AGR 2.0

General Relativistic N-body Problem:

Multi-chart approach, BD multipoles, Equations of motion

Relevant background references:

[BK88] V.A. Brumberg, S.M. Kopeikin, Nuovo. Cim. B 103, 63 (1988)

[DSX1] T. Damour, M. Soffel, C. Xu, 'General relativistic celestial mechanics I. Method and definition of reference systems' Phys. Rev. D 43, 3273 (1991)

[DSX2] T. Damour, M. Soffel, C. Xu 'General relativistic celestial mechanics II. Translational equations of motion', Phys. Rev. D 45, 1017 (1992)

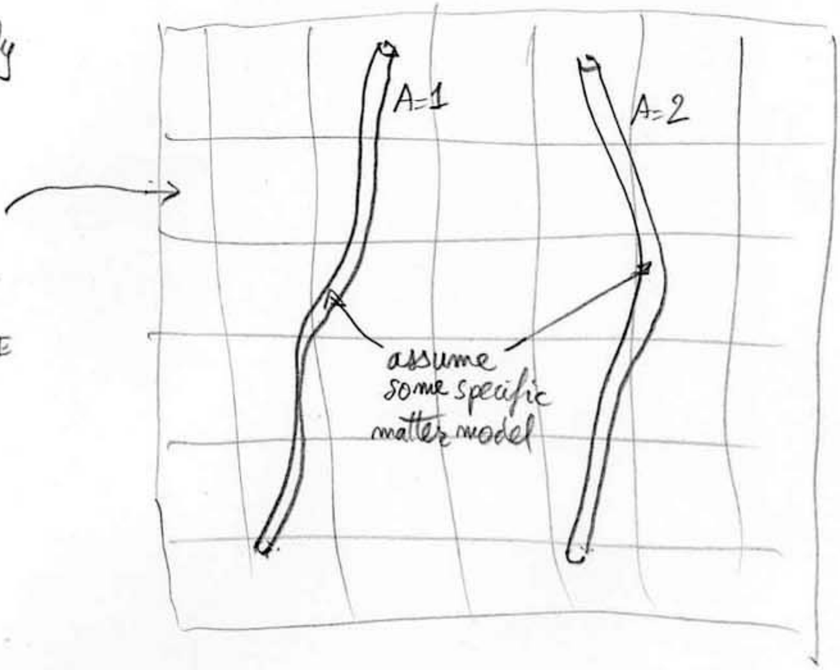
Review of problem of motion

[D87] T. Damour, 'The problem of motion in Newtonian and Einsteinian gravity', in "300 Years of Gravitation", ed. S.W. Hawking and W. Israel, Cambridge U. Press, 1987, pp 128–198.

2.1 Traditional (one chart) approach to the N-body problem

Traditionally

ONE
GLOBAL
COORDINATE
CHART
x^μ



Usual strategy: Try to solve
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$
$$\Downarrow$$
together with
$$\nabla_\nu T^{\mu\nu} = 0$$

assuming

specific $T^{\mu\nu} \overset{e.g}{=} (\varepsilon + p) u^\mu u^\nu + p g^{\mu\nu}$

and global expansion: $g_{\mu\nu}(x^\lambda) = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \dots$

together, may be, with global PN approximation ansätze

$$\partial_0 h_{\mu\nu} = \frac{1}{c} \partial_t h_{\mu\nu} \ll \partial_i h_{\mu\nu}$$
$$v \ll c$$
$$p \ll \varepsilon$$

Inconvenient aspects of global, one-chart approach:

- technically : e.g. a body which is (essentially) spherically symmetric w.r.t. its own rest frame, will appear, in global x^μ, as some deformed ellipsoid (with time-dependent deformation). Deformation important w.r.t. accuracy of modern techniques: VLBI, laser tracking, ...
- the gravitational field, and multipole moments, of body A in common x^μ chart are not directly related to the relevant observable quantities
- conceptually : PN approximation tends to reintroduce Newtonian way of thinking (absolute space and time) which can lead to errors or confusions.

$\boxed{2.2}$ Multi-chart approach to the N-body problem

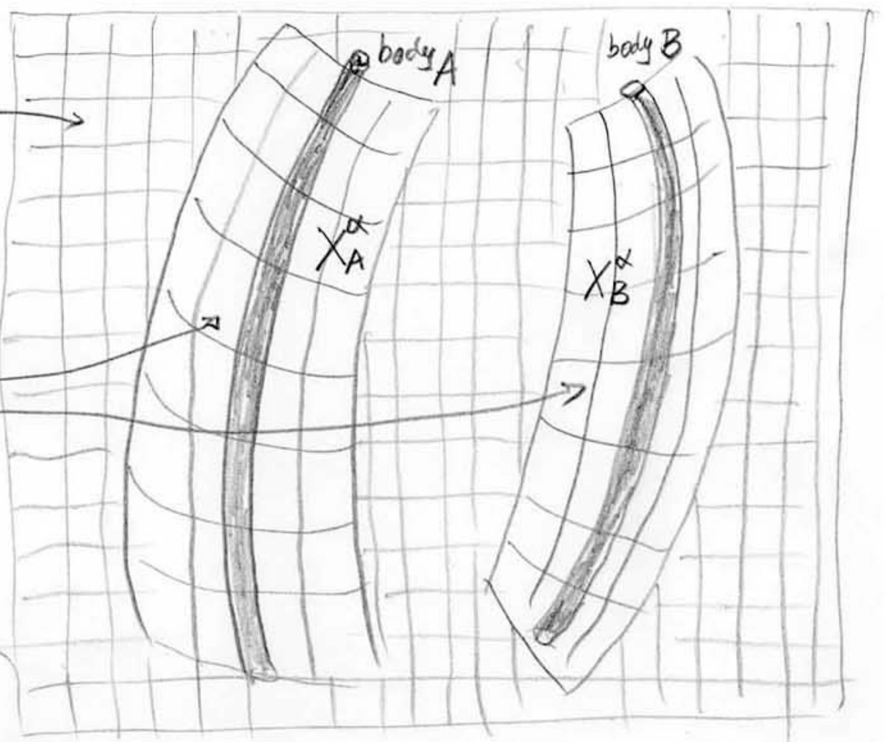
Use N+1 charts

one global chart x^μ

$+$

N local (body related) charts X^α_A

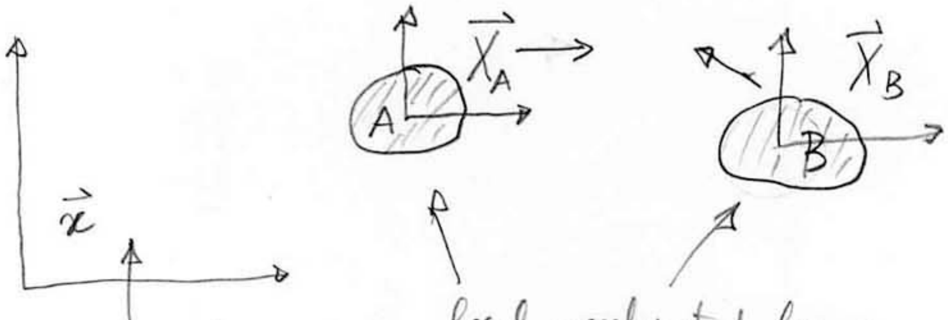
$A = 1, 2, \dots N$



Basic ideas:

