101 Questions, Problems and Conjectures around Scalar Curvature. (Incomplete and Unedited Version)*

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Between Two Worlds.

Positive scalar curvature mediates between the world of *rigidity* one finds around *convexity theory* and the realm of *softness* charactersitic of topology such as the *cobordism theory*.

The overall picture of the domain $Sc > 0$ is reminiscent of what one sees in the symplectic geometry but the former has not reached the level of maturity achieved by the latter. The main results and problems about $Sc > 0$ still rotate around a few simple minded questions.

[?1] ∘ What are *possible topologies* of manifolds which *admit Riemannin metrics* with scalar curvatures $Sc > 0$?

[?2] ∘ What are topologies of *spaces of metrics g* with Sc(g)>0?

[?3] ∘ What are geometries of *individual manifolds* with $Sc > \sigma$?

[?4] ∘ What are effect of lower bounds $Sc \geq \sigma$ on the topology and geometry of *maps* between manifolds?

Although these questions are rather naive, as compared, for instance, with what is pursued in the *symplectic field theory*, the answers which have been already achieved points to intersting new structure(s) behind $Sc > 0$.

For instance:

⋆ *All closed simply connected manifolds of dimensions* $n = 3, 5, 6, 7 \mod 8$ *admit metrics with* $Sc > 0$ by a 1992 theorem of Stolz [119].

But according to [68]-Hitchin Harmonic Spinors 1974] *there are smooth manifolds homeomorphic* (but not diffeomorphic!) *to the spheres of dimensions* 9 *and 10 which admit no such metrics.* (See section 4)

⋆ *If m is much greater than k then the kth homotopy group of the space of metrics with* $Sc > 0$ *on the sphere* S^{4m-k-1} *is infinite.* (See [63]-Hanke+ The space of metrics 2014] and section 31.)

⋆ *If the scalar curvature of a Riemannin metric g in the torical band (cylinder)* $\mathbb{T}^{n-1} \times [-1, +1]$ *satisfies*

$$Sc(g) \geq n(n-1) = Sc(S^n),$$

*A final version of an extract from this paper can be found on `http://www.ihes.fr/~gromov/PDF/dozen_problems_Sept_2017.pdf`.

then the distance between the two boundary components in this band is bounded by

$$[\mathbb{T}_\pm] \qquad dist_g(\partial_-, \partial_+) < \frac{2\pi}{n}.$$

(See [51]-Gromov Metric Inequalities 2017] and section 21, while the justification of the normalisation $Sc(S^n) = n(n-1)$ is given in section 1 along with the definition of the scalar curvature.)

[?5] An optimist would expect similar inequalities

$$dist_g(\partial_-, \partial_+) < \delta = \delta(Y) < \infty$$

(ideally with $\delta = \frac{2\pi}{dim(Y)+1}$) for metrics g on $Y \times [-1, +1]$ with $Sc(g) \geq n(n-1)$ *for all closed manifolds* Y of dimensions $\neq 4$, which themselves *admit no metrics with* $Sc > 0$. [1] This is proven for certain non-torical Y in [51] but no present day technics is applicable to *simply connected* manifolds Y. (The existence of such Y follows from Lichnerowicz' and Hitchin's results which rely on the Atiyah-Singer index theorem, see sections 3 and 4.)

On the other hand, it is not hard to see that

$$\delta(Y) \geq \frac{2\pi}{dim(Y)+1} \text{ for all } Y,$$

which implies that $[\mathbb{T}_\pm]$ is sharp for $Y = \mathbb{T}^{n-1}$ and also shows that

$[\mathbb{T}_\pm]$ *implies non-existence of metrics with* $Sc > 0$ *on tori*[2] .

⋆ Let X be a closed orientable Riemannian n-manifold X with $Sc(X) \geq n(n-1) = S^n$, which satisfies the following topological condition.

[$\mathbb{S}$] *The restrictions of the tangent bundle* $T(X)$ *to all closed surfaces* $S \subset X$ *are trivial.*[3]

Then X *is no larger area-wise than the unit sphere* S^n.

Namely,

X *admits* no *smooth map to* S^n *with non-zero degree, which strictly decreases the areas of the surfaces* S *in* X. *Moreover, if a* C^1*-smooth map* $f : X \to S^n$ *satisfies*

$$area(f(S)) \leq area(S) \text{ for all } S \subset X \text{ and } deg(f) \neq 0,$$

then f *is an isometry,* (see [81]-Llarull Sharp estimates 1998] and sections 17, 18).

Notice that [$\mathbb{S}$] is satisfied, for instance, if $H^2(X; \mathbb{Z}_2) = 0$, while the basic examples where [$\mathbb{S}$] is violated are complex projective spaces of even complex dimensions.

[?6] *Conjecturally*, [$\mathbb{S}$] is redundant anyway.

About the Methods. There is a intriguing interplay between two techniques involved in the study of manifolds X with $Sc(X) > 0$, which are unlike to what

[1] If Y admits a metric g_0 with $Sc(g_0) > 0$, then the metric $g_0 + dt^2$ on the cylinder $Y \times \mathbb{R}$ satisfies $Sc(g_0 + dt^2) = Sc(g_0) > 0$, which makes $\delta(Y) = \infty$.

[2] This is not how non-existence of metrics with $Sc > 0$ on tori was originally proved, see sections 5,6,7

[3] For those who knows this implies that X *is spin;* in fact, if $dim(X) \geq 3$ this is equivalent to X being spin.

is commonly practiced for manifolds with *positive sectional* and *positive Ricci* curvatures.

The first (historically the second) class of techniques introduced in 1979 by Schoen and Yau (see sections 5, 6) depends on *descending chains of minimal hypersurfaces in X*, while the second method, originated in the 1963 paper by Lichnerowicz relies on *linear elliptic* PDE, namely on the *index and vanishing theorems for the Dirac operators* on X, see sections 3, 7, 9.

Notice that the original Lichnerowicz and Hitchin non-existence theorems of metric with $Sc > 0$ apply to *spin* manifolds X, on which the Dirac operators D have $ind(D) \neq 0$ for *all* metrics on X, where this "non-equality" is derived from non-vanishing of certain topological invariants of X via the index theorem, where

these invariants (for n>4) *essentially depend on the differential structure of X.*

On the other hand, minimal hypersurfaces provide *homotopy theoretic* obstruction to $Sc > 0$ and, as far as topological non-existence theorems are concerned, they apply only to manifolds with large fundamental groups.

Interestingly enough, twisting Dirac operators with suitable vector bundles over X also shifts their essential topological applications to highly non-simply connected manifolds (see [54]-Gromov-Lawson, Positive scalar curvature 1983] and sections 4,16).

Thus, the "tree" of $Sc > 0$ has three distinct branches where II and III have a common Dirac operator origin, while I and III partly share their habitats.

[?7] But deeper structures (if they exist at all) that lie at the roots of Dirac operators and of minimal hypersurfaces are yet to be revealed.

In what follows, we explain the basic results and techniques concerning spaces with $Sc \geq \sigma$ and compile a list of open, many of them long standing, questions on geometry and, up to a lesser extent, on topology of these spaces. We formulate some questions as *conjectures* to direct one's thought in a fruitful direction, rather than to champion a particular yes/no outcome.

Contents

1 Definition of Scalar Curvature via Volumes of Balls, Examples, Formulas and the Central Problem.

The scalar curvature of a C^2-smooth Riemannian manifold $X = (X, g)$, denoted $Sc = Sc(X) = Sc(X, g) = Sc(g)$ is a continuous function on X, which is uniquely characterised by the following four properties.

$\bullet_1$ *Additivity under Cartesian-Riemannian Products* .

$$Sc(X_1 \times X_2, g_1 \oplus g_2) = Sc(X_1, g_1) + Sc(X_2, g_2).$$

$\bullet_2$ *Quadratic Scaling.*

$$Sc(\lambda \cdot X) = \lambda^{-2} Sc(X), \text{ for all } \lambda > 0,$$

where

$$\lambda \cdot X = \lambda \cdot (X, dist_X) =_{def} (X, dist_{\lambda \cdot X}) \text{ for } dist_{\lambda \cdot X} = \lambda \cdot dist(X)$$

for all metric spaces $X = (X, dist_X)$ and where $dist \mapsto \lambda \cdot dist(X)$ corresponds to $g \mapsto \lambda^2 \cdot g$ for the Riemannian quadratic form g.

(This makes the Euclidean spaces scalar-flat: $Sc(\mathbb{R}^n) = 0$.)

$\bullet_3$ *Volume Comparison.* If the scalar curvatures of n-dimensional manifolds X and X' at some points $x \in X$ and $x' \in X'$ are related by the strict inequality

$$Sc(X)(x) < Sc(X')(x'),$$

then the Riemannian volumes of small balls around these points satisfy

$$vol(B_x(X, \varepsilon)) > vol(B_{x'}(X', \varepsilon))$$

for all sufficiently small $\varepsilon > 0$.

Observe that this volume inequality is *additive under Riemannian products*: if

$$vol(B_{x_i}(X, \varepsilon)) > vol(B_{x'_i}(X'_i, \varepsilon)), \text{ for } \varepsilon \leq \varepsilon_0,$$

and *for all points* $x_i \in X_i$ and $x'_l \in X'_i$, $i = 1, 2$, then

$$vol_n(B_{(x_1,x_2)}(X_1 \times X_2, \varepsilon_0)) > vol_n(B_{(x'_1,x'_2)}(X'_1 \times X'_2, \varepsilon_0)$$

for all $(x_1, x_2) \in X_i \times X_2$ and $(x'_1, x'_2) \in X'_1 \times X'_2$.

This follows from the Pythagorean formula

$$dist_{X_1 \times X_2} = \sqrt{dist^2_{X_1} + dist^2_{X_2}}.$$

and the Fubini theorem applied to the "fibrations" of balls over balls:

$B_{(x_1,x_2)}(X_1 \times X_2, \varepsilon_0)) \to B_{x_1}(X_1, \varepsilon_0)$ and $B_{(x'_1,x'_2)}(X'_1 \times X'_2, \varepsilon_0)) \to B_{x_1}(X'_1, \varepsilon_0)$,

where the fibers are balls of radii $\varepsilon \in [0, \varepsilon_0]$ in X_2 and X'_2.

$\bullet_4$ *Normalisation/Convention for Surfaces with Constant Sectional Curvatures.* The unit spheres $S^2(1)$ have constant scalar curvature 2 and the hyperbolic plane $H^2(-1)$ with the sectional curvature -1 has scaler curvature -2

It is an elementary exercise. to prove the following.

$\star_1$ *The function* $Sc(X, g)(x)$ *which satisfies* $\bullet_1$-$\bullet_4$ *exists and unique*;

$\star_2$ *The unit spheres and the hyperbolic spaces with* $sect.curv = -1$ *satisfy*

$$Sc(S^n(1)) = n(n-1) \text{ and } Sc(H^n(-1)) = -n(n-1).$$

Thus,

$$Sc(S^n(1) \times H^n(-1)) = 0 = Sc(\mathbb{R}^n),$$

which implies that the volumes of the small balls in $S^n(1) \times (H^n(-1)$ are "very close" to the volumes of the Euclidean $2n$-balls.

$\star_3$ *The scalar curvature of a Riemannin manifold* X *is equal to the sum of the values of the sectional curvatures at the bivectors of an orthonormal frame in* X,

$$Sc(X)(x) = \sum_{i,j} c_{ij}, \ i, j = 1, ..., n.$$

(This agrees with $Sc = n(n-1)$ for $X = S^n$.)

$\star_4$ *The scalar curvature of a smooth hypersurface* $X \subset \mathbb{R}^{n+1}$ *with the principal curvatures* c_i, $i = 1, 2, .., n$, *is*

$$Sc(X) = \sum_{i \neq j} c_i \cdot c_j = \left(\sum_i c_i\right)^2 - \sum_i c_i^2.$$

$\star_5$. *The warped product metrics*

$$g = dy^2 + f^2 ds^2$$

on the product of Riemannian manifolds $Y \times S$ *and smooth functions* $f > 0$ *on* Y *satisfy*

$$Sc(g)(y, s) = Sc(Y)(y) + \frac{1}{f^2} Sc(S)(s) - \frac{m(m-1)}{f^2(y)} ||\nabla f(y)||^2 - \frac{2m}{f} \Delta f(y),$$

where $m = dim(S)$ *and* $\Delta = \sum \nabla_{i,i}$ *is the Laplace operator on* Y.

For instance, if $S = S^1$ *is the unit circle, then*

$$Sc(g) = Sc(Y) - \frac{2}{f} \Delta f.$$

$\star_6$. *The values of the Ricci curvature on the unit vectors $\nu = \nu_y \in T_y(X)$, normal to a smooth hypersurface $Y \subset X$ satisfy. .*

$$Ricci(\nu) = \frac{1}{2}\left(Sc(X) - Sc(Y) - \sum_{i\neq j} c_i \cdot c_j\right).$$

where $c_i = c_i(y)$, $y \in Y$, $i = 1, ..., n-1$, are the principal curvatures of Y in X.

(However simple, the last two formulae plays play a key role in the study of the scalar curvature of n-dimensional manifolds X via *minimal* $(n-1)$-dimensional submanifolds $Y \subset X$.)

Lower Bounds on $Sc(X)$. We are not so much concerned with the scalar curvature $Sc(X)$ per se, but rather with the effect of *lower scalar curvature bounds* on the geometry of X, where, for instance, the inequality "$Sc(X) > 0$" can be **defined** by saying that

all sufficiently small balls $B_x(\varepsilon) \subset X$, $\varepsilon \le \varepsilon_0(x) > 0$, have the volumes smaller than the volumes of the equividimensional Euclidean ε-balls.

Then "$Sc(X) \ge 0$" is defined as

$$Sc(X) > -\varepsilon\text{" for all } \varepsilon > 0.$$

Similarly

"$Sc(X) \ge \sigma$", $\sigma > 0$, is equivalent the volumes of $B_x(\varepsilon)$ in X being smaller than the volumes of the ε-balls in the Euclidean spheres $S^n(R)$ of radii $R > \sqrt{(n(n-1)/\sigma)}$,

and $Sc(X) \ge -\sigma$ is expressed by

the bound on the volumes of $B_x(\varepsilon)$ by those of the ε-balls in the hyperbolic spaces with constant the sectional curvatures $< -\sigma/n(n-1)$.

Also "$Sc(X) \ge -\sigma$" can be defined with *no reference* to hyperbolic spaces by *the reduction to the case $\sigma = 0$*

$$Sc(X \times S^m(R)) \ge 0 \text{ for } R = \sqrt{(m(m-1)/\sigma},$$

where one may use any $m \ge 2$ one likes.

[?8] **Question** A. *What is* Scalar Curvature*?* The above definition via the volumes of balls masks the true nature of the scalar curvature – it has proven useless (so far) for proving anything "global" about $Sc \ge \sigma$.

The vanishing Dirac operator techniques turns your mind toward *spectral invariants of small balls* instead of their volumes, where such an invariant should integrate from the local to the global scale similarly to how it works in the (*Borel*)-*Garland combinatorial cohomology vanishing theorems.* extension Alternatively – this may lead to a different class of objects – one may look for a definition of $Sc > \sigma$ based on a localisation of the method of minimal hypersurfaces (see section 32 for a version of this approach.) Such a definition could be better adapted for an extension of the concept $Sc > 0$ to singular spaces: calculus of variations, unlike Dirac operators, does not shy away from "bad singularities". [4]

[4]Minimal varieties make perfect sense, for instance, in manifolds with *continuous* Riemannian metric g, where the definition of the Dirac operator is problematic.

A local geometric definition of $Sc > \sigma$, if successful, will bring us a step closer to resolving the following.

[?9] **Problem A.** *Identify the most general class (classes?) of "geometric objects" with the properties similar to those of C^2-Riemannian manifold with $Sc \geq \sigma$.*

To make sense of this problem look at the available examples.

2 Symmetric Spaces, Fibrations and Surgeries with $Sc > \sigma$.

Compact Symmetric Spaces are the main representatives of manifolds with $Sc \geq 0$, where, all of them, except for flat tori, have $Sc > 0$.

The most prominent among these, besides spheres and product of spheres, are *complex projective spaces* $\mathbb{C}P^k$ of real dimensions $n = 2k$ and *quaternionic* $\mathbb{H}P^k$ of dimensions $n = 4k$.

Positivity of the scalar curvature is inherited by fibered spaces from the fibers.

Namely,

if the fibers $X_{\underline{x}} = f^{-1}(\underline{x}) \subset X$ of a smooth fibration $f : X \to \underline{X}$ between compact manifolds carry smooth Riemannian metrics which continuously depend on $\underline{x} \in \underline{X}$ and which all have $Sc > 0$, then X also supports a metric with $Sc > 0$.

Proof. Start with an arbitrary smooth metric g on X with $Sc(X_{\underline{x}}) > 0$ for all $\underline{x} \in \underline{X}$, orthogonally split the tangent bundle as

$$T(X) = T_\tau \oplus T_\nu \text{ for } T_\tau = ker(Df) \text{ and } T_\nu = ker(Df)^\perp,$$

accordingly split $g = g_\tau + g_\nu$ and observe that the scalar curvatures of the metrics

$$g_\varepsilon = \varepsilon \cdot g_\tau + g_\nu = \varepsilon(g_\tau + \varepsilon^{-1} g_\nu), \quad \varepsilon > 0,$$

for small positive ε are dominated by the scalar curvatures the fibers

$$Sc(g_\varepsilon) \asymp Sc(\varepsilon \cdot X_{\underline{x}}) \asymp \varepsilon^{-2} \text{ for } \varepsilon \to 0.$$

Thus, for instance, the (total spaces of)

fibrations with compact simply connected Riemannian symmetric fibers carry metrics with $Sc > 0$.

Similarly, according to Lawson-Yau,

manifolds X with non-trivial actions of compact connected non-Abelian Lie groups, e.g. compact homogeneous spaces different from tori, carry (fairly natural) metrics with $Sc > 0$.

(See [125]-Wiemeler Circle actions 2015] and references therein.)

Also such metrics exist on

manifolds with circle actions where the fixed point sets have codimensions two [125].

Calabi-Yau-Fano[5] *Kähler manifolds* provide yet another remarkable source of manifolds with $Sc > 0$, where the most beautiful among them carry *Kähler-Einstein* metrics. For instance – this follows from Yau's solution of Calabi conjecture,

smooth complex hypersurfaces in $\mathbb{C}P^{m+1}$ (of real dimension $n = 2m$) *of degrees* $d \leq m+1$ *carry Kähler Riemannian metrics with Ricci* > 0, which is much stronger than $Sc > 0$.

Moreover,

smooth hypersurfaces in $\mathbb{C}P^{m+1}$ *given by equations of the form*

$$z_0^{m+1} + f(z_1, ..., z_{m+1}) = 0,$$

where f *is a homogeneous polynomial of degree* $m+1$, *support Kähler-Einstein metrics with* $Sc > 0$. (See [27]-Dervan On K-stability 2015] and references therein.)

[?10] **Problem** *Extend the concept of* $Sc > 0$ *to singular Fano Varieties.*

For example, work out a definition of $Sc(X)$ along the lines suggested in Question 1 of the previous section, such that the non-strict positivity $Sc(X) \geq 0$ of generalised scalar curvature would be stable under deformations of smooth Fano varieties, e.g. of hypersurfaces $X_{reg} \subset \mathbb{C}P^{m+1}$ of low degrees, to singular ones.

Notice that the volumes of small balls at singular points in algebraic varieties, especially where they are not normal (locally reducible) are significantly greater than the volumes of balls at the regular points which indicate of non-applicability of the definition of scalar curvature via volumes of balls to singular spaces. (Compare with [11]-Basilio+ Sewing Riemannian Manifolds 2017].)

Further example of manifolds with $Sc > 0$ are obtained with *codimension 2 surgery* on n-dimensional manifolds, i.e. surgery based on submanifolds (e.g embedded spheres) of dimensions $\leq n-3$, since

codimension 2 surgery can be (*rather naturally*) *performed in the Riemannian category of manifolds with* $Sc > \sigma$.

(See [122]-Walsh Metrics of positive scalar 2008] and references therein.)

For instance, *connected sums* of n-manifolds with $Sc > 0$ carry metrics with $Sc > 0$ for all $n \geq 3$ and all orientable manifolds with $Sc > 0$ of dimension ≥ 4 *can made simply connected* by attaching 2-handles, while keeping $Sc > 0$.

To see geometrically how this works, look at a small ε-neighbourhood $X_\varepsilon = U_\varepsilon(P)$ of a compact smooth submanifold P in a Riemannin manifold W.[6]

If $codim(P) \geq 3$ then the boundary $\partial U_\varepsilon(P)$ is fibered by ε-spheres of dimensions $k \geq 2$ the scalar curvatures of which are approximately $\frac{k(k-1)}{\varepsilon^2}$ which blows up to $+\infty$ for $\varepsilon \to 0$ and

$$Sc(X_\varepsilon) \asymp \varepsilon^{-2} \to +\infty$$

as well.

[5]A complex manifold X is *Fano* if the *anticanonical line bundle* $L_{ac}(X)$, i.e. the top exterior power of the tangent bundle $T(X)$, is *ample*: some power L_{ac}^N is generated by holomorphic sections.

[6]Topologically, a surgery over a manifold X results in a manifold W with two boundary components where the the first one is X and the second one is the result of the surgery. The geometric construction we describe may be performed in this W with a cylindrical Riemannian metric near $X \subset W$.

Now, more generally, let $P \subset W$ be a piecewise smooth polyhedral subset, which is the union, $P = P_0 \cap P_{thin}$, where P_0 – the "thick part" of P – is a domain with smooth boundary $X_0 = \partial P_0$, and let $\sigma(w)$ be a continuous function on W.

[●<]. *If $codim(P_{thin}) \geq 3$, and if $\sigma(p) < Sc(X_0)(p)$ for the induced Riemannian metric in X_0 and all $p \in X_0 = \partial P_0$, then P admits a regular neighbourhood*[7] *$V \subset W$ with smooth boundary $X = \partial V$, such that the scalar curvature of X with the induced Riemannian metric satisfies*

$$Sc(X)(x) \geq \sigma(x), \ x \in X,$$

where, moreover, one may take V equal to P_0 outside a given neighbourhood of P_{thin}.[8]

Observe that such manifolds $X = X_\varepsilon = \partial U_\varepsilon$ converge or *collapse* to P if we take U_ε inside smaller and smaller neighbourhoods $W_\varepsilon \subset W$ of P. This collapse $X_\varepsilon \rightsquigarrow P$ generalises the above $(X, g_\varepsilon) \rightsquigarrow \underline{X}$ in the case where the fibers $f^{-1}(\underline{x})$ are round spheres. (Later, we shall indicated a general construction which incorporates both examples.)

On the Proof of **[●<]**. The above estimates $Sc(X_\varepsilon) \asymp \varepsilon^{-2}$ holds for the ε-neighbourhoods of *non-compact* submanifolds P, where $\varepsilon = \varepsilon(p) > 0$ is a small positive *function* on P, such which tends to $\to 0$ for $p \to \infty$ the first and the second derivatives of which are dominated by ε^{-2}. (One gets a fair idea of what happens by looking at such neighbourhoods of a straight line P in $\mathbb{R}^4$.)

If P is non-smooth, one needs to properly regularise such neighbourhoods at the corners. This is done by the standard *induction by skeletons* which reduces the problem to the case where P_{thin} is a *smooth coorientable submanifold with boundary* (a smooth ball if you wish) such that this boundary is contained in X_0 and P_{thin} is transversal to X_0.

If P_0 is the unit ball in $\mathbb{R}^4$ and P_{thin} is a straight ray normal to the boundary of this P_0, then the construction of the required *rotationally symmetric* V is straightforward and it equally applies to rays normal to umbilical hypersurfaces X_0 in hyperbolic spaces which is needed for $Sc(X_0) \leq 0$.

Then the general case follows by the *weak flexibility* argument on p.111 in [42]-Gromov Partial differential 1986]

On Geometry Versus Topology. The diffeomorphism class of $X = X_\varepsilon$ does not depend on the specifics of the above construction, but the geometry of X depends on how and in which order surgeries are performed.

In fact, in oder to keep $Sc > \sigma$, e.g in the case of $\sigma = 0$, one needs *very thin* handles, where this "thinness" depends on "thinness" of the handles which were attached earlier; thus, each consecutive handle comes much thinner than all preceding ones.

Conceivably, metrics obtained in this manner by interchanging the order of surgeries may be non-homotopic in the space of all metrics with $Sc(g) > 0$ on X. (Compare [123]-Walsh The Space of Positive 2014].)

No Surgery for $Sc \geq \sigma$. It is fundamental that there is no surgery for $Sc \geq 0$; for instance

[7] Such a $V \supset P$ is an equidimensional submanifold with boundary which admits a continuous family of diffeomorphic embeddings $I_t : V \to V$, $0 \leq t < 1$, such that the $I_t | P = Id$ for all t and such that I_t uniformly converge to a piecewise smooth map $V \to P$ for $t \to 1$.

[8] Our "*regular* neighbourhoods" are not strictly speaking "neighbourhoods". For instance, P_0 is regarded as a regular neighbourhood of itself.

the connected sums of n-tori, for all $n \geq 2$ admit no metrics with $Sc \geq 0$.

This is amazing! Two tori of dimension $n \geq 3$ can be connected by an arbitrarily long tube (1-handle), which harbours vast amount of huge scalar curvature except for two tiny regions where the scalar curvature, albeit negative, is $\geq -\varepsilon$.

Yet there is no way to redistribute this curvature and to make it everywhere ≥ 0.

There are two different approaches of this: the first one, due to Schoen and Yau, is based on the inductive use of minimal hypersurfaces (see section 6) and another one depends on the Atiyah-Singer index theorem for a "twisted" Dirac operator (see sections 7, 9.)

Similarly the union P of a half-space $\mathbb{R}^n_+ \subset \mathbb{R}^n$ and a ray normal to it admits no small neighbourhood V with $Sc(\partial V) \geq 0$.

Albeit non-existence of these V, which are $O(n-1)$-rotationally symmetric and equal to $\mathbb{R}^n_+$ away from P follows by a simple computation, the only proof in the general case available at the present day relies on the Dirac operator.

[?11] **Question.** What could be a, possibly non-geometric, extension of the concept of $Sc \geq 0$, where one would be able perform symmetrization and reduce the case of general neighbourhoods V to that of $O(n-1)$-symmetric ones?

(Such symmetrization, albeit imperfect, is possible for hypersurfaces with positive *mean* , rather than scalar, curvatures, see section 24.)

3 Topological Obstructions to $Sc > 0$ Implied by Vanishing Theorems for Harmonic Spinors on Compact Manifolds.

The techniques based on geometry of geodesics, which provide strong bounds on the size of manifolds with positive sectional and positive Ricci curvatures and on their fundamental groups.

For example, according to *Myers' theorem* [1941]

$$Ricci(X) > 0 \textit{ implies that } \pi_1(X) \textit{ is finite}$$

and

if Ricci ≥ 0 and if, for instance, the universal covering of X is contractible, than X is Riemannian flat by the Cheeger-Gromoll splitting theorem [1971].

But geodesics do not tell you much about $Sc \geq \sigma > 0$. In fact, the only available result here is the following.

Green-Berger Integral Scalar Curvature Inequality [1958], [1963], [1965]. *Among all manifolds X with given $vol(X)$ and $\int Sc(x)dx > 0$ round spheres maximize the average distance between conjugate points*[9] *on geodesics in X* [12]-Berger Lectures on geodesics].

(If $dim(X) = 2$ this follows from the Gauss-Bonnet formula for infinitesimally narrow geodesic triangles and the general case reduces to this by applying the Fubini theorem to the the space of the 2-frames in X moved by the geodesic flow.)

[9]The points of a geodesic segment are *conjugate* if the geodesic is not even locally distance minimising, not even locally, beyond these points.

The following two theorems came as a quite a surprise to differential geometers of that period.

Lichnerowicz' $\hat{A}$-Vanishing Theorem [1963]. *Compact Riemannian orientable spin manifolds X of dimension $n = 4k$ with $Sc(X) > 0$ have $\hat{A}[X] = 0$.*

Let us recall relevant definitions.

Spin. Since the fundamental group of the special (i.e. orientation preserving) orthogonal group $SO(n)$ for $n \geq 3$ is $\mathbb{Z}/2\mathbb{Z}$, there are exactly *two* different orientable bundles of rank $n \geq 3$ over closed connected surfaces. The trivial bundle is, by definition, *spin* and the non-trivial one is *non-spin.*

An orientable manifold X is called *spin* if the restrictions of the tangent bundle $T(X)$ to all surfaces $S \subset X$ are spin (i.e trivial).[10]

For instance, all orientable hypersurfaces $X^n \subset \mathbb{R}^{n+1}$ are spin, all 3-manifolds are spin, all simply connected n-manifolds with trivial second homotopy groups are spin. The simplest non-spin manifolds are the complex projective spaces $\mathbb{C}P^n$ of even complex dimensions n.

$\hat{A}[X]$-Genus. is a certain combination of Pontryagin numbers of X which satisfy

$$\hat{A}[X_1 \times X_2] = \hat{A}[X_1] \cdot \hat{A}[X_2]$$

and which may be, a priori, non zero for $n = 4k$.

For instance, if $n = 4$ then $\hat{A}[X]$ is proportional to the *rational Pontryagin number* $p_1[X]$ (or to the *signature* of X if you wish).

This $\hat{A}[X^4]$ vanishes if and only if a multiple of X bounds an oriented 5-manifold or if a multiple of the tangent bundle, $T(X) \oplus T(X) \oplus ... \oplus T(X)$, is the trivial bundle.

If $n = 8$, then $\hat{A}(X)$ is proportional to $(7p_1^2 - 4p_2)[X]$.

The existence of manifolds X where the Lichnerowicz theorem applies is not fully trivial. For instance. all compact homogeneous spaces, except for tori, admit, as we know, invariant metrics with $Sc > 0$.

The simplest example of a spin manifold with $\hat{A} \neq 0$ is *the Kummer surface* $X \subset \mathbb{C}P^3$ given by the equation

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0.$$

More generally, all smooth complex hypersurfaces $X \subset \mathbb{C}P^{m+1}$ (of real dimension $2m$) of degrees $\geq m + 2$ also have $\hat{A} \neq 0$ for even m [19]-Brooks The $\hat{A}$-genus of complex hypersurfaces].

These X are simply connected and there is no hint of a "geodesic argument" ruling out positive Ricci or positive sectional curvature on them. Yet, if the degree of such an X satisfies $deg(X) = m \mod 2$, then X is X is spin and it admits no metric with $Sc > 0$ by Lichnerowicz' theorem.

Remarkably, Lichnerowicz' theorem is sharp for some of these X:

Yau's solution of the Calabi conjecture implies that

smooth complex hypersurfaces $X \subset \mathbb{C}P^{m+1}$ of degrees $d = m + 2$, such as the Kummer surface, carry metrics with $Sc = 0$, in fact, with $Ricci = 0$.

[10] "Spin" makes sense also for non-orientable bundles and manifolds but we do not need them at this point.

But if $deg(X) \geq m+3$, then $|\hat{A}(X)| \geq 2$ and, unlike the case of $deg = m+2$, these X, if they are spin, admit no metrics with $Sc = 0$ if m is even. (We shall explain this later on.)

Hitchin's $\hat{\alpha}$-Vanishing Theorem [1974]. *There is a smooth topological mod 2 invariant of spin manifolds denoted $\hat{\alpha}[X]$, such that*

$$Sc(X) > 0 \Rightarrow \hat{\alpha}(X) = 0.$$

This $\hat{\alpha}$, defined on p. 147 in [**?**]-Atiyah-Singer The Index IV 1971] and on p.41 in [68]-Hitchin Harmonic Spinors 1974], may be non-zero for $n = 8k+1, 8k+2$. For instance, if X is homeomorphic (but not necessarily diffeomorphic) to the n-sphere, then $\hat{\alpha}$ vanishes if and only if X bounds a spin $(n+1)$-manifold.[11]

Thus, combined with results by Milnor and Adams, Hitchin's theorem implies that there are

manifolds Σ homeomorphic to S^n for $n = 8k+1, 8k+2$, k=1,2,3,..., which admit no metrics with $Sc(\Sigma) > 0$.

Moreover,

every spin manifold X of dimension n=8k+1 or n=8k+2 either admits no metric with $Sc(X) > 0$ itself or the connected sum X' of X with Σ admits no such metric, where observe, X and X' are piecewise smoothly homeomorphic.

Discussion. No geodesic kind of geometry is present either in the statements or in the proofs of these theorems, where the two key ingredients are as follows.

Atiyah-Singer index theorem for the Dirac operator D [1963-1971]. This D is a distinguished first order elliptic differential operator associated to the Levi Civita connection which acts on certain *spinor vector bundles* over X. These bundles and D are locally defined for all X but in order to be defined globally on X, the manifold X must be spin.

If $n = dim(X)$ is even, the spin bundle where D acts splits: $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$. Accordingly, $D = D^+ \oplus D^-$, where the operators D^+ and D^- are mutually adjoint for $D^+ : C^\infty(\mathbb{S}^+) \to C^\infty(\mathbb{S}^-)$ and $D^- : C^\infty(\mathbb{S}^-) \to C^\infty(\mathbb{S}^+)$ and where $ind(D) = dim(ker D^+) - dim(ker D^-)$.

