NON-EQUILIBRIUM STATISTICAL MECHANICS OF TURBULENCE.

by David Ruelle†.

Abstract: *The macroscopic study of hydrodynamic turbulence is equivalent, at an abstract level, to the microscopic study of a heat flow for a suitable mechanical system [12]. Turbulent fluctuations (intermittency) then correspond to thermal fluctuations, and this allows to estimate the exponents* τ_p *and* ζ_p *associated with moments of dissipation fluctuations and velocity fluctuations. This approach, initiated in an earlier note [12], is pursued here more carefully. In particular we derive probability distributions at finite Reynolds number for the dissipation and velocity fluctuations, and the latter permit an interpretation of numerical experiments [13]. Specifically, if* $p(z)dz$ *is the probability distribution of the radial velocity gradient we can explain why, when the Reynolds number* $\mathcal{R}$ *increases,* $\ln p(z)$ *passes from a concave to a linear then to a convex profile for large* z *as observed in [13]. We show that the central limit theorem applies to the dissipation and velocity distribution functions, so that a logical relation with the lognormal theory of Kolmogorov [10] and Obukhov is established. We find however that the lognormal behavior of the distribution functions fails at large value of the argument, so that a lognormal theory cannot correctly predict the exponents* τ_p *and* ζ_p

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1. Introduction.

In the present paper we study a measure ϖ which describes the velocities in a turbulent 3-D fluid at different levels of a scale of spatial lengths, assuming vanishing average velocities. The definition of ϖ combines basic ideas of statistical mechanics, the scaling laws of inviscid hydrodynamics, and decorrelation requirements which can hold only approximately. In particular we assume that the scaling factor κ of our scale of spatial lengths can be chosen such that successive levels of the scale are dynamically decorrelated in a natural manner. For certain questions we supplement the measure ϖ with a dissipation cutoff involving the viscosity ν. Since statistical mechanics is involved, ϖ automatically leads to fluctuations of the local velocity (and also of the local energy dissipation, and the local velocity gradients). Such fluctuations deviate from the homogeneous and isotropic model of turbulence, and have received the name of *intermittency*. One may hope that intermittent fluctuations have a *universal* distribution (i.e., a distribution independent of the geometry of the specific turbulent system considered). In particular, recent numerical experiments [13] support the idea that universality does not require highly developed turbulence, and holds also at relatively small Reynolds number. The probability measure ϖ discussed in this paper is universal by definition, and the problem we address here is to what extent ϖ fits the data provided by (lab and computer) experiments. A numerical approach to this problem is certainly desirable, but we shall here proceed analytically, taking advantage of the very explicit form of ϖ (see equation (3) below). We shall make many approximations, based in particular on the fact that the scaling constant κ is "large" (between 20 and 25), and we shall make liberal use of the notation $\approx$ (approximately equal). The spirit of our approach is thus to start from something very robust: the basic ideas of statistical mechanics (usually ignored in turbulence theory) and see to what extent a connection with experiments is achieved in spite of the crude approximations that we shall make. The results appear rather encouraging: this reflects probably the fact that the quantities of interest are *rates* associated with exponentially behaving quantities (such rates are typically rather stable under approximations).

We have thus evidence that our approach to turbulence, based on ideas of statistical mechanics, is basically correct. This leads in particular to some useful conclusions concerning the lognormal turbulence theory of Kolmogorov and Obukhov. From ϖ one obtains convolution product expressions (4) and (14) below, to which one can apply the central limit theorem: this gives approximately lognormal distributions for the local dissipation and the radial velocity increment. But the convolution products just referred to involve functions α, ϕ, ψ which are explicitly known, and one can check that the asymptotic exponents τ_p, ζ_p associated with the local energy dissipation and velocity increment distributions are not those predicted by the lognormal theory. One could say that the lognormal theory contains an element of truth but has limited applicability.

We review now the theory of hydrodynamic turbulence as proposed here, following an earlier paper [12]. In this theory, the fluid system is represented as an interacting union of subsystems (n,i) with finite degrees of freedom, in such a way that the turbulent energy cascade corresponds to a heat flow through the collection of subsystems. Specifically, the fluid system is enclosed in a cube C_0, and for each positive integer n, C_0 is cut in κ^{3n}

subcubes C_{ni} with edge size $\ell_n = \ell_0\kappa^{-n}$ for some choice of κ. A wavelet representation of divergence-free velocity fields provides a description of the inviscid Hamiltonian for the fluid in C_0 as an interacting collection of subsystems (n, i) roughly localized in the cubes C_{ni}. For the study of the turbulent energy cascade, one assumes that energy is input at large spatial wavelengths and dissipated at small spatial wavelengths. In the description in terms of the systems (n, i) the turbulent cascade corresponds to a heat flow from $n = 0$ to $n = N$ for large N (we only discuss the Hamiltonian *inertial range*). This means roughly that the temperature* at $n = 0$ is kept larger than the temperature at $n = N$ (in fact the heat flow from $n = 0$ is fixed rather than the temperature at $n = N$). Translating the study of the turbulent cascade into a problem of heat flow replaces the question of velocity fluctuations (i.e., intermittency) by a question of fluctuations in non-equilibrium statistical mechanics (in principle a very hard question). This was handled in [12] by arguing that the subsystems (n, i) are in approximate thermal equilibrium so that one can use the Boltzmann distribution to study fluctuations.

Let us proceed with a more precise discussion. We take our fluid to be contained in a cube C_0 of side ℓ_0 and let the velocity field $\mathbf{v}$ satisfy $\int \mathbf{v} = 0$, $\operatorname{div} \mathbf{v} = 0$. (A description in terms of the vorticity might be preferred, but would not change the present discussion). We divide C_0 into cubes C_{ni} of side $\ell_n = \ell_0\kappa^{-n}$ (with $i = 1, \ldots, \kappa^{3n}$) and denote by ϕ_{ni} the homothety mapping C_0 to C_{ni}. Choosing $2(\kappa^3 - 1)$ real vector fields $\mathbf{U}_\alpha$ on $\mathbf{R}^3$ with $\int \mathbf{U}_\alpha = 0$, $\operatorname{div} \mathbf{U}_\alpha = 0$, we assume that $\mathbf{v}$ has a unique wavelet decomposition into components (roughly) localized in the cubes C_{ni}:

$$\mathbf{v} = \sum_{n=0}^{\infty} \sum_{i=1}^{\kappa^{3n}} \sum_{\alpha=1}^{2(\kappa^3-1)} c_{ni\alpha} \mathbf{U}_\alpha \circ \phi_{ni}^{-1}$$

with $c_{ni\alpha} \in \mathbf{R}$. We write $\mathbf{v}_{ni} = \sum_\alpha c_{ni\alpha} \mathbf{U}_\alpha \circ \phi_{ni}^{-1}$. Notice that the values of $\mathbf{v}_{ni}$ at the centers of the κ^3 cubes $C_{(n+1)j} \subset C_{ni}$ are not independent (there are only $\approx (2/3)(\kappa^3 - 1)$ independent values).

