to \mathcal{E}^\perp$ *such that* $G(x, y)$ *depends continuously on* $(x, y) \in \mathbf{T}^m \times \mathbf{T}$, $R(x, y, a)$ *on* $(x, y, a) \in \mathbf{T}^m \times \mathbf{T} \times (-\epsilon, \epsilon)$,

$$E_a(x + \gamma_x(y, a), y) = \{X + aG(x, y)X + R(x, y, a)X : X \in \mathcal{E}\}$$

and $||R(x,y,a)||$ *is* $o(a)$ *uniformly in* x,y. $\square$

Notice now that, if $\tilde{x} = \pi_a(x,y)$, then $x = \tilde{x} + \gamma_{\tilde{x}}(y,a)$, where $\gamma_{\tilde{x}}(y,a) = O(a)$. Now

$$E_a(x,y) = E_a(\tilde{x} + \gamma_{\tilde{x}}(y,a), y) = \{X + aG(\tilde{x},y)X + R(\tilde{x},y,a)X : X \in \mathcal{E}\}$$

differs from

$$E_a(x + \gamma_x(y,a), y) = \{X + aG(x,y)X + R(x,y,a)X : X \in \mathcal{E}\}$$

by the replacement $\tilde{x} \to x$ in the right-hand side, and since $\mathrm{dist}(\tilde{x},x) = O(a)$, we find that $\mathrm{dist}(E_a(x,y), E_a(x + \gamma_x(y,a), y)) = o(a)$. Therefore, changing the definition of R, we can again write:

8. Corollary.

There are linear maps $G(x,y), R(x,y,a) : \mathcal{E} \to \mathcal{E}^\perp$*, depending continuously on their arguments, such that*

$$E_a(x,y) = \{X + aG(x,y)X + R(x,y,a)X : X \in \mathcal{E}\}$$

and $||R(x,y,a)||$ *is* $o(a)$ *uniformly in* x,y. $\square$

We may write $T_\xi F = T_\xi(F_0 + aF') = D_0 + aD'(\xi)$ where D_0 does not depend on ξ and preserves the decomposition $T_\xi M = \mathcal{E} + \mathcal{E}^\perp$. If we apply TF to an element $X + aGX + RX$ of E_a (as in Corollary 8) we obtain $X_1 +$ element of $\mathcal{E}^\perp \in E_a$, with $X_1 \in \mathcal{E}$:

$$X_1 = D_0 X + aD'X + a^2 D'GX + aD'RX \qquad \text{projected on } \mathcal{E} \tag{1}$$

Under $(TF)^\wedge$, the volume element θ in $E_a(\xi)$ is multiplied by a factor $M(\xi,a)$, and the projection in $\mathcal{E}$ of $(TF)^\wedge\theta$ is equal to the projection in $\mathcal{E}$ of θ multiplied by a factor $N(\xi,a)$ such that

$$M(\xi,a) = N(\xi,a) + \ell_a(\xi) - \ell_a(F\xi)$$

for suitable ℓ_a. We may compute N from (1):

$$N(\xi,a) = N_{(0)} + aN_{(1)}(\xi) + a^2 N_{(2)}(\xi) + o(a^2)$$

To proceed we take now $E_a = E_a^r$, and assume $\lambda_r < 0$. We have then, writing $d\xi$ for the volume element in T^{m+1},

$$L_a = \sum_{\ell=1}^{m_1+\ldots+m_r} \lambda_a^{(\ell)} = \int d\xi \, \log M(\xi,a) = \int d\xi \, \log N(\xi,a)$$

$$= L_{(0)} + aL_{(1)}(\xi) + a^2 L_{(2)}(\xi) + o(a^2) \tag{2}$$

More precisely, we shall prove

9. Proposition.

If $\lambda_r < 0$, we have

$$\sum_{\ell=1}^{m_1+\ldots+m_r} \lambda_a^{(\ell)} = \sum_{k=1}^{r} m_k \lambda_k + a^2 L + o(a^2)$$

where

$$L = \frac{1}{2} \sum_{n=-\infty}^{\infty} \int d\xi \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(\xi)) \mathrm{Tr}_{\mathcal{E}}(D_0^{-1} D'(F_0^n \xi)) \geq 0$$

