MOTION OF STRONGLY SELF-GRAVITATING BODIES

Some relevant references

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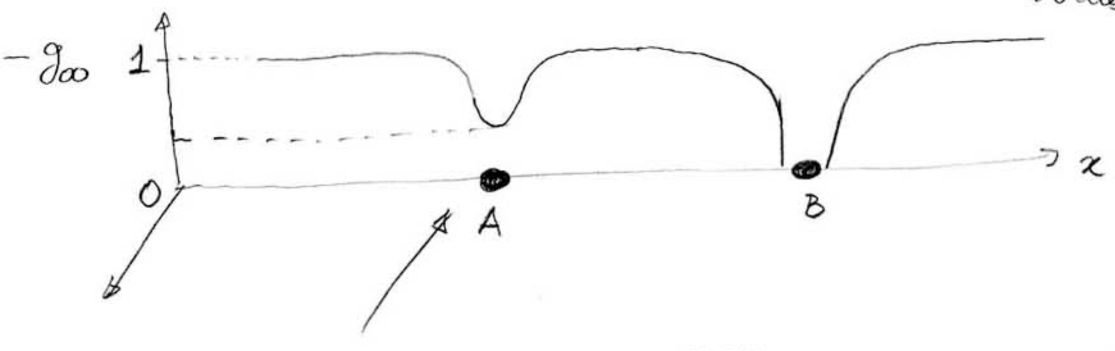
$\boxed{3.1}$ Matched Multi-chart approach

Problem with one-chart approach for strong self-gravity bodies



on surface of A : $-g_{00} \simeq 1 - \dfrac{2GM_A}{c^2 R_A}$

$$\boxed{\frac{GM_\odot}{c^2} \simeq 1.5\,\mathrm{km}}$$

for Neutron Star $\quad \dfrac{2G\overset{\simeq 1.5 M_\odot}{M_A}}{c^2 \underset{\simeq 10\,\mathrm{km}}{R_A}} \simeq \dfrac{2 \times 1.5 \times 1.5\,\mathrm{km}}{10\,\mathrm{km}} \simeq 0.45$

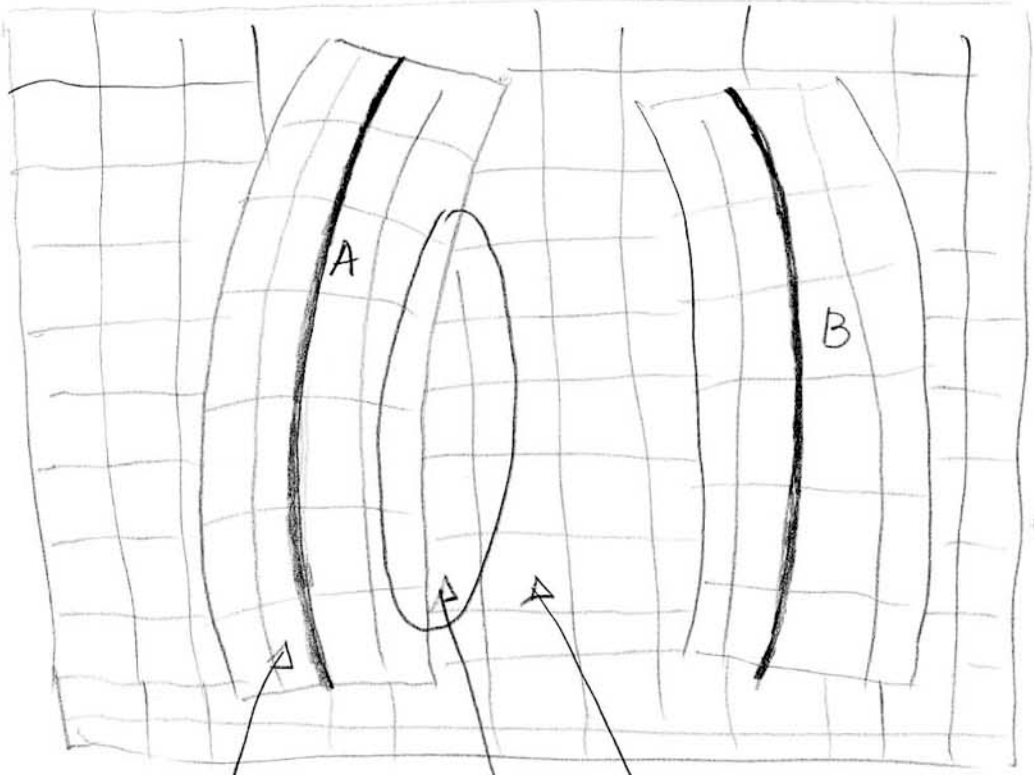
! comparable to 1

for Black Hole : by definition $\dfrac{2GM_A}{c^2 R_A} = 1$

Not (a priori) justified to use perturbation theory

$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}$, $h_{\mu\nu} \ll 1$ in <u>one chart x^μ</u>

$\Rightarrow$ Need to use
- multi-chart
- multiple expansions
- relate the various expansions



LOCAL CHART OF BODY A

X_A^α

perturbation of the metric of an isolated body

$$G_{\alpha\beta}(X_A^\gamma) = G^{(0)}_{\alpha\beta}(X_A; m_A) + H^{(1)}_{\alpha\beta}(X; m_A, m_B) + \dots$$

exact isolated A body

perturbation due to far away bodies + inertial effects

INTERNAL PERTURBATION SCHEME

GLOBAL CHART

x^μ

usual (PN or PM)

weak-field expansion

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + h^{(3)}_{\mu\nu} + \dots$$

EXTERNAL PERTURBATION SCHEME

'MATCHING' OF TWO EXPANSIONS IN INTERMEDIATE DOMAIN

$$\frac{Gm_A}{c^2} \lesssim R_A \ll |x - z_A| \ll d$$

3.2 Matching

Like in 1PN multi-chart approach, relate the coordinate systems by

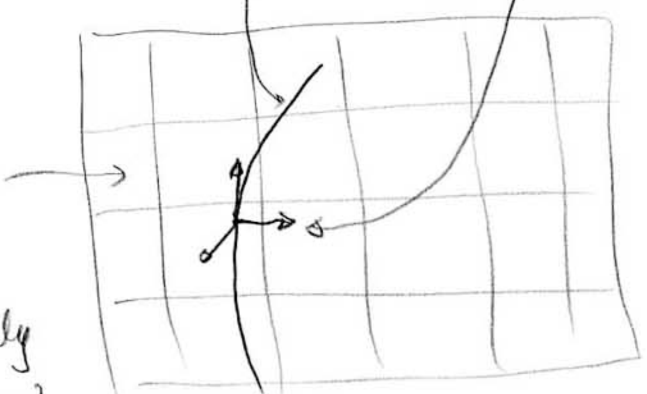
$$\boxed{x^\mu(X^\alpha_A) = z^\mu_A(T_A) + e^\mu_a(T_A)\,X^a_A + \tfrac{1}{2} f^\mu_{ab}(T_A)\,X^a_A X^b_A + \dots}$$

defines a worldline in x^μ chart

defines a spatial frame in x^μ chart

Taylor expansion in $\left(\frac{|\vec{X}_A|}{d}\right)^n$

d: distance to other bodies



x^μ global formally continued inside local A region

MATCHING CONDITION

$$\boxed{g^{\mu\nu}(x(X_A)) = \frac{\partial x^\mu}{\partial X^\alpha_A}\,\frac{\partial x^\nu}{\partial X^\beta_A}\; G^{\alpha\beta}(X_A)}$$

external expansion in intermediate domain $R_A \ll |\vec{x} - \vec{z}_A| \ll d$

expansion $z + eX + \dots$

internal expansion

3.3 Internal perturbation scheme

For simplicity, consider case where $\boxed{A = \text{NON-ROTATING Black Hole}}$

Schwarzschild mass of isolated body

$$G_{\alpha\beta}(\vec{X},T) = G^{(0)}_{\alpha\beta}(X; m) + H^{(1)}_{\alpha\beta} + \cdots$$

Schwarzschild

linear perturbation of Schwarzschild

Regge-Wheeler-Zerilli formalism

PARITY: even or odd

$$\boxed{H^{(1)}_{\alpha\beta}(\vec{X},T) = \sum_{l\geq 0} \sum_{m=-l}^{+l} \sum_{I=1}^{10} \sum_{\pi} H^{(I,\pi)}_{lm}(\hat{R},T)\; Y^{lm(I,\pi)}_{\alpha\beta}(\theta,\phi)}$$

radial (+time) functions

with $\boxed{\hat{R} \equiv \dfrac{R}{(Gm/c^2)}}$

basis of 10 tensorial spherical harmonics (in general gauge, or $\underline{\underline{6}}$ in special gauge)