Note that according to Kolmogorov theory $|\mathbf{v}_{ki}| \sim (\epsilon\ell_k)^{1/3}$, where ϵ is the mean energy dissipation per unit volume.

Energy conservation is expressed (as in the multifractal approaches, see [5], [3], [11], [6]) by

$$\frac{|\mathbf{v}_{ni}|^3}{\ell_n} = \frac{|\mathbf{v}_{(n+1)j}|^3}{\ell_{n+1}} \qquad \text{or} \qquad |\mathbf{v}_{(n+1)j}|^3 = \frac{|\mathbf{v}_{ni}|^3}{\kappa} \tag{1}$$

This is because the kinetic energy $|\mathbf{v}|^2/2$ in a given spatial range is weighted by the inverse of the time t spent in this range, and t scales like $\ell/|\mathbf{v}|$. We interpret (1) to mean that, corresponding to a given $\mathbf{v}_{ni}$, there corresponds a fluctuating $\mathbf{v}_{(n+1)j}$ such the the average

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* A different concept of temperature in the study of turbulence has been introduced by B. Castaing [4], as kindly pointed out by F. Bouchet, but Castaing's temperature is conserved along the cascade, contrary to the situation considered here.

of $V_{n+1} = |\mathbf{v}_{(n+1)j}|^3$ is $V_n/\kappa = |\mathbf{v}_{ni}|^3/\kappa$. It is natural to assume that the distribution of $\mathbf{v} = \mathbf{v}_{(n+1)j}$ in $\mathbf{R}^3$ maximizes the entropy, and is thus the Boltzmann distribution

$$\sim \exp\Big(-\frac{|\mathbf{v}|^3}{V_n\kappa^{-1}}\Big)d^3\mathbf{v}$$

i.e., $V = V_{n+1}$ has the distribution

$$\frac{1}{V_n\kappa^{-1}}\exp\Big(-\frac{V}{V_n\kappa^{-1}}\Big)dV \tag{2}$$

This is the choice that was made in [12]. It is a canonical ensemble expression expected to be reasonable* for moderately large V and large κ.

Note that we consider only interactions between subsystems (n,i), $(n+1,j)$ such that $C_{ni} \supset C_{(n+1)j}$. This is a strong form of the locality (in k space) usually assumed in turbulence theory. The lack of interaction between (n,i), (n,i') for $i \neq i'$ means that momentum conservation is not properly taken into account. In this respect we are following the multifractal approaches [5], [3], [11], [6]. The multifractal approaches are purely ad hoc, but physically motivated modifications have recently been introduced [15], [14]. The approach presented here has physical justification but makes the limiting assumption that the local average velocity of the fluid vanishes. For the study of concrete problems it will be necessary to make more general assumptions, taking into account the geometry of the situation considered. This can in principle be done, and will hopefully lead to more concrete studies of turbulence based on the physical ideas of nonequilibrium statistical mechanics.

In [12] we applied the ideas sketched above to the study of the exponents ζ_p associated with the moments $\langle|\Delta_\mathbf{r}\mathbf{v}|^p\rangle$ of velocity increments. In the present paper we study, in Section 2, the fluctuations of the scale dependent energy dissipation ϵ_ℓ and the corresponding exponents τ_p. In Section 3 we study the probability distribution of the the radial velocity increments $\Delta_r v$. Note that the exponents τ_p, ζ_p are large Reynolds number limits, while the distributions of the fluctuations of the energy dissipation and the radial velocity increments obtained in Sections 2 and 3 may be compared with experiments at finite Reynolds number. We discuss small radial velocity increments in Section 4, and large velocity increments in Section 5, obtaining an explanation of profiles obtained in [13]. The relation of the present paper with the lognormal theory of Kolmogorov [10] and Obukhov, is discussed in Section 6.

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* While the $|\mathbf{v}_{(n+1)j}|^3/\ell_{n+1}$ fluctuate, their sum over j is fixed ($\sim$ total energy flux in C_{ni}) so that very large values of V are forbidden in (2). This is related to the fact that fixing the average of the "energies" $|\mathbf{v}_{(n+1)j}|^3$ corresponds to a microcanonical ensemble, which is only asymptotically equivalent to the canonical ensemble (2), but the approximation should be good for moderate V and large κ.

Acknowledgments.

I am indebted to Christian Beck and Victor Yakhot for useful interaction during the preparation of this paper, and to Giovanni Gallavotti for earlier discussions; VY communicated reference [13] which is essential here. This work was initiated while I visited the Isaac Newton Institute in Cambridge, UK, at the end of 2013.

2. Fluctuations of the energy dissipation.

We follow (1) and (2), and write $V_{ni} = |\mathbf{v}_{ni}|^3$. Then the normalized distribution of $V = V_{(n+1)j}$ is given by (2) (we omit henceforth the index i). We find thus that, corresponding with the chain $C_0 \supset C_1 \supset \cdots \supset C_n$, with fixed V_0, there is a probability distribution

$$\frac{\kappa}{V_0}e^{-\kappa V_1/V_0}\frac{\kappa}{V_1}e^{-\kappa V_2/V_1}\cdots\frac{\kappa}{V_{n-1}}e^{-\kappa V_n/V_{n-1}}dV_1\cdots dV_n \tag{3}$$

and this extends to a probability measure ϖ on the space of infinite sequences $(V_n)_1^\infty$. In agreement with (1) we define the average energy dissipation ϵ_{ℓ_n} at size ℓ_n to be V_n/ℓ_n (this is an approximation of the definition in [10]). Writing $w_k = \kappa V_k/V_{k-1}$ (so that $\kappa^k V_k = V_0 w_1 \cdots w_k$) we have thus

$$\langle\epsilon_{\ell_n}^p\rangle = \int\frac{\kappa\, dV_1}{V_0}e^{-\kappa V_1/V_0}\int\frac{\kappa\, dV_2}{V_1}e^{-\kappa V_2/V_1}\cdots\int\frac{\kappa\, dV_n}{V_{n-1}}e^{-\kappa V_n/V_{n-1}}\Big(\frac{V_n}{\ell_n}\Big)^p$$

$$= \Big(\frac{V_0}{\ell_0}\Big)^p\int dw_1\, e^{-w_1}\int dw_2\, e^{-w_2}\cdots\int dw_n\, e^{-w_n}(w_1w_2\cdots w_n)^p$$

$$= \Big(\frac{V_0}{\ell_0}\Big)^p\Big(\int w^p e^{-w}\, dw\Big)^n = \Big(\frac{V_0}{\ell_0}\Big)^p\Big(\Gamma(1+p)\Big)^n$$

so that for large n

$$\frac{\ln\langle\epsilon_{\ell_n}^p\rangle}{\ln\ell_n} = \frac{p\ln(V_0/\ell_0)+n\ln\Gamma(1+p)}{\ln\ell_0 - n\ln\kappa} \approx -\frac{\ln\Gamma(1+p)}{\ln\kappa}$$

and we obtain $\langle\epsilon_{\ell_n}^p\rangle \approx \ell_n^{\tau_p}$ with

$$\tau_p = -\frac{\ln\Gamma(1+p)}{\ln\kappa} \qquad \text{so that} \qquad \zeta_p = \frac{p}{3}+\tau_{p/3}$$

where we have used the value of ζ_p obtained in [12].

