Ultra-high-accuracy computation of gravitational self-force observables in binary black holes

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Different analytical and numerical techniques to study binaries

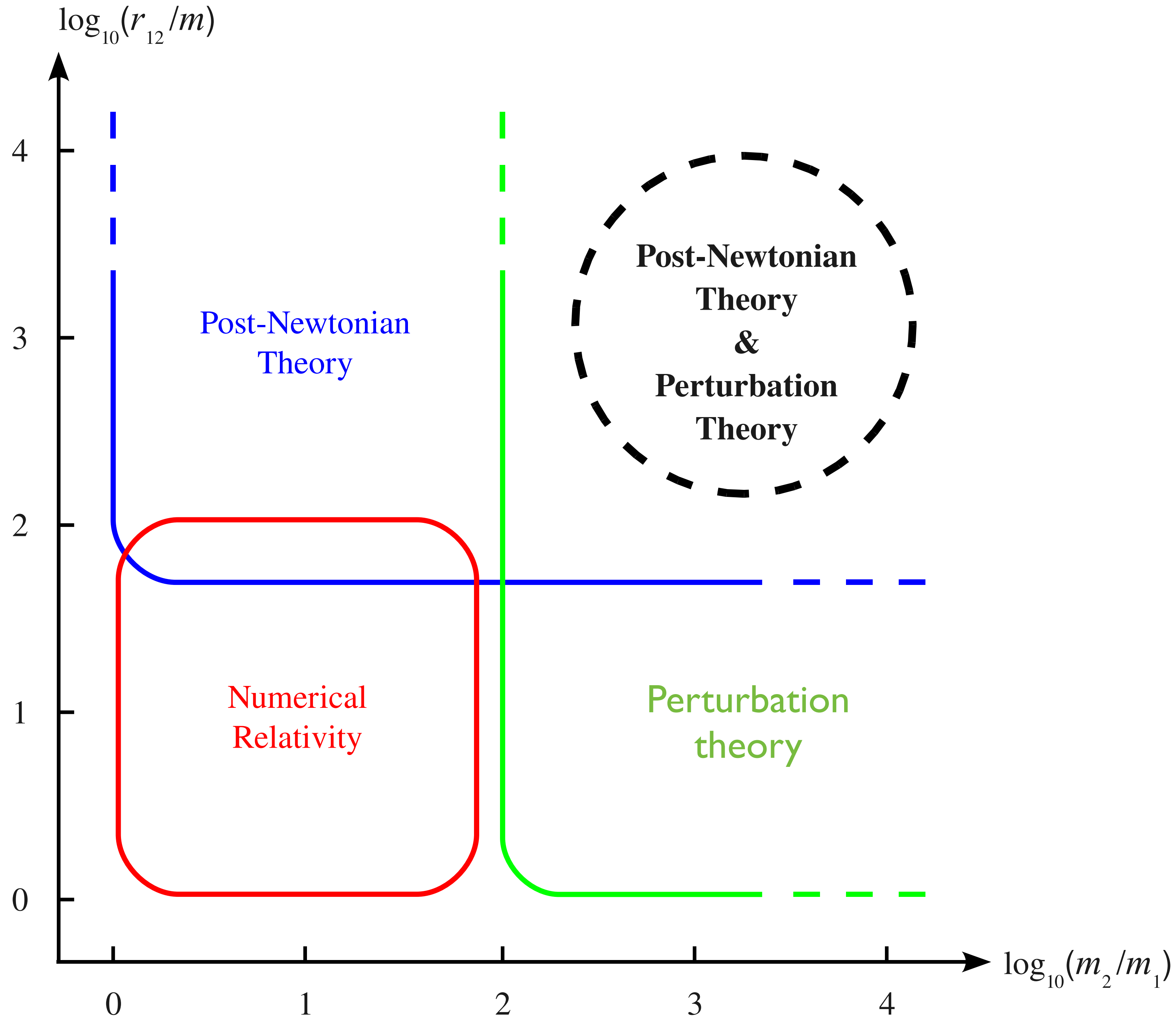
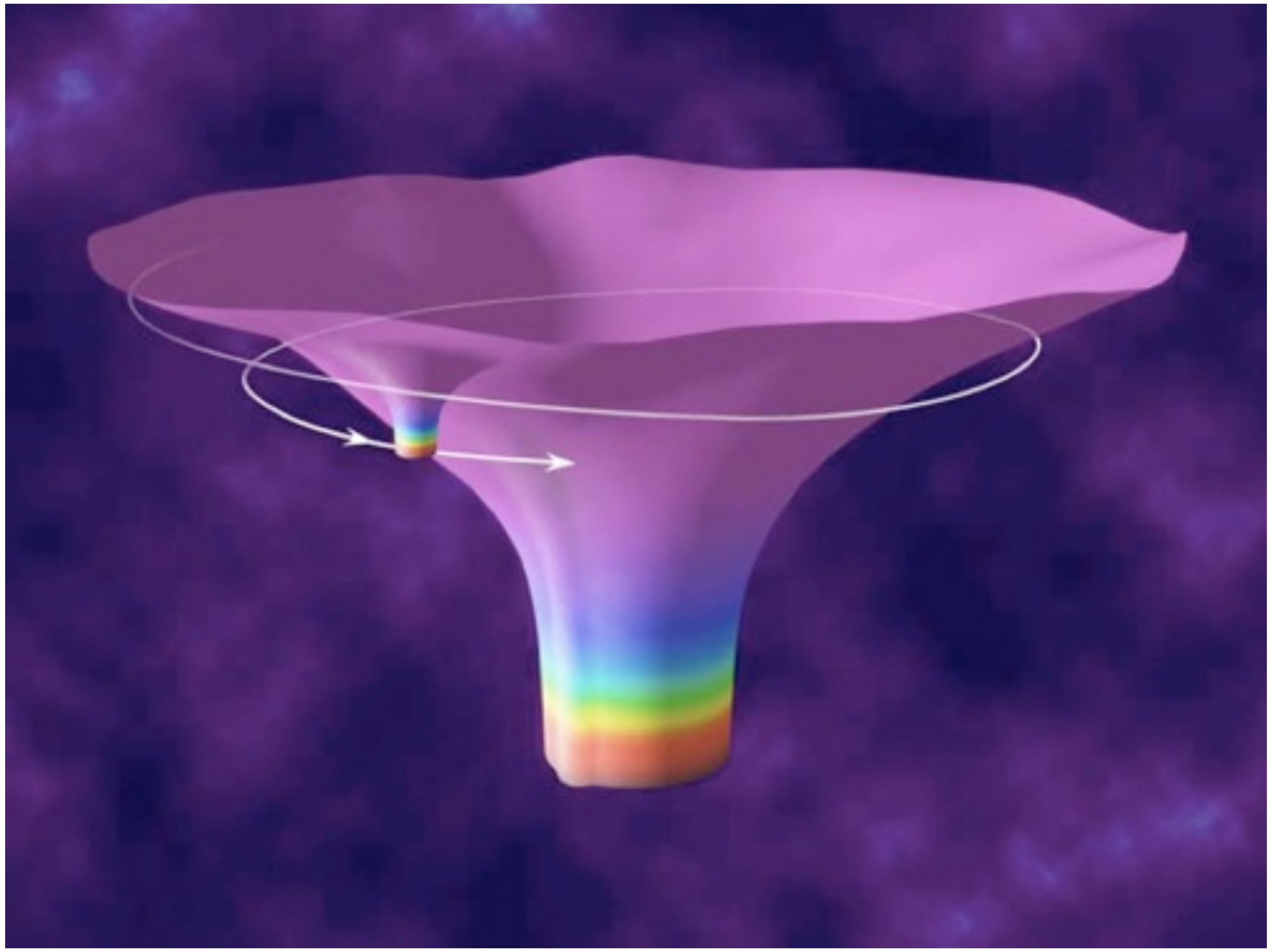


Fig: Blanchet-Detweiler-LeTiec-Whiting

Perturbation theory or self-force formalism

- A small body is orbiting a large body where the mass ratio is ~10^3 and higher (a stellar-mass black hole or neutron star orbiting a galactic supermassive black hole).



Perturbation theory or self-force formalism

- A small body is orbiting a large body where the mass ratio is $\sim 10^3$ and higher (a stellar-mass black hole or neutron star orbiting a galactic supermassive black hole).
- In the limit the mass of the small body goes to zero $\longrightarrow$ test-mass $\longrightarrow$ geodesic of background.

