Black Holes in Loop Quantum Gravity
Microstates and Hawking radiation

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From Loop Quantum Gravity...

Context : Loop Quantum Gravity

At the classical level

▷ Hamiltonian quantization of gravity : locally $\mathcal{M} = \Sigma \times T$

▷ Formulation : Ashtekar-Barbero first order gravity

▷ Partial gauge fixing (similar to ADM) : $SL(2,\mathbb{C})$ reduced to $SU(2)$

At the quantum level

▷ Hypothesis : states are one-dimensional excitations

▷ Consequences : non standard quantization but Diff-invariance

▷ Kinematical theory : the geometry (area and volume) is discrete

▷ Physical consequences : minimal length (UV cut-off), singularities resolution, and also statistical description of Black Holes

Open questions

▷ Quantum Dynamics ? Spin-Foams from TQFT...

▷ Barbero-Immirzi parameter γ ? Relevance at the quantum level...

▷ Vacuum ? How classical geometry emerges from LQG ?

... To Quantum Black Holes

Relation to Chern-Simons theory

Classical correspondence between CS and (spherical) BH

- ▷ Symplectic geometry is those of a Chern-Simons theory
- ▷ $SU(2)$ Gauge group and the level (coupling constant) $k \propto a_H$
- ▷ Manifold : a two-sphere with arbitrary number of punctures

Quantization is very well-known

- ▷ Hilbert space of quantum states from quantum group $U_q(su(2))$
- ▷ Dimension is finite and explicit (rather simple) formula

Thermodynamics of Black Holes

- ▷ Black Hole entropy : $S = a_H/4 - 3/2 \log a_H$ in Planck units
- ▷ Problems : γ fixed at quantum level and distinguishable punctures !
- ▷ No Hawking radiation, no temperature... Up to recent results

Our recent results

- ▷ γ is no more relevant : $\gamma = \pm i$ and $SL(2,\mathbb{C})$ gauge group
- ▷ Quantum version of Hawking (local) radiation

Overview

1. Loop Quantum Gravity in a nut shell

- *Why does ADM canonical quantization fail ?*
- *From Ashtekar gravity...*
- *... To kinematical quantum states*
- *Physical interpretation : discrete geometry*

2. Black Holes in LQG: a quick review

- *Heuristic picture*
- *Relation to Chern-Simons theory*

3. Complex variables and Hawking radiation

- *Back to complex variables*
- *The new Black Hole partition function*
- *Hawking radiation*

Why does ADM canonical quantization fail ?

Lagrangian formulation : M is the 4D space-time

▷ Einstein-Hilbert action : functional of the metric g

$$S_{EH}[g] = \int d^4x \sqrt{|g|} R$$

Hamiltonian formulation : $M = \Sigma \times T$ ('61)

▷ ADM variables : $ds^2 = N^2 dt^2 - (N^a dt + h_{ab} dx^b)(N^a dt + h_{ac} dx^c)$

▷ ADM action : (h, π) canonical variables

$$S_{ADM}[h, \pi; N, N^a] = \int dt \int d^3x (\dot{h}\pi + N^a H_a[h, \pi] + NH[h, \pi])$$

▷ Constraints $H = 0 = H_a$ generate the diffeomorphisms

What about the quantization ?

▷ Highly non linear constraints : quantum ambiguities and no solutions

▷ Huge symmetry group : how to take it into account ?

Starting point : first order formulation of gravity

- A tetrad e^I_μ (4×4 matrix) such that $g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ}$
- a $so(3,1)$ spin-connection ω^{IJ}_μ related to Levi-Civitta connection
- First order Hilbert-Palatini action

$$S_{HP}[e,\omega] = \int \langle \star(e \wedge e) \wedge F(\omega) \rangle$$

- Canonical analysis leads to second class constraints : problematic !