and $\mathrm{Tr}_{\mathcal{E}}$ *is defined as follows. Let* $\mathcal{E}$ *be the spectral subspace of the matrix* D_0 *(defining* $(\Phi, 1)$ *in* $\mathbf{R}^{m+1}$*) corresponding to the smallest* $m_1 + \ldots + m_r$ *eigenvalues (in absolute value, and repeated according to multiplicity). Also let* $\mathcal{E}^\perp$ *be the complementary spectral subspace. We define* P *to be the projection on* $\mathcal{E}$ *parallel to* $\mathcal{E}^\perp$*, and write* $\mathrm{Tr}_{\mathcal{E}} \ldots = \mathrm{Tr}_{\mathbf{R}^{m+1}} P \ldots P$*.*

The convergence of the series defining L is exponential as will result from the proof. We postpone showing that $L \geq 0$ until Remark 15(b).

The proposition is obtained by comparing formula (2) with the formula (5) below, which we shall obtain by a second order perturbation calculation.

To first order in a we have

$$F^n = (F_0 + aF')^n = F_0^n + a \sum_{j=1}^{n} F_0^{n-j} \circ F' \circ F_0^{j-1}$$

hence

$$T_\xi F^n = D_0^n + a \sum_{j=1}^{n} D_0^{n-j} D'(F^{j-1}\xi) D_0^{j-1}$$

If we apply TF^n to $X + aGX + RX \in E_a$ we obtain X_n+ element of $\mathcal{E}^\perp \in E_a$, with $X_n \in \mathcal{E}$. To zero-th order in a, $X_n = D_0^n X$, so we may write to first order $X_n = D_0^n X + aY_n(\xi)$. Therefore, to first order in a,

$$D_0^n X + aY_n(\xi) + aG(F^n\xi) D_0^n X = D_0^n X + a \sum_{j=1}^{n} D_0^{n-j} D'(F^{j-1}\xi) D_0^{j-1} X + a D_0^n G(\xi) X$$

and, taking the components along $\mathcal{E}^\perp$,

$$G(F^n\xi) D_0^n X = \sum_{j=1}^{n} D_0^{n-j} D'_\perp(F^{j-1}\xi) D_0^{j-1} X + D_0^n G(\xi) X$$

where $D'_\perp(.)$ is $D'(.)$ followed by taking the component along $\mathcal{E}^\perp$, or

$$\sum_{j=1}^{n} D_0^{-j} D'_\perp(F^{j-1}\xi) D_0^{j-1} X + G(\xi) X = D_0^{-n} G(F^n\xi) D_0^n X$$

When $n \to \infty$, the right-hand side tends to zero (exponentially fast, remember that $X \in \mathcal{E}$, $GX \in \mathcal{E}^\perp$). Therefore (to order 0 in a)

$$G(\xi)X = -\sum_{j=1}^{\infty} D_0^{-j} D'_\perp(F^{j-1}\xi) D_0^{j-1} X$$

which we shall use in the form

$$G(\xi)X = -\sum_{n=0}^{\infty} D_0^{-n-1} D'_\perp(F_0^n\xi) D_0^n X \tag{3}$$

where we have written F_0^n instead of F^n since G is evaluated to order 0 in a. (The right-hand side is an exponentially convergent series).

Returning to (1) we see that, to second order in a,

$$X_1 = D_0 X + aD'(\xi)X + a^2 D'(\xi)G(\xi)X \qquad \text{projected on } \mathcal{E}$$

$$= D_0(1 + aD_0^{-1}D'(\xi) + a^2 D_0^{-1}D'(\xi)G(\xi))X \qquad \text{projected on } \mathcal{E}$$