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- In the limit the mass of the small body goes to zero $\longrightarrow$ test-mass $\longrightarrow$ geodesic of background.
- For non-zero but small mass, it perturbs the background, this perturbation acts back on itself, it experiences a self-force and its world-line deviates from the background geodesic.

Different analytical and numerical techniques to study binaries

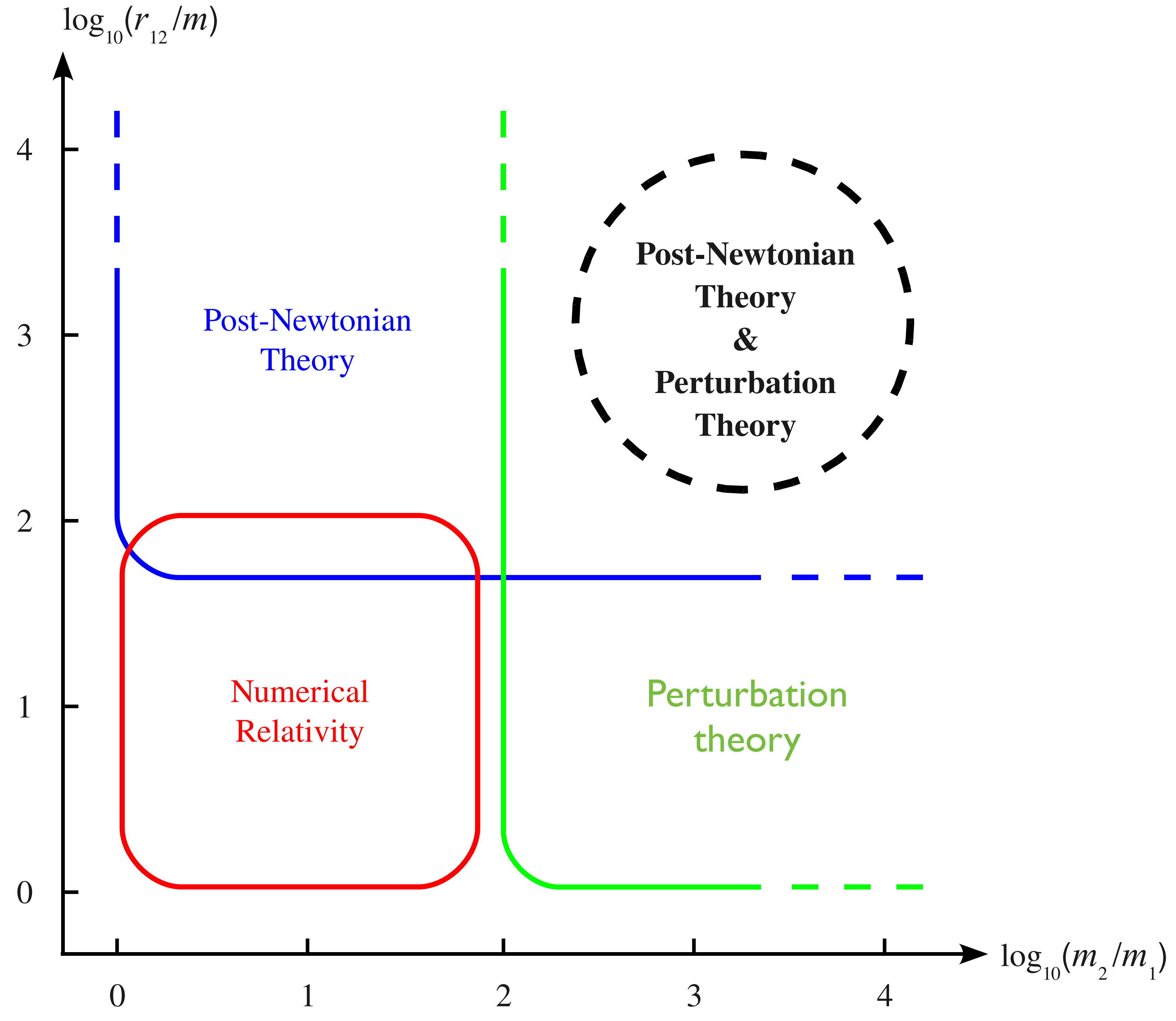
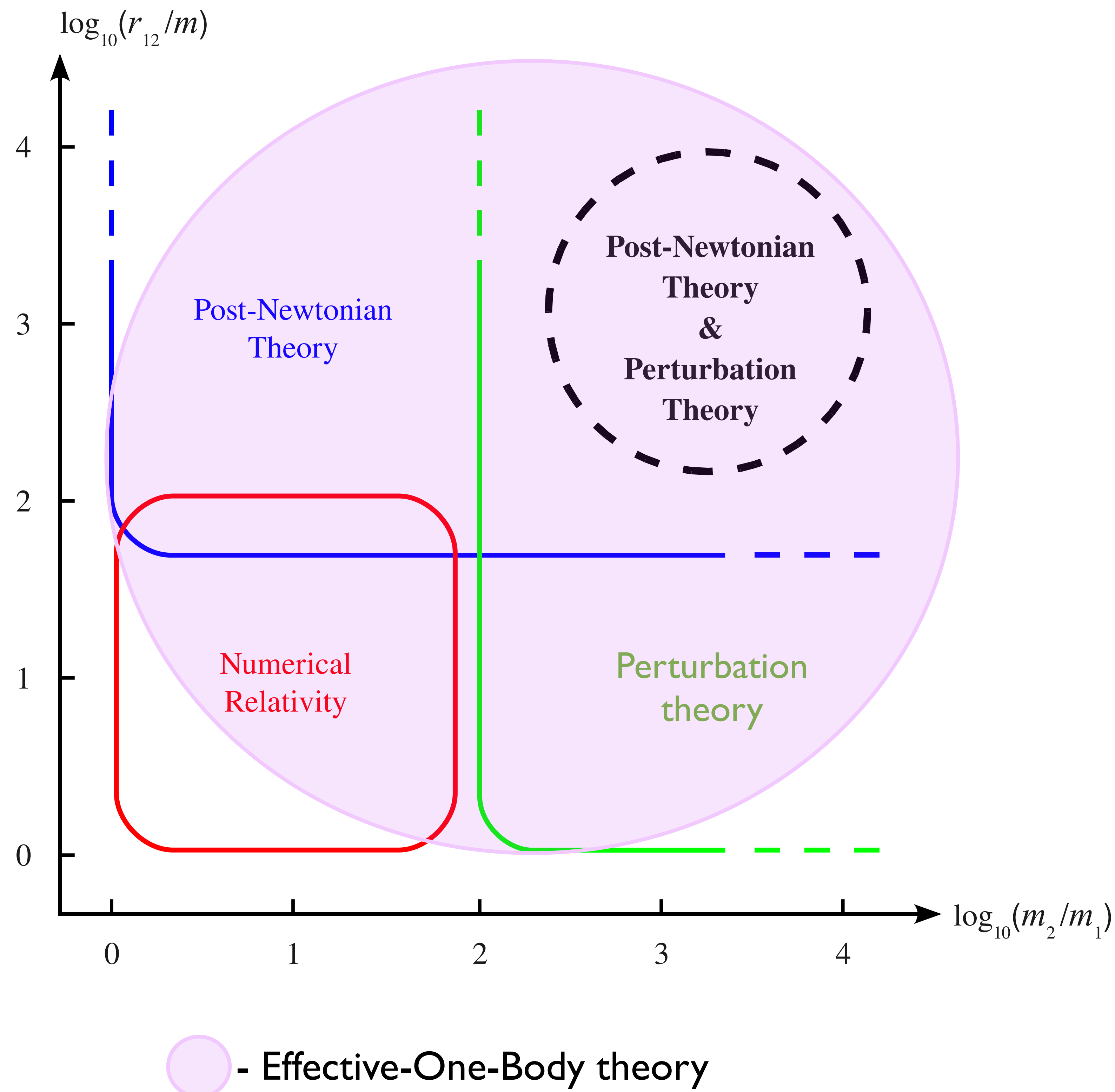
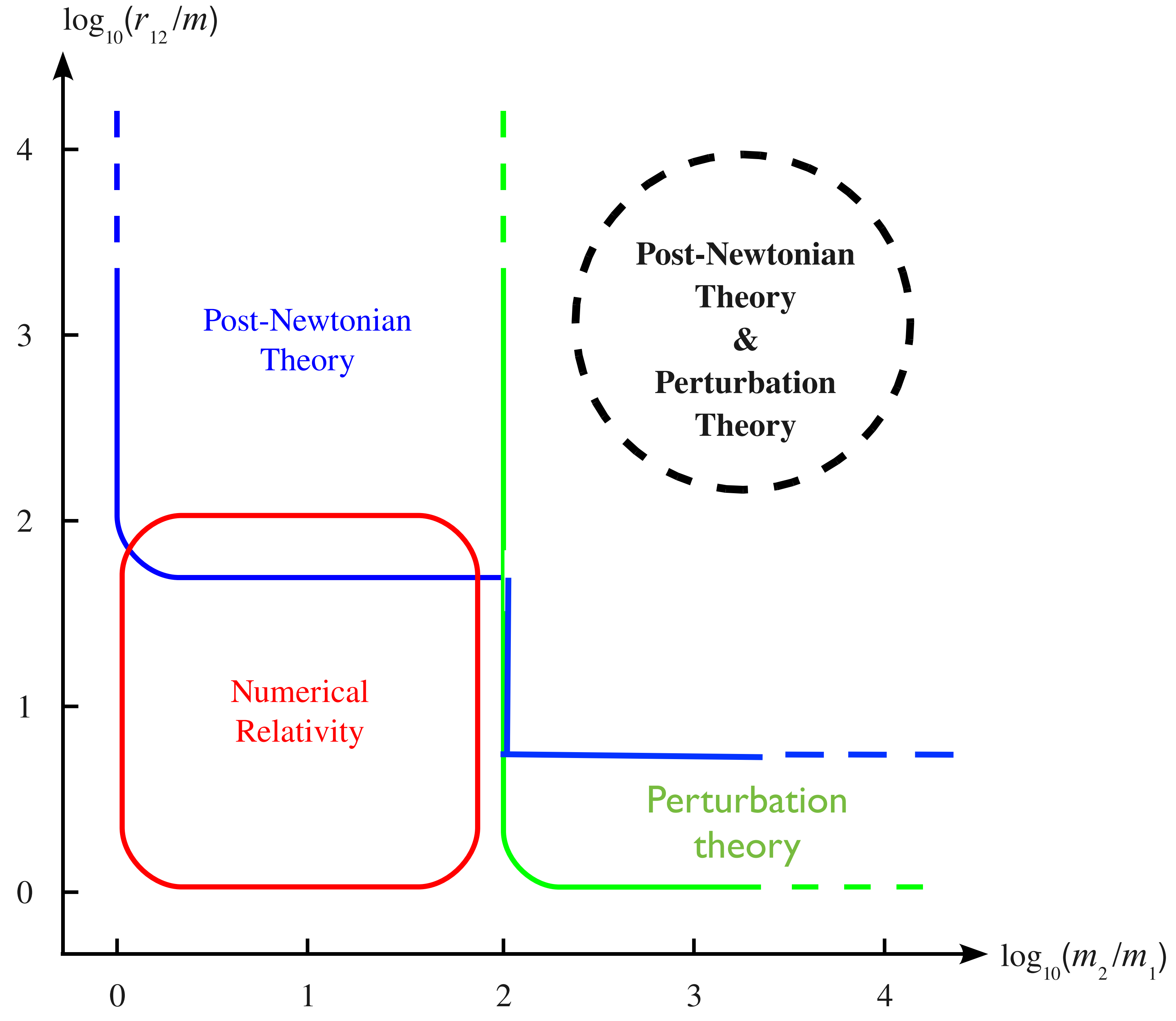


