A PRELIMINARY NOTE ON ALPÖGE'S PROPOSED PROOF OF THE GENERALIZED RIEMANN HYPOTHESIS

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0. Statement from the author

The main purpose of this preliminary report is to begin looking for some of the sources, provenance, and mathematical history behind the proposed proof of the generalized Riemann hypothesis announced by Levent Alpöge using a Claude model [Alp26] on November 20, 2026. The mathematics involved reflects many lifetimes of work, so I would like to recognize as many as possible of the people whose work may have contributed to it: the mathematicians working today, our teachers, and all of our predecessors. This is only a beginning, and I apologize for all the people, papers, and other contributions I may have missed.

The ideas used in modern mathematical proofs inevitably come from people, and I think that sharing them carries obligations to those people; see Harris [Har26a] for a great account of this notion of mathematics as a "gift economy". I also wonder what becomes of the labor and generosity behind the ideas, and who is expected to do the further work of understanding any proof announcement. While I share Michael Harris's [Har26b] objection to mathematicians acting as unpaid labor for AI companies, I am attempting to interpret this AI artifact because I want to recognize some of the mathematical origins of [Alp26] and the credit that is due to the people involved (without claiming to offer a complete history or to settle any questions of ownership).

Date: November 22, 2026.

I hope we will continue to remember and celebrate the people whose ideas have allowed the rest of us to see a little further.

The scale of the material involved in [Alp26] makes this an unusual exercise in exposition: the Lean code has roughly 17 million lines of code, while Alpöge [Alp26, p. 2, footnote 2] describes an AI-generated exposition of more than 250,000 pages that he chose not to release. Neither offers a practicable prospect of reading the argument from beginning to end in its present form. As such, I have also used AI to begin writing this report: I prompted GPT 6 Pro to digest the Lean code on Alpöge's website, infer a human-readable mathematical proof, to search the literature carefully for earlier sources and analogues of the ideas in the argument, and then to produce a written summary. These prompts were an attempt to obtain something that people could begin to read, discuss, and eventually improve. I then spent November 20, 2026 rewriting the AI output into the current form. Given the impossibility of any human checking natural language text against 17 million lines of Lean code in a short timeframe, I cannot claim that this summary is a faithful translation of the Lean code in [Alp26], but an incredible collective effort by the Mathlib and Lean communities [Lea26, Sut26] already resulted (after just one day!) in the verification of the Lean proof of the generalized Riemann hypothesis using independent implementations of the Lean kernel and in the verification that the Lean definitions agree with standard mathematical definitions. I also apologize to the reader for the condensed and imperfect exposition here, as most of the text from Section 2 onwards was largely written by AI.

Part of the pleasure of mathematics, for me, is that an idea can pass from one person to another and take on a life that neither could have anticipated. There is an art in making that possible, through the patient work of explanation and the equally patient work of learning, rooted in the earliest oral traditions of human culture. I hope that mathematicians will come to understand both proposed proofs of the generalized Riemann hypothesis that were announced on November 20, 2026,[1] discover what is new in them, and explain what further questions they make possible. I would be very glad if this small report helped someone write such an account. Such understanding should be available to anyone who wishes to learn, regardless of access to powerful AI systems.

Acknowledgments. I thank Michael Harris, Manvir Jaswal, Drew Sutherland, and Bartłomiej Żogała for pointing out additions and updates for this note.

[1] Ravi Vakil's Math 216 course website announced a proposed proof of GRH by Alessandro Masullo [Mas26] on November 20, 2026. Vakil's announcement reported that no AI was used in Masullo's work, and the paper seems to have been the result of nearly twelve years of work with a premature release prompted by the announcement of [Alp26] (not unlike the series of events that led to the release of [Sta26]). As a (monumental!) human effort that acknowledges previous ideas, it does not have the same need for this kind of AI-assisted historiographical exercise.

1. Mathematical overview

1.1. **Zeros of L-functions.** The connection between prime numbers and L-functions began with a question about arithmetic progressions in the early 19th century. Dirichlet [Dir37] wanted to show that every arithmetic progression

$$a_1, a_1 + \delta, a_1 + 2\delta, \dots$$

with relatively prime first term a_1 and common difference δ contained infinitely many primes. Certain periodic multiplicative functions $\chi : \mathbb{Z} \longrightarrow \mathbb{C}$, now called Dirichlet characters, allowed him to distinguish the arithmetic progressions, and his L-functions for $\mathrm{Re}(s) > 1$,

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

made it possible to study the primes in each progression. The functions $L(s, \chi)$ have meromorphic continuations to $\mathbb{C}$, and their zeros below refer to these continuations. For the zeta function $\zeta(s) = \sum_{n \geq 1} n^{-s}$ in $\mathrm{Re}(s) > 1$, Riemann [Rie59, p. 674] connected the distribution of primes with its zeros. In 1859 he used the assertion that the zeros in $0 < \mathrm{Re}(s) < 1$ lay on the critical line $\mathrm{Re}(s) = 1/2$ as a working hypothesis. The extension of the Riemann hypothesis to Dirichlet L-functions already appeared in the work of Piltz [Pil84, pp. 6, 25] in 1884, and is the generalized Riemann hypothesis considered in [Alp26].

Conjecture 1.1 (Generalized Riemann hypothesis for Dirichlet L-functions)**.** For every primitive Dirichlet character χ, every zero of $L(s, \chi)$ in the strip $0 < \mathrm{Re}(s) < 1$ lies on the line $\mathrm{Re}(s) = \frac{1}{2}$.

Throughout the rest of the note, $\mathbb{N}$ denotes the positive integers and $\mathbb{Z}_{\geq 0}$ the nonnegative integers. We will use the von Mangoldt function

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^k \text{ for a prime } p \text{ and } k \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

and Euler's constant $\gamma = 0.577\dots$. We will also assume from now on that χ is a primitive Dirichlet character, since removing Euler factors to obtain an imprimitive character introduces no new zeros of the corresponding L-function in the region $\mathrm{Re}(s) > 0$.

1.2. **The logarithmic derivative and sums over prime powers.** To exclude zeros of $L(s, \chi)$ to the right of the critical line, [Alp26] studies its logarithmic derivative. For $\mathrm{Re}(s) > 1$, the classical identity for the logarithmic derivative

$$-\frac{L'(s, \chi)}{L(s, \chi)} = \sum_{n \geq 1} \Lambda(n)\chi(n)n^{-s}$$

connects the zeros of $L(s,\chi)$ with prime powers: zeros give poles after meromorphic continuation of the left side, while $\Lambda(n)$ vanishes unless n is a prime power. To make this relation quantitative, [Alp26] uses the weighted sums, for $x \geq 1$,

$$\begin{aligned} S_\chi(x) &:= \sum_{n \leq x} \Lambda(n)\chi(n)\Big(\frac{x}{n} - 1\Big) \\ &= \sum_{p} \sum_{\substack{k \geq 1 \\ p^k \leq x}} (\log p)\chi(p^k)\left(\frac{x}{p^k} - 1\right); \end{aligned}$$

this is the first-order Riesz mean with coefficients $\Lambda(n)\chi(n)/n$ dating back essentially to de la Vallée-Poussin [dlVP96] (as attributed by [MV07, §12.1]; see [IMS09] for an example of its use in studying $-L'(s,\chi)/L(s,\chi)$). The weight $x/n - 1$ makes the Mellin transform of S_χ particularly simple:

$$\int_1^\infty S_\chi(x)x^{-s-1}\,dx = -\frac{L'(s,\chi)}{s(s-1)L(s,\chi)} \qquad \text{for } \mathrm{Re}(s) > 1.$$

The integral converts information about the growth of $S_\chi(x)$ into information about the poles of the expression on the right. A zero of $L(s,\chi)$ in the critical strip produces such a pole, so bounds for $S_\chi(x)$ can constrain the location of the zeros.

The displayed Mellin transform may also have a pole at $s = 1$ that does not come from a zero of $L(s,\chi)$. For a nonprincipal primitive character, [Alp26] cancels the possible pole from the factor $1/(s-1)$ by setting

$$G_\chi(x) := S_\chi(x) + xL'(1,\chi)/L(1,\chi).$$

The principal primitive character, denoted by $\mathbb{1}$, has conductor 1, and $L(s,\mathbb{1}) = \zeta(s)$. The pole of $\zeta(s)$ at 1 requires the different correction

$$G_{\mathbb{1}}(x) := \sum_{n \leq x} \Lambda(n)\Big(\frac{x}{n} - 1\Big) - x\log x + (1+\gamma)x.$$

The Mellin transforms of the corrected functions G_χ are regular at $s = 1$, while retaining the poles contributed by zeros in the critical strip.

[Alp26] bounds a positive linear combination of the corrected sums attached to powers of a character. For a primitive Dirichlet character ψ of order r, let $\mathcal{X}_\psi$ be the set of primitive characters inducing $\mathbb{1}, \psi, \dots, \psi^{r-1}$, and put

$$\mathcal{G}_\psi(x) := G_{\mathbb{1}}(x) + \frac{1}{3r+1} \sum_{\substack{\chi \in \mathcal{X}_\psi \\ \chi \neq \mathbb{1}}} \mathrm{Re}(G_\chi(x)). \tag{1.1}$$

All powers of ψ enter because [Alp26] uses character orthogonality to impose a condition on Frobenius elements. Indeed, for n coprime to the conductor of ψ, $r^{-1}\sum_{j=0}^{r-1} \psi(n)^j$ equals 1 if $\psi(n) = 1$ and 0 otherwise. In Section 3, [Alp26] uses averages over subgroups of an abelian Galois group to retain the terms G_χ with $\chi \in \mathcal{X}_\psi$.

To see why a bound for $\mathcal{G}_\psi$ can still detect zeros of the individual L-functions, notice that $G_{\overline{\chi}}(x) = \overline{G_\chi(x)}$. Since $\mathcal{X}_\psi$ is stable under complex conjugation, the real parts can be omitted from the sum in Equation 1.1. At a zero ρ of one of the $L(s,\chi)$ with $\chi \in \mathcal{X}_\psi$ in the critical strip, the residue of the Mellin transform of $\mathcal{G}_\psi$ is therefore $1/(\rho(1-\rho))$ times the sum of the multiplicities of ρ in the $L(s,\chi)$ with $\chi \in \mathcal{X}_\psi$, weighted by the positive coefficients in Equation 1.1; see Equation 2.3. The sum of these weighted multiplicities is positive, so ρ remains a pole even when several of the L-functions vanish there.

To exclude poles to the right of the critical line, [Alp26] proves the following upper bound for $\mathcal{G}_\psi$.

Proposition 1.2. *Let ψ be a primitive Dirichlet character. For every $\eta > 0$ and every $A \geq 2$, there is $Y \geq A$ such that*

$$\mathcal{G}_\psi(x) \leq x^{1/2+\eta} \qquad \text{for every } x \in [Y, Y^{3/2}]. \tag{1.2}$$

Notice that Proposition 1.2 bounds $\mathcal{G}_\psi$ only from above; the function itself may take either sign. Moreover, the intervals $[Y, Y^{3/2}]$ need not cover every sufficiently large x. Section 2 explains why these upper bounds on arbitrarily large intervals nevertheless suffice to exclude nontrivial zeros off the critical line.

As I will now attempt to summarize, Proposition 1.2 is where results about automorphic forms, Galois representations, Arakelov geometry, and integral p-adic Hodge theory enter the proposed proof in [Alp26].

1.3. **How automorphic periods give bounds for $\mathcal{G}_\psi$.** Why should automorphic periods enter a question about the sums G_χ? The logarithmic derivatives $L'(s,\chi)/L(s,\chi)$ in Section 1.2 already have a classical connection with periods of abelian varieties with complex multiplication. Chowla and Selberg [CS49, §4] expressed periods of CM elliptic curves through gamma values. Anderson [And82, Theorem 2.1] extended this relation to abelian varieties with CM by an abelian extension of $\mathbb{Q}$, expressing their periods, up to algebraic factors, through combinations of logarithmic derivatives of Dirichlet L-functions at 0. The functional equation for Dirichlet L-functions then relates the logarithmic derivatives for the odd characters in these formulas to values of L'/L at 1.

In [Alp26], CM abelian varieties enter through the theta functions used in the constructions below. Their periods compare algebraic differentials with the analytic coordinates in which the theta functions are written. These factors enter the normalization and archimedean estimates for automorphic periods, which are integrals of automorphic forms on unitary groups. The normalized automorphic periods take values in a number field, so their valuations at finite places can also be studied.

The lower bounds for these finite valuations involve prime powers satisfying conditions on Frobenius elements. After expanding those conditions in Dirichlet characters, [Alp26] obtains expressions involving the sums S_χ from

Section 1.2. For any number field K and any $z \in K^\times$, the Artin–Whaples product formula states that

$$\sum_{\sigma : K \hookrightarrow \mathbb{C}} \log|\sigma(z)| = \sum_{\mathfrak{p}} \operatorname{ord}_{\mathfrak{p}}(z) \log \#(\mathcal{O}_K/\mathfrak{p}),$$

where $\mathfrak{p}$ runs over the nonzero prime ideals of $\mathcal{O}_K$, and $\operatorname{ord}_{\mathfrak{p}}(z)$ is the exponent of $\mathfrak{p}$ in the principal fractional ideal (z). In [Alp26], the implication that an upper bound for the left side bounds the weighted sum of their finite valuations is applied to the algebraic numbers obtained by normalizing the automorphic periods.

The formulas for CM periods, together with formulas for the norms and dimensions of spaces of automorphic forms, account for the terms removed in passing from S_χ to G_χ. The task is then to bound the remaining terms closely enough to obtain the estimate for $\mathcal{G}_\psi$ in Proposition 1.2.

Automorphic periods and theta lifts. The unitary groups in [Alp26] are defined over the maximal totally real subfield E^+ of an abelian CM field E containing the field cut out by ψ. For a nondegenerate Hermitian E-space V, write $U(V)$ for its unitary group over E^+. Using $\mathbb{A}_F$ and $\mathbb{A}_{F,f}$ for the adeles and finite adeles of a number field F, write $[G] := G(E^+)\backslash G(\mathbb{A}_{E^+})$ for an algebraic group G over E^+.

On these unitary groups, [Alp26] begins with a pairing of cusp forms f on $[U(X_0)]$ and g on $[U(Y_0)]$, for an inclusion of Hermitian spaces $X_0 \subset Y_0$ with X_0 totally definite. It is a Bessel period of the kind studied by Gan, Gross, and Prasad [GGP12, §12 and §23]: [Alp26] first integrates g over the adelic quotient of a unipotent subgroup against the conjugate of a character to obtain a Fourier coefficient, then pairs its restriction to $[U(X_0)]$ with $\bar{f}$. For the comparison at finite primes, [Alp26] chooses models of the spaces of automorphic forms over a number field K and normalizes the periods to obtain K-bilinear pairings, with the choices described in Section 4.1.

To obtain lower bounds for the valuations of these normalized periods at finite primes of K, [Alp26] repeatedly applies the theta correspondence of Weil [Wei64] and Howe [How79]. Integration against a theta kernel carries an automorphic form from one unitary group to another. Following both arguments of the pairing gives two sequences of m theta lifts, on groups $U(X_j)$ and $U(Y_j)$ for $0 \le j \le m$. Every X_j is totally definite; the signatures of Y_j vary until Y_m is also totally definite. Kudla's [Kud84] seesaw identities relate the initial Bessel period to the pairings after these theta lifts, which include Fourier–Jacobi periods before ending in a restriction pairing for $U(Y_m) \subset U(X_m)$.

The successive theta lifts contribute terms to the lower bounds for these finite valuations. The normalizations introduce correction terms, and the theta lifts also change the Petersson norms of the automorphic forms and the dimensions of their spaces. To keep track of the norms, [Alp26] uses the formulas of Rallis [Ral87] and Gan, Qiu, and Takeda [GQT14, §11],

which express Petersson norms of theta lifts through special values of L-functions and local integrals. For the dimensions and multiplicities, [Alp26] uses Weyl's [Wey46] dimension formula and weighted counts of Hermitian lattices of the kind appearing in Siegel's [Sie35] mass formula.

With the CM periods included in the normalization, [Alp26] combines the formulas for Petersson norms, dimensions, and weighted counts of Hermitian lattices. The resulting expression agrees with the main term subtracted in Proposition 3.1. This agreement is the identity in Equation 4.5. After subtracting the main term, [Alp26] bounds the remaining terms to estimate $\mathcal{G}_\psi$.

Lattices of automorphic forms. To compare finite valuations with archimedean norms, [Alp26] works with lattices over the ring of integers $\mathcal{O}_K$ of K. Since $U(X_0)$ is compact at every real place, its irreducible representations at those places are specified by integral highest weights. For highest weights α, [Alp26] uses an $\mathcal{O}_K$-lattice M_α of algebraic automorphic forms on $U(X_0)$, in the sense of Gross [Gro99]. These algebraic automorphic forms take values in the algebraic representation of $\mathrm{Res}_{E^+/\mathbb{Q}}U(X_0)$ with highest weights α. The integral structure in this representation and the additional local conditions on the automorphic forms are described in Sections 4.1 and 4.4. On the other side of the pairing, downward theta lifts from $U(Y_m)$ give a lattice M_W in a finite-dimensional K-space W of test automorphic forms on $[U(Y_0)]$, with test vectors at the real places and local conditions at finite places chosen in [Alp26].

For the normalized K-bilinear period pairings P_α, [Alp26] constructs a finite-index sublattice $M'_\alpha \subseteq M_\alpha$ and a fractional ideal $\mathfrak{a}_\alpha$ such that $P_\alpha(M'_\alpha, M_W) \subseteq \mathfrak{a}_\alpha$. This is the precise meaning of divisibility of period values in the argument. Writing $\mathrm{ord}_\mathfrak{p}$ for the valuation that takes the value 1 on a local uniformizer at $\mathfrak{p}$, the inclusion says that

$$\mathrm{ord}_\mathfrak{p}(P_\alpha(f, g)) \geq \mathrm{ord}_\mathfrak{p}(\mathfrak{a}_\alpha)$$

for every finite prime $\mathfrak{p}$ of K, every $f \in M'_\alpha$, and every $g \in M_W$. A positive exponent of $\mathfrak{p}$ in $\mathfrak{a}_\alpha$ therefore specifies a power of $\mathfrak{p}$ dividing every such value locally; a negative exponent bounds an allowed denominator. The exponents depend on $\mathfrak{p}$ and on the parameters of the construction. Thus, after rescaling to $N_\alpha := \mathfrak{a}_\alpha^{-1} M'_\alpha$, the functionals $P_\alpha(f, -)$ take integral values on M_W for $f \in N_\alpha$. Writing $C := \mathrm{Hom}_{\mathcal{O}_K}(M_W, \mathcal{O}_K)$, [Alp26] obtains integral maps $b_\alpha : N_\alpha \to C$ defined by $b_\alpha(f)(g) = P_\alpha(f, g)$. In particular, $C \otimes_{\mathcal{O}_K} K = W^\vee$.

With Petersson norms on the automorphic forms and dual norms on C, [Alp26] uses the period maps b_α to bound the covolume of C from above, or equivalently its mean slope $\mu(C)$ from below. But the comparison also needs covolume bounds for the domains N_α. Those bounds begin with M_α, before accounting for the index $[M_\alpha : M'_\alpha]$ and the rescaling by $\mathfrak{a}_\alpha^{-1}$.

This is why the geometry of abelian varieties enters. To bound the covolumes of M_α, [Alp26] also uses holomorphic automorphic forms as

sections of bundles on a moduli scheme $\mathcal{M}$ of principally polarized abelian varieties with a compatible $\mathcal{O}_E$-action, given signature, and level. The bundles are constructed from invariant differentials of the universal abelian scheme over $\mathcal{M}$, and their integral sections are studied on arithmetic compactifications, as in Faltings and Chai [FC90]. The arithmetic Hilbert–Samuel formulas of Gillet and Soulé [GS88] and Zhang [Zha95] relate the covolumes of lattices of integral sections to arithmetic self-intersection numbers. To compare these numbers with heights of points, [Alp26] uses Zhang's [Zha95] inequalities for successive minima and Moriwaki's [Mor09] extension of Zhang's criterion for small sections, developed using ideas from Yuan [Yua08]. The contributions of finite primes to those heights require control of the integral modules of invariant differentials. For that control, [Alp26] uses Fargues's [Far11, §5.3.2] estimates for Hodge–Tate maps and the prismatic Dieudonné theory of Anschütz and Le Bras [ALB23]. Section 4.3 explains how these results give the required covolume bounds.

Even with covolume bounds for N_α, the norms of the period maps b_α alone cannot bound the covolume of C: if all their images lay in one line, they would say nothing about the other directions in $W^\vee$. So [Alp26] proves that the period functionals span nearly all of $W^\vee$, and still do so after small losses of dimension in their domains. The restrictions arise from the Harder–Narasimhan filtration for Hermitian lattices, developed by Stuhler [Stu76] and Grayson [Gra84] from the construction for vector bundles by Harder and Narasimhan [HN75]. Since its successive quotients have decreasing slopes, [Alp26] can retain the parts of N_α above a chosen slope threshold and use Bost's [Bos01, §4.1.4] inequality to control the slopes of their images in C. The bounds for inner products also measure how much of $W^\vee$ these restricted images can miss: by Lemma 4.2, the codimension of their span is bounded in terms of the dimensions removed from the domains. Varying the threshold then gives the lower bound for $\mu(C)$ in Proposition 4.3.

Choosing complex places avoiding three-term arithmetic progressions. The lower bounds for valuations of period values involve powers of rational primes satisfying Frobenius conditions. By character orthogonality, [Alp26] expresses these conditions through Dirichlet characters and obtains terms $G_\chi(a_t/w_{j,\rho})$, as in Proposition 3.1. The positive parameters a_t are attached to a set T of complex places of E contributing to the valuations. For $U(X_j)$ at the real place corresponding to a complex place ρ of E, the jth theta lift adds a block of Harish–Chandra parameters centred at $-w_{j,\rho}$. The positive integers $w_{j,\rho}$ thus determine these added blocks, as in Section 4.1. By choosing $w_{j,\rho}$ and averaging over subgroups of $\mathrm{Gal}(E/\mathbb{Q})$, [Alp26] retains the characters inducing powers of ψ with enough of their arguments close to x to recover $\mathcal{G}_\psi(x)$.

The complex places at which this choice is made form a set T_2. It must be large, but the valuation bounds also lose precision when too many of its places lie in a short segment of an orbit of Frobenius. To satisfy both

requirements, [Alp26] adapts Behrend's [Beh46] construction of large sets of integers not containing any nonconstant three-term arithmetic progressions. The version used in Section 4.6 limits each relevant intersection to at most two places. At the remaining complex places, the arguments $a_t/w_{j,\rho}$ are larger, and [Alp26] bounds their contributions using regions without zeros of the Dirichlet L-functions.

Hecke eigenvalue congruences and Galois cohomology. The formulas for Petersson norms are applied to automorphic forms with given Hecke eigenvalues, so [Alp26] projects onto sums of simultaneous Hecke eigenspaces. Since a decomposition over K need not split an $\mathcal{O}_K$-lattice, these projections can introduce denominators and cancel part of the divisibility of the period values. To bound the denominators introduced by projection onto Hecke eigenspaces, [Alp26] studies congruences between Hecke eigenvalues using Galois representations.

Base change first transfers the representations of the definite unitary groups to general linear groups over $\mathbb{A}_E$. For the regular algebraic polarized cuspidal constituents, after the twists described in Section 4.2, the results of Clozel, Harris, and Labesse [CHL11], Shin [Shi11], and Chenevier and Harris [CH13] attach representations of $\mathrm{Gal}(\overline{E}/E)$ whose Frobenius eigenvalues are determined by the Hecke eigenvalues in the normalization of Theorem 4.1. Congruences between the automorphic forms can then be studied through extension classes retained by integral lattices in these Galois representations.

The one-dimensional summands produced by theta lifting contribute groups $\mathcal{S}_\ell(E, j)$, for rational primes ℓ and integers $j \geq 1$. These are kernels of restriction from first Galois cohomology, unramified away from ℓ, to the completions of E above ℓ, with coefficients in the Tate twist $\mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)$. The duality theorems of Tate [Tat63] and Poitou [Poi67] relate their orders to second Galois cohomology, with corrections from local and global invariants. To bound part of this second cohomology, [Alp26] compares deformations of Galois representations with congruences between automorphic forms, following Wiles [Wil95], Taylor and Wiles [TW95], and Clozel, Harris, and Taylor [CHT08]. For certain reducible three-dimensional Galois representations, the deformation argument also involves Bloch–Kato Selmer groups of two-dimensional modular representations and their quadratic twists; to show that these Selmer groups vanish, [Alp26] uses Kato's [Kat04] Euler system.

These bounds must allow the highest weights, and hence the range of Tate twists, to grow. So [Alp26] also varies E through cyclotomic towers, using Iwasawa's [Iwa59] theory and Ferrero and Washington's [FW79] vanishing of the μ-invariant for abelian fields. Coarser bounds are needed at certain exceptional twists. The freedom to choose intervals in Proposition 1.2 is used here: [Alp26] chooses an interval on which every contributing exceptional index is small relative to x. The resulting bound for the orders of $\mathcal{S}_\ell(E, j)$ is recorded in Proposition 5.3.

The number of theta lifts and the exponent $1/2$. After combining the bounds for period values, norms, covolumes, and denominators, the dependence on the number m of theta lifts explains the exponent $1/2$. The two principal errors in the bound for $\mathcal{G}_\psi(x)$ have sizes x/m and m, up to factors that are arbitrarily small powers of x. Taking m close to $\sqrt{x}$ therefore makes both errors close to $\sqrt{x}$. The number fields, highest weights, and sets of complex places must be chosen together with m; Proposition 6.3 records the simultaneous choice in [Alp26]. With these choices, Lemma 6.1 gives Proposition 1.2, and the nonvanishing argument in Section 2 then gives Conjecture 1.1.

Figure 1 records the overarching ingredients in the proof of Conjecture 1.1 claimed in [Alp26]. Figures 2, 3, and 4 give more detail on the fundamental ingredients from Arakelov theory, integral p-adic Hodge theory, theta correspondences, CM theory, and Galois representations.

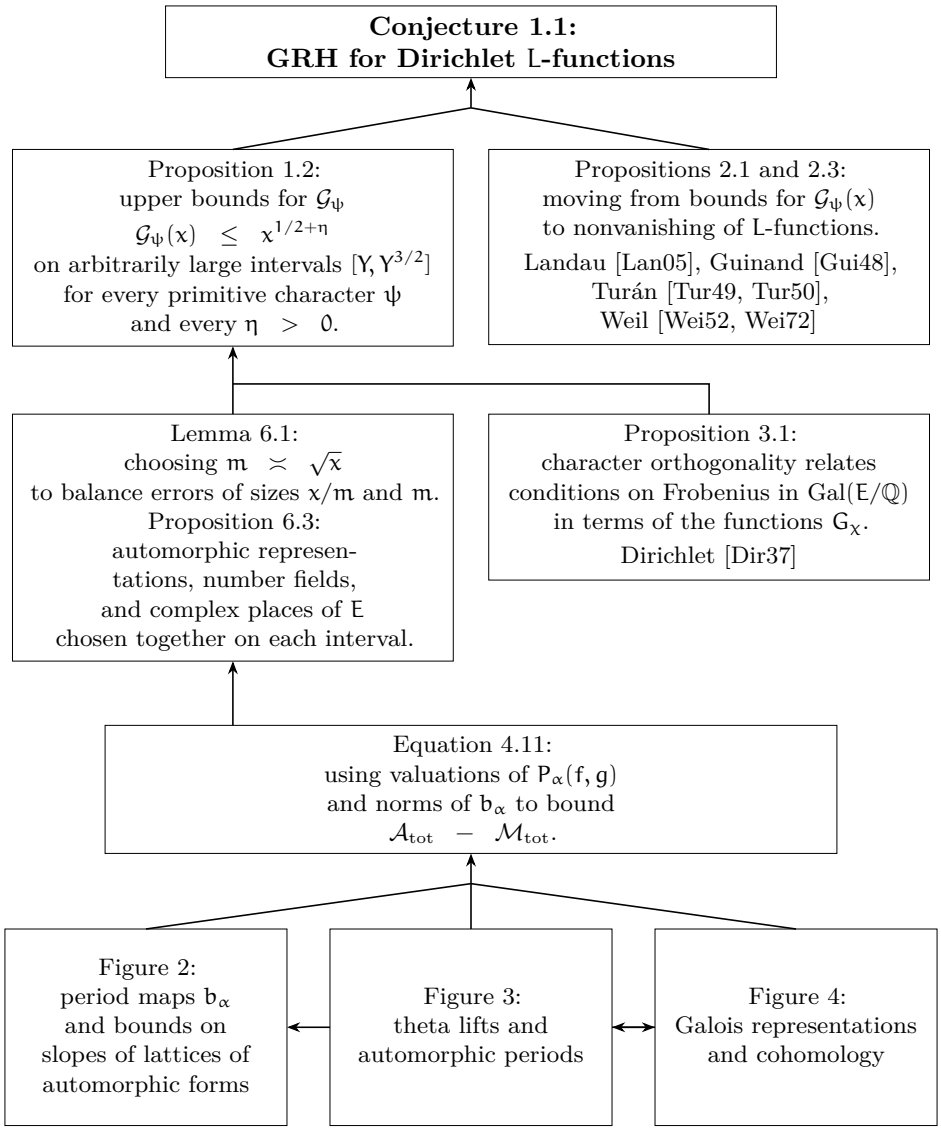


FIGURE 1. The main ingredients in [Alp26], with selected historical references.