Let now $(u^{(i)})$ and $(u^{(i)\perp})$ be conjugate bases of $\mathcal{E}$. Also let $\delta^{(i)}$ for $i = 1, \ldots, m_1 + \ldots + m_r$ be the eigenvalues of D_0 restricted to $\mathcal{E}$. Then, to second order in a,

$$N(\xi, a) \wedge_1^{m_1+\ldots+m_r} u^{(\ell)}$$

is, up to a factor of absolute value 1,

$$\Big(\prod_{\ell=1}^{m_1+\ldots+m_r} \delta^{(\ell)}\Big)[1 + a \sum_{i=1}^{m_1+\ldots+m_r} (u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(i)})$$

$$+a^2 \sum_{i<j}((u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(i)})(u^{(j)\perp}, D_0^{-1}D'(\xi)u^{(j)})$$

$$-(u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(j)})(u^{(j)\perp}, D_0^{-1}D'(\xi)u^{(i)})) + a^2 \sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)G(\xi)u^{(i)})] \wedge_\ell u^{(\ell)}$$

so that

$$N(\xi, a) = \Big(\prod_{\ell=1}^{m_1+\ldots+m_r} |\delta^{(\ell)}|\Big)[1 + \{a \sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(i)})$$

$$+a^2 \sum_{i<j}((u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(i)})(u^{(j)\perp}, D_0^{-1}D'(\xi)u^{(j)})$$

$$-(u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(j)})(u^{(j)\perp}, D_0^{-1}D'(\xi)u^{(i)})) + a^2 \sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)G(\xi)u^{(i)})\}]$$

Since $\log|\delta^{(\ell)}| = \lambda_0^{(\ell)}$ we obtain, to second order in a,

$$L_a = \int d\xi \, \log N(\xi, a) = m_1\lambda_1 + \ldots + m_r\lambda_r + \int d\xi \, [\{\ldots\} - \frac{a^2}{2}(\sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(i)}))^2]$$

where $\{\ldots\}$ has the same meaning as above. Write

$$\Psi_i(\sum_\ell \xi_\ell u^{(\ell)}) = (u^{(i)\perp}, D_0^{-1}F'(\sum_\ell \xi_\ell u^{(\ell)}))$$

The first term of $\int d\xi \, \{\ldots\}$ is

$$a\sum_i \int d\xi \, (u^{(i)\perp}, D_0^{-1}TF'(\xi)u^{(i)}) = a\sum_i \int d\xi \, \frac{\partial}{\partial\xi_i}\Psi_i$$

which vanishes because $\int d\xi \, \frac{\partial}{\partial\xi_i}\ldots = 0$. The next term in $\int d\xi \, \{\ldots\}$ is

$$a^2\sum_{i<j}\int d\xi \, ((\frac{\partial\Psi_i}{\partial\xi_i})(\frac{\partial\Psi_j}{\partial\xi_j}) - (\frac{\partial\Psi_i}{\partial\xi_j})(\frac{\partial\Psi_j}{\partial\xi_i})) = a^2\sum_{i<j}\int d\xi \, (\frac{\partial}{\partial\xi_i}(\Psi_i\frac{\partial\Psi_j}{\partial\xi_j}) - \frac{\partial}{\partial\xi_j}(\Psi_i\frac{\partial\Psi_j}{\partial\xi_i}))$$

which vanishes as above. Thus we are left with

$$L_a - (m_1\lambda_1 + \ldots + m_r\lambda_r)$$

$$= a^2\int d\xi \, [\sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)G(\xi)u^{(i)}) - \frac{1}{2}(\sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(i)}))^2] \qquad (4)$$

and we may write, using (3),

$$\sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)G(\xi)u^{(i)}) = -\sum_{n=0}^{\infty}\sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)D_0^{-n-1}D'_\perp(F_0^n)D_0^n u^{(i)})$$

$$= -\sum_{n=0}^{\infty}\sum_i \sum_j^{*} (u^{(i)\perp}, D_0^{-1}D'(\xi)u^{(j)})(u^{(j)\perp}, D_0^{-n-1}D'(F_0^n\xi)D_0^n u^{(i)})$$

where we have introduced conjugate bases $(u^{(j)})$, $(u^{(j)\perp})$ of $\mathcal{E}^\perp$, indexed by $j = m_1 + \ldots + m_r + 1, \ldots, m+1$, and $\sum_i$ is over $i \le m_1 + \ldots + m_r$, $\sum_j^*$ is over $j \ge m_1 + \ldots + m_r + 1$. The above expression is also