Fig: Blanchet-Detweiler-LeTiec-Whiting

Different analytical and numerical techniques to study binaries



Different analytical and numerical techniques to study binaries



Topics that will be covered

Gauge-invariant quantity, ΔU, (reciprocal of the red-shift invariant, z), the 1^{st} order (in mass-ratio) change in u^t.

$$u^\alpha u^\beta (g_{\alpha\beta} + h^{\mathrm{R}}_{\alpha\beta}) = 1$$

$$u^\alpha = [u_0^t + u_1^t + O(\mu^2)] k^\alpha$$

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ΔU for circular orbits or $\langle \Delta U \rangle$ for eccentric orbits uses the renormalized metric perturbation, $h^R_{\alpha\beta}$, dotted with background geodesic 4-velocity.

ΔU

1. Useful quantity to check the computations done in different gauges.

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4. ISCO shift (from ΔU and its derivatives) is used as a reference point for comparison with analytical and numerical methods.

Metric perturbation ($h_{\alpha\beta}$) in a modified radiation gauge

System of 10 coupled, 2^{nd} order PDEs in Lorenz gauge

↓

Here we only need to solve one, separable, 2^{nd} order PDE.

The PDE for $\dot{\psi}_0$ or $\dot{\psi}_4$.

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Radiative part of the full $h_{\alpha\beta}$ is extracted from the perturbed Weyl scalars, $\dot{\psi}_0/\dot{\psi}_4$.

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$$\dot{\psi}_0 = -\dot{R}_{\alpha\beta\gamma\delta}\ell^\alpha m^\beta \ell^\gamma m^\delta \qquad \dot{\psi}_4 = -\dot{R}_{\alpha\beta\gamma\delta}n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$$

$(\ell^\alpha, n^\alpha, m^\alpha, \bar{m}^\alpha)$ make a null-tetrad.

(ℓ^α, n^α) are real, outgoing- and ingoing-null vectors.

m^α is a complex null vector orthogonal to (ℓ^α, n^α).

These perturbed Weyl scalars, $\dot{\psi}_0/\dot{\psi}_4$ are invariant under gauge transformations and infinitesimal tetrad rotations.

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And the non-radiative part, which corresponds to the change in mass (δM) and angular-momentum (δJ) of spacetime, is then added in a convenient gauge.

Teukolsky equation

The equation that describes the dynamical perturbation, the Teukolsky equation, has the form

$$\mathcal{O}\,\mathcal{T}(h) = \mathcal{S}\,\mathcal{E}(h).$$

And $h_{\alpha\beta}$ is what we want to extract.

Teukolsky equation

$$\mathcal{O}\,\boxed{\mathcal{T}(h)} = \mathcal{S}\,\mathcal{E}(h)$$

$\dot{\psi}_0$ or $\dot{\psi}_4$

($\dot{\psi}$ for brevity)

Teukolsky equation

$$\mathcal{O}\,\mathcal{T}(h) = \mathcal{S}\,\boxed{\mathcal{E}(h)}$$

Einstein operator acting on $h = 8\pi\, T_{\mu\nu}$

Teukolsky equation

$$\mathcal{O}\,\mathcal{T}(h) = \boxed{\mathcal{S}}\,\mathcal{E}(h)$$

2^{nd} order derivative operator acting on $T_{\mu\nu}$

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2^{nd} order derivative operator acting on $\dot{\psi}$

Teukolsky equation

Newman-Penrose equations (Ricci & Bianchi identities):

Derivative operators acting on Weyl scalars
= Derivative operators acting on Ricci tensor

A combination of them gives us:

Derivative operators acting on $\dot{\psi}$
= Derivative operators acting on $R_{\mu\nu} \propto T_{\mu\nu}$

$$\mathcal{O}\,\mathcal{T}(h) = \mathcal{S}\,\mathcal{E}(h)$$

Separability of the Teukolsky equation

$$\dot{\psi} = R(r)\, e^{-i\omega t} S(\theta)\, e^{im\phi}$$

$$_s\dot{\psi} = \sum_{\ell,m,\omega} {}_sR_{\ell,m,\omega}(r)\, e^{-i\omega t}\; {}_sS^{a\omega}_{\ell,m}(\theta)\, e^{im\phi}$$

Ordinary 2^{nd} order ODE for
the radial part and the angular part

The solution to both these equations can be
written as a sum over known analytical functions.

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$R(r)$ is written as a sum over hypergeometric, and confluent hypergeometric functions

$S(\theta)$ is written as a sum over spin-weighted spherical harmonics.

Finding h from $\dot{\psi}$

Theorem: Suppose $\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}$ holds,

and suppose Ψ satisfies $\mathcal{O}^\dagger \Psi = 0$.

If $\mathcal{E}$ is self-adjoint, then $\mathcal{S}^\dagger \Psi$ satisfies $\mathcal{E}(f) = 0$.