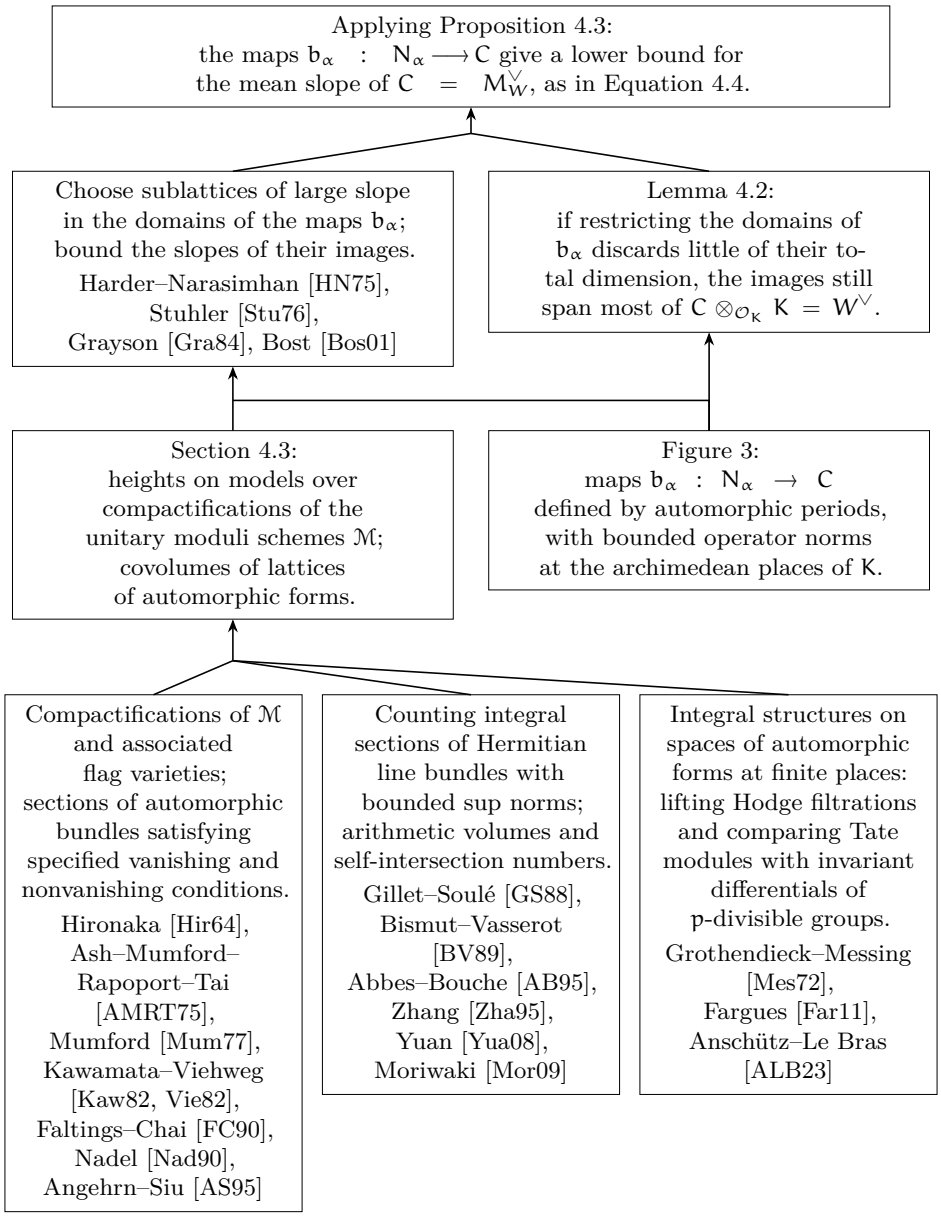


FIGURE 2. The maps defined by automorphic periods and the bounds for covolumes of lattices of automorphic forms in [Alp26], with selected historical references.

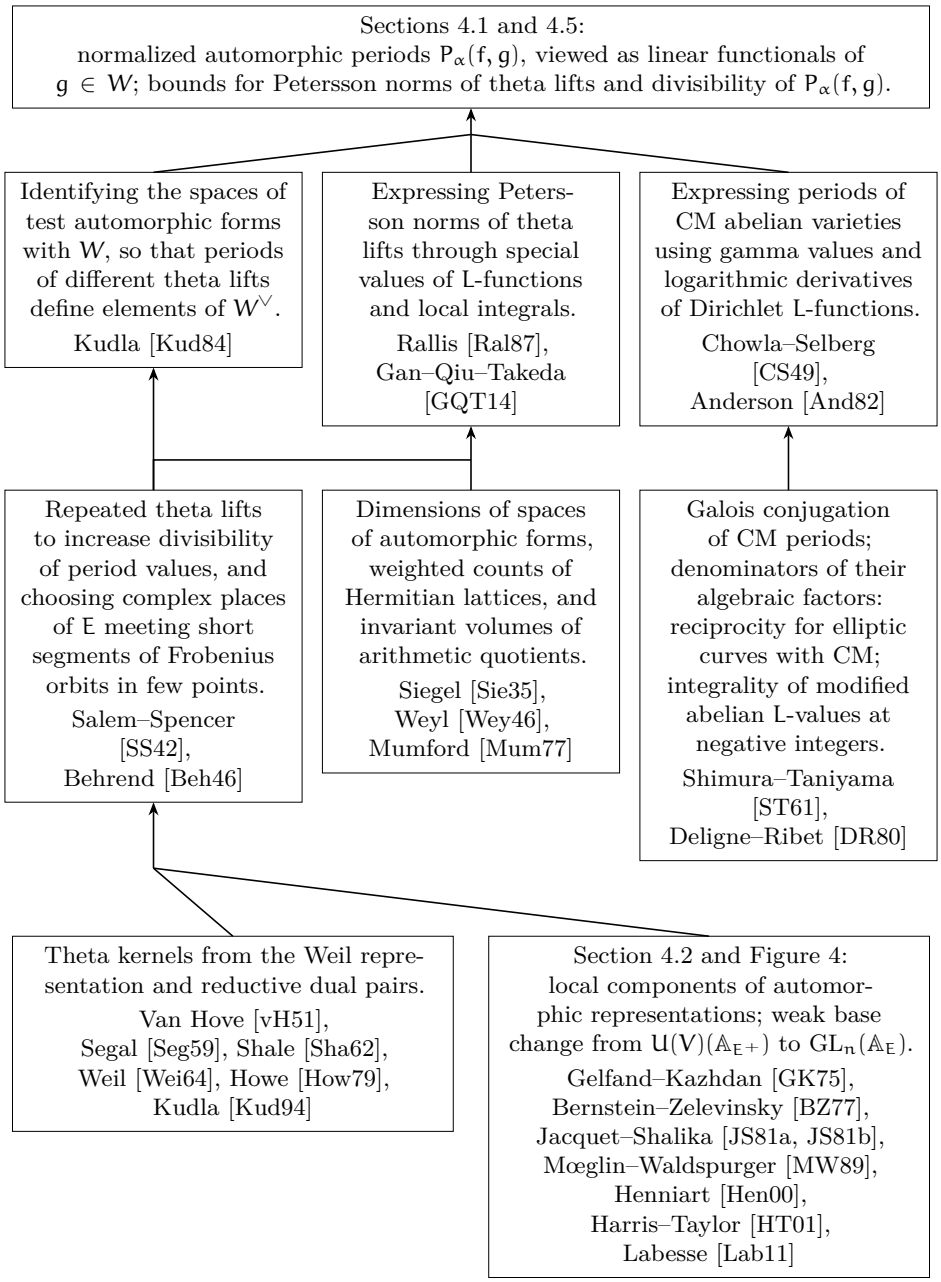


FIGURE 3. Theta lifts and CM periods in [Alp26], with selected historical references.

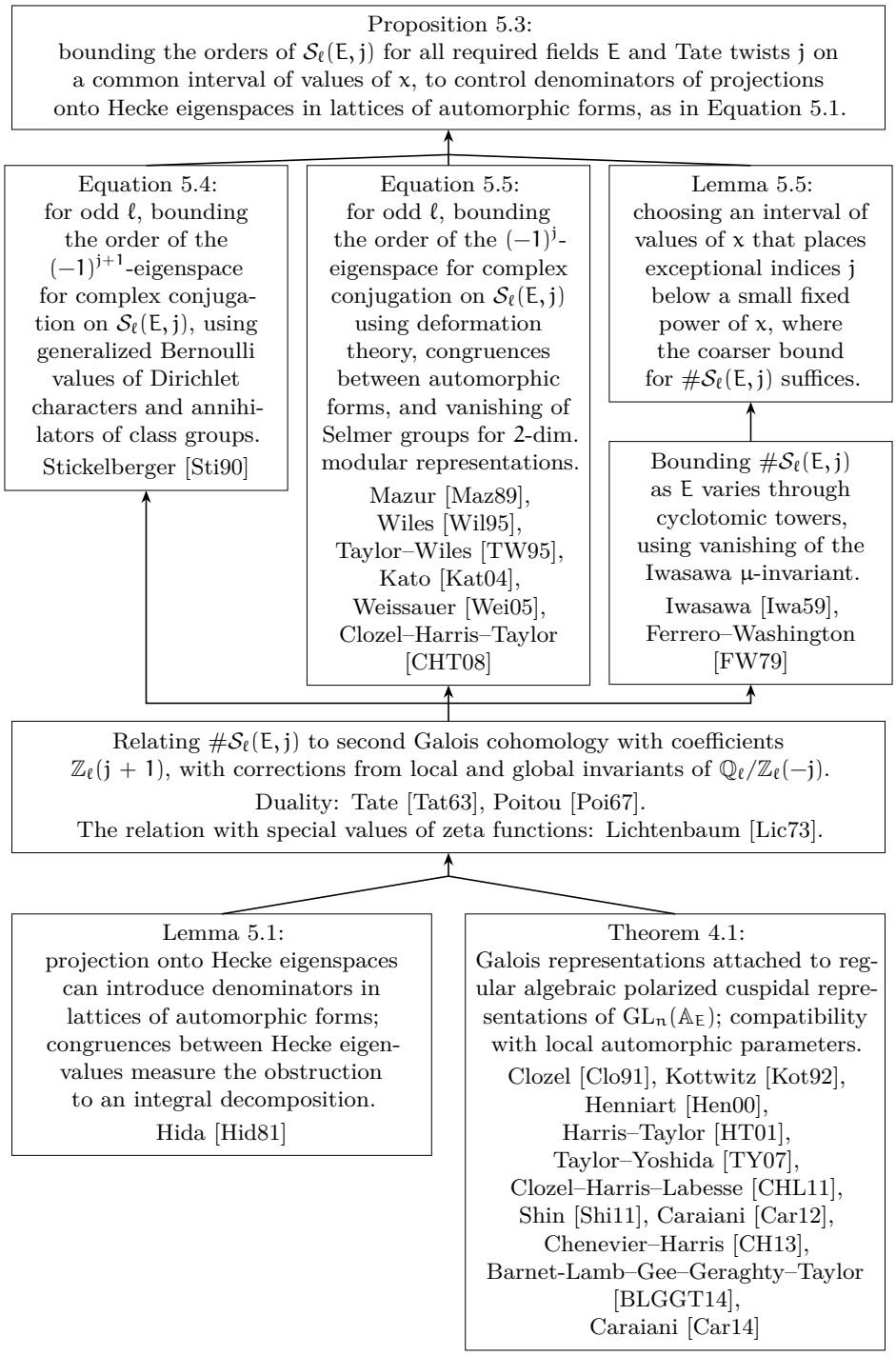


FIGURE 4. Galois cohomology and the denominators introduced in projecting lattices of automorphic forms in [Alp26], with selected historical references.

2. Detecting zeros from sums over prime powers

The estimate for $\mathcal{G}_\psi$ sought in [Alp26] has two limitations: it holds only on certain intervals $[Y, Y^{3/2}]$, with Y arbitrarily large, and even there it gives only an upper bound. To see why this suffices for GRH, consider the effect of a zero of an L-function on the Mellin transform of $\mathcal{G}_\psi$. The positive coefficients in the linear combination $\mathcal{G}_\psi$ of the corrected sums G_χ prevent contributions from a shared zero from canceling. By differentiating the Mellin transform many times, [Alp26] detects a pole using the upper bound for $\mathcal{G}_\psi$ on a single interval. The freedom to choose these intervals is useful in Section 6, where [Alp26] chooses auxiliary number fields and weights of automorphic representations subject to several arithmetic conditions.

For this part of the argument, the important feature of Equation 1.1 is the positivity of its coefficients. The nonvanishing criterion can therefore be stated for a more general linear combination. Let $\mathcal{X}$ be a finite family of primitive Dirichlet characters, possibly including $\mathbb{1}$. For positive real numbers a_χ, consider the combination

$$M_{\mathcal{X}}(x) := \sum_{\chi \in \mathcal{X}} a_\chi \mathrm{Re}(G_\chi(x)),$$

where the functions G_χ are defined in Section 1.2. Since the coefficients are positive, the multiplicities of a zero in different L-functions will add in the Mellin transform of $M_{\mathcal{X}}$. Proposition 2.1 states the upper bound for $M_{\mathcal{X}}$ that will suffice.

Proposition 2.1. *Let $0 < \vartheta < 1$, $\delta > 0$, and $C \in \mathbb{R}$. Suppose that there are arbitrarily large $Y \geq 2$ such that for every $x \in [Y, Y^{1+\delta}]$,*

$$M_{\mathcal{X}}(x) \leq C x^{\vartheta}.$$

Then every $L(s, \chi)$ with $\chi \in \mathcal{X}$ is nonzero in $\mathrm{Re}(s) > \vartheta$, away from the pole of $\zeta(s)$ at $s = 1$.

There is a useful classical precedent for recovering information about singularities from a condition on signs. Landau [Lan05] proved that a Dirichlet series with nonnegative coefficients has a singularity at its finite real abscissa of convergence. In the present situation, however, the upper bound is available only on certain intervals, so Landau's theorem does not apply directly. To concentrate an integral on one interval where the bound holds, [Alp26] differentiates the Mellin transform of $M_{\mathcal{X}}$ many times. Between these intervals, a rough bound for $M_{\mathcal{X}}$ will suffice.

2.1. How the Mellin transform detects zeros.

The connection between primes and zeros begins with the logarithmic derivative of an L-function: its Dirichlet series involves prime powers, whereas its poles record zeros with their multiplicities. This is the starting point of the explicit formulas to which [Alp26, p. 3] refers. Guinand [Gui48] related weighted sums over prime powers to weighted sums over zeros, under the Riemann hypothesis.

Weil [Wei52, Wei72] developed explicit formulas for L-functions over number fields, keeping track of the contributions from every place. For the weighted sums S_χ used in [Alp26], the identity follows directly by taking a Mellin transform, without assuming the Riemann hypothesis. The factor $x/n - 1$ contributes $1/(s(s-1))$, and the corrections in G_χ remove the singularity at 1.

Indeed, the sum defining S_χ in Section 1.2 can be integrated term by term for $\mathrm{Re}(s) > 2$. Since

$$\int_n^\infty \Big(\frac{x}{n} - 1\Big) x^{-s-1}\,dx = \frac{n^{-s}}{s(s-1)},$$

the Mellin transform of S_χ is

$$\int_1^\infty S_\chi(x) x^{-s-1}\,dx = -\frac{1}{s(s-1)}\frac{L'}{L}(s,\chi).$$

Write $F_\chi(s)$ for the Mellin transform of G_χ. For a nonprincipal character, the term $xL'(1,\chi)/L(1,\chi)$ added in the definition of G_χ therefore gives

$$F_\chi(s) = -\frac{1}{s(s-1)}\frac{L'}{L}(s,\chi) + \frac{1}{s-1}\frac{L'}{L}(1,\chi). \tag{2.1}$$

For $\chi = \mathbb{1}$, the same calculation gives

$$F_{\mathbb{1}}(s) = -\frac{1}{s(s-1)}\frac{\zeta'}{\zeta}(s) - \frac{1}{(s-1)^2} + \frac{1+\gamma}{s-1}. \tag{2.2}$$

It is useful to check the cancellation at $s = 1$ explicitly. For a nonprincipal character, L is holomorphic and nonzero at 1, so the two terms in Equation 2.1 have opposite residues there. In Equation 2.2, the cancellation follows from the expansion $\zeta'/\zeta(s) = -(s-1)^{-1} + \gamma + O(s-1)$. Thus both transforms are holomorphic at 1.

Logarithmic coordinates make the intervals in Proposition 2.1 easier to handle. Put $g(u) := M_{\mathcal{X}}(e^u)$ for $u \geq 0$. The interval $[Y, Y^{1+\delta}]$ becomes $[\log Y, (1+\delta)\log Y]$. For $\mathrm{Re}(s) > 2$, the Mellin transform can therefore be written as the Laplace transform

$$F(s) := \int_0^\infty g(u) e^{-su}\,du = \frac{1}{2}\sum_{\chi\in\mathcal{X}} a_\chi (F_\chi(s) + F_{\overline{\chi}}(s)).$$

The pairing of χ with $\overline{\chi}$ accounts for the real part in the definition of $M_{\mathcal{X}}$. By Equations 2.1 and 2.2, F extends meromorphically to $\mathrm{Re}(s) > 0$. The residues now show why the coefficients were chosen to be positive. Let $m_\chi(\rho)$ denote the multiplicity of a zero ρ of $L(s,\chi)$, with multiplicity 0 if ρ is not a zero. If $\mathrm{Re}(\rho) > 0$ and $L(s,\chi)$ or $L(s,\overline{\chi})$ vanishes at ρ for some $\chi \in \mathcal{X}$, then

$$\begin{aligned} \operatorname*{res}_{s=\rho} F(s) &= \frac{\nu_\rho}{\rho(1-\rho)}, \\ \nu_\rho &:= \frac{1}{2}\sum_{\chi\in\mathcal{X}} a_\chi (m_\chi(\rho) + m_{\overline{\chi}}(\rho)). \end{aligned} \tag{2.3}$$

Notice that ν_ρ is positive even when several characters have a zero at the same point. Thus no such zero disappears from F. Hadamard [Had96, §10] proved nonvanishing on $\mathrm{Re}(s) = 1$, and the Euler product excludes zeros for $\mathrm{Re}(s) > 1$. The poles of F in $\mathrm{Re}(s) > 0$ therefore form a discrete subset of $0 < \mathrm{Re}(s) < 1$. In particular, the residue at a real pole is positive; this sign is used in the proof of Proposition 2.1.

The poles of F have now been identified, but its integral expression involves all $u \geq 0$, including the gaps between the intervals in Proposition 2.1. To control the contribution from those gaps, observe that $|\Lambda(n)\chi(n)| \leq \log n$, so the corrected sums $G_\chi(x)$ are $O(x \log^2(2x))$. Consequently $|g(u)| \ll e^{2u}$ for $u \geq 0$. The derivatives of F will concentrate the integral on one interval, leaving this rough bound to control its complement.

2.2. **Using a bound for $M_{\mathcal{X}}$ on one interval.** Each zero of $L(s,\chi)$ in the critical strip, for $\chi \in \mathcal{X}$, gives a pole of F. The remaining difficulty is to detect one of these poles from a bound for $M_{\mathcal{X}}$, despite possible cancellation between contributions from different poles. Turán [Tur49, Tur50] addressed precisely this difficulty in studying errors in prime counting, using lower bounds for power sums. The discussion in [Alp26, pp. 3–4] explicitly mentions this method. In [Alp26], repeated differentiation of F serves a related purpose. The point of differentiation is chosen so that one pole is nearer than all the others; its contribution then dominates the derivatives of high order.

There is also a reason to use derivatives of high order in the integral expression for F. Differentiating k times introduces a factor u^k. After division by $k!$, the factor multiplying $g(u)$ in the integral is a constant multiple of a gamma density. When k is proportional to $\log Y$, most of the mass of this density lies inside the interval where the upper bound for g holds. The following lemma bounds the integral outside that interval.

Lemma 2.2. *Fix $\delta > 0$. There is a $\sigma > 2$ such that, for every sufficiently large U, there is an integer $k \geq U$ with $k \asymp U$ and*

$$\int_{[0,\infty)-[U,(1+\delta)U]} \frac{u^k}{k!} e^{-(\sigma-2)u}\, du \ll \sigma^{-k}.$$

The constants depend only on δ and σ.

Proof. Set $\tau := 1 + \delta/2$. For a sufficiently large $q > 0$, choose $k := \lfloor q\tau U \rfloor$. The measure $q^{k+1}u^k e^{-qu}\, du/k!$ has total mass 1 and exponential moments $(q/(q-t))^{k+1}$ for $t < q$. The exponential Markov inequality bounds its two tails using the function $I(\nu) := \nu - 1 - \log \nu$. The ratios of the two endpoints to the mean tend to $1/\tau < 1$ and $(1+\delta)/\tau > 1$, respectively. Since I is positive at both values, there is a constant $c > 0$, depending only on δ, such that the mass outside $[U, (1+\delta)U]$ is at most $2e^{-ck}$ for sufficiently large U.

Choose q large enough that $q\tau > 1$ and $\log((q+2)/q) < c$, and set $\sigma := q + 2$. The integral in the statement is then at most $2q^{-k-1}e^{-ck} \ll \sigma^{-k}$. This choice ensures that $k \geq U$ and $k \asymp U$ for sufficiently large U. □

Lemma 2.2 controls the integral outside one interval. Inside that interval, a different difficulty remains: an upper bound for g alone does not bound the absolute value of its transform. To use precisely the bound available, [Alp26] combines $|F^{(k)}(\sigma + it)|$ with $(-1)^k F^{(k)}(\sigma)$, for real t. After division by $k!$, this sum can be bounded using only the positive part of g. The proof below compares that upper bound with the faster growth forced by a pole of real part greater than ϑ.

Proof of Proposition 2.1. Increase C if necessary so that $C \geq 0$. In logarithmic coordinates, the hypothesis on $M_{\mathcal{X}}$ provides arbitrarily large U such that, for every $u \in [U, (1+\delta)U]$,

$$g(u) \leq Ce^{\vartheta u}.$$

Choose σ as in Lemma 2.2, and set $R := \sigma - \vartheta$. The factor $u^k/k!$ from Lemma 2.2 appears after differentiating k times and dividing by $k!$. Absolute convergence justifies differentiation under the integral for F. For real t, write

$$A_k(t) := \frac{(-1)^k}{k!} F^{(k)}(\sigma + it) = \int_0^\infty \frac{u^k}{k!} g(u) e^{-\sigma u} e^{-itu}\, du.$$

Write $g(u)_+ := \max(g(u), 0)$. Since g is real-valued,

$$|A_k(t)| + A_k(0) \leq 2 \int_0^\infty \frac{u^k}{k!} g(u)_+ e^{-\sigma u}\, du.$$

On $[U, (1+\delta)U]$, the assumed upper bound for g and positivity of the integrand imply

$$\int_U^{(1+\delta)U} \frac{u^k}{k!} Ce^{-Ru}\, du \leq C \int_0^\infty \frac{u^k}{k!} e^{-Ru}\, du = CR^{-k-1}.$$

On the complement, the rough bound $|g(u)| \ll e^{2u}$ is sufficient by Lemma 2.2. Consequently, along an unbounded sequence of integers k,

$$|A_k(t)| + A_k(0) \ll R^{-k}, \tag{2.4}$$

uniformly for real t. The implied constant is independent of the interval and of t.

It remains to compare Equation 2.4 with the contribution of a pole of F. Suppose that F has a pole with real part greater than ϑ. The argument separates into two cases, according to the poles nearest to the real point σ.

First, suppose that the unique nearest pole is real, say β, and that $d := \sigma - \beta < R$. Choose r with $d < r < R$ such that the disk of radius r centered at σ contains no other pole. By Equation 2.3, the residue c at β is positive. After subtracting $c/(s - \beta)$, the remainder is holomorphic on the disk of radius r, so that Cauchy's estimates imply

$$A_k(0) = \frac{c}{d^{k+1}} + O(r^{-k}).$$

Since $d < R$, the resulting lower bound for $A_k(0)$ contradicts Equation 2.4 at $t = 0$.

In every other case, there is a real t and a unique pole ρ nearest to $z := \sigma + it$ such that, on writing $d := |z - \rho|$,

$$d < R,$$
$$d < \operatorname{dist}(\sigma, \{\text{poles of } F\}).$$

To see this, suppose first that a nonreal pole is nearest to σ at distance less than R. Move the center a small distance vertically towards this pole, thereby decreasing its distance to the center. If no pole has distance less than R from σ, instead move the center to the imaginary part of a pole with real part greater than ϑ. In either case, the displayed inequalities are strict and persist under a further sufficiently small perturbation. The perturbation can be chosen so that the nearest pole is unique. Indeed, only finitely many poles are relevant in the compact region under consideration, and equality of the distances to two different poles excludes at most one point of the vertical line. Poles with the same imaginary part cannot be equidistant from points on $\operatorname{Re}(s) = \sigma$, since their real parts are less than σ.

Choose $r > d$ smaller than R, the distance from σ to every pole, and the distance from z to every pole other than ρ. Let $c \neq 0$ be the residue at ρ. Subtracting its principal part makes Cauchy's estimates applicable at z; at σ, the function F is already holomorphic on the disk of radius r. Consequently,

$$A_k(t) = \frac{c}{(z-\rho)^{k+1}} + O(r^{-k}),$$
$$A_k(0) = O(r^{-k}).$$

Hence

$$|A_k(t)| + A_k(0) \geq |c| d^{-k-1} - O(r^{-k}),$$

which again contradicts Equation 2.4. Thus F has no pole in $\operatorname{Re}(s) > \vartheta$. By Equation 2.3, $L(s, \chi)$ has no zero in $\operatorname{Re}(s) > \vartheta$ for any $\chi \in \mathcal{X}$. □

2.3. **Excluding zeros on both sides of the critical line.** To apply Proposition 2.1 to $\mathcal{G}_\psi$, recall that Equation 1.1 expresses it as a positive linear combination of the real parts of the corrected sums G_χ, including G_ψ. Thus the upper bound for $\mathcal{G}_\psi$ sought in [Alp26] rules out a zero of $L(s, \psi)$ to the right of the critical line. The functional equation then treats the other half of the strip. For completeness, the following deduction also includes the passage from primitive characters to all Dirichlet characters.

Proposition 2.3. *If Equation 1.2 holds for every primitive character ψ and every $\eta > 0$, then GRH holds for all Dirichlet L-functions.*

Proof. Suppose that $L(s, \psi) = 0$ with $\operatorname{Re}(s) > 1/2$. Choose ϑ with $1/2 < \vartheta < \min(\operatorname{Re}(s), 1)$, and set $\eta := \vartheta - 1/2$. By Proposition 2.1, with $\delta = 1/2$, the bound in Equation 1.2 excludes this zero. Thus every primitive Dirichlet L-function is nonzero to the right of the critical line.

To treat the other half of the critical strip, let χ be primitive of conductor q, and let $\kappa_\chi \in \{0,1\}$ satisfy $\chi(-1) = (-1)^{\kappa_\chi}$. The completed function

$$\Lambda(s,\chi) := \left(\frac{q}{\pi}\right)^{(s+\kappa_\chi)/2}\Gamma\left(\frac{s+\kappa_\chi}{2}\right)L(s,\chi)$$

satisfies the functional equation $\Lambda(s,\chi) = \varepsilon_\chi\Lambda(1-s,\overline{\chi})$, where $|\varepsilon_\chi| = 1$. Since the gamma factors have neither poles nor zeros in $0 < \mathrm{Re}(s) < 1$, a zero to the left of the critical line would give a zero to its right for $\overline{\chi}$. Thus every zero in the critical strip has real part $1/2$.

Finally, let χ be any Dirichlet character modulo N, induced by a primitive character χ^* of conductor q. Then

$$L(s,\chi) = L(s,\chi^*)\prod_{\substack{p\mid N\\ p\nmid q}}(1-\chi^*(p)p^{-s}).$$

For $\mathrm{Re}(s) > 0$, each added factor is nonzero because $|\chi^*(p)p^{-s}| < 1$. The conclusion follows for all Dirichlet characters, including the Riemann hypothesis in conductor 1. □

3. Recovering sums over prime powers from divisibility of automorphic periods

Proposition 2.1 turns upper bounds for $\mathcal{G}_\psi$ into nonvanishing of Dirichlet L-functions. To obtain the sums over prime powers that occur in $\mathcal{G}_\psi$, [Alp26] bounds the valuations of the normalized automorphic periods $P_\alpha(f,g)$ from Section 1.3 at finite primes of their coefficient field K. Character orthogonality provides the connection: just as Dirichlet [Dir37] separated residue classes using characters, the argument here expresses a condition on Frobenius as a sum of character values. Proposition 3.1 makes this relation explicit, including the main term subtracted to obtain the functions G_χ. The choices of the integers $w_{j,\rho}$ from Section 1.3, together with averages over characters, then recover $\mathcal{G}_\psi$.

3.1. **Expressing the Frobenius condition with characters.** A quadratic character distinguishes the coset in an abelian Galois group whose Frobenius elements contribute to the divisibility of period values in [Alp26]. The relation between the Frobenius condition and character values can be stated in a finite group, before choosing the abelian extension of $\mathbb{Q}$. Let Γ be a finite abelian group, written additively, and let $\kappa : \Gamma \longrightarrow \{\pm 1\}$ be a nontrivial quadratic character. Set $\Phi := \ker\kappa$ and $f := |\Phi|$, so that $|\Gamma| = 2f$, and write $\widehat{\Gamma}$ for the character group of Γ.

There is a small complication at ramified primes: a Frobenius element is defined only modulo the inertia subgroup. Averaging the character values over inertia removes the choice of a representative and produces the coefficients of the associated primitive Dirichlet L-function. To make this explicit, for each prime p, take a subgroup $I_p \subseteq \Gamma$ and a class $F_p \in \Gamma/I_p$. To evaluate a

character without choosing a preferred lift of F_p, for each $j \geq 1$ and $\chi \in \widehat{\Gamma}$, consider the average

$$a_\chi(p^j) := \frac{1}{|I_p|} \sum_{h \in I_p} \chi(jF_p + h).$$

If χ is trivial on I_p, the average equals $\chi(F_p)^j$; otherwise it is zero. In particular, it is independent of the chosen lift. Extend a_χ multiplicatively, with $a_\chi(1) := 1$. When Γ, I_p, and F_p come from an abelian extension of $\mathbb{Q}$, the numbers $a_\chi(n)$ are the coefficients of the primitive Dirichlet L-function associated to χ, including at ramified primes.

Summing over characters recovers the indicator of the Frobenius condition: for $g \in \Gamma$, character orthogonality gives

$$\frac{1}{f} \sum_{\chi \in \widehat{\Gamma}} \overline{\chi(g)} a_\chi(p^j) = \frac{2}{|I_p|} \mathbf{1}_{\{g - jF_p \in I_p\}}. \tag{3.1}$$

Indeed, the inner sum over characters is $|\Gamma|$ precisely when $jF_p + h = g$, and is zero otherwise. Although individual characters may take complex values, their sum on the left is therefore real and nonnegative. This is the identity that expresses the Frobenius condition at p^j as a sum of character values.