$$= -\sum_{n=0}^{\infty}\sum_i \sum_j^{*} \frac{\partial}{\partial\xi_j}(u^{(i)\perp}, D_0^{-1}F'(\sum_\ell \xi_\ell u^{(\ell)}))\frac{\partial}{\partial\xi_i}(u^{(j)\perp}, D_0^{-n-1}F'(F_0^n\sum_\ell \xi_\ell u^{(\ell)}))$$

and integration by part gives thus

$$\int d\xi \, \sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)G(\xi)u^{(i)})$$

$$= -\sum_{n=0}^{\infty}\int d\xi\,\sum_i \frac{\partial}{\partial\xi_i}(u^{(i)\perp}, D_0^{-1}F'(\sum_\ell \xi_\ell u^{(\ell)}))\sum_j^{*}\frac{\partial}{\partial\xi_j}(u^{(j)\perp}, D_0^{-n-1}F'(F_0^n\sum_\ell \xi_\ell u^{(\ell)}))$$

$$= -\sum_{n=0}^{\infty}\int d\xi\,\mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathcal{E}^\perp}(D_0^{-n-1}D'(F_0^n\xi)D_0^n)$$

$$= -\sum_{n=0}^{\infty}\int d\xi\,\mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathcal{E}^\perp}(D_0^{-1}D'(F_0^n\xi))$$

(here $\mathrm{Tr}_{\mathcal{E}^\perp} = \mathrm{Tr}_{\mathbf{R}^{m+1}} - \mathrm{Tr}_{\mathcal{E}}$). The fact that $F = F_0 + aF'$ is volume preserving (to first order in a) is expressed by $\mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(\xi)) = 0$ hence

$$\int d\xi\,\sum_i (u^{(i)\perp}, D_0^{-1}D'(\xi)G(\xi)u^{(i)})$$

$$= \sum_{n=0}^{\infty}\int d\xi\,\mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(F_0^n\xi))$$

and introducing this in (4) yields

$$L_a - (m_1\lambda_1 + \ldots + m_r\lambda_r)$$

$$= a^2[\sum_{n=1}^{\infty}\int d\xi\,\mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(F_0^n\xi)) + \frac{1}{2}\int d\xi\,(\mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi)))^2]$$

$$= \frac{a^2}{2}\sum_{n=-\infty}^{\infty}\int d\xi\,\mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(F_0^n\xi)) \tag{5}$$

where the last step used the invariance of $d\xi$ under F_0^n. □

10. Proof of Theorem 1.

We use Proposition 9, the corresponding result with F replaced by F^{-1}, and the fact that $\sum_{\ell=1}^m \lambda_a^{(\ell)} = 0$ (because F is volume preserving). This gives an estimate of all the sums of $\lambda_a^{(\ell)}$ that occur in Theorem 1. □

11. Corollary.

In the situation of Theorem 1, the central Lyapunov exponent is

$$\lambda^c = \frac{a^2}{2}\sum_{-\infty}^{\infty}\int d\xi\,[\mathrm{Tr}^u(D_0^{-1}D'(\xi))\mathrm{Tr}^u(D_0^{-1}D'(F_0^n\xi)) - \mathrm{Tr}^s(D_0^{-1}D'(\xi))\mathrm{Tr}^s(D_0^{-1}D'(F_0^n\xi))]$$

$$= \frac{a^2}{2}\sum_{-\infty}^{\infty}\int d\xi\,[\mathrm{Tr}^s(D_0^{-1}D'(\xi)) - \mathrm{Tr}^u(D_0^{-1}D'(\xi))]\mathrm{Tr}^c(D_0^{-1}D'(F_0^n\xi))$$

where Tr^s, Tr^u, Tr^c *denote the traces over the spectral subspaces* $\mathcal{E}^s$, $\mathcal{E}^u$, $\mathcal{E}^c$ *of* D_0 *corresponding to eigenvalues* < 1, > 1, *or* $= 1$ *in absolute value* ($\mathcal{E}^c$ *is one dimensional).*

Since F preserves the volume, the sum of all Lyapunov exponents vanishes. Therefore λ^c is minus the sum of the negative Lyapunov exponents, given by (5), minus the sum of the positive Lyapunov exponents. Note that replacing F by F^{-1}, $\mathcal{E}^s$ by $\mathcal{E}^u$ (and, to the order considered, $D'(\xi)$ by $-D'(\xi)$) replaces the sum of the negative Lyapunov exponents by minus the sum of the positive exponents. This gives the first formula for λ^c.