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Proof: Taking adjoint of $\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}$, gives us

$$\mathcal{E}^\dagger\mathcal{S}^\dagger = \mathcal{T}^\dagger\mathcal{O}^\dagger$$

$$\mathcal{E}\mathcal{S}^\dagger = \mathcal{T}^\dagger\mathcal{O}^\dagger$$

If $\mathcal{O}^\dagger\Psi = 0$, then $\mathcal{E}(\mathcal{S}^\dagger\Psi) = 0$, i.e., $h = \mathcal{S}^\dagger\Psi$

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Lets substitute $\mathcal{S}^\dagger\Psi$ back into the Teukolsky equation,

$$\begin{aligned} \mathcal{S}\mathcal{E}(\mathcal{S}^\dagger\Psi) &= \mathcal{O}\mathcal{T}(\mathcal{S}^\dagger\Psi) \\ 0 &= \mathcal{O}[\mathcal{T}\mathcal{S}^\dagger\Psi] \end{aligned}$$

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↑ Intermediate Hertz potential ↑

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↑ Intermediate Hertz potential ↑

First step: Solve for $\dot{\psi}$ (Teukolsky equation).

Second step: Invert to find Ψ

$$\dot{\psi} = \mathcal{T}\mathcal{S}^{\dagger}\Psi \qquad h = \mathcal{S}^{\dagger}\Psi$$

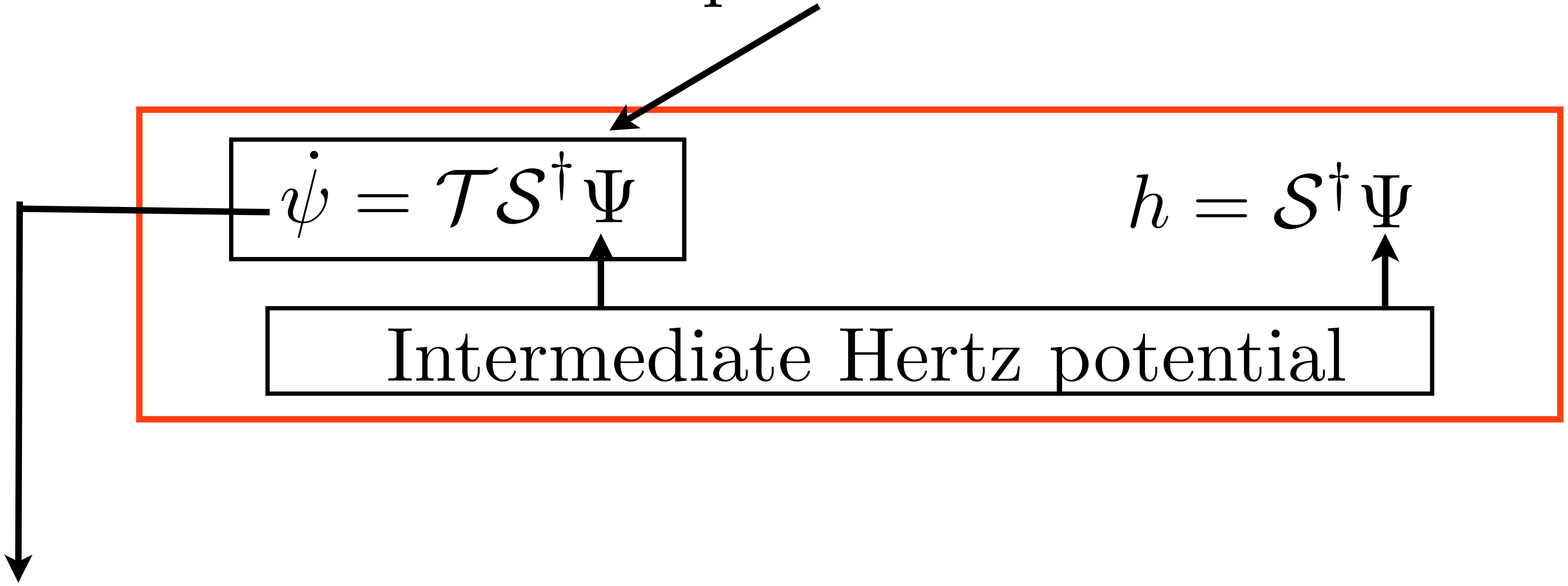
Intermediate Hertz potential

In frequency-domain the operator $\mathcal{T}\mathcal{S}^{\dagger}$ are almost the same as the ones in Teukolsky-Starobinsky identities

For circular/spherical orbits, the inversion is algebraic.

Radial mode of Ψ = constant $\times$ radial mode of $\dot{\psi}$

Second step: Invert to find Ψ



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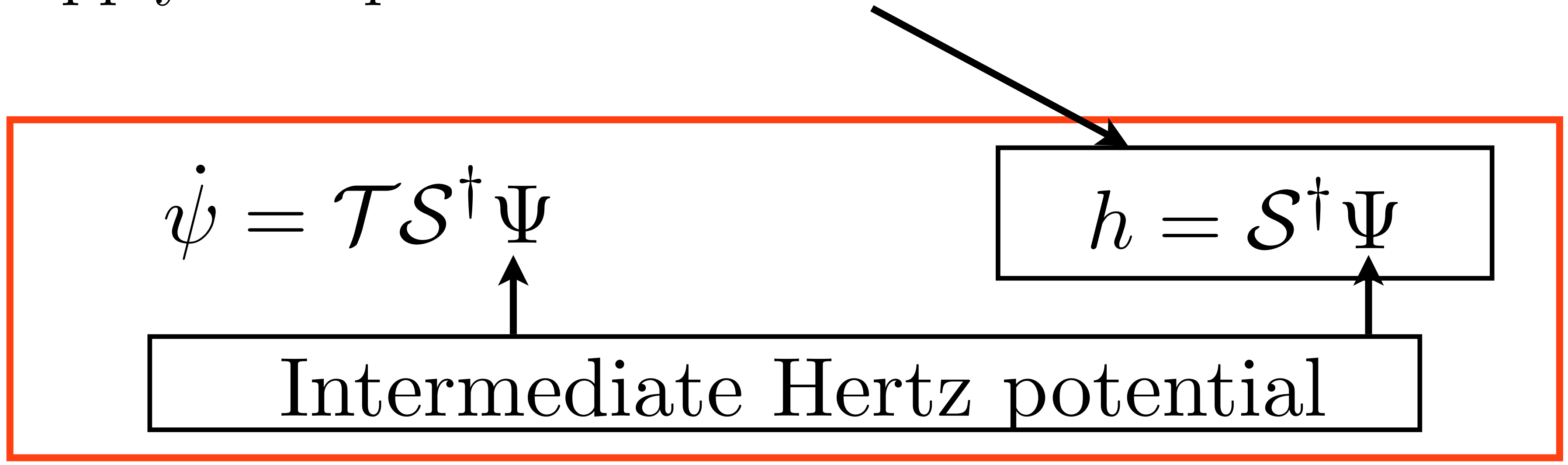
For generic orbits, the formalism for inversion in frequency-domain has been developed by A. Ori.

And its application is in progress for eccentric, equatorial orbits in Kerr (MVD Meent and AG Shah)

Third step: Apply the operator $\mathcal{S}^\dagger$ on Ψ to recover the radiative $h_{\alpha\beta}$



$$\dot{\psi} = \mathcal{T}\mathcal{S}^{\dagger}\Psi \qquad\qquad h = \mathcal{S}^{\dagger}\Psi$$

Intermediate Hertz potential

Fourth step: Add to this the $h_{\alpha\beta}$ that corresponds to the change in mass and angular momentum of the spacetime.

$$\dot{\psi} = \mathcal{T}\mathcal{S}^{\dagger}\Psi \qquad\qquad h = \mathcal{S}^{\dagger}\Psi$$

Intermediate Hertz potential

Fourth step: Add to this the $h_{\alpha\beta}$ that corresponds to the change in mass and angular momentum of the spacetime.