The Atiyah-Singer index theorem implies, in particular, that if either $\hat{A}[X] \neq 0$ or $\hat{\alpha}(X) \neq 0$ then $ind(D) \neq 0$; therefore,

X must support non-zero harmonic spinors s,

where "harmonic" means $D(s) = 0$.

Schroedinger-Lichnerowicz-Weitzenboeck algebraic identity [1932], [1963]. [117]-Schroedinger Diracsches Elektron im Schwerefeld 1932], [76]-Lichnerowicz Spineurs harmoniques 1963].

$$D^2 = \nabla_s \nabla_s^* + \frac{1}{4} Sc,$$

where $\nabla_s \nabla_s^*$ in this formula is a (non-strictly) positive operator (coarse Laplacian on spinors) which make D^2 strictly positive for $Sc(X) > 0$. Thus,

(a) *if $Sc(X) > 0$, then there is no non-zero harmonic spinors on X,*

(b) *if $Sc(X) \geq 0$ and X carries a non-zero harmonic spinor, then $Sc(X) = 0$ and all harmonic spinors on X are parallel.* [12]

[11] There is much more in [68] than the $\hat{\alpha}$-vanishing theorem.

[12] X is assumed connected.

This implies the theorems of Lichnerowicz and Hitchin, where Lichnerowicz uses the 1963 version of the index theorem for $4k$-dimensional manifolds [7]-Atiyah-Singer The Index 1963], while Hitchin needs a more sophisticated mod 2 index formula in the dimensions $n = 8k + 1, 8k + 2$ from the 1971-Atiyah-Singer paper [8].)

4 Simply Connected and non-Simply Connected Manifolds with Positive Scalar and Ricci Curvatures.

By looking closer at what can be obtained by codimension 2 surgery of compact symmetric spaces and fibrations with symmetric fibers one finds out that

(i) *all simply connected non-spin manifolds of dimensions $n \geq 5$ admit metrics with $Sc > 0$* [53].

And – this is more difficult– [119]-Stolz, Simply connected manifolds 1992],

(ii) *Simply connected spin manifold X of dimension $n \geq 5$ admit metrics with $Sc > 0$* unless $n = 4k$ and $\hat{A}[X] \neq 0$ or $dim(X) = 8k+1, 8k+2$ and $\hat{\alpha}[X] \neq 0$.

For instance,

⋆ *Closed simply connected manifolds of dimensions $\neq 0, 1, 2, 4 \mod 8$* [13] *carry metrics with positive scalar curvatures.*

Thus, theorems of of Lichnerowicz and Hitchin fully account for *possible topologies of compact simply connected* manifolds X of dimensions $n \geq 5$ with $Sc > 0$.

It is not so in dimension 4, where the Witten-Seiberg invariant provide additional obstructions for $Sc > 0$. For instance none of smooth complex surfaces $X \subset \mathbb{C}P3$, of degree $d \geq 5$ admits a Riemannin metric with $Sc \geq 0$, be it spin (for even d) or non-spin (odd d) [75] [121]

On Non-simply Connected Manifolds. If X is a smooth closed, possibly, non-simply connected manifold, then the codimension two surgery, say in the category of oriented spin manifolds, shows that existence/non-existence of a metric with $Sc > 0$ on X depends only the *spin bordism class* of X in the *classifying space of the fundamental group* $\pi_1(X)$ for the canonical map $X \to \mathsf{B}\pi_1(X)$,

$$[X]_{spin} \in \Omega_n^{spin}(\mathsf{B}\pi_1(X)).$$

It is believed that

if X admits a metric with $Sc > 0$, then $[X]_{spin} \otimes \mathbb{Q} = 0$.

More generally, one has the following

[?12] **Conjecture: $\mathbb{Q}$-Non-Essentiality of Manifolds with $Sc > 0$.** *No rational homology class*[14] *in the classifying space $\mathsf{B}\Gamma$ of a discrete group Γ can be realised by a continuous map from a closed oriented (spin or non-spin) Riemannian manifold X with $Sc(X) > 0$ to $\mathsf{B}\Gamma$. In particular,*

[13] Notice that the set $\{0, 1, 2, 4\}$ is multiplicatively closed mod 8.

[14] This is false for *integer homology* where the simplest examples are lens spaces.

[?13] [$*$] *no closed aspherical*[15] *manifold admits a metric with* $Sc > 0$.

This $\mathbb{Q}$-non-essentiality conjecture was proven for 3-manifolds and [$*$] was confirmed for 4-manifolds which contain *incompressible surfaces* [54]. Also solution of [$*$] for all 4-manifolds was announced in [110], [114].

There are several classes of groups, starting from free Abelian ones, for which the $\mathbb{Q}$-non-essentiality conjecture has been established; we review some of the corresponding results in sections 29

On Scalar and Ricci Flat Manifolds. If a compact manifold X admits no metric with $Sc > 0$ then every metric with $Sc(g) = 0$ on X must be *Ricci flat*, which means vanishing of the Ricci tensor $Ricci(g)$ (see [71], where this is proven for complete manifolds as well).

This does not immediately constrains the topology of X by itself, since there is no known topological obstructions to $Ricci = 0$ on simply connected manifolds of dimension $n \geq 5$ unless X is spin and either $\hat{A}[X] \neq 0$ or $\hat{\alpha}[X] \neq 0$. (I might have missed some paper, most probably for $n = 4$.)

But if one of X is spin and one of this invariants does not vanish, then X carries a non-zero spinor which is parallel if $Sc(X) = 0$. Then the existence of a parallel spinor, implies that not only X is Ricci flat but it must have a special holonomy group (see [104], [26] and references therein.)

This, along with Berger's classification of special holonomy groups, rules out, for instance, metrics with $Sc = 0$ on Hitchin's spheres.

And it is even easier to see that connected sums of 2^n copies of spin manifolds with $\hat{A} > 0$ carry no metrics with $Sc = 0$, since the number of linearly independent parallel spinors on an n-dimensional manifold is bounded by 2^n. (Locally irreducible manifolds can not carry even *two linearly independent* parallel spinors as, probably, all "parallel spinor people" know, compare [126]

On the other hand if X does carry a metric with $Sc \geq 0$ and $dim(X) \geq 3$ then it seems not hard to prove (I suspect this must in the literature) it also carries lots of metric with $Sc = 0$.

But the following question remains widely open.

[?14] **Question.** How common are Ricci flat metrics on compact simply connected

manifolds X which admit metrics with positive scalar curvatures?

One may conjecture on the basis of of known examples (unless I miss some) that (almost) *all metrics* g *with* $Ricci(g) = 0$ *on compact manifolds* must have *special holonomy* or to be *special* in a similar respect, e.g. would satisfy (strong) additional partial differential equations that are not formal differential-algebraic corollaries of $Ricci(g) = 0$.

An opposite conjecture, which is based on the absence of apparent obstructions, would be that *every compact simply connected* manifold X of dimension ≥ 5 which admits a metric with $Sc > 0$ *carries infinitely many* "geometrically different"[16] g with $Ricci(g) = 0$.

All we can say with certainty is that

we do not know what the right question about $Ricci = 0$ *is.*

[15] A locally contractible topological space X is *aspherical* if the universal covering $\tilde{X}$ is contractible.

[16] Two metrics g_1 and g_2 on X may be regarded as geometrically different if (X, g_1) is not isometric to (X, g_2).

Recall at this point, we mentioned this earlier, that complex hypersurfaces $X \subset \mathbb{C}P^{m+1}$ of degrees $\geq m+3$ have $\hat{A}(X) \geq 2$ even m.

Therefore, they must support pairs of linearly independent g-harmonic, hence, g-parallel, spinors for all Riemannin metrics g on X. This (most probably) rules out metrics g with $Sc(g) = 0$.

On the other hand, apart from Lichnerowicz' case (spin + $\hat{A} \neq 0$), the complex projective hypersurfaces, except maybe(?) for some $X \subset \mathbb{C}P^m$, where $m = 2$ mod 4 and $deg(X) = m \mod 2$, [17] carry metrics with $Sc > 0$ according to the classification of simply connected manifolds with $Sc > 0$ (see above).

However, there is no apparent candidates for counterexamples to the following tentative conjectures.

[?14$\frac{1}{2}$] **Conjecture.** Smooth complex projective hypersurfaces of dimensions $dim_{\mathbb{C}} = m$ and degrees $d \geq m+3$, be they spin or non-spin, admit no metrics with $Ricci \geq 0$.

[?15] **Conjecture**. *Connected sums of sufficiently many copies of compact manifolds which are not homotopy spheres carry no metrics with Ricci* $= 0$.

Yet,

connected sums of products of spheres,

$$\underbrace{S^m \times S^n \# ... \# S^m \times S^n}_{k},$$

for all $m, n \geq 2$, *and* $k = 1, 2, 3, ...$ *do carry metrics with Ricci* > 0, [107], [25]

5 Asymptotically Flat and Periodic metrics on $\mathbb{R}^3$ with $Sc \geq 0$.

Somewhen in early 70s, Robert Geroch, motivated by the *positive mass problem* in general relativity, conjectured (see [111]-Schoen Yau On the proof of the positive mass 1979]) that the 3-torus admits no non-flat metric with $Sc > 0$.[18]

This, along with the positive mass conjecture, was solved by Schoen and Yau [112]Schoen Yau Existence of incompressible 1979] with a use of minimal surfaces instead of geodesics as follows.[19]

Schoen-Yau Rendition of the Second Variation Formula. Let Y be a smooth closed coorientable surface in Riemannian 3-manifold X and let $Y_t \subset U_\varepsilon(Y) \subset X$, $-\varepsilon < t < \varepsilon$, be equidistant surfaces in the ε-neighbourhood of $Y_0 = Y$. By elementary calculus,[20]

$$A' = \frac{d(Area(Y_t))}{dt}(0) = \int_Y \mu(y)dy,$$

[17] It must be known to some people if $\hat{\alpha}(X) \neq 0$ for any of these X.

[18] I could not find this on the web, but according to [111], Geroch made a weaker conjecture: *the Euclidean Riemannian metric on* $\mathbb{R}^3$ *admits no* $Sc > 0$ *perturbations with compact supports.*

[19] Earlier, minimal surfaces were similarly used in [22]-Burago Toponogov On 3-dimensional riemannian] in the study of 3-manifolds with positive sectional curvatures. Also surfaces were used in the work by astrophysicists on the positive mass conjecture, e.g. in [36]-Geroch Energy extraction 1973]

[20] This goes back to Bonnet, I guess.

where $\mu(y)$ is *the mean curvature* of Y that is $\mu(y) = c_1(y) + c_2(y)$ for the principal curvatures c_1 and c_2 of Y in X, and

$$A'' = \frac{d^2(Area(Y)}{d^2t}(0) = \int_Y \left(-c_1(y)^2 - c_2(y)^2 + \mu(y)^2 - Ricci_\nu(y)\right) dy$$
$$= \int_Y (2c_1(y)c_2(y) - Ricci_\nu(y))\, dy$$

Next, following [112]Schoen Yau Existence of incompressible 1979], observe that

$$Ricci_\nu = \frac{1}{2}(Sc(X)|Y - Sc(X|Y))$$

where (in the present case of dim(X)=3) $Sc(X|Y))$ is twice the sectional curvature of X on the planes tangents to Y, and that – this is a crucial algebraic step,

$$Sc(Y) = Sc(X|Y) + 2c_1c_2$$

by Gauss theorema egregium.[21] Thus,

$$A'' = \int_Y \frac{1}{2}(Sc(Y)(y) - Sc(X)(y))\, dy + \int_Y c_1(y)c_2(y)dy,$$

and by the Gauss Bonnet formula,

[SY]$_3$ $$A'' = 2\pi\chi(Y) - \frac{1}{2}\int_Y Sc(X)(y)dy + \int_Y c_1(y)c_2(y)dy.$$

Schoen-Yau solution of the Geroch $3D$ **Conjecture [1979].** *Let X be a compact orientable 3-dimensional Riemannian manifold with $Sc > 0$. Then all homology classes $h \in H_2(X)$ are spherical. Consequently, the 3-torus admits no metric with $Sc > 0$.*

Proof. By the geometric measure theory every h is representable by an area minimising minimal surface $Y \subset X$ smoothness of which is guaranteed by a theorem by Fleming-Almgren's regularity theorem [**?**, **?**, **?**].[22] Since it is minimal $A' = 0$ and it has zero mean curvature: $\mu = c_1 + c_2 = 0$ and $c_1c_2 \le 0$, and since it is area minimising, $A'' \ge 0$; hence, the Euler characteristic $\chi(Y)$ is *positive* by the **[SY]**$_3$ inequality. QED. [23]

Commentary. There is more to **[SY]**$_3$, than the above corollary. In fact, **[SY]**$_3$ says that if $Sc(X) \ge \sigma$ the function $Y \mapsto Area(Y)$ is *concave* in certain directions on *saddle* surfaces Y (i.e. those with $c_1c_2 \le 0$) such that

$$\frac{2\pi\chi(Y)}{area(Y)} \ge \sigma,$$

which tells you something about manifolds with $S(X) \ge \sigma \le 0$ for $\sigma \le 0$. as well as for $\sigma > 0$.

[21]However simple, this is a remarkable identity, since the intrinsic curvature of Y should, a priori, depend on the *derivatives* of the curvature of X, but these derivatives cancel due to some symmetries, similarly to what happens in the derivation of (Bochner)-Schroedinger-Lichnerowicz-Weitzenboeck formula.

[22]The bulk of the paper [112] by Schoen and Yau is dedicated to an independent proof of the existence and regularity of minimal Y.

[23]It takes a little extra effort to show that all metrics on $\mathbb{T}^3$ with $Sc \ge$ are flat.

For instance, the above Schoen-Yau argument shows that

every surface $Y \subset X$ with positive genus ($\chi \leq 0$) and such that

$$area(Y) < 2\pi|\sigma| \cdot |\chi(Y)|$$

is homologous to a surface of smaller genus; hence, it is homologous to zero if it is genus minimising in its homology class.

Synge's 1936 Second Variation Argument for Manifolds with Positive Curvatures. The Schoen-Yau proof vaguely resembles the following.

Let γ be a geodesic in a Riemannian manifold X and ∂ be a unit vector field in a neighbourhood of γ which is normal to γ and is parallel along γ.

Since the curve γ is geodesic, the derivative of the length of γ by the flow ∂_t is zero and if X has *strictly curvatures* $\geq \kappa > 0$, then the second derivative of of the length of γ is negative,

$$\frac{\partial_t^2}{dt^2} length(\gamma)_{|t=0} < 0.$$

It follows that

if $sect.curv(X) > 0$ and γ is a closed geodesic which is *length minimising* in its homotopy class in X, then it admits *no parallel normal field.*

On the other hand a closed geodesic γ in an *even dimensional orientable* manifold, always carries a parallel normal field, since the holonomy operator in the normal space $H : N_{x_0}(\gamma) \circlearrowleft$, $x_0 \in \gamma$, has a non-zero fixed vector, because $dim(N)$ is even and because H is an orthogonal operator with determinant +1 and N is even dimensional.

Thus, $sect.curv(X) > 0$ rules out closed minimal geodesics in X; therefore,

closed orientable even dimensional Riemannian manifolds with strictly positive sectional curvatures are simply connected.

Similarly,

odd dimensional Riemannian manifolds with strictly positive sectional curvatures are orientable.

6 Conformal Laplacian and Codimension 1 Descent with Positive Scalar Curvature.

Kazdan-Warner's Conformal Modification of Scalar Curvature [72, 73]. Let (X_0, g_0) be a compact Riemannian manifold of dimension $n \geq 3$ where the scalar curvature function is denoted $S_0(x) = Sc(g_0)(x)$.

If the conformal Laplace operator $L = L_{g_0}$, defined as

$$f(x) \mapsto L(f(x)) = -\Delta f(x) + \gamma_n S_0(x) f(x) \text{ for } \Delta = -d^*d \text{ and } \gamma_n = \tfrac{n-2}{4n-1}$$

is positive, i.e.

$$\int_X \langle Lf(x), f(x)\rangle dx = \int_X \left(||df(x)||^2 + \gamma_n S_0(x) f(x)^2\right) dx \geq 0$$

for all smooth functions $f(x)$, then g_0 is conformal to a metric g_1 with positive scalar curvature. And if L is strictly positive, then one can get $Sc(g_1) > 0$ as well.

Proof. A straightforward computation shows that

$$◐ \qquad Sc\left(f^{\frac{4}{n-2}}g_0\right) = \gamma_n^{-1}L(f)f^{-\frac{n+2}{n-2}}$$

for all smooth positive functions f.[24]

Let $L(f) = \lambda f$, where λ is the minimal eigenvalue of L. If L is positive (strictly positive), then $\lambda \geq 0$ ($\lambda > 0$), while $f(x)$ doesn't vanish on X; otherwise, the non-negative function $f_1 = f + |f|$ (a priori, non-smooth at its zero set) would be yet another (a posteriori, smooth by elliptic regularity) solution of the equation $L(f) = \lambda f$. But such a relation can't hold near the boundaries of the zero sets of positive functions on X by a simple (and standard) symmetrisation argument.

Therefore, ◐ applies to f (chosen > 0) and the proof follows.

[$15\frac{1}{2}$] **Problem.** Work out the necessary and sufficient condition on (the Ricci curvature of?) g on a (possibly incomplete) manifold X for the existence of a small (conformal?) perturbation g' of g such that $g' > g$ and $Sc(g') > Sc(g)$.[25]

One knows in this respect that if $Sc(g) = 0$, then *Ricci* $\neq 0$ is known to be sufficient for the existence of such perturbations g' [71] and if $Sc(g) > 0$ and g is *Einstein*, i.e. $Ricci(g) = \lambda g$, then no small perturbation $g' > g$ with $Sc(g') > Sc(g)$ exists, see [39]-Goette Semmelmann Scalar curvature estimates 2002 where this is proven for some non-Einstein metrics as well.

Schoen-Yau Codimension 1 Descent Theorem [113]-Shoen Yau Structure 1979]. *Let X be a compact orientable n-manifold with $Sc > 0$. If $n \leq 7$, then every non-zero homology class $h \in H_{n-1}(X)$ is realisable by a map from a compact oriented $(n-1)$-manifold to X, where this manifold admits a metric with $Sc > 0$.*

In fact, there exists a codimension one submanifold $X_{[-1]} \subset X$, such that the conformal Laplacian $L = L_{g_{[-1]}}$ on $X_{[-1]}$ with the induced Riemannian metric $g_{[-1]}$ is strictly positive which can be conformally modified to have $Sc > 0$ by the above Kazdan-Warner theorem.

Proof. Take the volume minimising subvariety $Y \subset X$ in the class of h for $X_{[-1]}$ and observe that Y is a smooth cooriented submanifold by the *Almgren-Simons regularity theorem* which guaranties smoothness of Y for $n \leq 7$.

Let ν be the unit vector field normal to Y, let f be a smooth function on Y and let A'' be the second derivative of the $(n-1)$-volume of Y with respect to the field $\partial = f\nu$. [26] By the same calculation as in the derivation of the above $\mathbf{[SY]}_3$, one obtains:

$$\mathbf{[SY]}_f \quad A'' = \int_Y ||df(y)||^2 - \frac{1}{2}\left(||curv(Y)||^2(y) - Sc(Y)(y) + Sc(X)(y)\right)f^2 dy,$$

where $||curv(Y)||^2 = \sum c_i^2$ for the principal curvatures c_i of $Y \subset X$.

Since Y is volume minimising, $A'' \geq 0$ for all f and since $||curv(Y)||^2 \geq 0$ and $Sc(X) > 0$,

$$\int_Y ||df(y)||^2 + \frac{1}{2}Sc(Y)(y))f^2 dy > 0$$

[24]There must be a conceptual derivation of this formula.

[25]If $Sc(g) < 0$, then interesting perturbations g' of g are such that $g' < g$ and $Sc(g') > Sc(g)$.

[26]In general, the second derivatives $\partial^2 vol_{n-1}(Y)$ is defined with fields ∂ extended to a *neighbourhood* of Y in X. But if Y is minimal, then $\partial vol_{n-1}(Y) = 0$ for all fields ∂; hence, $\partial^2 vol_{n-1}(Y)$ depends only on $\partial|Y$.

for all functions $f \neq 0$ on Y. This means that the operator $-\Delta + \frac{1}{2}Sc(Y)$ is strictly positive and since $\gamma_n = \frac{n-2}{4n-1} < \frac{1}{2}$ the conformal Laplacian $-\Delta + \gamma_n Sc(Y)$ is strictly positive as well. QED.

$\bigstar_n$ **Corollary: No $Sc > 0$ on Schoen-Yau-Schick n-Manifolds.** A compact orientable n-manifold is *Schoen-Yau-Schick ,* if there exist $n-2$ integer homology classes $h_1, h_2, ..., h_{n-2} \in H_1(X)$, such that their intersection $h_1 \frown h_2 \frown ... \frown h_{n-2} \in H_2(X)$ is *non-spherical*, i.e. it is *not contained* in the image of the *Hurewicz homomorphism* $\pi_2(X) \to H_2(X)$.

Alternatively, a homology class $h \in H_n(K)$, where $K = K(\Pi, 1)$ is the Eilenberg-MacLane space be for an Abelian group Π, is called *SYS*, if its consecutive *cap-producs* with some cohomology classes $h_1, h_2, ..., h_{n-2} \in H^1(K, \mathbb{Z}$ are non-zero,

$$(...((h \cap h_1) \cap h_2) \cap .... \cap h_{n-2}) = h \cap (h_1 \smile, ..., \smile h_{n-2}) \neq 0 \in H_2(K).$$

(Geometrically speaking, generic 2-dimensional intersections of the n-cycles $C \subset K$ representing h with $(n-2)$-codimensional pullbacks of generic points of, some, say piecewise linear, maps $K \to \mathbb{T}^{n-2}$ are non-homologous to zero.)

Then a manifold X is SYS if the Abel classifying map $X \to K(\Pi, 1)$ for $\Pi = H_1(X)$ sends the fundamental class $[X] \in H_n(X)$ to a SYS class in this $K(\Pi, 1)$.

(Recall that, by definition, the spaces $K(\Pi, 1)$ have *contractible* universal coverings and fundamental groups *isomorphic* to Π. The standard finite dimensional approximations to these K are products of tori and *lens spaces* $L_i = S^N/\mathbb{Z}_{l_i}$, where the latter, observe, carry natural metrics with $Sc > 0$.

Abel's $X \to K$ maps are uniquely up-to homotopy, are characterised by inducing isomorphisms on the 1-dimensional homology groups.)

It is obvious that non homologous to zero 1-codimensional hypersurfaces in SYS manifolds are SYS as well; hence, their homologically non-trivial intersections are also SYS.

Thus, arguing by induction on n starting from the obvious case of $n = 2$, Schoen and Yau prove $\bigstar_n$ for $n \leq 7$ in their 1979 paper:

SYS manifolds of dimensions ≤ 7 admit no metrics with $Sc > 0$.

The primary examples of SYS manifolds are the n-torus $\mathbb{T}^n$ and the *overtorical* n-manifolds X which admit maps $X \to \mathbb{T}^n$ with *non-zero* degrees.

If $n = 2, 3$, then all Schoen-Yau-Schick n-manifolds admit such maps to $\mathbb{T}^2$ and $\mathbb{T}^3$. But starting from $n = 4$ there are Schoen-Yau-Schick n-manifolds X, such that all continuous maps $X \to \mathbb{T}^n$ are contractible the $(n-1)$-skeleton of $\mathbb{T}^n$.

Example. Let $P : \mathbb{T}^n \to \mathbb{T}^{n-1}$ be a coordinate projection and let $X = \mathbb{T}^n_\bullet$ be obtained by the 2D surgery of $\mathbb{T}^n$ over the circular fiber $\mathbb{T}^1_{t_\circ} = P^{-1}(t_\circ)$ for some point $t_\circ \in T^{n-1}$.

Let $T_i^{n-2} \subset \mathbb{T}^{n-1} \setminus \{t_\circ\}$, $i = 1, ...n-1$ be subtori which intersect at a single point, say at $t \neq t_\circ$ and observe that the surgery does not affect the P-pullbacks of these tori. Thus we have

$$P^{-1}(T_i^{n-2}) \subset X$$

where these tori intersect in X over the circle

$$P^{-1}(t) = \bigcap_{i=1,...n-1} P^{-1}(T_i^{n-2}).$$

Since this circle represents *non-zero* (torsion) element in $H_1(X)$ the intersection of the $n-2$ homology classes $h_i = [P^{-1}(T_i^{n-2})] \in H_{n-1}(X)$, $i = 1,2,...,n-2$. is aspherical and X is SYS. On other other hand, since $rank_{\mathbb{R}} H_1(X) = n-1 < n$, all maps $X \to \mathbb{T}^n$ have degrees zero.

Conclude by observing that

products of SYS manifolds by overtorical manifolds are SYS

while

products of three or more non-overtorical SYS manifolds are not SYS.

Also notice that

finite coverings of non-overtorical SYS manifolds may be non-SYS.

[?16] **Conjecture. Singularities are Unstable.** Brian White told me about 30 years ago that he believed that

Volume minimising hypersurfaces in generic Riemannian manifolds X are non-singular: singularities disappear under arbitrarily small smooth perturbations of metrics in X.

This was confirmed in 1993 by Nathan Smale, [118] for $n = 8$, which extends the Schoen-Yau theorem to $n = 8$.

[?17] **Conjecture 6. ISC: Singularities are Irrelevant.** Schoen and Yau announced 35 years ago [110], [114] that their descent metod extends to singular minimal subvarieties

In a series of papers over the last decade, Lohkamp suggested an approach to the solution of this conjecture(see [86]-Lohkamp The Higher Dimensional Positive Mass Theorem II 2016 where one can find references to his earlier papers). Recently, Schoen and Yau published an alternative proof of a version of the irrelevance conjecture [115]. [27]

7 Flatly Twisted Spinors over Tori and their Precursors in Algebraic Topology.

The kernels of D^+ and D^- on the flat even dimensional torus consist of parallel spinors, where both spaces have *equal dimensions* ($= 2^{n-1}$); hence the ordinary index of the Dirac operator vanishes on the torus.

However, these parallel spinors make the following *K-theoretic index* of D non zero not only on tori, but also on what we call *over-torical* manifolds X, which admit maps to the n-torus of non-zero degrees, or equivalently, admits n homology classes with *non-zero* intersection index.

The index of the Dirac operator $D = D^+ \oplus D^-$ on a Riemannian manifold X of even dimension n we speak about takes the values in the K-theory of the torus $\mathbb{T}^m$ which comes about as the space of the *unitary characters* of $\pi_1(X)$, i.e. of homomorphisms $\tau : \pi_1(X) \to \mathbb{T}$, which is the m-torus for $m = rank_{\mathbb{Q}} H_1(X)$.

These homomorphisms define flat unitary bundles l_τ, $\tau \in \mathbb{T}^m$, over X which are used to "twist" the spinor bundle $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ over X by taking the tensor products $\mathbb{S}^\pm \otimes l_\tau$ and accordingly twist the Dirac operator on the sections of these bundles,

$$D^\pm_\tau = D^\pm_{\otimes l_\tau} : C^\infty(\mathbb{S}^\pm \otimes l_\tau) \to C^\infty(\mathbb{S}^\mp \otimes l_\tau)$$

[27] I have not studied the papers by Lohkamp and Schoen-Yau in depth.

The kernels of D^+_τ and D^-_τ, even though their dimensions, in general, depend on τ, may be regarded as vector bundles over $\mathbb{T}^m$ the difference of which defines the K-theoretic index of Atiyah-Singer.

$\star$ *If X admits homology classes $h_1, h_2, ..., h_n \in H_1(X)$ with non zero intersection index,*

$$h_1 \frown h_2 \frown ... \frown h_n \neq 0$$

or, equivalently, if X admits a continuous map to the n-torus with non-zero degree, then this K-theoretic index is non-zero.

This was shown in 1972 by Lustig [?] for the *signature operator* in the context of the *Novikov conjecture* and adapted to $Sc > 0$ in [52].)

$\star\star$ **Corollary. No $Sc > 0$ on Over-torical Spin Manifolds [1980]**. *Let X be a compact n-dimensional Riemannian spin manifold. If X admits a map of non-zero degree to the n-torus $\mathbb{T}^n$, e.g. if X s homeomorphic to $\mathbb{T}^n$, then X admits no metric with positive scalar curvature.*

Proof. Since the bundles l_τ are flat, the twisted Dirac operators D_τ are locally equal to the untwisted D; hence they satisfy the Schroedinger-Lichnerowicz-Weitzenboeck formula. Therefore, if the scalar curvature were positive, the kernel of of D_τ would be zero for all $\tau \in \mathbb{T}^m$.

This proves our assertion for even n by contradiction with $\star$ and the odd dimensional case follows by multiplying X with the circle.[28] QED.

Twisted Elliptic Operators, $Sc > 0$, and Higher Signatures. The original arguments by Lichnerowicz and Hitchin link positivity of the scalar curvature of X to rather subtle differential-topological invariants of X, which, are *not homotopy invariants* for $n \geq 5$ and in the Hitchin case they are *not topological, not even combinatorial,* invariants of X.

In the twisted case, the emphasis is shifted from the smooth topology of X to the homotopy theoretic properties of manifolds X concerning the structure of vector bundles over them where these properties in the case of infinite, especially torsionless, groups $\pi_1(X)$ are often (always) reflected in the coarse geometry of the universal covering $\tilde{X}$ of X. Eventually, the "twisted road", you may expect, will bring you far from smooth spaces and regular metrics.

The origin of of "twisted paradigm shift" resides in the work by Lustig and Mishchenko who proved *Novikov's higher signature conjecture* for fundamental groups of manifolds of non-positive sectional curvatures.

Prior to their work (and, of course, the work by Novikov) the basic "untwisted" result of this kind was the (obvious) homotopy invariance of the ordinary signature $sign(X)$ (related to Pontryagin classes by the Thom-Hirzebruch L-formula[29]).

But then Lustig and, in a significantly more general form, Mishchenko brought forth the full power of the (generalised) twisted Atiyah-Singer index theorem for the signature operator and arrive the following

Theorem [1974]. Let $f : X \to B$ be a smooth maps between closed manifolds, where $dim(X) = dim(B) + 4k$, and let $sign(X, [f]_B)$ be the the signature of the pullback of a generic point $b \in B$.

[28] This product argument is manifestly immoral.

[29] Recall that $sign(X)$ of a $4k$ dimensional oriented manifold X is the signature, (the number of positive minus the number of negative terms in the diagonalization) of the quadratic intersection form on the $2k$-dimensional homology of X.

It is obvious (if you are fluent in elementary differential topology) that this $\sigma(X,[f]_B)$ is independent of $b \in B$ and of a representative of f in its homotopy class $[f]$. What is by no means obvious, however – this was proven by Mishchenko – is that

if B supports a metrics with non-positive curvature, then $\sigma(X,[f]_B)$ is a homotopy invariant of X.[30]

(This invariance is dramatically false for simply connected manifolds B and remains conjectural for general aspherical B.)

The corresponding "twisted story" with the scalar curvature, which started a few years later, remains, in many respects, close to that with the signatures.

For instance, Jonathan Rosenberg showed (see [101] and references therein) that $\mathbb{Q}$-non-essentiality of $Sc > 0$ would follow from the *strong Novikov conjecture.*

But there is an additional "geometric twist" to the scalar curvature story which makes it more intriguing and interesting than the Novikov conjecture.

8 Harmonic Stability of Parallel Spinors and the Positive Mass Theorem.

Since the index of the Dirac operator on the torus is zero the harmonic spinors (which are parallel on the flat torus) may, a priori, disappear [31] under small generic perturbations of the flat metric.

However – this lies at the core of the above argument –

harmonic spinors on $\mathbb{T}^n$ are stable in the τ-family.[32]

Namely, whenever the Dirac operator D loses harmonic spinors under a deformation/perturbation of the metric, some τ-twisted perturbed D necessarily acquires a non-zero harmonic spinor.[33]

The proof of this doesn't even need the full power of the index theorem, but only a simple topological argument applicable to general families of *Fredholm operators* + *Fourier analysis* on the flat torus. And the case of X non-diffeomorphic to $\mathbb{T}^n$ needs only the easy part of the proof of the Atiyah-Singer theorem.

If one passes from the torus to the Euclidean space $\mathbb{R}^n$, that is the universal covering $\mathbb{R}^n$ of $\mathbb{T}^n$, then the above can be reformulated in terms of "twisted stability" of (parallel) harmonic spinors under $\mathbb{Z}^n$*-periodic* perturbations of the Euclidean metric.

Then, instead of $\mathbb{Z}^n$*-periodic* perturbations one may look at perturbations with *compact supports* and/or at perturbations with fast decay at infinity.

It was proven by Witten in this regard that the parallel spinors on $\mathbb{R}^n$ are harmonically stable under suitably weighted L_2 perturbations of the Euclidean metric.

[30]This was preceded by results proven by topological techniques starting from Novikov's work on the topological invariance of Pontryagin classes, who proved this theorem for n-tori B in many cases, see [33, 102] and reference therein.

[31]It may take an effort to actually prove this.

[32]This is vaguely similar to the *stability* of volume minimising hypersurfaces exploited by the Schoen-Yau argument.