Result of Regge-Wheeler-Zerilli:

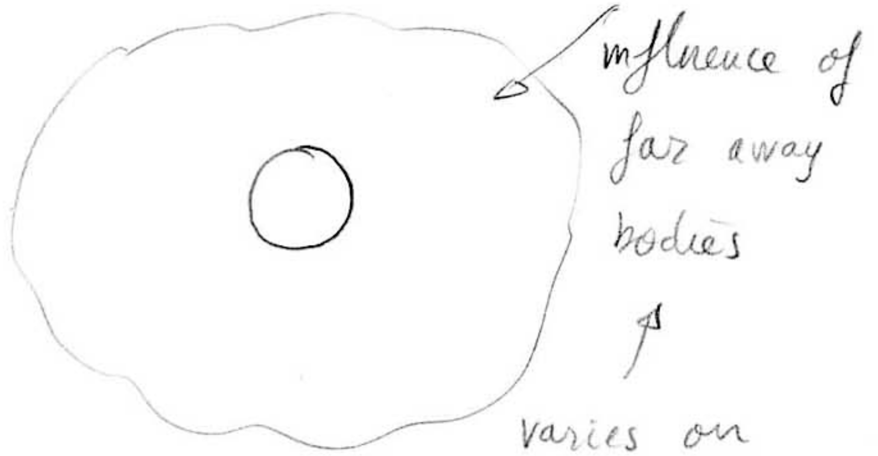
the terms $\boxed{l = 0 \text{ or } l = 1 \text{ can be gauged away}}$

strong-field analog of fixing local frame in DSX formalism by setting to zero the first tidal moments

$$\begin{cases} G(T) = 0 & \leftarrow \text{fixing } T \text{ scale} \\ G_a(T) = 0 & \leftarrow \text{fixing motion of origin of } A \text{ frame} \\ H_a(T) = 0 & \leftarrow \text{fixing rotation of } A \text{ frame} \end{cases}$$

Leading perturbation: $\ell = 2$, $\pi =$ even parity

Idea:



SLOW SCALE compared to $\frac{Gm}{c^3}$

Leading effect: neglect $\boxed{\frac{\partial^2}{\partial T^2} \ll \partial_{x^a}}$

$\Rightarrow$ stationary (even) perturbations of Schwarzschild

Regge-Wheeler '57, Zerilli '70, Edelstein Vishveshwara '70

reduced to a decoupled eq. for one metric component: H

$$\hat{R}(\hat{R}-2)\frac{d^2}{d\hat{R}^2}\left(\frac{H}{\hat{R}(\hat{R}-2)}\right) + 3(2\hat{R}-2)\frac{d}{d\hat{R}}\left(\frac{H}{\hat{R}(\hat{R}-2)}\right)$$

$$- (\ell-2)(\ell+3)\frac{H}{\hat{R}(\hat{R}-2)} = 0$$

simplifies for $\ell = 2$

General solution for $l=2$

$$\boxed{H_{lm} = D_{lm}\left[\hat{R}(\hat{R}-2) + k_l \hat{R}(\hat{R}-2)\int_{\hat{R}}^{\infty} \frac{5\,dx}{x^3(x-2)^3}\right]}$$

$\uparrow \boxed{l=2}$

two arbitrary constants: $D(T)$, k

$D(T)$: slowly varying

determined by going to the intermediate zone

$$1 \ll \hat{R} \ll d$$

$$\boxed{\hat{R} \equiv \frac{R}{Gm/c^2}}$$

$$\Downarrow$$

$$H_{lm} = D_{lm}\left[\hat{R}^2 - 2\hat{R} + \frac{k}{\hat{R}^3} + O\left(\frac{1}{\hat{R}^4}\right)\right]$$

grows like $\hat{R}^2$ as $\hat{R} \gg 1$

MATCHING

to $l=2$ tide in metric

in DSK: $\overline{W} = \frac{1}{2} X^{\langle ij \rangle} G^A_{ij}(T) + \cdots$

$l=2$ external tidal multipole

k: fixed by the condition that the perturbation be regular 'within body A'

in Black Hole case regularity at HORIZON $\hat{R}=2$

$$\Rightarrow \boxed{k_{BH} = 0}$$

for a Neutron Star one will have

$$\boxed{k_{NS} = \text{function of eq of state} = O(1)}$$

k depends on the internal structure of body A

3.4 'Effacement of internal structure' AGR 3.7

Matching $\rightarrow$ $$\boxed{\frac{D^A_{\ell m}}{(Gm^A/c^2)^2} \sim G^A_{<ij>} \simeq \partial_{ij}\frac{\overline{U}}{c^2} \sim \sum_{B\neq A}\partial_{ij}\left(\frac{Gm_B}{c^2|\vec{z}_A-\vec{z}_B|}\right)}$$

$\Rightarrow$ The internal perturbation $H^{(1)}_{\alpha\beta}$ contains two types of contributions

(1) one $\propto D_{\ell m}\,\hat{R}(\hat{R}-2)$ which does not depend on the internal structure of A (and which is determined by matching)

(2) another one $\propto D_{\ell m}\,k_\ell\,\hat{R}(\hat{R}-2)\displaystyle\int_{\hat{R}}^{\infty}\frac{5\,dx}{x^3(x-2)^3} \sim D_{\ell m}\,k_\ell\,\frac{1}{\hat{R}^3}$
which depends on the internal structure of A (BH or NS ...)

$$H^{\text{structure dependent}}_{\substack{\ell m \\ \ell=2}} \underset{\hat{R}\gg 1}{\simeq} k_\ell \frac{D_{\ell m}}{\hat{R}^3} \sim \underset{\ell=2}{k_\ell}\, G^A_{ij}\frac{\partial}{\partial\hat{X}^i}\frac{\partial}{\partial\hat{X}^j}\frac{1}{\hat{R}}\times\left(\frac{Gm^A}{c^2}\right)^2$$

after matching to global coordinate this is

because of $\hat{R} = \dfrac{R}{\left(\frac{Gm^A}{c^2}\right)}$

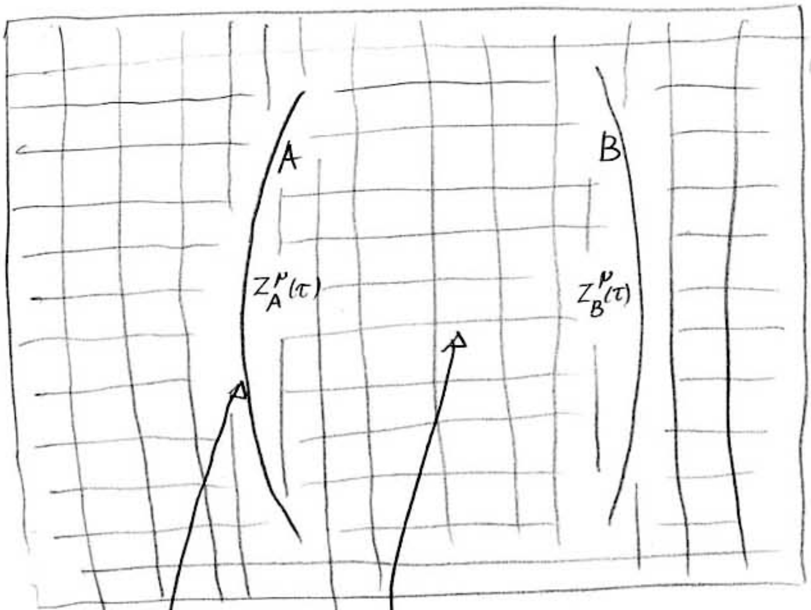
$$\boxed{\delta^{\text{structure dependent}} g_{\mu\nu} \sim k_2\,\partial_{ij}\frac{\overline{U}}{c^2}\;\partial_{ij}\frac{1}{|\vec{x}-\vec{z}_A|}\times\left(\frac{Gm^A}{c^2}\right)^5}$$

EFFACEMENT OF INTERNAL STRUCTURE MODULO $\dfrac{1}{c^{10}} \sim \left(\dfrac{v}{c}\right)^{10}$ beyond Newton

5PN

3.5 External perturbation scheme

Below 5PN we have



$$R_{\mu\nu}(g) = 0$$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \cdots$$

