DIFFERENTIATION OF SRB STATES FOR HYPERBOLIC FLOWS.

by David Ruelle*.

William Parry in memoriam

Abstract. *Let the* $\mathcal{C}^3$ *vector field* $\mathcal{X} + aX$ *on* M *define a flow* (f_a^t) *with an Axiom A attractor* Λ_a *depending continuously on* $a \in (-\epsilon, \epsilon)$. *Let* ρ_a *be the SRB measure on* Λ_a *for* (f_a^t). *If* $A \in \mathcal{C}^2(M)$, *then* $a \mapsto \rho_a(A)$ *is* $\mathcal{C}^1$ *on* $(-\epsilon, \epsilon)$ *and* $d\rho_a(A)/da$ *is the limit when* $\omega \to 0$ *with* $\mathrm{Im}\omega > 0$ *of*

$$\int_0^\infty e^{i\omega t} dt \int \rho_a(dx)\, X(x) \cdot \nabla_x (A \circ f_a^t)$$

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1. Introduction.

Given a time evolution $(x,t) \mapsto f^t x$, with $x \in$ manifold M, $t \in \mathbf{R}$, it is often possible to find a set $S \subset M$ and an invariant probability measure ρ on M such that lebesgue$(S) > 0$ (*i.e.*, S has positive Lebesgue measure), and

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T A(f^t x)\, dt = \rho(A) \qquad \text{if } x \in S \tag{1}$$

whenever $A : M \to \mathbf{R}$ is continuous. Such measures ρ are called SRB measures or SRB states. (In the case of a discrete time dynamical system, the integral in (1) is replaced by a sum).

SRB measures were defined and studied by Ya. Sinai, D. Ruelle and R. Bowen for uniformly hyperbolic* systems [31], [24], [8]. Then the concept was extended to general smooth dynamical system by F. Ledrappier, J.-M. Strelcyn and L.-S. Young [18], [19]. Later it was found that, in a number of situations where specific geometric information is available, one can prove detailed properties of SRB measures (see in particular L.-S. Young [33], and the monograph by C. Bonatti, L. Diaz and M. Viana [3]).

The SRB measures describe the statistical properties of physical systems, in particular in nonequilibrium statistical mechanics [28]. It is therefore desirable to study how these measures depend on parameters (*i.e.*, on the dynamical system (f^t)). For the large systems of statistical mechanics, a *linear response* is often observed experimentally when parameters are varied. This means that the expectation value $\rho(A)$ of an *observable* A should depend differentiably on parameters. It is not clear at present how to reconcile the concept of linear response with the fact that typical dynamical systems depend very discontinuously on parameters (and may exhibit a dense set of bifurcations). The uniformly hyperbolic case is however amenable to discussion (in physical situations, this amounts to accepting the *chaotic hypothesis* of G. Gallavotti and E.G.D. Cohen [16]). A formula for the derivative of SRB states with respect to parameters has been obtained in the case of Axiom A diffeomorphisms in [27]. Here we shall study Axiom A flows.

A precise statement of our results is given as Theorem A and Theorem B below. The general idea of the proofs is to use the symbolic dynamics for hyperbolic flows to study their SRB states, also applying methods of the thermodynamic formalism**.

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* We call uniformly hyperbolic the Anosov systems [1] and the more general Axiom A systems introduced by Smale [32] (see also Bowen [7]).

** Ya. Sinai introduced Markov partitions, symbolic dynamics, and studied the ergodic theory for Anosov diffeomorphisms [29], [30], [31]. A partial generalization to flows was given by M. Ratner [23]. Then R. Bowen gave a general definition of Markov partitions for Axiom A diffeomorphisms [4] and flows [5]. The ergodic theory for Axiom A flows was studied by R. Bowen and D. Ruelle [8], introducing what are here called SRB states on attractors for Axiom A flows. Some abstract results applicable to SRB states originate from equilibrium statistical mechanics and are subsumed in the so-called thermodynamic formalism [6], [25].

It will be convenient to use the following notation for the derivative at x of a function A on the manifold M in the direction of the vector field X:

$$X(x) \cdot \nabla_x A = (D_x A) X(x)$$

If f is a diffeomorphism of M we have thus

$$X(x) \cdot \nabla_x (A \circ f) = (D_{fx} A)(T_x f) X(x)$$

Note.

Since this paper was written in 2004, the following relevant reference has appeared:

O. Butterley and C. Liverani. "Smooth Anosov flows: correlation spectra and stability." The Journal of Modern Dynamics **1**,301-322(2007).

Also, the old monograph of Parry and Pollicott still deserves to be cited:

W. Parry and M. Pollicott. *Zeta functions and the periodic orbit structure of hyperbolic dynamics.* Astérisque **187-188**, Soc. Math. de France, Paris, 1990.

2. Differentiability of SRB states for hyperbolic systems.

Let $r \geq 3$, and (f_a^t) be a $\mathcal{C}^r$ hyperbolic dynamical system (diffeomorphism or flow) depending smoothly on a parameter a, with an SRB measure ρ_a. There are a number of results on the smoothness of $a \mapsto \rho_a$ as a distribution, *i.e.*, of $a \mapsto \rho_a(A)$ when A is smooth. See [21], [17], [10], [11], [2].

For applications to statistical physics it is desirable to have an explicit expression for $d\rho_a(A)/da$. In the case of an Axiom A diffeomorphism f_a, writing $X_a = (\frac{d}{da} f_a) \circ f_a^{-1}$, we obtain by a formal calculation

$$\frac{d}{da} \rho_a(A) = \sum_{k=0}^{\infty} \int \rho_a(dx)\, X_a(x) \cdot \nabla_x (A \circ f_a^k)$$

If f_a is mixing, this result holds with an exponentially convergent sum over k, as shown in [27]. The proof is more difficult than one might anticipate. (For other differentiability results see [14]).

In the present paper we tackle the case of an Axiom A flow (f_a^t) defined by a vector field $\mathcal{X} + aX$. Here a formal calculation yields

$$\frac{d}{da} \rho_a(A) = \int_0^{\infty} dt \int \rho_a(dx)\, X(x) \cdot \nabla_x (A \circ f_a^t)$$

What we shall show is that the Fourier transform

$$\int_0^{\infty} e^{i\omega t} dt \int \rho_a(dx)\, X(x) \cdot \nabla_x (A \circ f_a^t)$$

(defined as a distribution) extends to a holomorphic function of ω near $\omega = 0$ such that its value at 0 is $\frac{d}{da} \rho_a(A)$.

While the proofs presented here are relatively straightforward, they make detailed use of the references [5], [8], [25], [26], and lead to somewhat heavy formulas. (The author has tried without success to find simpler and more direct arguments).

3. Theorem A.

Let $\mathcal{X}$ and X be $\mathcal{C}^r$ vector fields ($r \geq 3$) on the compact manifold M, and let (f_a^t) be the flow defined by $\mathcal{X} + aX$. We assume that for small a the flow (f_a^t) has a nontrivial Axiom A attractor Λ_a (depending continuously on a) with SRB measure ρ_a.*

If $A \in \mathcal{C}^{r-1}(M)$, the function $a \mapsto \rho_a(A)$ is $\mathcal{C}^{r-2}$ and $\frac{d}{da}\rho_a(A)|_{a=0}$ is the value at $\omega = 0$ of the function defined for $\mathrm{Im}\omega > 0$ by

$$\omega \mapsto \int_0^\infty e^{i\omega t} dt \int \rho_0(dx)\, X(x) \cdot \nabla_x (A \circ f_0^t)$$

which extends meromorphically to $\{\omega : \mathrm{Im}\omega > -\delta\}$ for some $\delta > 0$, without pole at $\omega = 0$.

Note that the theorem does not assume the flow (f_a^t) to be mixing. If $\int_0^\infty dt\, |\rho_0((A \circ f_0^t).C)| < \infty$, where $C = \mathrm{div}_v^{cu}(X^c + X^u)$ is defined in Section 7 below, we have

$$\frac{d}{da}\rho_a(A)|_{a=0} = \int_0^\infty dt \int \rho_0(dx)\, X(x) \cdot \nabla_x (A \circ f_0^t)$$

[There are a number of results on decay of correlations for hyperbolic flows, see in particular Chernov [9], Dolgopyat [12], [13], Liverani [20], Fields, Melbourne and Török [15]. Since C is Hölder but not smooth in general, only [20] applies directly in the present situation].

A proof of Theorem A will be obtained from Theorem B below.