- to describe the metric $ds^2 = G^A_{\alpha\beta}(X^\gamma_A)\, dX^\alpha_A\, dX^\beta_A$ in each local chart X^α_A in terms mainly of quantities observed in this system ($T^{\alpha\beta}_A(X_A)$; multipole moments measured by satellites of A,...), plus some extra 'tidal-like' terms coming from the influence of far away bodies
- to obtain the equations of motion (in common x^μ chart) of each body as generalizations of the 'd'Alembert approach', i.e. reducing dynamics in x^μ to an 'equilibrium' problem in the local frame X^α_A
- to end up by expressing the global eqs of motion in terms of the locally measured multipole moments of each body
- framework is rather simple and fully explicit at the 1PN approximation (weakly self-gravitating bodies and v^2/c^2 beyond Newton), but the general idea is usefully extended to more general cases (notably strongly self-gravitating bodies)
- NB: We are considering general deformable bodies, with unspecified equation of state.

$\boxed{2.3}$ Reminder: d'Alembert approach to the Newtonian N-body problem



inertial frame

local accelerated frames associated with each body

in inertial frame

$i = 1,2,3$

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x^i}(\rho v^i) = 0 \\ &\frac{\partial}{\partial t}(\rho v^i) + \frac{\partial}{\partial x^j}[\rho v^i v^j + \overset{\text{material stresses}}{t^{ij}}] = \rho \frac{\partial U}{\partial x^i} \\ &\Delta_x U(\vec{x},t) = -4\pi G \rho(\vec{x},t) \end{aligned}}$$

formal solution $U(\vec{x},t) = \sum_{A=1}^{N} U^A(\vec{x},t)$

with $U^A(\vec{x},t) \equiv G \int_A d^3x' \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}$

yields non local (and non linear) evolution system for $\rho(\vec{x},t)$, $\vec{v}(\vec{x},t)$, which cannot be solved exactly.

Introduce transformation to some <u>accelerated</u> reference frames associated with each body

$$\boxed{x^i = z_A^i(t) + X_a^i}$$

assuming for simplicity no time-dependent rotation

arbitrary accelerated motion to be determined later

in accelerated A-frame

$$\rho_A \equiv \rho(\vec{X}_A, t) = \rho(\vec{z}_A + \vec{X}_A, t)$$

$$V_A^i \equiv v^i - \frac{dz_A^i}{dt}$$

$$\boxed{\begin{gathered} \frac{\partial \rho_A}{\partial t} + \frac{\partial}{\partial X_A^i}\left(\rho_A V_A^i\right) = 0 \\ \frac{\partial}{\partial t}\left(\rho_A V_A^i\right) + \frac{\partial}{\partial X_A^j}\left(\rho_A V_A^i V_A^j + t_A^{ij}\right) = \rho_A \frac{\partial U_A^{eff}}{\partial X_A^i} \end{gathered}}$$

<u>effective</u> gravitational potential (modified by inertial forces)

$$\boxed{U_A^{eff}(\vec{X}_A) = U(\vec{z}_A + \vec{X}_A) - C(t) - \frac{d^2\vec{z}_A}{dt^2}\cdot\vec{X}_A}$$

can be decomposed as

$$\boxed{U_A^{eff} = U^A + \bar{U}^A}$$

locally generated

$$\boxed{U^A(t, \vec{X}_A) = G\int d^3X'_A \frac{\rho_A(\vec{X}'_A, t)}{|\vec{X}_A - \vec{X}'_A|}}$$

externally generated + inertia effects

$$\boxed{\bar{U}^A = \sum_{B\neq A} U^B(\vec{z}_A + \vec{X}_A) - C(t) - \frac{d^2\vec{z}_A}{dt^2}\cdot\vec{X}_A}$$

Introduce (mass) multipole moments of body A wrt local A frame

$$\boxed{m^A_L(t) \equiv \int_A d^3X_A \; X_A^{\langle L\rangle} \rho_A} \qquad \ell = 0,1,2,\dots$$

+ local spin vector

$$\boxed{s^A_i(t) \equiv \int_A d^3X_A \; \varepsilon_{iab} \, X_A^a \, \rho \, V_A^b}$$

Using Action and Reaction principle, one has

$$\boxed{\begin{aligned} &\frac{d}{dt} m^A(t) = 0 \\ &\frac{d^2}{dt^2} m^A_i(t) = \int_A d^3X_A \, \rho_A \frac{\partial \bar{U}^A}{\partial X_A^i} \\ &\frac{d}{dt} s^A_i(t) = \varepsilon_{iab} \int d^3X_A \, \rho_A \, X_A^a \frac{\partial \bar{U}^A}{\partial X_A^b} \end{aligned}}$$

ONLY $\bar{U}^A$ enters here

Expand $\bar{U}^A(\vec{X}_A)$ in <u>tidal series</u>

$$\boxed{\bar{U}^A(\vec{X}_A, t) = g^A(t) + g^A_i(t) X_A^i + \frac{1}{2!} g^A_{ij}(t) X_A^{\langle i} X_A^{j\rangle} + \dots + \frac{1}{\ell!} g^A_L(t) X_A^{\langle L\rangle} + \dots}$$

$$g^A(t) = \sum_{B \neq A} U^B(\vec{z}_A(t)) - C(t)$$

$$g^A_i(t) = \sum_{B \neq A} \partial_i U^B(\vec{z}_A(t)) - \frac{d^2 z_A^i}{dt^2}$$

crucial inertial contribution at <u>dipole</u> level

$$g^A_L(t) = \sum_{B \neq A} \partial_L U^B(\vec{z}_A(t)) \quad \text{for } \ell \geq 2$$

- One can choose arbitrary $C(t)$ to set $g^A(t) = 0$; $l=0$ term
- One could also think that it is a good idea to choose the so far arbitrary $\frac{d^2\vec{z}_A}{dt^2}$ to set tidal dipole

  $$g^A_i(t) \overset{?}{=} 0$$

  In fact, NO!