To obtain the second formula, express $\mathrm{Tr}^u\mathrm{Tr}^u - \mathrm{Tr}^s\mathrm{Tr}^s$ in terms of $\mathrm{Tr}^u \pm \mathrm{Tr}^s$, and remember that (because F preserves the volume) $\mathrm{Tr}^s + \mathrm{Tr}^u + \mathrm{Tr}^c = 0$ when applied to $D_0^{-1}D'(\xi)$. $\square$

The above formula (5) takes a particularly simple form in a special case described in the next theorem.

12. Theorem.

Let Φ *be a hyperbolic automorphism of* $\mathbf{T}^m$, *with stable and unstable dimensions* m^s *and* $m^u = m - m^s$, *and with entropy* λ_0^u. *Let* $J : y \to y + \alpha \pmod 1$ *be a translation of* $\mathbf{T}$, *and* $\phi : \mathbf{T}^m \to \mathbf{T}$ *a group homomorphism* $\neq 0$. *Finally let* $\psi : \mathbf{T} \to \mathbf{R}^m$ *be a nullhomotopic* C^2 *function.*

Define $h, g_a : \mathbf{T}^m \times \mathbf{T} \to \mathbf{T}^m \times \mathbf{T}$ *by*

$$h\binom{x}{y} = \binom{\Phi x}{Jy + \phi\Phi x - \phi x} \qquad , \qquad g_a\binom{x}{y} = \binom{x + a\psi(y) \pmod 1}{y}$$

and let $f_a = g_a \circ h$.

Denote by λ_a^s *(resp.* λ_a^u*) the sum of the smallest* m^s *(resp. the largest* m^u*) Lyapunov exponents for* (f_a, volume). *Also let* $\lambda_a^c = -\lambda_a^s - \lambda_a^u$ *be the "central exponent". Then* λ_a^s, λ_a^u, λ_a^c *have expansions of order 2 in a:*

$$\lambda_a^s = -\lambda_0^u + \frac{a^2}{2}\int_{\mathbf{T}} dy\,((\nabla\phi)\psi'^s(y))^2 + o(a^2)$$

$$\lambda_a^u = \lambda_0^u - \frac{a^2}{2}\int_{\mathbf{T}} dy\,((\nabla\phi)\psi'^u(y))^2 + o(a^2)$$

$$\lambda_a^c = \frac{a^2}{2}\int_{\mathbf{T}} dy\,[((\nabla\phi)\psi'^u(y))^2 - ((\nabla\phi)\psi'^s(y))^2] + o(a^2)$$

Here $\psi'^s(y)$ *and* $\psi'^u(y)$ *are the components of the derivative* $\psi'(y) \in \mathbf{R}^m$ *in the stable and unstable subspaces* $\mathcal{E}^s$ *and* $\mathcal{E}^u$ *for* Φ. *Also, we have used* $\nabla\phi : \mathbf{R}^m \to \mathbf{R}$ *to denote the derivative of the map* $\phi : \mathbf{T}^m \to \mathbf{T}$ *with the obvious identifications.*

This theorem is a simple (but nontrivial) extension of the result proved by Shub and Wilkinson [17]. In the situation that they consider $\Phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $J =$identity, $\phi = (1, 0)$,

$\psi' = \psi'^u$. [Remark that, in the notation of [17], $u_0 = ((1,1).v_0)/(m-1) = ((1,0).v_0)$ so that the formula given in Proposition II of [17] agrees with our result above].

Notation.

We shall henceforth omit the (mod 1). We shall keep ∇ to denote the derivative in $\mathbf{T}^m$. With obvious abuses of notation, the reader may find it convenient to think of Φ or $\nabla\Phi$ as an $m \times m$ matrix (with integer entries and determinant ± 1), and ϕ or $\nabla\phi$ as a row m-vector (with integer entries not all zero).