The Ashtekar variables ('86)

- Restrict ω to be (anti) self-dual : $\star\omega^{\pm} = \pm i\omega^{\pm}$ and $S_A = S_{HP}[e,\omega^{\pm}]$
- No more second class constraints : right number of d.o.f.
- Classically equivalent to Einstein-Hilbert theory
- Complex variables ($\gamma = \pm i$) : $E^a = \epsilon^{abc} e_b \times e_c$ and $A^i_a = \omega^i_a + \gamma\omega^{0i}_a$
- Pair of canonical variables :

$$\{A^i_a(x), E^b_j(y)\} = (8\pi\gamma G)\delta^b_a \delta^i_j \delta^3(x,y)$$

The Barbero-Immirzi parameter

The Constraints become polynomials of A and E

- ▷ Gauss constraint $G = D_a E^a$: complex $SL(2,\mathbb{C})$ gauge symmetry
- ▷ Vectorial constraint $H_a = E^b \cdot F_{ab}$: space diffeomorphisms
- ▷ Scalar constraint $H = E^a \times E^b \cdot F_{ab}$: time reparametrizations
- ▷ BUT... No one knows how to deal with complex variables

The Immirzi-Barbero parameter γ

- ▷ Real γ : parametrizes a family of canonical transformations
- ▷ Now an $SU(2)$ connection : Ashtekar-Barbero connection
- ▷ Everything formally unchanged but H is no more a polynomial

$$H = E^a \times E^b \cdot (F_{ab} + (\gamma^2 + 1) R_{ab})$$

- ▷ Lagrangian formulation : the Holst action

$$S_{HP}[e,\omega] = \int \langle \star(e \wedge e) \wedge F(\omega)\rangle + \frac{1}{\gamma}\langle e \wedge e \wedge F(\omega)\rangle$$

- ▷ Kind of "Wick" rotation : gauge group becomes compact $SU(2)$

Polymer states hypothesis

Classical phase space of Ashtekar gravity :

- Phase space : $\mathcal{P} = T^*(\mathcal{A})$ with $\mathcal{A} = \{SU(2)\, connections\}$
- Holonomy-flux algebra associated to edges e and surfaces S

$$A(e) = P\exp(\int_e A) \quad \text{and} \quad E_f(S) = \int_S \mathrm{Tr}(f \star E)\,.$$

- Cylindrical functions : $f \in \mathrm{Cyl}$ is a function of $A(e)$ with $e \subset \gamma$
- $E_f(S)$ acts as a vector field on f if $S \cap \gamma \neq 0$.

Action of symmetries : $\mathcal{S} = \mathcal{G} \ltimes Diff(\Sigma)$ **with** $\mathcal{G} = C^\infty(\Sigma, SU(2))$

- Gauss constraint : $f(A(e)) \longmapsto f(g(s(e))^{-1}A(e)g(t(e)))$
- Diffeomorphisms : $f(A(e)) \longmapsto f(A(\varphi(e)))$
- Similar action for the variables $E_f(S)$
- Symmetries are automorphisms of classical algebra

Unicity of a (space) Diff-invariant representation

Construction of the quantum algebra $\mathfrak{A}$

- $\triangleright$ Elements of $\mathfrak{A}_{cl}$ are $a = (f, u)$: $f \in$ Cyl and u derivatives
- $\triangleright$ Ideal $\mathfrak{I}$ of the algebra : $a_1 a_2 - a_2 a_1 - i\hbar\{a_1, a_2\}$
- $\triangleright$ $\mathfrak{A} = \mathfrak{A}_{cl}/\mathfrak{I}$ with action of automorphisms $\mathcal{S}$

Representation theory of $\mathfrak{A}$: GNS framework

- $\triangleright$ Any representation of $\mathfrak{A}$ is a direct sum of cyclic representations
- $\triangleright$ A cyclic representation is characterized by a positive state $\omega \in \mathfrak{A}^*$
- $\triangleright$ The representation : $(\mathcal{H}, \pi, \Omega)$ with Ω cyclic : $\pi(\mathfrak{A})\Omega$ dense in $\mathcal{H}$
- $\triangleright$ ω is required to be invariant under $\mathcal{S}$
- $\triangleright$ LOST Theorem : The representation is unique !