In the lower bounds for valuations of $P_\alpha(f, g)$ in [Alp26], a prime power p^j contributes only while a/p^j exceeds a positive parameter w_ρ. In the automorphic application, these parameters specify the blocks of Harish–Chandra parameters added by theta lifting, as described in Section 4.1. For the character identity below, however, a and the w_ρ may be arbitrary positive real numbers. The parameters w_ρ are indexed by Φ, whereas the Frobenius terms run over $\Gamma - \Phi$. To match the indices, choose $c \in \Gamma - \Phi$ and $t \in \Phi$, and use the bijection $\rho_t(g) := c + g + t$ from $\Gamma - \Phi$ to Φ. For the collection $w = (w_\rho)_{\rho \in \Phi}$, the relevant sum is

$$\mathcal{A}(a, t, w) := \frac{1}{f} \sum_{g \notin \Phi} \sum_{\chi \in \widehat{\Gamma}} \overline{\chi(g)} \sum_{n \geq 1} \Lambda(n) a_\chi(n) \Big(\frac{a}{n} - w_{\rho_t(g)} \Big)_+, \tag{3.2}$$

where $u_+ := \max(u, 0)$. Each sum over n is finite. Equation 3.1 expresses the same quantity as

$$\mathcal{A}(a, t, w) = \sum_p \log p \sum_{j \geq 1} \frac{2}{|I_p|} \sum_{\substack{g \notin \Phi \\ g - jF_p \in I_p}} \left(\frac{a}{p^j} - w_{\rho_t(g)} \right)_+.$$

In particular, $\mathcal{A}(a, t, w) \geq 0$. The second expression keeps the contributions to divisibility of period values visible at each prime, while Equation 3.2 expresses the same sum in terms of the characters of Dirichlet L-functions.

3.2. **Removing the main contribution.** The lower bounds for valuations involve $\mathcal{A}(a, t, w)$, whereas Proposition 2.1 concerns linear combinations of the corrected sums G_χ. To pass between these expressions, [Alp26] first collects the correction terms in the definitions of G_χ. To relate the character

calculation to the functions G_χ, take $\Gamma = \mathrm{Gal}(E/\mathbb{Q})$ for an abelian CM extension $E/\mathbb{Q}$ containing an imaginary quadratic field, and let κ be the quadratic character corresponding to that subfield. For each prime p, take I_p to be the inertia subgroup and F_p the Frobenius class in Γ/I_p. Use the same letter for a character of Γ and its associated primitive Dirichlet character. Then $a_\chi(n) = \chi(n)$.

For each nonprincipal χ, put $\ell_\chi := L'(1,\chi)/L(1,\chi)$. These logarithmic derivatives are defined because $L(1,\chi) \neq 0$. In particular, ℓ_κ is real, since κ is real-valued.

Set $S_\chi(y) := 0$ for $0 < y < 1$. The definitions in Section 1.2 then extend to every $y > 0$ by

$$\begin{aligned} G_\chi(y) &:= S_\chi(y) + y\ell_\chi \qquad \text{if } \chi \neq \mathbb{1}, \\ G_{\mathbb{1}}(y) &:= S_{\mathbb{1}}(y) - y\log y + (1+\gamma)y. \end{aligned}$$

Since $\mathcal{A}(a,t,w)$ sums over the nontrivial coset of $\ker\kappa$, only the corrections for the trivial character and κ survive.

The surviving corrections form the real main term

$$\mathcal{M}(a,w) := a\left(\log a - 1 - \gamma + \ell_\kappa - \frac{1}{f}\sum_{\rho\in\Phi}\log w_\rho\right). \tag{3.3}$$

After subtracting $\mathcal{M}(a,w)$, the resulting expression is a linear combination of the functions G_χ. More precisely, we obtain the following identity.

Proposition 3.1. *For $\mathcal{A}$ and $\mathcal{M}$ as in Equations 3.2 and 3.3,*

$$\mathcal{A}(a,t,w) - \mathcal{M}(a,w) = \frac{1}{f}\sum_{g\notin\Phi}\sum_{\chi\in\widehat{\Gamma}}\overline{\chi(g)}w_{\rho_t(g)}G_\chi\left(\frac{a}{w_{\rho_t(g)}}\right). \tag{3.4}$$

Proof. Since $w > 0$, $(a/n - w)_+ = w(a/(wn) - 1)_+$. Thus the inner sum over n in Equation 3.2 is $w_{\rho_t(g)}S_\chi\big(a/w_{\rho_t(g)}\big)$. For $\chi \neq \mathbb{1}$, the correction is $-a\ell_\chi$; for the trivial character, it is

$$a\log a - a\log w_{\rho_t(g)} - (1+\gamma)a.$$

To see which corrections survive, sum over the nontrivial coset:

$$\frac{1}{f}\sum_{g\notin\Phi}\overline{\chi(g)} = \begin{cases} 1 & \text{if } \chi = \mathbb{1}, \\ -1 & \text{if } \chi = \kappa, \\ 0 & \text{otherwise.} \end{cases}$$

Among the nonprincipal characters, only $\chi = \kappa$ therefore contributes a correction, namely $a\ell_\kappa$. Moreover, since ρ_t is a bijection, the terms involving the logarithms of the parameters w_ρ sum to $-(a/f)\sum_{\rho\in\Phi}\log w_\rho$. Together with the correction for $\mathbb{1}$, these terms give $\mathcal{M}(a,w)$ in Equation 3.3. □

Only logarithmic derivatives at 1 occur in this identity.

Remark 3.2. The correction uses logarithmic derivatives only at 1, where the nonprincipal Dirichlet L-functions are already known to be nonzero. Thus Equation 3.4 does not depend on nonvanishing inside the critical strip.

The quantity to be bounded by automorphic periods is therefore the excess $\mathcal{A}(a,t,w)-\mathcal{M}(a,w)$. To use such a bound for $\mathcal{G}_\psi(x)$, however, [Alp26] still has to retain the characters occurring in Equation 1.1 and arrange that their arguments are close to x.

3.3. **Retaining the required characters and arguments.** The identity in Equation 3.4 sums over all characters of Γ, while $\mathcal{G}_\psi$ involves only the primitive characters inducing powers of ψ. Moreover, the arguments of G_χ are the ratios a/w_ρ. To make the desired arguments close to x, [Alp26] chooses the integers w_ρ specifying the real automorphic representations at two different scales. At selected complex places of E, the argument a/w of G_χ is close to x; at the other complex places it is much larger. There, bounds for $G_\chi(a/w)$ follow from regions in which the associated L-functions have no zeros. Averaging over a subgroup of the Galois group retains the required characters. To describe both this average and the parameters w_ρ, [Alp26] chooses an identification

$$\Gamma=\mathrm{Gal}(E/\mathbb{Q})\simeq(\mathbb{Z}/2\mathbb{Z})^2\times H$$

for a finite abelian group H, such that $\kappa(\epsilon_1,\epsilon_2,h)=(-1)^{\epsilon_1}$. After fixing one complex embedding of E, the elements of Γ label all embeddings $E\to\mathbb{C}$. The complex places of E are pairs of conjugate embeddings, so their labels lie in the quotient of Γ by complex conjugation. Since κ corresponds to an imaginary quadratic subfield, it takes the value -1 on complex conjugation. Thus $\Phi=\ker\kappa$ contains one label from each conjugate pair. These representatives form a CM type of E, meaning a choice of one embedding from each complex-conjugate pair. They label the complex places indexing w_ρ. At unramified finite primes, Frobenius acts on the complex places. In the chosen representatives, this action includes complex conjugation whenever the transported embedding lies outside the CM type.

For each translate $u\in H$, [Alp26] chooses positive integers $w_{u,j,\rho}$, with $1\le j\le m$ and $\rho\in\Phi$. At the real place of E^+ corresponding to ρ, these integers specify the blocks of Harish–Chandra parameters added to the representations in the jth theta lift. Thus $w_{u,j,\rho}$ is the parameter denoted by $e_{j-1,\rho}$ in Section 4.1, for the choice indexed by u. The sets T and T_2 label two selected sets of complex places of E. Each $t\in T$ has a positive parameter a_t. The places labeled by T_2 must occur sparsely in short segments of Frobenius orbits. In Section 4.6, [Alp26] obtains this property by adapting Behrend's [Beh46] construction. For a small $\varepsilon>0$, set $b:=\exp(x^\varepsilon)$ and $b':=xb/\exp(x^{\varepsilon/3})$. The parameters satisfy $a_t/b=x+O(1)$. At the places labeled by T_2, the integers $w_{u,j,\rho}$ lie between b and $b+\eta_w+1$, where η_w is much smaller than b. Thus $a_t/w_{u,j,\rho}$ is close to x at these places. At every remaining complex place, including those labeled by T, [Alp26] chooses these

integers so that $w_{u,j,\rho} \le 2b'$ for sufficiently large x. Since $a_t/b = x + O(1)$, it follows that $a_t/w_{u,j,\rho} \gg \exp(x^{\varepsilon/3})$. This much larger argument is the reason for the second choice of scale: estimates for L-functions in regions without zeros control the terms $G_\chi(a_t/w)$ at these places. The choice of parameters accounts for the dependence of those estimates on the characters and conductors. The terms whose arguments are close to x are the ones retained.

It remains to select the characters. For a subgroup $H_0 \subseteq H$, orthogonality gives

$$\sum_{u \in H_0} \chi(u) = \begin{cases} |H_0| & \text{if } \chi|_{H_0} = \mathbb{1}, \\ 0 & \text{otherwise,} \end{cases}$$

so that averaging over H_0 retains precisely the characters trivial on H_0. In [Alp26], the subgroup H_0 and the selected complex places are chosen so that the primitive characters inducing powers of ψ survive this average.

The terms defining $\mathcal{G}_\psi$ are now present in the averaged identity, but two errors remain: the contributions of other characters that survive the average, and the replacement of arguments close to x by x itself. In Section 6, [Alp26] bounds both errors as the fields, conductors, and weights vary.

It remains to obtain the upper bound for $\mathcal{A}(a, t, w) - \mathcal{M}(a, w)$ to be used in the averaged form of Equation 3.4. In Section 4, [Alp26] relates $\mathcal{A}(a, t, w)$ to the valuations of the normalized periods $P_\alpha(f, g)$ at finite primes of K, and compares those valuations with Petersson norms of theta lifts and dimensions of spaces of automorphic forms. The constants in that comparison reproduce $\mathcal{M}(a, w)$, leaving a bound for the required excess.

4. Bounding sums over prime powers with automorphic periods

After the character calculation in Section 3, the remaining problem is to bound $\mathcal{A}(a, t, w) - \mathcal{M}(a, w)$. At first, this may seem rather far from automorphic forms. The connection in [Alp26] comes from the normalized automorphic pairings $P_\alpha(f, g)$ over the coefficient number field K. The valuations of $P_\alpha(f, g)$ at finite places of K contain the summands of $\mathcal{A}(a, t, w)$, while bounds for the absolute values at the archimedean places limit how divisible $P_\alpha(f, g)$ can be, as in the product formula. The comparison is useful only if its constants also recover $\mathcal{M}(a, w)$: the desired bound concerns $\mathcal{A}(a, t, w) - \mathcal{M}(a, w)$.

The comparison of finite valuations with archimedean norms requires two further steps. To obtain enough divisibility of period values relative to the errors in their archimedean estimates, [Alp26] repeats the theta correspondence. It also varies the spaces of automorphic forms being paired with the fixed test space W. Pairing with elements of W gives period maps into the dual lattice C introduced in Section 1.3. If the images of the period maps were confined to a small subspace of $C \otimes_{\mathcal{O}_K} K$, they would say little about the covolume of C. The linear algebra in Section 4.4 is what allows [Alp26] to use enough of these images to bound the mean slope of C. Figures 2

and 3 show how the construction of the periods and the bounds for covolumes of lattices of automorphic forms enter the estimate for $\mathcal{A}(\mathfrak{a}, \mathfrak{t}, w) - \mathcal{M}(\mathfrak{a}, w)$.

4.1. **Increasing divisibility of period values by repeated theta lifts.** Before describing the Bessel and Fourier–Jacobi pairings used in the successive theta lifts, it helps to recall where the theta correspondence came from. Its underlying representations grew out of questions in quantum mechanics: a linear change of position and momentum preserving the commutation relations should, when possible, be realized by a unitary operator. Van Hove [vH51] had already studied this problem for canonical transformations. In the real case, Segal [Seg59] recognized the resulting projective representation, and Shale [Sha62] developed the theory in his study of the linear symmetries of free boson fields. In his later account, Segal [Seg63, pp. 31–33] explained the relation to quantum mechanics and to van Hove's work.

The passage to number theory required an extension to local fields and adeles. In making this extension, Weil [Wei64, pp. 143–144] sought to understand the modularity of theta series and the appearance of symplectic groups in Siegel's theory of quadratic forms. He explicitly credited Segal [Seg63] for the principle of his existence proof and for the starting point in locally compact abelian groups. Howe [How79] then organized the representation theory around reductive dual pairs: two reductive groups acting as mutual centralizers in a symplectic group. Restricting the Weil representation to such a pair made it possible to pass between representations of its two members.

The unitary groups and the theta integral. Let E be the auxiliary CM field from Section 3, with maximal totally real subfield E^+. For a nondegenerate Hermitian space (V, h_V) over E, write $U(V)$ for the algebraic group over E^+ preserving h_V. Its E^+-points are the E-linear automorphisms g satisfying $h_V(gv, gw) = h_V(v, w)$ for every $v, w \in V$. At a real place τ of E^+, a Hermitian form of signature (p_τ, q_τ) gives the real group $U(p_\tau, q_\tau)$. Thus a totally definite Hermitian space gives a compact unitary group at every real place.

To write a theta kernel, first take a Hermitian space V and a skew-Hermitian space V' over E. The trace of the tensor product of their forms makes $V \otimes_E V'$ a symplectic space over E^+, on which $U(V) \times U(V')$ acts as a reductive dual pair. Multiplying a Hermitian form by an element $\delta \in E^\times$ with $\bar{\delta} = -\delta$ makes it skew-Hermitian without changing its unitary group. Thus the theta correspondence for a Hermitian space and a skew-Hermitian space also applies when both unitary groups are described using Hermitian spaces; see Gan [Gan23, § 1.7 and § 2.2] for these conventions.

Write $\mathbb{A}_F$ and $\mathbb{A}_{F,f}$ for the adeles and finite adeles of a number field F. The Weil representation also depends on a nontrivial additive character ψ_0 of $E^+\backslash\mathbb{A}_{E^+}$ and two unitary Hecke characters η_V and $\eta_{V'}$ of $E^\times\backslash\mathbb{A}_E^\times$. For the

quadratic character ω_{E/E^+} associated to E/E^+, their restrictions satisfy

$$\eta_V|_{\mathbb{A}_{E^+}^\times} = \omega_{E/E^+}^{\dim_E V},$$
$$\eta_{V'}|_{\mathbb{A}_{E^+}^\times} = \omega_{E/E^+}^{\dim_E V'}.$$

The role of the Hecke characters is to specify lifts of the unitary groups to the metaplectic group, the central extension of the adelic symplectic group by the unit circle S^1; see Gan [Gan23, §2.3 and §3.2]. Kudla [Kud94] constructed these splittings. Pulling back the Weil representation along the splitting gives a representation ω of $U(V)(\mathbb{A}_{E^+}) \times U(V')(\mathbb{A}_{E^+})$.

Choose a polarization $V \otimes_E V' = \mathbb{X} \oplus \mathbb{Y}$ into maximal isotropic subspaces. In the corresponding Schrödinger model, ω acts on the Schwartz–Bruhat space $\mathcal{S}(\mathbb{X}(\mathbb{A}_{E^+}))$. For φ in this space, the theta kernel is

$$\theta_\varphi(g,h) = \sum_{x \in \mathbb{X}(E^+)} (\omega(g,h)\varphi)(x).$$

For an algebraic group G over E^+, write $[G] := G(E^+)\backslash G(\mathbb{A}_{E^+})$. For a cusp form f on $[U(V)]$, its theta lift is given by

$$\Theta_\varphi f(h) = c_V \int_{[U(V)]} \theta_\varphi(g,h) f(g)\, dg.$$

The factor $c_V > 0$ records the normalization of the Petersson pairing. With this definition, $f \mapsto \Theta_\varphi f$ is $\mathbb{C}$-linear, as are the theta lifting maps used in [Alp26]. In the convention of Gan, Qiu, and Takeda [GQT14, §11.2], the integral contains $\bar{f}$; replacing f by its conjugate passes between the two conventions. For cuspidal f the integral converges absolutely and defines an automorphic form on $[U(V')]$.

Which groups occur in the successive theta lifts. To follow both arguments of the period pairing through theta lifting, [Alp26] uses two sequences of Hermitian spaces X_j and Y_j. Write $k := \dim_E X_0$. Let m denote the number of theta lifts in each sequence. Both k and m are even. Put $r := k + m - 1$, and choose an odd integer $h \geq r + 5$ to govern the lengths of the blocks of Harish–Chandra parameters added by theta lifting. For $0 \leq j \leq m$, the dimensions are

$$\dim_E X_j = k + j(h+1),$$
$$\dim_E Y_j = r + jh.$$

At every real place, X_j is positive definite and Y_j has signature

$$\left(r + jh - \left\lfloor \frac{m-j}{2} \right\rfloor, \left\lfloor \frac{m-j}{2} \right\rfloor\right).$$

Thus Y_0 has signature $(k + m/2 - 1, m/2)$, whereas Y_m is totally definite. The isometric embeddings are $X_j \hookrightarrow Y_j$ for $j < m$ and $Y_m \hookrightarrow X_m$. Notice that the direction reverses at the last stage: Y_m has codimension one in X_m.

The choice of representations at infinity is especially explicit for the definite groups $U(X_j)$. At a real place, each local component is an irreducible representation of the compact group $U(\dim_E X_j)$. Recall that an irreducible representation of $U(n)$ has an integral highest weight $\lambda_1 \geq \cdots \geq \lambda_n$. Its Harish–Chandra parameters are the shifted numbers $\lambda_i + (n+1)/2 - i$ for $1 \leq i \leq n$. These parameters describe the infinitesimal character, meaning the character by which the centre of the enveloping algebra of the complexified Lie algebra of $U(n)$ acts.

Label the real places by embeddings τ in the chosen CM type of E. For the theta lift from $U(X_j)$ to $U(X_{j+1})$, [Alp26] retains the old Harish–Chandra parameters and adds the block

$$-e_{j,\tau} + \frac{h}{2}, \quad -e_{j,\tau} + \frac{h}{2} - 1, \quad \ldots, \quad -e_{j,\tau} - \frac{h}{2},$$

where $e_{j,\tau}$ is a given integer. The scalar parameters in Section 3 are $w_{j+1,\tau} = e_{j,\tau}$: the centre of the added block is $-w_{j+1,\tau}$. The real local representations for $U(Y_j)$ instead acquire blocks of length h, together with twists by characters. Thus the two sequences of theta lifts add different numbers of Harish–Chandra parameters. The initial automorphic representations, auxiliary Hecke characters of $E^\times \backslash \mathbb{A}_E^\times$, finite levels, and integral lattices are chosen along with these parameters. In particular, [Alp26] imposes additional conditions at an auxiliary rational prime on the initial spaces of automorphic forms on $[U(X_0)]$. These conditions select the subspaces used as domains of the maps b_α, and are written explicitly in Section 4.4.

Notice why the theta lift is repeated. The favorable terms in the lower bounds for finite valuations are summed over the m theta lifts. Repeating the construction therefore supplies the factor m in the principal contribution, while the changes in normalization and integral lattices also have to be accounted for. The analogy proposed in [Alp26] is with the method of Stepanov [Ste69]. To bound the number of points on a curve over a finite field, Stepanov constructed a nonzero auxiliary polynomial that vanished to high order at those points. He then bounded the number of points by comparing the degree with the orders of vanishing. Schmidt [Sch73] extended this approach to general absolutely irreducible plane curves. The number of theta lifts m plays the part of the order of vanishing in this analogy. But taking more theta lifts also enlarges the spaces of automorphic forms. In choosing m, [Alp26] therefore compares the additional contributions to the lower bounds for finite valuations with the errors in estimating the Petersson norms and dimensions of these spaces.

From Bessel periods to linear functionals. The embeddings between X_j and Y_j allow automorphic forms on $[U(X_j)]$ and $[U(Y_j)]$ to be paired. The initial pairing is a Bessel period for the inclusion $U(X_0) \subset U(Y_0)$ of algebraic groups over E^+. The orthogonal complement of X_0 in Y_0 consists of $(m-2)/2$ hyperbolic planes and a nonisotropic line. Choose a maximal isotropic flag in this complement, and let N be the unipotent radical of its stabilizer in $U(Y_0)$.

Choose a $U(X_0)$-invariant Bessel character ν of $N(\mathbb{A}_{E^+})$, trivial on $N(E^+)$, from the flag and the nonisotropic line as in Gan, Gross, and Prasad [GGP12, § 12 and § 23]. For cusp forms f on $[U(X_0)]$ and g on $[U(Y_0)]$, normalize $[N]$ to have volume one and use the inclusion $U(X_0) \subset U(Y_0)$ to define the Bessel period

$$\mathcal{B}_\nu(f, g) = c_{X_0} \int_{[U(X_0)]} \int_{[N]} \overline{f(h)} g(nh) \overline{\nu(n)} \, dn \, dh.$$

The inner integral extracts the Fourier coefficient of g attached to (N, ν); the outer integral pairs its restriction to $[U(X_0)]$ with f. Thus $\mathcal{B}_\nu$ is linear in g and conjugate-linear in f.

The point of the two sequences of theta lifts is that these pairings can be followed from one stage to the next. Kudla [Kud84] proved seesaw identities by integrating a theta kernel in two orders. In [Alp26], the resulting identities relate the successive periods to the final restriction pairing for $U(Y_m) \subset U(X_m)$, where the unipotent integral has disappeared because the codimension is one. At the intermediate stages, the difference $\dim_E Y_j - \dim_E X_j = m - 1 - j$ alternates in parity. The periods are Bessel periods for even j and Fourier–Jacobi periods for odd j; the latter include an additional theta function in the integrand, as in Gan, Gross, and Prasad [GGP12, §23].

The spaces and lattices for the period pairings. The seesaw identities compare integrals of complex-valued automorphic forms. To compare the archimedean sizes of period values with their finite valuations, [Alp26] also specifies models over K and integral lattices in the spaces being paired. On the $U(Y_0)$ side, the space of test automorphic forms depends on choices at both real and finite places. Fix the embedding $K \to \mathbb{C}$ used to realize the periods. For the real places τ of E^+, let π_τ be the local representations used in [Alp26], and put $\pi_\infty := \bigotimes_\tau \pi_\tau$, a representation of $U(Y_0)(E^+ \otimes_{\mathbb{Q}} \mathbb{R})$. Choose a unit vector ν_τ in each π_τ, and put $\nu_\infty := \bigotimes_\tau \nu_\tau$.

At finite places, fix the compact open subgroup $\mathcal{K}_f \subset U(Y_0)(\mathbb{A}_{E^+,f})$ and the locally constant unitary character ξ_f of $\mathcal{K}_f$ used in [Alp26]. Normalized averaging of right translations against $\overline{\xi_f}$ gives a projector R_f onto functions F satisfying $F(xk) = \xi_f(k)F(x)$ for $k \in \mathcal{K}_f$. Write $\widetilde{W}$ for the finite-dimensional complex space of functions $\overline{\iota(\nu_\infty)}$, where $\iota : \pi_\infty \to L^2([U(Y_0)])$ is continuous and linear, is equivariant for $U(Y_0)(E^+ \otimes_{\mathbb{Q}} \mathbb{R})$, and satisfies $R_f \circ \iota = \iota$. The complex conjugation is the convention used to make the normalized period bilinear.

At the places of E^+ above two auxiliary rational primes, [Alp26] chooses representations of finite groups obtained by reducing $U(Y_0)$ modulo those places. Let e_f^0 be the projector onto the corresponding isotypic components of $\widetilde{W}$; it is given by a central idempotent in the group algebra of the product of these finite groups. Let W be the K-model of its image supplied by [Alp26], so that $W \otimes_K \mathbb{C} = e_f^0 \widetilde{W}$.

With this conjugate realization of the test automorphic forms, the associated bilinear pairing is $(f, g) \mapsto \overline{\mathcal{B}_v(f, \overline{g})}$. The periods P_α also include the translates and normalizations used in [Alp26].

For the initial definite group $U(X_0)$, let α index the highest weights at the real places, and write $\mathcal{U}_\alpha$ for the corresponding K-space of algebraic automorphic forms, as in Gross [Gro99]. They are functions on $U(X_0)(\mathbb{A}_{E^+,f})$ with values in the algebraic representation of the restriction of scalars $\mathrm{Res}_{E^+/\mathbb{Q}} U(X_0)$ specified by α. The functions are equivariant under $U(X_0)(E^+)$ and invariant on the right under a compact open subgroup $\mathcal{K}_{\alpha,f}$, also chosen for each α. The period pairing uses their realization as scalar-valued functions on $[U(X_0)]$. Since X_0 is definite, the double quotient $U(X_0)(E^+)\backslash U(X_0)(\mathbb{A}_{E^+,f})/\mathcal{K}_{\alpha,f}$ is finite. A form is therefore determined by coefficient vectors at representatives of these double cosets. To specify which coefficient vectors are integral, [Alp26] realizes each coefficient representation as a determinant twist of a representation on polynomials in matrix entries. A highest-weight vector in the polynomial representation is a product of leading principal minors. Replacing one row by itself plus t times another and taking the coefficient of t^k defines a divided power operator. The $\mathbb{Z}$-span generated from the highest-weight vector by repeated applications of these operators is the integral Weyl module. Tensoring these modules over $\mathbb{Z}$, one for each real place, gives the coefficient lattice.

At each finite prime of K, [Alp26] extends this lattice to the local ring and transports it using a local integral basis of the Hermitian lattice associated with the double coset. A form belongs to $\widetilde{M}_\alpha$ when its coefficient vector at every double coset lies in the resulting lattices at every finite prime. These conditions define the $\mathcal{O}_K$-lattice $\widetilde{M}_\alpha \subset \mathcal{U}_\alpha$. At an auxiliary rational prime T_α splitting completely in E^+, [Alp26] takes the intersection of the kernels of averaging operators for the multiplicative groups of the residue fields. This defines $\mathcal{U}_\alpha^\circ$ and hence $M_\alpha := \widetilde{M}_\alpha \cap \mathcal{U}_\alpha^\circ$. The averaging operators and the resulting dimension are written out in Section 4.4. For the lower bounds on valuations of period values, [Alp26] then constructs a finite-index sublattice $M'_\alpha \subseteq M_\alpha$, with the same K-span $\mathcal{U}_\alpha^\circ$.

The downward theta lifts on the test automorphic forms also supply the integral structure in W. For a tuple $\mu = (\mu_1, \ldots, \mu_m)$ of auxiliary Hecke characters, write Θ_μ for the composite of the theta lifts from $[U(Y_m)]$ down to $[U(Y_0)]$, with the normalizations of [Alp26]. For algebraic automorphic forms on the definite group $U(Y_m)$, the same construction from integral Weyl modules in the coefficient representations gives the lattice $\widetilde{M}_\mu$. Let V_μ^0 be the image of the finite-place projector corresponding to e_f^0. The image of $\widetilde{M}_\mu \cap V_\mu^0$ under Θ_μ lies in W.

For each finite prime $\mathfrak{p}$ of K, write $\mathcal{O}_{K,\mathfrak{p}}$ for the localization of $\mathcal{O}_K$. The tuple $\mu(\mathfrak{p})$ chosen in [Alp26] defines the local lattice $(M_W)_\mathfrak{p}$ as the $\mathcal{O}_{K,\mathfrak{p}}$-span

of the downward theta images:

$$(M_W)_{\mathfrak{p}} = \mathcal{O}_{K,\mathfrak{p}}\,\Theta_{\boldsymbol{\mu}(\mathfrak{p})}\Big(\widetilde{M}_{\boldsymbol{\mu}(\mathfrak{p})} \cap V^0_{\boldsymbol{\mu}(\mathfrak{p})}\Big).$$

The construction in [Alp26] shows that these local lattices define a lattice in W. Equip M_W with its Petersson norms and $C = \mathrm{Hom}_{\mathcal{O}_K}(M_W, \mathcal{O}_K)$ with the dual norms.

After the identifications in the seesaw identities and the normalization by periods of CM abelian varieties, [Alp26] obtains K-bilinear pairings $P_\alpha : \mathcal{U}^\circ_\alpha \times W \to K$. For the fractional ideal $\mathfrak{a}_\alpha$ in Section 4.5, the valuations of these pairings satisfy $P_\alpha(M'_\alpha, M_W) \subseteq \mathfrak{a}_\alpha$. Consequently $N_\alpha = \mathfrak{a}_\alpha^{-1} M'_\alpha$ has the same K-span as M_α, and $b_\alpha : N_\alpha \to C$, defined by $b_\alpha(f)(g) = P_\alpha(f, g)$ for $g \in M_W$, is integral.

To compare the operator norms of b_α with the divisibility of the period values $P_\alpha(f, g)$, [Alp26] estimates Petersson norms of the theta lifts at the archimedean places of K. Rallis [Ral87] developed formulas expressing such norms through special values of L-functions and local integrals. Gan, Qiu, and Takeda [GQT14] treated the unitary case in their work on the regularized Siegel–Weil formula.

To bound the norms of b_α, [Alp26] uses the following identity on subspaces of W with fixed Hecke eigenvalues. Fix a complex embedding of K. Let B be a simultaneous Hecke eigensystem occurring in $W \otimes_K \mathbb{C}$, and write $W[B]$ for its eigenspace. At an unramified finite prime $\mathfrak{p}$ of E, let $B_\mathfrak{p}$ be its multiset of r Satake parameters, in the arithmetic normalization of [Alp26]. For the tuple $\boldsymbol{\mu}$ of auxiliary Hecke characters, let $S_{\boldsymbol{\mu}}$ be the finite set of primes omitted from the Euler products.

With $q_\mathfrak{p}$ the norm of $\mathfrak{p}$ and $\mu_j(\mathfrak{p})$ its unramified character value, put

$$L^{S_{\boldsymbol{\mu}}}(s, B \otimes \mu_j^{-1}) = \prod_{\mathfrak{p} \notin S_{\boldsymbol{\mu}}} \prod_{\beta \in B_\mathfrak{p}} (1 - \beta \mu_j(\mathfrak{p})^{-1} q_\mathfrak{p}^{-s})^{-1}.$$

The parameters satisfy $|\beta| \le q_\mathfrak{p}^{(r-1)/2}$. Since $h \ge r+5$, the point $s = (h+1)/2$ lies in the region of absolute convergence.