Fifth step: Subtract the singular piece and sum over all the multipoles.

$$H^{\mathrm{REN}}(x_p) = \lim_{x \to x_p} \left[H^{\mathrm{ret}}(x) - H^{\mathrm{S}}(x) \right]$$

$$H^{\mathrm{REN}}(x_p) = \sum_{\ell=0}^{\infty} \left[H_\ell^{\mathrm{ret}}(x_p) - H_\ell^{\mathrm{S}}(x_p) \right]$$

$$H_\ell^{\mathrm{ret}}(x_p) = \sum_{m=-\ell}^{+\ell} H_{\ell,m}^{\mathrm{ret}} Y_{\ell,m}$$

$$H_\ell^{\mathrm{S}}(x_p) = E_0$$

$$H^{\mathrm{REN}} = \sum_{\ell=0}^{\ell_{max}} \left(H_\ell^{\mathrm{ret}} - E_0 - \frac{E_2}{(\ell - \frac{1}{2})(\ell + \frac{3}{2})} - \frac{E_4}{(\ell - \frac{3}{2})(\ell - \frac{1}{2})(\ell + \frac{3}{2})(\ell + \frac{5}{2})} - \cdots \right)$$

Summary

$$u^\alpha u^\beta (g_{\alpha\beta} + h_{\alpha\beta}) = 1$$

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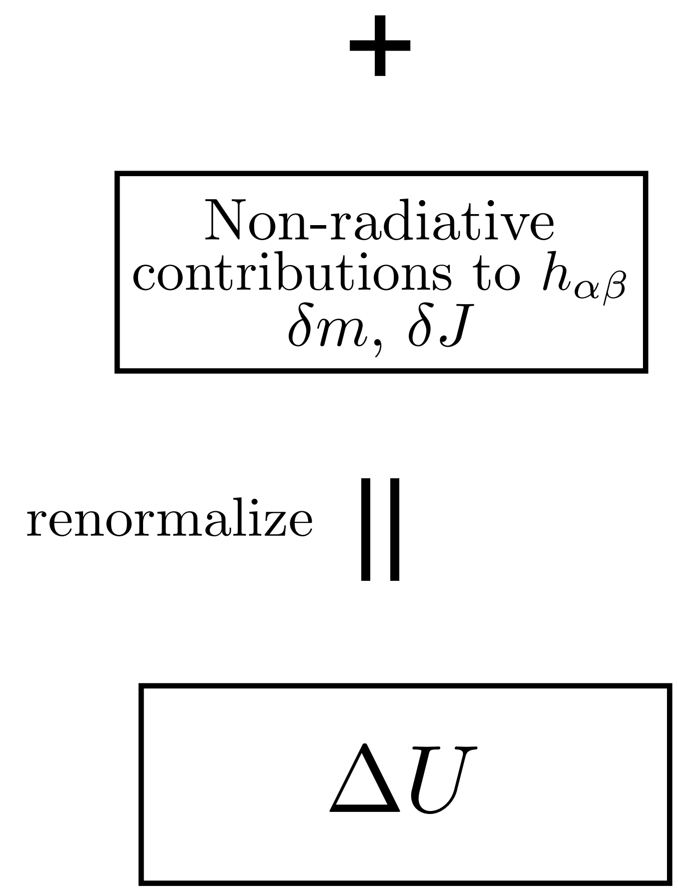

How its done?

MST (Mano-Suzuki-Takasugi) algorithm for the radial harmonics as a sum over known analytic functions with good convergence

Analytical form of the angular harmonics, ${}_sY_{\ell m}(\frac{\pi}{2}, 0)$

We use *Mathematica* which can handle very high precision computations

The accuracy is

about 1 *part in* 10^{227} for $r = 10^{20}M$
about 1 *part in* 10^{242} for $r = 10^{25}M$
about 1 *part in* 10^{252} for $r = 10^{30}M$

(Expected) pN expansion of u_1^t

$$\Delta U = \frac{\alpha_0}{r} + \frac{\alpha_1}{r^2} + \frac{\alpha_2}{r^3} + \frac{\alpha_3}{r^4} + \frac{\alpha_4}{r^5} + \frac{\beta_4 \log(r)}{r^5} + \frac{\alpha_5}{r^6} + \frac{\beta_5 \log(r)}{r^6} + \frac{\alpha_6}{r^7} + \frac{\beta_6 \log(r)}{r^7} + \cdots$$

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Notice the absence of 5.5-pN term

no $\frac{\alpha_{5.5}}{r^{6.5}}$

Analytically known pN coefficients

(from literature)

$$u_1^t = \frac{-1}{r} + \frac{-2}{r^2} + \frac{-5}{r^3} + \frac{\left(\frac{-121}{3} + \frac{41}{32}\pi^2\right)}{r^4}$$

$$+ \frac{\frac{-592384 - 196608\gamma + 10155\pi^2 - 393216\log(2)}{7680}}{r^5}$$

$$+ \frac{\frac{64}{5}\log(r)}{r^5} + \frac{\frac{956}{105}\log(r)}{r^6} + \cdots$$

Bini & Damour '13

Lets look at ΔU at $r = 10^{30} M$

One might wonder,
why are we going to such high radius?

ΔU at $r = 10^{30} M$

-1.00000000000000000000000000000200000000000000000
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94437592602951641738972170966870515028095201627087
89906276810574330937660318994188027816361571144500
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282 X 10^{-30}

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57568607408406791412850807911130634310792849252423
57201651745159732237405685006553928059167553391925
01074771963130952638292729369464281889351107271980
91444305209149244377004135440730531355680584679346
282 X 10^{-30}

$$\Delta U = \frac{-1}{r} + \frac{-2}{r^2} + \frac{-5}{r^3} + \frac{\left(\frac{-121}{3} + \frac{41}{32}\pi^2\right)}{r^4} + \cdots$$

-1.00000000000000000000000000000200000000000000000
00000000000050000000000000000000000000000276879026
94437592602951641738972170966870515028095201627087
89906276810574330937660318994188027816361571144500
463025987555823873 … 773 … 049 … 41365999702778 6
5756860740840 … 49252423
5720165174515 … 553391925
0107477196 … 7271980
914443 … 346
282 X 10^{-30}

Read off the first 4 coefficients to 30 places of accuracy

ΔU at $r = 10^{30} M$

Lets look at

$$\Delta U - \left(\frac{-1}{r} + \frac{-2}{r^2} + \frac{-5}{r^3} + \frac{\left(\frac{-121}{3} + \frac{41}{32}\pi^2\right)}{r^4} + \frac{\frac{64}{5}\log(r)}{r^5} + \frac{\frac{956}{105}\log(r)}{r^6}\right)$$

= -114.34895136757260295204000244483653876441286528440703886923484809292559636928276659763437619372125523054160542189937045693826002042714825538690979057075189... X 10^{-150}

ΔU at $r = 10^{30} M$

Lets look at

$$\Delta U - \left(\frac{-1}{r} + \frac{-2}{r^2} + \frac{-5}{r^3} + \frac{\left(\frac{-121}{3} + \frac{41}{32}\pi^2\right)}{r^4} + \frac{\frac{64}{5}\log(r)}{r^5} + \frac{\frac{956}{105}\log(r)}{r^6}\right)$$

= -114.34895136757260295204000244483653876441286
528440703886923484809292559636928276659763437 6
19372125523054160542189937045693826002042714 82
5538690979057075189... X 10^{-150}

$$u_1^t = \frac{-1}{r} + \frac{-2}{r^2} + \frac{-5}{r^3} + \frac{\left(\frac{-121}{3} + \frac{41}{32}\pi^2\right)}{r^4}$$
$$+ \frac{\frac{-592384 - 196608\gamma + 10155\pi^2 - 393216\log(2)}{7680}}{r^5}$$
$$+ \frac{\frac{64}{5}\log(r)}{r^5} + \frac{\frac{956}{105}\log(r)}{r^6} + \cdots$$

ΔU at $r = 10^{30} M$

Lets look at

$$\Delta U - \left(\frac{-1}{r} + \frac{-2}{r^2} + \frac{-5}{r^3} + \frac{\left(\frac{-121}{3} + \frac{41}{32}\pi^2\right)}{r^4} + \frac{\frac{64}{5}\log(r)}{r^5} + \frac{\frac{956}{105}\log(r)}{r^6}\right)$$

= -114.34895136757260295204000244483653876441286
5284407038869234 … 48092 … 596369 … 282766597634376
1937212552 … 204271482
553869 …

Again read off α_4 to 30 decimal places

$$+ \frac{\frac{-59238 \ldots \gamma \ldots \pi^2 - 393216 \log(2)}{7680}}{r^5}$$

$$+ \frac{\frac{64}{5}\log(r)}{r^5} + \frac{\frac{956}{105}\log(r)}{r^6} + \cdots$$

Puzzle?

pN theory didn't expect n.5-pN terms.

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And we got those terms starting at 5.5 pN in our numerical matching.