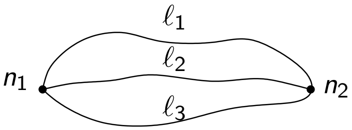
Properties of the representation :

- $\triangleright$ Irreducible unitary infinite dimensional representation
- $\triangleright$ Hilbert space non separable : non countable basis
- $\triangleright$ Action of $Diff(\Sigma)$ is not weakly continuous
- $\triangleright$ Stone : Infinitesimal generators of $Diff(\Sigma)$ do not exist

Spin-Network basis

Kinematical states : basis of spin-networks

▷ They are generalizations of Wilson loops with nodes



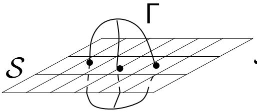
ℓ_i are oriented links

n_i are nodes

▷ Harmonic analysis on $SU(2)$: $\ell \to$ irreps and $n \to$ intertwiners

Geometric operators : area and volume become operators

▷ Area acts on edges and Volume on vertices

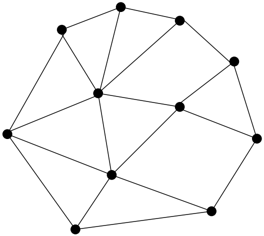


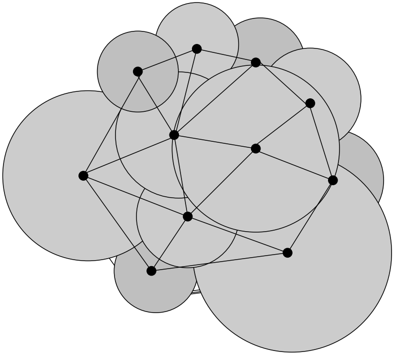
$$\mathcal{A}(\mathcal{S})|S\rangle = \frac{8\pi\gamma\hbar G}{c^3} \sum_{P\in\mathcal{S}\cap\Gamma} \sqrt{j_P(j_P+1)}|S\rangle$$

▷ The spectra are discrete : existence of a minimal length

Picture of space at the Planck scale

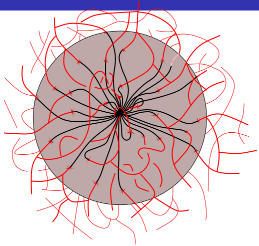
From the kinematics, Space is discrete...





▷ Edges carry quanta of area, nodes carry quanta of volume

Heuristic picture : model for real γ



$a_H = 8\pi\gamma\ell_P^2 \sum_j \sqrt{j(j+1)}$
Edges crossing spherical BH

Only spins $1/2$ contribute to the area

- Number of edges : $a_H = 8\pi\gamma\ell_P^2 \times \mathbf{N} \times \frac{\sqrt{3}}{2}$
- Number of states : number of singlets in $(1/2)^{\otimes N} \Longrightarrow \boldsymbol{\Omega} \sim 2^N$
- Bekenstein-Hawking formula for the entropy when $a_H \gg \ell_P^2$

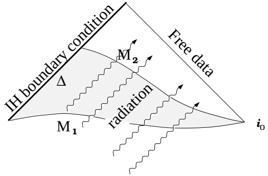
$$S \;=\; \log(\Omega) \;\sim\; N\log(2) \;=\; \frac{2\log(2)}{8\pi\gamma\ell_P^2\sqrt{3}} a_H \;\Longrightarrow\; \gamma = \frac{\log(2)}{\pi\sqrt{3}} \,.$$

Refined models : all spins contribute

- The value of γ changes. Why Is γ relevant at the quantum level ?