Write $\Theta^*_{\boldsymbol{\mu}}$ for the adjoint with respect to the Petersson inner products. With its chosen normalizations, [Alp26] expresses the change in norms through the identity

$$\Theta_{\boldsymbol{\mu}} \Theta^*_{\boldsymbol{\mu}}\big|_{W[B]} = \kappa_0 \prod_{j=1}^{m} L^{S_{\boldsymbol{\mu}}}\left(\frac{h+1}{2}, B \otimes \mu_j^{-1}\right) \mathrm{Id}_{W[B]}.$$

The positive factor κ_0 is independent of the tuple $\boldsymbol{\mu}$ and the Hecke eigensystem B. The factor κ_0 contains the archimedean factors from the formulas for Petersson norms, together with the constants from the chosen measures and levels. Thus the adjoint of $\Theta_{\boldsymbol{\mu}}$ multiplies squared norms on $W[B]$ by the displayed scalar. In Section 4.4, [Alp26] uses this identity to bound the norms of the period maps b_α when comparing the mean slopes of N_α and C.

The bound for the operator norms of b_α retains the constants in this identity. Together with the dimensions of the spaces of automorphic forms and the normalization of the periods of abelian varieties with complex multiplication, they recover the main term $\mathcal{M}(a, w)$ that was subtracted in Section 3. The equality responsible for this agreement is written out in Section 4.5.

4.2. **From Hecke eigenvalues to Galois representations.** The formulas for Petersson norms in Section 4.1 use individual Hecke eigenspaces. Projection onto these eigenspaces can introduce denominators in the lattices of automorphic forms. To control the denominators, [Alp26] studies congruences between Hecke eigenvalues by attaching Galois representations to them, as used in Section 5. The passage from automorphic to Galois representations has two steps; keeping them separate explains the Tate twists that occur in the cohomology groups. First, base change transfers automorphic representations of $U(V)(\mathbb{A}_{E^+})$, with V totally definite, to automorphic representations of $\mathrm{GL}_n(\mathbb{A}_E)$, where $n = \dim_E V$. The resulting representations need not be cuspidal. Instead, [Alp26] keeps track of pairs (Π_a, d_a), where Π_a is a unitary conjugate self-dual cuspidal representation of $\mathrm{GL}_{n_a}(\mathbb{A}_E)$ and $d_a \geq 1$. For the nontrivial automorphism c of E/E^+, conjugate self-duality is the isomorphism $\Pi_a^c \simeq \Pi_a^\vee$ with the contragredient. For a unitary group in n variables, their dimensions satisfy $\sum_a n_a d_a = n$. At a finite place w of E, write rec_{E_w} for the local Langlands correspondence from irreducible admissible representations of general linear groups over E_w to Weil–Deligne representations, and write $|\cdot|$ for the normalized absolute value on E_w. At an unramified place, the parameter of the base change is

$$\bigoplus_a \bigoplus_{j=0}^{d_a-1} \mathrm{rec}_{E_w}(\Pi_{a,w}) \otimes |\cdot|^{(d_a-1)/2-j}.$$

Thus even a single cuspidal representation can contribute several different powers of the norm character. Mœglin and Waldspurger [MW89] classified the discrete spectrum of the general linear groups in these terms. The weak base change from unitary groups used here is of the kind established by Labesse [Lab11, Corollary 5.3].

The second step attaches a Galois representation to a cuspidal constituent, after the appropriate norm twist. The Frobenius eigenvalues and the Tate twists in Section 5 depend on the normalization of this construction. Fix a rational prime ℓ. Write $G_E := \mathrm{Gal}(\overline{E}/E)$, fix an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$, and normalize local reciprocity to send a uniformizer to geometric Frobenius. For Hodge–Tate weights, use the convention in which the cyclotomic character ϵ_ℓ has weight -1; a Tate twist $V(j)$ means $V \otimes \epsilon_\ell^j$. Consider a regular algebraic polarized cuspidal pair (Π, ω), with Π a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ whose archimedean component Π_∞ has the infinitesimal character of an algebraic representation of $\mathrm{Res}_{E/\mathbb{Q}}\mathrm{GL}_n$. The character ω of $(E^+)^\times \backslash \mathbb{A}_{E^+}^\times$

has the same sign at every real place, and polarization means

$$\Pi^{c} \simeq \Pi^{\vee} \otimes (\omega \circ \mathrm{N}_{E/E^{+}} \circ \det).$$

For each embedding $\sigma : E \hookrightarrow \mathbb{C}$, write $a_{\sigma,1} \geq \cdots \geq a_{\sigma,n}$ for the highest weights of the algebraic representation whose dual has the infinitesimal character of Π_∞. Write G_{E_w} for the absolute Galois group of E_w. For an ℓ-adic representation ρ of G_{E_w}, assumed potentially semistable when $w \mid \ell$, write $\mathrm{WD}(\rho)$ for its associated Weil–Deligne representation. At $w \nmid \ell$, this is Deligne's [Del73, §8] construction; at $w \mid \ell$, it is Fontaine's [Fon94, §2.3.7] construction. The superscript F-ss means Frobenius semisimplification, which retains the monodromy operator. Write HT_τ for the multiset of Hodge–Tate weights at τ.

The passage from Π to a Galois representation is part of the global Langlands correspondence for GL_n. The following theorem brings together the constructions of Clozel, Harris, and Labesse [CHL11], Shin [Shi11], and Chenevier and Harris [CH13] with the further results on local–global compatibility; the argument in [Alp26] uses a weaker form, only needing compatibility at unramified places and the assertions about inertia, monodromy, and ℓ-adic Hodge theory needed for the particular local representations it chooses. We use conventions that agree with Barnet-Lamb, Gee, Geraghty, and Taylor [BLGGT14, §1].

Theorem 4.1. *There is a continuous semisimple representation* $r_{\ell,\iota}(\Pi) : G_E \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ *such that, at every finite place* w*,*

$$\iota\,\mathrm{WD}\big(r_{\ell,\iota}(\Pi)|_{G_{E_w}}\big)^{\text{F-ss}} \simeq \mathrm{rec}_{E_w}\Big(\Pi_w \otimes |\det|^{(1-n)/2}\Big). \tag{4.1}$$

At $w \mid \ell$*, the representation is de Rham, with*

$$\mathrm{HT}_\tau(r_{\ell,\iota}(\Pi)) = \{a_{\iota\tau,j} + n - j : 1 \leq j \leq n\}$$

for each $\tau : E \hookrightarrow \overline{\mathbb{Q}}_\ell$ *inducing* w*. If* Π_w *is unramified, the representation is crystalline at* w*.*

The construction of $r_{\ell,\iota}(\Pi)$ began with Clozel [Clo91], Kottwitz [Kot92], and Harris and Taylor [HT01], who used the cohomology of Shimura varieties under additional hypotheses at finite places. Clozel, Harris, and Labesse [CHL11] used the stable trace formula for unitary groups to remove the discrete-series hypothesis at a finite place, under an additional regularity condition on the weights. Independently, Shin [Shi11] obtained the construction for all regular algebraic weights in odd dimension, and under a weaker extra regularity condition in even dimension. Chenevier and Harris [CH13] then used ℓ-adic families of automorphic forms to treat the remaining weights and established the crystalline assertion in Theorem 4.1. These developments matter in [Alp26], where the dimensions and weights vary with the parameters.

The identification of local parameters in Equation 4.1 required further work, which we also briefly summarize. Harris and Taylor [HT01] (with

different proofs by Henniart [Hen00] and Scholze [Sch13]) proved the local correspondence for $\mathrm{GL}_n(F)$ over a nonarchimedean local field F of characteristic zero. For the Galois representations in Theorem 4.1, compatibility away from ℓ was established under successively more general hypotheses by Taylor and Yoshida [TY07] and Caraiani [Car12]. At places above ℓ, Barnet-Lamb, Gee, Geraghty, and Taylor [BLGGT14] proved compatibility up to semisimplification, and up to Frobenius semisimplification for the weights covered by Shin's construction. Caraiani [Car14, Theorem 1.1] identified the monodromy in the remaining cases.

At a place $w \nmid \ell$ where Π_w is unramified, the content of Equation 4.1 is especially concrete. If $\alpha_{w,1}, \dots, \alpha_{w,n}$ are the Satake parameters of Π_w and q_w is the cardinality of the residue field, then

$$\det(X - \iota r_{\ell,\iota}(\Pi)(\mathrm{Frob}_w)) = \prod_{j=1}^{n} \left(X - q_w^{(n-1)/2} \alpha_{w,j}\right).$$

The same normalization explains how [Alp26] chooses the twists of the cuspidal constituents. Let Π_0 be one of the conjugate self-dual cuspidal constituents above, now on $\mathrm{GL}_n(\mathbb{A}_E)$. Suppose the parameters z of its infinitesimal character are distinct at every complex embedding. Choose a real number a_0 for which every $a_0 - z$ is an integer. Applying Theorem 4.1 to $\Pi_0 \otimes |\det|^{(n-1)/2-a_0}$ gives Hodge–Tate weights $a_0 - z$ and unramified Frobenius eigenvalues $q_w^{a_0} \alpha_{w,j}$. The parameter a_0 fixes these integral Hodge–Tate weights and the powers of q_w in the Frobenius eigenvalues. The resulting twists are used when [Alp26] assembles the Galois representations attached to the different cuspidal constituents.

Beyond the unramified characteristic polynomials, [Alp26] uses the following consequences of Theorem 4.1 for its chosen local representations. At the ramified places selected in [Alp26], the local Langlands parameters have zero monodromy and controlled tame inertia; Equation 4.1 also identifies the characteristic polynomials of inertia elements. At places above ℓ where the automorphic representation is unramified, crystallinity and the specified Hodge–Tate weights allow [Alp26] to compare Galois-stable lattices in these representations. When the automorphic representation at a place above ℓ is ramified but has zero monodromy, the comparison of lattices uses potential crystallinity. The distinctions are necessary: knowing the Frobenius eigenvalues at almost all places would not supply the local conditions on extension classes used in Section 5.

Recognizing the cuspidal constituents from their local parameters also uses older results. Jacquet and Shalika [JS81a, JS81b] proved the required uniqueness from almost all local components, as well as the bounds $q_w^{-1/2} < |\alpha| < q_w^{1/2}$ for unramified Satake parameters of unitary cuspidal representations. When [Alp26] compares the local components of a cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_E)$ at conjugate finite places, admissibility from Bernstein and Zelevinsky [BZ77, Theorem 2.5] allows the use of Gelfand and

Kazhdan [GK75]: the contragredient is obtained by composing with inverse transpose.

Finally, relating congruences between Hecke eigenvalues to Galois cohomology requires more than a semisimple representation over $\overline{\mathbb{Q}}_\ell$. After enlarging the coefficient field, [Alp26] chooses Galois-stable lattices and compares their reductions modulo a coefficient prime. The congruences between Hecke eigenvalues then constrain how the residual constituents can be joined by extensions. These extension classes are the objects used to bound the denominators of the projections onto Hecke eigenspaces in Section 5. Notice that the construction of $\mathsf{r}_{\ell,\iota}(\Pi)$ starts with automorphic forms and attaches Galois representations to them; the automorphy lifting arguments in that section start with Galois representations and go in the opposite direction.

4.3. **Obtaining bounds for lattices of automorphic forms.** The maps $\mathsf{b}_\alpha : \mathsf{N}_\alpha \to \mathsf{C}$ compare lattices, so their norms and denominators are only part of the comparison. To bound the mean slope of C, [Alp26] also needs bounds for the covolumes of the initial lattices M_α as levels and highest weights vary. For this purpose, [Alp26] uses holomorphic automorphic forms on auxiliary unitary groups and their realization as sections of bundles on unitary moduli schemes. The argument calls for integral sections with controlled archimedean norms, and for sections that do not vanish at specified points. Figure 2 records the results from algebraic and arithmetic geometry used here.

Automorphic bundles on unitary moduli schemes. Fix the dimension, the action of $\mathcal{O}_\mathsf{E}$, its signature, and the level structure in one of the moduli problems used in [Alp26], and write $\mathcal{M}$ for the scheme parametrizing principally polarized abelian varieties with these data. The signature determines the dimensions of the eigenspaces for $\mathcal{O}_\mathsf{E}$ in the tangent space of each abelian variety. Let $\pi : \mathcal{A} \to \mathcal{M}$ be the universal abelian scheme, with zero section e. Its Hodge bundle $\omega := \mathsf{e}^*\Omega^1_{\mathcal{A}/\mathcal{M}}$ is the bundle of invariant differentials on the fibres of π. The $\mathcal{O}_\mathsf{E}$-action decomposes ω into eigensubbundles after extending the field of definition so that the embeddings of E split. For general highest weights, [Alp26] also uses flag varieties of the Hodge eigensubbundles and their duals. A point of such a flag variety specifies a filtration with one-dimensional successive quotients.

To apply intersection theory to these bundles, [Alp26] works on compactifications of $\mathcal{M}$ and the associated flag varieties. Ash, Mumford, Rapoport, and Tai [AMRT75] developed toroidal compactifications of arithmetic quotients over $\mathbb{C}$. To retain the integral lattices of holomorphic automorphic forms, [Alp26] also extends $\mathcal{M}$ and its bundles over rings of integers or their localizations, and then compactifies the resulting integral models. Integrality of a section is checked at all finite primes. For the Siegel moduli schemes of principally polarized abelian varieties with level structure, Faltings and Chai [FC90, Chapters IV–V] constructed arithmetic compactifications through degenerations to semi-abelian varieties, which are extensions of abelian varieties by

algebraic tori. In [Alp26], the boundary charts on compactifications of $\mathcal{M}$ allow cusp forms to be treated as sections vanishing along the boundary. On the associated flag varieties, the highest weights specify tensor powers of the quotient lines and their duals. The degrees of these line bundles enter the bounds for covolumes of lattices of holomorphic automorphic forms. The argument uses resolution and principalisation in characteristic zero, going back to Hironaka [Hir64], to work on smooth models and to make the base ideals of selected spaces of sections into Cartier divisors on a common modification. To compare the degrees of the determinant line bundles with invariant volumes, [Alp26] uses Mumford's [Mum77] study of metrics at the boundary. The same degrees and volumes enter the formulas for dimensions of spaces of automorphic forms in Section 4.5.

For sections of automorphic bundles on a compactification of $\mathcal{M}$, there are two questions to answer.

- First, how large are the spaces of sections?
- Second, can a section be chosen nonzero at a given point?

For the first question, [Alp26] uses the theorems of Kawamata [Kaw82] and Viehweg [Vie82] on vanishing of higher coherent cohomology, so that the dimensions can be computed from Euler characteristics. For the second, it uses multiplier ideals and vanishing in the sense of Nadel [Nad90], together with effective generation of sections of the kind proved by Angehrn and Siu [AS95], which bounds the positivity needed to obtain a section that does not vanish at that point. Together, these results are used in [Alp26] to bound the dimensions of spaces of sections, the orders of vanishing of sections, and the covolumes of the integral lattices of sections.

Small sections and arithmetic intersection numbers. The integral sections of the automorphic bundles carry norms at all archimedean places of their field of definition. To compare the covolume of a lattice of sections with degrees of line bundles on a compactification of $\mathcal{M}$, [Alp26] uses arithmetic intersection numbers. It is useful to state the relevant facts for a regular, integral, projective scheme X flat over $\mathcal{O}_K$. Write $d := \dim X$, so that $\dim X_K = d - 1$, and let $\overline{L}$ be a line bundle with smooth Hermitian metrics invariant under complex conjugation. The number of integral sections of $L^{\otimes k}$ with sup norm at most 1 is measured by

$$\widehat{h}^0(k\overline{L}) := \log \#\{s \in H^0(X, L^{\otimes k}) : ||s||_{\sup} \leq 1\},$$

where the sup norm is taken over the complex fibres at all archimedean places of K. The leading growth of $\widehat{h}^0(k\overline{L})$ is measured by the arithmetic volume

$$\widehat{\mathrm{vol}}(\overline{L}) := \limsup_{k\to\infty} \frac{d!}{k^d} \widehat{h}^0(k\overline{L}).$$

Positive arithmetic volume means that $\overline{L}$ is arithmetically big; in particular, some positive power has a nonzero integral section with sup norm at most 1.

The results used to produce small sections have different positivity hypotheses. Nefness of L on the geometric fibres means that its degree on every curve in a geometric fibre is nonnegative. Semipositivity concerns the curvature of the Hermitian metric at the archimedean places. Arithmetic nefness requires, in addition, nonnegative degrees on horizontal arithmetic curves, the closures in X of closed points of X_K. Equivalently, the corresponding heights of algebraic points of X_K must be nonnegative. For an algebraic point P defined over a finite extension F/K, properness extends P over $\mathcal{O}_F$, and its height is

$$h_{\overline{L}}(P) := \frac{\widehat{\deg}(P^*\overline{L})}{[F : \mathbb{Q}]}.$$

The arithmetic degree of $P^*\overline{L}$ combines the finite valuations of a nonzero rational section with the logarithms of its archimedean norms. This is the usual normalization of heights.

For positive Hermitian line bundles, Gillet and Soulé [GS88] related the asymptotic covolumes of lattices of sections to arithmetic intersection numbers, using the estimates for analytic torsion of Bismut and Vasserot [BV89]. Abbes and Bouche [AB95] subsequently gave another proof. Zhang [Zha95, Theorem 1.4] extended this arithmetic Hilbert–Samuel formula to the case where L_K is ample, L is nef on geometric fibres, and its metric is smooth and semipositive. To state the formula, write $\chi_{\sup}(H^0(X, L^{\otimes k}))$ for the logarithm of the volume of the unit ball for the sup norm divided by the covolume of the lattice $H^0(X, L^{\otimes k})$. Both volumes use the same Lebesgue measure on $H^0(X, L^{\otimes k}) \otimes_{\mathbb{Z}} \mathbb{R}$. With $\widehat{c}_1$ denoting the first arithmetic Chern class and $\widehat{\deg}(\widehat{c}_1(\overline{L})^d)$ its top arithmetic self-intersection, the formula is

$$\chi_{\sup}(H^0(X, L^{\otimes k})) = \frac{\widehat{\deg}(\widehat{c}_1(\overline{L})^d)}{d!} k^d + o(k^d).$$

Zhang [Zha95, Corollary 1.9] applied Minkowski's theorem to deduce that positive arithmetic self-intersection gives a nonzero strictly small integral section of some positive tensor power, meaning that its sup norm is less than 1.

The argument in [Alp26] also needs this conclusion when L_K is only nef. The further work of Yuan and Moriwaki matters at precisely this point. Yuan [Yua08, Theorem 2.2] proved an arithmetic analogue of the criterion for bigness of Siu [Siu93, §1]. For arithmetically ample Hermitian line bundles $\overline{A}$ and $\overline{B}$, the inequality

$$\widehat{\deg}\Big(\widehat{c}_1(\overline{A})^d\Big) > d\,\widehat{\deg}\Big(\widehat{c}_1(\overline{A})^{d-1}\widehat{c}_1(\overline{B})\Big)$$

ensures that $\overline{A} \otimes \overline{B}^{-1}$ is arithmetically big. The mixed intersection measures the cost of subtracting $\overline{B}$ from $\overline{A}$. Moriwaki [Mor09] adapted Yuan's estimates for changes in the number of small sections to prove continuity of arithmetic

volume. His results give

$$\widehat{\mathrm{vol}}(\overline{L}) \geq \widehat{\deg}\Big(\widehat{c}_1(\overline{L})^d\Big)$$

when L is nef on every geometric fibre and its metric is smooth and semipositive. Under the stronger assumption that $\overline{L}$ is arithmetically nef, Moriwaki [Mor09] proved the arithmetic Hilbert–Samuel formula for the number of small sections:

$$\widehat{h}^0(k\overline{L}) = \frac{\widehat{\deg}(\widehat{c}_1(\overline{L})^d)}{d!}k^d + o(k^d).$$

The distinction between fibrewise nefness and arithmetic nefness matters when a metric is rescaled: multiplying the metric by a constant preserves its curvature, but can change the signs of the heights on horizontal arithmetic curves.

From small sections to heights of points. A small section allows [Alp26] to compare heights of points with the arithmetic self-intersection of the line bundle. Suppose L is nef on every geometric fibre, its metric is smooth and semipositive, and $\delta := \deg_{L_K}(X_K) > 0$. Normalize the arithmetic self-intersection by putting

$$h_{\overline{L}}(X) := \frac{\widehat{\deg}(\widehat{c}_1(\overline{L})^d)}{d[K:\mathbb{Q}]\delta}.$$

For a real number c, let $\overline{L}(-c)$ denote the same line bundle with norms multiplied by e^c. Arithmetic intersection theory gives

$$h_{\overline{L}(-c)}(X) = h_{\overline{L}}(X) - c,$$
$$h_{\overline{L}(-c)}(P) = h_{\overline{L}}(P) - c.$$

If $c < h_{\overline{L}}(X)$, the rescaled bundle still has positive arithmetic self-intersection. Moriwaki's theorem therefore supplies a nonzero integral section s of some $L^{\otimes k}$ with $||s||_{\sup,\overline{L}} \leq e^{-kc}$. For P outside the zero divisor of s, the finite valuations of $s(P)$ are nonnegative, while its archimedean norms satisfy this bound. Consequently $h_{\overline{L}}(P) \geq c$.

Zhang [Zha95, proof of Theorem 5.2, p. 210] used this rescaling and evaluation of small sections in his inequalities for successive minima. The version in [Alp26] uses Moriwaki's result to obtain the section under the stated nefness hypotheses. The two comparisons needed here are particularly simple.

- If $h_{\overline{L}}(P) \geq a$ for every algebraic point P, then $h_{\overline{L}}(X) \geq a$.
- If a Zariski dense set of algebraic points satisfies $h_{\overline{L}}(P) \leq b$, then $h_{\overline{L}}(X) \leq b$.

For the lower bound $h_{\overline{L}}(X) \geq a$, $\overline{L}(-a)$ is arithmetically nef. Applying Moriwaki's [Mor09] arithmetic Hilbert–Samuel formula to this bundle shows that its top arithmetic self-intersection is nonnegative: the quantities $\widehat{h}^0(k\overline{L}(-a))$ are nonnegative because the zero section is included in the count. Since

$h_{\overline{L}(-a)}(X) = h_{\overline{L}}(X) - a$, the desired bound follows. For the upper bound $h_{\overline{L}}(X) \leq b$, suppose instead that $h_{\overline{L}}(X) > b$, and choose c strictly between b and $h_{\overline{L}}(X)$. The section just constructed forces every point of height at most b into its zero divisor, contradicting Zariski density. Thus a lower bound for heights valid at every algebraic point and an upper bound for heights on a Zariski dense set give the same bounds for the normalized arithmetic self-intersection.

For the application to Hodge bundles, there is a difficulty at the boundary: the natural metrics need not extend smoothly to the compactification. Let X now be one of the projective arithmetic models over a compactification of $\mathcal{M}$ used in [Alp26], including the auxiliary models carrying flags of the Hodge bundles. Again write $d := \dim X$, and let $\mathcal{L}$ be the extended line bundle formed from the chosen Hodge quotients and their duals. For a local nonvanishing section e of $\mathcal{L}$, write $\phi := -\log||e||$ for the metric potential. The modification in [Alp26] replaces ϕ by $\phi + U/2$. For a point z in the interior of a complex fibre, let A_z be the principally polarized abelian variety with $\mathcal{O}_E$-action represented by z. Write $\Lambda_z := H_1(A_z, \mathbb{Z})$ for its first integral homology group, viewed inside $\mathrm{Lie}(A_z)$ by integration of invariant differentials. Thus Λ_z is the period lattice of the complex abelian variety A_z. The function U is defined by

$$U(z) := \max\left\{0, \sup_I(-\log N_z(I))\right\},$$

where I runs over the nonzero saturated $\mathcal{O}_E$-sublattices of Λ_z isotropic for the alternating form supplied by the polarization. To define $N_z(I)$, choose an $\mathcal{O}_E$-linearly independent family in I whose span has finite index. Project these vectors in $\mathrm{Lie}(A_z)$ to its $\mathcal{O}_E$-eigenspaces corresponding to the chosen CM type of E, with the Hermitian metrics given by the principal polarization. Then $N_z(I)$ is the product of the resulting Gram determinants divided by the index of the span in I. Replacing ϕ by $\phi + U/2$ therefore multiplies the norm of the local frame e by $e^{-U/2}$.

The corrected potential is locally bounded near the boundary. To smooth it, [Alp26] convolves the local potentials with mollifiers and glues the results using a regularized maximum. The original Hodge potential is regularized separately: the local potentials are truncated from below before convolution, with the truncation level tending to $-\infty$ as the smoothing parameter tends to 0. Fix a sufficiently large cutoff B for rational primes. Write $\mathcal{L}_B$ for $\mathcal{L}$ tensored with the line bundles of the chosen vertical divisors above primes at most B; these divisors are supported in the finite fibres. Denote the two resulting smooth metrized line bundles by $\overline{\mathcal{L}}^+_{B,t'}$ and $\overline{\mathcal{L}}_{B,t}$, respectively, where $t', t > 0$ are the smoothing parameters. To apply the arithmetic Hilbert–Samuel formula, [Alp26] also uses an auxiliary line bundle $\mathcal{A}$ on X, with a smooth metric of positive curvature, and, for integers $1 \leq p \leq q$, considers

$$\overline{\mathcal{M}}_{p,q,t'} = (\overline{\mathcal{L}}^+_{B,t'})^{\otimes q} \otimes \overline{\mathcal{A}}^{\otimes p}.$$

For fixed p and q and sufficiently small t', these bundles are nef on the geometric fibres, ample on the generic fibre, and have semipositive curvature and metrics invariant under complex conjugation.

For a fixed cutoff B, [Alp26] obtains a constant a independent of p, q, t, and t' such that $h_{\overline{\mathcal{M}}_{p,q,t'}}(P)/q \geq a - (p/q)K_{\mathcal{A}}$ for every algebraic point P, using a constant $K_{\mathcal{A}}$ satisfying $h_{\overline{\mathcal{A}}}(P) \geq -K_{\mathcal{A}}$. The estimate includes boundary and interior points, so the comparison between heights of points and arithmetic self-intersection applies to all of X and gives the same lower bound for $h_{\overline{\mathcal{M}}_{p,q,t'}}(X)/q$.

It remains to remove the correction $U/2$ to the Hodge metric potential. A Stokes formula compares the arithmetic self-intersections for the potentials $\phi + U/2$ and ϕ. In [Alp26], the limits are taken first as $t' \to 0$, then as $t \to 0$, and finally with $p = 1$ and $q \to \infty$ to remove the auxiliary bundle. Write $\langle U \rangle$ for the average of U with respect to the $(d-1)$st power of the curvature form of the original Hodge metric on $\mathcal{L}$. More precisely, sum the integrals over the interiors of the complex fibres and divide by the same sum with U replaced by 1. The resulting inequality for the normalized height of X is

$$\lim_{t \to 0} h_{\overline{\mathcal{L}}_{B,t}}(X) \geq a - \frac{1}{2}\langle U \rangle.$$

The bounds for the Hodge metrics near the boundary in [Alp26] justify these limits and the integration of U. An upper bound for $\langle U \rangle$ then turns the lower bounds for heights of points into a lower bound for the arithmetic self-intersection of the Hodge bundle. This is the bound used to estimate covolumes of lattices of automorphic forms.

From heights to lines in Hermitian lattices. A Zariski dense set of points of bounded height gives an upper bound for normalized arithmetic self-intersection. For a Grassmannian with its Plücker line bundle, the points needed for this comparison are quite explicit. For $0 < j < r$, let $\mathrm{Gr}(j, r)$ parametrize the j-dimensional subspaces of K^r, equipped with its standard Hermitian metrics. On the standard affine chart, represent a subspace by the rows of $[I_j \; Z]$, where Z is a $j \times (r-j)$ matrix. For the Plücker line bundle with its Fubini–Study metric, the squared norm of the resulting Plücker vector is $\det(I_j + ZZ^*)$, where Z^* is the conjugate transpose. Choose the entries of Z to be roots of unity. The Plücker coordinates are integral and include a coordinate equal to 1, so the finite places contribute nothing to their height. At every complex embedding of the field of definition of the point, Hadamard's inequality gives

$$\frac{1}{2}\log\det(I_j + ZZ^*) \leq \frac{j}{2}\log(r - j + 1).$$

These points are Zariski dense in the Grassmannian. The normalized arithmetic self-intersection of the Plücker line bundle is therefore at most $\frac{j}{2}\log(r-j+1)$. For the bounds on lattices of automorphic forms, [Alp26]

also uses products of Grassmannians; the heights add for the product Plücker line bundle.

The comparison between heights of points and arithmetic self-intersection also produces lines of large degree in Hermitian lattices. To see this, let M be a Hermitian $\mathcal{O}_K$-lattice of positive rank r, with K totally imaginary. Writing Δ_K for the discriminant of K, the normalization used in this report is twice the usual Arakelov slope:

$$\mu(M) := -\frac{2}{[K:\mathbb{Q}]r}\log \operatorname{covol}(M) - \log 2 + \frac{\log|\Delta_K|}{[K:\mathbb{Q}]}.$$

The discriminant term and $-\log 2$ make $\mu(\mathcal{O}_K^r) = 0$ for the standard metrics. Thus a lower bound for $\mu(M)$ gives an upper bound for the covolume, with the rank and the field fixed. The maximum slope $\mu_{\max}(M)$ is the largest mean slope of a nonzero saturated sublattice of M; the minimum slope $\mu_{\min}(M)$ is the smallest mean slope of a nonzero torsion-free quotient of M with its quotient metrics.