- Insert tidal expansion of $\bar{U}^A$ in eqs for m^A, m^A_i, s^A_i

$$\begin{aligned}
&\frac{d\, m^A(t)}{dt} = 0 \\
&\frac{d^2 m^A_i(t)}{dt^2} = m^A g^A_i + m^A_j\, g^A_{ij} + \frac{1}{2!} m^A_{jk}\, g^A_{ijk} + \cdots + \frac{1}{l!} m^A_L\, g^A_{iL} + \cdots \\
&\frac{d s^A_i(t)}{dt} = \varepsilon_{iab}\, m^A_a\, g^A_b + \varepsilon_{iab}\, m^A_{aj}\, g^A_{bj} + \cdots + \frac{1}{l!} \varepsilon_{iab}\, m^A_{aL}\, g^A_{bL} + \cdots
\end{aligned}$$

- Let us now fix the motion of the local A frame by imposing that it follows, for all times, the overall motion of body A by setting

$$\boxed{0 = m^A_i(t) = \int_A d^3X_A\, \rho_A\, X^i_A = \int_A d^3x\, \rho(\vec{x},t)\, [x^i - z^i_A(t)]}$$

i.e. saying that the local A frame stays centered at the center of mass of body A

This condition of 'equilibrium' in A frame yields

$$\frac{d^2 m_i^A}{dt^2} = 0 = \underset{\swarrow}{m^A g_i^A} + \underset{\underset{0}{\parallel}}{m_j^A} g_{ij}^A + \frac{1}{2!} m_{jk}^A g_{ijk}^A + \cdots$$

$$\boxed{0 = m^A \Big(\underbrace{\sum_{B\neq A} \partial_i U^B(\vec{z}_A) - \frac{d^2 z_A^i}{dt^2}}_{\text{'d'Alembert'}} \Big) + \frac{1}{2!} m_{jk}^A g_{ijk}^A + \cdots + \frac{1}{l!} m_L^A g_{iL}^A + \cdots}$$

But we have fixed $z_A^i \equiv z_{cmA}^i$, hence we get the following (global) inertial-frame translational eqs of motion for body A

$$\boxed{m^A \frac{d^2 z^i_{cm.A}}{dt^2} = \sum_{B\neq A} \Big\{ m^A \partial_i U^B(\vec{z}_{cmA}) + \frac{1}{2!} m_{jk}^A \partial_{ijk} U^B(z_{cmA}) + \cdots + \frac{1}{l!} m_L^A \partial_{iL} U^B(z_{cmA}) \Big\}}$$

Replacing U^B by its multipole expansion

$$U^B(\vec{x},t) = \frac{Gm^B}{|\vec{x}-\vec{Z}_B|} - \partial_i\Big(\frac{Gm_i^B}{|\vec{x}-\vec{Z}_B|}\Big) + \frac{1}{2!}\partial_{ij}\Big(\frac{Gm_{ij}^B}{|\vec{x}-\vec{Z}_B|}\Big) \cdots + \frac{(-)^l}{l!}\partial_L \frac{Gm_L^B}{|x-z_B|} + \cdots$$

Finally we get a double series in the multipoles of all bodies

$$\boxed{m^A \frac{d^2 z^i_{cmA}}{dt^2} = G \sum_{B\neq A} \sum_{l\geq 0} \sum_{k\geq 0} \frac{(-)^k}{l!k!} m_L^A m_K^B \partial^A_{iLK}\Big(\frac{1}{|\vec{z}_{cmA} - \vec{z}_{cmB}|}\Big)}$$

Useful facts in Newtonian N-body problem:

- Eq. for U was linear $\Rightarrow$ allowed linear decomposition $U = \sum_A U^A$
- inertial forces in local frame $\Rightarrow$ linear addition to U
- $\exists$ multipole expansion for each U^A
- $\exists$ tidal expansion for U_A^{eff}, including inertial effects, in A frame

The 1PN generalrelativistic N-body problem is NON LINEAR

and the transformation $x^\mu = f^\mu(X_A^\alpha)$ is also NON LINEAR

BUT the problem can be reformulated (at 1PN) in a quasi-linear way which allows one to physically parallel the Newtonian treatment

One makes use of a hidden linearity in Einstein's equations at 1PN level. Essentially, after using at 1PN:

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$; $h_{ij} \simeq h_{00}\delta_{ij}$, $h_{0i} \ll h_{00}$, $\partial_0 \ll \partial_i$, the cubic piece in Einstein's action is

$$\mathcal{L}^{\text{Ein}} \sim (\partial_i h_{00})^2 + \overset{\text{cubic vertex}}{\overline{\alpha\, h_{00}(\partial_i h_{00})^2}} + \text{contact terms}$$

$$h_{00} = \varphi \quad : \quad (\partial\varphi)^2 + \alpha\varphi(\partial\varphi)^2 = (\partial\psi)^2$$

$$\text{if} \quad d\psi = \sqrt{1+\alpha\varphi}\, d\varphi \Rightarrow \boxed{\psi \simeq \varphi + \frac{1}{4}\alpha\varphi^2 = h_{00} + \frac{1}{4}\alpha h_{00}^2}$$

i.e. change of variable : $h_{00} \to \psi \Rightarrow \simeq$ linear theory $\mathcal{L} \sim (\partial\psi)^2$

$\boxed{2.4}$ Quasi-linear formulation of 1PN relativity

In *each* of the N+1 coordinate charts one can write 1PN-accurate metric in terms of

a scalar potential $w \propto \overset{\text{non-linear function of } g_{00}}{\ln(-g_{00})}$ cf 4 above

a vector potential $w_i \propto g_{0i}$

with

$g_{\mu\nu}(x^\lambda)$ in global chart $\rightarrow$

$$\boxed{\begin{aligned} g_{00} &= -\exp\left(-\frac{2}{c^2} w\right), \\ g_{0i} &= -\frac{4}{c^3} w_i, \\ g_{ij} &= \delta_{ij} \exp\left(+\frac{2}{c^2} w\right) + \overset{= \mathcal{O}(\frac{1}{c^4})}{\mathcal{O}(4)} \end{aligned}}$$