Hamiltonian formulation of Black Holes

The Horizon considered as a boundary



- ▷ Geometric conditions : null-surface, no expansion, $F \propto E$
- ▷ Restriction (here) : spherically symmetric black holes

Sympletic structure with a horizon boundary

- ▷ Requirement : conservation of symplectic structure
- ▷ In terms of Ashtekar-Barbero variables

$$8\pi G\gamma\,\omega(\delta_1,\delta_2) = \int_M \delta_{[1}E^i \wedge \delta_{2]}A_i - \frac{a_H}{\pi(1-\gamma^2)}\int_H \delta_1 A_i \wedge \delta_2 A^i$$

- ▷ Symplectic structure of $SU(2)$ Chern-Simons theory with $k \propto a_H$

$SU(2)$ Chern-Simons Theory

The Black Hole in LQG

- ▷ Described by a CS theory on a N-punctured sphere

Topological Gauge Field Theory of a connection A

- ▷ Action : $S(A) = \frac{k}{2\pi}\int \langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \rangle$
- ▷ Gauge symmetries : $A \longmapsto g^{-1}Ag + g^{-1}dg$
- ▷ Equations of motion : $F(A) = dA + A \wedge A = 0$

Generalization to the presence of punctures

- ▷ Each puncture associated to a world-line
- ▷ The world-line carries a "momentum" : $j \in \mathfrak{su}(2)$
- ▷ Singularities : $F(A) = j\,\delta_P(x)$

Path integral quantization

- ▷ $Z = \int [\mathcal{D}A] e^{iS[A]} \Longrightarrow k \in \mathbb{Z}$
- ▷ Deep relation with 3-manifolds and knots invariants
- ▷ Relation to CFT and Quantum Groups : $U_q(\mathfrak{su}(2))$ $q = e^{i\pi/(k+2)}$

Physical Hilbert Space

Classical Hamiltonian analysis : Space-time $\Sigma \times \mathbb{R}$

- Action : $S[A] = \frac{k}{2\pi} \int (\epsilon^{ab} A_a^i \partial_0 A_{bi} + A_0^i F_{abi})$
- Symplectic structure : $\{A_a^i(x), A_b^j(y)\} = \frac{2\pi}{k} \epsilon_{ab} \delta^{ij} \delta(x,y)$
- First class constraints : spatial curvature $F = 0$
- In the presence of a puncture P : $F = p\delta_P(x)$

Physical Hilbert space : sphere with N punctures

- Representation theory of $U_q(\mathfrak{su}(2))$: spin $j \leq k/2$ with $d_j = 2j+1$
- Hilbert space (Combinatorial quantization or CFT) :

$$\mathcal{H}(j_1, \cdots, j_n) \; = \; \mathrm{Inv}(V_{j_1} \otimes \cdots \otimes V_{j_N}) \, .$$

- Finite dimension (Verlinde formula)

$$\frac{2}{k+2} \sum_{\ell=0}^{k/2} (\sin(\frac{\pi d_\ell}{k+2}))^{2-N} \prod_{i=1}^{N} \sin(\frac{\pi d_\ell d_{j_i}}{k+2}) \, .$$

Black Holes with real γ

Black Hole micro states

- They are $U_q(\mathfrak{su}(2))$ invariant tensors (recoupling incoming spins)
- Physical Hilbert space of dimension $N(a_H)$

$$\mathcal{H}_{BH} = \{\bigotimes_n \bigotimes_{j_1,\cdots,j_n} \mathcal{H}(j_1,\cdots,j_n) | a_H = 8\pi\gamma\ell_P^2 \sum_i \sqrt{j_i(j_i+1)}\}$$

Recovering Bekenstein-Hawking entropy

- Punctures are distinguishable... WHY ?
- The Laplace transform can be easily computed :

$$\tilde{N}(s) = \int_0^\infty da\; e^{-sa} N(a)$$

- Studying the structure of its singularities gives :

$$S(a_H) = \frac{a_H}{4\ell_P^2} - \frac{3}{2}\log(\frac{a_H}{\ell_P^2}) + \mathcal{O}(1)$$

- Provided that γ fixed to some special value... WHY ?

