SINGULARITIES OF THE SUSCEPTIBILITY OF AN SRB MEASURE IN THE PRESENCE OF STABLE-UNSTABLE TANGENCIES. *

by David Ruelle†.

Abstract. *Let ρ be an SRB (or "physical"), measure for the discrete time evolution given by a map f, and let $\rho(A)$ denote the expectation value of a smooth function A. If f depends on a parameter, the derivative $\delta\rho(A)$ of $\rho(A)$ with respect to the parameter is formally given by the value of the so-called susceptibility function $\Psi(z)$ at $z = 1$. When f is a uniformly hyperbolic diffeomorphism, it has been proved that the power series $\Psi(z)$ has a radius of convergence $r(\Psi) > 1$, and that $\delta\rho(A) = \Psi(1)$, but it is known that $r(\Psi) < 1$ in some other cases. One reason why f may fail to be uniformly hyperbolic is if there are tangencies between the stable and unstable manifolds for (f, ρ). The present paper gives a crude, nonrigorous, analysis of this situation in terms of the Hausdorff dimension d of ρ in the stable direction. We find that the tangencies produce singularities of $\Psi(z)$ for $|z| < 1$ if $d < 1/2$, but only for $|z| > 1$ if $d > 1/2$. In particular, if $d > 1/2$ we may hope that $\Psi(1)$ makes sense, and the derivative $\delta\rho(A) = \Psi(1)$ has thus a chance to be defined.*

* An early version of this work was presented at the France-Brazil mathematical conference at IMPA in Sep. 2009, at the LAWNP conference in Buzios in Oct. 2009, and at the meeting *Progress in Dynamics* at the IHP in Nov. 2009.

† Math. Dept., Rutgers University, and IHES, 91440 Bures sur Yvette, France. email: ruelle@ihes.fr

0. Introduction.

Let f be a diffeomorphism of the compact manifold M, and ρ an SRB measure[1] for f. The derivative $\delta_X\rho(A)$ of the map $f \mapsto \rho$ in the direction of the smooth vector field[2] $\mathbf{X}$, evaluated at the smooth real function A, can be formally computed to be the value at $z = 1$ of

$$\Psi(z) = \sum_{n=0}^{\infty} z^n \int \rho(dx)\,\mathbf{X}(x) \cdot \partial_x(A \circ f^n) \tag{1}$$

We shall call Ψ the *susceptibility*[3].

In the uniformly hyperbolic case (i.e., if the support of ρ is a mixing Axiom A attractor for f), Ψ has a radius of convergence $r(\Psi) > 1$. One can furthermore prove that $f \mapsto \rho$ is differentiable and that its derivative is given by $\Psi(1)$[4]. In nonuniformly hyperbolic situations these assertions may fail: $r(\Psi)$ may be < 1, and $f \mapsto \rho$ is presumed to be nondifferentiable[5].

The above results suggest two problems: I. proving that $r(\Psi) < 1, \geq 1$, or > 1 in cases of some generality, and II. relating the derivative of $f \mapsto \rho$ to $\Psi(1)$ when this quantity is defined. The present note is about the first problem, and presents a nonrigorous study of the singularities of Ψ which may occur as a result of *tangencies*, i.e., tangencies of stable and unstable manifolds for the system (f, ρ), assumed to have no zero Lyapunov exponent. (The existence of tangencies excludes uniform hyperbolicity). Our study is not rigorous, but suggests that $r(\Psi) < 1$ if the partial Hausdorff dimension d of ρ in the stable direction is $< \frac{1}{2}$, while $r(\Psi) \geq 1$ if $d \geq 1/2$. This opens the possibility that, for some *fat* tangencies (d sufficiently large), $\Psi(1)$ is well defined. In that case, a derivative of $f \mapsto \rho$ may exist, with applications to the physical theory of *linear response*[6].

---

[1] For a discussion of SRB measures (Sinai-Ruelle-Bowen) see for instance [10], [27], [2], and references given there. For recent work analyzing SRB measures for a class of noninvertible maps, see [1].

[2] If we replace $f : x \mapsto fx$ by $x \mapsto fx + X(fx)$, then ρ is replaced by $\rho + \delta_X\rho$ to first order in X. The derivative of $f \mapsto \rho$ in the direction of X, evaluated at A, is $\delta_X\rho(A)$.

[3] The physical susceptibility is defined for a continuous time dynamical system, and is a function of the frequency ω. For the discrete time dynamics considered here, the susceptibility would be $\omega \mapsto \Psi(e^{i\omega})$, but for simplicity we call Ψ the susceptibility.

[4] The differentiability of $f \mapsto \rho$ has been established in [9], the inequality $r(\Psi) > 1$ and the identity $\delta_X\rho(A) = \Psi(1)$ are proved in [16]. There are corresponding results for hyperbolic flows [18], [3], and generalizations to partially hyperbolic systems [5].