13. Reformulation of the problem.

Note that $f_a^{-1} = h^{-1} \circ g_a^{-1}$ where h^{-1}, g_a^{-1} are obtained from h, g_a by the replacements $\Phi, J, \phi, \psi \to \Phi^{-1}, J^{-1}, \phi, -\psi$. These replacements also interchange the stable and unstable subspaces for Φ and replace λ^s, λ^u by $-\lambda^u$, $-\lambda^s$. Therefore the formula for λ^u in the theorem follows from the formula for λ^s. And the formula for $\lambda^c = -\lambda^s - \lambda^u$ also follows. To complete the proof of the theorem we turn now to the formula for λ^s.

Define

$$\hat\phi\binom{x}{y} = \binom{x}{y + \phi x}$$

then

$$F_0\binom{x}{y} = \hat\phi^{-1} h \hat\phi\binom{x}{y} = \binom{\Phi x}{Jy}$$

$$\hat g_a\binom{x}{y} = \hat\phi^{-1} g_a \hat\phi\binom{x}{y} = \binom{x + a\psi(y + \phi x)}{y - a(\nabla\phi)\psi(y + \phi x)}$$

so that

$$F\binom{x}{y} = \hat\phi^{-1} f_a \hat\phi\binom{x}{y} = \hat g_a F_0\binom{x}{y} = \binom{\Phi x + a\psi(Jy + \phi\Phi x)}{Jy - a(\nabla\phi)\psi(Jy + \phi\Phi x)}$$

Finally, $F = F_0 + aF'$ with

$$F_0\binom{x}{y} = \binom{\Phi x}{Jy} \qquad , \qquad F'\binom{x}{y} = \binom{\psi(Jy + \phi\Phi x)}{-(\nabla\phi)\psi(Jy + \phi\Phi x)}$$

Since F is conjugate (linearly) to f_a, we may compute λ^s from F instead of f_a.

14. Proof of Theorem 12.

Write $\mathbf{R}^{m+1} = \mathcal{E}^s + \mathcal{E}^u + \mathbf{R}$. We shall apply Proposition 9 with $\mathcal{E} = \mathcal{E}^s$, $\mathcal{E}^\perp = \mathcal{E}^u + \mathbf{R}$. Using $\xi = (x, y)$ and $X \in \mathcal{E}^s$, $Y \in \mathcal{E}^u$, $Z \in \mathbf{R}$ we may write

$$D_0\binom{X+Y}{Z} = \binom{(\nabla\Phi)(X+Y)}{Z}$$

$$D'(\xi)\binom{X+Y}{Z} = \binom{\psi'(Jy + \phi\Phi x)((\nabla\phi\Phi)(X+Y) + Z)}{-(\nabla\phi)\psi'(Jy + \phi\Phi x)((\nabla\phi\Phi)(X+Y) + Z)}$$

where ψ' denotes the derivative of ψ. Therefore

$$\mathrm{Tr}_{\mathcal{E}}(D'(\xi)D_0^{-1}) = (\nabla\phi)\psi'^s(Jy + \phi\Phi x)$$

and (5)contains the integrals

$$\int d\xi\,\mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(F_0^n\xi))$$

$$=\int d\xi\,[(\nabla\phi)\psi'^s(Jy+\phi\Phi x)][(\nabla\phi)\psi'^s(J^{n+1}y+\phi\Phi^{n+1}x)]$$

Performing a change of variables $\bar{x}=\Phi x$, $\bar{y}=Jy+\phi\Phi x$ we find that this is

$$=\int d\bar{x}\,d\bar{y}\,[(\nabla\phi)\psi'^s(\bar{y})][(\nabla\phi)\psi'^s(J^n\bar{y}+\phi\Phi^n\bar{x}-\phi\bar{x})]$$

We claim that this last integral vanishes unless $n=0$. This is because, if $n\neq 0$,

$$\int d\bar{x}\,\psi'(J^n\bar{y}+\phi\Phi^n\bar{x}-\phi\bar{x})=0$$

Indeed, $\phi\Phi^n\bar{x}-\phi\bar{x}$ is a linear combination with integer coefficients of the components $\bar{x}_1,\ldots,\bar{x}_m$ of $\bar{x}$, and the coefficients do not all vanish because $\phi\Phi^n=\phi$ is impossible (Φ is hyperbolic and $\phi\neq 0$). Integrating the derivative ψ' with respect to a variable $\bar{x}_j$ really occuring in $\phi\Phi^\ell\bar{x}-\phi\bar{x}$ gives zero as announced.