4. Corollary.

Suppose that the vector field X_t is constant in t and equal to X when $t \leq t_0$ for some time t_0, but that X_t may depend (smoothly) on t for $t \geq t_0$. Write $f_a^{(t,t_0)} x_0 = x(t)$ where $\frac{d}{dx}x(t) = \mathcal{X}(x(t)) + aX_t(x(t))$ and $x(t_0) = x_0$. One can then define a time dependent SRB state $\rho_a^t = f_a^{(t,t_0)}\rho_a$ so that it reduces to ρ_a for $t \leq t_0$. With this definition, if $\int_0^\infty dt\, |\rho_0((A \circ f_0^t).C)| < \infty$,

$$\frac{d}{da}\rho_a^t(A)|_{a=0} = \int_{-\infty}^t d\tau \int \rho_0(dx)\, X_\tau(x) \cdot \nabla_x (A \circ f_0^{t-\tau})$$

The Corollary follows directly from Theorem A when $t < t_0$. To obtain the general case differentiate both sides with respect to t.

Before we formulate Theorem B, we need some facts and definitions.

5. Correlation functions.

* The attractor Λ_a is nontrivial if it is not a fixed point or a periodic orbit.

If B, B' are smooth functions on a neighborhood of Λ_0 in M, their correlation function is $t \mapsto \rho_{BB'}(t) = \rho_0((B \circ f_0^t).B') - \rho_0(B)\rho_0(B')$. Multiplying by the characteristic function χ^+ of $[0, +\infty)$ we obtain $\rho_{BB'}^+(t) = \rho_{BB'}(t)\chi^+(t)$, and taking the Fourier transform

$$\hat{\rho}_{BB'}^+(\omega) = \int_0^\infty e^{i\omega t} dt [\rho_0((B \circ f_0^t).B') - \rho_0(B)\rho_0(B')]$$

This is a distribution, boundary value of a holomorphic function in the upper half complex plane, which furthermore extends to a meromorphic function in $\{\omega : \mathrm{Im}\omega > -\delta'\}$ for some $\delta' > 0$, with no pole at $\omega = 0$, as discussed in [22], [26]. Actually, the discussion in [26] uses a *symbolic* representation of Λ: points have a description (ξ, t) where ξ belongs to a Cantor set Σ, and t to an interval of $\mathbf{R}$. Instead of smooth B, B' one takes $B, B' \in \mathcal{C}^\sharp$, where $\mathcal{C}^\sharp$ is a Banach space of functions $t \mapsto B(\cdot, t)$, continuous: interval of $\mathbf{R} \to \mathcal{C}^\alpha(\Sigma)$. [To make the connection with the formalism of [26], it is useful to know that if $t \mapsto B(\cdot, t), \zeta(\cdot, t)$ are continuous: interval $\to \mathcal{C}^\alpha(\Sigma)$, and $t \mapsto B(\cdot, t)$ is $\mathcal{C}^2$: interval $\to$ bounded functions on Σ, then $t \mapsto B(\cdot, \zeta(\cdot, t))$ is continuous: interval $\to \mathcal{C}^\alpha(\Sigma)$].

For our purposes the function $B' = C$ to be introduced below will belong to $\mathcal{C}^\sharp$ rather than being smooth.

6. The volume elements $\tilde{v}$ and v.

Let $\mathcal{V}^u$ denote a strong unstable manifold for the flow (f_0^t). We have thus $\mathcal{V}^u \subset \Lambda_0$, and $\mathcal{V}^u$ is u-dimensional. There is a natural volume element $\tilde{v}$ on each such $\mathcal{V}^u$ so that, for all t, the natural volume element on $f_0^t\mathcal{V}^u$ is the image by f_0^t of the measure $\tilde{v}$, up to a multiplicative constant. This is seen in the same way as for the existence of a natural volume element on unstable manifolds contained in an attractor for an Axiom A diffeomorphism (see [27]). Here again $\tilde{v}$ has $\mathcal{C}^{r-1}$ density, and is uniquely defined up to a multiplicative constant.

If $\tilde{\mathcal{V}}^u$ is a u-dimensional manifold contained in a center-unstable manifold, and transversal to the flow (f_0^t), we can define a volume element $\tilde{v}$ on $\tilde{\mathcal{V}}^u$ as the image of $\tilde{v}$ on a strong unstable manifold $\mathcal{V}^u$ by a Poincaré map. In this manner we obtain a natural volume element $\tilde{v}$, defined up to a multiplicative constant and corresponding to Poincaré maps acting on manifolds $\tilde{\mathcal{V}}^u$ transversal to (f_0^t).

Let now $\mathcal{W}^{cu}$ denote a center-unstable manifold for the flow (f_0^t). We have thus $\mathcal{W}^{cu} \subset \Lambda_0$, and $\mathcal{W}^{cu}$ is $(u+1)$-dimensional. Take a chart $S \times I$ of M such that $\mathcal{X}$ is the unit vector in the last coordinate direction, and I is an interval of $\mathbf{R}$. Assuming also $\tilde{\mathcal{V}}^u \subset S$ we may write locally $\mathcal{W}^{cu} = \tilde{\mathcal{V}}^u \times I$ and define

$$v = \tilde{v} \times \text{Lebesgue}$$

A volume element v is thus given on the center-unstable manifolds $\mathcal{W}^{cu}$, and is unique up to a multiplicative constant. Note that v has $\mathcal{C}^{r-1}$ density and that f_0^t sends v to v up to a multiplicative constant. [We shall see in Section 10 that v is (up to a multiplicative constant) the conditional probability of the SRB measure ρ_0 on the (local) center-unstable manifold $\mathcal{W}^{cu}$].

7. The function $C = \mathrm{div}_v^{cu}(X^c + X^u)$.

For $x \in \Lambda_0$, let $T_xM = E_x^c + E_x^s + E_x^u$, where E_x^c is 1-dimensional containing $\mathcal{X}(x)$, and E_x^s, E_x^u are the strong stable and unstable subspaces at x for (f_0^t). We write

$$X(x) = X^c(x) + X^s(x) + X^u(x)$$

with $X^c(x) \in E_x^c$, $X^s(x) \in E_x^s$, $X^u(x) \in E_x^u$. If we take again a chart $S \times I$ of M such that $\mathcal{X}$ is that unit vector in the last coordinate direction, we see that E_x^c is independent of x, while E_x^s, E_x^u depend Hölder continuously on x, and are independent of the last coordinate of x. In particular $X^c(x), X^s(x), X^u(x)$ have $\mathcal{C}^r$ dependence on the last coordinate of x (while depending Hölder continuously on x).

The divergence of $X^c + X^u$ with respect to the volume element v on the manifold $\mathcal{W}^{cu}$ is denoted by $\mathrm{div}_v^{cu}(X^c + X^u)$. It is, *a priori*, a distribution, but we shall show that it is actually a Hölder continuous function on Λ_0 (note that this is a local question).

Let $f_0^t x \in \mathcal{W}^{cu}$, with $x \in S \cap \mathcal{W}^{cu} = \tilde{\mathcal{V}}^u$. We may write $X^c + X^u = X'^c + X'^u$ where $X'^c(f_0^t x) \in E_x^c$ and $X'^u(f_0^t x) \in T_xS \cap (E_x^c + E_x^u)$. We have then $\mathrm{div}_v^{cu}(X^c + X^u) = \partial X'^c + \mathrm{div}_{\tilde{v}} X'^u$ where $\partial X'^c$ denotes the derivative of X'^c with respect to the last coordinate (*i.e.*, $(\partial X)(f_0^t x) = \partial_t X(f_0^t x)$). Since ∂X is $\mathcal{C}^{r-1}$, $\partial X'^c$ is Hölder continuous. Note that we may also write $X = X''^c + X''^s + X'^u$ where $X''^c(f_0^t x) \in E_x^c$ and $X''^s(f_0^t x) \in T_xS \cap (E_x^c + E_x^s)$. The definition of $\mathrm{div}_{\tilde{v}}$ in $\mathcal{W}^{cu} \cap S$ is now very similar to that of div^u for the case of hyperbolic diffeomorphisms in [27], provided we replace the diffeomorphism f by Poincaré maps of (f_0^t). In fact, using a Markov partition for (f_0^t) we see that we need only a finite number of Poincaré maps $f_0^{T_{k\ell}}$ between sections S_k, S_ℓ. The stable and unstable directions for the system of Poincaré maps are $T_xS \cap (E_x^c + E_x^s)$, $T_xS \cap (E_x^c + E_x^u)$. One uses in [27] the *absolute continuity* result that the projection along stable manifolds from one transverse section to another one has Hölder continuous Jacobian, and one obtains that $\mathrm{div}_{\tilde{v}} X'^u$ is Hölder. Therefore $\mathrm{div}_v^{cu}(X^c + X^u)$ is a Hölder C function on Λ_0. [Integration by parts will show (in Section 10) that $\rho_0(C) = 0$ because boundary terms cancel out]. Instead of X we may use ∂X in the above argument, and find that

$$f_0^t x \mapsto \partial_t C(f_0^t x) = \mathrm{div}_v^{cu}(\partial X^c + \partial X^u)(f_0^t x)$$

is Hölder continuous on Λ_0. From this results that $t \mapsto (x \mapsto C(f_0^t x))$ defines a $\mathcal{C}^1$ function to $\mathcal{C}^\alpha(S)$.

