HYDRODYNAMIC TURBULENCE AS A PROBLEM
IN NON-EQUILIBRIUM STATISTICAL MECHANICS.

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Abstract: The problem of hydrodynamic turbulence is reformulated as a heat flow problem along a chain of mechanical systems which describe units of fluid of smaller and smaller spatial extent. These units are macroscopic but have few degrees of freedom, and can be studied by the methods of (microscopic) non-equilibrium statistical mechanics. The fluctuations predicted by statistical mechanics correspond to the intermittency observed in turbulent flows. Specifically, we obtain the formula

$$\zeta_p = \frac{p}{3} - \frac{1}{\ln\kappa}\ln\Gamma(\frac{p}{3}+1)$$

for the exponents of the structure functions ($\langle|\Delta_r v|^p\rangle \sim r^{\zeta_p}$). The meaning of the adjustable parameter κ is that when an eddy of size r has decayed to eddies of size r/κ their energies have a thermal distribution. The above formula, with $(\ln\kappa)^{-1} = .32 \pm .01$ is in good agreement with experimental data. This lends support to our physical picture of turbulence, a picture which can thus also be used in related problems.

Hydrodynamic turbulence is known to be a chaotic phenomenon [1,2,3]. This means that the time evolution (f^t) of a turbulent fluid system belongs to a much studied class of deterministic dynamics with sensitive dependence on initial conditions [4,5,6,7]. The statistical properties of turbulence are described by an ergodic invariant state ρ for (f^t) and, since chaotic dynamical systems have (uncountably) many ergodic states, a choice has to be made. A physically reasonable choice is that of so-called SRB states (see [8,9,10] and the references given there).

It is fair to say that the chaotic nature of turbulence has been largely ignored by the turbulence community, and that the choice of an ergodic state to describe the statistical properties of turbulence has been made by ad hoc assumptions (closure assumptions, Gaussianity, multifractal structure). Indeed, the study of SRB or "physical" states for the Navier-Stokes dynamics appears at first impossibly difficult. Nevertheless we propose here an approach of this sort: we bypass the mathematical problems of SRB states by using our understanding of the physics of a specific dynamical system, namely that corresponding to heat conduction, as seen from the point of view of nonequilibrium statistical mechanics. Our approach will thus use basic physical ideas, and approximations, rather than ad hoc assumptions. We shall in this manner obtain a surprisingly coherent view of the fluctuations in turbulence (intermittency).

We shall concern ourselves with incompressible fluids in 3 dimensions, described by the Navier-Stokes equation, but without paying too much attention to the specific form of the dissipative term. The fluid, with velocity field v, will be enclosed in a cube C_0 of side ℓ_0, which we may consider for simplicity to have periodic boundary conditions. We choose an integer $\kappa > 1$ and divide $C_0 = C_{01}$ into cubes C_{ni} of side $\ell_n = \ell_0\kappa^{-n}$, with $i = 1, \ldots, \kappa^{3n}$, where n is a positive integer. Let ϕ_{ni} be the homothety mapping C_0 to C_{ni}. One can choose $2(\kappa^3 - 1)$ real vector fields U_α on $\mathbf{R}^3$ with $\int U_\alpha = 0$, $\operatorname{div} U_\alpha = 0$, and such that if the velocity field v satisfies $\int v = 0$, $\operatorname{div} v = 0$, there is a unique representation

$$v = \sum_{n=0}^{\infty} \sum_{i=1}^{\kappa^{3n}} \sum_{\alpha=1}^{2(\kappa^3-1)} c_{ni\alpha} U_\alpha \circ \phi_{ni}^{-1}$$

with $c_{ni\alpha} \in \mathbf{R}$. This means that v has a wavelet decomposition into components (roughly) localized in the cubes C_{ni}.

We think now of the standard physical situation where energy is put into the fluid at large spatial wavelength (i.e., small n) and dissipated at small spatial wavelength (i.e., large n). Intermediate values of n correspond to the inertial range, where the time evolution should in some sense be Hamiltonian. Specifically, Arnold [11] has shown how an inviscid flow could be interpreted as geodesic flow on the group of volume preserving diffeomorphisms. The corresponding Hamiltonian is the kinetic energy of the velocity field.

We may thus think of the time evolution for (the finitely many) coefficients $c_{ni\alpha}$ as Hamiltonian, with external forces acting at low and high n. This is related to the physical concept of eddies as dynamical structures localized in space. However, instead of a cascade of eddies of smaller and smaller size, we think of a system of coupled Hamiltonian systems which we can label (n, i). If we assume that the different systems (n, i) are weakly coupled, we can reinterpret the global dynamics as a heat flow from the small n, where energy is input, to large n where it is dissipated, i.e., rapidly carried away to structures of molecular size of the fluid. Note that the multifractal description of eddy cascades [12,13,14,15] ignores interactions between (n, i), (n, i') except when these eddies are created from a common $(n-1, j)$. This corresponds to saying that the lateral interaction between the systems (n, i), (n, i') is weak, but this assumption does not appear to be essential in our approach.