[33]There is no twisted, i.e. with non-zero τ in the (Abelian) group $\mathbb{T}^m$, harmonic spinors on the flat torus, only the parallel untwisted ones.

Namely, Witten's argument shows that

• *if a complete Riemannian spin manifold $\tilde{X}$ is isometric to $\mathbb{R}^n$ at infinity, then it support non-zero harmonic spinors s such that $(s - s_0)_{|\infty} \in L_{2,wei}$ for all parallel spinors s_0 on $\mathbb{R}^n$.*

This, together with the Schroedinger-Lichnerowicz-Weitzenboeck formula with the boundary term, shows that *such an $\tilde{X}$ can not have $Sc > 0$.*

Of course, this conclusion also follows from ⋆⋆, since compactly supported perturbations of $\mathbb{R}^n$ obviously extend to $N \cdot \mathbb{Z}^n$-periodic ones, but Witten's • holds for a class of properly defined *asymptotically flat* perturbations, which delivers *an alternative proof of the Schoen-Yau positive mass theorem.* (See [95], [66] and references therein.)

Lokhamp's Deformation [1999] [83]. *Asymptotically flat metrics on $\tilde{X}$ with $Sc > 0$ and negative energy*[34] *can be deformed to asymptotically flat ones which have $Sc > 0$ on the non-flat part of $\tilde{X}$.*

Thus, the Witten (Bartnik for n>3) positive mass theorem for spin manifolds reduces to ⋆⋆ from the previous section.

Another instance of such reduction is as follows.

Semiperiodic Metrics. Let X be a complete Riemannian manifold X with $Sc(X) \geq 0$ which is isometric at infinity to a complete flat manifold X_{fl}.

☾ *If there exists a homomorphism between the fundamental groups $h : \pi_1(X) \to \pi_1(X_{fl})$ compatible in the obvious sense with the isometry between these manifolds at infinity,*[35] *then X is flat.*

Proof. Complete flat manifolds $X_{fl} = \mathbb{R}^n/\Gamma$ for $\Gamma = \pi_1(X_{fl})$ can be approximated by rational ones, $X'_{fl} = \mathbb{R}^n/\Gamma'$ where the rotation angles of all isometries in Γ' are rational, or equivalently, where some subgroups $\Gamma'' \subset \Gamma'$ of finite indexes consist of parallel translations.

It follows by a simple argument (with a use of conformal deformations as in the Kazdan-Warner theorem in section 6) that the metric in X admits an arbitrarily small smooth perturbation with $Sc \geq 0$ such that the corresponding X_{fl} becomes rational. Then ⋆⋆ applies to a finite covering of X and the proof follows.[36]

[☾ ?]. **Questions and Conjectures.** The existence of the homomorphism h may be (?), in general, essential. But if $\Gamma = \pi(X_{fl})$ acts on $\mathbb{R}^n$ by *parallel translations*, then, *conjecturally*,

[?18] *all X with $Sc(X) \geq 0$ isometric to X_{fl} at infinity are flat.*
(These X can be simply connected, for instance.)

[?19] Probably, one can fully determine assumptions on $\pi_1(X)$ depending on X_{fl} needed for this conclusion for all flat manifolds X_{fl}.

[?20] Also one can possibly relax the *isometry at infinity* condition by some "asymptotic flatness" and negativity of a suitable "energy at infinity".

[34]The *energy* or *mass* of an asymptotically flat metric is defined as the limit of certain curvature integrals over the spheres $S^{n-1}(R) \subset \tilde{X}$, $R \to \infty$.

[35]One may assume h is induced by a continuous map $X \to X_{fl}$ which is isometric at infinity.

[36]The proposition ⋆⋆ needs X to be spin, where the non-spin case needs a use of non-singular minimal hypersurfaces which are are available for $n \leq 8$, as it is explained in the previous section. Also one has to take special care of the case $Sc(X) = 0$.

9 Twisting Dirac with Moderately Curved Bundles.

The proof of the inequality $\inf_{x\in X} Sc(X)(x) \le 0$ for overtorical manifolds X (see ⋆⋆ in section 7) with a use of the Dirac operator twisted with *flat* bundles l_τ tells you nothing about the geometry of X, since the family l_τ originates from the fundamental group $\pi_1(X)$ which is not (directly) related to anything (geo)metric in X .

But the existence/non-existence of connections with *small* (rather than zero) curvatures on vector bundles over X visibly depends on such properties. An outcome of this is the *two curvatures inequality* ꝏ which is stated below and which shows that

> manifolds with scalar curvature $Sc \ge \sigma > 0$ can not be too large area-wise,

where the area-wise size of X is measured by the norms curvatures of vector bundles L over X defined as follows.

$||curv(L)||$ and Essential Bundles. Given a vector bundle (L,∇) with an orthogonal (unitary in the complex case) connection, over a Riemannian manifold X, let

$$||curv(L)||(x) = ||curv(\nabla)||(x) = ||curv(L,\nabla)||(x)$$

denote

the infimum of positive functions $C(x)$ such that the maximal rotation angles $\alpha \in [-\pi,\pi]$ of the parallel transports along the boundaries of smooth discs S in X satisfy

$$|\alpha| \le \int_S C(s)ds$$

(The holonomy operator splits into the direct sum of rotations $z \mapsto \alpha_i z$, $z \in \mathbb{C}$, $\alpha_i \in \mathbb{T} \subset \mathbb{C}$, $i = 1, 2, ..., rank(L)$, and our $\alpha = \max_i \alpha_i$.)

For instance, this norm of the tangent bundle (complexified if you wish) of the product of spheres, satisfies

$$\left\|curv\left(T\left(\bigtimes_i S^{n_j}(R_j)\right)\right)\right\| = \frac{1}{\min_j R_j^2}.$$

A complex vector bundle L with a on a manifold X is called $\mathbb{Q}$-*homologically essential*, if it is *trivial at infinity* and if some Chern number of L does not vanish, where this number is for non-compact X is defined with the corresponding Chern class in the *cohomology of X with compact supports.*

Two Curvatures Inequality. [1983] [54]. Let L be a $\mathbb{Q}$-homologically essential complex vector bundle with a unitary connection[37] over a complete Riemannian n-manifold X.

[37] "Unitary connection" means a connection which preserves a *Hermitian structure* in L and, similarly, "orthogonal connections" of real vector bundles are supposed to preserve Euclidean structures in the fibers.

If X is spin and if the scalar curvature $Sc(X)$ is uniformly positive (at infinity), $Sc \geq \sigma > 0$, then

ↀ $$\inf_{x\in X} \frac{Sc(X)(x)}{||curv(L)||(x)} \leq const_n.$$

Sketch of the Proof. The Schroedinger-Lichnerowicz-Weitzenboeck formula for the twisted Dirac operator $D_{\otimes L} : C^\infty(\mathbb{S}\otimes L) \to C^\infty(\mathbb{S}\otimes L)$ reads

$$D^2_{\otimes L} = \nabla_\otimes \nabla^*_\otimes + \frac{1}{4} Sc + \mathsf{R}_L$$

where $\nabla_\otimes$ denotes the covariant derivative operator in $\mathbb{S}\otimes L$ and R_L is a certain (zero order) operator which acts in the fibers of the twisted spin bundle $\mathbb{S}\otimes L$ and which is derived from the curvature of the connection in L.

If we are not concerned with the sharpness of constants , all we have to know is that R_L is controlled by

$$||\mathsf{R}_L|| \leq const \cdot ||curv(L)||$$

for $const = const(n, rank(L))$, where a little thought (no computation is needed) shows that, in fact, this constant depends only on $n = dim(X)$. (See [95, 74] for details and references.)

Since L is homologically essential, the index of D twisted with some bundle $L^\otimes$ associated to L[38] is non-zero by the Atiyah-Singer theorem and the proof for compact manifolds follows, and if X is non-compact one applies the *relative index theorem* for complete manifolds with uniformly positive scalar curvatures at infinity, [54]. .

⊘ **Corollary: Rough Non-Hypersphericity Inequality.**[39] *Let X be a complete orientated Riemannian manifold with $Sc(X) \geq \sigma > 0$. If X is spin, then it admits no locally constant at infinity*[40] *map f to the unit sphere $S^n(1)$, such that*

- *$f : X \to S^n(1)$ has positive degree,*
- *f is λ-Lipschitz with $\lambda \leq const'_n \sigma$.*

Moreover no map f with $deg(f) \neq 0$ decreases the areas of surfaces in X by a factor $\lambda^{-2} \geq (const'_n \sigma)^{-2}$.

Proof. If n is even, one applies ↀ to the bundle L on X induced from L_0 on S^n with $Chern_m(L_0) \neq 0$ and if n is odd one does this to $X \times \mathbb{R}$.

[+/−] Corollary to Corollary: Incompatibility of *sect.curv* ≤ 0 **and** $Sc > 0$**.** *If a compact manifold admits a metric g_0 with non-positive sectional curvatures, then it carries no metric g with $Sc(g) > 0$.*

Proof. The existence/nonexistence of λ-contracting maps of non-zero degrees from the universal covering $\tilde{X}$ of a *compact* Riemannin manifold X to $S^n(q)$ for *all* $\lambda > 0$ *does not* depend on the metric in X.

Since the universal coverings of manifolds with non-positive curvatures, being contractible, are spin and since they (obviously) admit such maps, ⊘ applies and the proof follows.

[38] The bundle $L^\otimes$ you need is the tensor product of certain exterior powers of L, [44].

[39] A sharp version of this due to M Llarull is presented in section 17

[40] Such a map is constant on the connected components of X minus a compact subset.

[?21] **Problem.** Remove the spin an the uniform positivity conditions, relax completeness and determine the sharp value of $const_n$ (depending on the K-theory class of L) in the inequality ⚭.

Remark. The method of minimal hypersurfaces (where for $n \geq 10$ one needs Brian White's or a version of Schoen-Yau's singularity conjectures from section 6) implies the non-spin version of ⦸ for λ-Lipschitz maps but it remains unclear if it ever works for *area* λ^2*-contracting* maps.

Also some bounds on on curvatures of vector bundles over non-spin manifolds X with $Sc(X) \geq \sigma > 0$ can be obtained with the Dirac operator as follows.

10 Dirac Twisted with $\frac{1}{2}$-Spin Bundles.

If ξ is a spin bundle over X then the spaces of spinors $\mathbb{S}_x(\xi)$, $x \in X$, make a bona fide vector bundle over X, denoted $\mathbb{S}(\xi)$ which can regarded as a *spinor square root* $\sqrt{\xi}$. But if ξ is non-spin, this bundle, which is defined everywhere locally, is not defined globally due to the inherent $\pm\sqrt{}$ ambiguity in spinors; we write $\mathbb{S}^{\pm}\surd(\xi)$ in this case and call it the *(virtual)* $\frac{1}{2}$*-spin bundle* associated to ξ.

If two, say orientable, bundles ξ_1 and ξ_2 of ranks $n \geq 3$ over a manifold X are *spin equivalent*, i.e. if their restrictions $\xi_{1,2}|S$ to all closed surfaces S in X are mutually isomorphic, i.e. simultaneously spin or non-spin, then their virtual square roots $\mathbb{S}^{\pm}\surd(\xi_1)$ and $\mathbb{S}^{\pm}\surd(\xi_2)$ have "equivalent ambiguities" which means in rigorous terms that their tensor product $\mathbb{S}^{\pm}\surd(\xi_1) \otimes \mathbb{S}^{\pm}\surd(\xi_2)$ is a well defined bundle.

Thus, although the Dirac operator D itself is not defined on non-spin manifolds, *the twisted Dirac operator* $D_{\otimes L^{[\pm]}}$ *is well defined for* $\frac{1}{2}$*-spin bundles* $L^{[\pm]}$ *which have the same* $\pm$ *ambiguities as* $\mathbb{S}^{\pm}\surd(X) = \mathbb{S}^{\pm}\surd(T(X))$, *e.g.* $L^{[\pm]} = S^{\pm}\surd(X) \otimes L_0$, *where* L_0 *is an arbitrary (true) vector bundle*

Now the proof of above inequality ⚭ automatically extends to *non-spin* manifolds X as follows.

Non-spin Two Curvatures Inequality. *Let X be a complete Riemannian (not necessarily spin) n-manifold X and $L^{[\pm]}$ be a $\mathbb{Q}$-homologically essential virtual complex vector bundle*[41] *defined over X which has the same $\pm$ambiguity as $\mathbb{S}S^{\pm}\surd(X)$.*

Then there exists a point $x \in X$, such that

$$Sc(X)(x) \leq const_n \|curv(L^{[\pm]})\|(x). \qquad (⚭_{\frac{1}{2}})$$

Dirac over Sphere Bundles. Let is give an alternative generalization of ⚭ to non-spin manifolds by passing to sphere bundles associated to non-spin bundles ξ over X as follows.

$X \rtimes_\xi S^m$ **and its Curvature.** If a bundle ξ over X is endowed with an *orthogonal structure*, i.e with Euclidean metrics in the fibers, let $X \rtimes_\xi S^m(R)$, $m = rank_{\mathbb{R}}(\xi) - 1$, denote the total space of the (sub)bundle of R-spheres in the fibers of ξ.

[41]Be $L^{[\pm]}$ a true or a virtual bundle, it is defined along with its connection everywhere locally on X. Therefore, its curvature $curv(L^{[\pm]})$, hence, its (real) Chern classes derived from $curv(L^{[\pm]})$, are *globally defined* on X, which also makes the concept of $\mathbb{Q}$-Essentiality defined for all these $L^{[\pm]}$.

Next, given a Riemannian metric g in X and a orthogonal connection ∇ in ξ let $\hat{g} = \hat{g}(\nabla, R)$ be the Riemannian metric in $\hat{X}(R) = X \rtimes_\xi S^m(R)$, such that

• $\hat{g}$ is equal to g lifted to the ∇-horizontal (sub)spaces in $T(\hat{X}(R))$,

• $\hat{g}$ is equal to the R-spherical metrics in the fibers $S^m_x(R) \subset \hat{X}(R)$, $x \in X$, of the sphere fibration $\hat{X}(R) \to X$,

• the ∇-horizontal (sub)spaces in $T(\hat{X}(R))$ are normal to the fibers $S^m_x(R)$.

Observe that the scalar curvature of this bundle satisfies

$$Sc(X(R)) \geq Sc(X) + m(m-1)R^{-2} - const_m \cdot ||curv(\xi)||.$$

Also notice that a bundle ξ over a manifold X is spin equivalent to the tangent bundle $T(X)$ if and only if the total space of the associated to ξ sphere bundle, denoted $X \rtimes_\xi S^m$, $m = rank(\xi)$, is spin.

Now, the inequity ↀ applied to $X \rtimes_\xi S^m$ implies the following.

Three Curvatures Inequality. Let ξ be an orientable vector bundle of rank $m+1$ over a complete Riemannian, not necessarily spin, n-manifold X with an orthogonal connection and let L be a $\mathbb{Q}$-homologically essential complex vector bundle over $\hat{X}(R) = X \rtimes_\xi S^m(R)$, $m = rank(\xi)$.

If ξ is spin equivalent to the tangent bundle $T(X)$, e.g. $\xi = T(X)$, then there exists a point (x,s) in the fiber $S^m(R)_x \in \hat{X}(R)$ for some $x \in X$, such that

$\hat{ↀ}_{nsp}$ $$Sc(X)(x) \leq const_{m+n}(||curv(L)||(x,s) + ||curv(\xi)||(x)) - const_m R^2.$$

(If X is spin and ξ is the trivial bundle of rank $m+1$, this follows follows from, ↀ applied to $X \times S^m$, which for $m \geq 1$ imposes an a priori stronger bound on $Sc(X)$, than ↀ for X itself.)

Discussion. Albeit the inequalities $ↀ_{\frac{1}{2}}$ and $\hat{ↀ}_{nsp}$ have *no topological impact on non-spin manifolds*, no corollaries like the above ⊘ and/or [+/−], they do provide non-trivial geometric information which can not be obtained by other means available at the present day.

For instance, given two smooth Riemannian metics g_0 and g on a closed n-manifold X, then the scalar curvature of g is bounded by the sectional curvature κ of g_0 as follows.

Two Metrics Inequality. *If g is area-wise greater than g_0, i.e. if*

$$area_g(S) \geq area_{g_0}(S)$$

for all smooth services $S \subset X$, then there exists a point $x \in X$, such that

$$Sc(g)(x) \leq const_n |\kappa(g_0)(\tau_x)|$$

for some tangent 2-plane $\tau_x \subset T_x(X)$ and a universal constant $const_n > 0$.

This, qualitatively speaking, well illustrates the principle that

positivity of curvature of a Riemannian manifold
brings about bounds on the geometric size of X,

but quantitatively, as well as topologically, it is a far cry from the expected inequalities of this kind.

11 Twisted Dirac Operators versus Minimal Hypersurfaces.

As far as overtorical manifolds X are concerned, Dirac operator method, albeit limited to spin manifolds X, has an advantage over the original Schoen-Yau theorem of being not limited to $n \leq 8$. But it is superseded by the solution of non-essentiality of singularities conjecture (see section 6).

Besides,

□ Minimal hypersurfaces do not mind non-spin manifolds, but there is *no single* (known) *instance* of obstruction to the existence of metrics with $Sc > 0$ on a *non-spin* manifold X of dimension ≥ 5 in terms of the *topology of* X by means of a Dirac operator.

For instance one can't rule out metrics with $Sc > 0$ on connected sums of $2n$-tori with complex projective spaces $\mathbb{C}P^n$ for even n with a use of Dirac operators.

□□ Spin or non-spin, "over-torical" is more topologically restrictive than the Schoen-Yau-Schick condition. Dirac operators are helpless for proving non-existence of metrics with $Sc > 0$ on these manifolds as the example X_{Sch} below shows.

○ If X is spin then the the *two curvatures inequality* ꝏ rules out many topological types of non-SYS compact manifolds X which admit metrics with $Sc > 0$, e.g. all those X which supports metrics with $sect.curv \leq 0$, which, are in general not Schoen-Yau-Schick.

□□□ The major advantage of minimal hypersurfaces over Dirac operators from a geometric perspective is their applicability(?) to the scalar curvature problems on *non-complete* manifolds and compact manifolds X *with non-empty boundaries.* (see section 20, 21)

○○□ The major advantage of twisted Dirac operators is their sensitivity to the *areas of surfaces* in manifolds X which allow bounds on the *area-wise* size of (spin and non-spin) complete manifolds with $Sc \geq \sigma$ (see sections 16, 17).

For instance, one proves with twisted Dirac operators (see the previous section) that given a Riemannin manifold $X = (X, g_0)$,

❍ *no complete Riemannin metric g on X* ("complete" is redundant if X is compact without a boundary) *which is area-wise greater than g can have $Sc(g) \geq \sigma_\circ$ for some positive constant $\sigma_\circ = \sigma_\circ(X, g_0)$*,

where, recall, *area-wise greater* means that all surfaces $S \subset X$ satisfy $area_g(S) \geq area_{g_0}(S)$.

Although method of minimal hypersurfaces fails short of proving this and can only deliver a less precise bound which is expressible in terms of *distances* and/or *lengths of curves* in X rather than areas of surfaces, this method applies to *non-complete* metrics and yields the following distance version of ❍, [42]

❒ [54] *no Riemannin metric g on an $X = (X, g_0)$ which is greater than g can have $Sc(g) \geq \sigma_\square$ for some positive constant $\sigma_\square = \sigma_\square(X, g_0)$.*

[42]If $dim(X) \geq 9$ the proof needs a suitable solution of the irrelevance of the singularities conjecture.

[?21] **Problem.** *Evaluate* $\sigma_\circ(X_0)$ *and* $\sigma_\square(X_0)$ *for "simple" Riemannian manifolds* $X_0 = (X, g_0)$.

◯◯◯□ Dirac operators, because they are invariant under isometries, often deliver *optimal geometric inequalities* for manifolds with $Sc \geq \sigma$.

For instance, $\sigma_\circ(X_0)$ and $\sigma_\square(X_0)$ are equal to $Sc(X_0)$ for many (conjecturally for all) compact symmetric spaces (X_0), see sections 17, but this seems hard, if possible at all, to prove with minimal hypersurfaces.

Even in the most transparent case, where we know by Llarull's theorem, that

if a smooth Riemannian metric g *on* $X = S^n$ *is greater than the spherical metric* $\underline{g}$ *(the difference* $g - \underline{g}$ *is positive semidefinite) then there is a point* $x \in X$*, where* $Sc(g)(x) \leq n(n-1) = Sc(S^n)$*,*

there is no proof of this by means of minimal hypersurfaces for $n \geq 3$.

(Maybe, such a proof is possible, it seems realistic for $n = 3$, but this is unlikely for complex and quaternionic projective spaces instead of S^n, where the Dirac operator works with no problem).

On the other hand, there are incomplete manifolds X_0 where a sharp evaluation is possible by means of minimal hypersurfaces, but not by the Dirac operator methods, see section 21.

Thomas Schick Example [1998]. [105]. Let $X = X_{Sch}$ be obtained from the n-torus $\mathbb{T}^n$) by attaching the 2-handle to the circle representing the triply-multiple of one of the generators of $\pi_1(\mathbb{T}^n)$. This is a spin Schoe-Yau-Schick manifold X for all $n \geq 4$ with the fundamental groups $\mathbb{Z}^{n-1} \times \mathbb{Z}/3\mathbb{Z}$, where all (known?) versions of indices of the Dirac operators vanish; thus,

no known Dirac operator argument, unlike the method of minimal hypersurfaces, can rule out metrics with $Sc > 0$ *on these* X *even in the spin case.*

12 Coarea, K-Area, K-Area$^+$: Definitions and Applications.

Let us bring out into the open geometric invariants behind the inequalities ꝏ, $ꝏ_{\frac{1}{2}}$ and $\hat{ꝏ}_{nsp}$ from sections 9-11.

Definition. The *coarea* of a complex vector bundle L over a compact Riemannian manifold as

the infimum of the norms of the curvatures of unitary connections ∇ *in* L,

$$coarea(L) = \inf_{\nabla} ||curv(L, \nabla)||_X.$$

for the curvature norm $||curv(L)||$) from section 9.

Hopf Example. Since the tangent bundle of the unit 2-sphere satisfies $coarea(T(S^2)) = 1$, *the Hopf bundle* $L_{Hopf} = \sqrt{T(S^2)}$, has $coarea(L_{Hopf}) = \frac{1}{2}$.

We adjust the definition of the K-area to the relative case, specifically where X is a *non-compact* manifold and define $coarea = coarea/\infty$ by only allowing connections in L which are trivial flat at infinity. [43]

[43] The relevance and significance of this "trivial", which means with *trivial monodromy*, will be discussed later on.

Whitney Sums and Tensor Products. Clearly,

[COAR⊕] $$coarea(L_1 \oplus L_2) \leq \max(coarea(L_1), coarea(L_2)).$$

and

[COAR⊗] $$coarea(L_1 \otimes L_2) \leq coarea(L_1) + coarea(L_2).$$

$[_!|^!]$-*Corollary;* The inequality [COAR⊕] implies that

the coarea is monotone non-decreasing for push-forwards of vector bundles $\tilde{L}$ over $\tilde{X}$ under finite covering maps between compact manifolds $f : \tilde{X} \to X$,

$$coarea(f_!(\tilde{L}) \leq coarea(\tilde{L}),$$

while the the (semi)invariance under pullbacks,

$$coarea(f^!(L) \leq coarea(L),$$

for vector bundles L over$X$ is fully obvious.

And if $\tilde{X}$ are open manifolds, X than the push-forward monotonicity holds if the covering map $f : \tilde{X} \to X$ is *trivial/split at infinity*, since triviality at infinity of a bundle $\tilde{L}$ on $\tilde{X}$ and triviality at infinity of f imply *triviality at infinity* of the push-forward bundle $f_!(\tilde{L})$ on X. [44]

Define *K-area(h)* of an *even dimensional* homology [44, 60, 79] class, $h \in H_{2i}(X)$, as the

supremum of $\frac{2\pi}{coarea(L)}$ *over all bundles L such that* $\langle h, L \rangle \neq 0$,

where this non-equality means that there is a cohomology class c in the subring generated by the Chern classes of L, such that $\langle c, h \rangle \neq 0$.

Odd Stabilisation. Extend the definition to the *odd dimensional* homology by taking the product of X with the line,

$$K\text{-}area(h) = K\text{-}area[h \otimes [\mathbb{R}]], \text{ for } h \in H_{2i-1}(X).$$

Thus, the *K-area* of (the fundamental class of) an even dimensional *Riemannian manifold*,

$$K\text{-area (X)}=K\text{-}area[X],$$

is equal to the supremum of the numbers A, such that X admits a complex vector bundle L, with unitary connection such that

- L is $\mathbb{Q}$-*homologically essential*, i.e. some *characteristic number* of L is *non-zero*, where this L is assumed *trivial flat at infinity* if X is non-compact;
- $||curv(L)|| \leq \frac{2\pi}{A}$.

And if X is *odd dimensional*, then this applies to the product of X by the real line, for

$$K\text{-}area(X) = K\text{-}area(X \times \mathbb{R}).$$

[44] A covering map $\tilde{A} \to A$ is *split/trivial* if it is one-to-one on all connected components of A and the triviality at infinity is understood accordingly. Thus, a covering map $f : \tilde{X} \to X$ is trivial at infinity if has trivial monodromy out of a compact subset in X (e.g. if X is simply connected at infinity).

Uncertain Remark. It follows from $_!|^!$ (and from the residual finiteness of the fundamental group $\pi_1(S^1) = \mathbb{Z}$, compare with $\tilde{\downarrow}$ below) that

$$K\text{-}area(X \times \mathbb{R}) \leq K\text{-}area(X \times S^1).$$

[?22] It seems not impossible, at least for compact X, that, in fact,

$$K\text{-}area(X \times \mathbb{R}) = K\text{-}area(X \times S^1).$$

K-Area$^+$. Let us extend coarea/∞ from vector bundles to the K-cohomology classes in X represented by formal differences of unitary vector bundles L_1, L_2 on X with connections ∇_1, ∇_2 and with

connection preserving isomorphisms between L_1 and L_2 at infinity,

where we agree that

$$||curv(L_1 - L_2) = \max(||curv(L_1)||, \max||curv(L_2)),$$

and where such a difference $L_1 - L_2$ is regarded *homologically essential* if the the Chern character of $L_1 - L_2$ does not vanish on $(X]$.

Accordingly, define K-$Area^+(X)$ as the

supremum of $\frac{2\pi}{||curv(L_1 - L_2)||}$ over all homologically essential differences of bundles.

It is obvious that

$$K\text{-}area^+(X) \geq K\text{-}area(X) \text{ for all } X,$$

$$K\text{-}area^+(X) = K\text{-}area(X) \text{ for } compact \text{ manifolds } X,$$

and we shall see below that K-$area^+(X)$ can be *significantly greater* than K-$area(X)$ for open manifolds X.

On the other hand, the definition of K-$area^+(X)$, as well as that of the K-area, is adapted to the relative index theorem from [54]; this allows the following strengthening (and conceptualisation) of the two curvatures inequality (ↂ from section 9).

Bound on the Scalar Curvature by Inverse K-Area$^+$. *All complete Riemannian spin manifolds satisfy*

$$\inf_{x \in X} Sc(X)(x) \leq \frac{const_n}{K\text{-}area^+(X)}.$$

Given a Riemannian manifold $X = (X, g)$ with scalar curvature $Sc(g)(x) > 0$ let us conformally modify g by the factor $Sc(g)(x)^{-1}$ and write $X/Sc = (X, Sc(g)^{-1}g)$ (which corresponds to multiplying the local distance function $dist_g$ by $\sqrt{Sc(g)}$).

Sc-Conformal K-Area Inequality. The two curvatures inequality ↂ from the previous section now says that

complete n-dimensional Riemannian spin manifolds with positive scalar curvatures satisfy

[K/Sc] $$K\text{-}area(X/Sc) \leq const_n$$

for some universal constant $const_n > 0$.

In particular,

if $Sc(X) \geq \sigma > 0$ then K-$area(X) \leq const_n \sigma^{-1}$.

13 K-Area and K-Area$^+$: Examples and Properties.

(a) 2-Spheres. Riemann surfaces X homeomorphic to S^2 satisfy

$$K\text{-}area(X) = area(X).$$

Proof. Since $chern(L_{Hopf}) \neq 0$ and $coarea(L_{Hopf}) = \frac{1}{2}$, the unit sphere S^2 satisfy $K\text{-}area(S^2) \geq 4\pi = area(S^2)$, while the inequality $K\text{-}area(X) \geq area(X)$ for $X = (S^2, g)$ for *all Riemannin metrics g* follows from the obvious *area-like scaling property* of the K-area and its (equally obvious) invariance under area preserving diffeomorphisms.

On the other hand, if a unitary L bundle over a closed surface X satisfies

$$\int_X ||curv(L) < 2\pi$$

then by (Chern-Weil) Gauss Bonnet theorem, $chern(L) = 0$.

The proof of the equality between K-areas and areas of spherical surfaces is thus concluded.

(b) Equidimensional Embeddings. Since bundles over open subsets $U \subset X$ which are trivial at infinity obviously extend to X,

$$K\text{-}area(U) \leq K\text{-}area(X)$$

for all manifolds X and all open subsets $U \subset X$.

(It is not so for the K-area$^+$. For instance, we shall see below that albeit

$$K\text{-}area^+(S^2) = K\text{-}area(S^2) = area(S^2) < \infty,$$

non-simply connected domains $U \subset S^2$ have $K\text{-}area^+(U) = \infty$.)

The above embedding inequality sometimes becomes an equality. For instance,

if the complement to an open subset $U \subset X$ is zero dimensional, then

$$K\text{-}Area(U) = K\text{-}Area(X).$$

(c) Surfaces of Genus Zero. If a surface X has infinite area then it follows from (e) below that its K-area is also infinite, while if $genus(X) = 0$ and $area(X) < \infty$, then X admits an area preserving diffeomorphism onto S^2 minus a zero dimensional subset. Hence,

All surfaces X of genus zero satisfy

$$K\text{-}area(X) = area(X).$$

On the other hand, by (d) below, all surfaces with *infinite fundamental groups*, regardless of their genera, have K-area$^+ = \infty$

(d) Invariance Under Covering Maps. If $f : \tilde{X} \to X$ is a finitely sheeted covering map, then the above !|! (obviously) implies that

$$[\tilde{\downarrow}]^+ \qquad K\text{-}area^+(\tilde{X}) = K\text{-}area^+(X).$$

Furthermore, if f is trivial/split at infinity, then

$$[\tilde{\downarrow}]. \qquad K\text{-}area(\tilde{X}) = K\text{-}area(X),$$

Surface Corollary/Example. *Orientable surfaces with infinite fundamental groups have K-areas$^+ = \infty$ and those with positive genera also have K-area $= \infty$.*

(The finiteness of K-areas of surfaces of genus zero with finite areas shows that triviality at infinity of f is essential.)

(e) Area Monotonicity of the K-Area. It is obvious that the K-Area *decreases* under equidimensional proper area decreasing maps $f : X_1 \to X_2$ of non-zero degrees.

Hyperspherical Corollary. If, given $\varepsilon > 0$, an orientable n-dimensional Riemannin manifold X admits a locally constant at infinity ε-Lipschitz map $X \to S^n$ of a non-zero degree, then

$$K\text{-}area^+(X) = \infty.$$

For instance the Euclidean space $\mathbb{R}^n$ and, consequently, all other *complete simply connected manifolds with sectional curvatures $\kappa \le 0$* have infinite K-areas.

This and the above (d) imply, that

compact manifolds X with $\kappa \le 0$ and with residually finite fundamental groups[45] *have K-area$(X) = \infty$.*

[?23] **Question.** Is the residual finiteness of the fundamental group essential? Can, for instance, a compact manifold X with *finite* K-area have the *universal covering with infinite* K-area?

A Skeptic's Response. This question is due to the provisional nature of the definition of the K-area. A satisfactory answer is available with the concept of the Fredholm K-area which is invariant under infinite coverings. (see section 27).

(f) K-Areas of Spheres. *The R-spheres $S^n(R)$ have K-areas $4\pi R^2$.*

Proof. Recall that the bound $K\text{-}area(S^n) \ge 4\pi R^2$ for $n = 2$ was derived from the existence of a connection with curvature $\frac{1}{2}$ on the Hopf bundle $L_{Hopf} = +\sqrt{T(S^2)}$.

If $n \ge 2$ is *even*, then one uses the spin bundle $\mathbb{S}^+(S^n)$ (or $\mathbb{S}^-(S^n)$, if you wish) in the role of L_{Hopf}, where it is easy to see that the spin bundle is $\mathbb{Q}$-*homologically essential* bundle and the curvature of the Levi-Civita connection on it satisfies

$$||curv(\mathbb{S}^\pm(S^n))|| = \frac{1}{2}.$$

If n is odd, then the lower bound of $K\text{-}area(S^n)$ by 4π is derived from that for S^{n+1} via an (obvious) area non-increasing map $S^n \times \mathbb{R} \to S^{n+1}$ of degree one and the case of the spheres of radii $R \ne 1$ follows by the scaling property of the K-area. Thus, the inequality

$$\mathbf{^K\bigcirc_{\ge 4\pi}}, \qquad K\text{-}area(S^n(R)) = area(S^2(R)) \ge 4\pi R^2,$$

[45]The fundamental group $\Pi = \pi_1(X)$ is residually finite if, given an $R > 0$, there is a finite covering $\tilde{X} \to X$ which is one-to-one on all R-balls in $\tilde{X}$. For instance, the fundamental groups of all locally symmetric spaces, e.g. of those with constant sectional curvatures, are residually finite by an old theorem of Selberg.

is established.