[5] $r(\Psi) < 1$ has been proved for certain (noninvertible) unimodal maps of the interval [17], [8], see also [19] and work in progress by Baladi and Smania. The analysis in [19] strongly suggests that for a certain class of unimodal maps, the function $f \mapsto \rho$ is nondifferentiable, even in the (weak) Whitney sense. There is also numerical evidence [4] that $r(\Psi) < 1$ for the classical Hénon attractor. For recent work on Hénon-like diffeomorphisms, see [14].

[6] A basic physical article on linear response is [21]. A review of linear response for dynamical systems is given in [20].

Acknowledgments.

I am indebted to Artur Àvila, Viviane Baladi, Bruno Cessac, Jean-Pierre Eckmann, and Jean-Christophe Yoccoz for valuable remarks.

1. Example: volume preserving diffeomorphisms.

Let ℓ be equivalent to Lebesgue measure on M, and let the f-invariant probability measure ρ be the restriction of ℓ to a certain open set $S \subset M$ [similarly, we may also consider the situation where f acts on $\mathbf{R}^m$, and S is a bounded open set in $\mathbf{R}^m$]. If either supp$X \subset S$, or supp$A \subset S$, we may write

$$\Psi(z) = \sum_{n=0}^{\infty} z^n \int_S \ell(dx)\, \mathbf{X}(x) \cdot \partial_x(A \circ f^n) = -\sum_{n=0}^{\infty} z^n \int_S \ell(dx)\, [\mathrm{div}_\ell \mathbf{X}(x)] A(f^n x)$$

and therefore $r(\Psi) \geq 1$.

If we furthermore suppose that either supp$X \subset S$, or supp$A \subset S$ and $\ell(A) = 0$, we have

$$(f, \rho)\,\mathrm{exponentially\,mixing} \quad \Rightarrow \quad r(\Psi) > 1$$

Volume preserving Anosov diffeomorphisms satisfy this condition, and the same is true of the time 1 map of an exponentially mixing volume preserving Anosov flow (which is not uniformly hyperbolic). Can exponential mixing happen for non-Anosov area preserving diffeomorphisms in 2 dimensions? We shall now see that mixing already implies that $\Psi(1)$ is well defined, and $\delta_X \rho(A) = \Psi(1)$ when the derivative δ_X is taken along diffeomorphisms preserving a (parameter dependent) volume.

For simplicity we discuss the case $S = M$. Let ρ be a probability measure equivalent to Lebesgue measure on the compact manifold M. Denote by $\tilde{\rho}$ the density of ρ with respect to Lebesgue measure on some charts. Thus $(f^*\rho)^\sim = \tilde{\rho} \circ f^{-1}/J \circ f^{-1}$ where $J(x) = |\det(D_x f)|$. Suppose now that $f, \tilde{\rho}$ depend smoothly on a parameter, and denote the derivative with respect to the parameter by a *prime*. In particular $f' = X \circ f$, $J'(x) = J(x)[\mathrm{div} X(fx)]$.

Writing $\rho_1 = f^*\rho$ we have $\tilde{\rho}_1 = (\tilde{\rho}/J) \circ f^{-1}$, or $\tilde{\rho}(x) = J(x)(\tilde{\rho}_1(fx))$, hence

$$\tilde{\rho}'(x) = J(x)[\tilde{\rho}_1'(fx) + \partial_{fx}\tilde{\rho}_1 \cdot X(fx)] + J(x)[\mathrm{div} X(fx)]\tilde{\rho}_1(fx)$$

or

$$(\tilde{\rho}'/J) \circ f^{-1} = \tilde{\rho}_1' + \partial\tilde{\rho}_1 \cdot X + [\mathrm{div} X]\tilde{\rho}_1 = \tilde{\rho}_1' + \mathrm{div}(\tilde{\rho}_1' X) = \tilde{\rho}_1' + [\mathrm{div}_{\tilde{\rho}_1} X]\tilde{\rho}_1$$

hence

$$\int dx\, \tilde{\rho}'(x) A(fx) = \int dx\, \tilde{\rho}_1'(x) A(x) + \int dx\, \tilde{\rho}_1(x)[\mathrm{div}_{\tilde{\rho}_1} X(x)] A(x)$$

Imposing the invariance condition $\rho = f^*\rho$, we have thus

$$\int dx\, \tilde{\rho}'(x) A(f^{N+1}x) = \int dx\, \tilde{\rho}'(x) A(x) + \sum_{n=0}^{N} \int dx\, \tilde{\rho}(x)[\mathrm{div}_{\tilde{\rho}} X(x)] A(f^n x) \tag{2}$$

Note that $\int dx\, \tilde\rho(x) = 1$ implies $\int \rho(dx)\,(\tilde\rho'/\tilde\rho)(x) = \int dx\, \tilde\rho'(x) = 0$. Therefore, imposing mixing gives that

$$\int dx\, \tilde\rho'(x) A(f^{N+1}x) = \int \rho(dx)\,(\tilde\rho'/\tilde\rho)(x) A(f^{N+1}x)$$

tends to 0 when $N \to \infty$. Equation (2) now implies that

$$\Psi(z) = -\sum_{n=0}^{\infty} z^n \int dx\, \tilde\rho(x)[\mathrm{div}_{\tilde\rho} X(x)] A(f^n x)$$

is well defined for $z = 1$, and $\int dx\, \tilde\rho'(x) A(x) = \Psi(1)$.