There is no hope for an exact study of the dynamics of the coupled systems (n, i), but we can get a first approximation from the Kolmogorov scaling theory of homogeneous turbulence [16]. Since this theory gives unique answers, the problem of selecting an SRB state does not occur here. According to the Kolmogorov theory, the fluid velocity corresponding to C_{ni} is $v_{ni} \sim (\epsilon \ell_n)^{1/3}$ where ϵ is the mean dissipation per unit volume, and the kinetic energy corresponding to C_{ni} is

$$\sim \frac{1}{2}\ell_n^3(\epsilon\ell_n)^{2/3} = \frac{\epsilon^{2/3}}{2}\ell_n^{11/3}$$

(we have put the fluid density equal to 1); the corresponding temperature is

$$T_n = \frac{1}{k}\frac{\epsilon^{2/3}\ell_n^{11/3}}{2(\kappa^3-1)} = \frac{1}{k}\frac{\epsilon^{2/3}\ell_0^{11/3}\kappa^{-11n/3}}{2(\kappa^3-1)} \tag{1}$$

where k is Boltzmann's constant. In view of the value of k, we see that T_n is huge for small n so that the flow of heat from high temperature to low temperature agrees with the energy cascade from small ℓ to large ℓ in the fluid. Notice that the heat resistance $(T_n - T_{n+1})/\epsilon$ is very large, which agrees with a weak coupling between the systems (n,i) for different values of n.

We see the situation as follows: a heat flow interpretation of the energy cascade in homogeneous turbulence is possible, using scaling laws, but ignores fluctuations (intermittency). To understand fluctuations we have to study the fluctuations of the energy flow in the Hamiltonian system of the coupled (n,i). This is a problem of nonequilibrium statistical mechanics, a problem known to be difficult [17,18,19]. In general one would need the systems (n,i) to be chaotic in some sense (this is physically reasonable for 3-dimensional hydrodynamics) but the Anosov assumptions of [18,19] are unreasonably strong. In the present situation, a rigorous analysis appears quite out of reach at this time. An approximate study is however possible, and will give more specific results than earlier multifractal approaches [12,13,14,15], and there will be a physical justification rather than ad hoc assumptions.

While we have at this time no detailed understanding of heat flow from the point of view of rigorous statistical mechanics, we expect that Fourier's law should hold under normal conditions. This is however no great help since Kolmogorov's theory yields the precise temperature distribution (1). As to the "microscopic" fluctuations, they are a difficult problem in nonequilibrium [20,21], being different in nature from the well-understood equilibrium fluctuations. Here we shall use the assumption that the systems (n,i) have weak mutual coupling to justify a Boltzmannian energy distribution for each Hamiltonian system (n,i).

Because of the large temperature gradient, the flow of energy is overwhelmingly from the system (n,i) to the systems $(n+1,j)$. To study this energy flow we use the conservation of energy and scaling as in the multifractal approaches [13] to write

$$|v_{ni}|^3/\ell_n = |v_{(n+1)j}|^3/\ell_{n+1} \qquad \text{or} \qquad |v_{(n+1)j}|^3 = |v_{ni}|^3\kappa^{-1} \tag{2}$$

Note that $|v|^3$ is proportional to the kinetic energy $\frac{1}{2}|v|^2$ with a weight 1/ time spent in a certain spatial frequency range. We interpret then (2) to mean that, given the energy $V_{ni} = |v_{ni}|^3$ in (n,i), the velocity $v = v_{(n+1)j}$ is fluctuating with Boltzmannian distribution

$$\sim \exp\big(-\frac{|v|^3}{V_{ni}\kappa^{-1}}\big)d^3v$$

Therefore the energy $V = V_{(n+1)j}$ has the normalized distribution

$$\frac{1}{V_n \kappa^{-1}} \exp\big(-\frac{V}{V_n \kappa^{-1}}\big) dV \tag{3}$$

where we write from now on V_n instead if V_{ni}, etc. We view (3) as an approximate, but physically motivated relation, the validity of which will be discussed below. Note that if we replace V_n by $\tilde{V}_n = \kappa^n V_n$ we have that $\tilde{V} = \tilde{V}_{n+1}$ is distributed according to

$$\frac{1}{\tilde{V}_n} \exp(-\frac{\tilde{V}}{\tilde{V}_n}) d\tilde{V}$$

We now discuss the structure functions, i.e., the moments

$$\langle |v_n|^p \rangle = \langle V_n^{p/3} \rangle$$

for positive integer p, and the exponents ζ_p such that

$$\langle |v_n|^p \rangle \sim \ell_n^{\zeta_p} \qquad \text{or} \qquad \zeta_p \ln \ell_n \sim \ln \langle V_n^{p/3} \rangle = -n \cdot \frac{p}{3} \ln \kappa + \ln \langle \tilde{V}_n^{p/3} \rangle$$