POST MINKOWSKIAN EXPANSION in powers of G

near each worldline $|x - z_A| \to 0$ (i.e. $|x - z_A| \ll d$)
one has some boundary conditions deduced from

$$g^{\mu\nu}(x(X_A)) = \frac{\partial x^\mu}{\partial X_A^\alpha}\frac{\partial x^\nu}{\partial X_A^\beta}\, G^{\alpha\beta}(X_A)$$

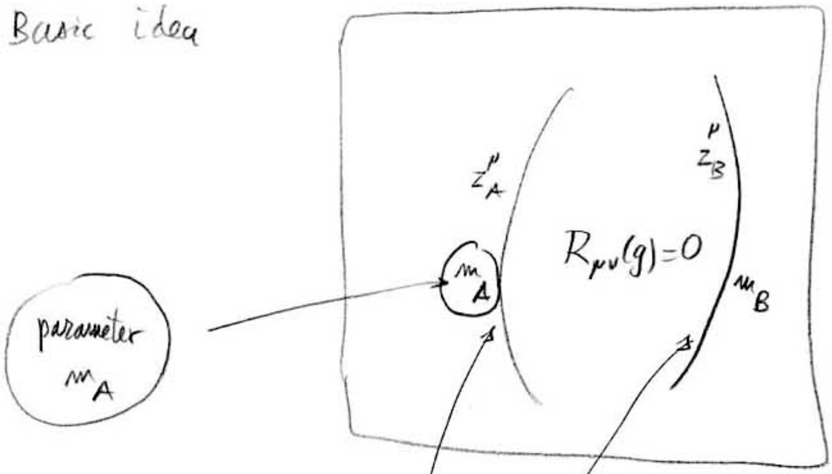
$$= G^{\alpha\beta}_{\text{Schwarzschild}} + \sum_{\ell \geq 2} \mathcal{O}(G_L \hat{X}^L)$$

insert $x^\mu = z_A^\mu(\tau_A) + e^\mu_{Aa} X_A^a + \frac{1}{2} f^\mu_{ab} X_A^a X_A^b + \cdots$

UNIQUELY DETERMINES THE MOST SINGULAR $\sim \frac{m_A^n}{|\vec{x} - \vec{z}_A|^n}$ TERMS AT EACH ORDER G^n IN POST-MINKOWSKIAN EXPANSION

3.6 'Skeletonization'

Basic idea



'Dominantly Schwarzschild' boundary conditions near worldlines

$\Rightarrow$ sufficient (at least at 3PM) to prove UNIQUENESS OF SOLUTION $g_{\mu\nu}(x^\lambda; m_A, m_B, z_A^\mu, z_B^\mu)$

$\Rightarrow$ it is enough to find ONE SPECIFIC METHOD of finding a solution $g_{\mu\nu}$ satisfying the required boundary conditions (including 'no incoming radiation condition')

$\Rightarrow$ construct a solution by using ANALYTIC CONTINUATION in some complex parameter

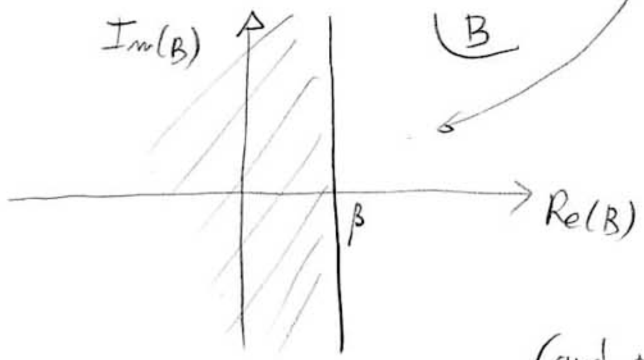
3.7 Analytic continuation

Let $F(x)$ be such that it has at most power-law singularity as $x \to 0$; i.e. $\exists \beta$: $x^\beta F(x) = \mathcal{O}(1)$ as $x \to 0$
and decays fast as $x \to \infty$ $\qquad \beta \in \mathbb{R}$

Consider function

$$\boxed{B \in \mathbb{C} \longrightarrow I_F(B) \equiv \int_0^{+\infty} x^B F(x)\, dx}$$

Complex B plane



integral converges if $Re(B) > \beta$
(and possibly $Re(B) < \gamma$)

$$\frac{d}{dB} I_F(B) = \int_0^\infty x^B \log x \, F(x)\, dx$$ also converges

$\Rightarrow$ $\boxed{I_F(B) \text{ is an analytic function of } B \in \mathbb{C},\ Re(B) > \beta}$

Assume: $$F(x) = \sum_{\substack{\text{finite} \\ -\alpha_i < \beta}} c_{\alpha_i} x^{\alpha_i} + x^N G_N(x)$$

where $G_N(x) = \mathcal{O}(1)$ as $x \to 0$

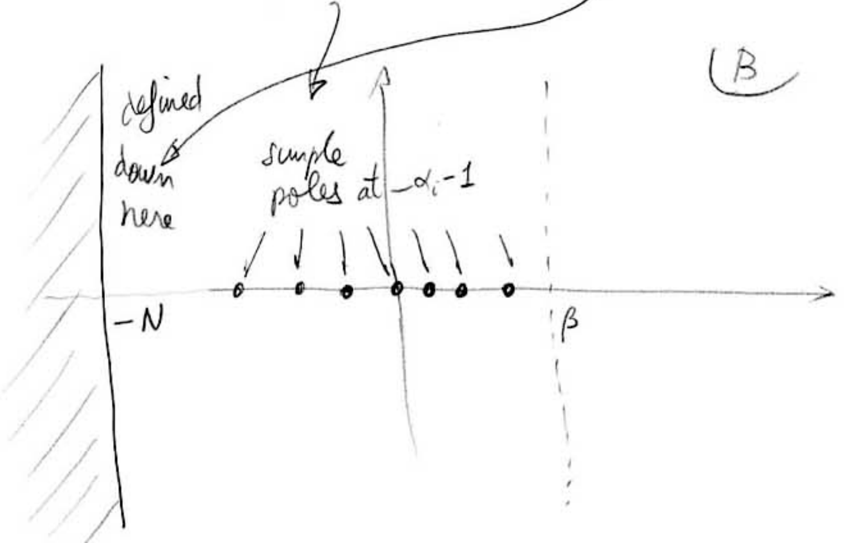
for $\mathrm{Re}(B) > \beta$

$$I_F(B) = \int_0^1 dx\, x^B F(x) + \int_1^\infty dx\, x^B F(x)$$

$$= \int_0^1 dx\, x^B \left(\sum c_{\alpha_i} x^{\alpha_i} + x^N G_N(x) \right) + \int_1^\infty dx\, x^B F(x)$$

$$= \sum c_{\alpha_i} \left[\frac{x^{B+\alpha_i+1}}{B+\alpha_i+1} \right]_0^1 + \int_0^1 dx\, x^{B+N} G_N(x) + \int_1^\infty dx\, x^B F(x)$$

$$= \sum \frac{c_{\alpha_i}}{B+\alpha_i+1} + \int_0^1 dx\, x^{B+N} G_N(x) + \int_1^\infty dx\, x^B F(x)$$



well define if $F(x)$ as very fast decay

Thm: <u>UNIQUENESS</u> of such an analytic continuation

$\Rightarrow$ $\boxed{\text{Uniquely defines } I_F(B) \text{ as a meromorphic function (with simple poles at } -\alpha_i - 1) \text{ all over } \mathbb{C}}$

3.8 Useful properties of analytic continuation

- $$\boxed{AC\left[\int_0^\infty dx\, x^B\,(F(x)+G(x))\right] = AC\left[\int_0^\infty dx\, x^B F(x)\right] + AC\left[\int_0^\infty dx\, x^B G(x)\right]}$$

- integrating by parts

in original domain

$$\int_0^\infty dx\, x^B F(x)\, dx = \left[\frac{x^{B+1}}{B+1} F(x)\right]_0^\infty - \frac{1}{B+1}\int_0^\infty dx\, x^{B+1} F'(x)\, dx$$

vanishes in original domain

$\Rightarrow$ vanishes always! (even when divergent!)

$$\Rightarrow \boxed{\begin{aligned} &\int_0^\infty dx\, x^B F(x)\, dx = -\frac{1}{B+1}\int_0^\infty dx\, x^{B+1} F'(x)\, dx \\ \text{or} \quad &\int_0^\infty dx\, x^B F'(x) = -B\int_0^\infty dx\, x^{B-1} F(x) \end{aligned}}$$

more generally: continuation of any (analytic) identities valid for B in some domain.