$G^A_{\alpha\beta}(X^\gamma_A)$ in local A chart $\rightarrow$

$$\boxed{\begin{aligned} G^A_{00} &= -\exp\left(-\frac{2}{c^2} W^A\right), \\ G^A_{0a} &= -\frac{4}{c^3} W^A_a, \\ G^A_{ab} &= \delta_{ab} \exp\left(+\frac{2}{c^2} W^A\right) + \mathcal{O}(4) \end{aligned}}$$

In each frame, the 'scalar' and 'vector' potentials satisfy <u>linear</u> ('Maxwell-like') equations

$$\boxed{\begin{aligned} &\Delta_x w + \frac{3}{c^2}\partial_t^2 w + \frac{4}{c^2}\partial_t\partial_i w_i = -4\pi G\sigma + O(4) \\ &\Delta_x w_i - \partial_{ij} w_j - \partial_t \partial_i w = -4\pi G \sigma^i + O(2) \end{aligned}}$$

where

$$\boxed{\begin{aligned} \sigma(\vec{x},t) &= \frac{T^{00} + T^{ss}}{c^2} \\ \sigma^i(\vec{x},t) &\equiv \frac{T^{0i}}{c} \end{aligned}}$$

Similarly for W^A, W_a^A: $\Delta_{X_A} W^A + \frac{3}{c^2}\partial_{T_A}^2 W^A + \cdots = -4\pi G \Sigma_A$

$$\Sigma_A(\vec{X},T) = \frac{T_A^{00}(X) + T_A^{ss}(X)}{c^2}$$

in X_A-chart

$\exists$ Maxwell-like <u>gauge invariance</u> of 1PN field eqs

$$\begin{cases} w' = w - \frac{1}{c^2}\partial_t \lambda \\ w_i' = w_i + \frac{1}{4}\partial_i \lambda \end{cases}$$

which corresponds to a shift of the time variable

$$\delta t = \frac{1}{c^4}\lambda(\vec{x},t)$$

This residual gauge invariance (after fixing the spatial gauge by the form of the metric) for the

'4 vector' $a_\mu = (c w, -4 w_i)$

suggests to introduce the gauge-invariant object

$$b_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$$

i.e.

'gravito electric field' $\quad e_i[w] \equiv \partial_i w + \frac{4}{c^2}\partial_t w_i$

'gravito magnetic field' $\quad b_{ij}[w) \equiv \varepsilon_{ijk} b_k \equiv -4[\partial_i w_j - \partial_j w_i]$

They satisfy Maxwell-like eqs

$$\begin{aligned}
\vec{\nabla}\cdot\vec{b} &= 0 \\
\vec{\nabla}\times\vec{e} &= -\frac{1}{c^2}\partial_t \vec{b} \\
\vec{\nabla}\cdot\vec{e} &= -\frac{3}{c^2}\partial_t^2 w - 4\pi G\sigma + \mathcal{O}(4) \\
\vec{\nabla}\times\vec{b} &= +4\,\partial_t\vec{e} - 16\pi G\vec{\sigma} + \mathcal{O}(2)
\end{aligned}$$

If the time coordinate x^0 is harmonic:

$$0 = \Box_g x^0 = -\frac{4}{c^3}\left(\partial_t w + \partial_i w_i\right) + \mathcal{O}(5)$$

we have the "Lorenz"-like gauge $\quad \boxed{\partial_t w + \partial_i w_i = 0}$

and

$$\Delta - \frac{1}{c^2}\partial_t^2 \longrightarrow \begin{aligned}
\Box w &= -4\pi G\sigma + \mathcal{O}(4) \\
\Delta w_i &= -4\pi G\sigma^i + \mathcal{O}(2)
\end{aligned}$$

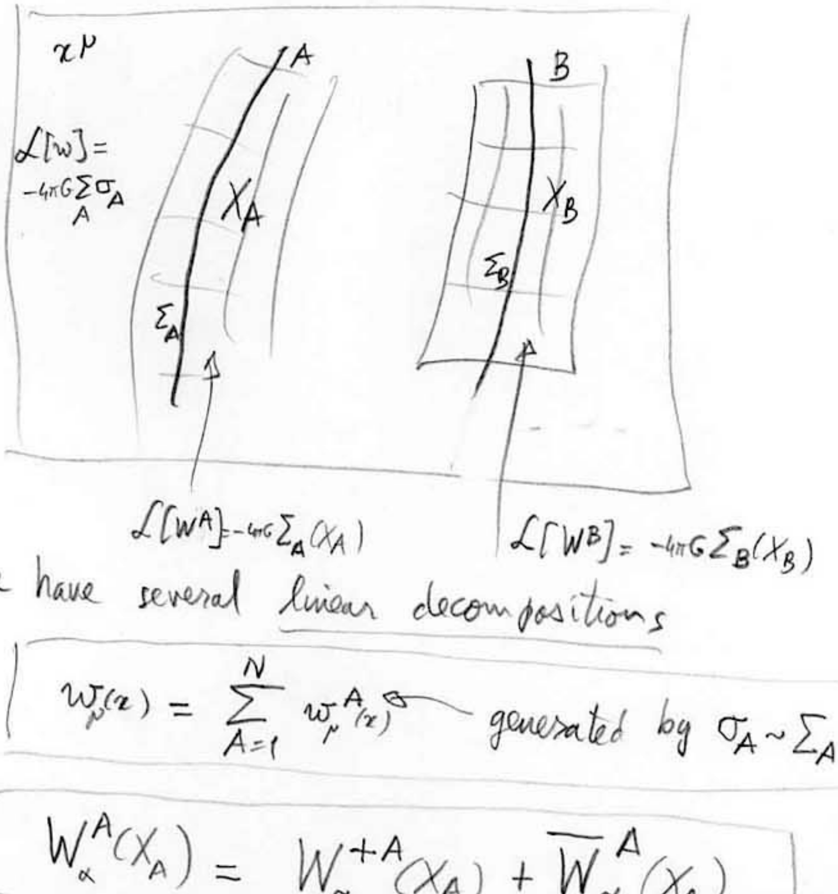
2.5 Structure of PN metric in various charts

In each frame $w_\mu \equiv (w, w_i)$ or $W^A_\alpha \equiv (W^A, W^A_a)$ satisfy some unhomogeneous linear equations

in global frame $\mathcal{L}^\mu[w_\nu] = -4\pi G \sum_{A=1}^{N} \sigma_A{}^\mu(x)$

in A-local frame $\mathcal{L}^\alpha[W^A_\beta] = -4\pi G \, \Sigma^\alpha_A(X_A)$

only one source term present in X_A frame.