We have here

$$\langle \tilde{V}_n^{p/3} \rangle = \int d\tilde{V}_1 \frac{e^{-\tilde{V}_1/\tilde{V}_0}}{\tilde{V}_0} \int \cdots \int d\tilde{V}_{n-1} \frac{e^{-\tilde{V}_{n-1}/\tilde{V}_{n-2}}}{\tilde{V}_{n-2}} \int d\tilde{V}_n \frac{e^{-\tilde{V}_n/\tilde{V}_{n-1}}}{\tilde{V}_{n-1}} \cdot \tilde{V}_n^{p/3}$$

$$\int_0^\infty d\tilde{V}_n \frac{e^{-\tilde{V}_n/\tilde{V}_{n-1}}}{\tilde{V}_{n-1}} \cdot \tilde{V}_n^{p/3} = \tilde{V}_{n-1}^{p/3} \int_0^\infty d\xi\, e^{-\xi} \xi^{p/3} = \tilde{V}_{n-1}^{p/3} \Gamma(\frac{p}{3}+1)$$

so that by induction we find

$$\langle \tilde{V}_n^{p/3} \rangle = [\Gamma(\frac{p}{3}+1)]^n \tilde{V}_0^{p/3}$$

$$\zeta_p \approx \frac{-n\frac{p}{3}\ln\kappa + \ln\langle \tilde{V}_n^{p/3}\rangle}{-n\ln\kappa} \approx \frac{p}{3} - \frac{1}{\ln\kappa} \ln\Gamma(\frac{p}{3}+1)$$

In conclusion we have the (approximate) prediction

$$\zeta_p = \frac{p}{3} - \frac{1}{\ln\kappa} \ln\Gamma(\frac{p}{3}+1) \tag{4}$$

Note that (4) gives $\zeta_3 = 1$.

Using either the heat propagation or the eddy cascade picture, we see that κ should be chosen such that the initial $\tilde{V}_n$-distribution concentrated on one value for (n, i) thermalizes to values of $\tilde{V}_{n+1}$ for the systems $(n + 1, j)$ distributed according to

$$\frac{1}{\tilde{V}_n} e^{-\tilde{V}_{n+1}/\tilde{V}_n} d\tilde{V}_{n+1}$$

This requires κ sufficiently large. However, if the value of κ is too large, several different temperatures will be present among the systems $(n + 1, j)$ connected with (n, j), and the $\tilde{V}_{n+1}$-distribution will not be Boltzmannian. The picture we have in mind is a situation in C_{ni} which depends on the spatial wavelength: at wavelength of the order of the size of $\mathrm{supp}(U_\alpha \circ \phi_{ni}^{-1})$ a single value of the kinetic energy is present; the distribution broadens as the wavelength diminishes and becomes a thermal distribution when it is divided by κ; at smaller wavelengths there are several patches with different temperatures. Of course a rigorous justification of this picture is well beyond the power of current mathematical methods. We can only claim this: κ should be such that when an eddy of size r has decayed to eddies of size r/κ their energies have a thermal distribution, after which the process can start again. In the dissipative range the distribution of V_n should be cut off at large V_n. Numerically, one finds that in the experimental range $p \leq 18$, formula (4) fits the small set of experimental data [22] well with $1/\log\kappa = .32 \pm .01$, i.e., κ between 20 and 25. (The fit of the data in [23] obtained by numerical simulation is less good). Note that the Boltzmannian distribution which we used can only be approximate in view of the way it has been obtained. In fact (4) must break down after about $p = 50$ because it gives decreasing ζ_p in contradiction with a Hölder inequality prediction. Also (2) may be valid as an average and acceptable in a mean field sense – see [24] – but would not hold for the very large velocity increments described by the moments ζ_p for large p.

From a physical point of view, one can try the following interpretation: the change of behavior as one passes from large to small wavelength corresponds to what is observed at the onset of turbulence. We use here the physical fact that transport is much faster in the turbulent than in the laminar regime. Therefore, when sufficiently small scales are reached and we are in the turbulent regime, thermalization takes place. We may compute the length ratio $\kappa = \ell_n/\ell_{n+1}$ in terms of the Reynolds number R_c for the onset of turbulence. If ϵ is the energy dissipation per unit volume and ν the kinematic viscosity, the Kolmogorov length is $\eta = (\nu^3/\epsilon)^{1/4}$, so that $\nu = (\epsilon\eta^4)^{1/3}$. The velocity corresponding to the length λ is given in the turbulent regime by $v_\lambda = (\epsilon\lambda)^{1/3}$. Therefore if the onset of turbulence corresponds to λ, we have

$$\kappa = \frac{\lambda}{\eta} = (\frac{\lambda^4\epsilon}{\nu^3})^{1/4} = (\frac{\lambda v_\lambda}{\nu})^{3/4} = R_c^{3/4}$$

The critical Reynolds number R_c is not defined with precision, but the value $R_c = \kappa^{4/3} \approx 60$ is not unreasonable. Clearly the calculation we have made is quite rough, but the exponent ζ_p should not be very sensitive to details, in particular because κ occurs only as its logarithm in (4). Notice also that the estimate $\kappa = R_c^{3/4}$ is proposed instead of a fundamental calculation which is beyond current possibilities. Altogether, the agreement of (4) with experiment, with a plausible value of κ, supports the physical picture of turbulence that we have presented. This picture can thus also be used in related problems.

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