8. Theorem B.

Under the conditions of Theorem A we have

$$\frac{d}{da}\rho_a(A)|_{a=0} = \int \rho_0(dx)\,(D_xA) \int_0^\infty dt\,(T_{f_0^{-t}x} f_0^t) X^s(f_0^{-t}x) - \hat{\rho}_{AC}^+(0)$$

[If $\int_0^\infty dt\,|\rho_0((A \circ f_0^t).C)| < \infty$, we have $\hat{\rho}_{AC}^+(0) = \int_0^\infty \rho_0((A \circ f_0^t).C)$]

The proof of Theorem B will occupy most of the rest of this paper. It is based on the study of SRB states with help of a Markov partition. We start with the unperturbed dynamics (*i.e.*, $a = 0$, the index a will be omitted until Section 11).

Let thus, for $r \geq 3$, Λ be an Axiom A attractor for the flow (f^t) defined on the manifold M by the C^r vector field $\mathcal{X}$:

$$\frac{df^t x}{dt} = \mathcal{X}(f^t x) \tag{2}$$

with $f^0 x = x$. There is a unique SRB measure ρ with support Λ for the flow (f^t). A perturbation $\delta\mathcal{X}$ of the vector field $\mathcal{X}$ causes a change $\delta\rho$ of the SRB state ρ and we have formally

$$\delta\rho(A) = \int_0^\infty ds \int \rho(dx)\delta\mathcal{X}(x) \cdot \nabla_x (A \circ f^s) \tag{3}$$

for smooth $A : M \to \mathbf{R}$. The main purpose of the present paper is to provide a proof of a modified version of (3), as described in Theorem A and Theorem B above.

9. Markov partition for the flow (f^t).

We introduce a Markov partition with data as follows (see [5]). A finite index set J is given, and an $J \times J$ matrix τ with entries 0 or 1 such that all entries of some power of τ are > 0. We denote by (Σ, σ) the mixing subshift of finite type defined by J, τ, and let

$$\Sigma_k = \{(\xi_j)_{j\in\mathbf{Z}} : \xi_0 = k\} \qquad , \qquad \Sigma_{k\ell} = \{(\xi_j)_{j\in\mathbf{Z}} : \xi_0 = k, \xi_1 = \ell\}$$

The construction of the Markov partition uses small pieces S_k of manifolds transversal to the flow (f^t) for $k \in J$ (the S_k are open codimension 1 smooth submanifolds of M). When $\tau_{k\ell} = 1$, an open subset $S_{k\ell}$ of S_k and a C^r real function $T_{k\ell} > 0$ on $S_{k\ell}$ are given such that $f^{T_{k\ell}} S_{k\ell} \subset S_\ell$. Finally, for some standard metric on Σ, there is a α-Hölder continuous map $\pi : \Sigma \to \cup_k (S_k \cap \Lambda)$ such that

$$\begin{array}{ccc} \Sigma_{k\ell} & \xrightarrow{\sigma} & \Sigma_\ell \\ \downarrow\pi & & \downarrow\pi \\ S_{k\ell} & \xrightarrow{f^{T_{k\ell}}} & S_\ell \end{array}$$

is commutative. A positive α-Hölder continuous function $\psi : \Sigma \to \mathbf{R}$ is defined by

$$\psi(\xi) = T_{k\ell}(\pi\xi) \qquad \text{when} \qquad \xi \in \Sigma_{k\ell}$$

Also, if A is Hölder continuous on Λ we define a γ-Hölder continuous function $\tilde{A}$ on Σ by

$$\tilde{A}(\xi) = \int_0^{\psi(\xi)} dt\, A(f^t \pi\xi) \tag{4}$$

(here $\gamma = \alpha$ if $A \in \mathrm{C}^1(M)$, otherwise we have to choose some $\gamma \leq \alpha$).

10. Equilibrium states.

We use here the formalism of [8], calling *equilibrium states* the invariant probability measures described elsewhere as *Gibbs states*. The *pressure* of a Hölder continuous function $\phi : \Lambda \to \mathbf{R}$ with respect to the flow (f^t) is

$$c = \sup_\nu \frac{h_\sigma(\nu) + \nu(\tilde\phi)}{\nu(\psi)}$$

where the sup is over σ-invariant probability measures ν on Σ, h_σ denotes the *entropy* with respect to the shift σ, and $\tilde\phi$ is defined according to (4). Let ν_0 be the unique equilibrium state for $\tilde\phi - c\psi$ on Σ. Then the unique equilibrium state μ_ϕ of ϕ for the flow (f^t) on Λ is given by

$$\mu_\phi(A) = \frac{\nu_0(\tilde A)}{\nu_0(\psi)} \tag{5}$$

We shall be interested in the case when $\phi = \phi^{(u)}$ is minus the time derivative of the unstable Jacobian:

$$\phi = \phi^{(u)} = -\frac{d}{dt}\lambda_t^+|_{t=0} = -\frac{d}{dt}\log\lambda_t^+|_{t=0}$$

with

$$\lambda_t^+(x) = ||(T_x f^t)^{\wedge(u+1)}|\text{volume element of } \mathcal{W}^{cu}|| = ||(T_x f^t)^{\wedge u}|\text{volume element of } \mathcal{V}^u||$$

Notice that we have

$$\phi^{(u)}(f^t x) = -\frac{d}{dt}\log\lambda_t^+(x)$$

For $\phi^{(u)}$ one can show that the pressure vanishes ($c = 0$) and $\mu_{\phi^{(u)}}$ is the SRB measure ρ on Λ for (f^t). Details and proofs of the above construction of the SRB measure ρ are given in [8]. Note that the function $\tilde\phi$ corresponding to $\phi = \phi^{(u)}$ is – up to a minus sign and composition with π – the unstable Jacobian $(\lambda^+_{T_{k\ell}})$ of $(f^{T_{k\ell}})$ acting on (S_ℓ). This reduces the study of ν_0 to the situation discussed in [27] for an Axiom A diffeomorphism f, with the replacement of f by $(f^{T_{k\ell}})$. In particular (5) shows that the conditional measures of ρ on $\mathcal{W}^{cu}$ are of the form $v = \tilde v \times$Lebesgue. We obtain thus $\rho(C) = \rho(\mathrm{div}_v^{cu}(X^c + X^u)) = 0$ because the integral with respect to v of the divergence div_v^{cu} yields a sum of boundary terms (for each element of the Markov partition); those terms cancel in the flow direction and then also in the unstable directions.

Let us summarize the situation. The "central" flow direction plays a trivial role, and we face here basically the same problems as for diffeomorphisms. The SRB measure ρ is smooth along unstable directions, *i.e.*, ρ has smooth conditional measures v (defined up to a multiplicative constant) on center-unstable manifolds, and the corresponding divergence div_v^{cu} therefore makes sense. The fact that $\mathrm{div}_v^{cu}(X^c + X^u)$, obtained by differentiating the Hölder continuous vector field $X^c + X^u$, is actually a Hölder function C results from absolute continuity of the map along stable manifolds from one transverse section to another. Finally, $\rho(C) = 0$ follows by integration by part and cancellation of boundary terms.

11. Flows depending on a parameter a.

If we replace $\mathcal{X}$ in (2) by $\mathcal{X}+aX$ for $a \in (-\epsilon, \epsilon)$ we may leave Σ, σ, S_k, $S_{k\ell}$ unchanged but replace (f^t), Λ, $T_{k\ell}$, π, ψ, ϕ, $\tilde{A}$ by (f^t_a), Λ_a, $T_{ak\ell}$, π_a, ψ_a, ϕ_a, $\tilde{A}_a$. Call π_* the map π introduced in Section 9. A hyperbolic fixed point argument shows that for suitable $\alpha > 0$ there is a α-Hölder $\pi_a : \Sigma \to \cup S_\ell$ such that

$$f_a^{T_{ak\ell}} \circ \pi_a \circ \sigma^{-1} = \pi_a \qquad \text{on} \qquad \sigma\Sigma_{k\ell}$$

and $a \mapsto \pi_a$ is $\mathcal{C}^{r-1} : (-\epsilon, \epsilon) \to \mathcal{C}^\alpha(\Sigma \to \cup S_\ell)$, reducing to π_* for $a = 0$.