The projective bundle $\mathbb{P}(M)$ has generic fibre $\mathbb{P}(M \otimes_{\mathcal{O}_K} K)$, parametrizing lines in $M \otimes_{\mathcal{O}_K} K$. With $\mathcal{O}(1)$ the dual of the tautological line and the induced Fubini–Study metric, the calculation in [Alp26] gives

$$h_{\overline{\mathcal{O}}(1)}(\mathbb{P}(M)) \le -\frac{1}{2}\mu(M) + \frac{1}{2}\log r.$$

The comparison between heights of points and normalized arithmetic self-intersection then supplies an algebraic point of $\mathbb{P}(M)$ whose height is at most $-\mu(M)/2 + (\log r)/2$ plus any given positive error. If the point is defined over K'/K and corresponds to the saturated line $L \subseteq M \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$, then $h_{\overline{\mathcal{O}}(1)}([L]) = -\mu(L)/2$. A point of small height therefore gives a line of large degree. Applying this construction of a line to a sublattice attaining $\mu_{\max}(M)$ shows that, for every $\epsilon > 0$, some finite extension K'/K contains such a line with

$$\mu(L) \ge \mu_{\max}(M) - \log r - \epsilon.$$

This is the absolute Siegel lemma used in the argument. Gaudron and Rémond [GR13, Theorem 1.1 and Lemma 3.2] deduced it, with a sharper constant, from Zhang's inequalities [Zha95]. The finite extension is natural here: the point supplied by the height comparison need not be K-rational.

For the maps b_α defined by the normalized automorphic pairings P_α, [Alp26] also needs to compare the slopes of a lattice and its image under an integral map. Bost's [Bos01, §4.1.4] method of slopes gives the relevant inequality. For a nonzero integral map $b : M \to N$ of Hermitian $\mathcal{O}_K$-lattices, let b_ν be its realization at a complex place ν, with the induced operator norm. Normalize the height of b with the same factor of two used in the definition of $\mu(M)$:

$$h(b) := \frac{2}{[K:\mathbb{Q}]}\sum_{\nu|\infty}\log\|b_\nu\|_{\mathrm{op}}^2.$$

Each complex place occurs once. With the induced metrics on the image, the inequality is

$$\mu_{\min}(b(M)) \geq \mu_{\min}(M) - h(b).$$

In particular, bounding the operator norms limits how far the minimum slope can fall under the map. In Section 4.4, [Alp26] uses this inequality after selecting sublattices of the domains N_α of b_α with the Harder–Narasimhan filtration.

Integral structures and covolumes. To apply the height comparisons to the automorphic bundles on $\mathcal{M}$, [Alp26] also controls their integral structures as the highest weights change. These structures determine the contributions of finite primes to the heights. For the principally polarized abelian schemes A parametrized by the integral models of $\mathcal{M}$, [Alp26] studies these integral structures through the p-divisible groups formed by their p-power torsion $A[p^n]$. Grothendieck–Messing theory, developed in Messing's [Mes72] study of crystals associated with p-divisible groups, describes deformations through lifts of the Hodge filtration over nilpotent divided-power thickenings. In [Alp26], this deformation theory enters the construction of integral points of $\mathcal{M}$ at good primes and the comparison of contributions of finite primes to heights as the highest weights change. For an abelian scheme $A/\mathcal{O}_F$ over the ring of integers of a p-adic field F, write $T_p(A) := \varprojlim_n A[p^n](\bar{F})$ for its integral Tate module. For these abelian schemes and their duals, [Alp26] also uses the estimates for Hodge–Tate maps in Fargues [Far11, §5.3.2]. After tensoring with the ring of integers in a completed algebraic closure of the local field, this map sends $T_p(A)$ to invariant differentials of $A^\vee$. Fargues's estimates control the failure of its image to fill the differential lattice and are compatible with the corresponding map for $A^\vee$. The later theory of Bhatt, Morrow, and Scholze [BMS18] compared integral p-adic cohomology theories through modules with Frobenius. Bhatt and Scholze [BS22] developed prismatic cohomology, and Anschütz and Le Bras [ALB23] used it to classify p-divisible groups over quasisyntomic rings by prismatic Dieudonné crystals. For these p-divisible groups, [Alp26] uses prismatic Dieudonné theory to identify the kernel of the Hodge–Tate map with the annihilator of the kernel of the logarithmic differential under the Tate pairing. The torsion terms in these comparisons affect the finite valuations in heights of points with respect to Hodge bundles, and hence the bounds for covolumes of lattices of automorphic forms.

For each choice of level and highest weight, [Alp26] first bounds the heights of the compactified components of $\mathcal{M}$ and compares their algebraic degrees with the dimensions of spaces of sections of the automorphic bundles. It then uses the arithmetic Hilbert–Samuel formula to bound the arithmetic degrees of the integral lattices of sections. Adding these degrees over the components and dividing by the total rank gives a lower bound for the mean slope of the resulting lattice of holomorphic automorphic forms, hence an upper bound for its logarithmic covolume. The estimates retain the losses

from changing highest weights, passing between levels of $\mathcal{M}$, and treating its boundary in the compactification. These lower bounds for mean slopes of lattices of automorphic forms enter Proposition 4.3. To transfer the bounds through the maps b_α to a bound for $\mu(C)$, [Alp26] also uses information about the subspace of $C \otimes_{\mathcal{O}_K} K$ spanned by their images.

4.4. **How the maps defined by periods control the mean slope.** Recall that $b_\alpha(f)$ is the functional $g \mapsto P_\alpha(f, g)$ on the space W of test automorphic forms, with $C \otimes_{\mathcal{O}_K} K = W^\vee$. There are many such maps b_α, but their number alone does not show that their images span nearly all of $W^\vee$: the images could all lie in the same small subspace. Moreover, [Alp26] restricts the maps to sublattices of their domains chosen according to slope. An estimate for the span of the original images would not suffice. The linear algebra below bounds how much of the target can be missed after those restrictions, in terms of the dimensions removed from the domains. The hypotheses measure how nearly the maps preserve lengths and how widely their images span the target.

Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space, and let $B_i : \mathbb{C}^{r_i} \to \mathcal{H}$ be finitely many linear maps. Put $D := \dim_{\mathbb{C}} \mathcal{H} > 0$ and $R := \sum_i r_i > 0$. Write B_i^* for the adjoint and $||B_i||_{HS}^2 := \mathrm{Tr}(B_i^* B_i)$ for the squared Hilbert–Schmidt norm. Choose positive real coefficients γ_i and set $K_i := \gamma_i B_i^* B_i$ and $Q := \sum_i \gamma_i B_i B_i^*$. Assume that $0 \le e \le 1/4$ and $\eta \ge 0$ satisfy both of the following bounds:

$$\sum_i ||K_i - I||_{HS}^2 \le eR, \tag{4.2}$$

$$\mathrm{Tr}(Q^2) \le (1+\eta)\frac{\mathrm{Tr}(Q)^2}{D}. \tag{4.3}$$

Equation 4.2 says that the weighted maps nearly preserve lengths, on average over the maps B_i. To interpret Equation 4.3, observe that Cauchy–Schwarz always gives $\mathrm{Tr}(Q^2) \ge \mathrm{Tr}(Q)^2/D$, with equality precisely when Q is scalar. Equation 4.3 therefore requires Q to be nearly scalar. In particular, the images of the maps B_i cannot all be concentrated in a small part of $\mathcal{H}$.

The equality case suggests a useful analogy with finite tight frames: collections of vectors for which the sum of the associated rank-one operators is a positive scalar multiple of the identity. In their study of such frames, Benedetto and Fickus [BF03] characterized finite normalized tight frames by minimizing the sum of squared inner products of their vectors. The maps B_i have domains of varying dimensions, and Equations 4.2 and 4.3 allow small errors. Nevertheless, the comparison with a scalar operator serves the same purpose: it measures how evenly the images of the maps B_i occupy $\mathcal{H}$.

Under these hypotheses, restricting the domains of the B_i loses only the following proportion of $\mathcal{H}$.

Lemma 4.2. *Let* $E_i \subseteq \mathbb{C}^{r_i}$ *be subspaces such that* $\sum_i \operatorname{codim} E_i \le \epsilon R$*, where* $\epsilon \ge 0$*. Then*

$$\operatorname{codim}_{\mathcal{H}} \sum_i B_i E_i \le (\eta + 4\epsilon + 4\sqrt{e})D.$$

Proof. The point is to estimate the trace of Q on the part of the target missed by the restricted maps. Write P_U for orthogonal projection onto a subspace U, and put $T := (\sum_i B_i E_i)^{\perp}$, $w := \dim T$, and $t := \operatorname{Tr} Q$. By Cauchy–Schwarz and Equation 4.2, $|t - R| \le R\sqrt{e}$, so $t \ge R/2$. Since $B_i^* T \subseteq E_i^{\perp}$,

$$\begin{aligned}\operatorname{Tr}(P_T Q) &\le \sum_i \operatorname{Tr}(P_{E_i^{\perp}} K_i) \\ &\le \sum_i \operatorname{codim} E_i + \left(\sum_i \operatorname{codim} E_i\right)^{1/2} \left(\sum_i ||K_i - I||_{HS}^2\right)^{1/2}.\end{aligned}$$

Since $\sum_i \operatorname{codim} E_i \le \epsilon R$, Equation 4.2 now gives $\operatorname{Tr}(P_T Q) \le R(\epsilon + \sqrt{\epsilon e})$. On the other hand, by Equation 4.3,

$$\left|\left| Q - \frac{t}{D} I \right|\right|_{HS}^2 \le \eta \frac{t^2}{D}.$$

Since P_T has Hilbert–Schmidt norm $\sqrt{w}$, the trace on T consequently satisfies

$$\operatorname{Tr}(P_T Q) \ge \frac{tw}{D} - t\sqrt{\frac{\eta w}{D}}.$$

Since $t \ge R/2$, the two bounds imply

$$w - \sqrt{\eta D w} \le 2D(\epsilon + \sqrt{\epsilon e}).$$

The inequality $\sqrt{\eta D w} \le (w + \eta D)/2$ implies $w \le D(\eta + 4\epsilon + 4\sqrt{\epsilon e})$. For $\epsilon \le 1$, this is the desired estimate; for $\epsilon \ge 1$, the bound $w \le D$ suffices. □

To use Lemma 4.2 for the period maps, [Alp26] chooses the subspaces according to the slopes of the lattices N_α, using a filtration whose construction came from vector bundles on curves. Harder and Narasimhan [HN75] introduced a filtration of a vector bundle whose successive quotients were semistable and had decreasing slopes. For lattices, Arakelov degree plays the role of the degree of a vector bundle. Stuhler [Stu76] carried the construction over to Euclidean lattices, and Grayson [Gra84] developed the corresponding reduction theory over rings of integers of number fields. The successive quotients of the resulting filtration have decreasing mean slopes and are semistable: no saturated sublattice has larger mean slope than the quotient itself.

For a given slope threshold, [Alp26] retains the Harder–Narasimhan sublattice of each N_α with slopes above that threshold, adjusted by the bound for the height of b_α. Bost's inequality from Section 4.3 places the image of this retained sublattice in the Harder–Narasimhan sublattice of C above the unadjusted threshold. Lemma 4.2 bounds the codimension of the span of these images in $C \otimes_{\mathcal{O}_K} K$. Only the resulting proportion of slopes of C can

lie below the threshold. Varying the threshold and integrating then gives a lower bound for the mean slope of C.

For Hermitian lattices over a totally imaginary field K, use the normalization of slopes and heights in Section 4.3. The same factor of two applies to all the Harder–Narasimhan slopes.

Consider full Hermitian $\mathcal{O}_K$-lattices C and N_i of ranks $d > 0$ and $r_i > 0$, together with integral maps $b_i : N_i \to C$. Assume that $h(b_i) \le h_i$ whenever $b_i \neq 0$, and put $R := \sum_i r_i > 0$. Choose $s \in \mathbb{R}$, $u \ge 0$, and $H \ge 0$ such that

$$\begin{aligned} Rs &\le \sum_i r_i(\mu(N_i) - h_i), \\ \mu_{\max}(N_i) - h_i &\le s + u, \\ s - \mu_{\min}(C) &\le H. \end{aligned}$$

The first inequality bounds the mean slope of the domain lattices from below by s, after subtracting the bounds h_i for the heights of the maps. The bound $s+u$ limits how far the largest adjusted slope can lie above s; together, these two bounds limit the proportion of adjusted slopes below a given threshold. The number H bounds how far a slope of C can lie below s.

At one archimedean place of K, assume also that, after choosing invertible coordinates, the maps b_i are projections to $\mathbb{C}^d$ of maps $B_i : \mathbb{C}^{r_i} \to \mathbb{C}^d \oplus \mathbb{C}^{d_1}$ satisfying the hypotheses of Lemma 4.2, with positive coefficients γ_i, $0 \le e \le 1/4$, and $\eta \ge 0$. Put $D := d + d_1$ and assume that $(1-\delta)D \le d$ for some $0 \le \delta < 1$. Thus Lemma 4.2 applies in a slightly larger space before projection to the complex realization $\mathbb{C}^d$ of C. Proposition 4.3 bounds the mean slope of C in terms of the slopes of the domain lattices.

Proposition 4.3. *With the lattices, maps, and bounds just described,*

$$\mu(C) \ge s - \frac{4u\log(1 + H/u) + (\eta + 4\sqrt{e})H}{1-\delta}. \tag{4.4}$$

When $u = 0$, interpret $u\log(1 + H/u)$ as zero.

Proof. List the slopes of all N_i, with rank multiplicities, after subtracting h_i in the ith summand. Their average is at least s, and each is at most $s+u$. For $\tau > 0$, let $k(\tau)$ count the adjusted slopes below $s - \tau$. Since

$$Rs \le k(\tau)(s-\tau) + (R - k(\tau))(s+u),$$

it follows that $k(\tau) \le Ru/(u+\tau)$. If $u = 0$, every adjusted slope equals s, so the same conclusion holds.

Let $N_i(\tau)$ be the Harder–Narasimhan sublattice with slopes at least $s - \tau + h_i$. Since b_i is integral and has height at most h_i, the slope inequality of Bost [Bos01, §4.1.4], applied after quotienting by its kernel, shows that each nonzero image $b_i N_i(\tau)$ has minimum slope at least $s - \tau$. Thus the images $b_i N_i(\tau)$ lie in the Harder–Narasimhan part $C_{\ge s-\tau}$.

By Lemma 4.2, the images of the complex spans of the $N_i(\tau)$ have corank at most

$$\left(\eta + 4\frac{u}{u+\tau} + 4\sqrt{e}\right)D$$

in $\mathbb{C}^d \oplus \mathbb{C}^{d_1}$. Projection to $\mathbb{C}^d$ cannot increase corank. Since $d \geq (1-\delta)D$, the proportion of slopes of C below $s-\tau$ is therefore at most

$$\frac{1}{1-\delta}\left(\eta + 4\frac{u}{u+\tau} + 4\sqrt{e}\right).$$

Write $\lambda_1, \ldots, \lambda_d$ for the slopes of C. Each is at least $s - H$, so integration recovers the total deficit below s:

$$s - \mu(C) \leq \frac{1}{d}\sum_{j=1}^{d}(s-\lambda_j)_+ = \int_0^H \frac{\#\{j : \lambda_j < s-\tau\}}{d}\,d\tau.$$

Since $\int_0^H u/(u+\tau)\,d\tau = u\log(1+H/u)$, the bound for the proportion of slopes of C below $s-\tau$ proves Equation 4.4. □

Proposition 4.3 bounds the mean slope of C from below, and hence its covolume from above, in terms of the slopes of N_i and the heights of the maps b_i.

Remark 4.4. The logarithm in Equation 4.4 comes from integrating the proportion of slopes that were discarded from the domains. In particular, the estimate uses the distribution of the images of all the maps, beyond what an estimate for any one map could provide.

Applying the inequality to the period maps. For each α, let h_α bound the height of b_α, and let s be the mean of $\mu(N_\alpha) - h_\alpha$, weighted by the ranks of N_α. These are the quantities used for h_i and s in Proposition 4.3. Recall from Section 3 that x is the argument at which $\mathcal{G}_\psi$ is to be bounded, $\varepsilon > 0$ is a small parameter, and $b = \exp(x^\varepsilon)$.

For the period maps, [Alp26] proves the bounds in Equations 4.2 and 4.3 before restricting the source spaces to $\mathcal{U}_\alpha^\circ$ and the test space to W. Fix the complex embedding of K used to realize the periods. In the following comparison of matrices, $\mathcal{U}_\alpha$ denotes its complexification, with the Petersson inner product. Recall from Section 4.1 that $\widetilde{W}$ is the space of test automorphic functions with the chosen vectors at the real places, before applying the projector e_f^0. Let $\mathcal{H}$ be the conjugate Hilbert space of $\widetilde{W}$. The Petersson pairing identifies $\mathcal{H}$ linearly with $\widetilde{W}^\vee$; projecting to the part retained by e_f^0 therefore gives $W^\vee \otimes_K \mathbb{C}$.

For each α, the integrations in the Bessel pairing $\mathcal{B}_\nu$ of Section 4.1, with the chosen translates and conjugation convention, define an operator $\mathcal{P}_\alpha : \mathcal{U}_\alpha \to \mathcal{H}$ through the Petersson pairing. In orthonormal bases $(e_{\alpha,j})$ and (f_a), its matrix has entries $\langle f_a, \mathcal{P}_\alpha e_{\alpha,j}\rangle$. Call this matrix B_α^{full}. The positive coefficients γ_α in Lemma 4.2 are fixed by Schur orthogonality for these operators. More precisely, recall from Section 4.1 that averaging over

$\mathcal{K}_f \subset U(Y_0)(\mathbb{A}_{E^+,f})$ against the character $\overline{\xi_f}$ defines the projector R_f. Let N be the unipotent subgroup occurring in the Bessel pairing of Section 4.1, and put $N_\infty := N(E^+ \otimes_{\mathbb{Q}} \mathbb{R})$ and $U_\infty = U(X_0)(E^+ \otimes_{\mathbb{Q}} \mathbb{R})$. The finite adelic translate x_α used in the period specifies, with n_f denoting the finite adelic component,

$$\Gamma_\alpha := \{n \in N(E^+) : n_f \in x_\alpha \mathcal{K}_f x_\alpha^{-1}\}.$$

If π_α is the representation of the compact group U_∞ determined by the weights, the normalizing constant is

$$c_\alpha = \frac{\operatorname{vol}(\mathcal{K}_f)\operatorname{vol}(\Gamma_\alpha \backslash N_\infty)\operatorname{vol}(U_\infty)}{\dim \pi_\alpha},$$

and $\gamma_\alpha = c_\alpha^{-1}$. The scalar in the normalization of the bilinear pairing $\overline{\mathcal{B}_\nu(f, \overline{g})}$ is included in the change of coordinates allowed in Proposition 4.3.

Since the auxiliary prime T_α splits completely in E^+, there are $f = [E^+ : \mathbb{Q}]$ places ν above T_α. At each such place, let $\rho_{\alpha,\nu}$ denote the action on $\mathcal{U}_\alpha$ of the multiplicative group of the residue field $\mathbb{F}_{T_\alpha}$. The subspace is

$$\mathcal{U}_\alpha^\circ = \bigcap_{\nu \mid T_\alpha} \ker\left(\sum_{c \in \mathbb{F}_{T_\alpha}^\times} \rho_{\alpha,\nu}(c) \right).$$

In other words, the component invariant under any of these multiplicative groups is removed. If $r_\alpha^{\mathrm{full}} := \dim \mathcal{U}_\alpha$ and r_α is the retained dimension, the calculation in [Alp26] gives

$$r_\alpha = r_\alpha^{\mathrm{full}} \left(1 - \frac{1}{T_\alpha - 1}\right)^f.$$

For $T_\alpha \geq b \geq 3$, the proportion removed is at most $2f/b$. Choose an isometric inclusion V_α of $\mathcal{U}_\alpha^\circ$ in $\mathcal{U}_\alpha$ and set $B_\alpha := B_\alpha^{\mathrm{full}} V_\alpha$. After the changes of coordinates just described, projecting B_α to $W^\vee \otimes_K \mathbb{C}$ gives the complex realization of b_α.

The estimates compare the maps B_α^{full} for distinct α and measure how far each normalized map is from an isometry. Write D_* for the main term in the dimension of the full space of test automorphic forms. Its factors are an invariant volume, indices of level subgroups, and formal degrees of the representations at infinity. Put $D := \dim \mathcal{H}$. With nonnegative error bounds η_0, η_1, and e_0, [Alp26] proves, for distinct α and β,

$$\gamma_\alpha \gamma_\beta \left|\left| (B_\alpha^{\mathrm{full}})^* B_\beta^{\mathrm{full}} \right|\right|_{\mathrm{HS}}^2 \leq (1 + \eta_0) \frac{r_\alpha^{\mathrm{full}} r_\beta^{\mathrm{full}}}{D_*}.$$

For each map B_α^{full} and for the dimension $D = \dim \mathcal{H}$ of the space of test automorphic forms, [Alp26] proves

$$\left|\left| \gamma_\alpha (B_\alpha^{\mathrm{full}})^* B_\alpha^{\mathrm{full}} - I \right|\right|_{\mathrm{HS}}^2 \leq e_0 r_\alpha^{\mathrm{full}},$$
$$D \leq (1 + \eta_1) D_*.$$

The first of these two bounds says that each normalized operator nearly preserves lengths. The second compares $\dim \mathcal{H}$ with the main term D_* used in the bound for $\left|\left|(B_\alpha^{\mathrm{full}})^* B_\beta^{\mathrm{full}}\right|\right|_{\mathrm{HS}}^2$ when $\alpha \neq \beta$. The errors η_0, η_1, and $\sqrt{e_0}$ can be bounded by $\exp(-x^{\varepsilon/8})$.

There are enough choices of α to make $\sum_\alpha r_\alpha^{\mathrm{full}} \geq D_*/\sigma$ for a chosen $0 < \sigma \leq 1/20$. Once η_0, η_1, and $\sqrt{e_0}$ are at most $\sigma/2$, replacing the spaces $\mathcal{U}_\alpha$ by the retained domains of B_α and removing at most a proportion $\delta_0 \leq 1/4$ of $\sum_\alpha \dim \mathcal{U}_\alpha$ yields

$$e = 2e_0 \qquad \text{and} \qquad \eta = 5\sigma + 3\sqrt{\delta_0}$$

in Lemma 4.2. The proportion of dimension removed by projecting $\mathcal{H}$ to $W^\vee$ is bounded separately by δ. Restricting the domains $\mathcal{U}_\alpha$ changes e and η, whereas projecting $\mathcal{H}$ to $W^\vee$ changes the denominator $1-\delta$ in Proposition 4.3.

To integrate the proportions of slopes below a given threshold, the proof of Proposition 4.3 also uses a lower bound for $\mu_{\min}(C)$. Since $C = M_W^\vee$, its smallest slope is $\mu_{\min}(C) = -\mu_{\max}(M_W)$. The estimates for M_W give, for a fixed $C_H > 0$,

$$s + \mu_{\max}(M_W) \leq b x^{C_H}.$$

Thus [Alp26] can take $H = b x^{C_H}$ in Proposition 4.3. The number u can be taken to be $\max_\alpha(\mu_{\max}(N_\alpha) - h_\alpha) - s$. The bounds for slopes of M_α and for the indices arising from changes of integral lattices at finite primes also bound u. To combine Proposition 4.3 with the bounds for norms of period maps, [Alp26] keeps track of how s, u, and H vary with the automorphic weights and levels. In particular, the choice of T_α relates the dimensions of the spaces of automorphic forms to the norms of the period maps.

4.5. **Recovering the main term from normalized automorphic periods.** To bound $\mathcal{A}(a,t,w) - \mathcal{M}(a,w)$, [Alp26] combines the bounds for the norms of b_α with those for the valuations of $P_\alpha(f,g)$. Equation 3.4 expresses the difference $\mathcal{A}(a,t,w) - \mathcal{M}(a,w)$ in terms of the corrected sums G_χ. A bound for $\mathcal{A}(a,t,w)$ alone would not suffice. The bounds for finite valuations of period values and for archimedean norms in [Alp26] recover $\mathcal{M}(a,w)$ with exactly the constants in that equation, leaving an error small enough for the proposed proof.

The extension from CM elliptic curves to abelian varieties in Anderson [And82] drew on Shimura's [Shi79] relations between periods, together with calculations of periods of Fermat curves by Weil [Wei76] and by Rohrlich in the appendix to Gross's [Gro78] paper. Anderson [And82, §4] also formulated and proved a result of Deligne, unpublished at the time, relating CM periods to gamma values. The algebraic factors left undetermined by these formulas also matter in [Alp26], because they affect the finite valuations of the normalized automorphic periods.

Accounting for these factors requires more than their algebraicity. The argument also needs to know how the periods change under Galois conjugation. For this, [Alp26] uses reciprocity for elliptic curves with complex multiplication: the conjugate elliptic curve and the action on its torsion are described through the Artin map. This is the elliptic case of the theory of Shimura and Taniyama [ST61]; see Milne [Mil07, Theorem 3.13] for a statement in terms of complex uniformization. The algebraic factors in the formulas for CM periods can also have denominators at finite places. To control these, [Alp26] uses the integrality of suitable modified values of abelian L-functions at negative integers over totally real fields, proved by Deligne and Ribet [DR80]. The reciprocity law and the integrality of the modified L-values control the algebraic factors in the expressions for CM periods; identities involving only absolute values would not determine the valuations of these factors at finite places.

The comparison of periods also involves dimensions and multiplicities, because [Alp26] averages the periods over spaces of automorphic forms. The dimensions of these spaces vary with the highest weights, and the multiplicities of their automorphic representations affect the average as well. To compute the dimensions and multiplicities, [Alp26] uses Weyl's [Wey46] formulas for dimensions of representations at infinity together with the mass formulas associated with Siegel [Sie35]. In Siegel's theory, weighted counts of lattices were related to products of local densities. Together with the Chowla–Selberg formula, the dimension and mass formulas are singled out in [Alp26] in its explanation of the cancellation between the main terms from norms of b_α and valuations of $P_\alpha(f, g)$.

The leading terms in these dimensions also have a geometric meaning. Hirzebruch's proportionality principle related characteristic numbers of compact locally symmetric spaces to those of their compact duals. Mumford [Mum77] extended this comparison to noncompact arithmetic quotients by controlling Hermitian metrics near the boundary. In [Alp26], the resulting relation between intersection numbers and invariant volumes identifies the leading growth of the dimensions of spaces of automorphic forms with a polynomial in the highest weights of the representations at infinity. The intersection numbers used to bound covolumes in Section 4.3 thus also determine the leading terms in these dimensions.

To see how the leading terms in [Alp26] cancel, fix one element $u \in H$ indexing the integers $w_{u,j,\rho}$ in Section 3, and suppress u from the notation. Recall from Section 3 that each selected complex place $t \in T$ has a positive parameter a_t. Write $w_{j,\rho} > 0$ for the integers specifying the blocks of Harish–Chandra parameters at the real places of E^+ in the m theta lifts, as in Section 4.1. The indices $\rho \in \Phi$ represent these places in the chosen CM type, and $f = |\Phi| = [E^+ : \mathbb{Q}]$, as in Section 3. Recall from Section 3.2 that κ is the character of the chosen imaginary quadratic subfield of E and $\ell_\kappa = L'(1, \kappa)/L(1, \kappa)$. Let Δ_E be the discriminant of E, and put $\nu := |\Delta_E|^{1/[E:\mathbb{Q}]}/2$, half its root discriminant. Also put $A := \sum_{t \in T} a_t$.

Write $\log \Omega$ for the mean of the logarithms of the CM period factors in the theta construction. The normalization includes a correction from the different of E over the imaginary quadratic field attached to κ. With this correction, the Chowla–Selberg formula gives

$$\log \Omega = \gamma + \log(\pi/\nu) - \ell_\kappa.$$

Thus the logarithmic derivative in this expression comes from the CM periods used to normalize the automorphic construction. The lower bounds for the valuations of period values contribute the term D_{div} below. The quantity L_w records the logarithms of the integers $w_{j,\rho}$:

$$D_{\mathrm{div}} := m \sum_{t \in T} (a_t \log a_t - a_t - a_t \log \Omega),$$

$$L_w := \sum_{j=1}^{m} \frac{1}{f} \sum_{\rho \in \Phi} \log(w_{j,\rho} \nu/\pi).$$

The sum of the main terms from Equation 3.4 is

$$\mathcal{M}_{\mathrm{tot}} := \sum_{j=1}^{m} \sum_{t \in T} a_t \left(\log a_t - 1 - \gamma + \ell_\kappa - \frac{1}{f} \sum_{\rho \in \Phi} \log w_{j,\rho} \right).$$

Since $\log(w\nu/\pi) = \log w + \log \nu - \log \pi$, the contributions from $\log \nu - \log \pi$ cancel against the corresponding terms in $\log \Omega$. Consequently,

$$D_{\mathrm{div}} - A L_w = \mathcal{M}_{\mathrm{tot}}. \tag{4.5}$$

Observe that $\mathcal{M}_{\mathrm{tot}}$ is exactly the main term from Equation 3.4, summed over the theta lifts and the complex places indexed by T. Thus the contributions from finite valuations and archimedean norms account for precisely the main term subtracted from the sums over prime powers.

To see how Equation 4.5 enters the comparison of norms of b_α with valuations of $P_\alpha(f, g)$, consider first the choice of T_α. This is an auxiliary rational prime in the level of the initial automorphic forms, and [Alp26] chooses its size using the dimensions of the spaces of automorphic forms. Write d_α^* for the product of the mass, the indices of the fixed level subgroups, and the dimensions of the representations at infinity for the initial group. The full space at level T_α, before imposing the additional conditions at this prime, has dimension $d_\alpha^*(T_\alpha^k - 1)^f$. The factor $T_\alpha^k - 1$ counts nonzero vectors over the residue field, and there are f places above T_α in E^+.

Recall the corresponding main term D_* for the dimension of the space of test automorphic forms from Section 4.4. The parameter $\Gamma_P > 1$ controls the dimensions of the spaces of initial automorphic forms relative to the dimension of the space of test automorphic forms: [Alp26] chooses T_α so that

$$\left(\frac{D_*}{d_\alpha^* \Gamma_P^2} \right)^{1/(fk)} \le T_\alpha \le \left(1 + \frac{1}{fk} \right) \left(\frac{D_*}{d_\alpha^* \Gamma_P^2} \right)^{1/(fk)}.$$

The prime also satisfies the required splitting and congruence conditions. Thus the estimates from Weyl's formula and the mass formula determine a bound for $\log T_\alpha$. The term $-A \log T_\alpha$ below comes from the period map at this auxiliary prime.