AGR 3.13

$\boxed{3.9}$ 'Dimensional regularization' ('t Hooft Veltman '72)

i.e. analytic continuation in the dimension of space d

Consider

$$\boxed{S = \int \frac{d^{d+1}x}{c} \sqrt{g} \, \frac{c^4}{16\pi G} R(g) - \sum_A m_A c \int \sqrt{-g_{\mu\nu}(z_A^\lambda) dz_A^\mu dz_A^\nu}}$$

POINT MASSES

or

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} \sum_A T^A_{\mu\nu}(x)$$

$$\boxed{T_A^{\mu\nu}(x) = \int m_A c^2 \frac{ds_A}{\sqrt{g(z_A)}} \frac{dz_A^\mu}{ds_A} \frac{dz_A^\nu}{ds_A} \delta^{d+1}(x^\lambda - z_A^\lambda(s_A))}$$

$$ds_A = \sqrt{-g_{\mu\nu}(z_A^\lambda) dz_A^\mu dz_A^\nu}$$

or

$$\boxed{R_{\mu\nu} = \frac{8\pi G}{c^4} \sum_A \left(T^A_{\mu\nu}(x) - \frac{1}{d-1} g_{\mu\nu} T^A(x) \right)}$$

d-independent in harmonic gauge

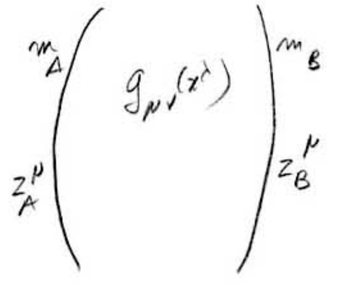
$$g^{\mu\nu} \Gamma^\lambda_{\mu\nu} = 0 .$$

introduces some explicit d-dependence in solving Einstein's eqs

$$2 R_{\mu\nu}^{\text{harmonic}} = - g^{\alpha\beta} \partial_{\alpha\beta} g_{\mu\nu} + g^{\cdot\cdot} g^{\cdot\cdot} (\partial g \partial g)$$

$\boxed{3.10}$ Relativistic gravitational interaction of two point masses

AGR 3.14



explicit form of Einstein's eqs in harmonic coords

$$-g^{\alpha\beta}\partial_{\alpha\beta}g_{\mu\nu} + g^{\alpha\beta}g^{\gamma\delta}\left(\partial_\gamma g_{\mu\alpha}\,\partial_\delta g_{\nu\beta} - \partial g\partial g + \partial g\,\partial g + \partial g\partial g - \frac{1}{2}\,\partial g\,\partial g\right)$$

$$= \frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{d-1}\,g_{\mu\nu}\,T_\lambda{}^\lambda\right)$$

$$T_{\mu\nu}(x) = \sum_A m_A c^2 \int ds_A\, \frac{g_{\mu\alpha}(z_A)\,g_{\nu\beta}(z_A)}{\sqrt{g(z_A)}}\,\frac{dz_A^\alpha}{ds_A}\,\frac{dz_A^\beta}{ds_A}\,\delta^{(d+1)}(x^\lambda - z_A^\lambda(s_A))$$

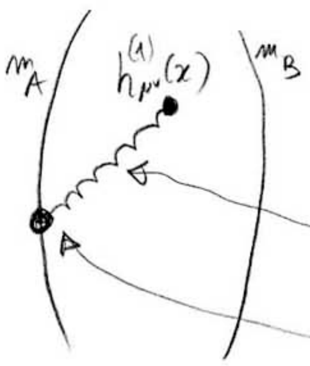
$$g_{\mu\nu}(x^\lambda) = \eta_{\mu\nu} + h^{(1)}_{\mu\nu}(x) + h^{(2)}_{\mu\nu}(x) + \cdots$$

$$\boxed{\begin{aligned}&\Box\, h^{(1)}_{\mu\nu}(x) = -\frac{16\pi G}{c^4}\left(T^{(1)}_{\mu\nu} - \frac{1}{d-1}\,\eta_{\mu\nu}\,T^{(1)}\right)\\ &\qquad\qquad ds_A^{(0)} = \sqrt{-\eta_{\mu\nu}\,dz_A^\mu dz_A^\nu}\\ &T^{(1)}_{\mu\nu} = \sum_A m_A c^2\,\eta_{\mu\alpha}\eta_{\nu\beta}\int ds_A^{(0)}\,\frac{dz_A^\alpha}{ds_A^{(0)}}\,\frac{dz_A^\beta}{ds_A^{(0)}}\,\delta^{(d+1)}(x - z_A)\end{aligned}}$$

3.11 Diagrams

inverse of $\Box$, i.e. scalar Green function $G_0(x,y)$

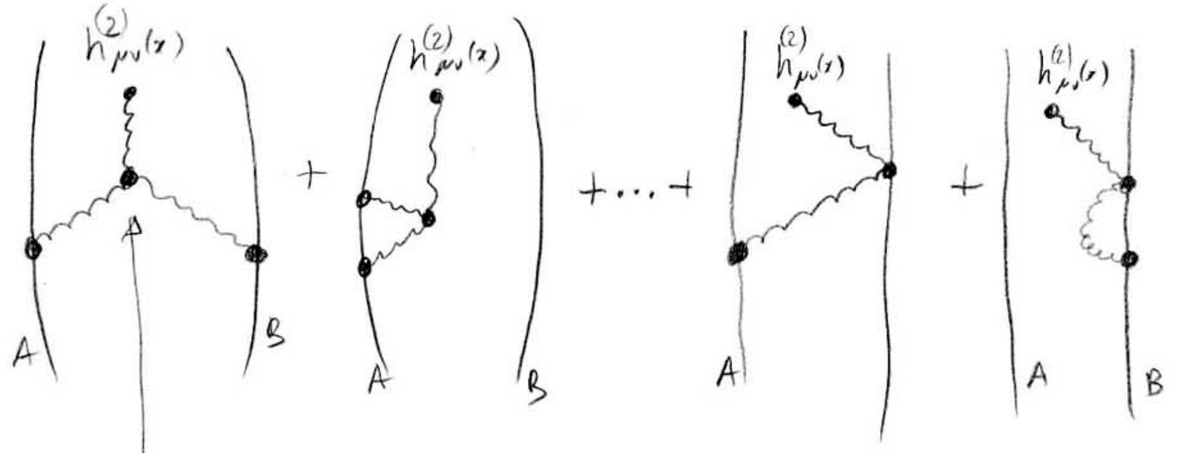
$$\Box G_0(x-y) = \delta^{(d+1)}(x-y)$$



$$h^{(1)}_{\mu\nu}(x) = \underbrace{-\frac{16\pi G}{c^4}\left(\eta_{\mu\alpha}\eta_{\nu\beta} - \frac{1}{d-1}\eta_{\mu\nu}\eta_{\alpha\beta}\right)\Box^{-1}}_{\text{PROPAGATOR}} \underbrace{T_A^{\alpha\beta}}_{\text{SOURCE}}$$

Next approximation

$$\Box h^{(2)} \sim h^{(1)}\partial^2 h^{(1)} + \partial h^{(1)}\partial h^{(1)} + h^{(1)} T^{(1)}$$



'CUBIC VERTEX'

i.e. non linearities

$\sim h\partial^2 h + \partial h \partial h$

in RHS

3.12 Reduced ('Fokker') action for interacting particles

$$S[z_A^\lambda; g_{\mu\nu}(x)] = -\sum_A m_A c \int \sqrt{-g_{\mu\nu}(z_A) dz_A^\mu dz_A^\nu} + \int \frac{d^{d+1}x}{c} \sqrt{g} \frac{c^4}{16\pi G} R(g)$$

+ surface term

+ gauge-fixing term $\propto \begin{pmatrix} \text{gauge} \\ \text{condition} \end{pmatrix}^2$

replace $g_{\mu\nu}(x)$ by gauge-fixed solution

$$g_{\mu\nu}[x; z_A^\lambda] = \eta_{\mu\nu} + h^{(1)}_{\mu\nu}[z_A^\lambda] + \cdots$$