Conclusion: *Suppose that ρ is f-ergodic, with density $\tilde\rho$, and that (f,ρ) is mixing on a function space $\mathcal{S}$ containing $\tilde\rho'/\tilde\rho$ and A, then $\Psi(1)$ is well defined, and $\delta_X\rho(A) = \int dx\, \tilde\rho'(x)A(x) = \Psi(1)$.*

2. Computer simulations.

It is accepted that, using a computer, one can approximate numerically an SRB measure by a time average:

$$\frac{1}{N_1 - N_0} \sum_{n=N_0+1}^{N_1} \delta_{f^n x}$$

for large $N_1 - N_0$ (and N_0 moderately large); the idea is that the computed orbit $f^n x$ is noisy because of roundoff errors, and that this noisy orbit has an SRB time average[7]. The Lyapunov exponents, and the coefficient L_+ introduced below, can also in principle be determined numerically. It is therefore possible to estimate $r(\Psi)$ in particular cases, and to test the relations proposed above between the stable dimension d of ρ and the convergence radius $r(\Psi)$ in the presence of tangencies. For example, let $\dim M = 2$, and let the Lyapunov exponents λ_-, λ_+ of (f,ρ) satisfy $\lambda_- < 0 < \lambda_+$, so that[8] $d = \lambda_+/|\lambda_-|$. Does the presence of tangencies together with $\lambda_+/|\lambda_-| < 1/2$ imply $r(\Psi) < 1$? (This appears to be the case for the classical Hénon attractor). Does $\lambda_+/|\lambda_-| \geq 1/2$ imply $r(\Psi) \geq 1$? Are there examples with tangencies and $r(\Psi) > 1$?

3. Singularities of Ψ in the presence of tangencies.

It is readily seen that the power series (1) defining the susceptibility has a radius of convergence $r(\Psi) > 0$. Tangencies between stable and unstable manifolds for (f,ρ) are expected to produce singularities of Ψ, thus limiting $r(\Psi)$. A difficulty of the problem is that the set of points of tangency has measure zero. Note in this respect that the angle between stable and unstable manifolds is defined ρ-a.e., and that the a.e. range of this angle determines if tangencies are allowed or not. A similar comment can be made for higher order contacts of the stable-unstable manifolds. It is reasonable to exclude

[7] See [15] and, for example, Eckmann and Ruelle [6].
[8] See L.-S. Young [24].

those higher order contacts which (given the dimension of M) are nongeneric if the stable and unstable manifolds are regarded as independent. At a generic tangency point O, the unstable manifold is folded in a way which is basically 2-dimensional (corresponding to variables x, y introduced below). Along the orbit $(f^n O)$ we have folds which are sharper and sharper as $n \to \infty$. This exponential sharpening of the folds, combined with the derivative ∂_x in (1), produces the singularities of $\Psi(z)$ with $|z| < 1$ which we want to study.

One can prove that $r(\Psi) < 1$ for certain unimodal maps of the interval[9]; these maps are non-invertible and give a degenerate example of tangencies that is relatively accessible to mathematical study. In what follows we discuss a crude imitation of the 1-dimensional situation for higher-dimensional diffeomorphisms. In the case of unimodal maps of the interval with an ergodic measure ρ absolutely continuous with respect to Lebesgue, the density of ρ has *spikes* $\sim |x - f^n c|^{-1/2}$ on one side of the points $f^n c$ of the postcritical orbit. These spikes are at the origin of the singularities of $\Psi(z)$ inside of the unit circle. Instead of an individual postcritical point, we find for higher dimensional diffeomorphisms a family of tangencies of stable and unstable manifolds: think of a pile of (local) unstable manifolds (with tangencies) carrying part of the measure ρ. Morally, this means that the spikes are "spread out" or "smoothed" (corresponding to integration over a measure transverse to the unstable manifolds). This smoothing may give weaker singularities of Ψ (i.e., larger $r(\Psi)$).

Let us choose local coordinates $(x, X, y, Y) \in \mathbf{R} \times \mathbf{R}^{s-1} \times \mathbf{R} \times \mathbf{R}^{u-1}$ such that the s-dimensional stable manifolds are given by $(y, Y) =$ const., and the local unstable manifold U through O is given by $x = ay^2, X = 0, Y = 0$ (for definiteness we take $a > 0$). The conditional measure of ρ on U is thus $\sim \Delta(dx\,dX\,dy\,dY) = \delta(x - ay^2)\delta(X)\delta(Y)dx\,dX\,dy\,dY$. One can argue that the variable Y does not play an important role in the present discussion, and we shall omit it, which amounts to taking $u = 1$. Using similar local coordinates near fO, we assume that the map f has the form

$$(x, X, y) \mapsto (e^{L_+}x, e^{\Lambda}X, e^{L_-}y)$$

where $L_- < 0, L_+ > 0$, and e^{Λ} is a contraction (stronger than that given by e^{L_-}).