Here are details. Define $\Psi_a = (\Psi_{ak\ell})$ where

$$\Psi_{akl}\pi = f_a^{T_{ak\ell}} \circ \pi \circ \sigma^{-1} \qquad \text{on} \qquad \sigma\Sigma_{k\ell}$$

for (a, π) close to $(0, \pi_*)$. Then Ψ_a maps a neighborhood of π_* in the Hölder space $\mathcal{C}^\alpha(\Sigma \to \cup_{k\ell} S_{k\ell})$ to $\mathcal{C}^\alpha(\Sigma \to \cup_{k\ell} S_{k\ell})$. We assume that we have charts identifying the $S_{k\ell}$ with open subsets of $\mathbf{R}^{\dim M-1}$, so that $\mathcal{C}^\alpha(\Sigma \to \cup_{k\ell} S_{k\ell}) \subset \mathcal{C}^\alpha(\Sigma \to \mathbf{R}^{\dim M-1})$. Note that $(a, \pi) \mapsto \Psi_a\pi$ is $\mathcal{C}^{r-1}$ hence $\mathcal{C}^1$ from a neighborhood of $(0, \pi_*)$ in $\mathbf{R} \times \mathcal{C}^\alpha(\Sigma \to \mathbf{R}^{\dim M-1})$ to $\mathcal{C}^\alpha(\Sigma \to \mathbf{R}^{\dim M-1})$. Taking $a = 0$ we see that π_* is a fixed point of Ψ_0 (see the commutative diagram in Section 9 above). The derivative $D_{\pi_*}\Psi_0$ is a bounded linear operator on $\mathcal{C}^\alpha(\Sigma \to \mathbf{R}^{\dim M-1})$. Let $V^s_{\pi_*\xi}, V^u_{\pi_*\xi} \subset \mathbf{R}^{\dim M-1}$ denote the stable and unstable subspaces at $\pi_*\xi$. (When $\xi \in \Sigma_\ell$ these are the intersections with $T_{\pi_*\xi}S_\ell$ of the center-stable and center-unstable spaces at $\pi_*\xi$ for (f^t_0), or the stable and unstable spaces for the $f_0^{T_{0k\ell}}$ acting on $\cup_\ell S_\ell$). We have chosen $\alpha > 0$ such that π_* is α-Hölder, and we may assume that also $\xi \mapsto V^{s,u}_{\pi_*\xi}$ is α-Hölder. The spaces $V^{s,u}_*$, defined to consist of the α-Hölder maps $\xi \to V^{s,u}_{\pi_*\xi}$ are closed linear subspaces of $\mathcal{C}^\alpha(\Sigma \to \mathbf{R}^{\dim M-1})$, and $\mathcal{C}^\alpha(\Sigma \to \mathbf{R}^{\dim M-1}) = V^s_* \oplus V^u_*$.

We show now that $D_{\pi_*}\Psi_0$ is a hyperbolic operator with respect to the direct sum decomposition $V^s_* \oplus V^u_*$, provided α has been chosen small enough, *i.e.*, if α is replaced by a suitable β (with $0 < \beta < \alpha$) which we shall now determine. It suffices to prove that $D_{\pi_*}\Psi_0$ induces a contraction on V^s_*, where $D_{\pi_*}\Psi_0$ is the map

$$u \mapsto (Tf_0^{T_{0k\ell}})(u \circ \sigma^{-1})$$

Using an "adapted metric" on M we may assume for the uniform norm

$$||Tf_0^{T_{0k\ell}}|\text{stable direction}||_0 \leq \lambda < 1$$

In the definition of the $\mathcal{C}^\beta$ norm

$$||\Phi|| = \max\big(\sup_\xi |\Phi(\xi)|, \sup_{\xi\neq\eta} \frac{|\Phi(\xi) - \Phi(\eta)|}{d(\xi, \eta)^\beta}\big)$$

we take the second sup only over pairs (ξ, η) such that $d(\xi, \eta)^\beta) < \epsilon$, where the constant ϵ will be fixed later (small but > 0).

Write $T_\xi = T_{\pi_*\xi} f_0^{T_{0k\ell}}$, $\delta = d(\xi, \eta)$. Given $u \in V^s_*$ (with $\mathcal{C}^\beta$ norm $||u||$) we may for each pair (ξ, η) with small δ choose $v \in V^s_{\pi_*\xi}$ with $|v - u(\eta)| \leq ||u||O(\delta^\alpha)$. We have

$$T_\xi u(\xi) - T_\eta u(\eta) = T_\xi(u(\xi) - v) + T_\xi v - T_\eta v + T_\eta(v - u(\eta))$$

$$|T_\xi(u(\xi)-v)| \le \lambda|u(\xi)-v| \le \lambda|u(\xi)-u(\eta)| + ||u||O(\delta^\alpha)$$
$$|T_\xi v - T_\eta v| \le ||u||O(\delta^\alpha)$$
$$|T_\eta(v-u(\eta))| \le ||u||O(\delta^\alpha)$$

hence

$$|T_\xi u(\xi) - T_\eta u(\eta)| \le ||u||(\lambda\delta^\beta + O(\delta^\alpha))$$

Since $d(\sigma\xi,\sigma\eta) \ge C\delta$ we have

$$\frac{|T_\xi u(\xi) - T_\eta u(\eta)|}{d(\sigma\xi,\sigma\eta)^\beta} \le ||u||\frac{\lambda\delta^\beta + O(\delta^\alpha)}{C^\beta\delta^\beta} = ||u||(\frac{\lambda}{C^\beta} + O(\delta^{\alpha-\beta}))$$

For small β we have $\lambda/C^\beta < 1$, and we may take ϵ such that

$$\lambda/C^\beta + O(\delta^{\alpha-\beta}) < 1 \qquad \text{if} \qquad 0 < \delta < \epsilon$$

This concludes the proof that $D_{\pi_*}\Psi_0$ is hyperbolic for suitable β, *i.e.*, when α is chosen small enough. We may thus apply the implicit function theorem to obtain the existence of π_a with the properties indicated above.

12. Smooth dependence of SRB state with respect to a.

Let $\phi_a = \phi_a^{(u)}$ be minus the time derivative of the unstable Jacobian for (f_a^t) and ν_a the unique equilibrium state for $\tilde\phi_a$ on Σ, where

$$\tilde\phi_a(\xi) = \int_0^{\psi_a(\xi)} dt\, \phi_a(f_a^t\pi_a\xi)$$

Then, according to Section 10, the SRB measure ρ_a for (f_a^t) on Λ_a is given by

$$\rho_a(A) = \frac{\nu_a(\tilde A_a)}{\nu_a(\psi_a)}$$

Assuming $A \in \mathcal{C}^r(M)$ we find that $a \mapsto \psi_a, \tilde A_a$ are $\mathcal{C}^{r-1} : (-\epsilon,\epsilon) \to \mathcal{C}^\alpha(\Sigma)$ because we know that $a \mapsto \pi_a$ is $\mathcal{C}^{r-1}$, and

$$\psi_a(\xi) = T_{ak\ell}(\pi_a\xi) \qquad \text{for} \qquad \xi \in \Sigma_{k\ell}$$

$$\tilde A_a(\xi) = \int_0^{\psi_a(\xi)} dt\, A(f_a^t\pi_a\xi)$$

The set $\hat\Lambda_a = E^u_{\Lambda_a}$ of unstable subspaces is an Axiom A attractor for the $\mathcal{C}^{r-1}$ action of (Tf_a^t) on the Grassmannian $\widehat{M} \to M$. Therefore if $\hat\pi_a : \Sigma \to \hat\Lambda_a$ makes the diagram

$$\begin{array}{ccc} & & \hat\Lambda_a \\ & \overset{\hat\pi_a}{\nearrow} & \downarrow \\ \Sigma & \overset{\pi_a}{\to} & \Lambda_a \end{array}$$

commutative, we see that $a \mapsto \hat{\pi}_a$ is $\mathcal{C}^{r-2} : (-\epsilon, \epsilon) \to \mathcal{C}^\alpha$ (where we may again have to replace the current value of α by a lower one). Note that

$$\tilde{\phi}_a(\xi) = -\log \lambda^+_{\psi_a(\xi)}(\pi_a \xi)$$

where $\lambda^+_t(\pi_a\xi)$ is the unstable Jacobian $||(T_{\pi_a\xi} f^t_a)^{\wedge u}$|volume element of $\hat{\pi}_a\xi||$. Note that $\lambda^+_{\psi_a(\xi)}(\pi_a\xi)$ is a $\mathcal{C}^{r-1}$ function of $a, \psi_a(\xi), \hat{\pi}_a\xi$, hence $a \mapsto \tilde{\phi}_a(\cdot)$ is $\mathcal{C}^{r-2} : (-\epsilon, \epsilon) \to \mathcal{C}^\alpha(\Sigma \to \mathbf{R})$. Therefore $a \mapsto \nu_a$ is $\mathcal{C}^{r-2} : (-\epsilon, \epsilon) \to (\mathcal{C}^\alpha(\Sigma \to \mathbf{R}))^*$. [We use here the thermodynamic formalism to obtain the $\mathcal{C}^\omega$ dependence of ν_a (considered as an element of the Banach space dual of $\mathcal{C}^\alpha$) on $\tilde{\phi}_a$ (considered as an element of $\mathcal{C}^\alpha$), see [25], Theorem 5.26]. Thus if $A \in \mathcal{C}^{r-1}(M)$, the function $a \mapsto \rho_a(A) = \nu_a(\tilde{A}_a)/\nu_a(\psi_a)$ is $\mathcal{C}^{r-2}$.