With T_α chosen, [Alp26] compares the integral lattices in $\mathcal{U}_\alpha^\circ$, keeping track of how imposing divisibility of period values changes their mean slopes. The construction starts with $M_\alpha = \widetilde{M}_\alpha \cap \mathcal{U}_\alpha^\circ$. The restriction to $\mathcal{U}_\alpha^\circ$ has already removed the invariant components at T_α. The subsequent construction in [Alp26] gives a finite-index sublattice $M'_\alpha \subseteq M_\alpha$ using local operators at the finite primes. At the primes where valuations of period values are bounded below, the construction combines images under iterates of these operators with powers of the prime and intersects with the chosen integral submodules. Thus M'_α still spans $\mathcal{U}_\alpha^\circ$ over K. The change from M_α to M'_α is measured by local lengths and does not remove further dimensions from the space of automorphic forms.

At a finite prime $\mathfrak{p}$ of K, let $e_\mathfrak{p}$ be the ramification index over the rational prime below $\mathfrak{p}$. Measure local exponents in units in which that rational prime has valuation 1. For the jth theta lift, write $g_{j,\mathfrak{p}}$ for its contribution to the lower bound for valuations of period values, $F_{j,\mathfrak{p}}$ for the change from the local choice of Hecke characters, and $E_{\alpha,j,\mathfrak{p}}$ for the correction from normalizing the theta lift. Set $g_{j,\mathfrak{p}} = 0$ at primes where the lower bound for valuations of period values is not used. All these exponents have finite support. The fractional ideal recording these lower bounds is

$$\mathfrak{a}_\alpha := \prod_{\mathfrak{p}} \mathfrak{p}^{\left\lfloor e_\mathfrak{p} \sum_{j=1}^{m} (g_{j,\mathfrak{p}} + F_{j,\mathfrak{p}} - E_{\alpha,j,\mathfrak{p}}) \right\rfloor}.$$

The valuation bounds say that $P_\alpha(M'_\alpha, M_W) \subseteq \mathfrak{a}_\alpha$. Consequently $N_\alpha = \mathfrak{a}_\alpha^{-1} M'_\alpha$ satisfies $P_\alpha(N_\alpha, M_W) \subseteq \mathcal{O}_K$. When an exponent in $\mathfrak{a}_\alpha$ is positive, this rescaling divides elements of M'_α by the corresponding prime ideal power; the divisibility of the period values is precisely what keeps their images integral. The lattice N_α remains in the original space $\mathcal{U}_\alpha^\circ$, whose dimension is unchanged.

Passing from local exponents to slopes means weighting a valuation by the logarithm of the norm of its prime ideal. In the normalization of Section 4.3, put $c_\mathfrak{p} := 2e_\mathfrak{p} \log(N\mathfrak{p})/[K : \mathbb{Q}]$. If $\tau_\mathfrak{p}$ records the allowance for restricting M_α to M'_α at $\mathfrak{p}$, the contributions to the change in mean slope are

$$F := \sum_{\mathfrak{p}} c_\mathfrak{p} \sum_{j=1}^{m} F_{j,\mathfrak{p}},$$

$$G := \sum_{\mathfrak{p}} c_\mathfrak{p} \left(\sum_{j=1}^{m} g_{j,\mathfrak{p}} - \tau_\mathfrak{p} \right),$$

$$E_\alpha := \sum_{\mathfrak{p}} c_{\mathfrak{p}} \sum_{j=1}^{m} E_{\alpha,j,\mathfrak{p}}.$$

The local exponents in G explain how sums over prime powers enter the change in mean slope from M_α to N_α. Consider a rational prime p unramified in E, of the inert type selected in Section 3, and at which the lower bound for valuations of period values is used. The lower bound applies for $p \le |H| x \exp(x^{\varepsilon/3})$. Fix a prime $\mathfrak{p}$ of K above p. Let $T_{\mathfrak{p}} \subseteq T$ consist of the selected places for which the divisibility conditions on period values at $\mathfrak{p}$ hold, and let τ_t be the representative of the place indexed by t in the chosen CM type. For an odd integer ν, the relevant weight is $w_{j,\mathrm{cFrob}_p^\nu \tau_t}$, where c is complex conjugation. The conjugation returns the transported place to the chosen CM type. These are the parameters in the blocks introduced in Section 4.1: $w_{j,\rho} = e_{j-1,\rho}$ for a representative ρ in the CM type. With $R_{\mathfrak{p}}$ collecting the losses in the local estimate, [Alp26] obtains

$$\sum_{j=1}^{m} g_{j,\mathfrak{p}} - \tau_{\mathfrak{p}} \ge \sum_{j=1}^{m} \sum_{t \in T_{\mathfrak{p}}} \sum_{\substack{\nu \ge 1 \\ \nu \text{ odd}}} \left(\frac{a_t}{p^\nu} - w_{j,\mathrm{cFrob}_p^\nu \tau_t} \right)_+ - R_{\mathfrak{p}}.$$

The sum is finite because $w_{j,\rho} \ge 1$. The local condition on Frobenius accounts for the odd exponents. The coefficients $c_{\mathfrak{p}}$ satisfy $\sum_{\mathfrak{p}\mid p} c_{\mathfrak{p}} = 2 \log p$, since $\sum_{\mathfrak{p}\mid p} e_{\mathfrak{p}} \log(N\mathfrak{p}) = [K : \mathbb{Q}] \log p$. After accounting for the places of T omitted from $T_{\mathfrak{p}}$, summing the bounds for valuations over $\mathfrak{p} \mid p$ reproduces the terms $(a_t/p^\nu - w)_+$ in Equation 3.2. To keep the losses from the omitted places and the error $R_{\mathfrak{p}}$ small, [Alp26] controls the number of selected places in short segments of Frobenius orbits, as described in Section 4.6.

Recall from Section 4.4 that h_α bounds the height of b_α. Write L_α for the remaining allowance for a decrease in mean slope when passing from M_α to N_α. This allowance includes the loss from rounding the exponents in the fractional ideal down to integers. The local normalizations at two auxiliary primes g_0 and g_1 contribute $\ell_g := \log(g_0 g_1)$. The same term occurs in the estimate for $\log T_\alpha$. With nonnegative error bounds E_{div}, E_{sl}, E_{norm}, and E_{count}, the estimates in [Alp26] take the form

$$\mu(C) + h_\alpha - F \le D_{\mathrm{div}} - A \log T_\alpha + E_{\mathrm{div}}, \tag{4.6}$$

$$\mu(N_\alpha) \ge \mu(M_\alpha) + G + F - E_\alpha - L_\alpha, \tag{4.7}$$

$$\mu(M_\alpha) \ge -E_{\mathrm{sl}},$$

$$E_\alpha \le mA\ell_g + E_{\mathrm{norm}}, \tag{4.8}$$

$$\log T_\alpha \ge L_w + m\ell_g - E_{\mathrm{count}}. \tag{4.9}$$

Equation 4.6 compares the height h_α of the period map and the mean slope of C with the valuations of period values, while Equation 4.7 records the effect of restricting and rescaling the initial lattices. The remaining bounds concern the slopes of M_α, the normalization of the theta lifts, and the auxiliary prime T_α selected using the dimension and mass formulas. Notice that the change F

in the local normalizations occurs in both of the first two estimates, and that ℓ_g occurs both in E_α and in $\log T_\alpha$. These terms cancel when the inequalities are combined; the remaining expression $D_{\mathrm{div}} - AL_w$ is precisely the main term in Equation 4.5.

Recall that s is the mean of $\mu(N_\alpha) - h_\alpha$, weighted by the ranks of N_α. Let $\overline{L}$ be the corresponding mean of L_α. By Equations (4.6)– (4.9),

$$s \geq G + \mu(C) - \mathcal{M}_{\mathrm{tot}} - (E_{\mathrm{sl}} + E_{\mathrm{norm}} + E_{\mathrm{div}} + AE_{\mathrm{count}}) - \overline{L}.$$

The two occurrences of $mA\ell_g$ cancel, and Equation 4.5 identifies the remaining main term with $\mathcal{M}_{\mathrm{tot}}$. Write $\mathcal{A}_{\mathrm{tot}}$ for the sum of $\mathcal{A}(a_t, t, w_j)$ over $1 \leq j \leq m$ and $t \in T$, where w_j denotes the collection $(w_{j,\rho})_{\rho\in\Phi}$ of positive parameters in Equation 3.2, and suppose $\mathcal{A}_{\mathrm{tot}} - G \leq E_{\mathrm{loss}}$. Since Proposition 4.3 bounds $s - \mu(C)$, the excess of the arithmetic sum over its main term satisfies

$$\begin{aligned}\mathcal{A}_{\mathrm{tot}} - \mathcal{M}_{\mathrm{tot}} &\leq E_{\mathrm{sl}} + E_{\mathrm{norm}} + E_{\mathrm{div}} + AE_{\mathrm{count}} \\ &\quad + \overline{L} + E_{\mathrm{loss}} + \frac{4u\log(1 + H/u) + (\eta + 4\sqrt{e})H}{1 - \delta}.\end{aligned} \tag{4.10}$$

The left-hand side is the sum of the differences $\mathcal{A}(a_t, t, w_j) - \mathcal{M}(a_t, w_j)$ from Equation 3.4. Thus, by comparing the norms of b_α with the valuations of $P_\alpha(f, g)$, [Alp26] has removed exactly the main term in Equation 3.4. It remains to bound the errors on the right-hand side of Equation 4.10. Among them is $\overline{L}$, the mean allowance for a possible decrease in slope when the integral lattices of automorphic forms are changed.

4.6. **The bound for $\mathcal{A}_{\mathrm{tot}} - \mathcal{M}_{\mathrm{tot}}$.** To use Equation 4.10 in the character identity of Equation 3.4, [Alp26] chooses the automorphic forms, weights, and complex places of E so that the bounds for valuations of period values, norms of period maps, and slopes of lattices of automorphic forms hold together. Recall the group H, the sets T and T_2 of selected complex places of E, and the scale b of $w_{j,\rho}$ from Section 3. The chosen CM type supplies representatives for these places. At unramified finite primes, Frobenius acts on the places; in terms of the representatives, the action includes the complex conjugation needed to stay in the chosen CM type. The parameter k is the dimension of the initial Hermitian space X_0. Write $n := |H|$, $t := |T|$, and $\nu_2 := |T_2|$, and let a be an upper bound for the parameters a_t. The coefficient field is still denoted by K.

The choice of T_2 illustrates the competing requirements on the selected complex places. More places in T_2 give more divisibility of period values, but too many in a short segment of a Frobenius orbit increase the losses in the lower bounds for valuations. To choose a large set T_2 that meets each relevant short segment in few places, [Alp26] uses an idea from additive combinatorics. Salem and Spencer [SS42] studied large sets of integers containing no nontrivial three-term arithmetic progression. Behrend [Beh46] obtained his stronger construction by placing integer vectors on a sphere before encoding them as integers.

The modification of Behrend's construction used in [Alp26] excludes three-term linear relations among the selected integers with bounded nonzero coefficients whose sum is zero. After restricting and translating Behrend's set, [Alp26] uses its elements to select the places in T_2 so that a short segment of a Frobenius orbit meets T_2 in at most two points. Thus a method originally developed to avoid arithmetic progressions also selects enough complex places of E for the period values to acquire the required divisibility, while controlling how often the selected places occur in a short segment of a Frobenius orbit.

With these choices of automorphic forms, weights, and complex places, [Alp26] bounds $\mathcal{A}_{\mathrm{tot}} - \mathcal{M}_{\mathrm{tot}}$ as follows, for a fixed exponent $c_{\mathrm{err}} \geq 0$:

$$\mathcal{A}_{\mathrm{tot}} - \mathcal{M}_{\mathrm{tot}} \leq x^{c_{\mathrm{err}}\varepsilon}\left(ta\left(1 + \frac{m^2}{k}\right) + m\left(2nmbx^{\varepsilon} + \frac{\nu_2 kbx^{2\varepsilon}}{n^2}\right)\right). \tag{4.11}$$

The terms in Equation 4.11 record the errors in comparing the period values and slopes of lattices of automorphic forms as the weights, levels, and selected complex places of E vary.

Remark 4.5. Estimates for the inner products arising from pairs of period maps b_α control e and η in Proposition 4.3. Estimates for the changes in the integral lattices of automorphic forms control $\overline{L}$. The remaining terms account for divisibility of period values at finite places, the slopes of the initial lattices, and the dimensions of the spaces of automorphic forms. For the eventual choice of parameters, the crucial point is that c_{err} is independent of the small parameter ε.

Equation 4.11 is the bound used with the character identity in Equation 3.4. Before choosing the parameters, however, it remains to account for the loss of divisibility of period values when the integral lattices of automorphic forms are changed. The bound for $\overline{L}$ included in Equation 4.11 uses the Galois cohomology of Section 5.

5. Controlling denominators with Galois cohomology

The term $\overline{L}$ in Equation 4.10 allows for the decrease in mean slope when [Alp26] restricts and rescales the initial lattice of automorphic forms to record divisibility of the period values. One difficulty in bounding that decrease comes from projection onto chosen Hecke eigenspaces. A decomposition of a vector space need not split an integral lattice inside it, so projection onto a summand can introduce denominators even when the decomposition respects every Hecke operator.

To bound these denominators, [Alp26] uses integral operators that separate the chosen Hecke eigenspaces from the other eigenspaces. The elementary step is recorded in Lemma 5.1; the arithmetic enters in the divisibility of the operators' eigenvalues. The divisibility of these eigenvalues reflects congruences between automorphic forms. To measure those congruences, [Alp26] passes to extension classes of the associated Galois representations. The bounds for the orders of the groups of extension classes retain their

dependence on the number fields, weights, and representations, since all three change in the construction of the periods.

5.1. **Denominators introduced by projection.** Recall from Section 4.1 that the pairings P_α have values in the coefficient field K. Fix a rational prime ℓ and a place $\lambda \mid \ell$ of K. Write K_λ for the completion and $\mathcal{O}_\lambda$ for its ring of integers. The valuation $\mathrm{ord}_{\mathcal{O}_\lambda}$ takes the value 1 on a uniformizer. Consider a full $\mathcal{O}_\lambda$-lattice M in $V = V_1 \oplus V_2$ over K_λ. If e denotes projection onto V_1, then $M \cap V_1 \subseteq eM$; the quotient measures the denominators introduced by the projection. Suppose an endomorphism preserving M acts by a nonzero scalar on V_1 and vanishes on V_2. Dividing by that scalar recovers the projection, and gives the following bound.

Lemma 5.1. *Suppose $T \in \mathrm{End}_{\mathcal{O}_\lambda}(M)$ acts by a nonzero scalar $a \in \mathcal{O}_\lambda$ on V_1 and by zero on V_2. Then $aeM \subseteq M \cap V_1$, and*

$$\mathrm{length}_{\mathcal{O}_\lambda}(eM/(M \cap V_1)) \leq \dim_{K_\lambda}(V_1)\mathrm{ord}_{\mathcal{O}_\lambda}(a).$$

Proof. For $m \in M$, the equality $aem = Tm$ shows that $aem \in M \cap V_1$. Thus $eM/(M \cap V_1)$ is a quotient of eM/aeM. Since eM is free of rank $\dim_{K_\lambda} V_1$, the desired length bound follows. □

In the automorphic application, distinct systems of Hecke eigenvalues can become congruent modulo λ. An integral Hecke operator that vanishes on the eigenspace for one system of Hecke eigenvalues has an eigenvalue divisible by λ on every congruent eigensystem. Thus congruence ideals enter the bound for denominators introduced by projection. Suppose the integral Hecke algebra is a local subalgebra $\mathbb{T} \subseteq \mathcal{O}_\lambda^{\mathcal{X}}$ that separates the finite set $\mathcal{X}$ of eigensystems and preserves M. The coordinate t_π of $t \in \mathbb{T}$ is its eigenvalue on the eigenspace V_π corresponding to $\pi \in \mathcal{X}$. Write $V = \bigoplus_{\pi \in \mathcal{X}} V_\pi$ and $m_\pi := \dim_{K_\lambda} V_\pi$. For a subset $Z \subseteq \mathcal{X}$, let e_Z project onto $V_Z := \bigoplus_{\pi \in Z} V_\pi$. For each $\pi \in Z$, the relevant congruence ideal is

$$c_\pi := \{t_\pi : t \in \mathbb{T},\ t_{\pi'} = 0 \text{ for every } \pi' \notin Z\} \subseteq \mathcal{O}_\lambda.$$

The ideal records the divisibility forced on the eigenvalue at π when an integral Hecke operator kills every eigenspace outside Z. The estimate used in [Alp26] includes

$$\mathrm{length}_{\mathcal{O}_\lambda}(e_Z M/(M \cap V_Z)) \leq \sum_{\pi \in Z} m_\pi \,\mathrm{length}_{\mathcal{O}_\lambda}(\mathcal{O}_\lambda/c_\pi).$$

At two fixed auxiliary primes, [Alp26] also uses integral operators to retain only a subset of the eigenspaces indexed by Z. If such an operator acts by an integer $A \neq 0$ on the retained eigenspaces and by zero on the other eigenspaces in Z, the bound has an additional $\mathrm{ord}_{\mathcal{O}_\lambda}(A)$ for each retained eigenspace, counted with its dimension.

To bound the denominators introduced by e_Z, [Alp26] bounds the lengths of the quotients $\mathcal{O}_\lambda/c_\pi$. Hida [Hid81] had related congruences between cusp

forms to special values of their L-functions. In [Alp26], the Galois representations of Section 4.2 provide another way to measure the congruences: integral lattices can retain extension classes that disappear when the representations are semisimplified over K_λ. To identify the groups whose lengths enter the denominator bound, it helps to write out the coefficients of these extension classes and their restrictions at the finite places.

5.2. **Classes that vanish over the local fields.** The coefficients of the extension classes can be read from the Galois representations attached to a theta lift. Recall that h is the odd integer introduced in Section 4.1 to specify the sizes of the groups in the theta lifts; m is the number of theta lifts. Use $G_E = \mathrm{Gal}(\overline{E}/E)$ and the cyclotomic character ϵ_ℓ with the conventions of Section 4.2. The Tate twists in the denominator estimate come from an auxiliary theta lift between unitary groups of Hermitian E-spaces of dimensions $r = k + m - 1$ and $r + h$; the space of dimension $r + h$ is totally definite. For a retained Hecke eigensystem π on the larger group, the associated semisimple representation of G_E over K_λ in [Alp26] decomposes into an r-dimensional representation ρ_λ and one-dimensional representations $\xi_1, \dots, \xi_h$:

$$R_\lambda \simeq \rho_\lambda \oplus \bigoplus_{c=1}^{h} \xi_c,$$
$$\xi_c = \xi_1 \epsilon_\ell^{-(c-1)}.$$

The one-dimensional summands ξ_c of R_λ come from the additional characters in the base change of the auxiliary theta lift. After tensoring with ξ_1^{-1}, these summands become $1, \epsilon_\ell^{-1}, \dots, \epsilon_\ell^{-(h-1)}$. This explains both the negative Tate twists and the range $1 \le j < h$ that will occur below.

The summand ρ_λ also contributes to the bound for denominators. To describe that contribution, choose a Galois-stable lattice T_π in $\rho_\lambda \otimes \xi_1^{-1}$, and set $A_\pi := (T_\pi \otimes_{\mathcal{O}_\lambda} K_\lambda)/T_\pi$. Write $H^i(E, A_\pi)$ for continuous Galois cohomology $H^i(G_E, A_\pi)$, and use the same convention for completions E_w at finite places w of E. The corresponding term in [Alp26] is the length of a Selmer group: the subgroup of $H^1(E, A_\pi)$ whose restrictions to the local fields satisfy specified conditions. At $w \mid \ell$, take the finite, or crystalline, local condition of Bloch and Kato [BK90, §3]: the image of $H^1_f(E_w, T_\pi \otimes_{\mathcal{O}_\lambda} K_\lambda)$ in $H^1(E_w, A_\pi)$. Away from ℓ and the allowed places of ramification, the classes are required to be unramified; at the allowed places away from ℓ, no local restriction is imposed. The length of this Selmer group enters the denominator estimate alongside lengths of cohomology groups with coefficients $\mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)$, for $1 \le j < h$; the twist means that G_E acts on $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ through ϵ_ℓ^{-j}. Further terms measure the indices between the integral lattices used in the local Galois comparisons, the class number of E, and valuations of differences of roots of unity.

To bound the length of the Selmer group for A_π, [Alp26] varies the auxiliary characters before fixing the lattices. Write s_π for the $\mathcal{O}_\lambda$-length

of this group divided by the ramification index $e_\lambda := \mathrm{ord}_{\mathcal{O}_\lambda}(\ell)$. The chosen finite-order twists make the Selmer group finite and give $s_\pi \le \nu_\ell + \ell^{-2}$. With ν_ℓ normalized by $\nu_\ell(\ell) = 1$, put

$$\nu_\ell := \min_{g \in G_E} \nu_\ell\Big(\epsilon_\ell(g)^{hr} - 1\Big).$$

The integer ν_ℓ measures how close the hr-th powers in the image of the cyclotomic character are to 1. Recall from Section 4.6 that $n = |H|$, where $\mathrm{Gal}(E/\mathbb{Q}) \simeq (\mathbb{Z}/2\mathbb{Z})^2 \times H$. With this notation, [Alp26] obtains

$$\sum_\ell \nu_\ell \log \ell \le 2 \log n + \log h + \log r.$$

Since $\sum_\ell \ell^{-2} \log \ell$ also converges, [Alp26] can bound the contribution from ρ_λ after summing over the coefficient primes. It remains to bound the contributions from the one-dimensional summands of R_λ. These contributions involve the groups $\mathcal{S}_\ell(E, j)$, which can be defined directly in Galois cohomology. Let S_ℓ be the set of finite places of E above ℓ, and write G_{E,S_ℓ} for the Galois group of the maximal extension of E unramified away from S_ℓ and the archimedean places. For $j \ge 1$, the group $\mathcal{S}_\ell(E, j)$ is the kernel of

$$H^1(G_{E,S_\ell}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)) \longrightarrow \prod_{w \mid \ell} H^1(E_w, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)).$$

When $\mathcal{S}_\ell(E, j)$ has finite length, write $s_\ell(E, j) := \mathrm{length}_{\mathbb{Z}_\ell} \mathcal{S}_\ell(E, j)$.

Remark 5.2. The groups $\mathcal{S}_\ell(E, j)$ are Tate–Shafarevich groups for the Galois modules $\mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)$. Although the definition only asks for vanishing at $w \mid \ell$, these classes are trivial over every completion of E. Indeed, let $w \nmid \ell$ have residue field of order q_w. The classes are unramified at w, and geometric Frobenius acts on $\mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)$ by q_w^j. Since $j \ge 1$, multiplication by $q_w^j - 1$ is surjective on $\mathbb{Q}_\ell/\mathbb{Z}_\ell$, so the unramified first cohomology at w vanishes. The archimedean completions are complex because E is a CM field, so their first cohomology also vanishes. The study of groups of locally trivial Galois cohomology classes, such as $\mathcal{S}_\ell(E, j)$, goes back to the work of Tate [Tat63, §3, pp. 290–291] and Poitou [Poi67] on global duality.

The denominator bound adds the contributions from all coefficient primes and all Tate twists in the required range. For a finite ℓ-primary module, its length times $\log \ell$ is the logarithm of its order. To record this total, for an integer $h' \ge 1$, let $\mathcal{P}$ range over the finite sets of rational primes and put

$$\mathcal{V}(E, h') := \sup_{\mathcal{P}} \sum_{\ell \in \mathcal{P}} \log \ell \sum_{1 \le j < h'} s_\ell(E, j).$$

Set $\mathcal{V}(E, h') = \infty$ if $\mathcal{S}_\ell(E, j)$ has infinite length for any prime ℓ and any $1 \le j < h'$. A finite bound for $\mathcal{V}(E, h')$ therefore bounds the orders of the kernels $\mathcal{S}_\ell(E, j)$ for all primes ℓ and all $1 \le j < h'$ at once, and includes their finiteness. The denominator estimate uses this sum with $h' = h$.

For the theta lifts, [Alp26] uses $\mathcal{V}(E,h)$ to bound the indices of projected lattices. At a finite place $w \mid \ell$ of the coefficient field K, let $M_{j,w}^{\mathrm{pr}}$ be the lattice obtained by projecting onto the retained Hecke eigenspaces at the j-th theta lift, for $1 \le j \le m$. Let $M_{j,w}^{\mathrm{int}}$ be the intersection of those same eigenspaces with the original integral lattice of automorphic forms. Write $\mathcal{O}_w$ for the ring of integers of K_w, and set $e_w := \mathrm{ord}_{\mathcal{O}_w}(\ell)$ and

$$\eta_{j,w} := \frac{1}{e_w} \operatorname{length}_{\mathcal{O}_w}\left(M_{j,w}^{\mathrm{pr}}/M_{j,w}^{\mathrm{int}}\right).$$

Among the tuples of auxiliary Hecke characters in Section 4.1, [Alp26] fixes one tuple μ_0 as a reference. Sum over the finitely many coefficient places where [Alp26] uses auxiliary characters different from the fixed reference choice. Writing Nw for the cardinality of the residue field at w, put

$$\eta_j := \frac{1}{[K:\mathbb{Q}]} \sum_w e_w \eta_{j,w} \log Nw.$$

Let $Z_j(\mu_0)$ be the image of the auxiliary theta lift at step j, and put $d_j := \dim_{\mathbb{C}} Z_j(\mu_0)$. For the parameter choices in [Alp26], the reference theta lift is injective, so d_j is also the dimension of its source. These source dimensions are independent of j. For sufficiently large x, [Alp26] obtains, simultaneously for $1 \le j \le m$, the following bound in the parameters of Section 4.6, with $f = [E^+ : \mathbb{Q}]$:

$$\eta_j \le d_j\left(C_2 b x^{2\varepsilon}\left(fm + \frac{|T_2|k(\log x)^2}{n^2}\right) + \mathcal{V}(E,h)\right), \tag{5.1}$$

provided $\mathcal{V}(E,h) < \infty$. The constant C_2 is independent of the varying parameters. The factor e_w restores the $\mathcal{O}_w$-length in the definition of $\eta_{j,w}$ before summing over w. For the normalization of mean slope in Section 4.3, a quotient of lattices of rank d_j changes mean slope by twice its normalized logarithmic index divided by d_j. Using Equation 5.1, [Alp26] therefore bounds the change in mean slope caused by projection onto the retained Hecke eigenspaces. Together with the bounds for the other changes of integral lattices at finite primes, this gives the allowance for denominators in Equation 4.10.

The term $\mathcal{V}(E,h)$ in Equation 5.1 is therefore the part of the denominator bound still to be estimated. Its appearance also recalls the conjectures of Lichtenbaum [Lic73], which related zeta values to the orders of étale cohomology groups and algebraic K-groups; [Alp26] explicitly points to this perspective.

For the application to periods, the required estimate is $\mathcal{V}(E,h) \le \exp(x^{\varepsilon}/2)$ for every field E and odd integer h used in the theta lifts, uniformly for x in one selected long interval. For sufficiently large x, the parameter choices in [Alp26] satisfy $h \le \lfloor x^{401} \rfloor$. Since the lengths $s_\ell(E,j)$ are nonnegative, $\mathcal{V}(E,h) \le \mathcal{V}(E,\lfloor x^{401} \rfloor)$. In Proposition 5.3, [Alp26] obtains the required bound by estimating $\mathcal{V}(E,h')$ at the larger cutoff $h' = \lfloor x^{401} \rfloor$.

5.3. **Choosing number fields for bounds on the orders of Galois cohomology groups.** The freedom to choose E is useful for bounding the length $s_\ell(E,j)$ of the Tate–Shafarevich group $\mathcal{S}_\ell(E,j)$. To recover $\mathcal{G}_\psi$ by the character averages in Section 3, however, E must contain the field cut out by ψ. For a primitive Dirichlet character ψ of conductor N_ψ, denote this field by A_ψ. Thus A_ψ is the fixed field of $\ker\psi$ in $\mathbb{Q}(\zeta_{N_\psi})$, where ζ_N is a primitive N-th root of unity. To retain this condition while obtaining uniform bounds for $s_\ell(E,j)$, [Alp26] enlarges A_ψ by a real cyclotomic layer and two imaginary quadratic fields. More precisely, for $a \in \mathbb{Z}_{\geq 0}$ and primes $q \equiv 3 \pmod 8$ and $d \equiv 7 \pmod{16}$, set

$$E(\psi,a,q,d) := A_\psi \mathbb{Q}(\zeta_{2^{a+2}})^+ \mathbb{Q}(\sqrt{-q})\mathbb{Q}(\sqrt{-d}).$$

The compositum is taken inside a common cyclotomic field; its factors need not be linearly disjoint. The superscript $+$ denotes the maximal real subfield. Write σ_b for the cyclotomic automorphism that raises roots of unity to their b-th powers. Among these automorphisms, those fixing $E(\psi,a,q,d)$ satisfy exactly the following conditions:

(i) $b \equiv \pm 1 \pmod{2^{a+2}}$;
(ii) $\psi(b) = 1$;
(iii) $\left(\frac{b}{q}\right) = 1$;
(iv) $\left(\frac{b}{d}\right) = 1$.

The field A_ψ remains fixed, while a grows with the argument x of $\mathcal{G}_\psi(x)$.

The congruences imposed on q and d enter both the choice of local automorphic representations and the uniform bounds for $s_\ell(E,j)$. The conditions on their sizes, for $Z > 1$ and $\varepsilon > 0$, are

$$\tfrac{1}{2}Z^{\varepsilon/5} < \log q \leq Z^{\varepsilon/5},$$
$$\tfrac{1}{2}Z^{\varepsilon/5}(\log Z)^2 < \log d \leq Z^{\varepsilon/5}(\log Z)^2.$$

In addition, [Alp26] imposes $q \mid d+1$ and $p \mid d+1$ for every odd prime $p \leq Z^{\varepsilon/5}$. Since both q and d are 3 modulo 4, the fields E are CM. The existence of pairs of primes satisfying all these size and congruence conditions is proved in [Alp26].

The uniform bound for $\mathcal{V}(E,h')$ can now be stated. Fix $0 < \varepsilon < 1/52$, and choose a function $a(x) \in \mathbb{Z}_{\geq 0}$ such that, for every $x \geq 1$,

$$x^{2+8\varepsilon} \leq 2^{a(x)} \leq 2x^{2+8\varepsilon}.$$

In the following estimate from [Alp26], the auxiliary primes q, d remain fixed throughout the interval, while the cyclotomic layer varies with x. This is the uniformity needed for the application to periods.