Note by the way that the validity of (3) is limited by the dissipation due to the viscosity ν, i.e., we must have $V_n^{1/3}\ell_n > \nu$; this will be used later.

Let $D_n(x)\,dx$ be the distribution, given V_0, of $x = \epsilon_{\ell_n}/\epsilon_{\ell_0}$ on $\mathbf{R}_+$. According to (3) we have

$$D_n(x) = \int\frac{\kappa dV_1}{V_0}e^{-\kappa V_1/V_0}\int\frac{\kappa V_2}{V_1}e^{-\kappa V_2/V_1}\cdots\int\frac{\kappa dV_n}{V_{n-1}}e^{-\kappa V_n/V_{n-1}}\,\delta(x-\frac{V_n}{\ell_n}\cdot\frac{\ell_0}{V_0})$$

$$= \int dw_1\, e^{-w_1} \int dw_2\, e^{-w_2} \cdots \int dw_n\, e^{-w_n}\, \delta(x - w_1 \cdots w_n)$$

so that, writing $\alpha(t) = \exp(t - e^t)$, we have

$$e^t D_n(e^t) = e^t \int \cdots \int \delta(e^t - e^{t_1 + \ldots + t_n}) \prod_{k=1}^{n} \Big(\exp(t_k - e^{t_k})\, dt_k \Big)$$

$$= \int \cdots \int \delta(t_1 + \ldots + t_n - t) \prod_{k=1}^{n} (\alpha(t_k)\, dt_k) = \alpha^{*n}(t) \tag{4}$$

where $*$ denotes the convolution product. The central limit theorem implies thus that $e^t D_n(e^t)\, dt$ is asymptotically Gaussian for large n, i.e., that $D_n(x)\, dx$ is asymptotically a lognormal distribution. Therefore the distribution of ϵ_{ℓ_n} (given V_0) is asymptotically lognormal.

A word of caution is needed here: when $t \to -\infty$ we have $\alpha(t) \approx \eta_1(-t) = \eta_1(|t|)$, where we have written $\eta_m(t) = \theta(t) e^{-mt}$ and we have $\eta_m^{*n}(t) = \theta(t) t^{n-1} e^{-mt}/(n-1)!$ Therefore, when $t \to -\infty$ we have $e^t D_n(e^t) \approx \eta_1^{*n}(|t|)$ and, for small x, $D_n(x)$ diverges like $|t|^{n-1}/(n-1)! = |\ln x|^{n-1}/(n-1)!$ This divergence is however cut off by the condition $V_n^{1/3} \ell_n > \nu$, or $\epsilon_n = V_n/\ell_n = (V_n^{1/3}\ell_n)^3/\ell_n^4 > \nu^3/\ell_n^4 = \kappa^{4n}\nu^3/\ell_0^4$, i.e.,

$$V_n^{1/3} \ell_n > \nu \qquad \Leftrightarrow \qquad \frac{\epsilon_n}{\epsilon_0} = x > \frac{\kappa^{4n}\nu^3}{V_0 \ell_0^3} = \kappa^{4n} \mathcal{R}^{-3} \tag{5}$$

where $\mathcal{R} = V_0^{1/3} \ell_0/\nu$ is the Reynolds number.

We have

$$\alpha^{*n}(t) = \int dt_1 \cdots \int dt_n\, \delta(t_1 + \ldots + t_n - t) \prod_{k=1}^{n} \exp(t_k - e^{t_k})$$

$$= e^t \int dt_1 \cdots \int dt_n\, \delta(t_1 + \ldots + t_n - t) \exp(-\sum_{k=1}^{n} e^{t_k})$$

$$= e^t \int dt'_1 \cdots \int dt'_n\, \delta(t'_1 + \ldots + t'_n) \exp(-e^{t/n} \sum_{k=1}^{n} e^{t'_k})$$

A quadratic approximation of $\sum e^{t'_k}$ gives

$$\sum_{k=1}^{n} e^{t'_k} \approx \sum_{k=1}^{n} (1 + t'_k + \frac{1}{2} {t'_k}^2) = n + \frac{1}{2} \sum_{k=1}^{n} {t'_k}^2 = n + \frac{1}{2} [\sum_{k=1}^{n-1} {t'_k}^2 + (\sum_{k=1}^{n-1} t'_k)^2] = n + \frac{1}{2} t'^T A t'$$

where $t' \in \mathbf{C}^{n-1}$ has components $t'_1, \ldots, t'_{n-1}$ and A is a $(n-1) \times (n-1)$ matrix with $A_{k\ell} = 2$ if $k = \ell$, $= 1$ otherwise. One finds $\det A = n$, and we have thus

$$\alpha^{*n}(t) \approx e^t \int dt'_1 \cdots \int dt'_{n-1} \exp[-e^{t/n}(n + \frac{1}{2} t'^T A t')]$$

$$= e^t \exp(-ne^{t/n})\Big(\frac{(2\pi)^{n-1}}{ne^{t(n-1)/n}}\Big)^{1/2} = \Big(\frac{(2\pi)^{n-1}}{n}\Big)^{1/2} e^{t(1/n+1)/2}\exp(-ne^{t/n})$$

$$D_n(x) \approx \Big(\frac{(2\pi)^{n-1}}{n}\Big)^{1/2} x^{(1/n-1)/2}\exp(-nx^{1/n})$$

and for large X

$$\int_X^\infty D_n(x)\,dx \approx \Big(\frac{(2\pi)^{n-1}}{n}\Big)^{1/2}(X^{(1-1/n)/2}+\ldots)\exp(-nX^{1/n})$$

Therefore, writing $S_j = \{(V_n)_1^\infty : V_j^{1/3}\ell_j > \nu\}$ we have, using (5),

$$\ln \varpi(S_n) = \ln \int_{\kappa^{4n}\mathcal{R}^{-3}}^\infty D_n(x)\,dx \approx -n\kappa^4\mathcal{R}^{-3/n} \tag{6}$$

so that $\varpi(S_n)$ decreases exponentially with n.