The assumption that the unstable manifolds are parallel affine manifolds is crude, and so is the assumption that L_-, L_+, and Λ are constant coefficients. [One might think of L_-, L_+, and Λ as Lyapunov exponents. But L_+, the only one of these coefficients to appear in the final formulas, is really the mean rate of expansion along a forward orbit $(f^n O)$ of tangencies]. These crude assumptions may be in part justified by the fact that we are looking for the leading singular behavior associated with a subset of unstable manifolds. We shall use informally the notation $\approx$ (approximately equal to) and $\sim$ (approximately proportional to) in trying to find the leading singular behavior.

The contribution of the conditional measure Δ of ρ on the piece U of unstable manifold is

$$\Psi^{\Delta}(z) \sim \sum_{n=0}^{\infty} z^n \int \Delta(dx\,dX\,dy)\,\mathbf{X}(x, X, y) \cdot \partial_{(x,X,y)}(A \circ f^n)$$

[9] See footnote 5.

Singularities for $|z| \leq 1$ can only come from the component $\mathbf{X}_1$ of $\mathbf{X}$ in the x-direction, giving

$$\Psi^\Delta(z) \sim \sum_{n=0}^\infty z^n \int dx\,dy\,\delta(x-ay^2)\mathbf{X}_1\partial_x(A_1\circ f^n)$$

$$\approx \sum_{n=0}^\infty (ze^{L_+})^n \mathbf{X}_1(0)\int (f^{*n}\delta(x-ay^2)dx\,dy)A_1'(x)$$

where $A_1'(x)$ is the derivative of $A_1(x)$, which is A evaluated in the coordinates $(x,0,0)$ centered at f^nO.

One has

$$\delta(x-ay^2) = \frac{1}{2\sqrt{ax}}[\delta(y-\frac{\sqrt{x}}{\sqrt{a}})+\delta(y+\frac{\sqrt{x}}{\sqrt{a}})]$$

hence

$$f^{*n}(\delta(x-ay^2)dx\,dy) = \frac{e^{-nL_+/2}}{2\sqrt{ax}}[\delta(y-\frac{\sqrt{x}}{\sqrt{a_n}})+\delta(y+\frac{\sqrt{x}}{\sqrt{a_n}})]dx\,dy$$

where $a_n = ae^{nL_+}e^{-2nL_-}$. Therefore

$$\Psi^\Delta(z) \approx \sum_{n=0}^\infty (ze^{L_+/2})^n \frac{\mathbf{X}_1(0)}{\sqrt{a}}\int_0^{\rm cutoff}\frac{dx}{\sqrt{x}}A_1'(x)$$

so that $r(\Psi^\Delta) = e^{-L_+/2} < 1$. This result is in agreement with that obtained with the spikes of the invariant density for unimodal maps in 1 dimension (which are limiting cases of diffeomorphisms with tangencies).

Remember however that Ψ is defined with the measure ρ rather than Δ. Let thus Γ be part of the measure ρ, carried by a pile of unstable manifolds (with tangencies) near O, and write

$$\Gamma(dx\,dX\,dy) = \int \gamma(d\xi\,dX)\,\delta(x-a(\xi,X)(y-b(\xi,X))^2-c(\xi,X))dx\,dy$$

where the integration is over the variable ξ, and $\gamma(d\xi\,dX)$ is a transverse measure of ρ in the stable direction, and we assume $a(\xi,X)>0$. It will turn out that we obtain the same estimate of $r(\Psi^\Gamma)$ for different Γ's, and we expect that the contributions Ψ^Γ to Ψ of different Γ's will add up convergently for $|z| < r(\Psi^\Gamma)$. [Such behavior was found for the contributions of different spikes in the unimodal case]. The most singular part of Ψ^Γ is of the form

$$\Psi_1^\Gamma(z) = \sum_{n=0}^\infty z^n \int \Gamma(dx\,dX\,dy)\mathbf{X}_1(x,X,y)\partial_x(A_1\circ f^n)$$

$$\approx \sum_{n=0}^\infty z^n \int \gamma(d\xi\,dX)\,\delta(x-a(\xi,X)(y-b(\xi,X))^2-c(\xi,X))\mathbf{X}_1(x,X,y)\partial_x A_1(e^{nL_+}x)dx\,dy$$

$$= \sum_{n=0}^{\infty} (ze^{L_+})^n \int \gamma(d\xi\, dX)\, \delta(x - a(\xi, X)(y - b(\xi, X))^2 - c(\xi, X)) \mathbf{X}_1(x, X, y) A_1'(e^{nL_+} x) dx\, dy$$

To define $A_1(x) = A(x, 0, 0)$ and $A_1'(x)$ we have again used coordinates $(x, 0, 0)$ centered at $f^n O$. Note that $A_1'(e^{nL_+} x) = A_1'([f^n(x, 0, 0)]_1)$ where $[\cdot]_1$ denotes the first component. Therefore, when n is large, the functions $x \mapsto [f^n(x, 0, 0)]_1, A_1'(e^{nL_+} x)$ oscillate many times, with a frequency $\sim nL_+$.