$\Rightarrow$ Reduced (Fokker) action for particles (with, say, $G_{sym} \equiv \frac{1}{2}(G_{ret} + G_{ad})$)

$$S^{reduced}[z_A^\lambda] = S[z_A^\lambda; g_{\mu\nu}[x; z_B^\lambda]]$$

$$= S^{(0)}_{FREE}[z_A^\lambda] + S^{(1)}[z_A] + \cdots$$

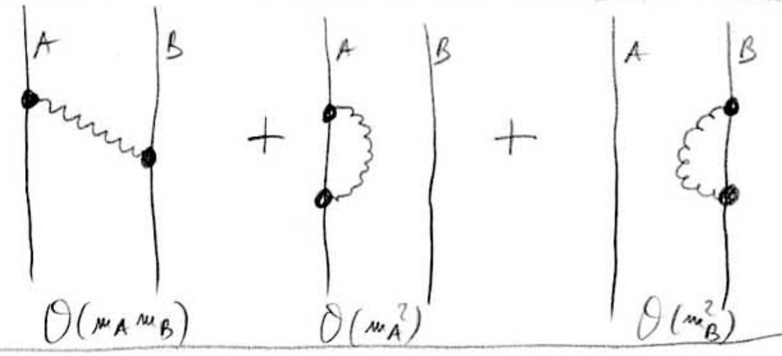
$$S^{(0)} = -\sum_A m_A c \int \sqrt{-\eta_{\mu\nu} dz_A^\mu dz_A^\nu} = -\int dt \sum m_A c^2 \sqrt{1 - \frac{\vec{v}_A^2}{c^2}}$$

$$\boxed{S^{(1)} = \frac{1}{4} \int \frac{d^{d+1}x}{c} h^{(1)}_{\mu\nu}(x) T^{\mu\nu}_{(1)}(x) = -\frac{4\pi G}{c^4} \int \frac{d^{d+1}x}{c} \frac{d^{d+1}y}{c} T^{\mu\nu}(x) \left(\frac{\eta_{\mu\alpha}\eta_{\nu\beta} - \frac{1}{d-1}\eta_{\mu\nu}\eta_{\alpha\beta}}{\Box} \right)_{xy} T^{\alpha\beta}(y)}$$

FIRST-ORDER INTERACTION — SOURCE — PROPAGATOR — SOURCE

AGR 3.17

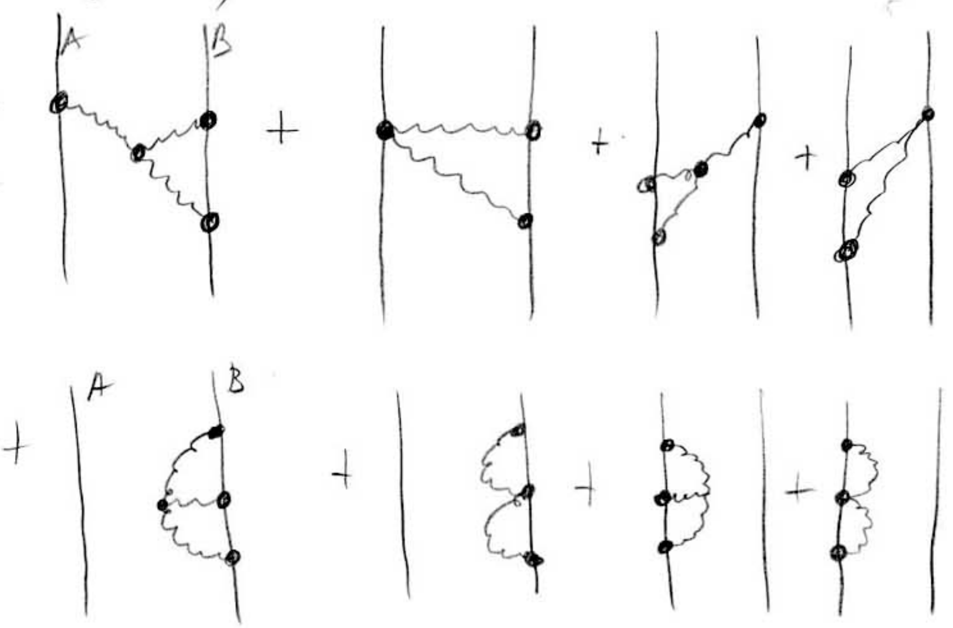
$S^{(1)} =$



higher-order interactions

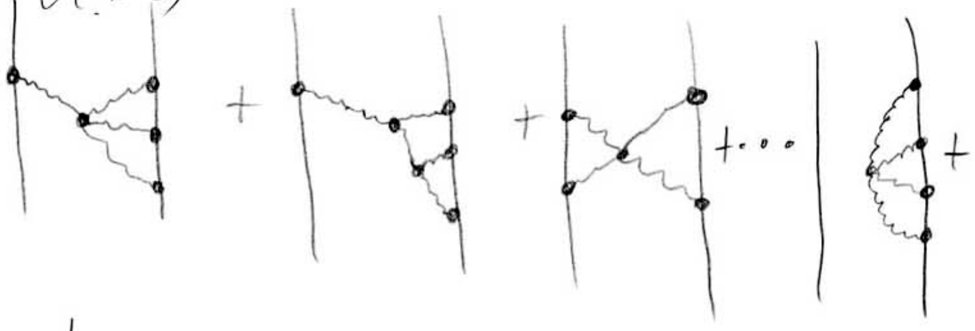
$\mathcal{O}(m_A m_B^2)$

$S^{(2)} =$



$\mathcal{O}(m_A m_B^3)$

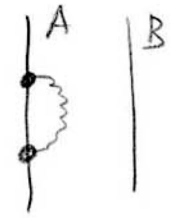
$S^{(3)} =$



$+ \ldots$

3.13 Self-gravity effects

Meaning of



?

Consider lowest (Newtonian) approximation

$$T^{00}(x) = \rho c^2 \quad \text{with} \quad \rho = \sum_A m_A\, \delta^{(d)}(\vec{x} - \vec{z}_A)$$

$$\text{with} \quad \Box = \Delta - \frac{1}{c^2}\partial_t^2 \simeq \Delta \quad ; \quad \Box^{-1} \simeq \Delta^{-1}$$

Define (Newtonian potential) $$\boxed{\Delta U = -4\pi G\, \frac{2(d-2)}{d-1}\, \rho}$$

$$S^{(1)}_{\text{Newton}} = \frac{1}{2}\, \rho \overset{\Delta^{-1}}{\bullet\!\!\sim\!\sim\!\sim\!\!\bullet} \rho = \int dt \left[\frac{1}{2} \int d^d x\, \rho(x)\, U(x) \right]$$

$$\boxed{U(x) = \sum_A \tilde{\tilde{k}}\, \frac{G m_A}{|\vec{x} - \vec{z}_A|^{d-2}}} \qquad \tilde{\tilde{k}} = \frac{2(d-2)}{d-1}\, \tilde{k} \underset{\text{in } d=3}{\longrightarrow} 1$$

$$\boxed{S^{(1)}_{\text{Newton}} = \int dt\, \frac{1}{2} \tilde{\tilde{k}}^2 \sum_A \sum_B \frac{G m_A m_B}{|\vec{z}_A - \vec{z}_B|^{d-2}}}$$

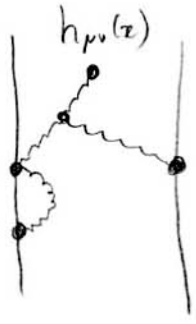
SELF-GRAVITY TERM

$$G m_A^2\, |\vec{z}_A - \vec{z}_A|^{2-d}$$

in $d=3$ infamous $\left[\frac{G m_A^2}{r}\right]_{r=0} = \infty$

Analytic Continuation in d: $\boxed{\underset{d}{AC}\left[r^{2-d}\right]_{r=0} = 0}$ because $= 0$ for $d<2$

More generally one checks that other self-gravity terms such as



are well-defined when $d \xrightarrow{AC} 3$ in spite of dangerous 'loop diagram' : 

- At the 2PN approximation, it has been shown (Damour 80, 83) that the analytic continuation $d \xrightarrow{AC} 3$ gives a finite result

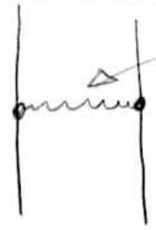
- At the 3PN approximation, if one uses ADM coordinates, it has been shown that the Hamiltonian is finite as $d \xrightarrow{AC} 3$ (Damour Jaranowski Schäfer 2001)

- At the 3PN approximation, and in harmonic coordinates, it has been shown that the AC $d \xrightarrow{AC} 3$ generates some poles $\propto \frac{1}{d-3}$, which can, however, be absorbed in a shift $\delta z_A^i \propto \frac{G^2 m_A^2}{c^6 (d-3)} \dddot{z}_A^i$ of the 'bare worldlines' (Blanchet Damour Esposito Farèse 2004)