We shall use later the results

$$\int t\alpha(t)\,dt = \int_0^\infty (\ln x)e^{-x}\,dx = -\gamma \quad , \quad \int t^2\alpha(t)\,dt = \int_0^\infty (\ln x)^2 e^{-x}\,dx = \gamma^2 + \frac{\pi^2}{6}$$

where γ is Euler's constant, so that $\mathrm{Var}(\alpha) = \pi^2/6$.

3. Fluctuations of velocity increments.

This Section contains the main technical machinery of the present paper. Contrary to the study of the exponents τ_p, ζ_p we shall not take the limit of infinite Reynolds number ($n \to \infty$).

Here we study the distribution of $\Delta_{\mathbf{r}}\mathbf{v} = \mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x})$, assuming that $\mathbf{x}$ and $\mathbf{x}+\mathbf{r}$ are in the same cube $C_{k,i(k)}$ for $k \le n$, and in different cubes for $k > n$. Then $\Delta_{\mathbf{r}}\mathbf{v} = \sum_k \Delta_k\mathbf{v}$ where

$$\Delta_k\mathbf{v} = \mathbf{v}_{k,i(k)}(\mathbf{x}+\mathbf{r}) - \mathbf{v}_{k,i(k)}(\mathbf{x}) \qquad \text{for } k \le n$$

$$\Delta_k\mathbf{v} = \mathbf{v}_{k,i'(k)}(\mathbf{x}+\mathbf{r}) - \mathbf{v}_{k,i(k)}(\mathbf{x}) \qquad \text{for } k > n$$

We see that $|\Delta_k\mathbf{v}|$ must tend to 0 when $|k-n|$ grows. A simple approximation is thus to replace $\Delta_{\mathbf{r}}\mathbf{v} = \sum_k \Delta_k\mathbf{v}$ by $\Delta_n\mathbf{v}$, and $\Delta_n\mathbf{v}$ by $\mathbf{v}_n = \mathbf{v}_{n,i(n)}$ as was done in [12] to estimate the exponents ζ_p of $\langle|\mathbf{v}_n|^p\rangle \approx \ell_n^{\zeta_p}$ (as earlier we write $\mathbf{v}_k$ instead of $\mathbf{v}_{k,i(k)}$).

We shall now study specifically the distribution of the *radial increment* $\Delta_r v$ obtained by choosing a coordinate axis in $\mathbf{R}^3$ (the x-axis), taking $\mathbf{r}$ along the x-axis, and letting r, $\Delta_r v$ be the components of $\mathbf{r}$, $\Delta_{\mathbf{r}}\mathbf{v}$ along the x-axis. The ideas will be the same as in [12], presented a bit more carefully, and will give interesting information on the distribution of large values of $\Delta_r v$. Our discussion will be approximate, in particular the distribution obtained will be symmetric with respect to the reflection $\Delta_r v \to -\Delta_r v$.

In the continuous limit we can think of the vector field $\mathbf{U}_\alpha$ used to represent $\mathbf{v}$ as given by $\mathbf{U}_\alpha(\mathbf{x}) = \mathbf{j}e^{i\mathbf{k}\cdot\mathbf{x}}$ with $\mathbf{j}\cdot\mathbf{k} = 0$. If $\mathbf{k}$ is uniformly distributed on the sphere $|\mathbf{k}| = 1$

and $\mathbf{j}$ on the circle $|\mathbf{j}| = 1$, the component u of $\mathbf{j}$ along a coordinate axis in $\mathbf{R}^3$ (the x-axis) has a distribution $\sim du/\sqrt{\alpha^2 - u^2}$ where $\alpha^2 = 1 - \beta^2$ and β is uniformly distributed on $[-1, 1]$. Therefore the x-component u of $\mathbf{U}_\alpha$ has a distribution

$$\sim du \int_0^{\sqrt{1-u^2}} \frac{d\beta}{\sqrt{1 - u^2 - \beta^2}} = du \int_0^1 \frac{d\gamma}{\sqrt{1 - \gamma^2}} \sim du \tag{7}$$

on $[-1, 1]$. We shall use this fact in a moment.

If we fix a velocity field distribution $\mu(d\mathbf{v}_0)$ in C_0 we can, using (3), compute the average of a function $\Phi((|\mathbf{v}_k|^3)_{k=0}^N)$ by

$$\langle \Phi \rangle = \int_0^\infty [\int \mu(d\mathbf{v}_0)\, \delta(|\mathbf{v}_0|^3 - V_0)] F(V_0)$$

$$F(V_0) = \Big(\prod_{k=1}^N \int_0^\infty \frac{\kappa dV_k}{V_{k-1}} e^{-\kappa V_k / V_{k-1}} \Big) \Phi(V_0, V_1, \ldots, V_N) \tag{8}$$

Given $V_1, \ldots, V_n$, the probability distribution of the x-components $v_1, \ldots, v_N$ of $\mathbf{v}_1$, $\ldots, \mathbf{v}_N$ is, in view of (7), given by

$$\Phi(v_1, \ldots, v_N; V_1, \ldots, V_N)\, dv_1 \cdots dv_N = \prod_{k=1}^N \Big(\frac{dv_k}{2V_k^{1/3}} \chi_{[-V_k^{1/3}, V_k^{1/3}]}(v_k) \Big) \tag{9}$$

where χ_A denotes the characteristic function of A. This particular choice of Φ corresponds by (8) to

$$F(v_1, \ldots, v_N; V_0) = \prod_{k=1}^N \Big(\int_0^\infty \frac{\kappa dV_k}{V_{k-1}} e^{-\kappa V_k / V_{k-1}} \cdot \frac{1}{2V_k^{1/3}} \cdot \chi_{[-V_k^{1/3}, V_k^{1/3}]}(v_k) \Big) \tag{10}$$

More generally, if $\lambda_1, \ldots, \lambda_N > 0$ we may study the probability that the x-components of $\lambda_1 \mathbf{v}_1, \ldots, \lambda_N \mathbf{v}_N$ are $u_1, \ldots, u_N$. Then (9) and (10) are replaced by

$$\Phi_\lambda(u_1, \ldots, u_N; V_1, \ldots, V_N)\, du_1 \cdots du_N = \prod_{k=1}^N \Big(\frac{du_k/\lambda_k}{2V_k^{1/3}} \chi_{[-V_k^{1/3}, V_k^{1/3}]}(u_k/\lambda_k) \Big)$$

$$F_\lambda(u_1, \ldots, u_N; V_0) = \prod_{k=1}^N \Big(\int_0^\infty \frac{\kappa dV_k}{V_{k-1}} e^{-\kappa V_k / V_{k-1}} \cdot \frac{1}{2\lambda_k V_k^{1/3}} \cdot \chi_{[-\lambda_k V_k^{1/3}, \lambda_k V_k^{1/3}]}(u_k) \Big) \tag{11}$$

We have denoted above by r, v_k the x-components of $\mathbf{r}$, $\mathbf{v}_k$; let also u_k be the x-components of $\Delta_k \mathbf{v}$. We claim that the probability distribution of the velocity increment u_k is roughly the same as the distribution of a suitable multiple $\lambda_k v_k$ of v_k, as follows:

if $k \leq n$: $\quad \lambda_k \approx \kappa \ell_{n+1}/\ell_k$

if $k \geq n$: $\qquad \lambda_k \approx 1$

This simply expresses the fact that $r \approx \ell_{n+1}$ and that the correlation length of v_k is $\approx \ell_k/\kappa$. We may thus apply (11) with λ_k as above.