We may replace $\mathbf{X}_1(x, X, y)$ by $\tilde{X}(\xi, X) = \mathbf{X}_1(c(\xi, X), X, b(\xi, X))$, and write

$$\int \delta(x - a(\xi, X)(y - b(\xi, X))^2 - c(\xi, X)) dy = \frac{1}{\sqrt{a(\xi, X)}} \cdot \frac{1}{\sqrt{x - c(\xi, X)}}$$

where the right-hand side is replaced by 0 if $x < c(\xi, X)$. Then

$$\Psi_1^\Gamma(z) \approx \sum_{n=0}^{\infty} (ze^{L_+})^n \int \frac{\gamma(d\xi\, dX) \tilde{X}(\xi, X)}{\sqrt{a(\xi, X)} \sqrt{x - c(\xi, X)}} A_1'(e^{nL_+} x) dx$$

If we let $\tilde{\gamma}(d\tilde{\xi})$ be the image of the measure $\gamma(d\xi\, dX) \tilde{X}(\xi, X) / \sqrt{a(\xi, X)}$ by $(\xi, X) \mapsto \tilde{\xi} = c(\xi, X)$, we obtain finally

$$\Psi_1^\Gamma(z) \approx \sum_{n=0}^{\infty} (ze^{L_+})^n \int h(x) A_1'(e^{nL_+} x) dx \qquad \text{where} \qquad h(x) = \int \frac{\tilde{\gamma}(d\tilde{\xi})}{\sqrt{x - \tilde{\xi}}}$$

4. Estimates when $\mathrm{supp}\tilde{\gamma}$ has zero Lebesgue measure.

We assume now that $\mathrm{supp}\tilde{\gamma}$ has Lebesgue measure $= 0$. Given $x \notin \mathrm{supp}\, \tilde{\gamma}$, let

$$\xi^* = \max\{\xi \in \mathrm{supp}\, \tilde{\gamma} : \xi < x\}.$$

and let $\gamma^*(d\eta)$ be the image, restricted to $\eta \geq 0$, of $\tilde{\gamma}(d\tilde{\xi})$ by $\tilde{\xi} \mapsto \eta = \xi^* - \tilde{\xi}$. We have then

$$h(x) = \int \frac{\gamma^*(d\eta)}{\sqrt{(x - \xi^*) + \eta}} = \int \frac{\phi'(\eta)\, d\eta}{\sqrt{(x - \xi^*) + \eta}}$$

where $\phi(\eta) = \int_0^\eta \gamma^*(dt)$ and $\phi(\eta) \sim \eta^d$ for small η. If $0 < \alpha < 1 - d$ we let

$$h_1(x) = \int ((x - \xi^*) + \eta)^{\alpha - 1} \phi'(\eta) d\eta = (1 - \alpha) \int_0^\infty ((x - \xi^*) + \eta)^{\alpha - 2} \phi(\eta) d\eta$$

where we have put an upper limit $+\infty$ to the integral because it does not need a cutoff. Therefore

$$h_1(x) \approx (1 - \alpha) \frac{(x - \xi^*)^{1+d}}{(x - \xi^*)^{2-\alpha}} \int_0^\infty \frac{\phi(t)\, dt}{(1 + t)^{1+\alpha}} = C_\alpha (x - \xi^*)^{d + \alpha - 1}$$

A. Assuming $d < 1/2$ and taking $\alpha = 1/2$ we may thus conclude that $h(x) \approx C(x-\xi^*)^{d-1/2}$, hence, if $I = (\xi^*, \xi^{**})$ is an interval of length ℓ of the complement of suppγ, we may estimate

$$\int_I |h(x)|^p dx \approx C^p \int_0^\ell dt\, t^{-p(1/2-d)} = C'\ell^{1-p(1/2-d)}$$

if $1 \le p < (1/2-d)^{-1}$. By scaling we assume that the number of intervals I with $|I| \approx \ell$ is $\sim \ell^{-d}$. We have thus

$$\int |h(x)|^p dx \sim \sum_\ell \ell^{1-d-p(1/2-d)}$$

If $p \ge 1$, and $1-d-p(1/2-d) > 0$, we have thus $h \in L_p$ (and the bound $p < (1-d)/(1/2-d)$ appears best possible). Let $1/p + 1/q = 1$, then the Fourier transform $\hat{h}$ is in L_q. We have

$$\frac{1}{q} < 1 - \frac{1/2-d}{1-d} = \frac{1/2}{1-d} \tag{3}$$

(and this bound appears best possible). If $|\hat{h}(s)| \sim s^{-t}$ for large s, we need $tq > 1$, i.e., $t > 1/q$ if $1/q$ satisfies (3), i.e.,