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pN theory didn't expect n.5-pN terms.
And we got those terms starting at 5.5 pN.

$$\text{5.5pN comes from } \ell = 2 \text{ multipole,}$$

$$\text{6.5pN comes from } \ell = 2, 3 \text{ multipole,}$$

$$\text{7.5pN comes from } \ell = 2, 3, 4 \text{ multipole} \cdots$$

Puzzle?

pN theory didn't expect n.5-pN terms.
And we got those terms starting at 5.5 pN.

This was very puzzling!!

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To confirm its presence, we performed the whole calculation in the RWZ gauge which agreed with the radiation gauge.

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A successful comparison with the UC-Dublin group (compared the source, $l=2$ multipole, of 5.5pN term):
23-24 digits of agreement at $r = 10^3 M$
43 digits of agreement at $r = 10^6 M$

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A successful comparison with the UC-Dublin group (compared the source, l=2 multipole, of 5.5pN term):
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43 digits of agreement at $r = 10^6 M$

To further confirm its presence, we computed analytical value of the 5.5pN coefficient using the self-force recipe in a radiation gauge, and found that the numerically extracted value agrees with it to **113** significant digits!!

Puzzle?

pN theory didn't expect n.5-pN terms.
And we got those terms starting at 5.5 pN.

It was later understood that the origin of these n.5pN terms come from the "tails-of-tails" terms in pN theory (Luc Blanchet).

Revised pN-series

$$\Delta U = \frac{\alpha_0}{r} + \frac{\alpha_1}{r^2} + \frac{\alpha_2}{r^3} + \frac{\alpha_3}{r^4} + \frac{\alpha_4}{r^5} + \frac{\beta_4 \log(r)}{r^5} + \frac{\alpha_5}{r^6} + \frac{\beta_5 \log(r)}{r^6} + \frac{\alpha_{5.5}}{r^{6.5}} + \frac{\alpha_6}{r^7} + \frac{\beta_6 \log(r)}{r^7} + \frac{\alpha_{6.5}}{r^{7.5}} + \frac{\alpha_7}{r^8} + \frac{\beta_7 \log(r)}{r^8} + \frac{\gamma_7 \log^2(r)}{r^8} + \cdots$$

Analytical value of other terms possible from high precision number?

$$\Delta U = \frac{\alpha_0}{r} + \frac{\alpha_1}{r^2} + \frac{\alpha_2}{r^3} + \frac{\alpha_3}{r^4} + \frac{\alpha_4}{r^5} + \frac{\beta_4 \log(r)}{r^5}$$
$$+ \frac{\alpha_5}{r^6} + \frac{\beta_5 \log(r)}{r^6} + \frac{\alpha_{5.5}}{r^{6.5}} + \frac{\alpha_6}{r^7} + \frac{\beta_6 \log(r)}{r^7}$$
$$+ \frac{\alpha_{6.5}}{r^{7.5}} + \frac{\alpha_7}{r^8} + \frac{\beta_7 \log(r)}{r^8} + \frac{\gamma_7 \log^2(r)}{r^8} + \cdots$$

Analytical value of other terms possible from high precision number?

$$\Delta U = \boxed{\frac{\alpha_0}{r}} + \boxed{\frac{\alpha_1}{r^2}} + \boxed{\frac{\alpha_2}{r^3}} + \boxed{\frac{\alpha_3}{r^4}} + \boxed{\frac{\alpha_4}{r^5}} + \boxed{\frac{\beta_4 \log(r)}{r^5}}$$
$$+ \frac{\alpha_5}{r^6} + \boxed{\frac{\beta_5 \log(r)}{r^6}} + \boxed{\frac{\alpha_{5.5}}{r^{6.5}}} + \frac{\alpha_6}{r^7} + \frac{\beta_6 \log(r)}{r^7}$$
$$+ \frac{\alpha_{6.5}}{r^{7.5}} + \frac{\alpha_7}{r^8} + \frac{\beta_7 \log(r)}{r^8} + \frac{\gamma_7 \log^2(r)}{r^8} + \cdots$$

What about other terms?

If possible, it will save us a number of long, tedious, analytical calculations, and help extract further terms with higher accuracy.

Analytical value of other terms possible from high precision number?

$$\Delta U = \frac{\alpha_0}{r} + \frac{\alpha_1}{r^2} + \frac{\alpha_2}{r^3} + \frac{\alpha_3}{r^4} + \frac{\alpha_4}{r^5} + \frac{\beta_4 \log(r)}{r^5} + \frac{\alpha_5}{r^6} + \frac{\beta_5 \log(r)}{r^6} + \frac{\alpha_{5.5}}{r^{6.5}} + \frac{\alpha_6}{r^7} + \frac{\beta_6 \log(r)}{r^7} + \frac{\alpha_{6.5}}{r^{7.5}} + \frac{\alpha_7}{r^8} + \frac{\beta_7 \log(r)}{r^8} + \frac{\gamma_7 \log^2(r)}{r^8} + \cdots$$

What about other terms?

YES!! INDEED!!

Lets look at the value of the 6-pN log term

$$\Delta U = \frac{\alpha_0}{r} + \frac{\alpha_1}{r^2} + \frac{\alpha_2}{r^3} + \frac{\alpha_3}{r^4} + \frac{\alpha_4}{r^5} + \frac{\beta_4 \log(r)}{r^5} + \frac{\alpha_5}{r^6} + \frac{\beta_5 \log(r)}{r^6} + \frac{\alpha_{5.5}}{r^{6.5}} + \frac{\alpha_6}{r^7} + \boxed{\frac{\beta_6 \log(r)}{r^7}} + \frac{\alpha_{6.5}}{r^{7.5}} + \frac{\alpha_7}{r^8} + \frac{\beta_7 \log(r)}{r^8} + \frac{\gamma_7 \log^2(r)}{r^8} + \cdots$$

Numerically extracted value

$$
\begin{aligned}
\beta_6 = -90.&398589065255731922398589065255731922 \\
&398589065255731922398589065255731922 \\
&398589065255731922398589048525187995 5\cdots
\end{aligned}
$$

Numerically extracted value

$$\beta_6 = -90.398589065255731922398589065255731922$$
$$398589065255731922398589065255731922$$
$$3985890652557319223985890485251879955\cdots$$

Numerically extracted value

$$\beta_6 = -90.398589065255731922398589065255731922398589065255731922398589065255731922398589065255731922398589048525187995 5\cdots$$

More than 5 repetition cycles

$$\beta_6 = \frac{-51256}{567}$$

Another example

$$\Delta U = \frac{\alpha_0}{r} + \frac{\alpha_1}{r^2} + \frac{\alpha_2}{r^3} + \frac{\alpha_3}{r^4} + \frac{\alpha_4}{r^5} + \frac{\beta_4 \log(r)}{r^5} + \frac{\alpha_5}{r^6} + \frac{\beta_5 \log(r)}{r^6} + \frac{\alpha_{5.5}}{r^{6.5}} + \frac{\alpha_6}{r^7} + \frac{\beta_6 \log(r)}{r^7} + \frac{\alpha_{6.5}}{r^{7.5}} + \frac{\alpha_7}{r^8} + \frac{\beta_7 \log(r)}{r^8} + \boxed{\frac{\gamma_7 \log^2(r)}{r^8}} + \cdots$$

Numerically extracted value

$$\gamma_7 = 52.1752380952380952380952380952380952380952380952380952380952380952380952380952375380434891643 31\cdots$$

Numerically extracted value

$$\gamma_7 = 52.17\boxed{523809}523809523809523809523809$$

$$523809523809523809523809523809$$

$$5238095237538043489164331\cdots$$

Numerically extracted value

$$\gamma_7 = 52.17\,523809\,523809\,523809\,523809\,523809$$
$$523809\,523809\,523809\,523809\,523809$$
$$523809\,523\,75380434891 64331\cdots$$

More than 11 repetition cycles

$$\gamma_7 = \frac{27392}{525}$$

6.5-pN term

$$
\begin{aligned}
\alpha_{6.5} = 69.&30909049662575956322060886698020553525276282\\
&80640692917511789688546478292427121038828291\\
&65780393465346820430681300807 64\cdots
\end{aligned}
$$

6.5-pN term

$$\alpha_{6.5} = 69.30909049662575956322060886698020553525276282$$
$$806406929175117896885464782924271210388282 91$$
$$657803934653468204306813008 0764\cdots$$

$$\frac{\alpha_{6.5}}{\pi} = 22.0617687074829931972789115646258503401360544 2$$
$$1768707482993197278911564625850340136054787080\cdots$$