Let $F_\Delta(u;V_0)\,du$ denote the probability distribution of $u = \Delta_r v$ for given V_0. We have thus

$$F_\Delta(u;V_0) = \Big(\prod_{k=1}^N \int_0^\infty \frac{\kappa dV_k}{V_{k-1}} e^{-\kappa V_k/V_{k-1}}\Big)\Phi_\Delta(u;V_1,\ldots,V_N)$$

where, with the notation indicated,

$$\Phi_\Delta(u;V_1,\ldots,V_N) = \Big(\prod_{k=1}^N \int \frac{du_k}{2\lambda_k V_k^{1/3}}\,\chi_{[-\lambda_k V_k^{1/3},\lambda_k V_k^{1/3}]}(u_k)\Big)\delta(u - \sum u_k)$$

In view of the values of the λ_k, a rough estimate of Φ_Δ is obtained by taking $\sum u_k = u_n$ so that*

$$\Phi_\Delta(u;V_1,\ldots,V_N) \approx \frac{1}{2V_n^{1/3}}\,\chi_{[-V_n^{1/3},V_n^{1/3}]}(u) = \frac{1}{2V_n^{1/3}}\,\chi_{[|u|^3,\infty)}(V_n)$$

and

$$F_\Delta(u;V_0) \approx F_\Delta(u) = \Big(\prod_{k=1}^n \int_0^\infty \frac{\kappa dV_k}{V_{k-1}} e^{-\kappa V_k/V_{k-1}}\Big)\frac{1}{2V_n^{1/3}}\,\chi_{[-V_n^{1/3},V_n^{1/3}]}(u)$$

Writing $W_k = \kappa^k V_k$ and $w_k = W_k/W_{k-1}$ (so that $W_k = V_0 w_1 \cdots w_k$) we find

$$F_\Delta(u) = \Big(\prod_{k=1}^n \int_0^\infty \frac{dW_k}{W_{k-1}} e^{-W_k/W_{k-1}}\Big)\frac{1}{2(\kappa^{-n}W_n)^{1/3}}\,\chi_{[|u|^3,\infty)}(\kappa^{-n}W_n)$$

$$= \frac{\kappa^{n/3}}{2}\Big(\prod_{k=1}^{n-1} \int_0^\infty \frac{dW_k}{W_{k-1}} e^{-W_k/W_{k-1}}\Big)\int_{\kappa^n|u|^3}^\infty \frac{dW_n}{W_{n-1}}\,e^{-W_n/W_{n-1}}\,\frac{1}{W_n^{1/3}}$$

$$= \frac{\kappa^{n/3}}{2}\Big(\prod_{k=1}^{n-1} \int_0^\infty dw_k\,e^{-w_k}\Big)\int_{\kappa^n|u|^3/V_0 w_1\cdots w_{n-1}} dw_n\,e^{-w_n}\,\frac{1}{(V_0 w_1\cdots w_n)^{1/3}}$$

$$= \frac{1}{2}\big(\frac{\kappa^n}{V_0}\big)^{1/3}\int\cdots\int_{w_1\cdots w_n > (\kappa^n/V_0)|u|^3} \prod_{k=1}^n \frac{dw_k\,e^{-w_k}}{w_k^{1/3}}$$

Since $F_\Delta(u)$ is an even function of u it is natural to consider the probability distribution $G_n(y)dy$ of $y = (\kappa^n/V_0)^{1/3}|u|$ corresponding to $F_\Delta(u)\,du$. We have

$$G_n(y) = \int\cdots\int_{w_1\cdots w_n > y^3} \prod_{k=1}^n \frac{dw_k\,e^{-w_k}}{w_k^{1/3}}$$

* Notice that Φ_Δ is a convolution product with respect to the variable u: replacing $\sum u_k$ by u_n replaces $N-1$ factors of this convolution product by a Dirac δ.

$$G_n(e^t) = \int \cdots \int_{t_1+\cdots+t_n>t} \prod_{k=1}^{n} (3\exp(2t_k - e^{3t_k})\, dt_k)$$

$$= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dt_1 \cdots dt_{n-1} \int_{t-t_1-\cdots-t_{n-1}}^{+\infty} dt_n \prod_{k=1}^{n} (3\exp(2t_k - e^{3t_k})) \qquad (12)$$

Define

$$\phi(t) = 3\exp(3t - e^{3t}) \qquad , \qquad \psi(t) = e^t \int_t^\infty 3\exp(2s - e^{3s})\, ds$$

so that

$$\phi(t) = -e^t \frac{d}{dt}(e^{-t}\psi(t)) \qquad , \qquad \psi(t) = e^t \int_t^\infty e^{-s}\phi(s)\, ds$$

and $\phi(t)dt, \psi(t)dt$ are probability distributions on $\mathbf{R}$. Note that

$$e^{-t}\psi(t) = \int_{e^t}^\infty 3v dv\, e^{-v^3} = \int_{e^{3t}}^\infty \frac{dw}{w^{1/3}} e^{-w} = \Gamma(\frac{2}{3}, e^{3t})$$

where $\Gamma(a,x)$ is the upper incomplete Gamma function, so that $\psi(t) \approx \Gamma(2/3)e^t$ for $t \to -\infty$, $\approx \exp(-e^{3t})$ for $t \to \infty$.