Now in order to prove that

$$^{\mathbf{K}}\bigcirc_{\leq 4\pi}, \qquad K\text{-}area(S^n(R)) = area(S^2(R)) \leq 4\pi R^2,$$

let us show that if a unitary) connection ∇ in a bundle L over S^n has $||curv|| < \frac{1}{2}$ then the bundle L is trivial.

To see this, let us parallelly transport an orthonormal frame from the north pole to the south pole along geodesics in S^n, $n \geq 2$ and let Φ be the resulting map from the unit $(n-1)$ sphere S^{n-1} – the tangent sphere at the south pole – to the unitary group $U(N)$, $N = rank_{\mathbb{C}}(L)$.

Let us endow $U(N)$ with the (standard bi-invariant) metric, such that the infimum of the lengths of the compact one-parameter subgroups, hence, of all closed geodesics, is 2π. Then the Lipschitz constant of Φ with respect to this metric satisfies

$$Lip(\Phi) \leq 2||curv(L)||$$

It follows that if a bundle L has $||curv(L) < \frac{1}{2}||$, then $Lip(\Phi) < 1$, and since the maps $S^{n-1} \to U(N)$ with $Lip < 1$ are contractible, L must be trivial and the inequality $^{\mathbf{K}}\bigcirc_{\leq 4\pi}$ for even n follows, from the product inequality below.

(g) K-Area of Products of Spheres. *The products of spheres satisfy*

$$K\text{-}area(S^{n_1}(R_1) \times ... \times S^{n_i}(R_i) \times ... \times S^{n_k}(R_k)) = 4\pi \min R_i^2,$$

where the minimum is taken over all i for which $n_i \geq 2$ and where we allow $S^{n_i}(R_i = \infty) =_{def} \mathbb{R}^{n_1}$.

Proof. The lower bound $K\text{-}area(\times_i S^{n_i}(R_i)) \geq 4\pi \min R_i^2$, follows from that for $S^{n_1+...+n_k}(\min R_i)$ via an obvious distance non-increasing map of degree one,

$$\bigtimes_i S^{n_i}(R_i) \to S^{n_1+...+n_k}(\min R_i).$$

To prove the opposite inequality, let us show that

The product of the unit n-spheres for $n \geq 2$ with all manifolds Y satisfy the following

S^n-Product Inequality: $\qquad K\text{-}area(S^n \times Y) \leq 4\pi.$

Indeed, let first $n = 2$ and suppose that the restrictions of a unitary connection ∇ in a bundle $L = (L, \nabla)$ over $S^2 \times Y$ to the spheres $S^2 \times y \subset S^2 \times Y$, $y \in Y$, have $||curv||, \frac{1}{2}$. Then the parallel north poles → south poles transport in these spheres defines maps $\Phi_y : S^1 = S^1_y \to U(N)$, such that $Lip(\Phi_y) < 1$.

By elementary (and obvious) Morse theory, this inequality implies that the family of maps Φ_y, $y \in Y$, is homotopic to a family of constant maps; therefore, the restrictions of L to the spheres $S^2 \times y$ are simultaneously trivializable *continuously in* $y \in Y$.

Hence, the bundle L is induced from (a bundle on) Y by the projection $S^2 \times Y \to Y$ and it *can't be homologically essential.*

Then the case of $n > 2$ follows via a distance non-increasing map of degree one, $S^2 \times S^{n-2} \times Y \to S^n \times Y$.

$R = \min R_i^2 = 1$ and let $n \geq 2$ be the dimension of the corresponding sphere. If $n = 2$, then the proof follows, by the reason we indicated above:

the maps $S^1 \to U(N)$ with $Lip < 1$ are canonically (and obviously) contractible to points as by the elementary Morse theory.

Then the case of $n > 2$ follows via a distance non-increasing map of degree one, $S^2 \times S^{n-2} \to S^n$. QED.

Remarks/Questions.

[?24] (i) It is unclear if the last step in the above argument is truly needed: conceivably, maps $\Phi : S^{n-1} \to U(N)$ with $Lip(\Phi) < \frac{1}{2}$ are contractible to constant ones *continuously in* Φ for all n.

(ii) The Product Inequality in the case $n = 2$ generalises to products $S^2 \times Y$ with arbitrary Riemannian metrics g on $S^2 \times Y$ as follows.

S^2-Product Inequality: $K\text{-}area(S^2 \times Y, g) \le \sup_{y \in Y} area_g(S^2 \times y)$.

Proof. The family of restrictions g_y of g from $S^2 \times Y$ to $S^2 \times y \subset S^2 \times Y$ can be replaced by a *continuous* family of metrics h_y on $S^2 \times y$ with *constant curvatures*, such that

$$area_{h_y}(U_y) = area_{g_y}(U_y) \text{ for all } U_y \subset S^2 \times y.$$

Then the above argument applies to the spheres $(S^2 \times y, h_y) = S^2(R_y)$, for

$$(S^2 \times y, h_y) = S^2(R_y) \text{ for } R_i = \sqrt{\frac{1}{4\pi} area_{h_y}(S^2 \times y)}$$

and the proof follows.

[?25] (iii) The S^n- and S^2-product inequalities seems to hold for *non-trivial* sphere fibrations, but I have not checked this carefully.

(iv) Much of the above generalises to (open) balls B^n with Riemannian metrics which admit *area non-increasing* diffeomorphisms $f : B^n \to S^n \setminus s_0$, $s_0 \in S^n$, such that the inverse maps $f^{-1} : S^n \setminus s_+ \to B^n$ are area preserving on the equatorial 2-spheres in S^n passing through s_0.

For instance, since the Euclidean balls $B^n(R) \subset \mathbb{R}^n$ admit such maps, we see that

$$K\text{-}area(B^n(R)) = 2\pi \cdot R^2,$$

and

$$K\text{-}area(B^n(R) \times Y) \le 2\pi \cdot R^2$$

for all Riemannin manifolds Y.

Warning. The infinite bands $X_1 = [0, a_1] \times \mathbb{R}$ and $X_2 = [0, a_2] \times \mathbb{R}$ have infinite K-areas, but their product satisfies

$$K\text{-}area(X_1 \times X_2) = a_1 \cdot a_2 < \infty$$

(This does not happen to *closed* manifolds X_1, X_2, where, in fact, $K\text{-}area(X_1 \times X_2) \le \max(K\text{-}area(X_1), K\text{-}area(X_2).$[46])

[46] The K-area, unlike the scalar curvature, is not truly additive/multiplicative under Riemannian products of manifolds. In fact, the best you can do, even you go Fredholm, is

$$K\text{-}area(X_1 \times X_2) = \max\,(K\text{-}area(X_1), K\text{-}area(X_2)).$$

One may tolerate this discrepancy between the two notions in-so-far as one does not care for sharp sharp inequalities. But the latter need a certain "normlization" of the K-area which we shall discuss elsewhere.

But

$$K\text{-}area^{+}(B^2(R)\times\mathbb{R}^n)=\infty \text{ for all } R>0 \text{ and } n\geq 1.$$

[?26] *Uncertain Remark.* Some versions of the above inequalities hold not only for $S^n\times X$ but also for *non-trivial* sphere and ball bundles over Riemannin manifolds X, but I have not worked this out in details

14 Spin-Area. and $K_{\frac{1}{2}}$-Area.

Denote by *spin-area*(X) of an orientable Riemannin manifold the supremum of the numbers A such that X supports a real vector bundle ξ over it with an orthogonal connection trivial flat at infinity, which is *spin equivalent* to the tangent bundle $T(X)$ and such that

$$||curv(\xi)||\leq\frac{2\pi}{A}.$$

Then define $K_{\frac{1}{2}}$-*area*(X) as the supremum of the numbers A' such that X supports a *virtual Q-homologically essential* complex vector bundle $L^{[\pm]}$ (as in the non-spin two curvatures inequality from previous section) with the same $\pm$-ambiguity as $T(X)$ and where $L^{[\pm]}$ admits a unitary connection trivial flat at infinity, such that

$$||curv(L^{[\pm]})||\leq\frac{\pi}{A'}.$$

Observe that since

$$||curv(L\otimes\pm\sqrt{\xi}||\leq(||curv(L)||+\frac{1}{2}||curv(\xi)||,$$

the $K_{\frac{1}{2}}$-area of (X) is bounded from below by

$$K_{\frac{1}{2}}\text{-}area(X)\geq[(K\text{-}area(X))^{-1}+(spin\text{-}area(X)^{-1}\;]^{-1}.$$

Examples.

• The spin areas of spin manifolds are infinite.

• The spin area of a simply connected non-spin manifold X is bounded by the area of the infimum of the areas of embedded spheres $S^2\subset X$ on which the tangent bundle $T(X)$ is non-spin.

• Even dimensional complex projective spaces $\mathbb{C}P^{2m}$ have spin areas $4\pi=area(S^2)$, since the canonical bundles L_{can} are non-spin and $||curv(L_{can})||=\frac{1}{2}$ for the standard metric in $\mathbb{C}P^{2m}$ which are normalised to have the maximum of the sectional curvatures equal 1.

• Let X be the total space of an orientable non-spin real vector bundle of rank 2 over an orientable closed surface X_0. Then

X is non-spin while spin-area$(X)=\infty$, unless X_0 is topologically the 2-sphere.

⋆ *Problem/Question.* It would be naive to believe that the spin areas of Riemannian n-manifolds are bounded from below by

$$spin\text{-}area(X)\geq b_n min.area(S_{\frac{1}{2}}\subset X),$$

where $min.area(S_{\frac{1}{2}} \subset X)$ denotes the infimum of areas of $T(X)$-*non-spin* surfaces S surfaces $S \subset X$ i.e. on which the tangent bundle $T(X)$ is non-spin, and where $b_n > 0$ is a universal constant.

In fact, the *stable systolic inequality* in conjunction with the *systolic freedom* of $\mathbb{C}P^2$ [9] suggest, for example, that the complex projective plane $\mathbb{C}P^2$ admits Riemannian metrics g with *arbitrarily small spin-areas* and *arbitrarily large* $min.area_g(S_{\frac{1}{2}} \subset \mathbb{C}P^2)$.

[?27] On the other hand, if the δ-neighbourhoods $U_\delta(S) \subset X$ of all $T(X)$-*non-spin* surfaces S in a Riemannian manifold X are "large" then the spin area of X must be also large.

[?28] For instance, let X be homomorphic to $\mathbb{C}P^2$ and let $vol(U_\delta(S)) \geq \delta^2$ for all $T(X)$-non-spin surfaces $S \subset X$ and $0 < \delta \leq 1$. Is then

$$spin\text{-}area(X) \geq 1/1\ 000\ 000?$$

Now let us reformulate the the non-spin two curvatures inequality from the previous section in the $K_{\frac{1}{2}}$-terms.

Sc-Conformal $K_{\frac{1}{2}}$-Area Inequality. *Complete n-dimensional Riemannian manifolds with positive scalar curvatures satisfy*

$$K_{\frac{1}{2}}\text{-}area(X) \leq const_n$$

for some universal constant $const_n > 0$.

In particular, *if $Sc(X) \geq \sigma > 0$, then*

$$\min\ (K\text{-}area(X),\ spin\text{-}area(X)) \leq const_n \cdot \sigma^{-1}.$$

15 Waist, Width, Filling Radius, Macroscopic Dimension and Uniform Contractibility.

We remind in in this section definitions of several invariants of Riemannian (and non-Riemannian) spaces X which characterise their "metric size" and which allow formulation of basic conjectures on manifolds with positive scalar curvatures $Sc(X) \geq \sigma > 0$ in section 10. (See [57] and references therein for more information.)

Slicings and Waists. An *m-sliced i-cycle* is an oriented i-dimensional psedomanifold $P = P^i$ partitioned into *m-slices* $P_q \subset P$, which are the pullbacks of the points of a simplicial map $\varphi : P \to Q$ where Q is an $(i-m)$-dimensional orientable pseudomanifold, where all pullbacks $P_q = \varphi^{-1}(q) \subset P$ have $dim(P_q) \leq m$, $q \in Q$, and where φ is required to be *proper*, hence, with compact pullbacks $\varphi^{-1}(q)$, if P is non-compact.

The m-waist, denoted $waist_m(h)$, of a homology class $h \in H_i(X)$ is

the infimum of the numbers w,

such that X receives a Lipschitz map from a compact m-sliced cycle, $\phi : P^n \to X$, which represent h, i.e.

$$\phi_*[P] = h$$

and the

the images of all slices in X have m-volumes $\leq w$,

where these "volumes of the images" are counted with multiplicities (which is unneeded for generically 1-1 maps.)

Clearly,

$$waist_m(h_1 + h_2) \leq \max(waist_m(h_1), waist_m(h_2))$$

and *m-waists are monotone decreasing under homology homomorphisms induced by m-volume contracting maps.*

Waists of Non-Compact Manifolds X. If X is *non-compact*, then $[X]$ is understood as the homology class with infinite supports; accordingly the corresponding *proper waists* are defined with *proper* Lipschitz maps $P \to X$.

Alternatively, one allows *non-proper* maps $P \to Q$ which slice P into *non-compact* sub-pseudomanifolds. Then proper maps $P \to X$ define what we called just *waist* denoted, $waist_m[X]$.

And if X a compact manifold with a boundary, we use the notation $[X]$ for the relative class in $H_n(X, \partial X)$ and set

$$waist_m[X] = waist_m[int(X)],$$

which can be also defined via slicing of *compact pseudomanifolds P with boundaries* and maps of pairs $(P, \partial P) \to (X, \partial X)$.

The m-waist of a compact connected orientable Riemannian n-manifolds X is defined as $waist_m[X]$ of the fundamental homology class $[X] \in H_n(X)$.

For example,

$$waist_n[X] = vol(X).$$

Waists of Spheres and Balls. It is obvious that the unit spheres S^n satisfy

$$waist_m(S^n) = vol(S^m)$$

for all $m = 1, 2, ..., n$.

Conversely, an application of *Almgren's min-max principle* yields the following.

Sharp Spherical Waist Inequality.

$$waist_m(S^n) \geq vol(S^m)$$

for all $m = 1, 2, ..., n$.

In fact, according to Almgen's Morse theory, the m-waists of a compact Riemannian manifold X is

bounded from below by the infimum of volumes of minimal m-dimensional subvarieties $Y^m_{min} \subset X$.

Then the lower bound

$$vol(Y_{min}) \geq vol(S^m)$$

for $Y^m_{min} \subset S^n$ follows from either of the following two inequalities.

$\bullet_{\kappa \geq 1}$ If the sectional curvatures of a compact Riemannian manifold X are bounded fro below by 1 then

$$vol(X)/vol(Y^m_{min}) \leq vol(S^m)/vol(S^m)$$

for all minimal subvarieties $Y^m_{min} \subset X$.

Indeed, according to *Levy-Bishop-Bujalo-Hentze-Karcher comparison inequality,* the volumes of the r-tubes (r-neighbourhoods) around $Y^m_{min} \subset X$ are bounded, as much as such tubes of $S^m \subset S^n$, where applicability of this inequality to an, a priori, singular Y^m_{min} relies on *the Almgren-Allard regularity theorem.*)

On Generalisations. This $\bullet_{\kappa\geq 1}$ yields *sharp lower bounds* on waists of many non-spherical manifolds, e.g. of the quotient spaces $X = S^n/\mathbb{Z}_i$ for isometric (free and non-free) actions of finite cyclic groups $\mathbb{Z}_i = \mathbb{Z}/i\mathbb{Z}$ on S^n, see [58, 1], section 3 in [**?**, **?**], [91] and references therein.

[?28] Besides, similar results are expected for all *singular Alexandrov spaces with lower curvature bounds* but the Almgren's regularity theory has not been developed even for Alexandrov spaces with *conical singularities*, where the only apparent, yet instructive, case is that of *isolated singularities.*

$\bullet_{\leq\kappa_0}$. If the injectivity radius of X at a point $x \in Y^m_{min} \subset X$ is $\geq R$ and if the sectional curvatures of X in the R-ball $B_x(R) \subset X$ are *bounded from above* by κ_0 then the m-volume $vol(Y^m_{min} \bigcap B_x(R))$ is bounded from below by the volume of the R-ball $B^m(R, \kappa_0)$ in the standard m-space with constant curvature κ_0 by the *monotonicity formula* for minimal subvarieties.

About the Balls. Neither $\bullet_{\kappa\geq 1}$ nor $\bullet_{\leq\kappa_0}$ directly apply to the balls $B^n(R, \kappa_0)$ with constant curvatures; yet, by comparing their waists to those of spheres by means of suitable $O(n)$-equivariant maps $B^n(R, \kappa_0) \to S^n$, one arrives at the expected values (see section 3 in [**?**, **?**, 2]),

$$waist_m(B^n(R, \kappa_0)) = vol(B^m(R, \kappa_0)).$$

Rectangular Example. the rectangular solid

$$X = [0, d_1] \times [0, d_2] \times ... \times [0, d_n] \subset \mathbb{R}^n, \; d_1 \leq d_2 \leq ... \leq d_m,$$

obviously satisfies

$$waist_m[X] \leq d_1 \cdot d_2 \cdot ... \cdot d_m,$$

and

$$waist_m[X] \geq waist_m[[0, d_0]^n] \text{ for } d_0 = \sqrt[m]{d_1 \cdot d_2 \cdot ... \cdot d_m},$$

where the latter implies that

$$waist_m[X] \geq wast_m[B^n(\frac{d_0}{2})] = vol_m(B^n(\frac{d_0}{2}])const_n \cdot d_1 \cdot d_2 \cdot ... \cdot d_m.$$

Also it is not hard to show that if $d_{m+1} >> d_m$, e.g. $d_{m+1} \geq (m+1)^m d_m$, then $waist_m[X] \leq d_1 \cdot d_2 \cdot ... \cdot d_m$. But – this was pointed out in [58] – deciding what happens for $d_1 = d_2 = ... = d_n$ remains a challenge.

However, one knows that, for instance, manifolds X with constant negative curvatures, e.g. Riemann surfaces of genus ≥ 2, admit finite i-sheeted coverings X_i with arbitrarily large i (which implement *expanders*), such that the rescaled manifolds $X_{i,\varepsilon} = \varepsilon^{\frac{1}{n}} i^{-1} X_i$, which have $vol(X_{i,\varepsilon}) = \varepsilon$, satisfy

$$waist_{n-1}[X_{i,\varepsilon}] \underset{i\to\infty}{\to} \infty \text{ for all } \varepsilon > 0.$$

Then Riemannian products of such manifolds,

$$X_{i_1\varepsilon_1} \times X_{i_2\varepsilon_2} \times ... \times X_{i_j\varepsilon_j},$$

which may cover different $X = X_1, X_2, ...X_j$, deliver a wide range of possible values of possible waists, but fail short of confirming (if it is true at all) the following.

[?29] **Waist Independence Conjecture.** *There exist compact Riemannian manifolds X of all dimensions n with arbitrarily prescribed $waists_m[X] > 0$ for all $m = 1, 2, ..., n$.*

Slice Area. What is most relevant for us in this paper is $waist_2[X]$, which is conceptually (and conjecturally) related to the K-area and which may be called *slice area* of X,

$$slice\text{-}area(h) = waist_2(h),\ h \in H_i(X), \text{ and } slice\text{-}area[X] = waist_2[X].$$

The slice area is close to the K-area for "simple" manifolds X.

For instance, it is easy to see that the ratio of the two is bounded from above and from below for rectangular solids $X = \times_i[0, d_i]$, $d_1, \leq d_2 \leq ... \leq d_n$, and that if $d_3 >> d_2$ then $K\text{-}area = slice\text{-}area = d_1 \cdot d_2$.

[?29] But it is unclear if $K\text{-}area = slice\text{-}area$ for *all* rectangular solids, where such equality does not seem 100% impossible even for products of general simply connected surfaces.

[?30] On the other hand, there probably exist compact simply connected n-dimensional manifolds for all $n \geq 4$ with arbitrarily prescribed (finite) values of the K-area and the slice-area.

Grassmannian Example/Exercise. Let the complex Grassmann manifold $Gr_N(\mathbb{C}^{N+M})$ be endowed with the standard Riemannian $U(N+M)$-invariant Riemannian metric which is normalised such that the maximum of its sectional curvatures is equal to one.

Then the slice area of this $Gr_N(\mathbb{C}^{N+M})$ satisfies

$$waist_2(Gr_N(\mathbb{C}^{N+M}) = 4\pi = area(S^2).$$

Uryson Width and Macroscopic Dimension. The *k-width* $width_k(X)$ of a metric space X is

the infimum of the numbers d,

such that X admits a continuous map to a *k-dimensional* polyhedral space P, say $\Delta : X \to P$, where

the diameters of the pull-backs of all points are bounded by d,

$$diam_X(\Delta^{-1}(p)) \leq d,\ p \in P.$$

.

It is obvious that $width_0(X) = diam(X)$ for connected X, that

$$width_0(X) \leq width_1(X) \leq ... \leq width_k(X) \leq width_{k+1}(X) \leq ...$$

and that

$$width_{n-1}(X) \leq waist_1(X).$$

It is also clear that the rectangular solids satisfy

$$width_{n-m}\left([0, d_1] \times [0, d_2] \times ... \times [0, d_n]\right) \leq \sqrt{d_1^2 + ... + d_m^2},$$

where we assume $d_1 \leq d_2 \leq ... \leq d_m$ as earlier, and the opposite inequality, called *Lebesgue Lemma*,

$$width_{n-m}([0,d_1] \times [0,d_2] \times ... \times [0,d_n]) \geq d_m$$

is also obvious my the modern standards.[47] But

[?31] *the sharp values of $width_{n-m}$ for these solids remains problematic for $m \geq 2$,*

(unless I missed some paper).

The macroscopic dimension of X is the minimal number k, such that $width_k(X) < \infty$.

For instance,

$$macr.dim(Y \times \mathbb{R}^k) = k$$

for all compact spaces Y, where the (intuitively obvious) inequality

$$macr.dim(\mathbb{R}^k) > k - 1$$

follows from Lebesgue's lemma.

Filling Radius. This radius, denoted $fil.rad(h)$, is defined for homology classes h of metric spaces X, e.g. for the fundamental classes $[X]$ of manifolds X, as

the infimum of numbers R such that $X = (X, dist_X)$ admits a metric extension $Y \supset X$, where

∘ $dist_Y(y, X) \leq R$ *for all* $y \in Y$;

∘ *the class h vanishes in the homology of Y.*

For example, if X is the unit n-sphere S^n with the usual Riemannin metric and $h = [S^n] \in H_n(S^n)$, then the hemisphere $S^{n+1}_+ \supset S^n$ is an instance of such a $Y \supset X$ with $R = \pi/2$.

It is obvious that

$$fil.rad[X] \leq width_{n-1}(X), \ n = dim(X).$$

What is less obvious and more interesting is the following

Filling-Waist Inequality [41, 55, 124].

$$fil.rad[X] \leq const_n \cdot waist_m[X]^{\frac{1}{m}}, \text{ for all } m = 1, ..., n.$$

In particular,

$$fil.rad[X] \leq const_n \cdot vol(X).$$

The latter was strengthened simultaneously in two respects by Larry Guth [59] as follows.

Width Small Ball Inequality. *There exits a positive constant $\varepsilon = \varepsilon(n) > 0$, such that the inequalities*

$$vol(B_x(1)) \leq \varepsilon, \ x \in X,$$

for all unit balls $B_x(1) \subset X$ imply that

$$width_{n-1}(X) \leq 1.$$

[47] This was proven by Brouwer who pointed out an error in the original paper by Lebesgue.

[?32] **Conjecture: Waist-Width Inequality.** *All complete Riemannian n-manifolds X satisfy*

$$width_{n-1}(X) \leq const_n \cdot waist_{n-k+1}(X).$$

Contractibility Radius. This "radius", denoted $contr(X,r)$, $r > 0$, of a metric space X is the infimum of the numbers R, such that every r-ball $B_x(r) \subset X$ is contractible within the concentric ball $B_x(R) \supset B_x(r)$ of radius R.

It is (almost) obvious that [41]:

A. *Complete Riemannian manifolds with cocompact isometry groups, e.g universal coverings of compact aspherical manifolds, have $contr(X,r) = \infty$ for all $r > 0$.*

B. *If a complete n-dimensional manifold X satisfies*

$$contr(X,r_i) \leq r_{i+1} \text{ for } i = 1,2,...,n \text{ and } r_1 \leq r_2 \leq ... \leq r_{n+1} < \infty,$$

then

$$fil.rad[X] \geq r_1/(n+1)!.$$

C. It follows that

the universal coverings $\tilde{X}$ of compact aspherical manifolds X have

$$fil.rad(\tilde{X}) = \infty.$$

Consequently,

these $\tilde{X}$ satisfy

$$waist_m(\tilde{X}) = \infty \ , m = 1,2,...,n = dim(X),$$

as well.

16 Standard Geometric and Topological Conjectures on Complete Manifolds with Positive Scalar Curvatures.

[?33] **Slice-Area Inequality.** *All complete Riemannian n-manifolds X with positive scalar curvatures satisfy*:

☻$_2$ $$waist_2[X/Sc] \leq b_n,$$

where, recall, "$/Sc$" signifies rescaling of the Riemannian metric g of X by the function $Sc(X)(x)^{-1}$, i.e. $X/Sc = (X, g(x)/Sc(X)(x))$.

Moreover

the (conjectural) optimal constant must be

$$b_n = 4\pi n(n-1),$$

which is taken from the equality

$$waist_2(S^n(1)) = 4\pi = 4\pi n(n+1)Sc(S^n(1))^{-1}.$$

Corollary to ☻$_2$.

☻$_2$ $$waist_2[X] \leq b_n(\inf_{x\in X} Sc(X)(x))^{-1}.$$

[?34] **Conjecture. Bounds on Width and on the Macroscopic Dimension.** *Complete n-dimensional Riemannian manifolds X with the scalar curvatures $Sc(X) \geq \sigma > 0$ satisfy*

$$macr.dim(X) \leq n-2.$$

Moreover,

⧗$_2$, $$width_{n-2}(X) \leq const_n \sigma^{-\frac{1}{2}}.$$

where, in fact proving even the weaker inequality

⧗$_1$ $$width_{n-1}(X) \leq const_n \sigma^{-\frac{1}{2}}.$$

would make one happy.

[?35] **Conjecture. Bound on the Filling Radius for $Sc \geq \sigma > 0$.**

☺ $$fil.rad[X] \leq const_n \cdot (\inf_{x\in X} Sc(X)(x))^{-2}.$$

This, in view of A,B,C from the previous section, yields the following.

[?36+1=37] **Conjecture 11. Constrained Contractibility for $Sc \geq \sigma > 0$.** *Complete manifolds X with $Sc(X) \geq \sigma > 0$ are not uniformly contractible.* = This yields non-asphercity conjecture [?13] from section 4: *There are no compact aspherical manifolds with positive scalar curvatures.*

Discussion. (a) Bounds on the filling radius stated in the previous section and the (easiest of all) $waist_2$-case of the conjectural waist-width inequality from the previous section show that

☻$_2$⇒ ☻$_2$ ⇒ ☺ ⇒ constrained contractibility ⇒ non-asphericity.
?⇓ ⇗
⧗$_2$ ⇒⧗$_1$

(b) The conjectures ☻$_2$ and ☻$_2$ are very strong with no serious evidence in their favour. They are not even known for manifolds with positive sectional curvatures, where only the case of convex hypersurfaces $X \subset \mathbb{R}^{n+1}$ seems not difficult.

(c) What is known about ⧗$_1$ and ⧗$_2$ is only (apparently non-sharp) ⧗$_2$ for complete 3-manifolds X^3 with $Sc(X^3) \geq \sigma > 0$ [54],

$$width_1(X^3) \leq 12\pi\sigma^{-\frac{1}{2}}.$$

But starting from $n = 4$ there is no convincing evidence even for $macr.dim(X) \leq n-1$. Yet some results are available for spin manifolds with cocompact isometry groups.[13, 28].

(d) The non-asphericity conjecture would follow from ⚭ from section 8 and the following conjectural algebra-topological property of universal coverings of compact spherical manifolds X, which, (an easy argument), depends only on the fundamental group of X.

[?38] **Conjecture. Asphericity** $\Rightarrow$ K**-**$Area = \infty$**.** *The universal coverings* $\tilde{X}$ *of compact aspherical manifolds* X *satisfy*

$$K\text{-}area(\tilde{X}) = \infty.$$

Notice that this inequality, even in a stabilised form, implies the *strong Novikov conjecture* for $\pi_1(X)$, which is stronger than the non-asphericity for $Sc \geq 0$ [101].

This makes it too good to be true; yet, no candidate for a counterexample is anywhere in sight.

(e) Recall that a more comprehensive form of the asphericity conjecture [?13] , that is $\mathbb{Q}$-Non-Essentiality conjecture [?12] from section 4. reads.

If a closed orientable manifold supports a metric with $Sc > 0$, *then the canonical (unique up to homotopy) map from* X *to the (classifying) Eilenberg-MacLane space of its fundamental group* $\pi_1 = \pi_1(X)$, *say* $\Phi : X \to \mathsf{B}\pi_1 = \mathsf{K}(\pi_1, 1)$, *vanishes on the fundamental class of* X,

$$\Phi_*[X] = 0 \in H_n(\mathsf{B}\pi_1),\ n = dim(X).$$

(Notice that $\Phi_*[X] = 0$ implies that $f_*[X] = 0$ for *all* continuous maps f from X to *all* aspherical spaces.)

This would follow from the following K-theoretic conjecture.

[?39] Let $\mathsf{B} = \mathsf{B}\Gamma$ be the classifying space of discrete countable group, let $f : X \to B$ be a continuous map from a Riemannian manifold X. Then there exits a compact subset $B_0 \subset B$ which contains the image $f(X) \subset \mathsf{B}$ and such that

the Fredholm coareas of certain non-zero multiples of the pullbacks bundles $f^*(L)$ *on* X *become arbitrarily small.*

Namely,

given $\varepsilon > 0$ *there exists an integer* $N \neq 0$ *and a Fredholm bundle* $(\mathcal{L}, \nabla)$ *over* X *(see section 27) with* $||curv(\nabla)|| \leq \varepsilon$, *such that* $\mathcal{L}$ *is* K*-theoretically equivalent to the* N*th Whitney power of* $f^*(L)$,

$$\mathcal{L} \underset{K}{\sim} \underbrace{f^*(\mathcal{L}) \oplus \ldots \oplus {}^*(\mathcal{L})}_{N}$$

17 Extremal Metrics with $Sc \geq 0$.

According to the *two metrics inequality* from section 9 there is an upper bound on the "area sizes" of metrics g on X with $Sc(g) \geq \sigma > 0$ on all compact manifolds without boundary. This suggests the following definitions.

Area Extremality. A metric $\underline{g}$ on a manifold X is called *area extremal/maximal* if all *area-wise larger* Riemannian metrics g on X, i.e. such that

$$area_g(S) \geq area_{\underline{g}}(S) \text{ for all smooth surfaces } S \subset X,$$

satisfy

[max] $$\inf_{x\in X}(Sc(g)(x) - Sc(\underline{g}(x)) \leq 0.$$

Area Extremality/Sc [78]. This is a stronger property of $\underline{g}$ where the integrals of the scalar curvatures of all Riemannin metrics g on X over all surfaces $S \subset X$ are *bounded by such integrals for* $\underline{g}$

[max]$_+$ $$\int_S Sc(\underline{g})(s)ds_{g_{\underline{g}}} \geq \int_S Sc(g)(s)ds_g.$$

Length Extremality. This is weaker than area extremality: all Riemannian metrics g, which are greater than $\underline{g}$ must satisfy the scalar curvature bound $\inf_{x\in X}(Sc(g)(x) - Sc(\underline{g}(x)) \leq 0$.

Length·Sc Extremality. All Riemannian metrics g on X must have the integrals of $Sc(g)$ over the curves in X *bounded by such integrals for* $\underline{g}$.

Rigidity in this context signifies that if g satisfies equalities in place of the inequalities [max], [max]$_+$, *etc., then* $g = \underline{g}$.

[?39$\frac{1}{2}$]**Question.** *Are* C^2*-limits of area extremal and of area extremal/Sc metrics on compact manifolds are area extremal? Is this true for length extremality?*

(Limits of rigid metric may be non-rigid as in [max 1] below.)

Examples

[max 1] *Surfaces.* Metrics with curvatures $\kappa \geq 0$ on closed surfaces are area·Sc extremal by the Gauss-Bonnet theorem. These surfaces are *area rigid* if and only if they contain *no domains isometric to cylinders* $[0, r_1] \times S^1(r_2)$ or to corresponding *Möbious bands,* e.g. if $\kappa > 0$.

On the other hand, these surfaces are never area·Sc rigid. In fact, there are plenty of diffeomorhisms between surfaces with positive curvatures which preserve the integrals of the curvatures over *all* open subsets in these surfaces.