$$|\hat{h}(s)| \sim s^{-t} \qquad \text{with} \qquad t \ge \frac{1/2}{1-d}$$

We come now to the estimation of $\int h(x) A_1'(e^{nL_+}x)\, dx$ where, for large n, $x \mapsto A_1'(e^{nL_+}x)$ is rapidly oscillating with frequency $\sim nL_+$. Since we are interested in the most singular part of h, we may replace it by a function with compact support. Because A_1' is a derivative, there is no zero-frequency contribution to the integral, and we have

$$\int h(x) A_1'(e^{nL_+}x)\, dx \sim \hat{h}(e^{nL_+})$$

with a negligible contribution of higher harmonics. Therefore

$$|\int h(x) A_1'(e^{nL_+}x)\, dx| \sim |\hat{h}(e^{nL_+})| \sim e^{-tnL_+} \le \exp[-n\frac{1/2}{1-d}L_+]$$

and the bound again appears best possible, so that $\Psi_1^\Gamma(z)$ converges for

$$|z| < \exp[-(1-\frac{1/2}{1-d})L_+] = \exp[-\frac{1/2-d}{1-d}L_+]$$

i.e., $r(\Psi_1^\Gamma) = \exp[-(1/2-d)L_+/(1-d)]$, and a reasonable guess would appear to be

$$r(\Psi) = \exp[-\frac{1/2-d}{1-d}L_+]$$

[hence $e^{-L_+/2} < r(\Psi) < \exp[(d-1/2)L_+] < 1$].

B. Assuming $1/2 \leq d < 1$, we write h (which is the convolution product $\tilde{\gamma} * (\cdot)^{-1/2}$) as

$$h \sim \tilde{\gamma} * (\cdot)^{\alpha-1} * (\cdot)^{\beta-1}$$

where $\alpha, \beta > 0, \alpha+\beta = 1/2, d+\alpha < 1$, or $\beta = 1/2-\alpha, 0 < \alpha < 1-d$. We have thus

$$h = h_1 * (\cdot)^{\beta-1} \qquad \text{where} \qquad h_1 = \tilde{\gamma} * (\cdot)^{\alpha-1}$$

and we have seen that $h_1(x) \approx C_\alpha (x-\xi^*)^{d+\alpha-1}$. We find as in **A.** that we can take $|\hat{h}_1(s)| \sim s^{-t}$ with $t \geq \alpha/(1-d)$, hence

$$\hat{h}(s) \sim s^{-\alpha/(1-d)} s^{-\beta} = s^{-1/2-\alpha d/(1-d)}$$

so that

$$|\int h(x) A_1'(e^{nL_+}x)\, dx| \sim |\hat{h}(e^{nL_+})| \leq \exp[-n(\frac{\alpha d}{1-d}+\frac{1}{2})L_+]$$

and $\Psi_1^\Gamma(z)$ converges for

$$|z| < \exp[(\frac{\alpha d}{1-d}+\frac{1}{2}-1)L_+] = \exp[(\frac{\alpha d}{1-d}-\frac{1}{2})L_+]$$

where we may let $\alpha \to 1-d$, hence we may estimate

$$r(\Psi_1^\Gamma) \geq \exp[(\frac{\alpha d}{1-d}-\frac{1}{2})L_+] = \exp[(d-1/2)L_+]$$

which is > 1. In fact $r(\Psi_1^\Gamma) = \exp[(d-1/2)L_+]$ is a reasonable guess.

The convergence radius $r(\Psi)$ now depends on the behavior of (f,ρ) away from tangencies, and we may expect that the derivative ∂_x in (1) plays a less important role. Therefore $r(\Psi)$ is expected to depend on the mixing properties of (f,ρ), over which one has some control [25], [26]. One may thus hope that $r(\Psi) \geq 1$, or even $r(\Psi) > 1$, and that $\Psi(1)$ is well defined. The situation where the set of tangencies is large ($d > 1/2$) reminds one of Newhouse's study of persistent tangencies (wild hyperbolic sets, infinitely many sinks, see [11], [12], [13]). While the situation considered by Newhouse has very discontinuous topology, it is not unthinkable that the particular measure ρ behaves differentiably in some sense.

C. It is plausible that the results of **A.** and **B.** remain true without the condition that $\text{supp}\tilde{\gamma}$ has zero Lebesgue measure. Furthermore, if the stable dimension d of ρ is ≥ 1, one can write d as a sum of partial dimensions[10], and use arguments as above. One expects thus that the formula $r(\Psi_1^\Gamma) \geq \exp[(d-1/2)L_+]$ will remain correct in that case and, as argued in **B.**, we may then have $r(\Psi) \geq 1$ or even $r(\Psi) > 1$.

If we have a continuous time dynamical system (a flow) instead of discrete time dynamics (a diffeomorphism), we expect similar results in the presence of tangencies: a

[10] See [6] Section IV.D, and references given there, in particular [10].

susceptibility function $\hat{\kappa}(\omega)$ with singularities in the upper half ω-plane if $d < 1/2$, no singularity if $d \geq 1/2$, and $\hat{\kappa}(0)$ hopefully well defined if $d > 1/2$. The continuous time dynamical situation is that most relevant for physical applications.