Proposition 5.3. *For the choices of ψ, ε, and $a(x)$ in Section 5.3, there are $J \geq 1$ and X_0 such that, for every $X \geq X_0$ and every pair of primes q, d satisfying the specified bounds and congruences with $Z = X^{1/J}$, there is*

$Y \geq X^{1/J}$ *with* $Y^2 \leq X$ *for which*

$$\mathcal{V}\Big(E(\psi, a(x), q, d), \lfloor x^{401} \rfloor\Big) \leq \exp\big(\tfrac{1}{2}x^{\varepsilon}\big) \tag{5.2}$$

for every $x \in [Y, Y^2]$.

Remark 5.4. The exponent 401 in [Alp26] leaves room for all the Tate twists arising from the repeated theta lifts; its precise value plays no further role. The interval $[Y, Y^2]$ works for every rational prime ℓ. The primes q, d remain fixed while $a(x)$, and hence E, may change; the bounds for $s_\ell(E, j)$ hold throughout that interval as both ℓ and j vary.

5.4. **The two eigenspaces for complex conjugation.** To estimate the length $s_\ell(E, j)$ of the Tate–Shafarevich group $\mathcal{S}_\ell(E, j)$, [Alp26] separates the two eigenspaces for complex conjugation on $\mathcal{S}_\ell(E, j)$. The two parts lead to different arithmetic arguments, while the prime 2 needs separate treatment because the usual projections involve division by 2. Let c be complex conjugation on E. For odd ℓ, split $\mathcal{S}_\ell(E, j)$ into the eigenspaces on which c acts as $(-1)^j$ and $(-1)^{j+1}$. Denote their lengths by $s_\ell^+(E, j)$ and $s_\ell^-(E, j)$, respectively. The labels $+$ and $-$ therefore depend on the parity of j. Since 2 is invertible in $\mathbb{Z}_\ell$, the projections $\big(1 \pm (-1)^j c\big)/2$ show that $s_\ell = s_\ell^+ + s_\ell^-$.

For a cutoff $h' \geq 1$, write B_2 for a bound on $\sum_{1 \leq j < h'} s_2(E, j)$, and write $B_\pm$ for bounds on the sums of $s_\ell^\pm(E, j)$ over odd primes ℓ and $1 \leq j < h'$, weighted by $\log \ell$. These bounds imply

$$\mathcal{V}(E, h') \leq B_2 \log 2 + B_- + B_+. \tag{5.3}$$

To estimate $s_\ell^+(E, j)$ and $s_\ell^-(E, j)$, [Alp26] uses global duality to relate the kernels $\mathcal{S}_\ell(E, j)$ to cohomology of positive Tate twists. Poitou [Poi67] and Tate [Tat63] established the duality theorems relating global cohomology to its local restrictions. In the application here, their theorems express $s_\ell(E, j)$ in terms of H^2, with corrections from local and global H^0.

When the cohomology groups have finite length, the identity relating $s_\ell(E, j)$ to the second Galois cohomology group in [Alp26] is

$$\begin{aligned} \operatorname{length}_{\mathbb{Z}_\ell} H^2(G_{E,S_\ell}, \mathbb{Z}_\ell(j+1)) + \operatorname{length}_{\mathbb{Z}_\ell} H^0(G_{E,S_\ell}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)) \\ = s_\ell(E, j) + \sum_{w \mid \ell} \operatorname{length}_{\mathbb{Z}_\ell} H^0(E_w, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j)). \end{aligned}$$

The lengths of the local groups $H^0(E_w, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-j))$ therefore contribute to the bound for $s_\ell(E, j)$.

[Alp26] bounds $s_\ell^-(E, j)$ through generalized Bernoulli numbers, with different arguments according to whether ℓ divides $[E : \mathbb{Q}]$. Write $n := [E : \mathbb{Q}]$ and $G := \operatorname{Gal}(E/\mathbb{Q})$, and let N_E be the conductor of E, the least positive integer such that $E \subseteq \mathbb{Q}(\zeta_{N_E})$. It is enough to consider $h' \geq 2$, since the sum defining B_- is empty when $h' = 1$.

For $j \geq 1$, let $\mathcal{X}_j$ consist of the characters $\eta : G \longrightarrow \mathbb{C}^\times$ satisfying $\eta(c) = (-1)^{j+1}$, where c is complex conjugation. Write χ_η for the corresponding

primitive Dirichlet character. To estimate the generalized Bernoulli numbers, write f_η for the conductor of χ_η. In terms of the Bernoulli polynomials $B_k(t)$, the normalization in Washington [Was97, Chapter 4] gives

$$B_{j+1,\chi_\eta} = f_\eta^j \sum_{a=1}^{f_\eta} \chi_\eta(a) B_{j+1}\left(\frac{a}{f_\eta}\right).$$

These numbers satisfy $L(-j,\chi_\eta) = -B_{j+1,\chi_\eta}/(j+1)$. To work with algebraic integers, [Alp26] sets

$$\beta_{j,\eta} := N_E\left(\prod_{\substack{p\le j+2\\ p \text{ prime}}} p\right) B_{j+1,\chi_\eta}.$$

The integer multiplying B_{j+1,χ_η} clears its denominator. The parity condition on η also ensures that B_{j+1,χ_η} is nonzero.

Suppose first that ℓ is an odd prime not dividing n. After embedding the character values in $\overline{\mathbb{Q}}_\ell$, the projectors $e_\eta := \frac{1}{n}\sum_{\sigma\in G}\eta(\sigma)^{-1}\sigma$ are integral over the ring of integers of an ℓ-adic field containing those values. Thus the character components can be treated separately. The arithmetic estimate proved in [Alp26] is

$$s_\ell^-(E,j) \le \sum_{\eta\in\mathcal{X}_j} \nu_\ell(\beta_{j,\eta}),$$

with $\nu_\ell(\ell) = 1$. For this estimate, [Alp26] constructs compatible elements in cyclotomic fields from Jacobi sums and Stickelberger elements, forms cohomology classes using Kolyvagin's [Kol90] derivative operators, and applies global reciprocity. Rubin [Rub91] explained the corresponding method for class groups using Gauss sums and Stickelberger ideals.

The passage from this estimate to a sum over ℓ is more elementary. The set $\mathcal{X}_j$ is stable under the Galois action on character values, so $M_j := \prod_{\eta\in\mathcal{X}_j}\beta_{j,\eta}$ is a nonzero rational integer. Consequently,

$$\sum_{\substack{\ell \text{ odd}\\ \ell\nmid n}} s_\ell^-(E,j)\log\ell \le \sum_\ell \nu_\ell(M_j)\log\ell = \log|M_j|.$$

It remains to bound the complex absolute values of the factors. The Fourier series of the Bernoulli polynomials gives $|B_k(t)| \le 4k!/(2\pi)^k$ for $k\ge 2$ and $0\le t\le 1$. Applying this inequality to the expression for B_{j+1,χ_η}, and using $f_\eta \le N_E$ and $\prod_{\substack{p\le j+2\\ p \text{ prime}}} p \le 4^{j+2}$, gives $|\beta_{j,\eta}| \le 16N_E^{j+2}(j+1)!$. Since $|\mathcal{X}_j| \le n$, it follows that $\log|M_j| \le n\log\left(16N_E^{j+2}(j+1)!\right)$. For $1\le j<h'$, the inequalities $16 \le (h'N_E)^4$, $N_E^{j+2} \le (h'N_E)^{h'+1}$, and $(j+1)! \le (h'N_E)^{h'}$ give $\log\left(16N_E^{j+2}(j+1)!\right) \le (2h'+5)\log(h'N_E)$. Summing over j and using

$(h'-1)(2h'+5) \le 3(h')^2$ therefore gives

$$\sum_{1\le j<h'} \sum_{\substack{\ell \text{ odd} \\ \ell\mid n}} s_\ell^-(E,j)\log \ell \le 3n(h')^2 \log(h' N_E).$$

At an odd prime $\ell \mid n$, the projectors e_η are no longer integral, so [Alp26] instead uses annihilators in $\mathbb{Z}_\ell[G]$. This part of the argument continues Stickelberger's [Sti90] construction of annihilators of class groups. Let $\mathcal{S}_\ell^-(E,j)$ denote the $(-1)^{j+1}$-eigenspace of $\mathcal{S}_\ell(E,j)$. Using a higher Stickelberger element at modulus $N_E\ell$, [Alp26] constructs $z \in \mathbb{Z}_\ell[G]$ annihilating $\varphi(N_E\ell)\mathcal{S}_\ell(E,j)$, where φ is Euler's totient function. Multiplication by z on $\mathbb{Z}_\ell[G]/\big(c-(-1)^{j+1}\big)$ has nonzero determinant Δ. The adjugate identity then shows that $\Delta\varphi(N_E\ell)$ annihilates $\mathcal{S}_\ell^-(E,j)$. Estimating the valuation of this scalar gives a bound for the exponent of $\mathcal{S}_\ell^-(E,j)$. Together with a separate bound for its ℓ-torsion, [Alp26] obtains an integer $r \ge 0$ such that

$$\begin{aligned} \ell^r \mathcal{S}_\ell^-(E,j) &= 0, \\ r\log \ell &\le C_0 n(j+1)\log(j\ell N_E), \\ \dim_{\mathbb{F}_\ell} \mathcal{S}_\ell^-(E,j)[\ell] &\le 8\ell n \log(\ell N_E), \end{aligned}$$

where C_0 is absolute. Filtering $\mathcal{S}_\ell^-(E,j)$ by the subgroups killed by successive powers of ℓ bounds its length by $r\dim_{\mathbb{F}_\ell}\mathcal{S}_\ell^-(E,j)[\ell]$. After summing over $1 \le j < h'$, this gives

$$\sum_{1\le j<h'} s_\ell^-(E,j)\log\ell \le 8C_0\ell n^2 (h')^2 \log(h'\ell N_E)^2.$$

Every odd prime dividing n divides $[A_\psi : \mathbb{Q}]$, since the cyclotomic and quadratic extensions adjoined to A_ψ have degrees that are powers of 2. Combining the two bounds therefore gives

$$B_- \le 3[E:\mathbb{Q}](h')^2\log(h'N_E) + C \sum_{\substack{\ell \mid [A_\psi:\mathbb{Q}] \\ \ell \text{ odd}}} \ell [E:\mathbb{Q}]^2 (h')^2 \log(h'\ell N_E)^2. \tag{5.4}$$

The constant C is absolute, so the same bound applies throughout the interval in Proposition 5.3 as $a(x)$ and h' vary.

For $s_\ell^+(E,j)$, [Alp26] instead uses congruences between automorphic forms and deformations of Galois representations. For the fixed character ψ, [Alp26] first sets aside a finite set of exceptional odd primes. At the remaining primes, it decomposes the coefficient modules of the Galois cohomology groups according to their prime-to-ℓ characters, parity, and congruences involving $\ell-1$ and j. The resulting bound for the total contribution B_+ in [Alp26] takes the form

$$B_+ \le (N_E h')^C + CB_{\mathrm{exc}}, \tag{5.5}$$

where B_{exc} is a common bound for the contribution to B_+ at each of finitely many exceptional cyclotomic layers, after summing over the twists indexed by j. The constant C includes the number of these layers.

It is worth pausing at Equation 5.5, since its proof in [Alp26] is one of the substantial arithmetic steps in the argument. The bound uses deformations of Galois representations and congruences between automorphic forms on the adelic quotients for unitary groups in three variables and for the group GSp_4 of symplectic similitudes in four variables, together with automorphy lifting and the associated local and global Galois representations. The representations in Section 4.2 were obtained from automorphic representations. In bounding B_+, [Alp26] also uses automorphy lifting to recognize suitable deformations of Galois representations as arising from automorphic forms. The connection with cohomology already appears in Mazur's [Maz89, §1.6] study of deformations. For the basic calculation, let F be a number field, let S contain the archimedean places, the places above ℓ, and the allowed places of ramification, and let k be a finite field of characteristic ℓ. Write $G_{F,S}$ for the Galois group of the maximal extension of F unramified outside S. For a continuous residual representation $\overline{\rho} : G_{F,S} \longrightarrow \mathrm{GL}_n(k)$, the adjoint module $\mathrm{ad}\overline{\rho}$ is $\mathrm{End}_k(k^n)$ with action $g \cdot X := \overline{\rho}(g) X \overline{\rho}(g)^{-1}$.

A lift to $k[\varepsilon]/(\varepsilon^2)$ has the form

$$\rho_\varepsilon(g) = (1 + \varepsilon c(g))\overline{\rho}(g).$$

The homomorphism condition makes c a 1-cocycle, and conjugation by a matrix congruent to the identity changes c by a coboundary. Thus $H^1(G_{F,S}, \mathrm{ad}\overline{\rho})$ parametrizes these deformations up to equivalence. For a small extension of coefficient rings with kernel I, the obstruction to lifting lies in $H^2(G_{F,S}, \mathrm{ad}\overline{\rho}) \otimes_k I$. Imposing local conditions on the lifts restricts $H^1(G_{F,S}, \mathrm{ad}\overline{\rho})$ to a Selmer subspace. In his proof of modularity for semistable elliptic curves, Wiles [Wil95] compared deformation rings with Hecke algebras, with Selmer groups controlling the deformations of Galois representations. The work of Taylor–Wiles [TW95] established the needed complete-intersection property of the Hecke algebras by varying auxiliary primes. Thus the attempt to show that Galois representations come from modular forms also gave a method for relating the size of a cohomology group to congruences between modular forms.

For higher-dimensional Galois representations, the comparison between deformation rings and Hecke algebras was developed by Clozel–Harris–Taylor [CHT08], who extended the method using automorphic forms on unitary groups. For the part involving GSp_4, [Alp26] also uses the four-dimensional Galois representations that Weissauer [Wei05] constructed from certain cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$.

For the bound on B_+, [Alp26] studies deformations of three-dimensional Galois representations over an auxiliary imaginary quadratic field $F_0/\mathbb{Q}$, with a polarization expressing conjugate self-duality. Automorphy lifting relates these deformations to automorphic forms on unitary groups attached to three-dimensional Hermitian spaces over $F_0/\mathbb{Q}$. Let $\theta : G_\mathbb{Q} \longrightarrow \{\pm 1\}$ be the quadratic character of $F_0/\mathbb{Q}$. At the reducible points relevant here, the representation of G_{F_0} is isomorphic to $\mathbb{1} \oplus V|_{G_{F_0}}$, where $\mathbb{1}$ denotes the trivial

representation and V is a two-dimensional representation of $G_{\mathbb{Q}}$ with $\det V = \theta$. The polarization gives an action of $G_{\mathbb{Q}}$ on the four-dimensional subspace of the adjoint module consisting of homomorphisms between the two summands. For this action, [Alp26] identifies that subspace with $V \oplus \left(V \otimes_{\overline{\mathbb{Q}}_\ell} \theta\right)$.

For a fixed Tate twist $-j$ with $j \geq 1$, automorphy lifting and descent in [Alp26] produce a classical cuspidal newform f of weight $2j+1$. Write $\rho_f : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ for the representation attached to f by Deligne [Del71], with the cohomological normalization giving Hodge–Tate weights $\{0, 2j\}$. In these conventions, [Alp26] identifies $V \simeq \rho_f(j)$.

The twist j is a noncentral critical twist. Kato [Kat04, Theorem 14.2(1), p. 235] proved finiteness of the corresponding Selmer groups with torsion coefficients. The Bloch–Kato Selmer groups with characteristic-zero coefficients therefore vanish. Applying this result to the two constituents, [Alp26] obtains

$$H^1_f(\mathbb{Q}, V) = 0 \quad \text{and} \quad H^1_f\left(\mathbb{Q}, V \otimes_{\overline{\mathbb{Q}}_\ell} \theta\right) = 0.$$

Consequently, the projections of an adjoint Selmer class to these two constituents vanish. This removes their contributions to the deformation calculation used to bound B_+.

The bound for B_+ involves all odd primes ℓ, so it remains to explain how [Alp26] sums the bounds obtained from these deformation problems. The identity used for the products M_j of generalized Bernoulli numbers occurs again, but the integer now comes from differences of Frobenius traces.

Consider first the primes ℓ outside the finite set of exceptional primes chosen for ψ, with $(\ell - 1) \nmid 2j$. Fix the Tate twist $j \geq 1$, a modulus N for the Dirichlet characters, and the corresponding conditions on the levels of the automorphic representations. Let $\mathcal{X}$ be one Galois orbit of these characters modulo N, fixed independently of ℓ. Choose an auxiliary prime $p \geq 7$ with $p \nmid N\Delta_{F_0}$, where Δ_{F_0} is the discriminant of F_0. For a place w of F_0 above p, put $Q := \#(\mathcal{O}_{F_0}/w)$, so Q is either p or p^2. For $\eta \in \mathcal{X}$, view η as an ℓ-adic character of $G_{\mathbb{Q}}$, and let ϵ_ℓ be the cyclotomic character. The argument in [Alp26] deforms the restriction to G_{F_0} of $\eta\epsilon_\ell^j \oplus \mathbb{1} \oplus \eta^{-1}\epsilon_\ell^{-j}$. Its trace at geometric Frobenius is

$$t_\eta := Q^{-j}\eta(Q)^{-1} + 1 + Q^j\eta(Q).$$

For the automorphic representations occurring in the deformation problem, [Alp26] constructs a finite list $a_1, \ldots, a_D$ of algebraic Frobenius traces that is independent of ℓ. This list includes the traces from irreducible three-dimensional representations and from the representations $\mathbb{1} \oplus \rho_f(j)|_{G_{F_0}}$ arising from modular forms. Galois conjugation permutes the indexed list, each $Q^j a_s$ is an algebraic integer, and the bounds for the complex absolute values give $|a_s| \leq \sqrt{Q} + 2$. The number D is bounded polynomially in N and $j+1$, with constants depending on F_0.

The product

$$M_{j,p} := \prod_{\eta \in \mathcal{X}} \prod_{s=1}^{D} Q^j (a_s - t_\eta)$$

is therefore a rational integer. It is nonzero because $\eta(Q)$ is a root of unity and $|t_\eta| \geq Q^j - 1 - Q^{-j} > \sqrt{Q} + 2$. The same bounds for the factors give $\log|M_{j,p}| \leq D\#\mathcal{X}(2j \log Q + 1)$.

For each character component, the deformation argument bounds the length of an auxiliary Selmer group over its coefficient ring, divided by the ramification index of that ring over $\mathbb{Z}_\ell$. After summing over $\eta \in \mathcal{X}$, the sum of these normalized lengths is at most $2v_\ell(M_{j,p})$ for the primes under consideration with $\ell \neq p$, where $v_\ell(\ell) = 1$. Because $M_{j,p}$ is independent of ℓ, [Alp26] can now use

$$\sum_{\ell} v_\ell(M_{j,p}) \log \ell = \log|M_{j,p}|$$

to sum the bounds over ℓ. Repeating the construction with a second auxiliary prime accounts for the contribution at $\ell = p$. The bounds for D and $\log|M_{j,p}|$, summed over the character orbits and Tate twists, then give the polynomial contribution to Equation 5.5.

The remaining character components require separate arguments. For certain components with $j \geq 2$, [Alp26] uses automorphic forms on $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ and introduces an auxiliary character whose conductor is a power of 2 depending on ℓ. A generalized Bernoulli number that is an ℓ-adic unit makes the additional Selmer contribution vanish. Other components at the excluded primes are treated by descent through cyclotomic extensions. There is also a trivial-character case in which the Selmer group vanishes. For $j = 1$ at the excluded primes, the argument uses unitary groups attached to Hermitian spaces over $\mathbb{Q}(i)/\mathbb{Q}$ or the coarser bounds for second Galois cohomology, according to the conductor.

These arguments leave the contribution B_{exc} in Equation 5.5. For Proposition 5.3, the constant C in that equation is fixed throughout the chosen interval, while the conductor N_E and cutoff h' may vary.

Equation 5.5 leaves the contribution B_{exc} from exceptional cyclotomic layers, while Equation 5.3 also includes B_2 from the prime 2. For these contributions, the bounds will depend on the location of the interval $[Y, Y^2]$, not just on how large Y is. This is why [Alp26] retains the freedom to choose Y in Proposition 5.3.

5.5. **Choosing intervals for the exceptional Tate twists.** To bound the contributions from the exceptional cyclotomic layers and from $\ell = 2$, [Alp26] studies how the lengths $s_\ell(E, j)$ vary through cyclotomic towers.

The variation through a tower brings the argument in [Alp26] back to the problem that originally motivated Iwasawa theory, namely the growth of class groups as the field runs up a tower. Iwasawa [Iwa59] studied that growth by treating inverse limits of class groups as modules with a continuous

Galois action. For abelian fields, Ferrero–Washington [FW79] proved that the Iwasawa μ-invariant vanishes. In the eventual formula for the ℓ-adic valuation of the class number, μ is the coefficient of ℓ^{ν} at the layer of degree ℓ^{ν} over the base field. The argument in [Alp26] uses this vanishing to estimate the lengths $s_{\ell}(E, j)$, keeping track of the dependence on the base field as the auxiliary primes and the cyclotomic layers vary.

To identify the indices j that require a coarser bound for $s_{\ell}(E, j)$, [Alp26] uses the following estimate along cyclotomic towers. Let L_0 be an abelian number field containing the ℓ-th roots of unity, and also i if $\ell = 2$. Let $\mu_{\ell^{\infty}}$ denote the roots of unity of ℓ-power order, and write L_{ν} for the layer of $L_0(\mu_{\ell^{\infty}})/L_0$ with $[L_{\nu} : L_0] = \ell^{\nu}$. There are a positive integer R, at most R elements s_i in an extension of $\mathbb{Q}_{\ell}$, and constants j_0, c_0 such that, for every $B \in \mathbb{Z}_{\geq 0}$, if $|s_i - j|_{\ell} \geq \ell^{-B}$ for every i, then

$$s_{\ell}(L_{\nu}, j) \leq R(\max(\nu, j_0) + c_0 + B).$$

The values s_i are the finitely many ℓ-adic parameters that the integer j has to avoid for this estimate to apply. The number B sets the lower bound ℓ^{-B} on the distances from these parameters. For $\ell = 2$, the base field in [Alp26] is $L_0 = A_{\psi}(i, \sqrt{-q}, \sqrt{-d})$. For this base field, [Alp26] bounds $s_2(E(\psi, a, q, d), j)$ by $1 + s_2(L_0(\zeta_{2^{a+2}}), j)$. The field $L_0(\zeta_{2^{a+2}})$ is a layer L_{ν} with $\nu \leq a + 2$. For each exceptional odd prime ℓ, [Alp26] applies the bound for $s_{\ell}(L_{\nu}, j)$ with $L_0 = \mathbb{Q}(\zeta_{\ell N_{\psi}})$ and the fixed layer $\nu = \lfloor \log_{\ell} R \rfloor + 3$. The number R is independent of the layer ν. For $\ell = 2$, [Alp26] also obtains a choice of R independent of q and d, using the congruences imposed on these primes in Section 5.3.

Although the parameters s_i are ℓ-adic, the indices j of the Tate twists are ordinary integers. A bounded interval of integers can therefore contain only a few indices j that are very close to some s_i in the ℓ-adic metric. To see why, fix $Z \geq 1$. For a fixed s_i, the condition $|s_i - j|_{\ell} < \ell^{-\lfloor \log_{\ell} Z \rfloor}$ can hold for at most one integer $1 \leq j \leq Z$. Indeed, if two distinct integers satisfy the condition, then the ultrametric inequality forces their difference to have ℓ-adic valuation greater than $\lfloor \log_{\ell} Z \rfloor$. This is impossible for a nonzero integer of ordinary absolute value at most Z. Thus there are at most R exceptional indices j between 1 and Z.

At the exceptional indices, [Alp26] has a much coarser bound for $s_{\ell}(E, j)$, which suffices only when j is small relative to x. It therefore chooses the interval of values of x so that every exceptional index in the range of Proposition 5.3 lies below a small fixed power of x. When $x \leq X$, every Tate twist included in $\mathcal{V}(E, \lfloor x^{401} \rfloor)$ has index $j < X^{401}$. The exceptional indices in this finite range determine the choice of interval, using the following elementary observation. Let $0 < \eta \leq C_0$, let $R \in \mathbb{Z}_{\geq 0}$, and let $\mathcal{E} \subseteq \mathbb{N}$ have cardinality at most R. Set $J := 2^{R+2}(C_0/\eta)^R$.

Lemma 5.5. *For every $X > 1$, there is $Y \geq X^{1/J}$ with $Y^2 \leq X$ such that, for every $x \in [Y, Y^2]$ and $j \in \mathcal{E}$,*

$$j \leq x^{C_0} \quad \Longrightarrow \quad j \leq x^{\eta}.$$

Proof. Let $L := \log X$. For $0 \leq i \leq R$, set $u_i := (L/2)(\eta/(2C_0))^i$. The $R+1$ intervals $(\eta u_i, 2C_0 u_i]$ are pairwise disjoint, since $2C_0 u_{i+1} = \eta u_i$ for $0 \leq i < R$. At least one of these intervals contains no point $\log j$ with $j \in \mathcal{E}$. Choose such an i and set $Y := e^{u_i}$. Since $u_i \leq L/2$ and

$$u_i \geq \frac{L}{2^{R+1}(C_0/\eta)^R} \geq L/J,$$

the required bounds on Y hold. If $x \in [Y, Y^2]$ and $x^{\eta} < j \leq x^{C_0}$, then $\eta u_i < \log j \leq 2C_0 u_i$, contrary to the choice of i. □

For the exceptional indices $j \leq x^{\eta}$, [Alp26] bounds second Galois cohomology with coefficients in $\mathbb{Z}_\ell(j+1)$ for the layers L_ν used for $\ell = 2$ and the exceptional odd primes. These layers are totally complex and contain the ℓ-th roots of unity, or the fourth roots of unity when $\ell = 2$. For such a layer L, let S_ℓ be its places above ℓ and Δ_L its discriminant.

The argument passes through bounds for torsion in the homology of the groups $\mathrm{GL}_N(\mathcal{O}_L)$. Soulé [Sou03] used this approach to bound torsion in the algebraic K-groups of rings of integers; Bayer-Fluckiger, Emery, and Houriet [BFEH15] subsequently improved the bounds. The estimate established in [Alp26], with a fixed constant C_K, is

$$\log \#H^2(G_{L,S_\ell}, \mathbb{Z}_\ell(j+1)) \leq \exp\Big((2j[L:\mathbb{Q}])^{C_K} \log(2|\Delta_L|)\Big).$$

The restriction $j \leq x^{\eta}$ is useful here because j occurs in the exponent. With η sufficiently small, [Alp26] can keep the sum of these coarse bounds within the estimate sought in Proposition 5.3. For indices j outside the exceptional set, the distances from the parameters s_i allow the bound for $s_\ell(L_\nu, j)$ that is linear in B, with B logarithmic in the upper bound X^{401} for the indices of the Tate twists. Consequently, for the cutoff $h' = \lfloor x^{401} \rfloor$, the contribution B_2 at $\ell = 2$ and the contribution B_{exc} from exceptional cyclotomic layers satisfy the following bounds for every x in the chosen interval $[Y, Y^2]$:

$$B_2 \leq \exp(x^{4\varepsilon/5}) \qquad \text{and} \qquad B_{\mathrm{exc}} \leq \exp(x^{4\varepsilon/5}).$$

The bounds for $H^2(G_{L,S_\ell}, \mathbb{Z}_\ell(j+1))$ also relate to special values of zeta functions.

Remark 5.6. Soulé [Sou03, §5.4] discussed the connection between bounds for these second cohomology groups and lower bounds for higher regulators through Lichtenbaum's formula. This gives another interpretation of the cohomological quantities bounded in this section.

5.6. **Completing the bound for the orders of Galois cohomology groups.** To recover Proposition 5.3, [Alp26] combines the bounds for B_-, B_+, and B_2. For the application in [Alp26], take $E := E(\psi, a(x), q, d)$ and $h' := \lfloor x^{401} \rfloor$. Since $N_E \le 2^{a(x)+2} N_\psi q d$, the bounds on $a(x)$, q, d, and h' imply $\log(N_E h') \le 2x^{\varepsilon/4}$ for sufficiently large x in the selected interval. Since $[E : \mathbb{Q}] \le \varphi(N_E) \le N_E$, with φ denoting Euler's totient function, the right side of Equation 5.4 is at most $C_1 \exp(8x^{\varepsilon/4})$ for a fixed C_1. Combining this estimate with Equations (5.3) and (5.5) gives

$$\begin{aligned}\mathcal{V}(E, h') &\le \exp(x^{4\varepsilon/5}) \log 2 + C_1 \exp(8x^{\varepsilon/4}) + (N_E h')^C + C \exp(x^{4\varepsilon/5}) \\ &\le \exp(x^{4\varepsilon/5}) \log 2 + C_1 \exp(8x^{\varepsilon/4}) + \exp(2Cx^{\varepsilon/4}) + C \exp(x^{4\varepsilon/5}).\end{aligned}$$

Each exponent is $o(x^\varepsilon)$. For fixed $C, C_1, \varepsilon > 0$ and sufficiently large x, the sum is therefore at most $\exp(\frac{1}{2}x^\varepsilon)$. This is Equation 5.2 in Proposition 5.3.

The bound for $\mathcal{V}(E, h')$ controls the contribution of locally trivial extension classes to the denominators introduced by projection onto Hecke eigenspaces. Together with the bounds for the other indices between integral lattices of automorphic forms, it bounds $\overline{L}$ in Equation 4.10 and keeps the loss caused by denominators within the error allowed in Equation 4.11. The remaining choice of parameters, including the number of theta lifts, is made in Section 6. With those choices, [Alp26] obtains a bound for $\mathcal{G}_\psi(x)$ with exponent as close to $1/2$ as required.

6. Choosing the parameters to obtain GRH

The bound for $\mathcal{V}(E, h')$ in Proposition 5.3 completes the estimates for denominators needed in Equation 4.11. That equation bounds $\mathcal{A}_{\mathrm{tot}} - \mathcal{M}_{\mathrm{tot}}$ using automorphic periods and integral lattices of automorphic forms, while the character expansion in Section 3 relates this difference to $\mathcal{G}_\psi(x)$. It remains to choose the parameters so that the resulting upper bound for $\mathcal{G}_\psi(x)$ has exponent arbitrarily close to $1/2$, as required in Proposition 1.2.