[max 2] *Scalar Flat Manifolds.* If $Sc(\underline{g}) = 0$ and if the underlying manifold X, assumed connected, admits no metric with $Sc > 0$ – this happens, as we know, to all Riemannin flat and certain (not all) Ricci flat manifolds, then $\underline{g}$ is area·Sc extremal. But, none of these metrics is rigid.

Despite the existence of a bound on the size of a manifold (X, g) by $\sigma = \inf_x Sc(g)(x)$, it does not apparently imply that extremal metrics on X exist:

families of strictly increasing metrics with positive strictly increasing scalar curvatures may be *unbounded* as the following example shows.

[max 3] *Mixed Curvature Products.* Let S_1 and S_2 be surfaces with metrics g_1 and g_2 which have constant scalar curvatures $\kappa_1 > 0$ and $-\kappa_2 < 0$ where $\kappa_2 = \frac{2}{3}\kappa_1$. Then the 4-manifolds

$$X(t) = (S_1, \times S_2, (2 - (1+t)^{-2})^2 g_1 \oplus (1+t)^2 g_2$$

have positive strictly increasing scalar curvatures starting from $Sc(X(0)) = \frac{1}{3}$ and blowing up to $X(\infty) = (S_1, 2g_1) \times \mathbb{R}^2$ with scalar curvature $Sc(X(\infty)) = \frac{1}{2}$.

It remains unclear if these manifolds X admit extremal metrics at all, but the existence of such metrics on the n-spheres is confirmed by a 1996 theorem by Llarull.

[max 4] **Extremality and Rigidity of S^n.** *The unit spheres S^n; hence, all compact manifolds with constant sectional curvatures, are area extremal/Sc and, if $n \geq 3$ Area rigid/Sc.*

Llarull only states the *area rigidity/extremality of S^n* in his paper but his argument presented in [81, 95] yields *area extremality/Sc and rigidity/Sc as well* as it was pointed out by Listing [78].)

[max 5] In fact, Llarull proves the following more general result (rendered extremal/Sc in[78][48]).

Area Spin Extremality/Rigidity Theorem. Let X be a complete Riemannian orientable spin manifold of dimension $n + 4k$ such that

$$Sc(X) \geq n(n-1) = Sc(S^n)$$

and let $f : X \to S^n$ be a smooth *non-strictly area decreasing* map,

$$area(f(S)) \leq \cdot area(f(S)) \text{ for all smooth surfaces } S \subset X.$$

If the pull back

$$Y_{\underline{s}} = f^{-1}(\underline{s}) \subset X$$

of a generic point $\underline{s} \in S^n$ satisfies

$$\hat{A}(Y_{\underline{s}}) \neq 0,$$

then $Sc(X) = n(n-1)$ and the map f is an isometric submersion, i.e. the differential $Df : T(X) \to T(Y)$ has $rank(Df) = n$ and Df is isometric on the vectors normal to the ($4k$-dimensional) kernel of Df.

In particular, if $k = 0$ and f has non-zero degree, e.g. it is a homeomorphism, then f is an isometry.

Goette and Semmelmann generalised the above to several classes of symmetric spaces with non-constant curvatures. (See [38, 39], where one finds references to earlier results by other authors). For instance, they prove the following.

[max 6] *Compact symmetric spaces X with non-zero Euler characteristics, e.g. the Grassmanians $X = Gr_{2n}(\mathbb{R}^{4n}) = SO(4n)/SO(2n) \times SO(2n)$, are area extremal and area rigid.*

More surprisingly, Goette and Semmelmann showed that there are *non-empty C^2-open* sets of area extremal metrics on the spheres and *open subsets of extremal metric in the spaces of Kähler metrics* on certain complex manifolds.

Namely the prove the following.

[max 7] **Persistent Extremality of Metrics with Constant Curvatures.** *Metrics on S^n with positive curvature operators, in particular, the metrics which are C^2-close to (one could say "moderately far from") the standard metric on S^n are area·Sc extremal* and rigid.

[48] Be careful with [78]: some statements in this paper, unless I misinterpret them, are incorrect.

[max 8] **Persistent Extremality of Kähler Metrics.** *Compact Kähler manifolds with Ricci > 0 are area extremal and rigid* [38]-Goette Semmelmann Spin-c Structures 2001].

This generalises an earlier theorem by Min-Oo [93] who proved

area extremality of compact Hermitian symmetric spaces.

[max 9] **Extremality of Products.** In all known cases – this is true for the spaces and metrics in the above examples [max 1][49], [max 2][50] and [max 4]-[max 8],

Riemannin products of area extremal manifolds are area extremal and products or area rigid manifolds are area rigid.

For instance,

the products of spheres,

$$X = \bigtimes_i S^{n_i}$$

are area extremal, in fact, area·Sc extremal, and if (and only if) *all* $n_i \geq 2$, *then these* X *are area rigid.*

[?40] **Conjecture Area Extremality and Rigidity of Symmetric and Einstein Spaces.** *All Riemannin manifolds with positive and parallel Ricci tensor, in particular all Symmetric and all Einstein Spaces* X *are area extremal and those of them which contain no local flat* factors are area rigid.

For Einstein spaces, this agrees with local extremality lemma in [39], while an essential class of examples where the available proofs do not work are compact *Lie groups* X with biinvarinat metrics $\underline{g}$, where the tangent bundles are trivial and can not force non-vanishing of indices of natural Dirac operators over these X.

[?41] For instance, if $\underline{X} = SO(n)$ with $n \geq 5$, then no known method can rule out metrics $g \geq \underline{g}$ on X with $Sc(g) > Sc(\underline{g})$.

All known area extremal metrics are supported on rather special but the following questions remains mainly unexplored.

Three Questions

[?41] *Are there compact manifolds X which support metrics g with $Sc(g) > 0$ but admit no area extremal or length extremal metrics $\underline{g}$?*

[?42] *Can one "effectively" evaluate the minimal constant $\lambda = \lambda(X, g)$,, such that a given Riemannin manifold $X = (X, g)$, e.g where $sect.curv(g) > 0$, would support an area extremal (or a length extremal) metric $\underline{g}$ which would be λ-biLipschitz equivalent to g, where such a λ were expressible in terms of the pinching constant in the case where $sect.curv(g) > 0$?*

[?43] *Would it be more prudent to replace the condition $Sc(g) > 0$ by Ricci > 0?*

[49] Metrics on S^2 with curvatures $\sigma > 0$ have positive curvature operators, call them KOP, and so they are included in [max 7]. This would also apply to $\sigma \geq 0$ if we knew that all metric on S^n with KOP≥ 0 were approximable by metrics with KOP>0.

[50] We do not know if *all* metrics with $Sc = 0$ from [max 2] are suitable to be factors of area extremal manifolds – available proofs apply only to *flat metrics* and to (necessarily Ricci flat) metrics which admit parallel spinors and where the underlying manifolds X are spin with $\hat{A}(X) \neq 0$, such as certain (not all) Calabi-Yau manifolds, for instance.

18 Logic of the Dirac Operator Proofs of Area Extremality Theorems.

All known proofs of area extremality of a manifold $\underline{X}$ apply to smooth non-strictly area contracting maps $f : X \to \underline{X}$, such that

A. On the one hand, the twisted Dirac operator $D_{\otimes L}$ on X, i.e. D with coefficients in some bundle L induced by f from a bundle $\underline{L}$ on $\underline{X}$ satisfies:

$$index(D_{\otimes L}) \neq 0.$$

B. On the other hand, the inequality $Sc(X) > Sc(\underline{X})$ makes the operator $D^2_{\otimes L}$ *positive*, thus, prohibiting L-twisted harmonic spinors on X.

A is a topological condition on f, which is tailored to be satisfied by the identity map $\underline{X} \underset{id}{\longrightarrow} \underline{X}$, while B makes the operator $D^2_{\otimes \underline{L}}$ on $\underline{X}$ *non-strictly positive.*

Thus,

A necessitates the existence of *non-zero harmonic* $\underline{L}$-twisted harmonic spinors on $\underline{X}$ while B implies that these spinors are *parallel.*

Apparently, this is possible only if the the bundle $\underline{L}$ with its connection is somehow derived from the tangent bundle $T(\underline{X})$ and so a manifold $\underline{X}$ can serve this purpose only *if it carries some parallel (spinorial) tensors.*

[44] **Problem** *Describe all such $\underline{X}$ and decide which of them are area extremal.* (compare [64]).

Let us see how this works for the above [max]-examples.

I. Extremality of Even Dimensional Spheres. The only parallel tensors on a generic manifolds are constant functions and scalar multiples of the Riemannian volume form. This doesn't look much but these two are what makes the indices of the Dirac operator twisted with the spinor bundles, $D_{\otimes \mathbb{S}^{\pm}}$, on $\underline{X} = S^{2m}$ non-zero.

In fact, according to the Atiyah-Singer formula, these indices, for all[51] manifolds X are expressed in terms of the Euler characteristic and the signature of X as follows.

$$ind(D_{\otimes \mathbb{S}^{\pm}}) = \frac{1}{2}(\chi(X) \pm sign(X)).$$

Also, the index theorem implies that if X is a spin manifold and $f : X \to S^n$ is a continuous map with *non-zero degree* then the Dirac operator on X twisted with the induced bundle $L = f^!(\mathbb{S}^+(S^n))$ satisfies

$$ind(D_{\otimes L}) \neq 0$$

and, more generally, the same remains true if the *$\hat{A}$-degree of f is non-zero.*

Thus, in order to prove extremality of the spheres S^n for even n, (the above [max 4] and [max 5]), that is

to rule out the existence of smooth area decreasing maps $f : X \to S^n$
of non-zero degrees (and/or $\hat{A}$-degrees)

one needs to show that

[51] This twisted Dirac operator is defined and satisfies the index formula even if X is non-spin as it is (essentially) explained in section 10.

if $Sc(X) \geq n(n-1) = Sc(S^n)$ and $f : X \to S^n$ is a smooth (non-strictly) area decreasing map, then the operator $D_{\otimes L}$ on X for $L = f^!(\mathbb{S}^+(S^n))$ is positive and if $Sc(X) > n(n-1)$, then $D^2_{\otimes L}$ is strictly positive, which makes

$$ind(D_{\otimes L}) = 0.$$

This is achieved by analysing the last term in the twisted Schroedinger-Lichnerowicz-Weitzenboeck formula,

$$D^2_{\otimes L} = \nabla_\otimes \nabla^*_\otimes + \tfrac{1}{4} Sc(X) + \mathsf{R}_L,$$

and showing that

if $f : X \to S^n$ is a smooth (non-strictly) area decreasing map and $L_\pm = f^!(\mathbb{S}^\pm(S^n))$, then the lowest eigenvalues of $\mathsf{R}_{L_\pm}$, are bounded from below by $-\frac{n(n-1)}{4}$.

(In general, R_L is a fiber-wise Hermitian operator which acts on the bundle $\mathbb{S}(X) \otimes L$ and which is related to the curvature ∇^L_{ij} of L as follows [52, 74].

$$\mathsf{R}_L(s \otimes l) = \frac{1}{2} \sum_{i,j} (e_i \cdot e_j \cdot s) \otimes (\nabla^L_{ij} l), s \in \mathbb{S}(X), l \in L,$$

where $\{e_i\}$ is an orthonormal frame in X and "$\cdot$" denotes Clifford multiplication. A convenient frame in the present case of $L = L_\pm = f^!(\mathbb{S}^\pm(S^n))$ is the one which diagonalises the f-pullback of the spherical Riemannin metric to X [81], [93].)

This concludes our outline of the proof the of area extremality of S^n for even n and the proof of the area·Sc extremality follows along the same lines.

II. Extremality via Positivity of the Curvature Operator. By looking at $\mathsf{R}_{L_\pm}$ closer, one can see following Goette and Semmelmann that

if $\underline{X}$ is a Riemannian manifold with (non-strictly) positive curvature operator and $f : X \to \underline{X}$ is (non-strictly) area decreasing map then the operators $\mathsf{R}_{L_\pm}$ for $L_\pm = f^!(\mathbb{S}^\pm(\underline{X})$ are bounded from below at all $x \in X$ by

$$[\geq -\tfrac{1}{4}] \qquad \mathsf{R}_{L_\pm, x} \geq -\frac{1}{4} Sc(\underline{X})(\underline{x}) \text{ for } \underline{x} = f(x).$$

This implies the above extremality statements [max 6] and [max 7], for spin manifolds $\underline{X}$ with $\chi(\underline{X}) \neq 0$ (these manifolds are denoted X in [max 6] and [max 7]), where the corresponding area·Sc extremality is proved similarly.

And if $\underline{X}$ is non-spin, then instead of insisting that X is spin, one requires that the bundle $f^!(T(\underline{(X)})$ is spin-equivalent to $T(X)$. Then the twisted $D_{L\pm}$ is defined and it satisfies all of the above properties (see section 10).

III. Extremality of Kähler Manifolds. The Kähler forms on Kähler manifolds $\underline{X}$ are *parallel* and the *canonical line bundles* $\underline{L} = L_{can}(\underline{X})$ (of holomorphic n-forms on X, $n = dim_{\mathbb{C}}(X)$) are *spin equivalent* to $T(\underline{X})$.

Because of this, if $f : X \to \underline{X}$ is a spin map then the twisted Dirac operator $D_{\otimes f^!(\underline{L})}$ is defined and – it follows from the index theorem – if $Ricci(X) > 0$ and $deg(f) \neq 0$, then

$$ind\left(D_{\otimes f^!(\underline{L})}\right) \neq 0.$$

Thus, the area extremality of Kähler $\underline{X}$ with $Ricci(\underline{X}) > 0$ reduces to proving the inequality $\left[\geq -\frac{1}{4}\right]$ for area decreasing maps f,

$$\mathsf{R}_{f^!(\underline{L})} \geq -\frac{1}{4}Sc(\underline{X}) \text{ for } \underline{L} = L_{can}(\underline{X});$$

this was done by Min-Oo for Hermitian symmetric spaces and by Goette and Semmelmann in general.

On Ricci = 0. The condition $Ricci(\underline{X}) > 0$ is unneeded for the area extremality (it is needed for the area rigidity) of compact Hermitian *symmetric* spaces:

flat factors are taken care with a use of families of flat bundles over $\underline{X}$ and the f-induced ones over X as in section 7 for $\underline{X} = \mathbb{T}^n$.

But there also exist simply connected (Calabi-Yau) Ricci flat Kähler, manifolds X_{CY} which may carry metrics with $Sc > 0$. In fact, by $\star$ in section 4, all simply connected X_{CY} of complex dimensions $4k+3$ carry such metrics.

Therefore,

neither these X_{CY}, nor their products with the above $\underline{X}$ are area extremal.

On the other hand, it is plausible, that the metrics $\underline{g}$ (which have $Ricci(\underline{g}) \geq 0$) on such manifolds admit *no* (small) area increasing deformations which also increase the scalar curvature (compare 0.4 in[39]). Then it would be interesting to evaluate the size of the gaps between the metrics $\underline{g}$ and g where $g \geq \underline{g}$ and $Sc(g) > Sc(\underline{g})$.

IV. Extremality of Products. Whenever the area extremality is proven for some manifolds $\underline{X}$ by the above A+B argument, it is (more or less) automatically extends to Riemannin products of these manifolds.

Furthermore, if an $\underline{X}$ is (A+B)-extremal in this sense, then also $(\underline{X}\times\mathbb{R}^n)/\mathbb{Z}^n$ also so extremal for all free proper actions of $\mathbb{Z}^n$ on $\underline{X}\times\mathbb{R}^n$, which follows by invoking the families of flat complex line bundles from section 7.

V. Extremality in Odd Dimensions. Area decreasing maps $f : X \to S^{n-1}$ can be (obviously) suspended to area area decreasing maps $X' = X \times S^1(r) \to \underline{X} = S^n$ for all sufficiently large radii r.

Exploiting splitting of X', Llarull [81] shows that the area decreasing property of f implies that

$$\mathsf{R}_{f^!(\mathbb{S}^{\pm}(\underline{X}))} \geq -\frac{(n-1)(n-2)}{4}.$$

This rules out twisted harmonic spinors on X' for $Sc(X') > (n-1)(n-2)$ and if n is even, the area extremality of S^{n-1} follows by the application of the index theorem to (even dimensional!) X' and $\underline{X} = S^n$ as earlier.

Alternatively, one can use the *spherical suspension* $\ominus(X)$, which satisfies away from the singularities at the poles,

$$Sc(\ominus(X)) = Sc(X) + 2n + 1.$$

One removes arbitrarily small neighbourhoods of these poles and attach a 1-handle to them, thus, producing X' homeomorphic to $X \times S^1$.

The key point here is is that this X' can be endowed with a metric g'_ε such that $Sc(g'_\varepsilon) \geq Sc(X') - \varepsilon$ with an arbitrarily small $\varepsilon > 0$ as it is done in surgery for smooth spaces with $Sc \geq \sigma$ [?] and such that (X', g'_ε) admits an area decreasing map $X' \to S^n$.

This reduces area extremality of S^{n-1} to that for S^n.

Both arguments strike one as something artificial – the adequate concepts are, obviously, missing.

A seemingly more conceptual approach to odd dimensional $\underline{X}$ is suggested in [39] but this does not seem to deliver extremality of products of odd dimensional spheres, for example,[52] while the above applies with no problems.

VI. Area Rigidity. This needs the following additional property of the curvature term in the twisted Schroedinger- Lichnerowicz-Weitzenboeck formula which sharpens the above $[\geq -\frac{1}{4}]$.

If $f : X \to \underline{X}$ is an equidimensional smooth area non-increasing map, then the bound

$$\mathsf{R}_{L_\pm, x} \geq -\frac{1}{4} Sc(\underline{X})(\underline{x}) \textit{ for } \underline{x} = f(x)$$

must be strict at all points $x_> \in X$ where the differential of $Df : T_x(X) \to T_{f(x)}(\underline{X})$ is not isometric,

$[> -\frac{1}{4}]$
$$\mathsf{R}_{L_\pm, x_>} > -\frac{1}{4} Sc(\underline{X})(f(x_+)),$$

see [81, 95, 39].

[?45] Spin Problem. All of the above only applies to spin maps $f : X \to \underline{X}$, for which the required twisted Dirac operator defined, and, as on similar occasions we met earlier, the necessity of the spin condition, say for equidimensional maps of degrees$\neq 0$, remains unsettled.

Extremality via Lipschitz Maps. If a manifold $\underline{X}$ admits no C^2-smooth length decreasing map $X \to \underline{X}$ of degree$\neq 0$ which also strictly decreases the scalar curvature, then it admits no such Lipschitz map either my a simple approximation argument.

[?46] But it is unclear if this remain true with "area" in place of "length".

19 Extremality of Open Manifolds.

If $\underline{X} = (\underline{X}, \underline{g})$ is a complete Riemannin manifold, then the metrics g on $\underline{X}$ which are length-wise greater [53] than $\underline{g}$ are complete but this is not so, in general for g which are area-wise greater than $\underline{g}$.

[52] I have not followed the arguments in [39] in detail.

[53] *Length-wise greater* means just *greater*. We add *length-wise* to emphasise the distinction from *area-wise*.

Because of this, we call $\underline{X}$ in this section *area extremal* if no *complete* area-wise greater metric on $\underline{X}$ can have $Sc(g) > Sc(\underline{g})$.

A companion concept, where the inequality $Sc(g) > Sc(\underline{g})$ is replaced by scalar $Sc(g) > Sc(\underline{g}) + \varepsilon$ for some $\varepsilon > 0$ will go under heading *area gap extremality*.

Similarly one defines "complete" length and length gap extremality and observe that if $\underline{X}$ is area or length extremal than it is area gap extremal or length area extremal correspondingly and that "gap" is automatic for compact manifolds $\underline{X}$.

Complete scalar flat $\underline{X}$ which admit no complete metrics with *positive* scalar curvature (i.e. $Sc > 0$) are instances of area extremal manifolds in this sense; they are area gap extremal if they admit no complete metrics with *uniformly positive* ($Sc \geq \varepsilon > 0$) scalar curvature. But the converse is not quite true.

Flat Examples (a) *All complete flat Riemannin manifolds are area gap extremal.*

(b) *A complete non-compact Riemannin flat manifold* $\underline{X}$ *is area extremal if and only if its soul* $\underline{X}_{soul} \subset \underline{X}$, *which is a maximal compact flat submanifold in* $\underline{X}$, *has* $codim(\underline{X}_0) = 1$.

Proof of (a). Since the universal covering of $\underline{X}$ is equal to $\mathbb{R}^n$ the general case reduces to non-existence of (not necessarily spin) manifolds with $Sc \geq \varepsilon > 0$ wich admit proper Lipschitz map to $\mathbb{R}^n$ of non-zero degrees.

Proof of (b) Since a a finite cover of $\tilde{\underline{X}}$ of $\underline{X}$ is homeomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}$, $n = dim(\underline{X})$, non-existence of complete metrics with $Sc > 0$ on $\tilde{\underline{X}}$; hence on $\underline{X}$ follows by a twisted Dirac operator argument similar to the case (a).

Actually, this argument applies to $\tilde{\underline{X}}$ times the 2-sphere $S^2(R)$ with large radius R [54] and the proof of **B** for $X_0 = S^n$ below) where this (annoyingly artificial) spherical factor is brought in to make the relevant Dirac operator *strictly positive* at infinity, which is needed for the applicability of the relative index theorem.

Notice, in the regard of (a) that

a complete flat Riemannin manifold $\underline{X}$ *admits a complete metrics* g *with* $Sc(g) \geq \varepsilon > 0$, *if and only if* $codim(\underline{X}_{sole}) \geq 3$.

What is non-trivial here is only if but this easily follows ([51]) from the inequality $width(X) < \frac{2\pi}{n}$ for torical bands in section 21.

[?47] **Conjecture: Stabilisation of Extremality.** *Let* X_0 *be a compact area extremal Riemannin manifold. Then*

A. $X_0 \times \mathbb{R}^m$ *is area gap extremal for all* m.

B. $X_0 \times \mathbb{R}$ is area extremal.

This can be confirmed for all known examples of X_0 the area extremality of which was established by a Dirac operator arguments sketched in the previous section. Let us do it in two cases.

Start with **A** for $X_0 = S^n$, let X be a complete orientable spin Riemannian manifold, let $f : X \to S^n \times \mathbb{R}^m$ be a smooth proper map.

All we shall need of $\mathbb{R}^m$ for our purpose is that $Sc(\mathbb{R}^m) = 0$ and $K\text{-}Area(\mathbb{R}^m) = \infty$.

Assume for simplicity's sake that m is and let L_{ε_0} be a $\mathbb{Q}$-homologically essential unitary vector bundle with ε_0-flat connection which is flat at infinity.

Assume for the same reason that n is even and let $\mathbb{S}^+$ be the "positive" spinor bundle on S^n.

Let $L^!_{\varepsilon_0}$ be the f-pullback of the tensor product $\mathbb{S}^+ \otimes L_{\varepsilon_0}$ and observe – this needs looking at the Llarull's computation for $\mathsf{R}_{L^!_{\varepsilon_0}}$ – that if

the scalar curvature of X at infinity is $\geq (n(n-1) + \varepsilon > 0$ and if $\varepsilon_0 > 0$ is much smaller than ε, then the twisted Dirac operator $D_{\otimes L_{\varepsilon^!_0}}$ on X is strictly positive at infinity and if $deg(f) \neq 0$ it has non-zero index relative the Dirac operator twisted with the pullback of $\mathbb{S}^+$.

Thus, X carries non-trivial twisted harmonic spinors.

On the other hand, if the scalar curvature of X at infinity is $\geq (n(n-1) + \varepsilon > 0$ on all of X and $\varepsilon_o < \varepsilon$, then again, by looking at looking at LLarull's $\mathsf{R}_{L^!_{\varepsilon_0}}$ and sees that X can't carry such spinors and the proof follows.

"Subcomplete" Extremality. Let $\underline{X}_o$ be an open, not necessarily complete Riemannin manifold.

[?48] **Question.** When does such an $\underline{X}_0$ is area extremal in the category of complete manifolds?

Namely, when do all complete metrics g on $\underline{X}$, which are area-wise larger than $\underline{g}$, necessarily have $Sc(g)(\underline{x}_0) \leq Sc(\underline{g})(x_0)$ at some point $\underline{x}_0 \in \underline{X}$?

It was pointed out by Llarull in [80] that punctured spheres $S^n \setminus \{s_o\}$ are extremal in this sense.

Moreover, Let $\underline{X}$ be one of compact area extremal [max] manifolds from section 17. e.g. the sphere S^n or a Hermitian symmetric space, and let $\Sigma_o \subset \underline{X}$ be a closed, say piecewise smooth, subset, such that

[//] *the parallel Levi-Civita transport in $T(\underline{X})$ is* trivial *along the curves in Σ_o, e.g. Σ_o is a union of disjoint trees.* Then

$\underline{X}_o = \underline{X} \setminus \Sigma_o$ is area extremal in the the category of complete manifolds.

In fact, the relative index theorem [54] allows an extension of the original proofs for compact $\underline{X}$ to manifolds $\underline{X}_o$ with complete metrics.

The condition $[/\!\!/]$ is far from being necessary.

For instance, there are "codimension two" obstructions coming from the fundamental group at infinity ([54, ?, 51]) which show that

$[\smallsetminus\bigcirc]$ *there are no complete metrics with $Sc \geq \sigma > 0$ on the complement of (possibly trivial) knots and links in S^3 as well on S^4 minus two or more equatorial S^2 in general position.*

On the other hand the complements to the m-skeleta of smooth triangulations of an n-dimensional manifolds $\underline{X}$, where $m \geq 2$, admit, by an easy surgery type argument, arbitrarily large complete metrics with $Sc > 0$. Thus, for instance,

the complements to the m-skeleta for $m \geq 2$ of triangulations of compact scalar flat *manifolds $\underline{X}$, e.g. for $\underline{X} = \mathbb{T}^n$, are* not area extremal *in this "complete" sense.*

However, we suggests the following.

[?49] **Conjecture.** *The sphere S^n minus Σ_o is area extremal in the "subcomplete" sense for all closed subsets $\Sigma_o \subset S^n$ of topological dimensions $k \leq 1$.*

20 Lengths, Widths and Areas of Non-Complete Manifolds with $Sc > 0$.

To develop an adequate picture of *complete manifolds* with scalar curvatures $\geq \sigma$ one needs to understand geometric constraints imposed by the inequality $Sc \geq \sigma$ on (bounded) domains in these manifolds.

Thus, we look at possible sizes of *incomplete manifolds X*, e.g. (compact) *manifolds with boundaries*, where we do not, a priori, assume they are contained in complete manifolds with lower bounds on Sc. (We shall say something about the *shapes* rather than mere sizes of these X in section 22)

A simple example showing what can and what cannot be expected in this regard is

the universal covering of the 2-sphere minus 2 opposite points times $\mathbb{R}^{n-2}$, denoted

$$\tilde{\Sigma}^n_\pi = \widetilde{S^2 \smallsetminus \{\cdot\cdot\}} \times \mathbb{R}^{n-2}$$

which satisfies:

- $Sc(\tilde{\Sigma}^n_\pi) = Sc(S^2) = 2$,
- $waist_1[\tilde{\Sigma}^n_\pi] = \pi$ and $width_1(\tilde{\Sigma}^n_\pi) \leq \pi$.
- $waist_m[\tilde{\Sigma}^n_\pi] = width_{n-m}(\tilde{\Sigma}^n_\pi) = \infty$ for $m = 2, ..., n$ and $K\text{-}area[\tilde{\Sigma}^n_\pi] = \infty$ as well.
- *The r-interior* $\tilde{(\Sigma^n_\pi)}_{-r} \subset \Sigma^n_\pi$, that is the set of points x which are r-far from infinity, i.e. such that the closed r'-balls $B_x(r') \subset \Sigma^n_\pi$ for $r' < r$ are compact, has *small* all $waist_m$ and K-area, actually, zero in the present example.

r-Interior and Completeness. Metric completeness of an X in this terms is equivalent to non-emptiness of the r-interior of X for $r = +\infty$.

[?50] **Conjecture.** *All of the above is satisfied, modulo constants, for all n-manifolds, possibly incomplete and/or with boundaries, with* $Sc(X) \geq \sigma > 0$. Namely

⧓$^{bnd}_1$ $$width_{n-1}(X) \leq const_n \sigma^{-\frac{1}{2}},$$

⧓$^{bnd}_2$ $$width_{n-2}(X_{-r}) \leq const_n \sigma^{-\frac{1}{2}} \text{ for } r \geq const_n \sigma^{-\frac{1}{2}},$$

☻$^{bnd}_2$ $$waist_2(X_{-r}) \leq const_n \sigma^{-1} \text{ for } r \geq const_n \sigma^{-\frac{1}{2}},$$

⚭^{bnd} $$K\text{-}area(X_{-r}) \leq const_n \sigma^{-1} \text{ for } r \geq const_n \sigma^{-\frac{1}{2}},$$

where, of course, all these $const_n$, especially the optimal ones, may be different.

Among the first three inequalities, which generalise the corresponding *conjectures* in section 16, a definite result is available only for ⧓$^{bnd}_2$ for 3-manifolds, which, similarly to ⧓$_2$, is proven with a use of minimal surfaces [54].

On the other hand, the inequality ⚭^{bnd}, which, in the case where X is *complete spin*, easily follows from the index theorem for the twisted Dirac operator, remains problematic for *non*-complete, let them be spin, n-manifolds starting from $n = 3$. (To make sense of this for $n = 3$ one should replace $K\text{-}area(X^3_{-r})$, which is zero by definition for odd n, by $K\text{-}area(X^3_{-r} \times \mathbb{R})$.)

It is tempting to try to reduce these conjectures, especially ⚭^{bnd}, to the case of complete manifolds by extending (the metric on) X (on $X \times \mathbb{R}^N$?) to a complete manifold which would also have the scalar curvature bounded from below by a positive constant. This raises the following question.

[?51] **Extension Problem.** Let X be a Riemannian n-manifold with $Sc(X) \geq \sigma > 0$ and let $\sigma_- \leq \sigma$, r and $r_+ \geq r$ be positive numbers.

When does there exist an n-dimensional manifold X_+ with $Sc(X_+) \geq \sigma_-$, such that the r-interior $X_{-r} \subset X$ isometrically embeds into the r_+-interior $(X_+)_{-r_+} \subset X_+$?

[?52] **Conjecture. Completion by Extension.** *If $\sigma > \sigma_-$ and $r \geq const_n(\sigma - \sigma_-)^{-\frac{1}{2}}$ for some (large) constant $const_n$, then the extension problem is solvable with $r_+ = \infty$:*

there exists a complete X_+ with $Sc(X_+) \geq \sigma_-$ which isometrically contains X_{-r}

Let us introduce geometrically more transparent geometric invariants which may serve as lower bound to the K-area.

[?53] **Conjecture. Sharp Spherical Length Comparison Inequality.** *Spheres with finitely many punctures are length extremal.* In fact – this is, probably equivalent–

All Riemannin n-manifolds X, possibly non-complete and with boundaries, which have $Sc(X) \geq Sc(S^n) = n(n-1)$ satisfy

$$co\text{-}s.leng(X) \leq 2\pi$$

In plain words, if $Sc(X) \geq n(n-1)$, then

there is no no strictly distance decreasing proper maps from X to S^n with non-zero degrees.[54]

[?54] **Conjecture: Extremality of Concave Spherical Balls.** *The balls $B(R) \subset S^n$ of radii $R \geq \frac{\pi}{2}$ are length extremal:*

no Riemannian metric g on such a ball which is greater than the spherical one (of constant curvature 1) can have $Sc(g) > n(n-1) = Sc(S^n)$.

[?54$\frac{1}{2}$] **Opposite Conjecture.** *No manifold of dimension $n \geq 3$ with smooth* non-empty *boundary is ever length extremal.*

What is known.

The sharp spherical length comparison inequality looks obvious for $n = 2$ and the concave length extremality is easy to prove for the hemisphere $S^2_+ (= B(\frac{\pi}{2}))$.

In fact, assume S^2_+ is *not* length extremal and let X be a surface with sectional curvature >1, which admits a a distance decreasing homeomorphism $f : X \to S^2_+$ of degree $\neq 0$.

Let $x \in X$ be the pullback of the center of S^2_+ and let $S \subset X$ be the shortest closed curve in the complement to the open ball

[54] Recall that "proper maps" $X \to S^n$ are supposed to be locally constant on the boundary of X as well as on the complement $X \setminus COMP$ for a sufficiently large compact subset in X.

$B_x(\frac{\pi}{2}) \subset X$ which is non-contractible in this complement. Then the contradiction follows by comparing the length of S with that of the boundary circle $S^1 \subset S^2_+$.

Probably, a similar argument applies to all concave balls in S^2. Also, there may be other *length extremal* surfaces with *concave boundaries*.

Warning. Let $X_0 = U_\delta(S^1) \subset S^2$ be the δ-neighbourhood of the equator $S^1 \subset S^2$ with the spherical metric g_0 on it. (The complement of X_0 is made of two opposite $\left(\frac{\pi}{2} - \delta\right)$-balls.)

Then X_0 *admits metrics* $g > g_0$ *with* $Sc(g) > Sc(g_0) = 2 = Sc(S^2)$.