13. Differentiating $a \mapsto \rho_a(A)$ at $a = 0$.

Writing $B = A - \rho_0(A)$ we have

$$\rho_a(A) = \rho_0(A) + \rho_a(B) = \rho_0(A) + \frac{\nu_a(\tilde{B}_a)}{\nu_a(\psi_a)}$$

where $\tilde{B}_a = \tilde{A}_a - \rho_0(A)\psi_a$. Therefore

$$\frac{d}{da}\rho_a(A)|_{a=0} = \frac{1}{\nu_0(\psi_0)} \frac{d}{da}(\nu_a(\tilde{B}_a))|_{a=0}$$

because $\nu_0(\tilde{B}_0) = 0$ (use the formula $\rho_0(A) = \nu_0(\tilde{A}_0)/\nu_0(\psi_0)$ from Section 12). In view of the above formula we shall now study $\nu_a(\tilde{B}_a)$ to first order in a.

14. Reparametrization: modifying the map π_a to first order in a.

A Markov partition parametrizes points of Λ in the form $f^t\pi\xi$ where $\xi \in \Sigma$ and $0 \leq t < \psi(\xi)$. We have taken $\pi\xi$ in a piece of smooth manifold S_k transversal to the flow. But we may just as well use a parametrization $f^t\pi^\sharp\xi$ of Λ, where $\pi^\sharp\xi = f^{\tau(\xi)}\pi\xi$ with continuous $\tau : \Sigma \to \mathbf{R}$.

We consider a first such reparametrization which consists in replacing S_k by a union of strong unstable manifolds (as is needed for the application of [26] in Section 20). This reparametrization corresponds to a Hölder continuous choice of $\xi \mapsto \tau(\xi)$, and replaces the S_k by non-smooth "manifolds" in general.

We return now to smooth S_k and write

$$\pi_a\xi = \pi_0\xi + a(U^c(\xi) + U^s(\xi) + U^u(\xi))$$

to first order in a, with $U^c(\xi) \in E^c_{\pi_0\xi}$, $U^s(\xi) \in E^s_{\pi_0\xi}$, $U^u(\xi) \in E^u_{\pi_0\xi}$. We may thus consider a second reparametrization:

$$\pi^\sharp_a\xi = \pi_0\xi + a(U^s(\xi) + U^u(\xi))$$

$$= \pi_a\xi - aU^c(\xi) = f_a^{-a\theta(\xi)}\pi_a\xi$$

where θ is defined by $U^c(\xi) = \theta(\xi)\mathcal{X}(\pi_0\xi)$. Note that the replacement of π_a by $\pi_a^\sharp$ replaces also $\psi_a(\xi)$ by $\psi_a(\xi) + a\theta(\xi) - a\theta(\sigma\xi)$, $\tilde{A}_a(\xi)$ by $\tilde{A}_a(\xi) + a\theta(\xi)A(\pi_a\xi) - a\theta(\sigma\xi)A(\pi_a\sigma\xi)$, and $\tilde{\phi}_a(\xi)$ by $\tilde{\phi}_a(\xi) + a\theta(\xi)\phi_a(\pi_a\xi) - a\theta(\sigma\xi)\phi_a(\pi_a\sigma\xi)$. Thus, the replacement of π_a by $\pi_a^\sharp$ changes ψ_a, $\tilde{A}_a$, $\tilde{\phi}_a$ by a coboundary. In particular ν_a and $\nu_a(\tilde{B}_a)$ are unchanged.

Let us now perform the first and then the second reparametrization, *i.e.*, first replacing S_k by a union of strong stable manifolds, and second taking

$$\pi_a^\sharp\xi = \pi_0\xi + a(U^s(\xi) + U^u(\xi))$$

Here we have

$$\pi_a^\sharp\xi = \pi_a\xi - U^c(a,\xi) = f_a^{-\theta(a,\xi)}\pi_a\xi$$

but because of the lack of smoothness of S_k, we cannot write $U^c(a,\xi) = aU^c(\xi)$, $\theta(a,\xi) = a\theta(\xi)$ in general. Nevertheless, the replacement of π_a by $\pi_a^\sharp$ changes $\psi_a, \tilde{A}_a, \phi_a$ by a coboundary, so that ν_a and $\nu_a(\tilde{B}_a)$ are unchanged. In view of this we shall from now on replace π_a by $\pi_a^\sharp$ and change ψ_a, $\tilde{A}_a$, $\tilde{\phi}_a$ accordingly, but without altering the notation.

15. Calculation of $\tilde{B}_a - \tilde{B}_0$.

We have

$$\tilde{B}_a(\xi) - \tilde{B}_0(\xi) = \int_0^{\psi_a(\xi)} d\tau\, B(f_a^\tau(\pi_0\xi + aU^s(\xi) + aU^u(\xi)) - \int_0^{\psi_0(\xi)} dt\, B(f_0^t\pi_0\xi)$$

Write $X^c(x) = \eta(x)\mathcal{X}(x)$, where η is Hölder continuous on Λ_0 (and $\eta(f_0^t\pi_0\xi)$ is a smooth function of t). We can then define a map $[0,\psi_a(\xi)] \to [0,\psi_0(\xi)]$ by $\tau \to t$ such that

$$\frac{dt}{d\tau} = 1 + a\eta(f_0^\tau\pi_0\xi)$$

Writing also $f_a^\tau = f_{a*}^t$ we obtain (to first order in a)

$$\tilde{B}_a(\xi) - \tilde{B}_0(\xi) = \int_0^{\psi_0(\xi)} dt[(1 - a\eta(f_0^t\pi_0\xi))B(f_{a*}^t(\pi_0\xi + aU^s(\xi) + aU^u(\xi)) - B(f_0^t\pi_0\xi)]$$

$$= a(Z' - Z'')$$

with

$$aZ' = \int_0^{\psi_0(\xi)} dt[B(f_{a*}^t(\pi_0\xi + aU^s(\xi) + aU^u(\xi)) - B(f_0^t\pi_0\xi)]$$

$$Z'' = \int_0^{\psi_0(\xi)} dt\,\eta(f_0^t\pi_0\xi)B(f_0^t\pi_0\xi)$$

The contributions of Z' and Z'' are evaluated in the Appendix.

From now on we shall write π, f, ψ, ν instead of π_0, f_0, ψ_0, ν_0. For $n \geq 0$, $\xi \in \Sigma$, we define

$$\Psi(-n,\xi) = -\psi(\sigma^{-n}\xi) - \ldots - \psi(\sigma^{-1}\xi)$$

$$\Psi(n,\xi) = \psi(\xi) + \ldots + \psi(\sigma^{n-1}\xi)$$

so that $\Psi(-n,\sigma^n\xi) = -\Psi(n,\xi)$, $\Psi(0,\xi) = 0$, $\Psi(1,\xi) = \psi(\xi)$, and $f^{\Psi(k,\xi)}\pi\xi = \pi\sigma^k\xi$. With this notation, the evaluation of Z', Z'' in the Appendix yields the following result.

16. Lemma. *We have*

$$\nu(\frac{d}{da}\tilde{B}_a)|_{a=0} = \nu(Z' - Z'')$$

$$= \sum_{k=-\infty}^{-1} \int \nu(d\xi) \int_0^{\psi(\xi)} dt\,(D_{f^t\pi\xi}B) \int_{\Psi(k,\xi)}^{\Psi(k+1,\xi)} d\theta\,(T_{f^\theta\pi\xi}f^{t-\theta})X^s(f^\theta\pi\xi)$$

$$+ \int \nu(d\xi) \int_0^{\psi(\xi)} dt\,(D_{f^t\pi\xi}B) \int_0^t d\theta\,(T_{f^\theta\pi\xi}f^{t-\theta})X^s(f^\theta\pi\xi)$$

$$- \sum_{k=-\infty}^{-1} \int \nu(d\xi) \int_0^{\psi(\xi)} dt\,B(f^t\pi\xi) \int_{\Psi(k,\xi)}^{\Psi(k+1,\xi)} d\theta\,(\mathrm{div}_v^{cu}X^c)(f^\theta\pi\xi)$$

$$- \int \nu(d\xi) \int_0^{\psi(\xi)} dt\,B(f^t\pi\xi) \int_0^t d\theta\,(\mathrm{div}_v^{cu}X^c)(f^\theta\pi\xi)$$

$$- \int \nu(d\xi) \int_0^{\psi(\xi)} dt\,(D_{f^t\pi\xi}B) \int_t^{\psi(\xi)} d\theta\,(T_{f^\theta\pi\xi}f^{t-\theta})X^u(f^\theta\pi\xi)$$

$$- \sum_{k=1}^{\infty} \int \nu(d\xi) \int_0^{\psi(\xi)} dt\,(D_{f^t\pi\xi}B) \int_{\Psi(k,\xi)}^{\Psi(k+1,\xi)} d\theta\,(T_{f^\theta\pi\xi}f^{t-\theta})X^u(f^\theta\pi\xi)$$

[The meaning of div_v^{cu} has been discussed in Section 7. The sums over k converge exponentially, by hyperbolicity (directly) for the X^s and X^u parts, and by exponential decay of correlations for the X^c part: see the Appendix for details].