Increasing the number m of theta lifts has two competing effects on the bound. After division by the coefficient of $\mathcal{G}_\psi(x)$, two of the terms have sizes x/m and m, up to fixed powers of x^ε. Increasing m decreases x/m and increases m. Taking m close to $\sqrt{x}$ keeps both terms close to $\sqrt{x}$, which explains the exponent $1/2$. The remaining terms depend on the dimension of the initial Hermitian space and the sizes of the selected sets of complex places of E. In [Alp26], those parameters are chosen together with m to bound every remaining term by $x^{1/2}$ times a fixed power of x^ε.

6.1. **The number of theta lifts and the size of the error.** Recall the bound in Equation 4.11. With H, T, and T_2 as in Section 4.6, put $n = |H|$, $t = |T|$, and $\nu_2 = |T_2|$, while m is the number of theta lifts. The number b is the scale of the integers $w_{j,\rho}$ for the complex places $\rho \in T_2$, as in Section 3; these integers specify blocks of parameters of the real unitary representations. The parameter $k = \dim_E X_0$ is the dimension of the initial Hermitian space,

and a bounds the positive parameters a_t indexed by the places in T. The estimates in [Alp26] allow multiplicative factors that are fixed powers of x^{ε}, so the parameters are chosen within ranges. The elementary consequence of those ranges is given in Lemma 6.1. The simultaneous choices in [Alp26] are recorded in Proposition 6.3.

Fix $x \geq 3$, $\varepsilon > 0$, and $C \geq 0$. Suppose n, t, ν_2, m are positive integers, $b, k \in \mathbb{R}_{>0}$, and $a \in \mathbb{R}_{\geq 0}$ such that

$$\begin{aligned} a &\leq 2xb, \\ \frac{n}{2}x^{-\varepsilon/8} &\leq t, \\ nx^{-\varepsilon} &\leq \nu_2 \leq n, \\ x^{1/2-\varepsilon} &\leq m \leq x^{1/2+\varepsilon/2}, \\ xm &\leq k \leq n^2 x^{1/2-\varepsilon}. \end{aligned}$$

Write

$$B := ta\left(1 + \frac{m^2}{k}\right) + m\left(2nmbx^{\varepsilon} + \frac{\nu_2 kbx^{2\varepsilon}}{n^2}\right)$$

for the expression in brackets in Equation 4.11. The calculation in Section 3 leads to a coefficient proportional to $tmb\nu_2/n$ for $\mathcal{G}_\psi(x)$. Dividing by this coefficient turns the bound for $\mathcal{A}_{\mathrm{tot}} - \mathcal{M}_{\mathrm{tot}}$ into a bound for $\mathcal{G}_\psi(x)$. In Lemma 6.1, Z represents $\mathcal{G}_\psi(x)$ after subtracting a bound for the terms from additional characters retained by the averaging over H_0 in Section 3.

Lemma 6.1. *Assume the parameter bounds in Section 6.1. If $Z \in \mathbb{R}$ satisfies*

$$\frac{tmb\nu_2}{2n}\left(Z - 4x^{1/2+\varepsilon}\right) \leq x^{C\varepsilon}B + \frac{2tmb}{x}, \tag{6.1}$$

then $Z \leq 26x^{1/2+(C+21/8)\varepsilon}$. In particular, if $26 \leq x^{3\varepsilon/8}$, then $Z \leq x^{1/2+(C+3)\varepsilon}$.

Proof. It is convenient to divide first by $Q := tmb$ and to write $d := \nu_2/n$. The effect of increasing m is visible in the summand ta/Q:

$$\frac{ta}{Q} = \frac{a}{mb} \leq \frac{2x}{m} \leq 2x^{1/2+\varepsilon}.$$

The summand $ta(m^2/k)/Q$ stays bounded because $k \geq xm$:

$$\frac{ta(m^2/k)}{Q} = \frac{am}{bk} \leq \frac{2xm}{k} \leq 2.$$

The summand $2nm^2bx^{\varepsilon}/Q$ also involves the ratio $t/n = |T|/|H|$. The lower bound on t/n ensures that

$$\frac{2nm^2bx^{\varepsilon}}{Q} = \frac{2nmx^{\varepsilon}}{t} \leq 4mx^{9\varepsilon/8} \leq 4x^{1/2+13\varepsilon/8}.$$

For the last term of B/Q, the upper bound on k is needed as well:

$$\frac{m\nu_2 kbx^{2\varepsilon}/n^2}{Q} = \frac{\nu_2 kx^{2\varepsilon}}{n^2 t} \leq \frac{2k}{n^2}x^{17\varepsilon/8} \leq 2x^{1/2+9\varepsilon/8}.$$

Since $x \geq 1$, the sum of the four terms is bounded by

$$\frac{B}{Q} \leq 10x^{1/2+13\varepsilon/8}.$$

Dividing Equation 6.1 by $Q/2 > 0$ and using the bound for B/Q gives

$$d\Big(Z - 4x^{1/2+\varepsilon}\Big) \leq 20x^{1/2+(C+13/8)\varepsilon} + \frac{4}{x} \leq 22x^{1/2+(C+13/8)\varepsilon}.$$

The last inequality uses $x \geq 3$ and $x^{1/2+(C+13/8)\varepsilon} \geq 1$. Since $d \geq x^{-\varepsilon} > 0$, it follows that

$$Z \leq 4x^{1/2+\varepsilon} + 22x^{1/2+(C+21/8)\varepsilon} \leq 26x^{1/2+(C+21/8)\varepsilon}.$$

If $26 \leq x^{3\varepsilon/8}$, the factor 26 is absorbed by $x^{3\varepsilon/8}$. □

Remark 6.2. Notice how the parameter bounds in Lemma 6.1 enter its proof. The bounds on the number m of theta lifts keep the terms x/m and m close to $\sqrt{x}$. The bounds on $k = \dim_E X_0$ control the two terms involving k in B/Q. The lower bounds on t/n and ν_2/n ensure that division by the coefficient of Z introduces at most a fixed power of x^{ε}. To satisfy these conditions simultaneously, [Alp26] chooses the Hermitian spaces, highest weights, and selected complex places of E together.

6.2. **Obtaining the bound for $\mathcal{G}_\psi$.** It remains to realize the ranges in Lemma 6.1 by choices of number fields, highest weights, and selected complex places in [Alp26]. The choices must work throughout arbitrarily large intervals $[Y, Y^{3/2}]$, since the argument about zeros in Section 2 needs a bound for $\mathcal{G}_\psi(x)$ at every point of such an interval.

Before applying Lemma 6.1, [Alp26] bounds the terms in the expansion of $\mathcal{A}_{\mathrm{tot}} - \mathcal{M}_{\mathrm{tot}}$ from characters retained by the averaging over H_0 in Section 3 but not used in $\mathcal{G}_\psi$. For the choices in [Alp26], the sum of the absolute values of these additional contributions is at most $4rx^{1/2+\varepsilon}$, where r is the order of ψ. The following proposition records the uniform choices in [Alp26], with the term $4rx^{1/2+\varepsilon}$ subtracted in the definition of Z.

Proposition 6.3. *Let ψ be a primitive Dirichlet character of order r. There is $C \geq 0$ such that, for every sufficiently small $\varepsilon > 0$, there are arbitrarily large $Y \geq 3$ for which the hypotheses of Lemma 6.1 can be satisfied at every $x \in [Y, Y^{3/2}]$ with this C and*

$$Z = \mathcal{G}_\psi(x) - 4rx^{1/2+\varepsilon}.$$

The constant C in Proposition 6.3 accounts for the errors in the estimates for period values and in the comparisons between integral lattices of automorphic forms. It is fixed before ε is chosen; the remaining parameters may depend on x. This order of choices allows the exponent in Lemma 6.1 to approach $1/2$.

Proof of Proposition 1.2. Fix ψ, $\eta > 0$, and $A \geq 2$. Let r be the order of ψ, and choose C as in Proposition 6.3. Choose $\varepsilon > 0$ small enough that

the proposition applies and $(C+4)\varepsilon \le \eta$. Since Proposition 6.3 permits arbitrarily large Y, choose one with

$$Y \ge \max\Big\{A, 3, 26^{8/(3\varepsilon)}, (1+4r)^{1/\varepsilon}\Big\}.$$

For each $x \in [Y, Y^{3/2}]$, choose the parameters in Proposition 6.3 with the same C. The choice of Y ensures that $26 \le x^{3\varepsilon/8}$, so Lemma 6.1 gives

$$\mathcal{G}_\psi(x) \le x^{1/2+(C+3)\varepsilon} + 4rx^{1/2+\varepsilon}.$$

Since $C+3 \ge 1$ and $1+4r \le x^{\varepsilon}$, it follows that

$$\mathcal{G}_\psi(x) \le (1+4r)x^{1/2+(C+3)\varepsilon} \le x^{1/2+(C+4)\varepsilon} \le x^{1/2+\eta}.$$

This holds throughout the chosen interval, with $Y \ge A$. □

With the interval bound of Proposition 1.2 in place, [Alp26] applies Proposition 2.3 to exclude zeros of the Dirichlet L-functions to the right of the critical line. The functional equation then excludes zeros in $0 < \mathrm{Re}(s) < 1/2$, completing the proposed argument for Conjecture 1.1.

References

[AB95] Ahmed Abbes and Thierry Bouche. Théorème de Hilbert–Samuel "arithmétique". *Ann. Inst. Fourier (Grenoble)*, 45(2):375–401, 1995.

[ALB23] Johannes Anschütz and Arthur-César Le Bras. Prismatic Dieudonné theory. *Forum Math. Pi*, 11:e2, 1–92, 2023. Erratum, *Forum Math. Pi* **13** (2025), e3.

[Alp26] Levent Alpöge. [Announcement concerning the Riemann hypothesis]. `https://alpo.ge/remark.pdf`, 2026. Announcement of an argument developed with Claude. The Lean formalization is available in fourteen archive parts, to be concatenated in the indicated order: part01, part02, part03, part04, part05, part06, part07, part08, part09, part10, part11, part12, part13, part14. Accessed November 20, 2026.

[AMRT75] Avner Ash, David Mumford, Michael Rapoport, and Yung-Sheng Tai. *Smooth Compactification of Locally Symmetric Varieties*, volume 4 of *Lie Groups: History, Frontiers and Applications*. Math Sci Press, Brookline, Massachusetts, 1975. Second edition, as *Smooth Compactifications of Locally Symmetric Varieties*, Cambridge University Press, 2010, with the collaboration of Peter Scholze.

[And82] Greg W. Anderson. Logarithmic derivatives of Dirichlet L-functions and the periods of abelian varieties. *Compos. Math.*, 45(3):315–332, 1982.

[AS95] Urban Angehrn and Yum-Tong Siu. Effective freeness and point separation for adjoint bundles. *Invent. Math.*, 122(2):291–308, 1995.

[Beh46] Felix A. Behrend. On sets of integers which contain no three terms in arithmetical progression. *Proc. Natl. Acad. Sci. USA*, 32(12):331–332, 1946.

[BF03] John J. Benedetto and Matthew Fickus. Finite normalized tight frames. *Adv. Comput. Math.*, 18(2–4):357–385, 2003.

[BFEH15] Eva Bayer-Fluckiger, Vincent Emery, and Julien Houriet. Hermitian lattices and bounds in K-theory of algebraic integers. *Documenta Mathematica*, Extra Volume Merkurjev:71–83, 2015.

[BK90] Spencer Bloch and Kazuya Kato. L-functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progress in Mathematics*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990.

[BLGGT14] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor. Local-global compatibility for $\mathfrak{l} = \mathfrak{p}$, II. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(1):165–179, 2014.

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Integral $\mathfrak{p}$-adic Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 128:219–397, 2018.

[Bos01] Jean-Benoît Bost. Algebraic leaves of algebraic foliations over number fields. *Publ. Math. Inst. Hautes Études Sci.*, 93:161–221, 2001.

[BS22] Bhargav Bhatt and Peter Scholze. Prisms and prismatic cohomology. *Ann. of Math. (2)*, 196(3):1135–1275, 2022.

[BV89] Jean-Michel Bismut and Éric Vasserot. The asymptotics of the Ray–Singer analytic torsion associated with high powers of a positive line bundle. *Comm. Math. Phys.*, 125(2):355–367, 1989.

[BZ77] I. N. Bernstein and A. V. Zelevinsky. Induced representations of reductive $\mathfrak{p}$-adic groups. I. *Ann. Sci. Éc. Norm. Supér. (4)*, 10(4):441–472, 1977.

[Car12] Ana Caraiani. Local-global compatibility and the action of monodromy on nearby cycles. *Duke Math. J.*, 161(12):2311–2413, 2012.

[Car14] Ana Caraiani. Monodromy and local-global compatibility for $\mathfrak{l} = \mathfrak{p}$. *Algebra Number Theory*, 8(7):1597–1646, 2014.

[CH13] Gaëtan Chenevier and Michael Harris. Construction of automorphic Galois representations, II. *Camb. J. Math.*, 1(1):53–73, 2013.

[CHL11] Laurent Clozel, Michael Harris, and Jean-Pierre Labesse. Construction of automorphic Galois representations, I. In Laurent Clozel, Michael Harris, Jean-Pierre Labesse, and Bao Châu Ngô, editors, *On the Stabilization of the Trace Formula*, volume 1 of *Stabilization of the Trace Formula, Shimura Varieties, and Arithmetic Applications*, pages 497–527. International Press, Somerville, MA, 2011.

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor. Automorphy for some $\mathfrak{l}$-adic lifts of automorphic mod $\mathfrak{l}$ Galois representations. *Publ. Math. Inst. Hautes Études Sci.*, 108:1–181, 2008.

[Clo91] Laurent Clozel. Représentations galoisiennes associées aux représentations automorphes autoduales de GL(n). *Publ. Math. Inst. Hautes Études Sci.*, 73:97–145, 1991.

[CS49] Sarvadaman Chowla and Atle Selberg. On Epstein's zeta function. I. *Proc. Natl. Acad. Sci. USA*, 35(7):371–374, 1949.

[Del71] Pierre Deligne. Formes modulaires et représentations ℓ-adiques. In *Séminaire Bourbaki, vol. 1968/69*, volume 179 of *Lecture Notes in Mathematics*, pages 139–172. Springer-Verlag, 1971. Exposé 355.

[Del73] Pierre Deligne. Les constantes des équations fonctionnelles des fonctions L. In *Modular Functions of One Variable II*, volume 349 of *Lecture Notes in Math.*, pages 501–597. Springer-Verlag, Berlin, 1973.

[Dir37] Peter Gustav Lejeune Dirichlet. Beweis des Satzes, daß jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält. *Abh. Königl. Preuss. Akad. Wiss. Berlin*, pages 45–71, 1837.

[dlVP96] C. J. de la Vallée Poussin. Recherches analytiques sur la théorie des nombres premiers, i–iii. *Annales de la Société Scientifique de Bruxelles*, 20:183–256, 281–362, 363–397, 1896.

[DR80] Pierre Deligne and Kenneth A. Ribet. Values of abelian L-functions at negative integers over totally real fields. *Invent. Math.*, 59(3):227–286, 1980.

[Far11] Laurent Fargues. La filtration canonique des points de torsion des groupes $\mathfrak{p}$-divisibles. *Ann. Sci. Éc. Norm. Supér. (4)*, 44(6):905–961, 2011. Avec la collaboration de Yichao Tian.

[FC90] Gerd Faltings and Ching-Li Chai. *Degeneration of Abelian Varieties*, volume 22 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1990.

[Fon94] Jean-Marc Fontaine. Représentations ℓ-adiques potentiellement semi-stables. In Jean-Marc Fontaine, editor, *Périodes* p*-adiques*, number 223 in Astérisque, pages 321–347. Société Mathématique de France, 1994. Exposé VIII.

[FW79] Bruce Ferrero and Lawrence C. Washington. The Iwasawa invariant μ_p vanishes for abelian number fields. *Ann. of Math. (2)*, 109(2):377–395, 1979.

[Gan23] Wee Teck Gan. Automorphic forms and the theta correspondence. arXiv:2303.14918, 2023.

[GGP12] Wee Teck Gan, Benedict H. Gross, and Dipendra Prasad. Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups. In *Sur les conjectures de Gross et Prasad. I*, number 346 in Astérisque, pages 1–109. Société Mathématique de France, Paris, 2012.

[GK75] I. M. Gelfand and D. A. Kazhdan. Representations of the group $\mathrm{GL}(n, K)$ where K is a local field. In I. M. Gelfand, editor, *Lie Groups and Their Representations*, pages 95–118. Halsted Press, New York, 1975.

[GQT14] Wee Teck Gan, Yannan Qiu, and Shuichiro Takeda. The regularized Siegel–Weil formula (the second term identity) and the Rallis inner product formula. *Invent. Math.*, 198(3):739–831, 2014.

[GR13] Éric Gaudron and Gaël Rémond. Minima, pentes et algèbre tensorielle. *Israel J. Math.*, 195:565–591, 2013.

[Gra84] Daniel R. Grayson. Reduction theory using semistability. *Comment. Math. Helv.*, 59(4):600–634, 1984.

[Gro78] Benedict H. Gross. On the periods of abelian integrals and a formula of Chowla and Selberg. *Invent. Math.*, 45(2):193–211, 1978. With an appendix by David E. Rohrlich.

[Gro99] Benedict H. Gross. Algebraic modular forms. *Israel J. Math.*, 113:61–93, 1999.

[GS88] Henri Gillet and Christophe Soulé. Amplitude arithmétique. *C. R. Acad. Sci. Paris Sér. I Math.*, 307:887–890, 1988.

[Gui48] A. P. Guinand. A summation formula in the theory of prime numbers. *Proc. London Math. Soc. (2)*, 50:107–119, 1948.

[Had96] Jacques Hadamard. Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques. *Bulletin de la Société Mathématique de France*, 24:199–220, 1896.

[Har26a] Michael Harris. Mathematics as gift economy. Silicon Reckoner, November 20, 2026. `https://siliconreckoner.substack.com/p/mathematics-as-gift-economy`.

[Har26b] Michael Harris. Now all mathematicians can be Mechanical Turks. Silicon Reckoner, October 8, 2026. `https://siliconreckoner.substack.com/p/now-all-mathematicians-can-be-mechanical`.

[Hen00] Guy Henniart. Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique. *Invent. Math.*, 139(2):439–455, 2000.

[Hid81] Haruzo Hida. Congruences of cusp forms and special values of their zeta functions. *Invent. Math.*, 63(2):225–261, 1981.

[Hir64] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2)*, 79:109–203, 205–326, 1964.

[HN75] Günter Harder and Mudumbai Seshachalu Narasimhan. On the cohomology groups of moduli spaces of vector bundles on curves. *Math. Ann.*, 212(3):215–248, 1975.

[How79] Roger Howe. θ-series and invariant theory. In Armand Borel and William Casselman, editors, *Automorphic Forms, Representations and* L*-Functions,*

Part 1, volume 33 of *Proceedings of Symposia in Pure Mathematics*, pages 275–285. American Mathematical Society, Providence, RI, 1979.

[HT01] Michael Harris and Richard Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[IMS09] Yasutaka Ihara, V. Kumar Murty, and Mahoro Shimura. On the logarithmic derivatives of Dirichlet L-functions at $s = 1$. *Acta Arith.*, 137(3):253–276, 2009.

[Iwa59] Kenkichi Iwasawa. On Γ-extensions of algebraic number fields. *Bull. Amer. Math. Soc.*, 65(4):183–226, 1959.

[JS81a] Hervé Jacquet and Joseph A. Shalika. On Euler products and the classification of automorphic representations. I. *Amer. J. Math.*, 103(3):499–558, 1981.

[JS81b] Hervé Jacquet and Joseph A. Shalika. On Euler products and the classification of automorphic forms. II. *Amer. J. Math.*, 103(4):777–815, 1981.

[Kat04] Kazuya Kato. p-adic Hodge theory and values of zeta functions of modular forms. *Astérisque*, 295:ix, 117–290, 2004. Cohomologies p-adiques et applications arithmétiques. III.

[Kaw82] Yujiro Kawamata. A generalization of Kodaira–Ramanujam's vanishing theorem. *Math. Ann.*, 261(1):43–46, 1982.

[Kol90] V. A. Kolyvagin. Euler systems. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progress in Mathematics*, pages 435–483. Birkhäuser Boston, Boston, MA, 1990.

[Kot92] Robert E. Kottwitz. On the λ-adic representations associated to some simple Shimura varieties. *Invent. Math.*, 108(3):653–665, 1992.

[Kud84] Stephen S. Kudla. Seesaw dual reductive pairs. In Ichirô Satake and Yasuo Morita, editors, *Automorphic Forms of Several Variables (Katata, 1983)*, volume 46 of *Progress in Mathematics*, pages 244–268. Birkhäuser, Boston, MA, 1984.

[Kud94] Stephen S. Kudla. Splitting metaplectic covers of dual reductive pairs. *Israel J. Math.*, 87(1–3):361–401, 1994.

[Lab11] Jean-Pierre Labesse. Changement de base CM et séries discrètes. In Laurent Clozel, Michael Harris, Jean-Pierre Labesse, and Bao Châu Ngô, editors, *On the Stabilization of the Trace Formula*, volume 1, pages 429–470. International Press, Somerville, MA, 2011.

[Lan05] Edmund Landau. Über einen Satz von Tschebyschef. *Math. Ann.*, 61:527–550, 1905.

[Lea26] Lean and Mathlib communities. Lean verification of the Riemann Hypothesis. Lean community blog, `https://leanprover-community.github.io/blog/posts/riemann-hypothesis/`, 2026. November 21, 2026.

[Lic73] Stephen Lichtenbaum. Values of zeta-functions, étale cohomology, and algebraic K-theory. In Hyman Bass, editor, *Algebraic K-Theory, II: "Classical" Algebraic K-Theory, and Connections with Arithmetic*, volume 342 of *Lecture Notes in Mathematics*, pages 489–501. Springer, Berlin, 1973.

[Mas26] Alessandro Maria Masullo. The Music of the Primes. `https://math216.wordpress.com/wp-content/uploads/2026/11/the-music-of-the-primes.pdf`, 2026. Draft released November 20, 2026. Announced by Ravi Vakil on November 20, 2026, on the Math 216 course website: `https://math216.wordpress.com/2026/11/20/hurried-announcement-alessandro-masullo-grh/`.

[Maz89] Barry Mazur. Deforming Galois representations. In Yasutaka Ihara, Kenneth A. Ribet, and Jean-Pierre Serre, editors, *Galois Groups over* $\mathbf{Q}$, volume 16

of *Mathematical Sciences Research Institute Publications*, pages 385–437. Springer, New York, 1989.

[Mes72] William Messing. *The Crystals Associated to Barsotti–Tate Groups: With Applications to Abelian Schemes*, volume 264 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1972.

[Mil07] James S. Milne. The fundamental theorem of complex multiplication. arXiv:0705.3446, 2007.

[Mor09] Atsushi Moriwaki. Continuity of volumes on arithmetic varieties. *J. Algebraic Geom.*, 18(3):407–457, 2009.

[Mum77] David Mumford. Hirzebruch's proportionality theorem in the non-compact case. *Invent. Math.*, 42:239–272, 1977.

[MV07] Hugh L. Montgomery and Robert C. Vaughan. *Multiplicative number theory. I. Classical theory*, volume 97 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.

[MW89] Colette Mœglin and Jean-Loup Waldspurger. Le spectre résiduel de $\mathrm{GL}(\mathfrak{n})$. *Ann. Sci. École Norm. Supér. (4)*, 22(4):605–674, 1989.

[Nad90] Alan Michael Nadel. Multiplier ideal sheaves and Kähler–Einstein metrics of positive scalar curvature. *Ann. of Math. (2)*, 132(3):549–596, 1990.

[Pil84] Adolf Piltz. *Über die Häufigkeit der Primzahlen in arithmetischen Progressionen und über verwandte Gesetze*. A. Neuenhahn, Jena, 1884. Habilitationsschrift, Universität Jena.

[Poi67] Georges Poitou, editor. *Cohomologie galoisienne des modules finis*. Number 13 in Travaux et Recherches Mathématiques. Dunod, Paris, 1967. Séminaire de l'Institut de mathématiques de Lille, 1962–1963.

[Ral87] Stephen Rallis. L-*Functions and the Oscillator Representation*, volume 1245 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

[Rie59] Bernhard Riemann. Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse. *Monatsber. Königl. Preuss. Akad. Wiss. Berlin*, pages 671–680, 1859.

[Rub91] Karl Rubin. Kolyvagin's system of Gauss sums. In Gerard van der Geer, Frans Oort, and Joseph Steenbrink, editors, *Arithmetic Algebraic Geometry*, volume 89 of *Progress in Mathematics*, pages 309–324. Birkhäuser Boston, Boston, MA, 1991.

[Sch73] Wolfgang M. Schmidt. Zur Methode von Stepanov. *Acta Arith.*, 24(4):347–367, 1973.

[Sch13] Peter Scholze. The local Langlands correspondence for $\mathrm{GL}_{\mathfrak{n}}$ over $\mathfrak{p}$-adic fields. *Invent. Math.*, 192(3):663–715, 2013.

[Seg59] Irving Ezra Segal. Foundations of the theory of dynamical systems of infinitely many degrees of freedom. I. *Mat.-Fys. Medd. Danske Vid. Selsk.*, 31(12):1–39, 1959.

[Seg63] Irving Ezra Segal. Transforms for operators and symplectic automorphisms over a locally compact abelian group. *Math. Scand.*, 13:31–43, 1963.

[Sha62] David Shale. Linear symmetries of free boson fields. *Trans. Amer. Math. Soc.*, 103(1):149–167, 1962.

[Shi79] Goro Shimura. Automorphic forms and the periods of abelian varieties. *J. Math. Soc. Japan*, 31(3):561–592, 1979.

[Shi11] Sug Woo Shin. Galois representations arising from some compact Shimura varieties. *Ann. of Math. (2)*, 173(3):1645–1741, 2011.

[Sie35] Carl Ludwig Siegel. Über die analytische Theorie der quadratischen Formen. *Ann. of Math. (2)*, 36(3):527–606, 1935.

[Siu93] Yum-Tong Siu. An effective Matsusaka big theorem. *Ann. Inst. Fourier (Grenoble)*, 43(5):1387–1405, 1993.

[Sou03] Christophe Soulé. A bound for the torsion in the K-theory of algebraic integers. *Documenta Mathematica*, Extra Volume Kato:761–788, 2003.

[SS42] Raphaël Salem and Donald C. Spencer. On sets of integers which contain no three terms in arithmetical progression. *Proc. Natl. Acad. Sci. USA*, 28(12):561–563, 1942.

[ST61] Goro Shimura and Yutaka Taniyama. *Complex Multiplication of Abelian Varieties and Its Applications to Number Theory*, volume 6 of *Publications of the Mathematical Society of Japan*. Mathematical Society of Japan, Tokyo, 1961.

[Sta26] Stark units for real quadratic fields and reciprocity laws, 4 October 2026. Available at `https://www.danrad.net/papers/stark-reciprocity.pdf`.

[Ste69] S. A. Stepanov. On the number of points of a hyperelliptic curve over a finite prime field. *Math. USSR-Izv.*, 3(5):1103–1114, 1969.

[Sti90] Ludwig Stickelberger. Ueber eine Verallgemeinerung der Kreistheilung. *Math. Ann.*, 37(3):321–367, 1890.

[Stu76] Ulrich Stuhler. Eine Bemerkung zur Reduktionstheorie quadratischer Formen. *Arch. Math. (Basel)*, 27(6):604–610, 1976.

[Sut26] Andrew V. Sutherland. Verification files for the Lean proof of the Riemann Hypothesis. `https://math.mit.edu/~drew/grh/`, 2026. Last updated November 21, 2026.

[Tat63] John Tate. Duality theorems in Galois cohomology over number fields. In *Proceedings of the International Congress of Mathematicians (Stockholm, 1962)*, pages 288–295, Djursholm, 1963. Institut Mittag-Leffler.

[Tur49] Pál Turán. On a new method in the analysis with applications. *Časopis pro pěstování matematiky a fysiky*, 74(2):123–131, 1949.

[Tur50] Paul Turán. On the remainder-term of the prime-number formula, I. *Acta Math. Acad. Sci. Hungar.*, 1:48–63, 1950.

[TW95] Richard Taylor and Andrew Wiles. Ring-theoretic properties of certain Hecke algebras. *Ann. of Math. (2)*, 141(3):553–572, 1995.

[TY07] Richard Taylor and Teruyoshi Yoshida. Compatibility of local and global Langlands correspondences. *J. Amer. Math. Soc.*, 20(2):467–493, 2007.

[vH51] Léon van Hove. Sur certaines représentations unitaires d'un groupe infini de transformations. *Acad. Roy. Belgique Cl. Sci. Mém. Coll. in-8° (2)*, 26(6):1–102, 1951.

[Vie82] Eckart Viehweg. Vanishing theorems. *J. Reine Angew. Math.*, 335:1–8, 1982.

[Was97] Lawrence C. Washington. *Introduction to Cyclotomic Fields*, volume 83 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1997.

[Wei52] André Weil. Sur les "formules explicites" de la théorie des nombres premiers. *Communications du Séminaire Mathématique de l'Université de Lund*, pages 252–265, 1952. Tome supplémentaire.

[Wei64] André Weil. Sur certains groupes d'opérateurs unitaires. *Acta Math.*, 111:143–211, 1964.

[Wei72] André Weil. Sur les formules explicites de théorie des nombres. *Math. USSR-Izv.*, 6(1):1–17, 1972.

[Wei76] André Weil. Sur les périodes des intégrales abéliennes. *Comm. Pure Appl. Math.*, 29(6):813–819, 1976.

[Wei05] Rainer Weissauer. Four dimensional Galois representations. In Jacques Tilouine, Henri Carayol, Michael Harris, and Marie-France Vignéras, editors, *Formes automorphes (II): le cas du groupe* GSp(4), volume 302 of *Astérisque*, pages 67–150. Société Mathématique de France, 2005.

[Wey46] Hermann Weyl. *The Classical Groups: Their Invariants and Representations*, volume 1 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, second edition, 1946. First published in 1939.

[Wil95] Andrew Wiles. Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.

[Yua08] Xinyi Yuan. Big line bundles over arithmetic varieties. *Invent. Math.*, 173:603–649, 2008.

[Zha95] Shouwu Zhang. Positive line bundles on arithmetic varieties. *J. Amer. Math. Soc.*, 8(1):187–221, 1995.

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