Returning to (5) we have thus

$$\lambda_a^s+\lambda_0^u=\frac{a^2}{2}\int d\xi\,(\mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi)))^2$$

$$=\frac{a^2}{2}\int d\bar{y}((\nabla\phi)\psi'^s(\bar{y}))^2$$

which is the formula given for λ_a^s in Theorem 12. And according to Section 13 this completes our proof. ∎

15. Final remarks.

(a) Shub and Wilkinson [17] showed that close to a diffeomorphism (hyperbolic automorphism Φ of $\mathbf{T}^2$) $\times$ (identity on $\mathbf{T}$) there is a C^1 open set of ergodic volume preserving C^2 diffeomorphisms of $\mathbf{T}^3$ with central Lyapunov exponent $\lambda^c>0$. They remark that their result extends to the situation where Φ is a hyperbolic automorphism of $\mathbf{T}^m$ with one-dimensional expanding eigenspace. More generally, if Φ is any hyperbolic automorphism of $\mathbf{T}^m$, Theorem 12 gives close to (Φ, rotation of $\mathbf{T}$) in $\mathrm{C}^2(\mathbf{T}^{m+1})$ a diffeomorphism F with $\lambda^c>0$. Since λ^c is given by an integral over the volume of a local "central" stretching exponent, we have $\lambda^c>0$ in a C^1 neighborhood of F. But by a result of Dolgopyat and Wilkinson [8] (Corollary 0.5), stable ergodicity is here C^1 open and dense in the C^2 volume preserving diffeomorphisms (C^1 is improved to C^k in [12]): we have center bunching and stable dynamical coherence because we consider perturbations of (Φ, rotation of $\mathbf{T}$) for which the center foliation is C^1, see [10], [13]. In conclusion, close to (hyperbolic automorphism Φ of $\mathbf{T}^m$)$\times$(rotation on $\mathbf{T}$) there is a C^1 open set V of ergodic volume preserving C^2

diffeomorphisms of $\mathbf{T}^{m+1}$ with central Lyapunov exponent $\lambda^c > 0$ (or also with $\lambda^c < 0$). In particular, if $F \in V$, the conditional measures of the volume on the circles $\pi^{-1}\{x\}$ are atomic, as discussed in [16].

(b) The coefficient L in Proposition 9 is ≥ 0. Consider indeed the unitary operator U defined by $U\psi = \psi \circ F$ on $L^2(\mathbf{T}^{m+1}, \text{volume})$, and let $E(.)$ be the corresponding spectral measure, so that

$$U = \int_{\mathbf{T}} e^{2\pi i\theta} E(d\theta)$$

If $\psi(\xi) = \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))$ we have a measure $\nu \geq 0$ on $\mathbf{T}$ defined by $\nu(d\theta) = (\psi, E(d\theta)\psi)$ and the Fourier coefficients

$$c_n = \int e^{2\pi n i\theta}\nu(d\theta) = \int d\xi \, \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))(D_0^{-1}D'(F_0^n\xi))$$

of this measure tend to zero exponentially. Therefore $\nu(d\theta) = \rho(\theta)d\theta$ has a smooth density ρ and

$$L = \frac{1}{2}\sum_{n=-\infty}^{\infty} c_n = \frac{1}{2}\rho(0) \geq 0$$

(c) Suppose now that F is not necessarily a volume preserving perturbation of F_0. We may still hope that F has an SRB measure ρ_a. If F_0 were hyperbolic, we would have an expansion

$$\rho_a = \rho_0 + a\delta + o(a)$$

(see [15]) with $\rho_0 =$ Lebesgue measure and δ a distribution. For smooth Ψ, $\delta(\Psi)$ is given (because ρ_0 is Lebesgue measure) by the simple formula (see [15])

$$\delta(\Psi) = -\sum_0^{\infty} \rho_0((\Psi \circ F_0^n).\mathrm{div}(F' \circ F_0^{-1}))$$