5. Physical discussion.

In this brief physically oriented discussion we shall, for simplicity, use the language of discrete time dynamical systems.

We have made above a nonrigorous analysis of how tangencies between stable and unstable manifolds may influence the radius of convergence $r(\Psi)$ of the susceptibility function. We have found two different regimes depending on whether the stable dimension d of the SRB measure ρ is $< 1/2$ or $\geq 1/2$.

If $d < 1/2$ we expect $r(\Psi) < 1$, i.e., the tangencies cause singularities of $\Psi(z)$ with $|z| < 1$. Such singularities reflect the exponential growth of small periodic perturbations of the dynamics (f, ρ) (see [20]). Experimentally, this may be visible as resonant behavior when a physical system is excited by a weak periodic signal: it would be of particular interest to study the case of hydrodynamic turbulence.

If $d \geq 1/2$ we expect $r(\Psi) \geq 1$ and, if $d > 1/2$, the value $\Psi(1)$ may be well defined. Since $\Psi(1)$ is formally related to the derivative of ρ with respect to f, we may hope that this derivative exists in some sense. This would apply to physical systems not too far from equilibrium (at equilibrium, ρ has a density, and $d \geq 1$ unless all Lyapunov exponents vanish) with obvious application to linear response in nonequilibrium statistical mechanics. For large physical systems ($\dim M$ large), when there is chaos and a density of Lyapunov exponents can be defined, one also expects d large by the Kaplan-Yorke formula[11], provided the degrees of freedom of the large system have a sufficiently strong effective interaction.

In view of the mathematical difficulty of analyzing dynamical systems with tangencies, a computer-experimental study would be desirable. The situation of choice would be that of 2-dimensional diffeomorphisms with an SRB measure ρ such that the Lyapunov exponents λ_-, λ_+ satisfy $\lambda_- < 0 < \lambda_+$. In that case we know [24] that $d = \lambda_+/|\lambda_-|$, and the radius of convergence $r(\Psi)$ is also accessible numerically. For the classical Hénon attractor we have $d < 1/2$, and it appears [4] that $r(\Psi) < 1$. In other cases, studied by Ueda and coworkers [22], [23], visual inspection of the computer plot of the attractor seems to indicate a large d, and it would be desirable to estimate $r(\Psi)$.

6. Infinitesimally stable ergodic measures.

Consider the general situation of a diffeomorphism f of the compact manifold M, and of an ergodic measure ρ for f on M. We want to study formally the stability of ρ under an infinitesimal change of f.

We shall use a space $\mathcal{D}$ of smooth functions on M, with dual $\mathcal{D}^*$, and a space $\mathcal{V}$ of smooth vector fields on M. If $X \in \mathcal{V}$, we write $\hat{X}(A) = \int \rho(dx)\, X(x) \cdot \partial_x A$, so that $\hat{X} \in \mathcal{D}^*$. Defining $T : \mathcal{D}^* \to \mathcal{D}^*$ and $Tf : \mathcal{V} \to \mathcal{V}$ by

$$(T\xi)(A) = \xi(A \circ f) \qquad , \qquad ((Tf)X)(fx) = (T_x f)X(x)$$

[11] See [6] Section IV.C, and references given there, in particular [7].

we find that $((Tf)X)^\wedge = T\hat{X}$.

Consider $\rho+\hat{X}$ as an infinitesimal perturbation of ρ (it corresponds to replacing ρ by its image under $x \mapsto x + X(x)$). The measure ρ is mapped to itself by f, while $\rho+\hat{X}$ is mapped to $\rho+T\hat{X}$. We say that ρ is *infinitesimally stable* (or attracting) if $(T^n\hat{X})(A) \to 0$ exponentially[12] with n whenever $X \in \mathcal{V}, A \in \mathcal{D}$. It is plausible that an infinitesimally stable measure must be SRB.

We perturb f to $\tilde{f} = f + X \circ f$, where $X \in \mathcal{V}$. If $\xi \in \mathcal{D}^*$, the $\tilde{f}$-invariance of $\rho+\xi$, i.e.,

$$(\rho+\xi)(A \circ (f + X \circ f)) = (\rho+\xi)(A)$$

is then given, to first order in X, by

$$\int \rho(dx)\,[A(fx) + X(fx) \cdot \partial_{fx}A] + \xi(A \circ f) = \rho(A) + \xi(A)$$

or $\hat{X} + T\xi = \xi$, hence $T^n\xi - T^{n+1}\xi = T^n\hat{X}$, hence $\xi - T^{N+1}\xi = \sum_{n=0}^N T^n\hat{X}$. Therefore, if ρ is infinitesimally stable, we obtain $\rho+\xi$ which is $\tilde{f}$-invariant to first order by taking

$$\xi(A) = \sum_{n=0}^{\infty}(T^n\hat{X})(A)$$

and ξ is unique such that $(T^n\xi)(A) \to 0$ for all $A \in \mathcal{D}$ when $n \to \infty$. This shows that the linear response $X \mapsto \xi$ is related to infinitesimal stability.