In fact,

given $\varepsilon \in (0, \frac{\pi}{2} - \delta)$, *there is a cyclic covering of* $U_{\delta+\varepsilon}(S^1)$ *which admits a strictly distance decreasing homeomorphism* $U_{\delta+\varepsilon} \to U_\delta(S^1)$.

In higher dimensions, only *non-sharp* bounds on the Lipschitz constants of proper maps $X \to S^n$ of non-complete manifolds X are available, see [54, 51]

[?55] **Conjecture 18. Interior Hemi-Spherical Area Inequality.** *The* r*-interiors of all compact Riemannin* n*-manifolds* X *with boundaries and with* $Sc(X) \geq Sc(S^n) = n(n-1)$ *satisfy for all* $r \geq \pi/2$,

$$co\text{-}s_+ar(X_{-r}) \leq 2\pi :$$

no strictly area decreasing proper map $X_{-r} \to S^n_+$ *of non-zero degree for* $r > \pi/2$ *exists.*

It is even unclear what happens in this regard to domains in closed spin manifolds X. For instance:

[?56] *What are possible values of the co-*s_+*areas of the complements of* r*-balls in compact spin manifolds* X *with* $Sc(X) \geq n(n-1)$*?*

21 Width of Bands with $Sc \geq n(n-1)$ and Curvature of Submanifolds in S^n.

Our *bands* are connected Riemannian manifolds with two boundary components the distances between which are regarded as the *widths* of the bands

[?57] **$\frac{2\pi}{n}$-Conjecture.** Let X be a compact n-dimensional band with $Sc(X) \geq n(n-1)$.

If **no** *smooth closed hypersurface* $Y \subset X$ *which separates the two boundary components admits a metric with* $Sc > 0$, *then*

$$width(X) < \frac{2\pi}{n}.$$

This is confirmed in [51] for certain bands with large fundamental, e.g. for the torical ones $X = \mathbb{T}^{n-1} \times [-1,1]$, but there is no approach in sight for simply connected bands X.

In fact, if Σ^{n-1} is a Hitchin sphere, for example,

[?58] there is no apparent non-trivial bound on the width of $X = \Sigma^{n-1} \times [-1,1]$ even we assume that the *sectional curvature* of X is = 1.

On the other hand it is easy to see that the suprema of the principal curvatures of immersions of Hitchin's Σ^{n-1} to S^{n+k} satisfy (see [51])

$$supcurv(\Sigma^{n-1} \hookrightarrow S^{n+k}) \geq \frac{\sqrt{n-2}}{k+1},$$

but one doesn't know

[?59] *what is the (asymptotically for $n \to \infty$ and/or for $k \to \infty$) sharp inequality for immersions of these Σ^{n-1} to spheres*

[?60] *Also it is unclear if there are (non-trivial) inequalities of this kind for other exotic spheres.*

Nor does one know what happens to other topologically complicated manifolds immersed into geometrically simple ones.

Specifically, let Y_j be diffeomorphic to the product of j spheres, of positive dimensions,

$$X_j = S^{n_1} \times \ldots \times S^{n_j}, .$$

[?61] *Is then every immersion from X_j to the unit ball in $\mathbb{R}^N$ satisfies*

$$supcurv(X_j \hookrightarrow B^N(1) \subset \mathbb{R}^N)) \geq \sqrt{k} \textit{ for all } N \geq n_1 + \ldots. + n_j + 1?$$

[?62] But it is also not impossible that all manifolds admit immersions into the unit ball in the Hilbert space $\mathbb{R}^\infty$ with principal curvatures bounded by a universal constant, say by 1 000 000.

22 Convex Polyhedra, Manifolds with Corners and Patterns of Concentration of Positivity of Scalar Curvature on Curves, Surfaces and Spaces.

Let $P \subset \mathbb{R}^n$ be a compact convex polyhedron with non-empty interior, let $Q_i \subset P$, $i \in I$, denote its $(n-1)$-faces and let

$$\angle_{ij}(P) = \angle(Q_i, Q_j)$$

denote its dihedral angles.

Say that P is *extremal* if all convex polyhedra P' which are combinatorially equivalent to P and which have

$$\angle_{ij}(P') \leq \angle_{ij}(P) \text{ for all } i, j \in I,$$

satisfy

$$\angle_{ij}(P') = \angle_{ij}(P).$$

It is known that

the simplices and and also all P with $\angle_{ij}(P) \leq \frac{\pi}{2}$, e.g. rectangular solids, are extremal.

[?63] **Problem.** Identify combinatorial types $\mathcal{P}_{extr}$ of convex polyhedra where all representative $P \in \mathcal{P}$ are extremal and also describe extremal P of non-extremal types $\mathcal{P}_{nonextr}$.

Next, call P *mean convexly extremal* if *there is no $P' \subset \mathbb{R}^n$ diffeomorphic to P* and such that

• the faces $Q'_i \subset P'$ corresponding to all $Q_i \subset P$ have $mean.curv(Q'_i) \geq 0$,

• the dihedral angles of P', that are the angles between the tangent spaces $T_{p'}(Q'_i)$ and $T_{p'}(Q'_j)$ at the points p' on the $(n-2)$-faces $Q'_{ij} = Q'_i \cap Q'_j$, satisfy

$$\angle_{ij}(P') \leq \angle_{ij}(P),$$

• this angle inequality is strict at some point, i.e. there exits $p'_0 \in Q'_{ij}$ in some Q'_{ij}, such that

$$\angle(T_{p'_0}(Q'_i), T_{p'_0}(Q'_j)) < \angle_{ij}(P).$$

[?64] **Question.** *Are all extremal convex polyhedra P are mean convexly extremal?*

It is not even known

[?65] *if the regular 3-simplex is mean convexly extremal.*
But

mean convex extremality of the n-cube

follows by developing the cube P into a complete (orbi-covering) manifold $\hat{P}$ homeomorphic to $\mathbb{R}^n$ by reflecting P in the faces, approximating the natural continuous Riemannin metric metric on $\hat{P}$ by a smooth one with $Sc \geq \varepsilon > 0$ [49]

And the same argument yields [49] the following

[$\ast$] *Let a Riemannin metric g on the n-cube P satisfy:*

$\ast_0$ $Sc(g) \geq 0$.

$\ast_1$ $mean.curv_g(Q_i) \geq 0$,

$\ast_2$ $\angle_{ij}(P,g) \leq \frac{\pi}{2}$.

Then, necessarily, $Sc(g) = 0$, $mean.curv_g(Q_i) = 0$ and $\angle_{ij}(P,g) = \frac{\pi}{2}$.

[?66] Probably, these equalities imply that P is *isometric to a Euclidean rectangular solid* but the approximation/smoothing is no good for proving this kind of rigidity.

The main merit of [$\ast$] is that it provides a test for $Sc \geq 0$ in all Riemannin manifolds X:

$Sc(X) \geq 0$ if and only if no cubical domain $P \subset X$ satisfies

$$[mean.curv_g(Q_i) > 0] \& [\angle_{ij}(P,g) \leq \frac{\pi}{2}].$$

[?67] This suggests a possibility of defining $Sc(X) \geq 0$ for some singular spaces, X, e.g. for manifolds with *continuous* (bounded measurable?) Riemannian metrics and for *Alexandrov spaces* with sectional curvatures bounded from below.

The following would add credulity to this suggestion.

[?68] **Conjecture.** Let $\tilde{X}$ be the universal covering of a Riemannin manifold X homeomorphic to the n-torus. Then

$\tilde{X}$ has *non-positive scalar curvature at infinity.*

This means that

$\tilde{X}$ can be exhausted by *overcubic*[55] domains $P_i \subset \tilde{X}$ with corners, such that all $n-1$ faces of P_i have positive mean curvatures and all dihedral angles of P_i are $\leq \frac{\pi}{2}$.

This would be especially pleasant to prove for X with continuous metrics g but the case of smooth g seems non-trivial either.

.... to be continued

[55] A manifold P with corners is called *overcubic* if it admits a map of degree one to the n-cube $\square^n$ such that the k-faces of P go to the k-faces of $\square^n$.

23 Symmetrization by Reflection with $Sc > \sigma$.

Let X be a compact Riemannian n-manifold, $Y_0 \subset X$ be a closed cooriented hypersurface, and let $U \supset Y$ be a (small) neighbourhood of Y_0 which is divided by Y_0 into halves, say $U_+, U_- \subset U$.

Let $Y_+ = Y_+(\varepsilon) \subset U_+$, $\varepsilon > 0$, be a closed hypersurface which is $(n-1)$-volume minimizing among all closed hypersurfaces $Y'_+ \subset U_+$ such the n-volume in the region between Y_+ and Y in U is equal to a given (small) number $\varepsilon > 0$, and similarly, we define $Y_- = Y_i(\varepsilon) \subset U_-$.

These $Y_\pm$ may intersect Y_0 as well as the boundaries of $U_\pm$ but this does not happen if Y_0 is *minimizing in* U, which means that all hypersurfaces $Y' \subset U$ which are homologous to Y satisfy

$$vol(Y') \geq vol(Y_0).$$

(We shall later meet Y_0 which are minimizing among all hypersurfaces which *divide the volume of* U *in a given proportion.*)

In fact, if $\varepsilon > 0$ is sufficiently small, these $Y_\pm(\varepsilon)$ are unique and they keep away from Y and from the boundary of U.

Furthermore the regions $V_\pm(\varepsilon) \subset U_\pm$ bounded by Y and $Y_\pm$ are strictly monotone increasing,

$$V_\pm(\varepsilon_1) \subset Int(V_\pm(\varepsilon_2)) \text{ for all small } \varepsilon_2 > \varepsilon_1.$$

And according to the standard regularity theory, the hypersurfaces $Y_\pm$ are smooth with strictly positive mean curvatures away from their singularities $\Sigma_0 \subset Y_0$ of codimensions ≥ 7.

(If $n \leq 8$, these $Y_\pm(\varepsilon)$ are *non singular* for an open dense set of small $\varepsilon > 0$ by Natan Smale's theorem [118]; conjecturally,this is true for all n.)

Symmetrization by Reflections. Assume Y_0 is *connected* and let

$$V(\varepsilon) = V_+ \cup V_-$$

and let $\hat{V}(\varepsilon)$ be obtained by reflecting $V(\varepsilon)$ across $\partial V(\varepsilon) = Y_+(\varepsilon) \cup Y_1(\varepsilon)$.

In other words, $\hat{V}(\varepsilon)$ is the *infinite cyclic covering of the double*

$$V_\varepsilon V_\varepsilon = V(\varepsilon) \underset{\partial V(\varepsilon)}{\cup} V(\varepsilon).$$

These $\hat{V}(\varepsilon)$ carry natural continuous Riemannian metrics $\hat{g}_\varepsilon$ away from Σ_0 which converge to an $\mathbb{R}$-*invariant* metric $\hat{g}_0$ on the Hausdorff limit

$$\hat{V}_{\downarrow 0} = \lim_{\varepsilon \to 0} \hat{V}(\varepsilon).$$

It is easy to see that $\hat{g}_0$ is *smooth Riemannian* away from the $\mathbb{R}$-orbit $\mathbb{R}(\Sigma) \subset \hat{V}_{\downarrow 0}$ and that the Riemannian metrics $\hat{g}_\varepsilon$ converge to $\hat{g}_0$ in the C^0-topology for the natural simultaneous (y,t)-coordinatisation of the spaces $\hat{V}(\varepsilon)$ and $\hat{V}_{\downarrow 0} \setminus \mathbb{R}(\Sigma) = (Y_0 \setminus \Sigma) \times \mathbb{R}$.

What is significant here is the following.

Preservation of Sc-Positivity. *If $Sc(X) \geq \sigma$ then also $Sc(\hat{g}_0) \geq \sigma$. Moreover,*

$$Sc(\hat{g}_0)(y,t) \geq Sc(X)(y) \text{ for all } y \in Y_0 \setminus \Sigma_0 \text{ and } t \in \mathbb{R}.$$

Proof. One may assume – slightly C^2-perturb the original Riemannian metric in X if necessary – that Y_0 is *strictly minimizing* and the subvarieties $Y_\pm(\varepsilon)$ have *strictly positive* mean curvatures at the regular points. These can be δ*-regularised* that is approximated by C^2*-smooth hypersurfaces*, say

$$[reg_\delta] \quad Y_\pm(\varepsilon,\delta) = \partial V_\pm(\varepsilon,\delta), \text{ where } V_\pm(\varepsilon) \supset V_\pm(\varepsilon,\delta) \supset V_\pm(\varepsilon-\delta),$$

where $\delta > 0$ can be chosen arbitrarily small and where the hypersurfaces $Y_\pm(\varepsilon,\delta)$ have *strictly positive mean curvatures*, [48]

Then the C^0-continuous Riemannian metrics in the corresponding reflection manifolds $\hat{V}_\pm(\varepsilon,\delta)$ can be approximated by smooth metrics with $Sc \geq Sc(X) - \sigma'$ with arbitrarily small σ' (see [49]) and the proof follows by

semicontinuity of the scalar curvature under C^0-convergence of Riemannian metrics [49], [10].

Alternative Proof. Rescale positive functions $f_\pm(y;\varepsilon)$ on Y_0 which represent $Y_\pm(\varepsilon)$ in normal coordinates and of let $f_\pm(y) = \pm f(y)$ be the limit functions for $\varepsilon \to 0$. Observe that

$$\hat{g}_0 = g_{Y_0} + f^2 dt^2,$$

where g_{Y_0} denotes the Riemannian metric on the non-singular locus in $Y_0 \subset X$ induced from X.

The positivity of the mean curvatures of $Y_\pm(\varepsilon)$ translates, in the limit for $\varepsilon \to 0$, to positivity of $L(f)$ for a certain linear differential (namely, stability) operator L on $Y_0 \setminus \Sigma_0$, where the latter positivity implies that

$$Sc(g_{Y_0} + f^2 dt^2) \geq Sc(X)$$

as in 11.14 of [54]. QED.

Examples and Conjectures. If X is a flat n-torus and $Y_0 \subset X$ is a saboteurs, then the doubles $V_\varepsilon V_\varepsilon = V(\varepsilon) \underset{\partial V(\varepsilon)}{\cup} V(\varepsilon)$ are also flat tori canonically homeomorphic to X (but not isometric) with

$$width_{n-1}(V_\varepsilon V_\varepsilon) \leq \varepsilon$$

while their cyclic coverings $\hat{V}(\varepsilon) = \widetilde{V_\varepsilon V_\varepsilon}$ are isometric to $Y_0 \times \mathbb{R}$.

Now, given a sequence of homology classes $h_0, h_1, ..., h_i, ... \in H_{n-1}(X)$ and a sequence of positive numbers $\varepsilon_0, \varepsilon_1, ..., \varepsilon_i$, let

$$X_0 = X,\ X_1 = V_{\varepsilon_0} V_{\varepsilon_0},\ X_2 = V_{\varepsilon_1} V_{\varepsilon_1}, ..., X_i = V_{\varepsilon_{i-1}} V_{\varepsilon_{i-1}}, ...$$

where $V(\varepsilon_i) \subset X_i$ is the ε_i-neighbourhood of the subtorus $Y_i \subset X_i$ in the class h_i, etc.

Clearly, if the classes h_i span $H_{n-1}(X)$ and $\varepsilon_i \to 0$, then

$$diam(X_i) \to 0 \text{ for } i \to \infty.$$

Moreover, let $X = X_0$ be a compact Riemannian n-manifold coming with a homotopy class of maps f_0 to the torus $\mathbb{T}^n$ and $Y_0 \subset X_0$ be a connected volume minimizing subvariety the homology class of which goes to a non-zero multiple of a class $h_0 \in H_1(\mathbb{T}^n)$ represented by a subtorus. Then the corresponding $X_1 = V_{\varepsilon_0} V_{\varepsilon_0}$ also comes with a homotopy class of maps f_1 to $\mathbb{T}^n$ such that

$$deg(f_1) \leq deg(f_0) \text{ and } deg(f_0) \neq 0 \Rightarrow deg(f_0) \neq 0.$$

(In fact, $deg(f_1)$ is a divisor of $deg(f_0)$.)

Thus, given $h_i \subset H_1(\mathbb{T}^n)$ and $\varepsilon_i > 0$ we construct spaces

$$X_i = X(\varepsilon_{i-1}) = V_{\varepsilon_{i-1}} V_{\varepsilon_{i-1}},$$

which, if we worry by about singularities, can be δ-regularised by setting

$$X'_i = X(\varepsilon_{i-1}, \delta_{i-1}) = V_{\varepsilon_{i-1},\delta_{i-1}} V_{\varepsilon_{i-1},\delta_{i-1}}$$

as in the above $[reg_\delta]$.

Observation. *If the classes h_i span $H_1(\mathbb{T}^n)$ and $deg(f_0) \neq 0$, then*

$$width_{n-7}(X'_i) \to 0, \textit{for } \varepsilon_i \to 0$$

because minimal hypersurfaces have codimensions ≥ 7.

[?69] **Conjecture. Shrinking of Singularities.** *Let X be a compact orientable Riemannin n-manifold, $f_0 : X \to \mathbb{T}^n$ be a continuous map of non-zero degree, h_i, i=0,1,2,..., be 1-dimensional homology classes in $\mathbb{T}^n$ which generate $H_1(\mathbb{T}^n)$ and let $\varepsilon_i \to 0$ be positive numbers. Then the above (regularised for prudence) spaces X_i satisfy*

$$diam(X'_i) \to 0 \textit{ for } i \to \infty.$$

Moreover, the minimal hypersurfaces $Y_i \subset X'_i$ become non-singular for sufficiently large i and the manifolds X'_i as well as X_i admit λ_i-bi-Lipschitz homeomorphisms to flat n-tori Y_i with $\lambda_i \to 1$ for $i \to \infty$.

This conjecture implies that overtorical manifolds X, (i.e. admitting maps to $\mathbb{T}^n$ with non-zero degrees) with $Sc(X) \geq 0$ are, in fact, flat, since the scalar curvature is semicontinuous under "Lipschitz limits". (The proof of this semicontinuity for C^0-convergence given in [49] automatically extends to the Lipschitz convergence.)

(The arguments used in in [86] and those in [115], for the proof of this "non-positivity" $Sc(X) \not> 0$ probably yield the proof of the above Conjecture as well.)

24 Scalar Curvature and Mean Curvature.

One may think of positive scalar curvature as *Riemannian internalisation* of the concept of *mean convexity*, where a Riemannian manifold Y with boundary, e.g. a smooth domain Y in a larger Riemannian manifold, is called *mean convex* if the boundary ∂Y has positive mean curvature.

This "internalisation": is motivated by the following.

Doubling Lemma [52]. *The natural C^0-Riemannian metric g_0 on the double*[56] *$X = Y +_\partial Y$ of a manifold $(Y, \partial Y)$ with $Sc(Y) \geq 0$ and with $mean.curv(\partial Y) > 0$ along the boundary can be C^0-approximated by C^2-metrics g with with $Sc(g) > 0$, where, moreover, this approximation is C^2 away from the "∂-edge" $Z_\partial \subset X$ where the two copies of Y meet in X.*

This is achieved by

(1) Smoothing g_0 with $Sc > 0$ close to Z_∂ by rescaling along the geodesics normal to Z_∂ (a five line argument)

$$+$$

(2) redistribution of positivity of Sc on all of X by a C^2-small conformal deformation (another 5 lines).

Also there is a similarity between the following "thin" spaces with $mean.curv > 0$ and with $Sc > 0$.

Thin$_{\mathbf{mean>0}}$. Given a closed subset Y in a Riemannian n-manifold X, define $vol_\partial(Y)$ as the infimum of the $(n-1)$-volumes of the boundaries of arbitrarily small neighbourhoods $U \supset Y$ of Y in X,

[56]The double of a manifold Y with boundary, is the result of gluing two copies of Y along the boundary.

i.e.

$$vol_\partial(Y) = \sup_{U'} \inf_{U} vol_{n-1}(\partial U), \text{ for all open } U' \supset U \supset Y.$$

Observe that this vol_∂ is bounded by the $(n-1)$-dimensional Hausdorff measure,

$$vol_\partial(Y) \leq mes_{n-1}(Y).$$

In particular,

closed subsets $Y \subset X$ *with vanishing* $(n-1)$*-dimensional Hausdorff measure*

have $vol_\partial(Y) = 0$ (but the converse, probably, is not true).

Next, write

$$mean.curv_\partial(Y) \geq \kappa$$

if for all $\varepsilon > 0$ all neighbourhoods $U' \supset Y$ contain smaller smooth neighbourhoods $U \supset Y$ such that

$$mean.curv(\partial U) \geq \kappa - \varepsilon$$

Implication $\mathbf{vol_\partial(Y) = 0 \Rightarrow mean.curv_\partial(Y) = \infty}$. To show this let $\mu = \mu(x)$ be a continuous function on $U' \setminus Y$ which is $\geq \kappa$ for a given κ and which may blow up at Y.

Let U_0 be a μ*–bubble* pinched between Y and U', i.e. U_0 minimises the following functional

$$U \mapsto vol_{n-1}(\partial U) - \int_U \mu(x)dx.$$

If Y is compact and μ is sufficiently large near X such a μ-bubble U_0 exists and

$\bullet_1$ the boundary ∂U_0 is smooth away from a possible singular subset $\Sigma \subset \partial U_0$ of codimension ≥ 7;

$\bullet_2$ $mean.curv(\partial U_0, x) = \mu(x)$ at the regular points $x \in \partial U_0$;

$\bullet_3$ U_0 can be approximated by domains U with *smooth* boundaries ∂U such that $mean.curv(\partial U) \geq \kappa - \varepsilon$ for a given $\varepsilon > 0$.

(These $\bullet_1$ and $\bullet_2$ are standard results of the geometric measure theory and $\bullet_3$ is an elementary exercise, see [48, 49] for details.[57])

Probably, the implication $\mathbf{vol_\partial(Y) = 0 \Rightarrow mean.curv_\partial(Y) = \infty}$ remains valid for all closed subsets in X, but the above argument, as it stands, delivers the following weaker property in the non-compact case.

[57]I apologise for referring to my own articles, but I could not find what is needed on the web.

If X has uniformly bounded geometry[58] *then every closed subsets $Y \subset X$ with $vol_\partial(Y) = 0$ is equal to the intersection of a decreasing family of domains $U_\kappa \subset X$, $\kappa \to \infty$, where $mean.curv(\partial U_\kappa) \geq \kappa$.*

Remarks/Conjectures. (a) The role of bounded geometry is to ensure a lower bound on the volumes of balls in the μ-bubble away from Y where μ is small and, thus, keep domains U which minimise the function $U \mapsto vol_{n-1}(\partial U) - \int_U \mu(x)dx$ within an ε-neighbourhood of Y.

(b) The doubles of thin mean convex domains $U \subset X$ provide examples of *collapsed manifolds with $Sc > 0$.* [59]

(c) All smooth domains $U \subset \mathbb{R}^n$ with $mean.curv(\partial U) \geq 0$ are diffeotopic to regular neighbourhoods of subpolyhedra $P \subset U$ with $codim(P) \geq 2$.

In fact, if Y is bounded, this follows by Morse theory applied to a linear function on Y, while unbounded Y can be exhausted by bounded $Y_i \subset Y$ with $min.curv(\partial Y_i) > 0$.

(d) All piecewise smooth subpolyhedra $P \subset X$ with $codim(P) \geq 2$ admit arbitrarily small *regular* neighbourhoods with arbitrarily large mean curvatures of their boundaries.

This is obvious for smooth submanifolds $Y \subset X$, e.g. for curved segments Y in the unit ball in $\mathbb{R}^n$, $n \geq 3$ and the general case follows by a geometric surgery similar to that for $Sc > 0$.[60]

(e) *Exercise.* Show that every bounded smooth domain $U_0 \subset \mathbb{R}^n$, $n \geq 3$ admits arbitrarily long simple curves (arcs) $C \subset U$, such that $curvature(C) \leq const = const(U)$.

Construct such curves with regular neighbourhoods $U \subset U_0$ which fill almost all of U_0 and such that the mean curvatures of the boundaries ∂U tend to infinity.

Apply this successively to $C_i \subset U_{i-1}$ and obtain

$$U_0 \supset U_1 \supset ... \supset U_{i-1} \supset U_i \supset ....$$

such that

$$mean.curv(\partial U_i) \to \infty \text{ and } vol(U_i) \geq const > 0/$$

Show that the intersection $Y_\infty = \cap_i U_i$ is a compact set such that

- Y has positive Lebesgue measure, $vol_n(Y) > 0$;
- $vol_\partial(Y_\infty) = \infty$;
- $mean.curv_\partial(Y) = \infty$;

[58] This means that there exist $\rho > 0$ and $\lambda > 0$, such that all ρ-balls $B_x(\rho) \subset X$ are λ-bi-Lipschitz homeomorphic to a Euclidean ball.

[59] The original metric in X can be perturbed to have $Sc > 0$ near Y, since our $Y \subset X$ have $codim(Y) \geq 2$.

[60] Despite the scaling difference between Sc and $mean.curv$, the iterated similarity argument does apply to the mean convex surgery, but I have always felt uneasy about it.

• the topological dimension of Y is one.

Construct similar $Y \subset \mathbb{R}^n$ with $dim_{top} = m$ for all $m \leq n - 2$.

II. Thin$_{\mathbf{Sc>0}}$ and Thin$_{\mathbf{mean.curv>0}}$ There is a direct simple construction, see [53], of arbitrarily small smooth regular neighbourhoods $U \supset Y$ of piecewise smooth subpolyhedra $Y \subset X$ in Riemannian manifolds X with $codim(Y) \geq 3$, such that the induced Riemannian metrics in the boundaries ∂U satisfy $Sc(\partial U) \geq \kappa$ for a given $\kappa > 0$.

By the same argument, if $V \subset X$ is a smooth domain with $Sc(\partial V) > \kappa$ and Y is transversal to ∂U, then the union $V \cup Y$ admits an arbitrarily small smooth regular neighbourhood $U \supset V \cup Y$ with $Sc(\partial U) > \kappa$.

For instance, the function $\lambda |\sin t|$, can be uniformly approximated, for all $\lambda > 0$, by C^2-functions $\varphi(t) > 0$, such that the hypersurface $H_\varphi \subset \mathbb{R}^{n+1}$ obtained by rotating the graphs of $\varphi(t)$ around the t-axis in $\mathbb{R}^{n+1}$ has $Sc(H_\varphi) > 0$ for all $n \geq 3$.

Similarly, subpolyhedra $Y \subset X$ with $codim(Y) \geq 2$ admit arbitrarily small smooth regular neighbourhoods U with arbitrary large $mean.curv(\partial U$.

Also, the above H_φ can be arranged with $mean.curv(H_\varphi) > 0$ for all $n \geq 2$.[61]

Intersection and Symmetrization for mean.curv>0. An essential feature of mean convexity which has no(?) counterpart for $Sc > 0$ is the following well known intersection property.

If closed subsets $Y_i \subset X$ have $mean.curv_\partial(Y_i) \geq \kappa$ then also their intersection satisfies

$$mean.curv_\partial(\bigcap_i Y_i) \geq \kappa.$$

This boils down to showing that if U_1 and U_2 are smooth domains with
$mean.curv(\partial U_{1,2}) > \kappa$, then the intersection $U_1 \cap U_2$ can be approximated by smooth domains U with $mean.curv(\partial U) > \kappa$. (See [48] for details and references.)

In particular, one can G-*symmetrise* mean convex subsets $V \subset X$ with isometry groups G of X,

$$V_{symm} = \bigcap_{g \in G} g(V)$$

.

[61] This is counterintuitive. I once have convinced myself in the validity of this (§$5\frac{5}{7}$ in [44]) but has not written it down.

Thick Mean Convex Domains and the Maximum Principle. The following properties of mean convex subsets in manifolds with lower bounds on their Ricci curvatures can be, probably, traced to the work by Paul Levy and S. B. Myers. [62]

Let X be a Riemannian n-manifold with $Ricci \geq \rho$, i.e. $Ricci(X) \geq \rho \cdot g_X$, and $Y \subset X$ have $mean.curv_\partial(Y) \geq n-1$.

(For instance, $X = \mathbb{R}^n$. where $Ricci = 0$. and $Y \subset X$ is a smooth domain with $mean.curv(\partial Y) \geq n-1$.)

Inball Inequality. *If $\rho > -n$, then the radius $Rad_{in}(Y)$ of the maximal inscribed ball in Y, i.e. $Rad_{in}(Y) = \sup_{y \in Y} dist(y, \partial Y)$, is bounded by the radius $R = R(n, \rho)$ of the ball $B(R, \rho)$ with $mean.curv(\partial B(R, \rho)) = n-1$ in the standard (complete simply connected) n-dimensional space X_ρ of constant curvature with Ricci $= \rho$ (i.e. with the sectional curvature ρ/n).* [63]

Indeed, the normal exponential map to ∂Y necessarily develops conjugate points on geodesic segments normal to ∂Y of length $> R(n, \rho)$.

Inball Equality *If $Rad_{in}(Y) = R = R(n, \rho)$ then Y, assuming it is connected, is isometric the R-ball in the standard space with constant curvature ρ/n.*

This is proven by fiddling at the boundary points of the regions in ∂Y where the maximal in-ball meets ∂Y.

+Inequality. Let a domain $Y \subset \mathbb{R}^n$ with $mean.curv \geq n-1$ contains two mutually normal balls balls $B^{n_1}(R_1)$ and $B^{n_2}(R_2)$, $n_1 + n_2 = n$, which meet at their centers,

Then $\min(R_1, R_2) \leq 1$.

Indeed, symmetrization reduces this to the case, where Y is $O(n_1) \times O(n_2)$-invariant and the problem becomes a computable one-dimensional one.

In fact, it is an exercise[64] to fully determine the range of possible values of (R_1, R_2), where one sees, for instance. that

$$R_1 \to \infty \Rightarrow R_2 \to \frac{n_2 - 1}{n-1}.$$

This suggests the following.

Moving Balls in Mean Convex Domains. Let $Y_{-R} \subset Y$ denote the set of the centers of the R-balls in Y, that is the set of points $y \in Y$ with $dist(y, \partial Y) \geq R$.

[62] $Ricci(X, g) \geq \rho$ stands for $Ricci(X, g) \geq \rho g$.

[63] If $\rho \leq -n$, no such ball exists and $Rad_{in}(Y)$ may be infinite.

[64] I have not done it.

[?70] **Conjecture** *Let a domain $Y \subset \mathbb{R}^n$ have $mean.curv(\partial Y) \geq n - k + \varepsilon$ for some $\varepsilon > 0$ and $k = 2, ..., n-1$. Then Y_{-1} admits a continuous map onto a $(k-1)$-dimensional polyhedral space space, say $\Delta : Y_{-1} \to P^{k-1}$, such that the pullbacks of all points are uniformly bounded,*

$$diam(\Delta^{-1}(p)) \leq const = const(n, \varepsilon).$$

Thus, the macroscopic dimension (see below) of Y_{-1} is $\leq k-1$.

(The extremal case where $\varepsilon = 0$ is seen in $Y = B^{n-k}(1) \times \mathbb{R}^k \subset \mathbb{R}^n$, where $\mathbb{R}^k$ admits *no continuous map* to any P^{k-1} with uniformly bounded pullbacks of all $p \in P$ by *Lebesgue Lemma*.)

In particular,

If $Y \subset \mathbb{R}^n$ is a connected domain with $mean.curv(\partial Y) > n-2+\varepsilon$ then the subset $Y_{-1} \subset Y$ is bounded.

(Recall that the macroscopic dimension of a metric space M is the minimal dimension of polyhedral spaces P, for which M admits a continuous map $\Delta : M \to P$, such that $diam_M(\Delta^{-1}(p)) \leq d$ for some constant $d = d(M)$.)

Now let us formulate the scalar curvature conjecture of the above conjecture. Start with a few definitions.

Length Metrics in Spaces of Maps. The space Φ of maps ϕ from X to S, where S is a metric space comes with the *sup-metric*

$$dist(\phi_1, \phi_2) = \sup_{x \in X} dist(\phi_1(x), \phi_2(x)).$$

Let us endow subsets in Φ with the corresponding *length metrics*: such a metric on a $\Psi \subset \Phi$ is

the supremum of the metrics which locally agree with the sup-metric on Ψ, which the same (except for irrelevant pathological cases) as the metric defined via the length of curves in Ψ.

What is interesting is that this length metric in a Ψ may be significantly greater than the sup-metric, which happens when Ψ is distorted inside Φ.

Examples. (A) *Continuous Maps* Let $\Psi = C(X{\to}S) \subset \Phi$ be the space of *continuous* maps $X \to S$ with the above length metric and let $\tilde{S} \to S$ be a locally isometric covering map.[65] Then the corresponding map $C(X{\to}\tilde{S}) \to C(X{\to}S)$ is an isometry. It follows that

the space of maps from the n-ball B^n to a compact space S with an *infinite* fundamental group has *infinite diameter.*

[65] Here and below we assume that our spaces are locally contractible, e.g. manifolds or cell complexes and that S is a *length* metric space, e.g. a Riemannian manifold.

But if S has *finite fundamental group*, then the space $C(B^n \to S)$ has *finite diameter*. For instance,

$$diam(C(B^n \to S^m(1))) = diam(S^m(1)) = \pi \text{ for } m > n.$$

Somewhat less obviously,

$$diam(C(B^n \to S^n(1))) \leq 3\pi,$$

which implies that the (*infinite* cyclic) universal covering of the space $C(S^{m-1} \to S^m(1))$ also has diameter $\leq 3\pi$.