Similarly (replacing F by F^{-1}, hence F_0, $D_0^{-1}D'(\xi)$ by F_0^{-1}, $-D'(F_0^{-1}\xi)D_0^{-1}$ we see that the anti-SRB state has an expansion

$$\bar{\rho}_a = \rho_0 + a\bar{\delta} + o(a)$$

with

$$\bar{\delta}(\Psi) = \sum_{n=1}^{\infty}\int d\xi \, \Psi(F_0^{-n}\xi)\mathrm{Tr}_{\mathbf{R}^{m+1}}(D'(F_0^{-1}\xi)D_0^{-1})$$
$$= \sum_{n=0}^{\infty}\int d\xi \, \Psi(F_0^{-n}\xi)\mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(\xi))$$

We can now estimate the Lyapunov exponents for (F, ρ_a) to second order in a even though we are not sure of the existence of the SRB measure ρ_a. We simply assume that we can use the formula for $\delta(\Psi)$. Going through the proof of Proposition 9 we have to replace $\int d\xi\, \log N(\xi,a)$ by $\rho_a(\log N(.,a))$ and (to second order in a) this adds to the right-hand side of (4) a term

$$-a^2\sum_{n=1}^{\infty}\int d\xi\ \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(\xi))$$

Taking into account the integrations by part we obtain now instead of (5) the formula

$$L_a-(m_1\lambda_1+\ldots+m_r\lambda_r)=\frac{a^2}{2}\sum_{n=-\infty}^{\infty}\int d\xi\ \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(F_0^n\xi))$$

$$-a^2\sum_{n=-\infty}^{\infty}\int d\xi\ \mathrm{Tr}_{\mathcal{E}}(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(F_0^n\xi)) \tag{6}$$

Let a^2L^s, a^2L^u, a^2L^c be the a^2 contributions to the sum of the noncentral negative, noncentral positive, and the central Lyapunov exponents for the SRB measure. We obtain a^2L^s from (6) when $n_r=n^s$. A similar calculation gives a^2L^u (it is convenient here to work via the anti-SRB measure, then replace F by F^{-1}). Estimating the average expansion coefficient gives $a^2(L^s+L^u+L^c)=\rho_a(\log\det(D_0+aD'(.))$, hence $L^s+L^u+L^c$, hence L^c. The results are

$$L^s=\frac{1}{2}\sum_{n=-\infty}^{\infty}\int d\xi\ \mathrm{Tr}^s(D_0^{-1}D'(\xi))\mathrm{Tr}^s(D_0^{-1}D'(F_0^n\xi))$$

$$-\sum_{n=-\infty}^{\infty}\int d\xi\ \mathrm{Tr}^s(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(F_0^n\xi))$$

$$L^u=-\frac{1}{2}\sum_{n=-\infty}^{\infty}\int d\xi\ \mathrm{Tr}^u(D_0^{-1}D'(\xi))\mathrm{Tr}^u(D_0^{-1}D'(F_0^n\xi))$$

$$L^c=-\frac{1}{2}\sum_{n=-\infty}^{\infty}\int d\xi\ \mathrm{Tr}^c(D_0^{-1}D'(\xi))\mathrm{Tr}^c(D_0^{-1}D'(F_0^n\xi))$$

$$-\sum_{n=-\infty}^{\infty}\int d\xi\ \mathrm{Tr}^c(D_0^{-1}D'(\xi))\mathrm{Tr}^u(D_0^{-1}D'(F_0^n\xi))$$

$$L^s+L^u+L^c=-\frac{1}{2}\sum_{n=-\infty}^{\infty}\int d\xi\ \mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(\xi))\mathrm{Tr}_{\mathbf{R}^{m+1}}(D_0^{-1}D'(F_0^n\xi))$$

which can be rewritten variously.

In view of recent work [4], [1], [6], it seems reasonable to conjecture that if the above L^c is $\neq 0$, then there exists an SRB measure for (small) finite a.

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