The above considerations apply to the uniformly hyperbolic situation where ρ is an SRB measure on an Axiom A attractor. The purpose of the present paper has been to make plausible the infinitesimal stability of SRB measures in a different situation where there are stable-unstable tangencies. [Note that $\Psi(z) = \sum_{n=0}^\infty z^n(T^n\hat{X})(A)$, so that $r(\Psi) > 1$ is equivalent to the infinitesimal stability condition that $(T^n\hat{X})(A) \to 0$ exponentially].

References.

[1] A. Àvila, S. Gouëzel, and M. Tsujii "Smoothness of solenoidal attractors." Discr. and Cont. Dynam. Syst. **15**,21-35(2006).

[2] C. Bonatti, L. Diaz, and M. Viana. *Dynamics beyond uniform hyperbolicity.* Springer, Berlin, 2005.

[3] O. Butterley and C. Liverani "Smooth Anosov flows: correlation spectra and stability." J. Modern Dynamics **1**,301-322(2007).

[4] B. Cessac "Does the complex susceptibility of the Hénon map have a pole in the upper half plane? A numerical investigation." Nonlinearity **20**,2883-2895(2007).

[5] D. Dolgopyat "On differentiability of SRB states for partially hyperbolic systems." Invent. Math. **155**,389-449(2004).

[6] J.-P. Eckmann and D. Ruelle "Ergodic theory of chaos and strange attractors." Rev. Mod. Phys. **57**,617-656(1985).

[12] An alternate (weaker) requirement would be that $\Psi(1) = \sum_{n=0}^\infty \rho(dx)\,X(x)\cdot\partial_x(A\circ f^n)$ converges whenever $X \in \mathcal{V}, A \in \mathcal{D}$.

[7] P. Frederickson, J.L. Kaplan, E.D. Yorke, and J.A. Yorke "The Lyapunov dimension of strange attractors." J. Diff. **49**,185-207(1983).

[8] Y. Jiang and D. Ruelle "Analyticity of the susceptibility function for unimodal Markovian maps of the interval." Nonlinearity **18**,2447-2453(2005).

[9] A. Katok, G. Knieper, M. Pollicott, and H. Weiss "Differentiability and analyticity of topological entropy for Anosov and geodesic flows." Invent. Math. **98**,581-597(1989).

[10] F.Ledrappier and L.S.Young. "The metric entropy of diffeomorphisms: I. Characterization of measures satisfying Pesin's formula, II. Relations between entropy, exponents and dimension." Ann. of Math. **122**,509-539,540-574(1985).

[11] S. Newhouse "Diffeomorphisms with infinitely many sinks." Topology **13**,9-18 (1974).

[12] S. Newhouse "The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms." Publ. Math. IHES **50**,102-151(1979).

[13] S. Newhouse "Lectures on dynamical systems." pp. 1-114 in: *CIME Lectures, Bressanone, Italy, June 1978.* Birkhäuser, Boston, 1980.

[14] J. Palis and J.-P. Yoccoz "Non-uniformly hyperbolic horseshoes arising from bifurcations of heteroclinic cycles." IHES Publications Mathématiques **110**,1-217(2010).

[15] D. Ruelle "What are the measures describing turbulence?" Progr. Theoret. Phys. Suppl. **64**,339-345(1978).

[16] D. Ruelle "Differentiation of SRB states." Commun. Math. Phys. **187**,227-241(1997); "Correction and complements." Commun. Math. Phys. **234**,185-190(2003).

[17] D. Ruelle "Differentiating the absolutely continuous invariant measure of an interval map f with respect to f." Commun.Math. Phys. **258**,445-453(2005).

[18] D. Ruelle "Differentiation of SRB states for hyperbolic flows." Ergod. Theor. Dynam. Syst. **28**,613-631(2008).

[19] D. Ruelle "Structure and f-dependence of the a.c.i.m. for a unimodal map f of Misiurewicz type." Commun. Math. Phys. **287**,1039-1070(2009).

[20] D. Ruelle "A review of linear response theory for general differentiable dynamical systems." Nonlinearity **22**,855-870(2009).

[21] J. Toll "Causality and the dispersion relation: logical foundations." Phys. Rev. **104**,1760-1770(1956).

[22] Y. Ueda "Explosion of strange attractors exhibited by Duffing's equation." Ann. NY Acad. Sci. **357**,422-434(1980).

[23] Y. Ueda and N. Akamatsu "Chaotically transitional phenomena in the forced negative resistance oscillator." IEEE Trans. CAS-**28**,217-223(1981).

[24] L.-S. Young "Dimension, entropy and Liapunov exponents." Ergod. Theory Dynam. Syst. **2**,109-124(1982).

[25] L.-S. Young "Statistical properties of dynamical systems with some hyperbolicity." Annals of Math. **147**,585-650(1998).

[26] L.-S. Young "Recurrence times and rates of mixing." Israel J. Math. **110**,153-188(1999).

[27] L.-S. Young "What are SRB measures, and which dynamical systems have them?" J. Statist. Phys. **108**,733-754(2002).