6.5-pN term

$$\alpha_{6.5} = 69.30909049662575956322060886698020553525276282$$
$$806406929175117896885464782924271210388282 91$$
$$65780393465346820430681300807 64\cdots$$

$$\frac{\alpha_{6.5}}{\pi} = 22.06\boxed{1768707482993197278911564625850340136054 42}$$
$$\boxed{1768707482993197278911564625850340136054}787080\cdots$$

Almost 2 repetition cycles here

$$\alpha_{6.5} = \frac{81077\pi}{3675}$$

One is not always lucky to have such repetition cycles...

$$\frac{\alpha_{7.5}}{\pi} = 176.49758751536529314307092084869862647640425418203195980973759881572859646496135482085019945242682178\cdots$$

One is not always lucky to have such repetition cycles...

$$\frac{\alpha_{7.5}}{\pi} = 176.4975875153652931430709208486986264764042541820319598$$
$$0973759881572859646496135482085019945242682178\cdots$$

Multiplying it with appropriate powers of the first few prime numbers clear things up

$$\frac{\alpha_{7.5}}{\pi} \times 5^2 \times 3^5 \times 7 \times 11 =$$
$$82561159.000000000000000000000000000000000000000000000000000005286027\cdots$$

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$$\frac{\alpha_{7.5}}{\pi} \times 5^2 \times 3^5 \times 7 \times 11 =$$
$$82561159.000000000000000000000000000000000000000000000000000005286027\cdots$$

Consistent with the accuracy with which we extracted this number

Final list of analytically known pN terms

$$\frac{-1}{r} + \frac{-2}{r^2} + \frac{-5}{r^3} + \frac{\frac{-121}{3} + \frac{41\pi^2}{32}}{r^4}$$
$$+ \frac{-592384 - 196608\gamma + 10155\pi^2 - 393216\log(2)}{7680\ r^5}$$
$$+ \frac{64}{5}\frac{\log(r)}{r^5} + \frac{-956}{105}\frac{\log(r)}{r^6} + \frac{-13696\pi}{525 r^{6.5}} + \frac{-51256}{567}\frac{\log(r)}{r^7}$$
$$+ \frac{81077\pi}{3675\ r^{7.5}} + \frac{27392}{525}\frac{\log^2(r)}{r^8} + \frac{82561159\pi}{467775\ r^{8.5}} + \frac{-27016}{2205}\frac{\log^2(r)}{r^9}$$
$$+ \frac{-11723776\pi}{55125}\frac{\log(r)}{r^{9.5}} + \frac{-4027582708}{9823275}\frac{\log^2(r)}{r^{10}} + \frac{99186502\pi}{1157625}\frac{\log(r)}{r^{10.5}}$$
$$+ \frac{23447552}{165375}\frac{\log^3(r)}{r^{11}}$$

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$$\frac{-1}{r}+\frac{-2}{r^2}+\frac{-5}{r^3}+\frac{\frac{-121}{3}+\frac{41\pi^2}{32}}{r^4}$$
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$$+\frac{23447552}{165375}\frac{\log^3(r)}{r^{11}}$$

* one later added by Nathan KJMcDaniel

* Bini-Damour later calculated everything analytically upto 8.5pN

Fluxes

- Motivated by our work on extracting analytical coefficients from numerical data for ΔU.

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- Motivated by Fujita's remarkable 22-pN work on fluxes at infinity in Schwarzschild.
- We calculated the 20-pN fluxes at infinity and event horizon for quasi-circular, equatorial orbits in Kerr spacetime.

Fluxes

We calculate the flux in Schwarzschild spacetime for 144 different values of Ω by placing the orbit at

$$R = 1, 2, 3, ..., 9 \times 10^{18,19,20,...,32,33}$$ 144 points

$$R := (M\Omega)^{-2/3}$$

We calculate $f_S^H(R_i)$ and $f_S^\infty(R_i)$.

Fluxes

For Kerr spin parameters

$$a/M = 1, 2, 3, 4, 5 \times 10^{-1,-2,-3,-4,-5}$$ 25 points

and different R's we place the orbit at different radii

$$r_K = \left((RM)^{2/3} - (aM)^{1/2} \right)^{2/3}$$

and calculate $f_K^H(r_{K_{ij}}, a_j)$ and $f_K^\infty(r_{K_{ij}}, a_j)$

25 X 144 = 3600 points

Fluxes

Perform a 2D fit by fitting $[f_K^{H(\infty)}(r_{K_{ij}}, a_j) - f_S^{H(\infty)}(R_i)]$ to a polynomial of the for $c_{i,j,k} a^k \log^j(R)/R^i$ twice.

(i) to extract as many analytical coefficients as possible,

In Kerr, at ∞, 17.5pN.

In Schwarzschild, at horizon, 19pN.

In Kerr, at horizon, 17pN.

Fluxes

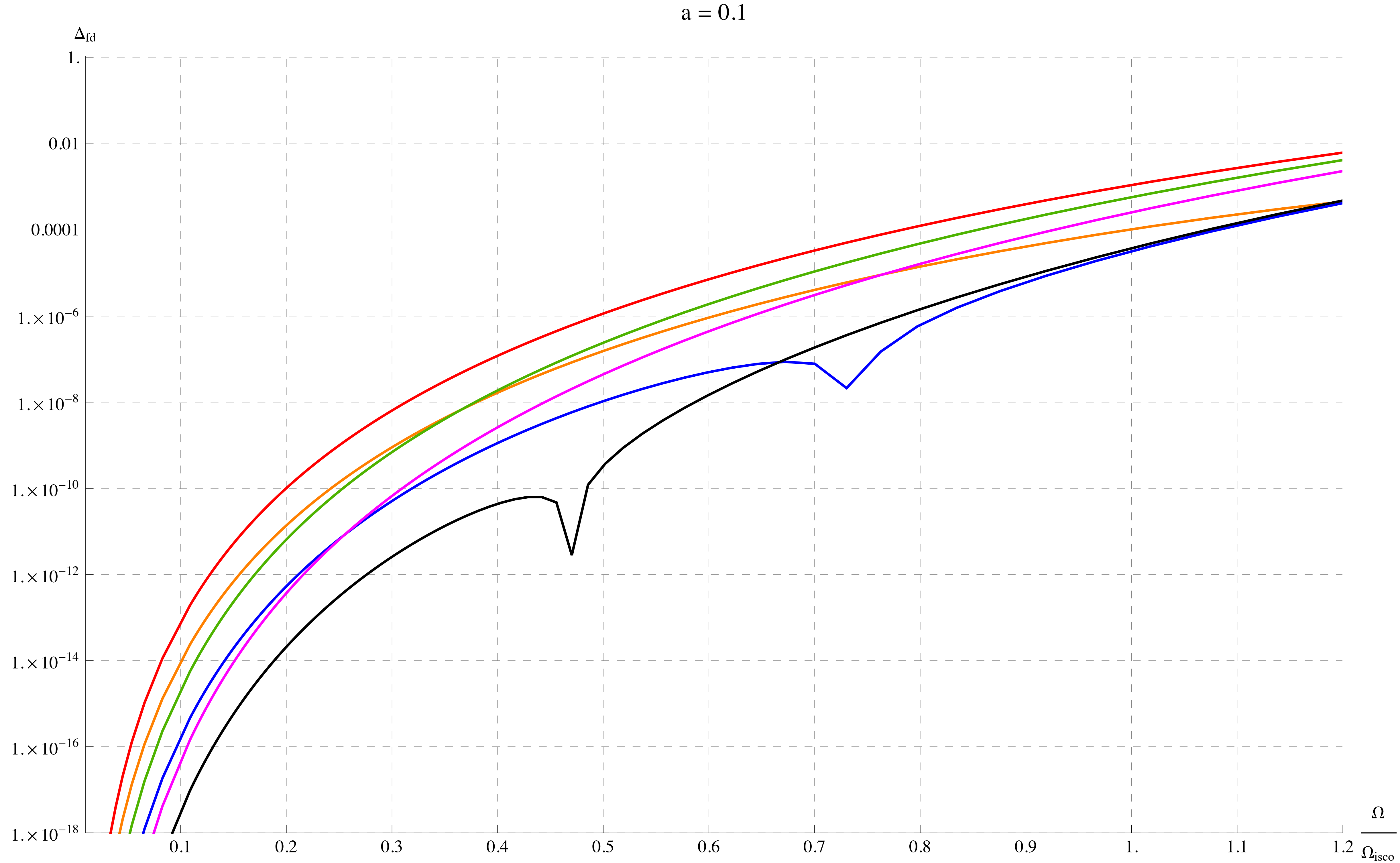
Perform a 2D fit by fitting $[f_K^{H(\infty)}(r_{K_{ij}}, a_j) - f_S^{H(\infty)}(R_i)]$ to a polynomial of the for $c_{i,j,k} a^k \log^j(R)/R^i$ twice.