Therefore we have several linear decompositions

$$\boxed{w_\mu(x) = \sum_{A=1}^{N} w_\mu{}^A(x) \leftarrow \text{generated by } \sigma_A \sim \Sigma_A}$$

$$\boxed{W^A_\alpha(X_A) = W^{+A}_\alpha(X_A) + \overline{W}^A_\alpha(X_A)}$$

locally generated by Σ_A

unknown homogeneous solution : $\mathcal{L}[\overline{W}^A] = 0$

2.6 Relation between the various linear decompositions

One proves that the transformation $x^\mu = f^\mu(X^\alpha)$ between global x^μ and any local X^α (index A suppressed here) is

$$x^\mu = z^\mu(X^0) + e^\mu_a(X^0)\left[X^a + \frac{1}{c^2}\left(\frac{1}{2}A_a\vec{X}^2 - X^a(\vec{A}\cdot\vec{X})\right)\right] + \eta^\mu$$

some worldline in x^μ chart representing the moving origin of X-frame

some triad of spacetime vectors which depends only on choice of some slowly changing rotation matrix $R^i{}_a(T)$

$$\eta^0 = \frac{1}{c^3}\,\eta(X^0, X^a) = \mathcal{O}(\vec{X}^2)$$
$$\eta^i = \Theta\left(\frac{1}{c^4}\right)$$

(Minkowski acceleration)

$$A_a \equiv \eta_{\mu\nu}\, e^\mu_a \frac{d^2 z^\nu}{d\tau^2}$$

with

$$d\tau^2 = -\eta_{\mu\nu}\, dz^\mu dz^\nu$$

Effect of coordinate transformation $x^\mu = f^\mu(X^\alpha)$:

$$g^{\mu\nu}(x) = \frac{\partial x^\mu(X)}{\partial X^\alpha}\frac{\partial x^\nu(X)}{\partial X^\beta}\, G^{\alpha\beta}(X)$$

non linear in terms of $w_\nu(x)$

non linear coordinate transf

non linear in terms of $W_\alpha(X)$

where $w_\mu = \sum_{A=1}^{N} w^A_\mu$

where $W = W^+ + \overline{W}$

locally generated

(homogeneous rest)

After non trivial analysis, simple links

locally generated pieces

$$\boxed{w^{A}_{\mu}(x) = \mathcal{A}^{A}_{\mu\alpha}(X^{0}_{A})\, W^{+A}_{\alpha}(X)}$$

'externally generated' pieces

$$\boxed{\sum_{B\neq A} w^{B}_{\mu}(x) = \mathcal{A}^{A}_{\mu\alpha}(X^{0}_{A})\, \overline{W}^{A}_{\alpha}(X_A) + \mathcal{B}^{A}_{\mu}(X_A)}$$

explicitly computed in terms of Z^{μ}_{A}, $e_{A}{}^{\mu}_{a}$

can be rewritten as

$$\boxed{\overline{W}^{A}_{\alpha} = \mathcal{A}^{A(-1)}_{\alpha\mu}\Big[\sum_{B\neq A} w^{B}_{\mu} - \mathcal{B}^{A}_{\mu}(X)\Big]}$$

'tidal grav. field in A frame'

generated by far away bodies

additional 'inertial contribution'

closely analogous to Newtonian result:

$$\boxed{\overline{U}^{A} = \sum_{B\neq A} U^{B} - C(t) - \frac{d^{2}\vec{Z}_{A}}{dt^{2}}\cdot\vec{X}_{A}}$$

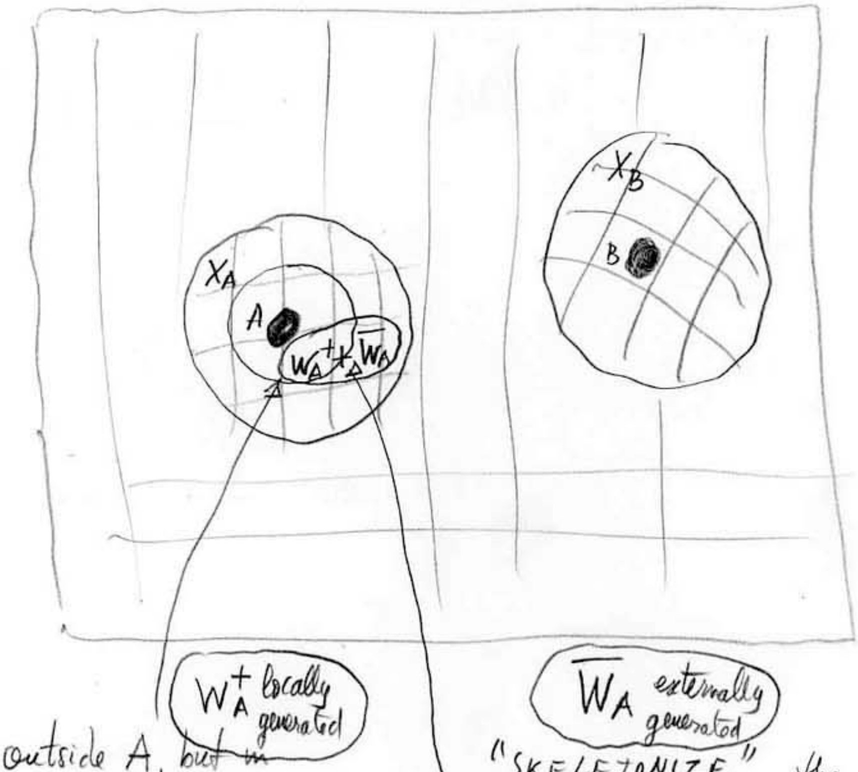
tidal potential

externally generated

inertia effects of accelerated frame

2.7 Introduction of relativistic multipole and tidal moments

Basic idea



outside A, but in
local A frame

SKELETONIZE A

by 1PN-accurate
multipoles

M_L^A

S_L^A

describing

W_A^+

"SKELETONIZE" the
effect of other bodies
+ inertial effects
in local A frame

by expanding

$\overline{W}_A$

in Taylor series in $\vec{X}_A$

relativistic multipole moments (1PN) : BD moments [BD89]

$$M^A_{a_1 a_2 \cdots a_\ell}(T) \equiv \int_A d^3X\, X^{\langle a_1} \cdots X^{a_\ell\rangle}\, \frac{T^{00}_A + T^{ss}_A}{c^2}$$