Using (12) we find the convolution expression:

$$-e^t \frac{d}{dt} G_n(e^t) = e^t \int \cdots \int \delta(t_1 + \ldots + t_n - t) \prod_{k=1}^{n} (3\exp(2t_k - e^{3t_k}) dt_k)$$

$$= \int \cdots \int \delta(t_1 + \ldots + t_n - t) \prod_{k=1}^{n} (3\exp(3t_k - e^{3t_k}) dt_k)$$

$$= \int \cdots \int \delta(t_1 + \ldots + t_n - t) \prod_{k=1}^{n} (\phi(t_k) dt_k) = \phi^{*n}(t) \qquad (13)$$

hence

$$e^t G_n(e^t) = e^t \int_t^\infty ds\, e^{-s} \phi^{*n}(s) = (\phi^{*(n-1)} * \psi)(t) \qquad (14)$$

We discuss now more precisely the distribution $G_n(y)dy$ obtained above. For $t \to -\infty$, $\phi(t) \approx 3\eta_3(|t|)$ [where $\eta_m(t) = \theta(t)e^{-mt}$ as defined in Section 2], and $\psi(t) \approx \Gamma(2/3)e^{-|t|}$, so that

$$\phi * \psi(t) \approx 3\Gamma(\frac{2}{3}) \int \theta(|t| - s)\theta(s) e^{-(|t|-s)} e^{-3s}\, ds = 3\Gamma(\frac{2}{3}) e^{-|t|} \int_0^{|t|} e^{-2s}\, ds$$

$$= \frac{3}{2}\Gamma(\frac{2}{3}) e^{-|t|}(1 - e^{-2|t|}) \approx \frac{3}{2}\Gamma(\frac{2}{3}) e^{-|t|}$$

By induction, $\phi^{*(n-1)} * \psi(t) \approx (3/2)^{n-1}\Gamma(2/3)e^{-|t|}$ when $t \to -\infty$. Therefore $G_n(y) \to (3/2)^{n-1}\Gamma(2/3)$ when $y \to 0$, i.e.,

$$G_n(0) = (3/2)^{n-1}\Gamma(2/3) = (3/2)^n\Gamma(5/3) \tag{15}$$

.

If we write $\tilde{\phi}(t) = e^{-t}\phi(t)$, (13) gives

$$-\frac{d}{dt}G_n(e^t) = \int\cdots\int \delta(t_1 + \ldots + t_n - t)\prod_{k=1}^{n}(\tilde{\phi}(t_k)dt_k) = \tilde{\phi}^{*n}(t) \tag{16}$$

so that

$$-\frac{d}{dy}G_n(y) = \frac{1}{y}\tilde{\phi}^{*n}(\ln y) \geq 0$$

i.e., $G_n(y)$ is a decreasing function of y.

We study now the behavior of $G_n(y)$ for large y. We have

$$\tilde{\phi}^{*n}(t) = \int dt_1 \cdots \int dt_n\, \delta(t_1 + \ldots + t_n - t)\prod_{k=1}^{n}(3\exp{(2t_k - e^{3t_k})})$$

$$= e^{2t}\int dt_1 \cdots \int dt_n\, \delta(t_1 + \ldots + t_n - t)\prod_{k=1}^{n}(3\exp{(-e^{3t_k})})$$

$$= 3e^{2t}\int d\tau_1 \cdots \int d\tau_n\, \delta(\tau_1 + \ldots + \tau_n - 3t)\exp{(-\sum_{k=1}^{n} e^{\tau_k})}$$

$$= 3e^{2t}\int d\tau_1' \cdots \int d\tau_n'\, \delta(\tau_1' + \ldots + \tau_n')\exp{(-e^{3t/n}\sum_{k=1}^{n} e^{\tau_k'})}$$

A quadratic approximation of $\sum e^{\tau_k'}$ gives

$$\sum_{k=1}^{n} e^{\tau_k'} \approx \sum_{k=1}^{n}(1 + \tau_k' + \frac{1}{2}{\tau_k'}^2) = n + \frac{1}{2}\sum_{k=1}^{n}{\tau_k'}^2 = n + \frac{1}{2}[\sum_{k=1}^{n-1}{\tau_k'}^2 + (\sum_{k=1}^{n-1}\tau_k')^2] = n + \frac{1}{2}\tau'^T A\tau'$$

where $\tau' \in \mathbf{C}^{n-1}$ has components $\tau_1', \ldots, \tau_{n-1}'$ and A is a $(n-1)\times(n-1)$ matrix with $A_{k\ell} = 2$ if $k = \ell$, $= 1$ otherwise. One finds $\det A = n$, and we have thus

$$\tilde{\phi}^{*n}(t) \approx 3e^{2t}\int d\tau_1' \cdots \int d\tau_{n-1}' \exp[-e^{3t/n}(n + \frac{1}{2}\tau'^T A\tau')]$$

$$= 3e^{2t}\exp(-ne^{3t/n})\Big(\frac{(2\pi)^{n-1}}{ne^{3t(n-1)/n}}\Big)^{1/2} = 3\Big(\frac{(2\pi)^{n-1}}{n}\Big)^{1/2} e^{t(3/n+1)/2}\exp(-ne^{3t/n})$$

Therefore, by (16),

$$-\frac{dG_n}{dy} \approx 3\Big(\frac{(2\pi)^{n-1}}{n}\Big)^{1/2} y^{(3/n-1)/2} \exp(-ny^{3/n})$$

and finally for large y

$$G_n(y) \approx \Big(\frac{(2\pi)^{n-1}}{n}\Big)^{1/2} (y^{(1-3/n)/2} + \ldots) \exp(-ny^{3/n}) \tag{17}$$

$$\int_Y^\infty G_n(y)\,dy \approx \frac{1}{3}\Big(\frac{(2\pi)^{n-1}}{n}\Big)^{1/2} (Y^{3(1-3/n)/2} + \ldots) \exp(-nY^{3/n})$$

In particular

$$G_1(y) \approx y^{-1}\exp(-y^3) \qquad , \qquad G_2(y) \approx \sqrt{\pi}\, y^{-1/4} \exp(-2y^{3/2})$$

$$G_3(y) \approx \frac{2\pi}{\sqrt{3}} \exp(-3y) \qquad , \qquad G_4(y) \approx \frac{(2\pi)^{3/2}}{2} y^{1/8} \exp(-4y^{3/4})$$

4. The distribution of small velocity increments.

Instead of $F_\Delta(u)du$, which approximates the distribution of the x-component u of $\Delta_r\mathbf{v}$, it is convenient to consider the distribution $p_n(z)dz$ of $z = u(\mathrm{Var}(F_\Delta))^{-1/2}$ so that $\mathrm{Var}(p_n) = 1$. We claim that in our case $y = ((1/3)\Gamma(5/3)^n)^{1/2}|z|$, so that

$$p_n(z) = \frac{1}{2}\Big(\frac{1}{3}\Gamma\Big(\frac{5}{3}\Big)^n\Big)^{1/2} G_n\left(\Big(\frac{1}{3}\Gamma\Big(\frac{5}{3}\Big)^n\Big)^{1/2} |z|\right) \tag{18}$$

[Since $\mathrm{Var}(p_n) = 1$ by definition we have to estimate $\mathrm{Var}(G_n)$. We find, using formula (16),

$$\begin{aligned} 3\mathrm{Var}(G_n) &= \int_0^\infty 3y^2 G_n(y)\,dy = -\int y^3 \frac{dG_n}{dy}\,dy = \int y^2 \tilde{\phi}^{*n}(\ln y)\,dy \\ &= \int_{-\infty}^\infty \tilde{\phi}^{*n}(t) e^{3t}\,dt = \Big(\int \tilde{\phi}(t) e^{3t}\,dt\Big)^n = \Big(\int \phi(t) e^{2t}\,dt\Big)^n \\ &= \Big(\int 3e^{5t}\exp(-e^{3t})\,dt\Big)^n = \Big(\int_0^\infty w^{2/3} e^{-w}\,dw\Big)^n = \Gamma\Big(\frac{5}{3}\Big)^n \end{aligned}$$

where $\Gamma(5/3) \approx 0.9027452929.$]