> Crucial point: the AC $d \xrightarrow{AC} 3$ defines, in $d = 3$, a finite solution of $R_{\mu\nu}(g) = 0$ in $d = 3$, which satisfies the 'Dominant Schwarzschild boundary conditions' deduced from MATCHING. The uniqueness of this solution ensures that the formal AC generates the physically relevant metric outside N strongly self-gravitating bodies

$\boxed{3.14}$ Explicit results available for the dynamics of 2 condensed bodies

$\boxed{\text{"one-graviton exchange"}}$

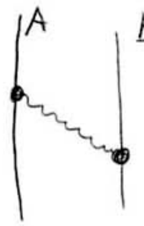
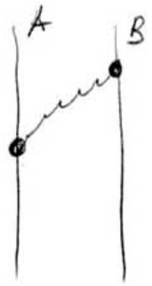
$S^{(1)} =$ 

NB: physically relevant retarded propagator, but must use

$\boxed{\text{time-symmetric} = \frac{1}{2}(\text{adv} + \text{ret})}$

for deriving conservative dynamics

$\Rightarrow$ $\boxed{\text{NON-LOCAL (in TIME) ACTION}}$

$S^{(1)} = \frac{1}{2}$  $+ \frac{1}{2}$ 

symmetric scalar propagator

in $\boxed{d=3}$

$$S^{(1)} \propto \iint ds_A\, ds_B \underbrace{\frac{dz_A^\mu}{ds_A}\frac{dz_A^\nu}{ds_A}\left(\eta_{\mu\alpha}\eta_{\nu\beta} - \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\right)\frac{dz_B^\alpha}{ds_B}\frac{dz_B^\beta}{ds_B}}_{\text{specific 'spin 2' velocity dependence}}\, \delta\left(\left(z_A^\mu(s_A) - z_B^\mu(s_B)\right)^2\right)$$

When using coordinate times

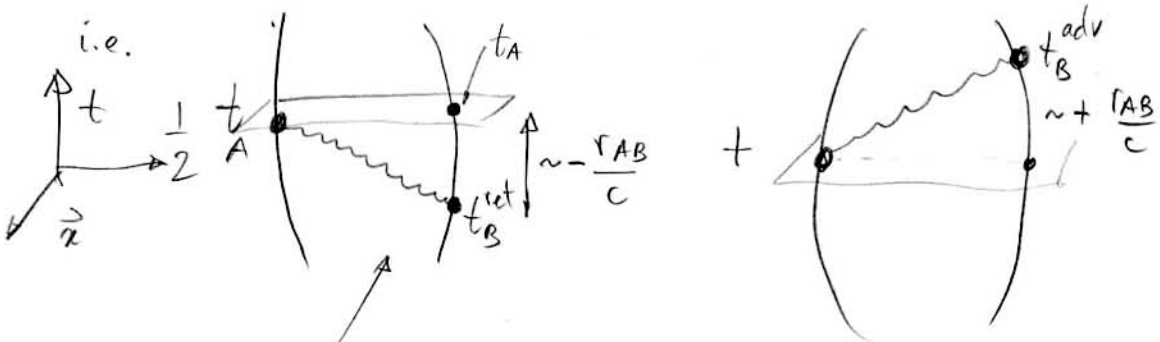
$$\iint dt_A\, dt_B \;\cdots\; \delta\left((t_A - t_B)^2 - \frac{1}{c^2}\left(\vec{z}_A(t_A) - \vec{z}_B(t_B)\right)^2\right)$$

$$\downarrow$$

$$\frac{1}{2}\,\frac{\delta\left((t_A-t_B) - \frac{1}{c}\left|\vec{z}_A(t_A) - \vec{z}_B(t_B)\right|\right)}{\left(1 - \vec{n}_{AB}\frac{\vec{v}_{AB}}{c}\right) r_{AB}} + \frac{1}{2}\,\frac{\delta\left((t_A-t_B) + \frac{r_{AB}}{c}\right)}{\left(1 - \vec{n}\frac{\vec{v}}{c}\right) r_{AB}}$$

To have an ordinary (time-local) action one needs to expand in retardation $\sim \frac{r_{AB}}{c}$

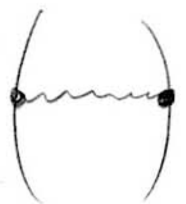
i.e.



essentially a Taylor (actually Lagrange) expansion

$\rightarrow$ introduces <u>higher time derivatives</u> of $\vec{z}_B(t)$: $\ddot{z}_B$, $\dddot{z}_B$, ...

At 1PN, one can integrate by parts to eliminate $\ddot{z}_B$

$\Rightarrow$ 

$$= \int dt \, \frac{1}{2} \sum_{A \neq B} \frac{G m_A m_B}{r_{AB}} \left[1 + \frac{3}{2c^2}(\vec{v}_A^{\,2} + \vec{v}_B^{\,2}) \overbrace{- \frac{7}{2c^2}(\vec{v}_A \cdot \vec{v}_B) - \frac{1}{2c^2}(\vec{n}_{AB} \cdot \vec{v}_A)(\vec{n}_{AB} \cdot \vec{v}_B)}^{\text{specific of spin 2 propagator}} + O\left(\frac{1}{c^4}\right) \right]$$

come from retardation

same as $L^{LD}_{2\text{-body}}\left[\vec{z}_A(t) - \vec{z}_B(t), \dot{\vec{z}}_A(t), \dot{\vec{z}}_B(t) \right]$

by effacement:
same as weakly self-gravitating case

At 2PN, ie. $\mathcal{O}\left(\frac{v^4}{c^4}\right)$ beyond Newton

one cannot get rid of higher-derivatives $\ddot{\ddot{z}}$

$$\Rightarrow \quad L_{2body}\left[\vec{z}_A(t)-\vec{z}_B(t),\ \dot{\vec{z}}_A(t),\ \dot{\vec{z}}_B(t),\ \ddot{\vec{z}}_A(t),\ \ddot{\vec{z}}_B(t)\right]$$

In principle, even higher derivatives at 3PN, $\frac{v^6}{c^6}$, etc...

However, by suitable use of some tricks ('double-zero', + integ. by parts)

one can arrange to have

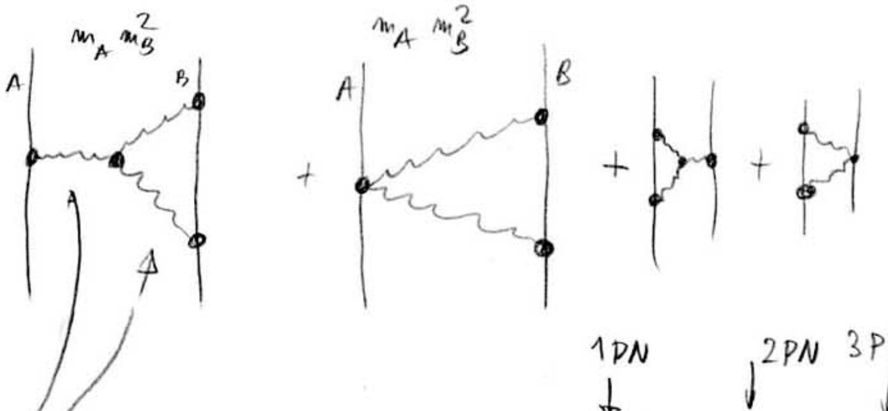
$$L\left[z_A - z_B,\ \dot{z}_A,\ \dot{z}_B;\ \underbrace{\ddot{z}_A,\ \ddot{z}_B}_{}\right]$$

LINEAR IN ACCELERATIONS

(see Damour Schäfer 1985 + ...)

$$\boxed{n\text{-graviton exchange with } n \geq 2}$$

e.g. $\mathcal{O}(m^3)$

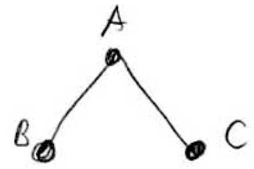


$$\begin{array}{ccc} \text{1PN} & \text{2PN} & \text{3PN} \\ \downarrow & \downarrow & \downarrow \end{array}$$

must expand $$\frac{1}{\square_{sym}} = \frac{1}{\Delta - \frac{1}{c^2}\partial_t^2} = \frac{1}{\Delta\left(1-\frac{1}{c^2}\partial_t^2\Delta^{-1}\right)} = \Delta^{-1} + \frac{1}{c^2}\partial_t^2\Delta^{-2} + \cdots$$

eg. 1PN

$$L_{\underset{\text{i.e. 3 masses}}{3\text{-body}}} = -\frac{1}{2}\sum_{B\neq A\neq C} \frac{G^2 m_A m_B m_C}{c^2 r_{AB} r_{AC}}$$