(i) to extract as many analytical coefficients as possible,

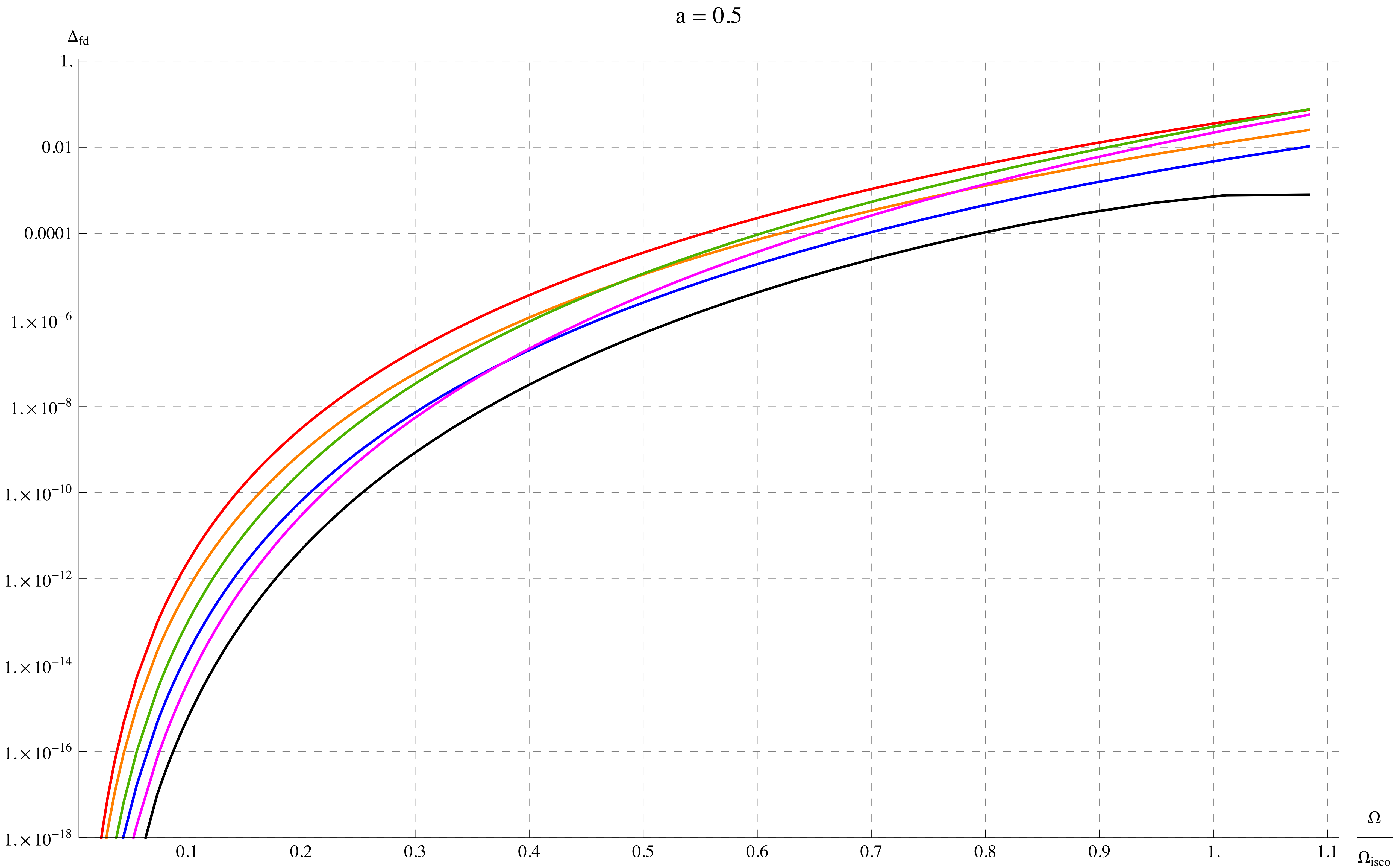
(ii) subtract the analytical coefs. to extract the numerical coefs.

When the pN-term is not a simple polynomial in a, we extract the coefficients of the power series in a.

Already known 4pN $\rightarrow$ 20pN



Black: 20pN. Magenta: 19pN. Blue: 18pN.
Green: 17pN. Orange: 16pN. Red: 15pN.



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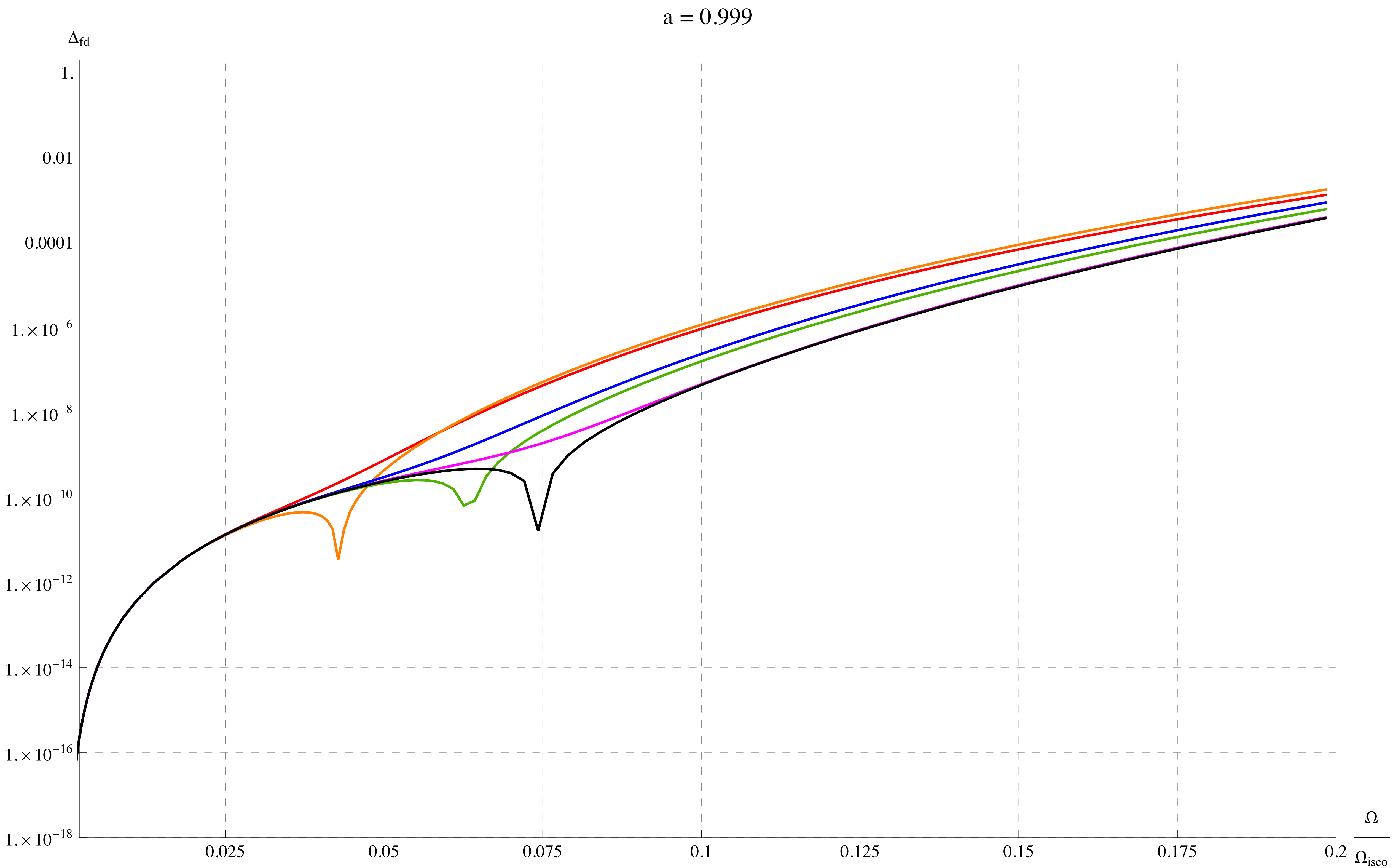

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Green: 17pN. Orange: 16pN. Red: 15pN.

Thank you!