Back to the complex Ashtekar variables

Continuation to $\gamma = i$

- The level k becomes imaginary
- It should correspond to CS theory with $SL(2,\mathbb{C})$ gauge group
- New partition function for CS theory with $\lambda = |k|$

$$\mathcal{Z} \simeq \frac{2}{\lambda} \sum_{d=1}^{\lambda} \sinh^2(\frac{\pi d}{\lambda}) \prod_{i=1}^{N} \frac{\sinh(\frac{\pi d}{\lambda}(2j_i+1))}{\sinh(\frac{\pi d}{\lambda})} \, .$$

The semi-classical limit is immediate

- Double scaling limit : large spin $j_i \to \infty$ and $\ell_P \to 0$ s.t. $\ell_P^2 j_i \to \ell_i$
- No more oscillations and $S = \log \mathcal{Z} = \frac{a_H}{4\ell_P^2} + \cdots$ with $a_H = 8\pi \sum_i \ell_i$

The degrees of freedom of the Black Hole are described in terms of

- (anti) Self-dual representations of the Lorentz group
- They may be continuous representations
- No need to invoke distinguishability anymore

Black Hole temperature and radiation

Evidences of the Hawking radiation

- Point of view of local observer (acceleration a) at vicinity of face i

$$P_i(j \to k) = \frac{\mathcal{Z}(j)}{\mathcal{Z}(j) + \mathcal{Z}(k)} \simeq \frac{1}{1 + \exp(\beta \Delta E)}$$

- Unruh temperature $\beta = 2\pi/a$ and Local energy $E = Aa/8\pi$

Partition function beyond the leading order term

$$\mathcal{Z} \simeq \frac{2\sinh^2 \pi}{\lambda} \prod_{i=1}^{N} \left(\sum_{m=0}^{\infty} \exp(-\beta E_m^{(j_i)}) \right)$$

- The energy spectrum is the energy of an accelerated observer

$$E_m^{(j)} = \langle j, m | aK | j, m \rangle = (m - j)a$$

- Locally, thermalized states at β : probably gravitational dof
- The vacuum has a negative energy and responsible for the huge

Pair creation from LQG point of view

Vacuum of Loop Quantum Gravity

- ▷ Our analysis : BH d.o.f. are self-dual representations
- ▷ From SF models, there is one vacuum given by

$$|(\rho, k); (j, m)\rangle \in \mathbb{V}_{\rho,k} \; SL(2,\mathbb{C}) \text{ principal series}$$

- ▷ For a fixed j, unique solution of the constraints : $\rho(j)$, $k(j)$ and $m = j$

Decomposition into self-dual anti-self-dual representation of $|0\rangle$

- ▷ Quantum analogue of L/R wedge decomposition : pair creations

$$|0\rangle = \sum_{m_\pm} C(m_\pm)|m_+\rangle \otimes |m_-\rangle$$

- ▷ Trace over one half leads to the density matrix

$$\hat{\rho} = Tr_-(|0\rangle\langle 0|) = \sum_{m_+} |C(m_+)|^2 |m_+\rangle\langle m_+|$$

- ▷ Where the complicated function $|C(m)|^2 \simeq \exp(-\beta E(m))$

Conclusion

LQG in a nut shell

- ▷ Kinematical States are labelled by topological graphs
- ▷ Geometrical operators have discrete spectra
- ▷ The quantum dynamics is still under construction

What do Quantum Black Holes teach us ?

- ▷ The Barbero-Immirzi parameter should return to $\gamma = i$
- ▷ The LQG dof contain the graviton (at least close to a BH horizon)

Effective description of a Quantum Black Hole

- ▷ Thermalized graviton at Unruh temperature for a local observer
- ▷ The vacuum has a negative energy
- ▷ The energy of the vacuum is responsible for the entropy : degeneracy

Next steps

- ▷ Relation to CFT (via CS Theory) : making sense to Cardy formula
- ▷ Far away point of view : true Hawking radiation ?