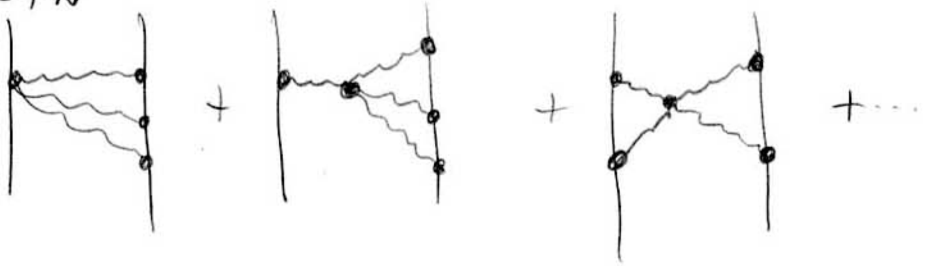
$$\Downarrow$$

$$\sim \frac{m_A m_B^2}{r_{AB}^2} + \frac{m_B m_A^2}{r_{BA}^2}$$

when $B = C$ i.e 2 bodies only

For computing the integrals that appear at 2PN and 3PN

eg



one needs to use:

① eliminate '1PN' cubic vertex $\sim h_{00} (\vec{\nabla} h_{00})^2$ by $(\ln - g_{00})$

② Riesz formula for

$$I(\alpha, \beta) = \int \frac{d^3x}{4\pi} |\vec{x} - \vec{z}_A|^\alpha \, |\vec{x} - \vec{z}_B|^\beta = (\text{product } \Gamma \text{ functions}) \times |\vec{z}_A - \vec{z}_B|^{\alpha+\beta+3}$$

Dynamics of two point masses in harmonic coordinates

2 PN level : $(v/c)^4$ Damour, Deruelle 81

2.5 PN level : $(v/c)^5$ Damour 82, 83

actually obtained from EQS OF MOTION

3 PN level : $(v/c)^6$ Blanchet - Faye '00, Blanchet Damour Esposito-Farèse '04, Itoh, Futamase '03 '04

3.5 PN $(v/c)^7$ Königsdörffer, Faye, Schäfer '03, Pati Will '02, Nissanke Blanchet '05

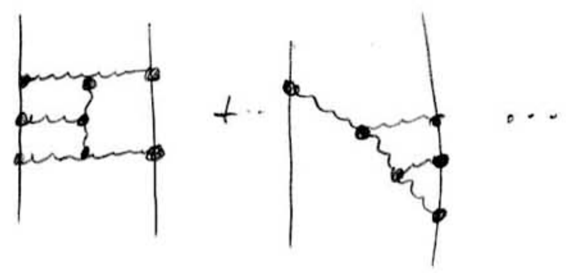
Dynamics of two point masses in Arnowitt Deser Misner coords

2 PN $(v/c)^4$ Schäfer '85 '86

2.5 PN $(v/c)^5$ [and Ohta, Okamura, ... 70's]

3 PN $(v/c)^6$ Jaranowski Schäfer '98 - Damour Jaranowski Schäfer '01

up to '3 loop diagrams'

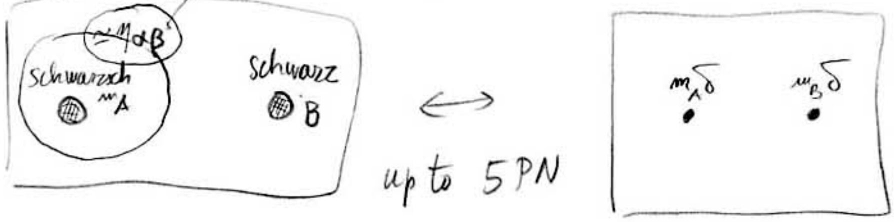


NB: AC $d \to 3$ $\Rightarrow$ finite results

Possible violation of 'effacement' outside GR

in GR

ONLY $\simeq \eta_{\alpha\beta}$ in INTERMEDIATE DOMAIN



[to which one must add SPIN EFFECTS for spinning bodies]

3.15 Tensor-scalar gravity

$$S = \frac{c^4}{16\pi G_*} \int \frac{d^4x}{c}\, \tilde{g}^{1/2} \left[F(\Phi) \tilde{R} - Z(\Phi) \tilde{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - U(\Phi) \right]$$
$$+ S_m [\psi_m ; \tilde{g}_{\mu\nu}]$$

'MATTER' 'PHYSICAL METRIC'

but redefinitions: $\Phi \longrightarrow \varphi = f(\Phi)$

$$\tilde{g}_{\mu\nu} \rightarrow g^*_{\mu\nu} = F(\Phi)\, \tilde{g}_{\mu\nu}$$

$$\boxed{S = \frac{c^4}{16\pi G_*} \int \frac{d^4x}{c}\, g_*^{1/2} \left[R_* - 2 g_*^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right] + S_m [\psi_m ; e^{2a(\varphi)} g^*_{\mu\nu}]}$$

'Einstein frame' POTENTIAL COUPLING FUNCTION

$$\boxed{\begin{aligned} R^*_{\mu\nu} &= 2\partial_\mu\varphi\,\partial_\nu\varphi + 8\pi G_*\left(T^*_{\mu\nu} - \tfrac{1}{2}T^* g^*_{\mu\nu}\right) \\ \Box_{g_*}\varphi &= -4\pi G_*\,\alpha(\varphi)\,T_* - \frac{1}{4}\frac{\partial V(\varphi)}{\partial\varphi} \end{aligned}}$$

ASSUMED HERE TO BE SMALL AND NEGLIGIBLE

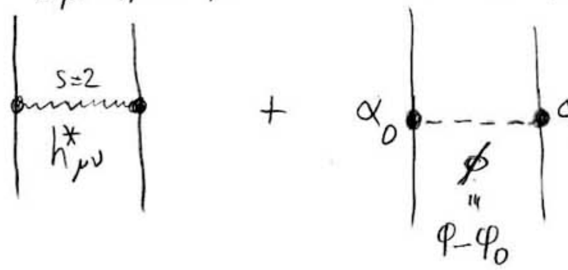
say $m_\varphi^2 \sim H_0^2$

crucial $\boxed{\alpha(\varphi) \equiv \dfrac{\partial a(\varphi)}{\partial\varphi}}$ and $\boxed{\beta(\varphi) \equiv \dfrac{\partial\alpha(\varphi)}{\partial\varphi} \equiv \dfrac{\partial^2 a(\varphi)}{\partial\varphi^2}}$

BACKGROUND VALUE OF φ

DIAGRAMS

$g^*_{\mu\nu} = \eta_{\mu\nu} + h^*_{\mu\nu}$ $\qquad$ $\varphi = \varphi_0 + \phi$



$\boxed{\alpha_0 \equiv \alpha(\varphi_0)}$

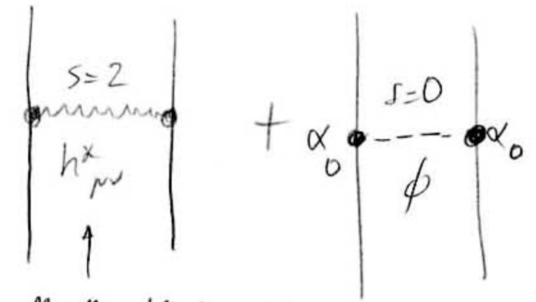
Newtonian level: $\dfrac{G_* m_A m_B}{r_{AB}} + \dfrac{G_*(\alpha_0 m_A)(\alpha_0 m_B)}{r_{AB}} \Rightarrow \boxed{\underset{\text{Cavendish}}{G} = G_*(1+\alpha_0^2)}$

in Einstein units

$\boxed{\tilde{G} = G_* e^{2a(\varphi_0)}(1+\alpha_0^2)}$

in physical units

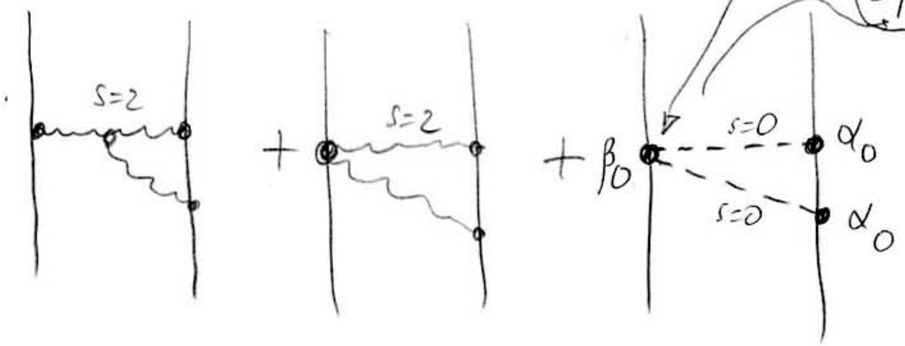
1PN level



$$T^{*}_{\mu\nu}\ \frac{\eta_{\mu\alpha}\eta_{\nu\beta}-\frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}}{\Box}\ T^{*}_{\alpha\beta} + T\ \frac{1}{\Box}\ T$$

$$\boxed{\bar{\gamma} \equiv \gamma^{PPN}-1 = -2\ \frac{\alpha_0^2}{1+\alpha_0^2}}$$