17. Evaluation of $\tilde{\phi}_a - \tilde{\phi}_0$.

We have seen in Section 10 that the function $\tilde{\phi}$ corresponding to $\phi = \phi^{(u)}$ is – up to a minus sign and composition with π – the unstable Jacobian $(\lambda^+_{T_{k\ell}})$ of $(f^{T_{k\ell}})$ acting on (S_ℓ). This reduces the study of ν to the situation discussed in [27] for an Axiom A diffeomorphism f, with the replacement of f by $(f^{T_{k\ell}})$. This remark remains true in the a-dependent situation, and reduces the evaluation of $\tilde{\phi}_a - \tilde{\phi}_0$ to the situation discussed in [27] for Axiom A diffeomorphisms. We shall thus simply quote Proposition 1 of [27]II,which takes here the form

$$-\frac{\tilde{\phi}_a - \tilde{\phi}_0}{\tilde{\phi}_0} \sim a(\mathrm{div}_{\tilde{v}}^u \tilde{X}^u) \circ \pi$$

In this formula the left-hand side is evaluated to first order in a, and we have used the following notation. The equivalence $\sim$ means that the integrals of both sides with respect to every σ-invariant measure on Σ coincide. We have written

$$\int_0^{T_{k\ell}(x)} dt(T_{f^tx}f^{T_{k\ell}(x)-t})X^u(f^tx) = \tilde{X}^u(f^{T_{k\ell}(x)}x)$$

Finally, the divergence $\mathrm{div}^u_{\tilde{v}}$ is computed, on the intersection $\mathcal{V}^u$ with S_k of a center unstable manifold $\mathcal{W}^{cu}$, with respect to a natural volume element $\tilde{v}$ defined earlier. (Note that, by our choice of S_k, $\mathcal{V}^u$ is a strong unstable manifold). As in [27], and as in Section 7, $\mathrm{div}^u_{\tilde{v}}\tilde{X}^u$ is a Hölder continuous function on $S_k \cap \Lambda$.

The relation between $\tilde{X}^u, X^u$ and $\tilde{v}, v$ also gives (see Section 7)

$$(\mathrm{div}^u_{\tilde{v}}\tilde{X}^u)(f^{T_{k\ell}(x)}(x)) = \int_0^{T_{k\ell}(x)} dt\,(\mathrm{div}^{cu}_v X^u)(f^tx)$$

Therefore we may write

$$\frac{d}{da}\log\tilde{\phi}_a(\xi)|_{a=0} \sim -\int_0^{\psi(\xi)} dt\,(\mathrm{div}^{cu}_v X^u)(f^t\pi\xi) = \gamma(\xi)$$

The right-hand side is a Hölder continuous function of ξ and, since ν_a is the equilibrium state for $\tilde{\phi}_a$, the thermodynamic formalism (see [25] Chapter 5, Exercise 5(b)) yields

$$\frac{d}{da}\nu_a(\tilde{B})|_{a=0} = \sum_{k=-\infty}^{\infty}[\nu(\tilde{B}.(\gamma\circ\sigma^k)) - \nu(\tilde{B})\nu(\gamma)]$$

where the sum converges exponentially and, since $\nu(\tilde{B}) = 0$, we find

$$\frac{d}{da}\nu_a(\tilde{B})|_{a=0} = -\sum_{k=-\infty}^{\infty}\int\nu(d\xi)\tilde{B}(\xi)\int_0^{\psi(\sigma^k\xi)} dt\,(\mathrm{div}^{cu}_v X^u)(f^t\pi\sigma^k\xi)$$

This yields the following result

18. Lemma. *We have*

$$\frac{d}{da}\nu_a(\tilde{B})|_{a=0} = -\sum_{k=-\infty}^{\infty}\int\nu(d\xi)\tilde{B}(\xi)\int_{\Psi(k,\xi)}^{\Psi(k+1,\xi)} d\theta\,(\mathrm{div}^{cu}_v X^u)(f^\theta\pi\xi)$$

$$= -\sum_{k=-\infty}^{\infty}\int\nu(d\xi)\int_0^{\psi(\xi)} dt\,B(f^t\pi\xi)\int_{\Psi(k,\xi)}^{\Psi(k+1,\xi)} d\theta\,(\mathrm{div}^{cu}_v X^u)(f^\theta\pi\xi)$$

where the sum over k converges exponentially.

The right-hand side above may be written as the sum of a part Z_- where $\theta \le t$ and a part Z_+ where $\theta > t$. In fact we claim that

$$\frac{d}{da}\nu_a(\tilde{B})|_{a=0} = Z_- + Z_+$$

$$Z_- = -\sum_{k=-\infty}^{-1}\int \nu(d\xi)\int_0^{\psi(\xi)} dt\, B(f^t\pi\xi)\int_{\Psi(k,\xi)}^{\Psi(k+1,\xi)} d\theta\,(\mathrm{div}_v^{cu}X^u)(f^\theta\pi\xi)$$

$$-\int \nu(d\xi)\int_0^{\psi(\xi)} dt\, B(f^t\pi\xi)\int_0^t d\theta\,(\mathrm{div}_v^{cu}X^u)(f^\theta\pi\xi)$$

$$Z_+ = \sum_{k=1}^{\infty}\int \nu(d\xi)\int_0^{\psi(\xi)} dt\,(D_{f^t\pi\xi}B)\int_{\Psi(k,\xi)}^{\Psi(k+1,\xi)} d\theta\,(T_{f^\theta\pi\xi}f^{t-\theta})X^u(f^\theta\pi\xi)$$

$$+\int \nu(d\xi)\int_0^{\psi(\xi)} dt\,(D_{f^t\pi\xi}B)\int_t^{\psi(\xi)} d\theta\,(T_{f^\theta\pi\xi}f^{t-\theta})X^u(f^\theta\pi\xi)$$

For the calculation of the term Z_+, notice that if we write $\pi\xi = x$, the integral over $\nu(d\xi)\,dt$ reduces on the manifolds $\mathcal{W}^{cu}$ to integration over $\tilde{v}(dx)\,dt = v(dx\,dt) = dx_1\dots dx_u\,dt$ for a suitable choice of coordinates. Then, writing $X^u = Y$,

$$B(f^tx)(\mathrm{div}_v^{cu}X^u)(f^\theta x) = B(x,t)\sum_{k=1}^{u}\partial_k Y^k(x,\theta)$$

An integration by parts transforms this to

$$-\sum_{k=1}^{u}\partial_k B(x,t)Y^k(x,\theta) = -(D_{f^tx}B)(T_{f^\theta x}f^{t-\theta})X^u(f^\theta x)$$

plus boundary terms involving $B(x,t)(T_{f^\theta x}f^{t-\theta})X^u(f^\theta x)$ with exponentially convergent integral over θ. The boundaries of pieces of $\mathcal{W}^{cu}$ are compact with zero measure, and it is readily seen that the boundary terms cancel.

Putting Lemma 16 and Lemma 18 together yields:

19. Proposition.

We have

$$\frac{d}{da}\nu_a(\tilde{B}_a)|_{a=0} = \sum_{k=-\infty}^{-1}\int \nu(d\xi)\int_0^{\psi(\xi)} dt\,(D_{f^t\pi\xi}B)\int_{\Psi(k,\xi)}^{\Psi(k+1,\xi)} d\theta\,(T_{f^\theta\pi\xi}f^{t-\theta})X^s(f^\theta\pi\xi)$$

$$+\int \nu(d\xi)\int_0^{\psi(\xi)} dt\,(D_{f^t\pi\xi}B)\int_0^t d\theta\,(T_{f^\theta\pi\xi}f^{t-\theta})X^s(f^\theta\pi\xi)$$

$$-\sum_{k=-\infty}^{-1}\int\nu(d\xi)\int_0^{\psi(\xi)}dt\,B(f^t\pi\xi)\int_{\Psi(k,\xi)}^{\Psi(k+1,\xi)}d\theta\,C(f^\theta\pi\xi)$$

$$-\int\nu(d\xi)\int_0^{\psi(\xi)}dt\,B(f^t\pi\xi)\int_0^t d\theta\,C(f^\theta\pi\xi)$$

where we have written $C=\mathrm{div}_v^{cu}(X^c+X^u)$.