A natural guess is that the distribution $G_n(y)dy$ corresponding to the distribution $p(z)dz$ measured at Reynolds number $\mathcal{R}$ is obtained when $\ell_n (= \kappa^{-n}\ell_0)$ is equal to the Kolmogorov dissipation length $\approx \ell_0\mathcal{R}^{-3/4}$ so that in our model $\mathcal{R}^{-3/4} \approx \ell_n/\ell_0 = \kappa^{-n}$ or $\kappa^n = \mathcal{R}^{3/4}$ or $n\ln\kappa = (3/4)\ln\mathcal{R}$, or

$$n = \frac{3}{4}\frac{\ln\mathcal{R}}{\ln\kappa} \approx 0.24\ln\mathcal{R} \tag{19}$$

with the estimate $(\ln\kappa)^{-1} = 0.32$ in [12]. This gives $n = 1.10$ for $\mathcal{R} = 96$ and $n = 2.03$ for $\mathcal{R} = 4638$.

Since $G_n(0) = (3/2)^n\Gamma(5/3)$ by (15), we obtain from (18)

$$p_n(0) = \frac{1}{2}\Big(\frac{1}{3}\Gamma\Big(\frac{5}{3}\Big)^n\Big)^{1/2}\Big(\frac{3}{2}\Big)^n\Gamma\Big(\frac{5}{3}\Big) = \frac{1}{2\sqrt{3}}\Gamma\Big(\frac{5}{3}\Big)\cdot\Big(\frac{3}{2}\Gamma\Big(\frac{5}{3}\Big)^{1/2}\Big)^n$$

Here we have

$$\frac{1}{2\sqrt{3}}\Gamma\Big(\frac{5}{3}\Big) = 0.260600118 \qquad \text{and} \qquad \frac{3}{2}\Gamma\Big(\frac{5}{3}\Big)^{1/2} = 1.425193638$$

The above prediction for $p(0)$ can be compared with experimental data (see [13]).

For small $y > 0$ we have

$$-G_n'(y) = \frac{1}{y}\tilde{\phi}^{*n}(\ln y) \approx \frac{1}{y}3^n\eta_2^{*n}(|\ln y|) = \frac{1}{y}3^n e^{-2|\ln y|}\frac{|\ln y|^{n-1}}{(n-1)!} = \frac{3^n}{(n-1)!}y|\ln y|^{n-1}$$

from which one can estimate

$$p_n'(z) = \frac{1}{6}\Gamma\Big(\frac{5}{3}\Big)^n G_n'\left(\Big(\frac{1}{3}\Gamma\Big(\frac{5}{3}\Big)^n\Big)^{1/2}|z|\right)$$

for small $z > 0$.

5. The distribution of large velocity increments.

We consider now the problem of comparing the distributions $p_n(z)dz$ with distributions obtained in (numerical) experiments [13] for large z. Note that $\ln p_n(z)$ corresponds via (18) to $\ln G_n(y) \approx -ny^{3/n}$ (see (17)). Superficially, the curves in [13] (for instance Fig. 6) look like $\ln p_2$, $\ln p_3$, $\ln p_4$, passing from a concave ($\ln p_2$) to a linear ($\ln p_3$) then a convex ($\ln p_4$) behavior at large z when $\mathcal{R}$ increases. Things are however not quite that simple.

The relation (19) gives $n = 2.03$ for $\mathcal{R} = 4638$ and this does not compare well with the data of [13]. In fact $n = 2$ gives a concave function $\log p_2$ while for $\mathcal{R} = 4638$ Fig. 6 of [13] gives $\log p$ linear or convex at large z. To resolve this conflict we shall be more careful and take into account the fluctuations of n at given $\mathcal{R}$.

In the presence of viscosity the absolute value of the radial velocity gradient is

$$\omega = \Big|\frac{\partial v_x}{\partial x}\Big| \approx \frac{\Delta_r v}{r} \text{ when } r \text{ is the dissipation length}$$

We fix ℓ_0, V_0, write $\omega_0 = V_0^{1/3}/\ell_0$, and define $P(\xi)\,d\xi$ to be the probability distribution of $\xi = \omega/\omega_0$.

As in Section 2 let ϖ denote the probability measure on sequences $(V_n)_1^\infty$ defined by (3). Then with high ϖ-probability the sequence $(V_1, \ldots, V_n, \ldots)$ decreases to 0. Writing again $S_j = \{(V_n) : V_j^{1/3}\ell_j > \nu\}$ we have thus with high probability $S_1 \supset \ldots \supset S_n \supset \ldots$ and (6) shows that $\varpi(S_n)$ decreases exponentially with n.

We use the near partition into the sets $S_{n-1}\backslash S_n$ to approximate $P(\xi)$ as

$$P(\xi) \approx \sum_n P_n^*(\xi)$$

with

$$P_n^*(\xi) = \ell_n\omega_0\Big(\prod_{k=1}^n \int \frac{\kappa\, dV_k}{V_{k-1}} e^{-V_k/V_{k-1}}\Big)^* \frac{1}{V_n^{1/3}} \chi_{[0,V_n^{1/3}]}(\ell_n\omega_0\xi)$$

where $(\ldots)^*$ means that the integrals are restricted to $S_{n-1}\backslash S_n$. [Note that there is some arbitrariness in computing the gradient ξ at level n for a viscosity cutoff between level $n-1$ and level n].

Suppose now that for some j we have

$$\frac{\nu}{\ell_j} \le \ell_j\omega_0\xi \qquad \text{i.e.} \qquad \kappa^{2j} \le \xi\mathcal{R}$$

Then if $n \le j$ we find $P_n^*(\xi) = 0$ because

$$\Big(\frac{\nu}{\ell_n} \le \ell_n\omega_0\xi \qquad \text{and} \qquad V_n^{1/3} \le \frac{\nu}{\ell_n}\Big) \qquad \Rightarrow \qquad \chi_{[0,V_n^{1/3})}(\ell_n\omega_0\xi) = 0$$

We have thus

$$P(\xi) \approx \sum_{n>j} P_n^*(\xi) \qquad \text{if} \qquad \kappa^{2j} \le \xi\mathcal{R}$$

Note that if we remove the $*$ in the definition of P_n^* we have

$$P_n(\xi) = \ell_n\omega_0.2F_\Delta(\ell_n\omega_0\xi) = \ell_n\omega_0(\kappa^n/V_0)^{1/3}G_n((\kappa^n/V_0)^{1/3}\ell_n\omega_0\xi)$$

$$= \kappa^{-2n/3}G_n(\kappa^{-2n/3}\xi) \approx \Big(\frac{(2\pi)^{n-1}}{n}\Big)^{1/2}\kappa^{1-n}\xi^{(1-3/n)/2}\exp(-n(\kappa^{-2n/3}\xi)^{3/n})$$

which shows that the variable ξ in P corresponds to $y = \kappa^{-2n/3}\xi$ in G_n so that the values of ξ corresponding to the abscissa z in [13] are quite large.