$$+ \frac{1}{2(2\ell+3)} \frac{1}{c^2} \frac{d^2}{dT^2} \int_A d^3X\, \vec{X}^2 X^{\langle a_1} \cdots X^{a_\ell\rangle} \frac{T^{00}_A}{c^2}$$

$$- \frac{4(2\ell+1)}{(\ell+1)(2\ell+3)c^2} \frac{d}{dT} \int_A d^3X\, X^{\langle b} X^{a_1} \cdots X^{a_\ell\rangle} \frac{T^{0b}_A}{c}$$

everything in the A-frame

$$S^A_{a_1 \cdots a_\ell}(T) \equiv \int_A d^3X\, \varepsilon_{bc\langle a_\ell} X_{a_1} \cdots X_{a_{\ell-1}\rangle} X_b \frac{T^{0c}_A}{c}$$

relativistic tidal moments (1PN)

gauge-invariant gravito-electric and -magnetic "tidal fields"

$$\bar{E}^A_a(X_A) = \partial_a \bar{W}^A + \frac{4}{c^2} \partial_T \bar{W}^A_a$$

$$\bar{B}^A_a(X_A) = \varepsilon_{abc}\, \partial_b (-4 \bar{W}^A_c)$$

externally generated, as seen in X_A frame

'electric-' and 'magnetic-type' relativistic tidal moments

$$G^A_{a_1 \cdots a_\ell}(T) \equiv \partial_{\langle a_1 a_2 \cdots a_{\ell-1}} \bar{E}^A_{a_\ell\rangle} \Big|_{\vec{X}_A = 0}$$

$$H^A_{a_1 \cdots a_\ell}(T) \equiv \partial_{\langle a_1 \cdots a_{\ell-1}} \bar{B}^A_{a_\ell\rangle} \Big|_{\vec{X}_A = 0}$$

- Expression of local gravitational field in terms of multipole and tidal moments

$$\boxed{\begin{aligned}
\overset{+}{W}{}^{A}(T,\vec{X})^{\text{outside}\,A} &= G\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_L\left(\frac{M^A_L(T\overset{\text{half-sum}}{\mp}R/c)}{R}\right)+\frac{1}{c^2}\partial_T(\Lambda^A-\lambda)+O(4)\\
\overset{+}{W}{}^{A}_a(T,\vec{X}) &= -G\sum_{l\geq 1}\frac{(-)^l}{l!}\left[\partial_{L-1}\left(\frac{\dot{M}^A_{aL-1}}{R}\right)+\frac{l}{l+1}\varepsilon_{abc}\partial_{bL-1}\left(\frac{S^A_{cL-1}}{R}\right)\right]\\
&\quad -\frac{1}{4}\partial_a(\Lambda^A-\lambda)+O(2)
\end{aligned}}$$

$$\boxed{\begin{aligned}
\overline{W}{}^{A}(T,\vec{X}) &= \sum_l\frac{1}{l!}\left[\hat{X}^L G^A_L(T)+\frac{1}{2(2l+3)c^2}\vec{X}^2\hat{X}^L\ddot{G}^A_L(T)\right]+\frac{1}{c^2}\partial_T\overline{\Lambda}{}^A+O(4)\\
\overline{W}{}^{A}_a(T,\vec{X}) &= \sum_l\frac{1}{l!}\left[-\frac{2l+1}{(l+1)(2l+3)}\hat{X}^{aL}\dot{G}^A_L+\frac{l}{4(l+1)}\varepsilon_{abc}\hat{X}^{bL-1}H^A_{cL-1}\right]\\
&\quad -\frac{1}{4}\partial_a\overline{\Lambda}{}^A+O(2)
\end{aligned}}$$

$\boxed{2.8}$ Relativistic tidally expanded equations of motion

d'Alembert-type approach

Local evolution equation for matter variables

$$0 = \nabla_\beta T^{\alpha\beta} = \frac{\partial}{\partial x^\beta} T^{\alpha\beta} + \Gamma^\alpha_{\sigma\beta} T^{\sigma\beta} + \Gamma^\beta_{\sigma\beta} T^{\alpha\sigma}$$

$$\boxed{\begin{aligned} &\partial_T \Sigma + \partial_{X^a} \Sigma^a = \frac{1}{c^2}\partial_T T^{ss} - \frac{1}{c^2}\Sigma\,\partial_T W + O(4) \\ &\partial_T\left[\left(1+\frac{4W}{c^2}\right)\Sigma^a\right] + \partial_{X^b}\left[\left(1+\frac{4W}{c^2}\right)T^{ab}\right] = \mathcal{F}^a(T,\vec{X}) + O(4) \end{aligned}}$$

$$\boxed{\mathcal{F}^a(T,\vec{X}) = \Sigma E_a + \frac{1}{c^2} B_{ab}\Sigma^b = \left(\Sigma\vec{E} + \frac{1}{c^2}\vec{\Sigma}\times\vec{B}\right)_a}$$

TOTAL $\vec{E}$ and $\vec{B}$ fields $E^+ + \bar{E}$ …

remarkable Lorentz form of relativistic force density

Note:

LINEAR in W

$$\boxed{\mathcal{F}^a = \mathcal{F}^a[W^{tot}_\alpha] = \mathcal{F}^a[\overset{+}{W}] + \mathcal{F}^a[\bar{W}]}$$

'self-force density' $\overset{+}{\mathcal{F}}{}^a$

'external force density' $\bar{\mathcal{F}}^a$

AGR 2.20

Consider evolution of $l=0$ and $l=1$ BD multipoles

$$\begin{cases} \dfrac{d}{dT}\overset{\ell=0}{M^A}(T) = F_0^A[W] \equiv -\frac{1}{c^2}\int_A d^3X\, \Sigma\, \partial_T W - \frac{1}{c^2}\frac{d}{dT}\int_A d^3X\, \Sigma\, X^b \partial_b W \\ \dfrac{d^2}{dT^2} M_a^A(T) = F_a^A[W] = \int_A d^3X\, \mathcal{F}^a[W] - \frac{1}{c^2}\frac{d}{dT}\int d^3X\,(4W\Sigma_+^a X^a \Sigma \partial_T W) \\ \qquad - \frac{1}{c^2}\frac{d^2}{dT^2}\int_A d^3X\,(X^a X^b - \frac{1}{2}\vec{X}^2\delta^{ab})\Sigma\,\partial_b W \end{cases}$$

again RHS's are <u>linear</u> in W

$$\Rightarrow \quad F_\alpha^A[W^A] = \overset{\overset{+}{W}}{F_\alpha^{+A}} + \overset{\bar{W}}{\bar{F}_\alpha^A}$$