20. Proof of Theorem A and Theorem B.

We may write

$$\sum_{k=-\infty}^{-1}\int\nu(d\xi)\int_0^{\psi(\xi)}dt\,(D_{f^t\pi\xi}B)\int_{\Psi(k,\xi)}^{\Psi(k+1,\xi)}d\theta\,(T_{f^\theta\pi\xi}f^{t-\theta})X^s(f^\theta\pi\xi)$$

$$+\int\nu(d\xi)\int_0^{\psi(\xi)}dt\,(D_{f^t\pi\xi}B)\int_0^t d\theta\,(T_{f^\theta\pi\xi}f^{t-\theta})X^s(f^\theta\pi\xi)$$

$$=\int\nu(d\xi)\int_0^{\psi(\xi)}dt\,(D_{f^t\pi\xi}B)\int_{-\infty}^t d\theta\,(T_{f^\theta\pi\xi}f^{t-\theta})X^s(f^\theta\pi\xi)$$

$$=\nu(\psi)\int\rho(dx)\,(D_xB)\int_0^\infty d\tau(T_{f^{-\tau}x}f^\tau)X^s(f^{-\tau}x)\qquad(6)$$

This gives the first term occuring in theorem B. In view of the exponential convergence of the integral over τ (and using the notation at the end of Section 1) this term is also the value at 0 of the expression

$$\omega\mapsto\int\rho(dx)\,(D_xB)\int_0^\infty e^{i\omega\tau}d\tau\,(T_{f^{-\tau}x}f^\tau)X^s(f^{-\tau}x)$$

$$=\int_0^\infty e^{i\omega t}dt\int\rho(dx)\,X^s(x)\cdot\nabla_x(B\circ f^t)$$

which is holomorphic in ω for $\mathrm{Im}\omega>-\delta$, for some $\delta>0$, as required for Theorem A.

As to the series

$$-\sum_{k=-\infty}^{-1}\int\nu(d\xi)\int_0^{\psi(\xi)}dt\,B(f^t\pi\xi)\int_{\Psi(k,\xi)}^{\Psi(k+1,\xi)}d\theta\,C(f^\theta\pi\xi)$$

$$-\int\nu(d\xi)\int_0^{\psi(\xi)}dt\,B(f^t\pi\xi)\int_0^t d\theta\,C(f^\theta\pi\xi)\qquad(7)$$

its sum is formally

$$-\nu(\psi)\int_0^\infty dt\int\rho(dx)\,B(x)C(f^{-t}x)$$

To obtain a rigorous estimate of (7) we consider the Fourier transform, as temperate distribution, of $\rho^+_{BC}(\cdot) = \rho_{BC}(\cdot)\chi^+(\cdot)$ where ρ_{BC} is the correlation function and χ^+ the characteristic function of $[0,\infty)$. This Fourier transform, *i.e.*,

$$\hat{\rho}^+_{BC}(\omega) = \int_0^\infty e^{i\omega t}dt \int \rho(dx)\, B(x)C(f^{-t}x)$$

is the boundary value on the real axis of a function of ω holomorphic for $\mathrm{Im}\omega > 0$. Furthermore, this function continues meromorphically to $\{\omega : \mathrm{Im}\omega > -\delta^*\}$ for some $\delta^* > 0$, and is regular at $\omega = 0$ (see [22], [26]). Our ambition is to prove that its value at 0 is, up to the factor $-\nu(\psi)$, equal to (7). To do this we follow the calculation in [26] Section 4 which expresses the Fourier transform as a series converging in the sense of distributions. Note that, in order to use [26], we need the reparametrization of Section 14 which replaces S_k by a union of stable manifolds. Up to an additive term holomorphic in ω near $\omega = 0$, one finds that $\hat{\rho}^+_{BC}(\omega)$ is

$$= \frac{1}{\nu(\psi)}\tilde{\nu}[\tilde{B}_\omega \sum_{n=0}^\infty (S^{-1}\mathcal{L}_{\Phi-i\omega\Psi}S)^n \tilde{C}_{-\omega}] \tag{8}$$

In this formula $\tilde{\nu}$ is the image of ν by the projection $\Sigma \to \Sigma_-$ where Σ_- is the semi-infinite subshift defined by $\Sigma_- = \{(\xi^-_j)_{j\leq 0} : \tau_{\xi^-_{j-1}\xi^-_j} = 1\}$, and $\tilde{B}_\omega$, $\tilde{C}_{-\omega}$ are Hölder continuous functions on Σ_- depending holomorphically on ω. The *interactions* Φ and Ψ are related to $\phi^{(u)}$ and ψ, and the *transfer operator* $\mathcal{L}_{\Phi-i\omega\Psi}$ acting on Hölder continuous functions on Σ_- depends holomorphically on ω. Specifically, one may write

$$\tilde{\phi}(\xi) = -\Phi_0(\xi_0) - \sum_{\ell=1}^\infty \Phi_{2\ell}(\xi_{-\ell}\ldots,\xi_\ell) \quad , \quad \psi(\xi) = -\Psi_0(\xi_0) - \sum_{\ell=1}^\infty \Psi_{2\ell}(\xi_{-\ell}\ldots,\xi_\ell)$$

where $|\Phi_{2\ell}|, |\Psi_{2\ell}| < \text{const} \times \alpha^\ell$. From the interaction $\Phi = (\Phi_{2\ell})_{\ell\geq 0}$ one defines a $\alpha^{1/2}$-Hölder function A^-_Φ on Σ_- by

$$A^-_\Phi(\xi^-) = -\sum_{\ell=0}^\infty \Phi_{2\ell}(\xi^-_{-2\ell-1},\ldots,\xi^-_0)$$

and an operator $\mathcal{L}_\Phi$ (transfer operator) on $\mathcal{C}^{\alpha^{1/2}}(\Sigma^- \to \mathbf{C})$ by

$$(\mathcal{L}_\Phi U)(\xi^-) = \sum_{\eta\in J} t_{\xi^-_0\eta}[\exp A^-_\Phi(\xi^- \vee \eta)]U(\xi^- \vee \eta)$$

where we have written $\xi^- \vee \eta = (\ldots,\xi^-_{-1},\xi^-_0,\eta) \in \Sigma_-$. Similarly one defines $\mathcal{L}_{\Phi-\omega\Psi}$; for small $|\omega|$ this operator is quasicompact: it has a simple eigenvalue $\lambda(\omega)$ with $\lambda(0) = 1$, $\lambda'(0) \neq 0$, and the rest of the spectrum is contained in a disc of radius < 1. The eigenfunction S of $\mathcal{L}_\Phi$ to the eigenvalue 1 is > 0, and we have denoted by S or S^{-1} the multiplication or division by that function. The derivation of the above formula is presented in [26] with slightly different notation, and one can also see that $\tilde{\nu}(\tilde{B}_0) = \tilde{\nu}(\tilde{C}_0) = 0$. We can, in the expression (8), evaluate the part corresponding to the eigenvalue $\lambda(\omega)$ of $\mathcal{L}_{\Phi-i\omega\Psi}$. This part is of the form $(1-\lambda(\omega))^{-1}$ times two factors, one corresponding to $\tilde{B}_\omega$ and the other to $\tilde{C}_{-\omega}$. Both of these factors vanish at $\omega = 0$ as can be seen from [26]. Since $(1-\lambda(\omega))^{-1}$ has a simple pole at $\omega = 0$, the above product vanishes there. The Fourier transform of $\rho_{BC}(\cdot)$ is thus a distribution in ω which reduces to an analytic function of ω for $|\omega|$ small, and this analytic function is given by a convergent series corresponding to the part of the spectrum of $\mathcal{L}_{\Phi-i\omega\Psi}$ strictly inside the unit circle. One can thus take $\omega = 0$ and obtain a convergent expression for the Fourier transform of $\rho^+_{BC}(\cdot)$ at $\omega = 0$. Manipulations as described in [26] then show that the Fourier transform of $\rho^+_{BC}(\cdot)$ at $\omega = 0$ is, up to a factor $-\nu(\psi)$, equal to (7). From Proposition 19, (6) and (7) we obtain thus Theorem B since $D_x B = D_x A$, and $\rho_{BC} = \rho_{AC}$.

Note now that

$$\rho_{AC}(t) = \rho((A \circ f^t).C) = \int \rho(dx)\, A(f^t x)(\mathrm{div}_v^{cu}(X^c + X^u))(x)$$

$$= -\int \rho(dx)\, (X^c(x) + X^u(x)) \cdot \nabla_x (A \circ f^t)$$

where we have used the fact that v is the conditional measure of ρ on center-unstable manifolds, and performed an integration by parts. Theorem A follows then readily from Theorem B.