More generally,

$$diam(C(B^n \to S)) \leq n \cdot const(S) \ ,$$

for all compact, say cellular, spaces S with finite fundamental groups and, probably, the universal coverings of all connected components of the spaces $C(X \to S)$ are similarly bounded by $dim(X) \cdot const(S)$.

The above linear bound on $diam(C(B^n \to S))$ is asymptotically matched by a lower bound for most (all?) compact non-contractible spaces S.

For instance, it follows from 1.4 in [40] that

$$diam(C(B^n \to S)) \geq n \cdot const(S) \text{ with } const(S) > 0$$

if S is the m-sphere $S^m(1)$, $m \geq 2$, or, more generally, if the iterated loop space $\Omega^k(S)$ for some $k \geq 1$ has non-zero rational homology groups $H_i(\Omega^k(S); \mathbb{Q})$ for i from a subset of positive density in $\mathbb{Z}_+$.

(B) *Lipschitz Maps.* Let X and S be metric spaces and $Lip_\lambda(X \to S)$ be the space of λ-Lipschitz maps with the above length metric which we now denote $dist_\lambda$ and where we observe that the inclusions

$$Lip_{\lambda_1}((X \to S), dist_{\lambda_1}) \subset Lip_{\lambda_2}((X \to S), dist_{\lambda_2}), \ \lambda_1 \leq \lambda_2,$$

are distance decreasing.

The simplest space here, as earlier is where X is the ball, but now the geometry of it is essential. For instance,

$$diam_\lambda(Lip_\lambda(B^n(R) \to S)) \leq \lambda R + diam(S),$$

where $B^n(R)$ is the Euclidean R-ball, where $diam_\lambda$ is the diameter measured with $dist_\lambda$ and where, observe,

$$Lip_{\lambda_1}(B^n(R_1) \to S) = Lip_{\lambda_2}(B^n(R_2) \to S) \text{ for } \lambda_1 R_1 = \lambda_2 R_2.$$

More interesting is the lower bound

$$diam_{c\lambda}(Lip_\lambda(B^n(R) \to S)) \geq const(S, c)\lambda R,$$

which holds whenever *the real homology* $H_i(S,\mathbb{R})$ *does not vanish for some* $i \le n$.

In fact,

$$diam_{c\lambda}(Lip_\lambda(B^n(R) \to S)) \ge diam_{c\lambda}(Lip_\lambda(B^i(R) \to S)) \text{ for } n \ge i,$$

while evaluation of a non-cohomologous to zero real i-cocycle h at 1-Lipschitz maps $f : B^i(R) \to S$ defines a $C \cdot R^{n-1}$*-Lipschitz map* from $Lip_1(B^i(R) \to S))$ to $\mathbb{R}$ for some $C = C(S,c))$. It follows that the 1-Lipschitz maps f with $h(f) \approx vol_i(B^i(R)) \approx R^n$ – these exist by the Hurewicz-Serre theorem for the *minimal* i where $H_i(S,\mathbb{R}) \ne 0$ – are within distance $\gtrsim R$ from the constant maps.

Questions.

[?71] [a] Are the diameters

$$diam_c(Lip_1(B^n(R) \to S)))$$

bounded for a large fixed c and $R \to \infty$ if $H_i(S,\mathbb{R}) = 0$ for $i = 1, 2, ..., n$.

(It is not hard to show that $diam_1(Lip_\lambda(B^1(R) \to S^m(1))) \le const \cdot \log(R)$)

[?72] [b] What is the asymptotics of the diameters

$$diam_c(Lip_1(B^n_H(R) \to S)))$$

for the hyperbolic balls $B^n_H(R)$ and $R \to \infty$?

[?73] [c] Let S be a Riemannian manifold homeomorphic to the connected sum of twenty copies of $S^2 \times S^2$. Are there 1-Lipschitz maps $f_R : B^4(R) \to S$, $R \to \infty$, such that $h(f_R) \ge const \cdot R^4$ for a cocycle h (e.g. a closed 4-form) which represents the fundamental cohomology class $[S] \in H^4(S;\mathbb{R})$, and some $const = const(S) > 0$?

[?74] **Conjecture. Parametric Hypersphericity.** Let X be a complete oriented Riemannian n-manifold and let $\Psi(X) \subset Lip_\lambda(X \to S^n(1)))$ be the space of 1-Lipschitz locally constant at infinity maps[66] of degree one from X to the unit sphere.

If $Sc(X) \ge m(m-1) + \varepsilon$, $m \ge 2$, $\varepsilon > 0$, *then the macroscopic dimension of* $\Psi(X)$ *is* $\le n - m - 1$.

Discussion. (a) If $m = n$ this is equivalent to Llarull's extremality theorem for S^n.

(b) If $m = n-1$ the conjecture says that all connected components of Ψ have diameters bounded by a constant.

(c) The manifold $X = S^m(1) \times \mathbb{R}^{n-m}$ has $Sc(X) = m(m-1)$ and

[66] This means such a map $X \to S^n$ is constant on connected components of X minus a compact subset.

$macr.dim(\Psi(X)) = n - m$.

[?75] If $m = n-1$ then, *conjecturally*, this is the only manifold with this property:

the inequalities

$$macr.dim(\Psi(X)) \geq 1 \text{ and } Sc(X) \geq (n-1)(n-2)$$

should imply that $X = S^{n-1} \times \mathbb{R}$.

But if $m \leq n-2$ one has lots of such manifolds:

• products $X = S^m \times X_0$ where $Sc(X_0) > 0$ and $macr.dim(X_0) = n - m$,

• • small perturbations of metrics and surgeries of these products keeping $Sc(X) \geq m(m-1)$ and do not disturbing 1-Lipschitz maps $X \to S^n(1)$.

Example. Let X be the unit sphere $S^n(1)$ and X be the geometric connected sum of two copies of $S^n(1)$ where these spheres are connected by a tube $S^{n-1}(r) \times [0,l]$ for some $r \leq 1$ and $l \geq \pi/4$.

There are two obvious 1-Lipschitz maps $X \to S^n(1)$ of degree one which are constant on one of the two spheres. Clearly,

these maps lie in the same connected component of $Lip_\lambda(X {\to} S^n(1))$ *if and only if* $\lambda \geq 1/r$

Furthermore,

if $\lambda = 1$*, and* $r < 1$*, then each of these components has diameter* $\leq 2\pi$.

But if $\lambda = r = 1$ *then the image of* $Lip_\lambda(X {\to} S^n(1))$ *in* $C(X \to S^n(1))$ *has diameter* $\asymp l$.

Unlike the case where $Ricci \geq \rho$ there is no bound on inscribed balls for n-manifolds with $Sc > \sigma$ for $n \geq 3$ and all σ, but the "in-filling inequality" below does have the scalar curvature counterpart(s).

Recall that the *filling radius* of an integer k-cycle represented by an oriented subpseudomanifold (e.g. submanifold) $Z \subset Y$ is the minimal R such that Z bounds in its R-neighbourhood $U_R(Z) \subset Y$.

Define $FillRad^k_{in}(Y)$ as the supremum of the filling radii of all k-cycles in Y *homologous to zero in* Y.

Observe that

$$FillRad^{n-1}_{in}(Y) = Rad_{in}(Y)$$

and that if Y is *mean convex* then

$FillRad^{n-2}_{in}(Y)$ is *bounded by the supremum of the inradii* $rad_{in}(M, Y)$ *of* $(n-1)$*-dimensional minimal subvarieties* $M \subset Y$ *with boundaries,* *for*

$$Rad_{in}(M,Y) =_{def} \sup_{y\in M} dist_Y(y,\partial M).$$

ii. Upper Bound on the In-Filling Radius. Start with the case where $Y \subset \mathbb{R}^n$ with $mean.curv(\partial Y) \geq n-1$ and where $M \subset Y$ is an $(n-1)$-ball of Radius R.

Let $U_\circ(M,r) \supset M$ be the lens-like region between two spherical caps of hight $r \in [0,R]$ and with the boundaries ∂M. If $R > 1$, the mean curvatures of these caps are $< n-1$; hence they do not meet ∂Y which makes the R-ball $B(R) = U_\circ(M,R)$ contained in Y. Since this contradicts **i**, we conclude that $R \leq 1$ that is

the sharp bound

$$FillRad_{in}^{n-2}(Y) \leq 1$$

for the Euclidean domains Y *with* $mean.curv(\partial Y) \geq n-1$.

Now let $M \subset Y$ be a minimal hypersurface in a Riemannian manifold Y and let $y_0 \subset M$ be a pont with $dist(y_0,\partial M) > R$. Define $U_\circ^\theta(M,r) \supset M$ as the subset of points $y \in Y$ such that

$$dist(y,M) \leq \theta(dist(y,y_0))$$

where $\theta(d)$, $d \geq 0$, is the function for which $U_\circ^\theta(M,r) = U_\circ(M,r)$ in the above model case of the $(n-1)$-ball $M \subset \mathbb{R}^n$.

It is not hard to show that

if Y *is a Euclidean domain and* R *is sufficiently large,* $R \geq R_0 = R_0(n)$, *then the mean curvature of the boundary* $\partial U_\circ^\phi(M,r)$ *is bounded by* $n-1 = mean.curv(\partial Y)$. (Such a bound makes sense despite possible singularities of $\partial U_\circ^\phi(M,r)$.)

Then, by the maximum principle,

$$FillRad_{in}^{n-2}(Y) \leq R_0(n).$$

Remarks. (a) The above argument yields similar inequality for manifolds Y with locally bounded geometries and it may(?) also extend to manifolds with $Ricci(Y) \geq 0$ as well as (in obviously modified form) to manifolds Y with $Ricci(Y) \geq \rho$ for negative $\rho > -n0$.

(b) Also one expects *sharp* inequalities of this kind, say $FillRad_{in}^{n-2}(Y) \leq 1$ for $Y \subset \mathbb{R}^n$ with $mean.curv(\partial Y) \geq n-1$.

Maximum Principle and the Half-space Theorem. There are also non-trivial geometric constrains on *non-strictly* mean convex hypersurfaces $Y \subset \mathbb{R}^n$, i.e. with $mean.curv(\partial Y) \geq 0$ derived by the maximum principle. For instance,

◖ *If such a $Y \subset \mathbb{R}^3$ with $mean.curv(\partial Y) \geq 0$ contains a plane, then, assuming ∂Y is non-empty and connected, Y is equal to a half-space.*

This follows from *the half-catenoid maximal principle* that was originally used by Hoffman and Meeks [69] to show that

properly embedded minimal surfaces $Y \subset \mathbb{R}^3_+$ are flat.

Symmetrization. Intersections Y_{int} of subsets Y with $mean.curv(\partial Y) \geq c$ have $mean.curv(\partial Y_{int}) \geq c$ with a properly defined generalised mean curvature for such Y_{int}. This allows *G-symmetrization* of $Y's$ under actions of isometry groups G acting on $\mathbb{R}^n \supset Y$,

$$Y \rightsquigarrow Y_{sym} = \bigcap_{g \in G} g(Y).$$

Thus, one can prove ◖ in some (all?) cases by symmetrization $Y \rightsquigarrow Y_{sym} \subset \mathbb{R}^n$ where Y_{sym} is equal to the intersection of the copies of Y obtained by rotations of Y around the axis normal to the hyperplane $\mathbb{R}^{n-1} \subset Y$.

Similarly, one proves the following

◐ **Sc > 0 ⇴ mean.curv > 0**. Certain constraints on the geometry of mean convex domains $Y \subset \mathbb{R}^n$ can be derived from *positive mass like* results applied to the double of Y.

For example, if ∂Y is obtained by a compact perturbation of a hyperplane, this perturbation can be $\mathbb{Z}^{n-1}$-invariantly extended to all of Y and then (the solution of the) Geroch conjecture applies to the double of the resulting $\mathbb{Z}^{n-1}$-periodic domain.

This however is not terribly impressive since either the maximum principle or $\mathbb{R}^{n-1}$-symmetrization painlessly yield the same result as well. symmetrisation

On the other hand, positive scalar curvature enters the proof of the Fischer-Colbrie-Schoen theorem

Fischer Colbrie-Schoen Planarity Theorem [34]. *Complete stable minimal surfaces in $\mathbb{R}^3$ are flat.*

This implies [87] what is called

Strong Half-space/Slab Theorem. *The only mean convex domains in $\mathbb{R}^3$ with disconnected boundaries are slabs between parallel planes.*

On the other hand, as it pointed out in [69] the spaces $\mathbb{R}^n$ for $n \geq 4$, contain, for instance mean convex domain bounded by pairs of n-dimensional catenoids.

Yet, there must be significant constrains on the size of mean convex domains, such as the following.

[?76] **Conjecture. Stability of Periodic Slabs.** *The only $\mathbb{Z}^{n-3}$-invariant mean convex domains in $\mathbb{R}^n$ with disconnected boundaries are slabs between parallel hyperplanes.*

Besides "thin and narrow" mean convex domains Y which surround codimension two subsets in Riemannian manifolds X, there are also "thick" ones. For instance,

let X be a complete Riemannian n-manifold, such that

$\bigcirc_\infty$ *X is connected at infinity.*

$\circledcirc_\infty$ $vol_{n-1}(\partial_\infty X) = \infty$, that is every proper continuous function $f : X \to \mathbb{R}_+$ satisfies

$$\limsup_{t\to\infty} vol_{n-1}(f^{-1}(t)) \to \infty.$$

$\bar{\bullet}$ *X is locally "$(n-2)$-thick"*: There exist $\varepsilon > 0$, $\alpha > 1$ and $c > 0$, such that all $(n-2)$-cycles $B \subset X$ with diameters $diam(B) \leq \varepsilon$ and with $vol_{n-2}(B) \leq \varepsilon^{n-2}$ bound $(n-1)$-chains C, i.e. $B = \partial C$, such that

$$vol_{n-1}(C) \leq c \cdot vol_{n-2}(B)^\alpha.$$

(An X which is *uniformly bi-Lipschitz* homeomorphic to $\mathbb{R}^n$, or to any Riemannian homogeneous space, satisfies these three conditions.)

Then, granted $\bigcirc_\infty$, $\circledcirc_\infty$, $\bar{\bullet}$, every compact subset $Y_0 \subset X$ is contained in a *smooth compact mean convex* domain $Y_1 \subset X$.

Sketch of the Proof. Minimise $vol_{n-1}(\partial Y)$ among all bounded domains $Y \subset X$ which contain Y_0. The conditions $\bigcirc_\infty$ and $\circledcirc_\infty$ guaranty that Y_{min} exists, but it may, a priori be unbounded with "thin and narrow spikes" going to infinity; $\bar{\bullet}$ rules out this possibility.

The boundary ∂Y_{min} may have singularities, but these are controlled by Almgren-Allard regularity theorem. This allows approximation of Y_{min} by *smooth* mean convex domains Y_1 [48].

Lament. The condition $\bar{\bullet}$, unlike $\circledcirc_\infty$ is *unstable* under *bounded measurable* perturbations of metrics, where such a perturbation of the standard metric on $\mathbb{R}^n$ in a small neighbourhood of a hyperplane $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ may turn this $\mathbb{R}^{n-1}$ into a minimal hypersurface $C \subset \mathbb{R}^n$ with $vol_{n-1}(C) < \infty$.

Let X be a smooth manifold with a C^0-continuous Riemannian metric g_0 and let $Y_1, Y_2 \subset X$ be smooth domain with common boundary

$$Z = \partial Y_1 = \partial Y_2 = Y_1 \cap Y_2$$

such that that the restrictions of g_0 to Y_1 and to Y_2 are C^2-smooth.

Denote $mean.curv_{1,2}(Z_\partial)$ the mean curvatures of the hypersurface Z_∂ in $Y_{1,2} \supset Z_\partial$ and say that the scalar curvature of (X, g_0) is (strictly) positive on Z and write $Sc(g)_{|Z_\partial} = Sc(X)_{|Z_\partial} > 0$ if

$$mean.curv_1(Z_\partial) + mean.curv_2(Z_\partial) > 0.$$

The doubling lemma along with its proof straightforwardly generalises to the following [3, 92, 49],

Edge Smoothing Lemma. *If $Sc(X)_{|Z_\partial} > 0$ than g_0 can be C^0-approximated by C^2-metrics g such that*

- *$Sc(g) \geq Sc(g_0)$ on $X \setminus Z_\partial$;*
- *$Sc(g, z) \geq \sigma(z)$ for a given continous function σ on Z_∂ and all $z \in Z_\partial$;*
- *approximating metrics g can be taken equal g_0 away from arbitrarily small neighbourhoods of Z_∂ in X.*

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25 Collapse of Hypersurfaces with Scalar Curvature Blow-up.

[?77] **Problem.** *Describe "Remnants of Collapse" of Hypersurfaces with Scalar Curvatures Blowing-up to $+\infty$.*

Namely, decide when a closed subset Y in a C^2-smooth Riemannian manifold (W, g) appears as a *limit* of smooth domains $V_i \subset W$, $i \in I$, with $Sc(\partial V_i) \to \infty$, where "limit" means that $Y = \bigcap_i V_i$, where, if Y is non-compact, one may additionally insist that $V_j \subset V_i$ for $j > i$, or moreover, that V_i eventually become smaller than any given neighbourhood $U \supset Y$. (For this one needs uncountable I.)

The following conjectures may give you a feeling of what this description might tell you.

[?77$\frac{1}{2}$] **Conjecture. No Collapse to Codimensions 0, 1 and 2.** *If*

$$Y = \bigcap_i V_i, \text{ where } Sc(\partial V_i) \geq \sigma_i \to \infty \text{ for } i \to \infty,$$

then the topological dimension of Y satisfies

$$dim(Y) \leq dim(W) - 3.$$

Discussion. In the simplest case, where $W = \mathbb{R}^{n+1}$ and Y, is the unit ball, Llarull's extrmality theorem implies that

no compact smooth domain $V \supset Y$ can have $Sc(\partial V) > n(n-1)$, where, recall, the scalar curvature of $X = \partial V \subset \mathbb{R}^{n+1}$ is expressed in terms of the principal curvatures c_i, $i = 1, 2, .., n$, of X as

$$Sc(X) = \sum_{i \neq j} c_i \cdot c_j = mean.curv(X)^2 - \sum_i c_i^2 \text{ for } mean.curv(X) = \sum_i c_i.$$

Also, there are similar bounds on the scalar curvatures of boundaries of domains V which contain more general *topological submanifolds* (and sub-pseudo-manifolds) of codimensions 0, 1 and 2. In fact, the present day techniques – minimal hypersurfaces and the Dirac operator – come close to ruling out all Y with $codim(Y) \leq 1$ as remnants of collapse with $Sc \to \infty$ and the case of $codim = 2$ also seems within reach.

[?78] **Conjecture. Subsets with Low Hausdorff Dimensions are Remains of Scalar Curvature Blow-ups.** *All closed subset $Y \subset W$ with*

$$dim_{Hau}(Y) < n - 1 = dim(W) - 2,$$

are intersections of decreasing families of smooth domains $V_i \subset W$ with $Sc(\partial V_i) \to +\infty$

This is obvious for $n = 2$. Also subsets $Y \subset W$ which are *contained in smooth hypersurfaces* $Z^n \subset W$ and which have *zero Hausdorff* mes_{n-1}, are representable as such intersections by a simple argument.

It is convenient at this point define $Sc_g(\partial_\downarrow Y)$ as the supremum of the numbers σ such that every neighbourhood $U \supset Y$ contains a smaller smooth $V \supset Y$ such that the scalar curvature of ∂V for the metric induced from the metric g in the ambient manifold W.

[?79] **Conjecture. Invariance and Non-invariance of $Sc_\cap(Y) = +\infty$.**

The inequality $Sc^{[n]}_{g\cap}(Y) = +\infty$ is independent of the Riemannian metric g in $W \supset Y$ Moreover it is a bi-Lipschitz invariant. But it is not a topological invariant.

There is no serious evidence here, but there are a few examples.

For instance, given $k = 1, 2, ..., n - 2$ and $\mu > 0$, one can arrange nested neighbourhoods V_{ε_i}, $i = 1, 2, ...$, of flat k-subtori in $W = \mathbb{T}^{n+1}$ such that $Sc(\partial V_{\varepsilon_i}) \to \infty$ and such that the Lebesgue measure of their (solenoidal) intersection Y will be equal $mes(W) - \mu$.

(With a little effort, one can make a similar construction with all V_ε homeomorphic to the k-ball.)

On the other hand, there, probably, exist *compact zero dimensional* (Cantor) sets $Y \subset \mathbb{R}^{n+1}$ with $Sc^{[n]}_\cap(Y) \neq +\infty$. (Compare 5.3 in [48].)

[?80] **Conjecture. Stabilisation under Cartesian Products.**

$$\left[Sc^{[n]}_{g\cap}(Y) = +\infty\right] \Leftrightarrow \left[Sc^{[n+k]}_{g\oplus g_k\cap}(Y\times X^k) = +\infty\right],$$

where $X^k = (X^k, g_k)$ is a compact Riemannian manifold of dimension k and where $W \times X^k \supset Y \times X^k$ is endowed with the metric $g \oplus g_k$.

Notice that the implication

$$\left[Sc^{[n]}_{g\cap}(Y) = +\infty\right] \Rightarrow \left[Sc^{[n+k]}_{g\oplus g_k\cap}(Y\times X^k) = +\infty\right]$$

is obvious for compact manifolds X^k *without boundary* as well as for complete non-compact manifolds with the scalar curvatures bounded from below where the leading example is $X^k = \mathbb{R}^k$. Possibly, this remains true for compact manifolds *with boundary*.

The reverse implication

$$\left[Sc^{[n+k]}_{g\oplus g_k\cap}(Y\times X^k) = +\infty\right] \Rightarrow \left[Sc^{[n]}_{g\cap}(Y) = +\infty\right],$$

probably, fails to be true for $n = 2$ and, possibly, for $n = 3,4$ but it is plausible for $n \geq 5$.

Relaxing C^2-Smoothness of ∂V_ε to C^1. The requirement for the existence metrics with $Sc > \sigma \to \infty$ on the boundaries of domains $V_\varepsilon \subset W$ which approximate $Y \subset W$ in the above definition of $Sc^{[n]}_{g\cap}(Y) = +\infty$ is compounded with the necessity of these boundaries to be isometrically embeddable to (W, g).[67]

This complication can be removed by applying to the Nash-Kuiper theorem on isometric C^1-embeddings and allowing V_ε to have C^1*-smooth* boundaries, yet with C^2*-smooth*, rather than continuous, induced metrics, where it would be even more natural to admit *continuous* metrics on ∂V_ε. However the extension of the inequality $Sc \geq \sigma$ to continuous metrics is a non-trivial matter.

26 Manifolds with Small Balls.

The definition of the scalar curvature in terms of the volumes of small balls in section 1 suggests the following.

Say that a *metric measure space* $X = (X, dist, vol)$ where $vol = vol_X$ is the measure on X, called here *volume*, is *locally volume-wise smaller* than another such space $X' = (X', dist', vol')$ and write

$$X \prec_{vol} X'$$

[67]There is variety of obstructions to embeddablity of surfaces to 3-spaces (see a brief overview of basic examples in section 3.2.3 in [42]) but amazingly little is known for $n \geq 3$.

if all ε-balls in X are smaller than the ε-balls in X',

$$vol(B_x(\varepsilon)) < vol'(B_{x'}(\varepsilon))$$

for all $x \in X, x' \in X'$ and some continuous positive function $\varepsilon = \varepsilon(x, x')$.

Cartesian Additivity. Observe that

$$X \prec_{vol} X' \text{ and } Y \prec_{vol} Y'$$

imply that

$$X \times Y \prec_{vol} X' \times Y',$$

where the product spaces are endowed with $vol_{X\times Y} =_{def} vol_X \oplus vol_Y$ and with the Pythagorean product metrics,

$$dist_{X\times Y} = \sqrt{dist_X^2 + dist_Y^2}.$$

(The Cartesian additivity obviously holds for all kind of metric products, say for $dist_{X\times Y} = (dist_X^p + dist_Y^p)^{1/p}$.)

Postulate $Sc(\mathbb{R}^n) = 0$ and say that the *n-volumic scalar curvature* of a metric space X is *(strictly) positive* if X is *locally volume-wise smaller* than the Euclidean n-space $\mathbb{R}^n$,

$$X \prec_{vol} \mathbb{R}^n.$$

Also postulate that the 2-sphere of radius R has

$$Sc(S^2(R)) = 2R^{-2}$$

and that

$$Sc(X \times \mathbb{R}) = Sc(X) \text{ for all } X.$$

Then declare that the *n-volumic scalar curvature* of X is $\leq \kappa$ for $\kappa \geq 0$ if X is *locally volume-wise smaller* than $S^2(R) \times \mathbb{R}^{n-2}$ for all $R > 1/\sqrt{\kappa}$,

$$X \prec_{vol} S^2(R) \times \mathbb{R}^{n-2},\ R > 1/\sqrt{\kappa}.$$

Next, motivated by the additivity $Sc(X \times Y) = Sc(X) + Sc(Y)$ of the classical scalar curvature, say that

the *n-volumic scalar curvature of X is $> \kappa$ for $\kappa < 0$* if $X \times S^2(R)$ for $R = \sqrt{-\frac{2}{\kappa}}$ has its $(n+2)$*-volumic scalar curvature* > 0.

Definition of the n-Volumic Scalar Curvature $Sc^{vol_n}(X)(x)$. This is the infimum of the numbers κ, such that $x \in X$ admits a neighbourhood $U \subset X$ such that the n-volumic scalar curvature of U is $> \kappa$.

Observe that Sc^{vol} properly scales as befits a true curvature,

$$Sc^{vol_n}(R\cdot X) = R^{-2}Sc^{vol_n}(X)$$

for all $R>0$, where

$$R\cdot X =_{def} (X, R\cdot dist_X, R^n\cdot vol_X),$$

and that

$$[+]_\times \qquad Sc^{vol_{n+m}}(X\times Y) = Sc^{vol_n}(X) + Sc^{vol_m}(Y).$$

by the Cartesian additivity of $\prec_{vol}$.

We know that if X is a C^2-*smooth* n-dimensional Riemannian manifold, then

$Sc^{vol_n}(X)$ *equals the classical scalar curvature* $Sc(X)$

with the conventional normalisation $Sc(S^n(1)) = n(n-1)$.

But it remains problematic if there are "non-smooth spaces" X *remote* from C^2-Riemannian manifolds where the lower bounds on Sc^{vol} *integrate to non-trivial global relations* for geometric (and topological) invariants of X similarly to how it happens in the C^2-case.

The apparent class of such non-smooth X is that of C^0-Riemannian manifolds where, despite an absence of serious evidence, we make the following conjectures.

[?79] **Conjecture. C^0-closeness of the spaces of C^0-metrics with Volumically Positive Scalar Curvatures.** *If a Riemannian C^0-metric g on an n-dimensional manifold X can be C^0-approximated by C^0-metrics g' with $Sc^{vol_n}(g')\geq\kappa$ then $Sc^{vol_n}(g)\geq\kappa$.*

This conjecture is motivated by the corresponding property of smooth metrics:

C^0-Closure Theorem. *If a Riemannian C^2-metric g on X can be C^0-approximated by C^2-metrics g' with $Sc(g')\geq\kappa$ then $Sc(g)\geq\kappa$* [49, 10].

Test. Check [?79] for continuous *piecewise smooth* metrics, e.g. which are obtained by gluing smooth manifolds by isometries between their boundaries as in the edge smoothing lemma in section 24.

[?80] **Conjecture. C^2-Smoothing of Continuous Metrics with Volumically Positive Scalar Curvatures.** *All continuous Riemannian metrics g on a smooth n-dimensional manifold X which satisfy*

$$Sc^{vol_n}(g) =_{def} Sc^{vol_n}(X, dist_g, vol_g) > \kappa$$

for a given $\kappa \in (-\infty, +\infty)$ can be uniformly (i.e. C^0) approximated by C^2-metrics g' with $Sc(g') > \kappa$.

An obvious topological corollary of this reads:

[?81] **Conjecture. Topological Equivalence of Different Scalar Curvatures.** *If a smooth n-manifold admits a* continuous *metric g_1 with $Sc^{vol_n}(g_1) > 0$ then it also admits a C^2*-smooth *metric g_2 with $Sc(g_1) > 0$.*

This says in other words that

if a closed n-manifold X carries no smooth *metric with $Sc > 0$, then every* continuous *Riemannian metric g on X admits balls $B(X, g; R)$ of* arbitrarily small *radii $R > 0$ such that $B(X, g; R) \geq vol(B^n_{Eucl}(R))$.*

In view of this, one has the following, probably unrealistic, strengthening of the $\mathbb{Q}$-non-essentiality conjecture [?12] in section 4.

[?82] **Conjecture. C^0-Continuous Guth-Geroch Lower Volume Bound for Balls in the Coverings of Essential Manifolds.** [57]. *The universal coverings $\tilde{X}$ of $\mathbb{Q}$-essential n-manifolds X with continuous Riemannian metrics contain R-balls $\tilde{B}(R)$ of all radii $R > 0$ such that*

$$vol(\tilde{B}(R)) \geq vol(B^n_{Eucl}(R)).$$

The main justification for **5** is the following rough version of it.

Corollary to Guth' Mesoscopic Filling Radius Theorem [55, 57]. *The universal coverings $\tilde{X}$ of $\mathbb{Q}$-essential Riemannian n-manifolds X contain R-balls $\tilde{B}(R)$ of all radii $R > 0$ such that*

$$vol(\tilde{B}(R)) \geq \varepsilon_n vol(B^n_{Eucl}(R))$$

for some universal constant $\varepsilon_n > 0$.

(d) The sharp bound here, i.e. with $\varepsilon_n = 1$ is available for *large balls* by the following result.

Burago-Ivanov Asymptotic Ball Volume Theorem [21].[68] *If the universal cover $\tilde{X}$ of an $\mathbb{Q}$-essential manifold X admits a sequence of balls $\tilde{B}(R_i) \subset \tilde{X}, R_i \to \infty$, (where R_i depend on the metric in X), such that*

$$vol(\tilde{B}(R_i)) \leq vol(B^n_{Eucl}(R_i)),$$

then X is flat.

[68] This paper is about X homeomorphic $\mathbb{T}^n$; the general case reduces to that by the classification of groups of polynomial growth.

(e) The Burago-Ivanov argument automatically extends to *non-Riemannian* manifolds and pseudomanifolds $(X, dist)$ with arbitrary metrics and suitably defined "volumes" on them, namely *Hilbert volumes* $\widetilde{Hil}.vol^n$ (see [46]). (These "volumes" are not, a priori, additive.) Guth' theorem also remains valid for all $(X, dist, \widetilde{Hil}.vol^n)$ which motivates the following.

[?83] **Conjecture. Non-Riemannian Guth-Geroch.** *Let X be an n-dimensional $\mathbb{Q}$-essential pseudomanifold (e.g. manifold) with an arbitrary metric. Then the universal covering $\tilde{X}$ of X contains balls of all radii R the Hilbert volumes of which are ≥ than these of the Euclidean R-balls,*

$$\widetilde{Hil}.vol^n(\tilde{B}(R)) \geq vol(B^n_{Eucl}(R)).$$

[?84] **Conjecture Non-Riemannian ε-Llarull.** *Let a compact n-dimensional pseudomanifold has the Hilbert volumes of all its balls of radii $\leq \varepsilon_0$ smaller than the volumes of such balls in S^n. Then all λ-Lipschitz maps from X to the sphere S^n are contractible, say, starting from $\varepsilon_0 = \frac{\pi}{4}$ and $\lambda \leq \frac{1}{2}$.*

The conjectures [79] - [84] are on the side of wishful thinking – something quite opposite may be true, e.g. the following.

[?85] **Conjecture. C^0-Density of C^0-metrics with Volumically Positive Scalar Curvatures.** *Continuous Riemannian metrics with $Sc^{vol_n} > 0$ on an X are dense in the spaces of all Riemannian metrics on X for all n-dimensional manifolds X for $n \geq 3$.*[69]

(This may be compared with *Lohkamp C^0-Approximation Theorem* [82]:

C^2-metrics with $Sc \leq -1$ are C^0-dense in the spaces of all Riemannian metrics

on n-manifolds X for $n \geq 3$.)

On the other hand, it seems probable that most known properties of smooth manifolds with $Sc > 0$ generalise to spaces with "benign singularities", e.g. to *Alexandrov spaces X with sectional curvatures bounded from below by -1.*

(A geodesic metric space[70] is *Alexander Kapovitch Petrunin Alexandrov geometry 2017* with *sect.curv* ≥ -1 if, for every quadruple of points $x_i \in X$, $i = 1, 2, 3, 4$, there exists a quadruple of points x'_i in the hyperbolic plane H^2 with sectional curvature -1, such that

[69] *Conformal representation* of metrics on surfaces makes this approximation unlikely for n=2. In fact, it may be safer to assume $n \geq 4$.

[70] "Geodesic" means that every two points x_1 and x_2 in X can be joined by a path of length $d = dist(x_1, x_2)$.

$dist_{H^2}(x'_i, x'_j) = dist_X(x_i, x_j)$ for $i, j = 1, 2, 3$, while $dist_{H^2}(x'_i, x'_4) \leq dist_X(x_i, x_4)$, see [4] and references therein.)