Theorem ('1PN Action and Reaction'): $\boxed{F_\alpha^{+A} = 0 \mod \mathcal{O}(4)}$

$$\Rightarrow \quad F_\alpha^A[W] = F_\alpha^A[\underset{\substack{\uparrow \\ \text{insert tidal expansion}}}{\bar{W}}]$$

$\swarrow$

complicated intermediate calculations
but nice simplifications allows one to express
the final result only in terms of <u>multipole</u> and
<u>tidal</u> moments

$$\boxed{\begin{aligned} \frac{d M^A}{d T_A} &= \bar{F}_0^A \left[M_L^A , G_L^A \right] + O(4) \\ \frac{d^2 M_a^A}{d T_A^2} &= \bar{F}_a^A \left[M_L^A , S_L^A , G_L^A , H_L^A \right] + O(4) \end{aligned}}$$

where, for instance,

$$\boxed{\bar{F}_a = \sum_\ell \frac{1}{\ell !} \left\{ M_L \, G_{aL} + \frac{1}{c^2} \frac{\ell}{\ell+1} \, S_L \, H_{aL} + \text{seven other } {}^1\!/\!{}_{c^2} \text{ terms} \right\}}$$

"Newtonian-like" term ↑   explicit 1PN corrections ↑

NB: $\bar{F}_0 = -\frac{1}{c^2} \sum_\ell \frac{1}{\ell !} \left\{ (\ell+1) M_L \dot{G}_L + \ell \dot{M}_L G_L \right\}$ is $\neq 0$ in general

$\boxed{2.9}$ Translational equations of motion à la d'Alembert

① Attach the spatial origin of A-frame to body A by requiring $\boxed{M_a^A(T) = 0}$ ← specific way of defining a relativistic center of mass for A

② Write the consequence of center of mass definition

$$\frac{d^2 M^A_a}{dT^2} = 0 = \bar{F}_a [M^A_L, S^A_L, G^A_L, H^A_L]$$

i.e.
$$0 = M^A G^A_a + \frac{1}{2!} M^A_{bc} G^A_{abc} + \dots + \frac{1}{l!} M^A_L G^A_{aL}$$

$$+ \frac{1}{c^2} \sum_l \frac{1}{l!} \frac{l}{l+1} S_L H_{aL} + \text{seven other } \frac{1}{c^2} f(MSGH)$$

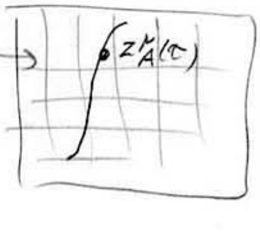
'tidal dipole'
$$\boxed{G^A_a = \sum_{B \neq A} G^{B/A}_a - A^A_a}$$

contributions from other bodies

$$\boxed{A^A_a = \eta_{\mu\nu} \, e^\mu_{Aa} \frac{d^2 z^\nu_A}{d\tau^2}}$$

Minkowski acceleration of z^μ_A worldline

x^μ global frame →



⇒ yields translational equations of motion for body A

$$\boxed{\begin{aligned} M^A A^A_a = \sum_{B \neq A} \Big(M^A G^{B/A}_a [M^B_K, S^B_K] + \sum_{l \geq 2} \frac{1}{l!} M^A_L \, G^{B/A}_{aL} [M^B_K, S^B_K] \Big) \\ + \frac{1}{c^2} F^{(1PN)}_a \big(M^A_L, S^A_L ; G^A_{L'} [M^B_K, S^B_K], H^A_{L'} [M^B_K, S^B_K] \big) + O(4) \end{aligned}}$$

RHS can be fully expressed in terms of <u>multipole</u> moments

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$\boxed{2.10}$ Rotational equations of motion

Similarly, with extra work (DSX3: PRD 47, 3124 (1993)) one can derive a 1PN-accurate law of evolution for a certain 1PN-accurate 'spin vector' of body A.

$$\boxed{\frac{d}{dT_A} S_a^{A(1PN)} = \overline{L}_a^A [M_L^A, S_L^A, G_L^A, H_L^A] + \mathcal{O}(4)}$$

$$\overset{(1PN)}{S_a^A} = S_a^{\substack{BD\\ \text{usual}}}[\Sigma_A^\alpha] + \frac{1}{c^2}\overline{S}_a[\overline{E}, \overline{B}]$$

tidal fields

$$\sum_l \frac{1}{l!}\left[\varepsilon_{abc} M_{bL}^A G_{cL}^A + \frac{1}{c^2}\frac{l+1}{l+2}\varepsilon_{abc} S_{bL}^A H_{cL}^A\right]$$

Newtonian torque

$\frac{1}{c^2}$ corrections

2.11 Applications of DSX formalism

- Allows one to control / correct the application of GR in, e.g., the relativistic description of the Earth environment: tests of GR on Earth, satellite motion around the Earth (DSX4: PRD 49, 618 (1994))

- Allows one to describe, with relativistic accuracy, the use of modern technologies in solar-system: VLBI, laser tracking of Moon etc ...

- Allows one to derive 1PN accurate equations of motion for solar-system, including all effects of higher multipoles (notably spin and quadrupole, which are the dominant corrections)

2.12 Application to a derivation of the Lorentz-Droste-Einstein-Infeld-Hoffmann equations of motion for N monopolar bodies

truncation to a 'monopolar model' for each body:

i.e. $\forall A=1,\dots,N$: $\ell \geqslant 1 \Rightarrow M^A_L = 0 = S^A_L$

i.e. keep only mass monopole $\ell = 0$ $\boxed{M^A \neq 0}$

- The general law $$\frac{dM^A}{dT} = -\frac{1}{c^2}\sum_{\ell} \frac{1}{\ell!}\left\{(\ell+1)\, M_L \dot{G}_L + \ell \dot{M}_L G_L\right\}$$

$$= -\frac{1}{c^2} M^A \dot{G}^A$$

actually 'monopole tidal moment' $\overset{A}{G}(T) = \overline{W}_A(T,\vec{0})$ $\swarrow \vec{X}_A = 0$