$a(\varphi_0+\phi) = a_0 + \alpha_0\phi + \frac{1}{2}\beta_0\phi^2$

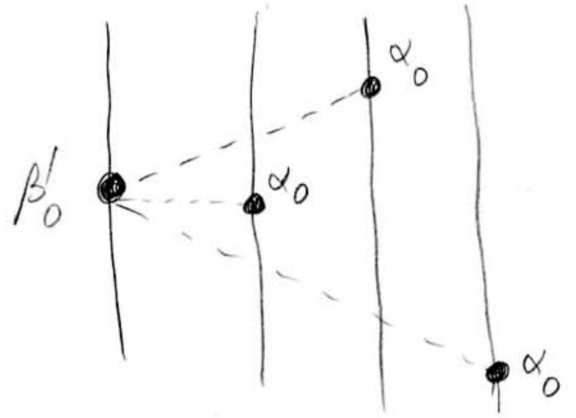


$$\boxed{\bar{\beta} \equiv \beta^{PPN}-1 = \frac{1}{2}\,\frac{\alpha_0\beta_0\alpha_0}{(1+\alpha_0^2)^2}}$$

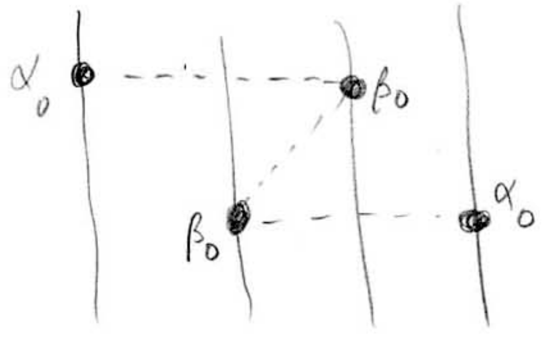
1PN 'POST-EINSTEIN' parameters

3.16 PARAMETRIZED 2PN POST-EINSTEIN

At 2PN need: $\boxed{a(\varphi) = \alpha_0 \phi + \frac{1}{2}\beta_0 \phi^2 + \frac{1}{6}\beta_0' \phi^3}$



$$\boxed{\epsilon \equiv \frac{\beta_0' \alpha_0^3}{(1+\alpha_0^2)^3}}$$



$$\boxed{\zeta \equiv \frac{\alpha_0 \beta_0 \beta_0 \alpha_0}{(1+\alpha_0^2)^3}}$$

DIAGRAMMATIC PROOF THAT

→ ONLY TWO 2PN POST-EINSTEIN PARAMETERS ϵ, ζ in tensor-scalar gravity

(Damour Esposito Farèse '96)

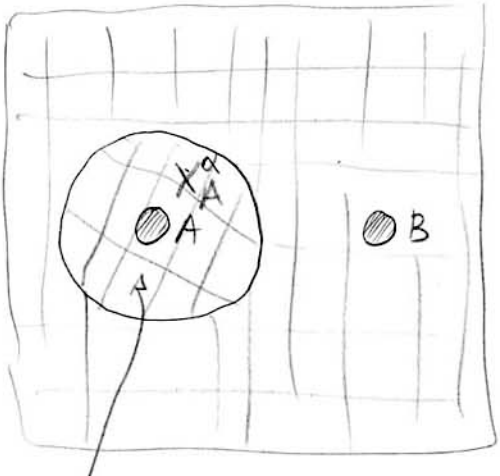
- Unmeasurable in solar system, even with parcsec light deflection
- Accessible through self-gravity effects in motion and radiation of binary pulsars

AGR 3.29

3.17 Violation of 'effacement' in tensor-scalar gravity

(Eardley '75 , Will Eardley '77)

MODIFICATION OF 'INTERNAL PROBLEM'



NEED BOUNDARY CONDITIONS ('FAR AWAY' $R_A \ll |\vec{X}_A| \ll d$) OF A BOTH FOR

$$G^A_{\alpha\beta}(X) \simeq \eta_{\alpha\beta}$$

AND

$$\varphi(X) \simeq \varphi_{asymptot_A}$$

$$\begin{cases} R^*_{\alpha\beta} = 2\partial_\alpha\varphi\,\partial_\beta\varphi + 8\pi G_*\left(\overset{A}{T}{}^*_{\alpha\beta} - \frac{1}{2}\overset{A}{T}{}^* G^A_{\alpha\beta}\right) \\ \Box_{g_*}\varphi = -4\pi G_*\,\alpha(\varphi)\,T_* \end{cases}$$

obtained by matching

$$\varphi_{as_A} = \varphi_0 + \text{contribution due to B at distance } r_{AB}$$

φ_{as_A} changes the solution, within body A

Leading internal solution when $R_A \ll |\vec{X}_A| \ll d$

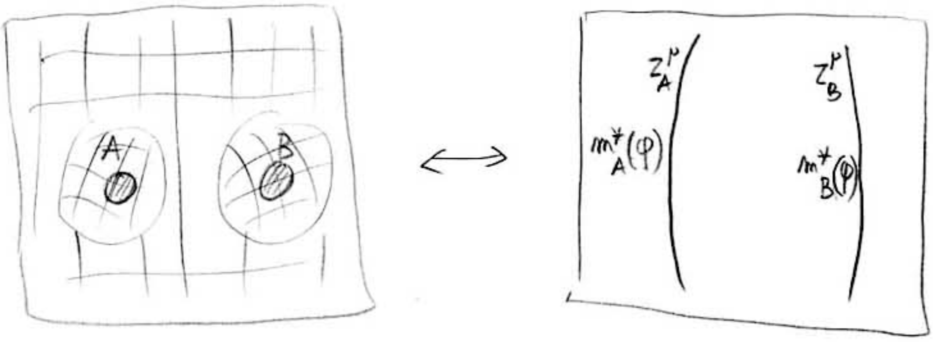
$$\boxed{\begin{aligned} G^*_{\alpha\beta}(X) &\simeq \eta_{\alpha\beta} + \frac{2G_*\, m^*_A(\varphi_{as_A})}{|\vec{X}|}\,\delta_{\alpha\beta} \\ \varphi &\simeq \varphi_{as_A} - G_*\,\alpha_A(\varphi_{as_A})\,\frac{m_A(\varphi_{as_A})}{|\vec{X}|} \end{aligned}}$$

Where (Damour Esposito Farèse '92)

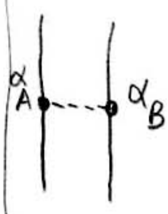
$$\boxed{\alpha_A(\varphi_{as}) = \frac{\partial \ln m^*_A(\varphi_{as})}{\partial \varphi_{as}}}$$

crucial 'master function': $\boxed{m^*_A = m^*_A(\varphi_{as})}$

MODIFIED SKELETONIZATION OF COMPACT BODIES :



$$\boxed{\begin{aligned} S^{\text{skeleton}} = & \frac{c^4}{16\pi G_*} \int \frac{d^{d+1}x}{c}\, g_*^{1/2} \left[R_* - 2 g_*^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right] \\ & - \sum_A c \int m^*_A(\varphi(z_A)) \sqrt{-g^*_{\mu\nu}(z_A)\, dz_A^\mu dz_A^\nu} \end{aligned}}$$



- → strong self-gravity of neutron stars can introduce significant violation of 'effacement' → significant deviations for GR predictions $\boxed{\propto \alpha_A \alpha_B \text{ etc...}}$
- On the other hand, for Black Holes: 'no scalar hair' → $\boxed{\begin{aligned} m^*_A &= \text{cst} \\ \alpha_A &= 0 \end{aligned}}$
  e.g. 2 BHs in tensor-scalar ≡ 2 BHs in GR!