The simplest among expected results is the following.

[?86] **Conjecture. Geroch for Alexandrov Spaces.** *If an n-dimensional Alexandrov space X with $sect.curv \geq -1$ and $Sc^{vol_n}(X) \geq 0$ admits a continuous map Φ with non-zero degree* (i.e. the homology homomorphism Φ_* does not vanish on $H_n(X)$) *to the n-torus, then the universal covering of X is isometric to $\mathbb{R}^n$.*

Moral in Conclusion. The definition of Sc^{vol} is not supposed to answer the question *"What is scalar curvature"*, but rather to inspire a quest for a true definition.

27 Fredholm Coarea and Stable K-Area.

Let X be a Riemannin manifold or a more general (metric) space where one may speak of length of curves and areas of surfaces and define the Fredholm K-area on its homology similarly to how it was done for K-area$^+$ in sections 12,13 but now allowing *infinite dimensional* bundles L over X.

Namely, given a complex Hilbertian vector bundle L over X, introduce the concept of a unitary connection ∇ in it via parallel transport over certain curves in X and define the norm of the curvature of ∇ as in section 9:

$||curv(\nabla)||(x)$ *is the infimum of positive functions $C(x)$ such that the maximal rotation angles $\alpha \in [-\pi, \pi]$ of the parallel transports along the boundaries of "nice" (smooth in the Riemannin case) surfaces S in X satisfy*

$$|\alpha| \leq \int_S C(s)ds$$

Represent elements κ of the *K-cohomology* group of X by pairs of complex Hilbertian vector bundle $\mathcal{L} = (L_1, L_2)$ over X with *Fredholm homomorphisms*[71]

and it is also useful for certain non-compact ones. $\Phi : L_1 \to L_2$ and define the coarea (norm) of κ as the infimum of numbers c, such that κ admits a representation by $\Phi : L_1 \to L_2$, where L_1 and L_2 are endowed with connections with $||curv|| \le c$.

Do the same for the *K-cohomology with compact supports* (say, on locally compact X), where the homomorphisms Φ are required to be unitary *isomorphisms at infinity*, i.e. outside compact subsets in X and such that Φ must be connection preserving at infinity

Define the coarea "norm" on κ in the so defined $K_{comp}(X)$ as the infimum of the numbers c, such that κ admits a representation by $\Phi : L_1 \to L_2$, where L_1 and L_2 are endowed with connections with $||curv|| \le c$ and such that Φ is connection preserving at infinity.

Define *Fredholm K-area* on the K-homology of X, on the ordinary homology and on the homology with infinite supports by linear duality: for instance, the value of this K-area on a homology class h, equal the the *reciprocal of the infimia* of the coareas of κ, such that $Chern(\kappa)(h) \neq 0$.

This is (obviously) related to the K-area and K-area$^+$ from section 12 by the inequalities

$$\text{Fredholm } K\text{-area} \ge K\text{-area}^+ \ge K\text{-area}$$

[?87] but there are no apparent examples (if any) where these inequalities are strict.

Everything we know about K-area$^+$ easily extends to the the Fredholm K-area where the main gain is *covariant functoriality* of the Fredholm K-area, such as follows.

Fredholm Push-forwards of under possibly infinite covering maps. Let $f : X \to Y$ be a covering between oriented Riemannin manifolds.

There is an obvious *push forward map* from Hilbertian bundles L over X to Hilbertian bundles over Y, say $L \mapsto M = f_*(L)$, where the fiber of the bundle M over $y \in Y$ equals the Hilbertian sum of the fibers of L over the pullback $f^{-1}(y) \subset X$,

$$M_y = \bigotimes_{x \in f^{-1}(y)} L_x.$$

For instance, if $X = Y \times \Sigma \to Y$ is the trivial covering with all fibers equal to a given countable set Σ and L is the trivial line bundle, then M is the trivial bundle with the fiber $l_2(\Sigma)$.

Now if $\Phi : L_1 \to L_2$ is a Fredholm homomorphism which is an isomorphism at infinity, then the corresponding homomorphism between the pushed forward bundles, say $\Psi : M_1 \to M_2$ is also Fredholm as well as isomorphic at infinity.

And since, clearly, the pushed bundles have the same curvatures,

$$||curv(M_{1,2})|| = ||curv(L_{1,2}), ||$$

[71]This is usually defined for compact spaces X where the operators are required to be bounded. We shall use below unbounded operators and our spaces may be sometimes non-compact.

the Fredholm K-areas of the fundamental homology classes of X and Y, assuming these are manifolds or pseudomanifolds, satisfy:

$$\text{Fredholm } K\text{-area}[X_1] \geq \text{Fredholm } K\text{-area}[X_2].$$

More generally let $f : X \to Y$ be a fibration where the fibers $Z_y \subset X$ have positive dimension and let them carry Riemannian structures continuous in $y \in Y$.

Let $\Theta(Z_y)$ be some bundle over Z_y associated with the tangent bundle $T(Z_y)$ and let the Θ-push-forward $L \underset{\Theta}{\mapsto} M = f_{*\Theta}(L)$ be defined by taking the space H_y of the square integrable sections of $L_{|Z_y} \otimes \Theta(Z_y)$, (where $L_{|Z_y}$ denotes the restriction of L to Z_y) for the fiber M_y.

Assume the fibers Z_y are closed even dimensional spin manifolds, let the restrictions $L_{|Z_y}$ be endowed with unitary connections continuous in $y \in Y$ and let

$$D_y : \mathbb{S}^+ \otimes M_y \to \mathbb{S}^- \otimes M_y$$

be the Dirac operators on Z_y twisted with $L_{|Z_y}$.

If L is a finite dimensional bundle then the operators D_y are Fredholm[72] and the resulting Fredholm bundle over Y serve as $f_*(L)$. (The spinor bundles $\mathbb{S}^+$ and $\mathbb{S}^-$ play here the role of Θ.)

And in general, where $\mathcal{L}$ is defined by $\Phi : L_1 \to L_2$, one gets a (quasi-commutative) diagram of Fredholm homomorphisms which after a little tinkering [73] defines the push forward Fredholm bundle

$$\mathcal{M} = f_*(\mathcal{L}).$$

Now let the fibration $f : X \to Y$ carry a connection Ξ, that is a parallel transport $Z_{y_1} \to Z_{y_1}$ along paths from y_1 to y_2 in Y and let a bundle L over X is is also endowed with a connection, call it ∇.

If the Ξ-transport preserves the Riemannin metrics in Z_y and if the connection ∇ is unitary, then it induces a connection in the Fredholm bundle $f_*(L)$,the curvature of which is bounded – this is obvious but significant – by the curvatures of the connection Ξ and ∇ on L. In particular , if the connection Ξ is flat, i. e. locally trivial, e.g. $X = Y \times Z$ with the trivial connection, then

$$||curv(f_*\nabla)|| \leq ||curv(\nabla)||.^{74}$$

1mm

Product Corollary. *If Z is a closed Riemannin manifold with a non-zero characteristic number, e.g. $\chi(Z) \neq 0$, then*

$$\textit{Fredholm } K\textit{-area}[Y \times Z] \leq \textit{Fredholm } K\textit{-area}[Y].$$

[72] The operators D_y are unbounded, but they are bounded Fredholm on suitable Sobolev spaces.

[73] This construction goes back to the work of Atiyah and Singer with its present form suggested in [?]-Mishchenko Infinite-dimensional representations 1974. See Ch 9 in [44]-Gromov Positive curvature macroscopic dimension 1996] for more about it.

[74] The Sobolev subspaces in in the Hilbert spaces H_y, on which the operators D_y are *bounded* Fredholm, are not, in general, invariant under the transport by $f_*(\nabla)$ but this causes no problem due to ellipticity of D_y.

Then this easily extends to open manifolds and implies for instance the following.

Subcorollary 1: Bott Periodicity The Fredholm K-area is stable under multiplication by the Euclidean spaces:

$$\text{Fredholm } K\text{-area}[X \times \mathbb{R}^{2m}] = \text{Fredholm } K\text{-area}[X].$$

Subcorollary 2: Stable Hypersphericity. Let Y be a complete Riemannian manifold of dimension n with $Sc(Y) \geq n(n-1)$, let Z be an orientable psedomanifold of dimension m.

Let $f : Y \times Z \to S^{n+m}$ be a continuous map which is locally constant at infinity and which is λ-Lipschitz on $Y \times \{z\} \subset Y \times Z$ for all $z \in Z$.

If X is orientable, if the universal covering is spin and if $\lambda < c_n$ for some universal constant $c_n > 0$. then the map f has zero degree.

[88] *Remark/Conjecture.* It is not hard to show that $c_n \geq 2^{-n}$ and,

conceivably, $c_n = 1$.

Subcorollary 2 implies, for instance, that

the filling radii of proper 1-Lipschitz embeddings from Y to (possibly infinite dimensional) manifolds W with non-positive sectional curvatures, e.g. to the Hilbert space, satisfy:

$$fil.rad(Y \subset W) \leq \frac{1}{c_n}.$$

In fact, if a submanifold $Y \subset W$ has $fil.rad(Y \subset W) > \lambda^{-1}$, then there exists a λ-Lipschitz map $Y \times Z \to S^{N-1}$, $N = dimW$, for some Z, which, if $N < \infty$, has *non-zero degree.* (See section 13 in [54]-Gromov Lawson Positive scalar curvature 1983] and section 8 in [41]-Gromov Filling Riemannian manifolds 1983].)

And since the above constant c_n is independent of the dimension of the sphere, the case $N = \infty$ reduces to that of $N < \infty$ by a straightforward approximation argument.[75]

[89] *Conjectures.* Probably, the present day techniques is sufficient to prove the above for maps to $CAT(0)$-spaces, i.e. possibly singular Alexandrov spaces W with non-positive curvatures, where the expected inequality is $fil.rad \leq 1$.

Also, in view of [70]-Kasparov Yu The coarse geometric Novikov 2005], this may be true for imbeddings to uniformly convex Busemann spaces W with non-positive curvatures with the bound on $fil.rad$ depending on the convexity modulus of W.

28 Manifolds with Scalar Curvature bounded from below by a negative constant: their Volumes, Spectra and Soap Bubbles.

Let X_0 be a complete Riemannian manifold (or an Alexandrov space) with the sectional curvatures ≤ -1 and let X be a compact n-dimension Riemannin manifold with $Sc \geq -n(n-1)$

[75] This was pointed out to me by Alexander Dranishnikov about 15 years ago who suggested a simple proof of G. Yu's theorem in [127]-Yu The coarse Baum-Connes 2000].

[?90] **Conjecture. Hyperbolic Volume Inequality.** *Then every continuous map $f_0 : X \to X_0$ is homotopic to a map f, such that*

$$vol_n(f(X)) \le vol(X)$$

where, moreover, this inequality can be made strict, unless X has a constant negative curvature and the map f_0 is homotopic to a locally isometric map.

Notice that if X as well as X_0 has *constant sectional curvature -1*, this conjecture, which generalises *the Mostow's rigidity theorem*, is proved by evaluating the *simplicial volume* of X.

Recall that $simpl.vol(X)$ is a non-negative numerical invariant of compact orientable topological manifolds X such that continuous maps between equidimensional manifolds, $X \to Y$ of degree d, satisfy

$$simpl.vol(X) \ge d \cdot simpl.vol(Y),$$

with the equality $simpl.vol(X) = d{\cdot}simpl.vol(Y)$ for d-sheeted coverings $X \to Y$.

The simplicial volume of an X is known to be non-zero if X admits a metric with negative sectional curvatures.

[?91] **Conjecture. Bound on the Simplicial Volume.** *The simplicial volume of a compact manifold X with $Sc(X) \ge -\sigma^2$, is proportionally bounded by the volume of X,*

$$simpl.vol(X) \le const_n \sigma^n vol(X).$$

There is little evidence for the conjectures 3 and 4. If $n \ge 4$ one doesn't even know if there are metrics g on n-manifolds (X_0, g_0) with negative curvatures, such that $Sc(g) \ge -\sigma^2$ for a fixed $\sigma > 0$ and such that $g \le \varepsilon g_0$ for an arbitrarily small $\varepsilon > 0$.[76]

Neither is one able to prove (or disprove) that manifolds with positive scalar curvatures have zero simplicial volumes.

Possibly, these conjectures need significant modifications to become realistic.

.... to be continued [92] [93] [94]... ???

29 Spectra of Dirac Operators, C^*-Algebras, Asymptotics of Infinite Groups, etc.

asymptotically flat and asymptotically periodic bundles. Dirac on products of non-compact manifolds

[95???]**Conjecture.** *The Dirac operators on the universal covering of aspherical manifolds contain zeros in their spectra.*

[23]-Chang+ Positive scalar curvature and a new index]

[97] Piotr W. Nowak. Zero-in-the-spectrum conjecture on regular covers of compact manifolds Comment. Math. Helv. 84 (2009), 213?222

[76] If n=3 this follows by the Schoen-Yau argument applied to (possibly infinite) minimal surfaces in X [43] while the decisive sharp volume bound follows from the properties of the Ricci flow on 3-manifolds proved by Perelman, see [128] and references therein.

Abstract. We prove the zero-in-the-spectrum conjecture for large, regular covers associated to amenable subgroups of fundamental group of a closed manifold N, provided that ˆ1.N/ is

.... to be continued [96]-[100]???

30 Foliations with $Sc > 0$

[101???]**Questions.** *When does a smooth manifold X admit an n-dimensional foliation $\mathcal{F}$ with a Riemannin metric on the leaves with $Sc > 0$?*

Would such an X itself (possibly, stabilised in some manner) admit a metric with $Sc > 0$.

Which geometric results and/or techniques concerning manifolds with $Sc > 0$ extend to foliations $\mathcal{F}$ with $Sc > 0$?

The index theoretic techniques were developed by Alain Connes for foliations in [24]-Connes Cyclic cohomology and the transverse fundamental class of a foliation 1986].

Connes' argument, which is adapted to the leaf-wise Dirac operator D twisted with the bundles associated to the normal bundle to $\mathcal{F}$ [24], also apples to D twisted with moderately curved bundles on X (this is explained in ch 9 in [44]-Gromov Positive curvature, macroscopic dimension 1996]), which implies, for instance, that

tori and, in general, compact manifolds with non-positive sectional curvatures, admit no smooth foliations with $Sc > 0$.

Further results were obtained by in [?]-Bernameur-Heitsch Enlargeability foliations] and in [129]-Zhang Positive scalar curvature on foliations 2017], where the technique developed by the latter author allows *non-spin* foliations, provided the ambient manifold is spin.

.... to be continued

31 Spaces of Metrics and Spaces of Spaces with $Sc > 0$

32 Non-smooth Spaces and Geometric Functors with $Sc \geq \sigma$.

33 Bibliography.

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We prove several new results around Gromov's waist theorem. We consider waists of different Riemannian manifolds, real and complex projective spaces, flat tori, convex bodies in the Euclidean space. We also make an effort to establish certain waist-type results in terms of the Hausdorff measure.

arXiv:1612.06926 [math.DG]

[2] Arseniy Akopyan, Roman Karasev

In this paper we find a tight estimate for Gromov's waist of the balls in spaces of constant curvature, deduce the estimates for the balls in Riemannian manifolds with upper bounds on the curvature (CAT(?)-spaces), and establish similar result for normed spaces.

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[9] Ivan Babenko, Mikhail Katz Systolic freedom of orientable manifolds

In 1972, Marcel Berger defined a metric invariant that captures the 'size' of k-dimensional homology of a Riemannian manifold. This invariant came to be called the k-dimensional SYSTOLE. He asked if the systoles can be constrained by the volume, in the spirit of the 1949 theorem of C. Loewner. We construct metrics, inspired by M. Gromov's 1993 example, which give a negative answer for large classes of manifolds, for the product of systoles in a pair of complementary dimensions. An obstruction (restriction on k modulo 4) to constructing further examples by our methods seems to reside in the free part of real Bott periodicity. The construction takes place in a split neighbourhood of a suitable k-dimensional submanifold whose connected components (rationally) generate the k-dimensional homology group of the manifold. Bounded geometry (combined with the coarea inequality) implies a lower bound for the k-systole, while calibration with support in this neighbourhood provides a lower bound for the systole of the complementary dimension. In dimension 4 everything reduces to the case of $S^2 \times S^2$

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We explore to what extent one may hope to preserve geometric properties of three dimensional manifolds with lower scalar curvature bounds under Gromov-Hausdorff and Intrinsic Flat limits. We introduce a new construction, called sewing, of three dimensional manifolds that preserves positive scalar curvature. We then use sewing to produce sequences of such manifolds which converge to spaces that fail to have nonnegative scalar curvature in a standard generalised sense. Since the notion of nonnegative scalar curvature is not strong enough to persist alone, we propose that one pair a lower scalar curvature bound with a lower bound on the area of a closed minimal surface when taking sequences as this will exclude the possibility of sewing of manifolds.

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[14] BORIS BOTVINNIK AND DEMETRE KAZARAS

Minimal hypersurfaces and bordism of positive scalar curvature metrics

Let (Y,g) be a compact Riemannian manifold of positive scalar curvature (psc). It is well-known, due to Schoen-Yau, that any closed stable minimal hypersurface of Y also admits a psc-metric. We establish an analogous result for stable minimal hypersurfaces with free boundary. Furthermore, we combine this result with tools from geometric measure theory and conformal geometry to study psc-bordism.

: arXiv:1609.09142 [math.DG] Sep 28, 2016

[15] Boris Botvinnik Manifolds with singularities accepting a metric, of positive scalar curvature

Submitted on 1 Nov 1999 (v1), last revised 27 Nov 2001

We study the question of existence of a Riemannian metric of positive scalar curvature metric on manifolds with the Sullivan-Baas singularities. The manifolds we consider are Spin and simply connected. We prove an analogue of the Gromov-Lawson Conjecture for such manifolds in the case of particular type of singularities. We give an affirmative answer when such manifolds with singularities accept a metric of positive scalar curvature in terms of the index of the Dirac operator valued in the corresponding

"K-theories with singularities". The key ideas are based on the construction due to Stolz, some stable homotopy theory, and the index theory for the Dirac operator applied to the manifolds with singularities. As a side-product we compute homotopy types of the corresponding classifying spectra. arXiv:math/9910177 [math.DG]

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arXiv:1008.3097v3 [math.DG] 21 Nov 2011

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proceedings of the american mathematical society Volume K7. Number 2. February 1983

Abstract. In this note, we classify the even-dimensional complex hypersurfaces and complete intersections which carry a metric of positive scalar curvature. This is done by computing the ,4-genus of these manifolds to eliminate all cases not known to carry such a metric.

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In this article, we develop a new index theory for noncompact manifolds endowed with an admissible exhaustion by compact sets. This index theory allows us to provide examples of noncompact manifolds with exotic positive scalar curvature phenomena.

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We show that every closed, smooth, simply-connected (n+2)-manifold with a smooth, effective action of a torus Tn admits an invariant Riemannian metric with positive Ricci curvature.

arXiv:1609.06125

[26] Xianzhe Dai, Xiaodong Wang, Guofang Wei

On the Stability of Riemannian Manifold with Parallel Spinors

[27] R. Dervan On K-stabiliy of finite covers

We show that certain Galois covers of K-semistable Fano varieties are K-stable. We use this to give some new examples of Fano manifolds admitting Kähler-Einstein metrics, including hypersurfaces, double solids and three-folds.

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K-area is an invariant for Riemannian manifolds introduced by Gromov as an obstruction to the existence of positive scalar curvature. However in general it is difficult to determine whether K-area is finite or not. though the definition of K-area is quite natural. In this paper, we study how the invariant changes under

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[55] Larry Guth Volumes of balls in large Riemannian manifolds

If (M^n, g) is a complete Riemannian manifold with filling radius at least R, then we prove that it contains a ball of radius R and volume at least $c(n)R^n$. If (M^n, hyp) is a closed hyperbolic manifold and if g is another metric on M with volume at most c(n)Volume(M,hyp), then we prove that the universal cover of (M,g) contains a unit ball with volume greater than the volume of a unit ball in hyperbolic n-space.

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[60] Hanke Positive scalar curvature, K-area and essentialness, arXiv:1011.3987 ,

[61] B. Hanke, T. Schick, Enlargeability and index theory,J. Differential Geom. 74 (2) (2006), 293-320.

[62] Hanke, Bernhard; Pape, Daniel; Schick, Thomas Codimension two index obstructions to positive scalar curvature, Annales de l'institut Fourier, Volume 65 (2015) no. 6 , p. 2681-2710.

[63] Bernhard Hanke, Wolfgang Steimle The space of metrics of positive scalar curvature Bernhard Hanke, Thomas Schick, Wolfgang Steimle (Bonn) (Submitted on 1 Dec 2012 (v1), last revised 12 Feb 2014 (this version, v3))

We study the topology of the space of positive scalar curvature metrics on high dimensional spheres and other spin manifolds. Our main result provides elements of infinite order in higher homotopy and homology groups of these spaces, which, in contrast to previous approaches, are of infinite order and survive in the (observer) moduli space of such metrics. Along the way we construct smooth fiber bundles over spheres whose total spaces have non-vanishing A-hat-genera, thus establishing the non-multiplicativity of the A-hat-genus in fibre bundles with simply connected base.

. arXiv:1212.0068

[64] Rafael Herrera Noemi Santana. Spinorially twisted Spin structures, II: pure spinors and holonomy

(Submitted on 25 Jun 2015)

We introduce a notion of pure spinor for spinorially twisted spin representations, analyze the geometrical consequences of their existence, and characterize special Riemannian holonomies by the existence of parallel (twisted) pure spinor fields.

arXiv:1506.07681

[65] M. Herzlich, Extremality for the Vafa-Witten bound on a sphere, Geom. Funct. Anal. 15 (2005), 1153? 1161,

arXiv:math.DG/0407530

[66] M. Herzlich, Universal positive mass theorems Marc Herzlich (IMAG) (Submitted on 23 Jan 2014 (v1), last revised 19 Aug 2016 (this version, v2))

In this paper, we develop a general study of contributions at infinity of Bochner-Weitzenböck-type formulas on asymptotically flat manifolds, inspired by Witten's proof of the positive mass theorem. As an application, we show that similar proofs can be obtained in a much more general setting as any choice of an irreducible natural bundle and a very large choice of first-order operators may lead to a positive mass theorem along the same lines if the necessary curvature conditions are satisfied.

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SCALAR CURVATURE AND CONFORMAL DEFORMATION OF RIEMANNIAN STRUCTURE DIFFERENTIAL GEOMETRY 10 (1975) 113-134

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[77] Mario Listing Scalar curvature rigidity of hyperbolic product manifolds

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Scalar curvature and vector bundles

[79] Mario Listing, Scalar curvature and vector bundles

In the first part we use Gromov's K–area to define the K–area homology which stabilizes into singular homology on the category of pairs of compact smooth manifolds. The second part treats the questions of certain curvature gaps. For instance, the L?–curvature gap of complex vector bundles on a compact manifold is positive if and only if the K–area homology coincides with the reduced singular homology in all even degrees. In the third part we give some upper bounds of the scalar curvature on compact manifolds. In particular, we generalize results by Llarull and Goette, Semmelmann.

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[88] Arjun Malhotra The Gromov-Lawson-Rosenberg conjecture for some finite groups

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[91] Yashar Memarian On Gromov's Waist of the Sphere Theorem

The goal of this paper is to give a detailed and complete proof of M. Gromov's waist of the sphere theorem.

arXiv:0911.3972 [math.MG]

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Abstract. We prove the zero-in-the-spectrum conjecture for large, regular covers associated to amenable subgroups of fundamental group of a closed manifold N, provided that ˆ1.N/ is

[98] SIMON RAULOT Rigidity of compact Riemannian spin Manifolds with Boundary

Abstract.

In this article, we prove new rigidity results for compact Riemannian spin manifolds with boundary whose scalar curvature is bounded from below by a non-positive constant. In particular, we obtain generalizations of a result of Hang-Wang based on a conjecture of Schroeder and Strake arXiv:0803.3108

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[100] J. Roe, Positive curvature, partial vanishing theorems, and coarse indices (Submitted on 23 Oct 2012 (v1), last revised 2 Nov 2012

Let M be a complete Riemannian manifold, D a Dirac-type operator on M whose Weitzenbock curvature is uniformly positive on the complement of a subset Z of M. We show that the coarse index of D is localized to the K-theory of the coarse $C^a st$-algebra of Z. Applications are discussed, including a coarse form of the relative index theorem. arXiv:1210.6100 [math.KT] (or arXiv:1210.6100v2 [math.KT]

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[102] Jonathan Rosenberg Novikov's Conjecture

We describe Novikov's "higher signature conjecture," which dates back to the late 1960's, as well as many alternative formulations and related problems. The Novikov Conjecture is perhaps the most important unsolved problem in high-dimensional manifold topology, but more importantly, variants and analogues permeate many other areas of mathematics, from geometry to operator algebras to representation theory

arXiv:1506.05408

[103] Jonathan Rosenberg, Stephan Stolz A "stable" version of the Gromov-Lawson conjecture (Submitted on 5 Jul 1994)

We discuss a conjecture of Gromov and Lawson, later modified by Rosenberg, concerning the existence of metrics of positive scalar curvature. It says that a closed spin manifold M of dimension $n \geq 5$ has such a metric if and only if the index of a suitable "Dirac" operator in , the real K-theory of the group C^*-algebra of the fundamental group of M, vanishes. It is known that the vanishing of the index is necessary for existence of a positive scalar curvature arXiv:dg-ga/9407002

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[108] Thomas Schick, Mostafa Esfahani Zadeh. Large scale index of multi-partitioned manifolds (Submitted on 3 Aug 2013 (v1), last revised 25 Jul 2015 (this version, v5))

Let M be a complete n-dimensional Riemannian spin manifold, partitioned by q two-sided hypersurfaces which have a compact transverse intersection N and which in addition satisfy a certain coarse transversality condition. Let E be a Hermitean bundle with connection on M. We define a coarse multi-partitioned index of the spin Dirac operator on M twisted by E. Our main result is the computation of this multi-partitioned index as the Fredholm index of the Dirac operator on the compact manifold N, twisted by the rest

arXiv:1308.0742

[109] Thomas Schick. The topology of positive scalar curvature (Submitted on 16 May 2014 (v1), last revised 25 Jun 2014 (this version, v2))

In this survey article, given a smooth closed manifold M we study the space of Riemannian metrics of positive scalar curvature on M. A long-standing question is: when is this space non-empty (i.e. when does M admit a metric of positive scalar curvature)? More generally: what is the topology of this space? For example, what are its homotopy groups? Higher index theory of the Dirac operator is the basic tool to address these questions. This has seen tremendous development in recent years, and in this survey we will discuss some of the most pertinent examples. In particular, we will show how advancements of large scale index theory (also called coarse index theory) give rise to new types of obstructions, and provide the tools for a systematic study of the existence and classification problem via the K-theory of C^*-algebras. This is part of a program "mapping the topology of positive scalar curvature to analysis". In addition, we will show how advanced surgery theory and smoothing theory can be used to construct the first elements of infinite order in the k-th homotopy groups of the space of metrics of positive scalar curvature for arbitrarily large k. Moreover, these examples are the first ones which remain non-trivial in the moduli space of such metrics.

arXiv:1405.4220 [math.GT]

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[111] R. Schoen and S.-T. Yau, *On the proof of the positive mass conjecture in general relativity,* Commun. Math. Phys. 65, (1979). 45-76.

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[115] R. Schoen and S. T. Yau, Positive Scalar Curvature and Minimal Hypersurface Singularities Richard Schoen, Shing-Tung Yau, arXiv:1704.05490 [math.DG]

[116] Benedikt von Seelstrang, Homological Invariance of Infinite K-area

(Submitted on 26 Jul 2016) Following ideas of Brunnbauer and Hanke, we construct a notion of infinite K-area for homology classes of simplicial complexes with finitely generated fundamental groups. As in Brunnbauer's and Hanke's work, the results concerning these homology classes will imply that Gromov's property ofhaving infinite K-area depends only on the image of the fundamental class under the classifying map of the universal cover. As acorollary we obtain another proof of a theorem of Fukumoto, that the property of infinite K-area is invariant under p-surgery with p?1. As a result of the proof of our main theorem, we will clarify a point left open in a paper of Mishchenko and Teleman about extensions of almost flat bundles. arXiv:1607.07820

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[119] S. Stolz. *Simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) 136 (1992), 511-540.

[120] Guangxiang Su, Weiping Zhang, Positive scalar curvature and connected sums

(Submitted on 1 May 2017 (v1), last revised 24 May 2017 (this version, v2))

Let N be a closed enlargeable manifold in the sense of Gromov-Lawson and M a closed spin manifold of equal dimension, a famous theorem of Gromov-Lawson states that the connected sum M#N admits no metric of positive scalar curvature. We present a potential generalization of this result to the case where M is nonspin. We use index theory for Dirac operators to prove our result.

Comments: 5 pages. Correct a mistake in the previous version. The result of the current version does not imply the original statement in the earlier version Subjects: Differential Geometry (math.DG) Cite as: arXiv:1705.00553 [math.DG]

[121] Taubes, C.: The Seiberg-Witten invariants and symplectic forms , Math. Res. Lett. 1, 809-822 (1994)

[122] Mark Walsh Metrics of positive scalar curvature and generalised Morse functions, part 1

(Submitted on 8 Nov 2008)

It is well known that isotopic metrics of positive scalar curvature are concordant. Whether or not the converse holds is an open question, at least in dimensions greater than four. We show that for a particular type of concordance, constructed using the surgery techniques of Gromov and Lawson, this converse holds in the case of closed simply connected manifolds of dimension at least five.

arXiv:0811.1245

[123] M Walsh, The Space of Positive Scalar Curvature Metrics on a Manifold with Boundary.

Abstract. We study the space of Riemannian metrics with positive scalar curvature on a compact manifold with boundary. These metrics extend a fixed boundary metric and take a product structure on a collar neighbourhood of the boundary. We show that the weak homotopy type of this space is preserved by certain surgeries on the boundary in co- dimension at least three. Thus, there is a weak homotopy equivalence between the space of such metrics on a simply connected spin manifold W of dimension $n \geq 6$ and with simply connected boundary, and the corresponding space of metrics of positive scalar curvature on the standard disk arXiv:1411.2423

[124] Stefan Wenger, A short proof of Gromov's filling inequality

(Submitted on 29 Mar 2007)

We give a very short and rather elementary proof of Gromov's filling volume inequality for n-dimensional Lipschitz cycles (with integer and Z_2-coefficients) in L^∞-spaces. This inequality is used in the proof of Gromov's systolic inequality for closed aspherical Riemannian manifolds and is often regarded as the difficult step therein.

arXiv:math/0703889 [math.DG]

[125] Michael Wiemeler Circle actions and scalar curvature

(Submitted on 10 May 2013 (v1), last revised 29 Jan 2015 (this version, v3))

We construct metrics of positive scalar curvature on manifolds with circle actions. One of our main results is that there exist S1-invariant metrics of positive scalar curvature on every S1-manifold which has a fixed point component of codimension 2. As a consequence we can prove that there are non-invariant metrics of positive scalar curvature on many manifolds with circle actions. Results from equivariant bordism allow us to show that there is an invariant metric of positive scalar curvature on the connected sum of two copies of a simply connected semi-free S1-manifold M of dimension at least six provided that M is not spin or that M is spin and the S1-action is of odd type. If M is spin and the S1-action of even type then there is a k>0 such that the equivariant connected sum of 2k copies of M admits an invariant metric of positive scalar curvature if and only if a generalized $\hat{A}$ -genus of M/S1 vanishes.

arXiv:1305.2288

[126] David J. Wraith Non-negative versus positive scalar curvature (Submitted on 3 Jul 2016)

We show that results about spaces or moduli spaces of positive scalar curvature metrics proved using index theory can typically be extended to non-negative scalar curvature metrics. We illustrate this by providing explicit generalizations of some classical results concerning moduli spaces of positive scalar curvature metrics. We also present the first examples of manifolds with infinitely many path-components of Ricci non-negative metrics in both the compact and non-compact cases.

arXiv:1607.00657

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[128] Wei Yuan, Volume comparison with respect to scalar curvature (Submitted on 28 Sep 2016)

In this article, we investigate the volume comparison with respect to scalar curvature. In particular, we show volume comparison hold for small geodesic balls of metrics near V-static metrics. As for global results, we give volume comparison for metrics near Einstein metrics with certain restrictions. As an application, we recover a volume comparison result of compact hyperbolic manifolds due to Besson-Courtois-Gallot, which provides a partial answer to a conjecture of Schoen on volume of hyperbolic manifolds.

arXiv:1609.08849

[129] W Zhang Positive scalar curvature on foliations: the enlargeability,

Inspired by a recent paper of Benameur and Heitsch, we generalize the famous result of Gromov and Lawson on the nonexistence of metric of positive scalar curvature on enlargeable manifolds to the case of foliations, without using index theorems on noncompact manifolds. arXiv:1703.04313 2017

[130] Rudolf Zeidler Positive scalar curvature and product formulas for secondary index invariants

the one of the complex hyperbolic space must be isometricall y biholomorphic to it. This result has been known for some time in odd c omplex dimension and we provide here a proof in even dimension. arXiv:1412.0685