Fix freedom in definition of local time T_A by requiring

$$\overline{W}_A(T,\vec{0}) = 0 \quad \rightarrow \quad G^A(T) = 0$$

$$\Rightarrow \qquad \boxed{\frac{dM^A}{dT} = 0}$$

Fix origin of A frame to be the 1PN 'center of mass'

i.e. $$\boxed{M^A_a(T) = 0}$$

$$\Rightarrow 0 = \frac{d^2 M^A_a}{dT^2} = M^A G^A_a + \underset{\substack{\downarrow \\ 0}}{\frac{1}{2!} M^A_{bc} G^A_{abc}} + \dots + \underset{\substack{\downarrow \\ 0}}{\frac{1}{c^2} \sum_\ell \frac{1}{\ell!} \frac{\ell}{\ell+1} S_L H_{aL}} + \dots$$

$$\Rightarrow \qquad \boxed{G^A_a = 0}$$

i.e. $$\boxed{\overline{E}_a^A \Big|_{\vec{X}_A = 0} = 0}$$

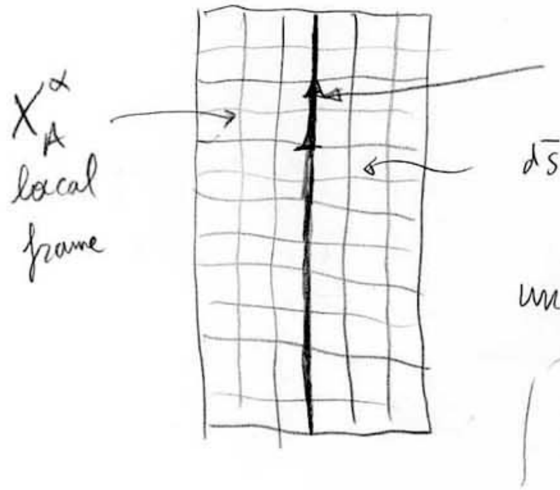
Let us define the following metric around body A

$$\boxed{\begin{aligned} \overline{G}^A_{00}(T,\vec{X}) &\equiv -e^{-\frac{2}{c^2}\overline{W}^A} \\ \overline{G}^A_{0a}(T,\vec{X}) &\equiv -\frac{4}{c^3}\overline{W}^A_a \\ \overline{G}^A_{ab}(T,\vec{X}) &\equiv \delta_{ab}\, e^{+\frac{2}{c^2}\overline{W}^A} \end{aligned}} \neq \left\{ \begin{aligned} G_{00} &= -e^{-\frac{2}{c^2}W} \\ G_{0a} &= -\frac{4}{c^3} W_a \\ G_{ab} &= \delta_{ab}\, e^{\frac{2}{c^2}W} \end{aligned} \right.$$

↑ defines an 'external metric' obtained by discarding the self-terms $\overset{+}{W}{}^A$ in the exponential parametrization

↑ real metric, with $W = \overset{+}{W}{}^A + \overline{W}^A$

Then consider the A worldline within metric $\bar G_{\alpha\beta}$



unit tangent vector

$$\boxed{\bar u_A^\alpha = e^{\bar W/c^2}\frac{\partial}{\partial T} = \left(e^{\bar W/c^2}, 0, 0, 0\right)}$$

$$\bar G^A_{\alpha\beta}\, \bar u_A^\alpha\, u_A^\beta = -1$$

coordinate vectorial basis: $\epsilon_0 \equiv \frac{1\partial}{c\partial T_A} \leftrightarrow \bar u_A = c e^{\bar W/c^2}\epsilon_0$

$$\epsilon_a \equiv \frac{\partial}{\partial X_A^a}$$

simple calculation

$$\boxed{\bar G_A\left(\epsilon_a, \bar\nabla_{\bar u_A}\bar u_A\right) \equiv \bar G^A_{\alpha\beta}\, \epsilon_a^\alpha\, \bar u_A^\sigma\, \overset{\text{with } \bar\Gamma_A}{\bar\nabla_\sigma}\, \bar u_A^\beta = -\left.\bar E^A_a\right|_{\vec X = 0}}$$

Therefore

$$G^A_a = 0 \rightarrow \bar E^A_a = 0 \rightarrow \boxed{\bar\nabla_{\bar u_A}\bar u_A = 0}$$

i.e. the A worldline is a geodesic of the A-external metric $\boxed{d\bar s^2_A = \bar G^A_{\alpha\beta}(X)\, dX^\alpha dX^\beta}$

by transforming to global coordinates $X^{\alpha}_{A} \rightarrow x^{\mu}$ one concludes that the A worldline is a geodesic of the following global-coords A-external metric

$$d\bar{s}^2 = \bar{g}^{A}_{\mu\nu}(x^{\lambda})\, dx^{\mu} dx^{\nu}$$ where

$$\begin{cases} \bar{g}^{A}_{00}(x) = -\exp\left(-\frac{2}{c^2} \sum_{B \neq A} w^{B}(x)\right) \\ \bar{g}^{A}_{0i}(x) = -\frac{4}{c^3} \sum_{B \neq A} w^{B}_{i}(x) \\ \bar{g}^{A}_{ij}(x) = \delta_{ij} \exp\left(\frac{2}{c^2} \sum_{B \neq A} w^{B}(x)\right) \end{cases}$$

with some extra work (using $w^{A}_{\mu} = \mathcal{A}^{A}_{\mu\alpha} \overset{+}{W}{}^{A}_{\alpha}$)

one can express $w^{B}(x)$ in terms of M_B and x-coordinates.

Finally one checks that the 1PN eqs of motion $\frac{d^2 z^i_A}{dt^2} = A^i_{A0} + \frac{1}{c^2} A^i_{A2}$ can be derived from the Lagrangian (Lorentz Droste '17; EIH '38)

$$\boxed{\begin{aligned} L(\vec{z}_A, \vec{v}_A) = {} & \sum_A \frac{1}{2} M_A \vec{v}_A^{\,2} + \frac{1}{2} \sum_{A \neq B} \frac{G M_A M_B}{r_{AB}} + \frac{1}{8c^2} \sum M_A \vec{v}_A^{\,4} \\ & + \frac{3}{2c^2} \sum_{A \neq B} \frac{G M_A M_B \vec{v}_A^{\,2}}{r_{AB}} - \frac{1}{4c^2} \sum_{A \neq B} \frac{G M_A M_B}{r_{AB}} \left[7 v_A \cdot v_B + (\vec{n}_{AB} \cdot \vec{v}_A)(\vec{n}_{AB} \cdot \vec{v}_B) \right] \\ & - \sum_A \sum_{B \neq A} \sum_{C \neq A} \frac{1}{2c^2} \frac{G^2 M_A M_B M_C}{r_{AB}\, r_{AC}} \end{aligned}}$$