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APPENDIX

1. Calculation of Z'.

We have

$$f^t_{a*}x = f^t x + a\mathcal{R}^t_x(X^s + X^u)$$

where we have defined, for a vector field Y,

$$\mathcal{R}^t_x Y = \int_0^t d\theta\,(T_{f^\theta x} f^{t-\theta})Y(f^\theta x)$$

Therefore

$$Z' = \int_0^{\psi(\xi)} dt\,(D_{f^t\pi\xi}B)[(T_{\pi\xi}f^t)(U^s + U^u) + \mathcal{R}^t_{\pi\xi}(X^s + X^u)]$$

Notice also that

$$(T_{\pi\sigma^{-1}\xi}f^{\psi(\sigma^{-1}\xi)})U^{s,u}(\sigma^{-1}\xi) + \mathcal{R}^{\psi(\sigma^{-1}\xi)}_{\pi\sigma^{-1}\xi}X^{s,u} = U^{s,u}(\xi)$$

Defining

$$(\mathcal{R}Y)(\xi) = \mathcal{R}^{\psi(\sigma^{-1}\xi)}_{\pi\sigma^{-1}\xi}Y$$

$$(\mathcal{T}V)(\xi) = T_{\pi\sigma^{-1}\xi}f^{\psi(\sigma^{-1}\xi)}V(\sigma^{-1}\xi) \qquad , \qquad (\mathcal{T}_{-}V)(\xi) = T_{\pi\sigma\xi}f^{-\psi(\xi)}V(\sigma\xi)$$

we find

$$U^s = (1-\mathcal{T})^{-1}\mathcal{R}X^s = \sum_0^\infty \mathcal{T}^n\mathcal{R}X^s$$

$$U^u = -\mathcal{T}_{-}(1-\mathcal{T}_{-})^{-1}\mathcal{R}X^u = -\sum_1^\infty \mathcal{T}_{-}^n\mathcal{R}X^u$$

where the series on the right-hand side converge exponentially, and

$$(T_{\pi\xi}f^t)\mathcal{T}^n\mathcal{R}X^s = (T_{\pi\sigma^{-n}\xi}f^{\psi(\sigma^{-n}\xi)+\cdots+\psi(\sigma^{-1}\xi)+t})\mathcal{R}^{\psi(\sigma^{-n-1}\xi)}_{\pi\sigma^{-n-1}\xi}X^s$$

$$= \int_0^{\psi(\sigma^{-n-1}\xi)} d\theta\,(T_{f^\theta\pi\sigma^{-n-1}\xi}f^{\psi(\sigma^{-n-1}\xi)+\cdots+\psi(\sigma^{-1}\xi)+t-\theta})X^s(f^\theta\pi\sigma^{-n-1}\xi)$$

$$= \int_{-\psi(\sigma^{-n-1}\xi)-\cdots-\psi(\sigma^{-1}\xi)}^{-\psi(\sigma^{-n}\xi))-\cdots-\psi(\sigma^{-1}\xi)} d\theta'(T_{f^{\theta'}\pi\xi}f^{t-\theta'})X^s(f^{\theta'}\pi\xi)$$

Similarly

$$(T_{\pi\xi}f^t)\mathcal{T}_{-}^n\mathcal{R}X^u = (T_{\pi\sigma^n\xi}f^{-\psi(\sigma^{n-1}\xi)-\cdots-\psi(\xi)+t})\mathcal{R}^{\psi(\sigma^{n-1}\xi)}_{\pi\sigma^{n-1}\xi}X^u$$

$$= \int_0^{\psi(\sigma^{n-1}\xi)} d\theta\,(T_{f^\theta\pi\sigma^{n-1}\xi}f^{-\psi(\sigma^{n-2}\xi)-\cdots-\psi(\xi)+t-\theta})X^u(f^\theta\pi\sigma^{n-1}\xi)$$

$$= \int_{\psi(\sigma^{n-2}\xi)+\cdots+\psi(\xi)}^{\psi(\sigma^{n-1}\xi))+\cdots+\psi(\xi)} d\theta' (T_{f^{\theta'}\pi\xi} f^{t-\theta'}) X^u(f^{\theta'}\pi\xi)$$

We have thus

$$(T_{\pi\xi} f^t)U^s + \mathcal{R}^t_{\pi\xi} X^s$$

$$= \sum_{n=0}^{\infty} \int_{\Psi(-n-1,\xi)}^{\Psi(-n,\xi)} d\theta\, (T_{f^\theta\pi\xi} f^{t-\theta}) X^s(f^\theta\pi\xi) + \int_0^t d\theta\, (T_{f^\theta\pi\xi} f^{t-\theta}) X^s(f^\theta\pi\xi)$$

$$(T_{\pi\xi} f^t)U^u + \mathcal{R}^t_{\pi\xi} X^u$$

$$= -\sum_{n=1}^{\infty} \int_{\Psi(n-1,\xi)}^{\Psi(n,\xi)} d\theta\, (T_{f^\theta\pi\xi} f^{t-\theta}) X^u(f^\theta\pi\xi) + \int_0^t d\theta\, (T_{f^\theta\pi\xi} f^{t-\theta}) X^u(f^\theta\pi\xi)$$

$$= -\sum_{n=2}^{\infty} \int_{\Psi(n-1,\xi)}^{\Psi(n,\xi)} d\theta\, (T_{f^\theta\pi\xi} f^{t-\theta}) X^u(f^\theta\pi\xi) - \int_t^{\psi(\xi)} d\theta\, (T_{f^\theta\pi\xi} f^{t-\theta}) X^u(f^\theta\pi\xi)$$

We can also write

$$(T_{\pi\xi} f^t)U^s + \mathcal{R}^t_{\pi\xi} X^s$$

$$= \sum_{k=-\infty}^{-1} \int_{\Psi(k,\xi)}^{\Psi(k+1,\xi)} d\theta\, (T_{f^\theta\pi\xi} f^{t-\theta}) X^s(f^\theta\pi\xi) + \int_0^t d\theta\, (T_{f^\theta\pi\xi} f^{t-\theta}) X^s(f^\theta\pi\xi)$$

$$(T_{\pi\xi} f^t)U^u + \mathcal{R}^t_{\pi\xi} X^u$$

$$= -\sum_{k=1}^{\infty} \int_{\Psi(k,\xi)}^{\Psi(k+1,\xi)} d\theta\, (T_{f^\theta\pi\xi} f^{t-\theta}) X^u(f^\theta\pi\xi) - \int_t^{\psi(\xi)} d\theta\, (T_{f^\theta\pi\xi} f^{t-\theta}) X^u(f^\theta\pi\xi)$$

These two formulas give the desired evaluation of Z'.

2. Calculation of $\nu(Z'')$.

We have

$$\int \nu(d\xi) \int_0^{\psi(\xi)} dt\, [\eta(f^t\pi\xi) - \eta(\pi\sigma^{-n}\xi)] B(f^t\pi\xi)$$

$$= \int \nu(d\xi) \int_0^{\psi(\xi)} dt\, B(f^t\pi\xi) \int_{\Psi(-n,\xi)}^t d\theta\, \frac{d}{d\theta}\eta(f^\theta\pi\xi)$$

Using charts where $\mathcal{X}$ is the unit vector in the last coordinate direction, we see that

$$\frac{d}{d\theta}\eta(f^\theta\pi\xi) = (\mathrm{div}^{cu}_v X^c)(f^\theta\pi\xi)$$

Since $\int \nu(d\xi)\eta(\pi\sigma^{-n}\xi) \int_0^{\psi(\xi)} dt\, B(f^t\pi\xi)$ tends to 0 for $n \to \infty$ (by exponential decay of correlations for (ν, σ)) we have

$$\nu(Z'') = \int \nu(d\xi) \int_0^{\psi(\xi)} dt\, \eta(f^t\pi\xi) B(f^t\pi\xi)$$

$$= \lim_{n\to\infty} \int \nu(d\xi) \int_0^{\psi(\xi)} dt\, B(f^t\pi\xi) \int_{\Psi(-n,\xi)}^{t} d\theta\, (\mathrm{div}_v^{cu} X^c)(f^\theta \pi\xi)$$

$$= \sum_{k=-\infty}^{-1} \int \nu(d\xi) \int_0^{\psi(\xi)} dt\, B(f^t\pi\xi) \int_{\Psi(k,\xi)}^{\Psi(k+1,\xi)} d\theta\, (\mathrm{div}_v^{cu} X^c)(f^\theta \pi\xi)$$

$$+ \int \nu(d\xi) \int_0^{\psi(\xi)} dt\, B(f^t\pi\xi) \int_0^{t} d\theta\, (\mathrm{div}_v^{cu} X^c)(f^\theta \pi\xi)$$

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