We consider in particular the case $\mathcal{R} = 4638$ in Fig. 6 of [13]. The condition $\kappa^{2j} \le \xi\mathcal{R}$ becomes $\xi \ge 34.5$ for $j = 2$ (with $\kappa = 20$) so that, except for z close to 0 only the P_n^* with $n > 2$ contribute. This gives a reasonable interpretation of the fact that $\log_{10} p(z)$ appears to be dominated by P_3, with some admixture of P_4 visible at the largest z.

The above discussion is somewhat qualitative, based on an analytic study of the measure ϖ defined by (3). Numerical studies, using (3) and the cutoff $V_n^{1/3}\ell_n > \nu$ are desirable, to test quantitatively how accurate a description of turbulence by the measure ϖ really is.

6. Comparison with the ideas of Kolmogorov and Obukhov.

We note that, with $\alpha(t) = \exp(t - e^t)$ as in Section 2, we have $\phi(t) = 3\alpha(3t)$ so that

$$\int t\phi(t)\,dt = -\frac{\gamma}{3} \qquad , \qquad \int t^2\phi(t)\,dt = \frac{1}{9}(\gamma^2 + \frac{\pi^2}{6})$$

and also

$$\int t\psi(t)\,dt = \int te^t\,dt \int_t^\infty e^{-s}\phi(s)\,ds = \int dt\,[e^t(t-1)]e^{-t}\phi(t) = \int dt\,(t-1)\phi(t) = -\frac{\gamma}{3} - 1$$

$$\int t^2\psi(t)\,dt = \int t^2e^t\,dt \int_t^\infty e^{-s}\phi(s)\,ds = \int dt\,[e^t(t^2-2t+2)]e^{-t}\phi(t) = \int dt\,(t^2-2t+2)\phi(t)$$

$$= (\frac{\gamma}{3} + 1)^2 + \frac{1}{9}\cdot\frac{\pi^2}{6} + 3$$

Therefore $\mathrm{Var}(\phi) = \pi^2/54$ and $\mathrm{Var}(\psi) = \pi^2/54 + 3$.

The central limit theorem applied to (14) implies then that $e^tG_n(e^t)$ is asymptotically Gaussian for large n:

$$e^tG_n(e^t) \approx \frac{1}{\sigma\sqrt{2\pi}} \exp\big(-\frac{(t+\bar{\mu})^2}{2\sigma^2}\big)$$

with $\bar{\mu} = n\gamma/3 + 1$, $\sigma^2 = n\pi^2/54 + 3$. This means that $G_n(y)\,dy$ is asymptotically a lognormal distribution for large n. Note however that $\alpha(t)$, $\phi(t)$, $\psi(t)$ have only exponential (not Gaussian) decay when $t \to -\infty$. The deviation of the G_n from lognormal explains why the exponents τ_p, ζ_p are not correctly predicted by lognormal theory.

The necessity to take fluctuations into account in Kolmogorov's classical theory of turbulence (Kolmogorov [7], [8], [9]) was pointed out by Landau. In [10], Kolmogorov used ideas of Obukhov to deal with Landau's remark, and presented a lognormal theory of fluctuations of the energy dissipation ϵ_ℓ on scale ℓ.

In our approach the probability Ansatz (3) is fundamental (based on a statistical mechanical understanding of turbulence) and the lognormality of ϵ_ℓ is an approximate deduction. This explains why the exponents τ_p, ζ_p obtained from (3) are different from those obtained from a lognormal theory (and give a better fit of the experimental data).

Christian Beck, who has studied a hierarchical model of turbulence [2], has conjectured that in such a hierarchical model one could derive a lognormal distribution for $\Delta_r v$ [private communication in December 2013 during a workshop at the Isaac Newton Institute in Cambridge, UK]. The present paper provides an example of such a derivation.

References.

[1] F. Anselmet, Y. Gagne, E.J. Hopfinger, and R.A. Antonia "High-order velocity structure functions in turbulent shear flows." J. Fluid Mech. **140**,63-89(1984).

[2] C. Beck. "Chaotic cascade model for turbulent velocity distributions." *Phys. Rev. E* **49**,3641-3652(1994).

[3] R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani "On the multifractal nature of fully developed turbulence and chaotic systems."J. Phys. A **17**,3521-3531(1984).

[4] B. Castaing. "The temperature of turbulent flows." *J. Phys. II France* **6**,105-114 (1996).

[5] U. Frisch and G. Parisi "On the singularity structure of fully developed turbulence" in *Turbulence and Predictability in Geophysical Fluid Dynamics* (ed. M. Ghil, R. Benzi, and G. Parisi), pp. 84-88. North-Holland, 1985.

[6] G. Gallavotti *Foundations of Fluid Mechanics.* Springer-Verlag, Berlin, 2005 (see Section 6.3).

[7] A.N. Kolmogorov. "The local structure of turbulence in incompressible viscous fluid for very large Reynolds number." Dokl. Akad. Nauk SSSR **30**,301-305(1941).

[8] A.N. Kolmogorov. "On degeneration (decay) of isotropic turbulence in an incompressible viscous liquid." Dokl. Akad. Nauk SSSR **31**,538-540(1941).

[9] A.N. Kolmogorov. "Dissipation of energy in locally isotropic turbulence." Dokl. Akad. Nauk SSSR **32**,16-18(1941).

[10] A.N. Kolmogorov. "A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number." *J. Fluid Mech.* **13**,82-85(1962).

[11] C. Meneveau and K.R. Sreenivasan "Simple multifractal cascade model for fully developed turbulence." Phys. Rev. Lett. **59**,1424-1427(1987).

[12] D. Ruelle. "Hydrodynamic turbulence as a problem in nonequilibrium statistical mechanics." PNAS **109**,20344-20346(2012).

[13] J. Schumacher, J. Scheel, D. Krasnov, D. Donzis, K. Sreenivasan, and V. Yakhot. "Small-scale universality in turbulence." Preprint (2014).

[14] R. Stresing and J. Peinke. "Towards a stochastic multi-point description of turbulence." *New J. of Phys.* **12**,103046+14(2010).

[15] V. Yakhot "Pressure-velocity correlations and scaling exponents in turbulence." J. Fluid Mech. **495**135-143(2003).