

BINARY PULSARS:

Motion, Timing and Use as Probes of
Relativistic Gravity

Some relevant references:

- 2.5PN dynamics: Damour, Deruelle PLA 87, 81 (1981); CRAS II 293, 537 (1981)
 Damour CRAS II 294, 1355 (1982); 293, 877 (1981)
 in Gravitational Radiation, ed. N. Deruelle, T. Piran, North Holland (1983), p. 59
PRL 51, 1019 (1983)
 see also Blanchet, Faye, Ponsot PRD 58, 124002 (1998)
 and for ADM approach: Schäfer, Ann Phys 161, 81 (1985); GRG 18, 255 (1986)
- Solving 1PN and 2PN motion: Damour PRL 51, 1019 (1983); Damour, Deruelle AJHP 43, 107 (1985)
 Damour, Schäfer Nuovo Cim. 101B, 127 (1988)
- Variation of constants: PRL 51, 1019 and Damour, Gopakumar, Iyer PRD 70, 064028 (2004)
- Timing Formula: Damour, Deruelle AJHP 44, 263 (1986)
- Parametrized Post-Keplerian Formalism: Damour, Taylor PRD 45, 1840 (1992)
- Binary Pulsar Tests: Damour, Schäfer PRL 66, 2549 (1991)
 Taylor et al. Nature 355, 132 (1992); Weisberg, Taylor astro-ph/0407149
 see also Stairs et al. ApJ 581, 501 (2002); Kramer et al. Science (2006)
 Stairs / Living Review → Damour, Esposito-Farèse, PRL 70, 2220 (1993); CQG 9, 2093 (1992);
Phys Rev D 54, 1474 (1996); 58, 042001 (1998); ... (2006)
- Short Review of Exp. Tests: see chapter 18 of Particle Data Group Review: <http://pdg.lbl.gov/>

4.1 Binary pulsars and relativistic gravity

EXPERIMENT	THEORY
<p>Discovery of PSR 1913+16 Hulse, Taylor '74</p>	<p>Spin-orbit effects Damour, Ruffini '74</p> <p>Energy loss in gravit. waves Wagoner '75</p> <p>Dipole radiation as test of gravity Eardley '75, Will, Eardley '77</p>
<p>Observation of secular acceleration of the orbital motion: \dot{P}_b Taylor, Fowler, McCullough '79 Taylor, Weisberg '82</p>	<p>Relativistic "Timing Formula" Blandford, Teukolsky '76 ... Damour, Deruelle '86</p> <p>Relativistic dynamics of compact objects (from) (from). $a = \frac{Gm}{r^2} \left(1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} + \frac{v^5}{c^5} \right)$ Damour, Deruelle '81; Damour '82 '83</p>
<p>Discovery of PSR 1534+12 Wolszczan '91</p>	<p>Strong-field tests of gravity and Parametrized Post-Keplerian formalism Damour '87; Damour, Taylor '92</p> <p>Spin-orbit and NS inertia moments Damour, Schäfer '88</p>
<p>Experimental constraints on strong-field relativistic gravity Taylor, Wolszczan, Damour, Weisberg '92</p>	<p>Tests of Strong Equivalence Principle Damour, Schäfer '91</p>
<p>Observation of spin-orbit effects Kramer '98; Weisberg, Taylor '02</p>	<p>Non-perturbative strong-field effects in tensor-scalar gravity Damour, Esposito-Farese '93</p>
<p>Discovery of Double Binary Psr 0737-3039AB Burgay et al '03; Lyne et al '04</p>	

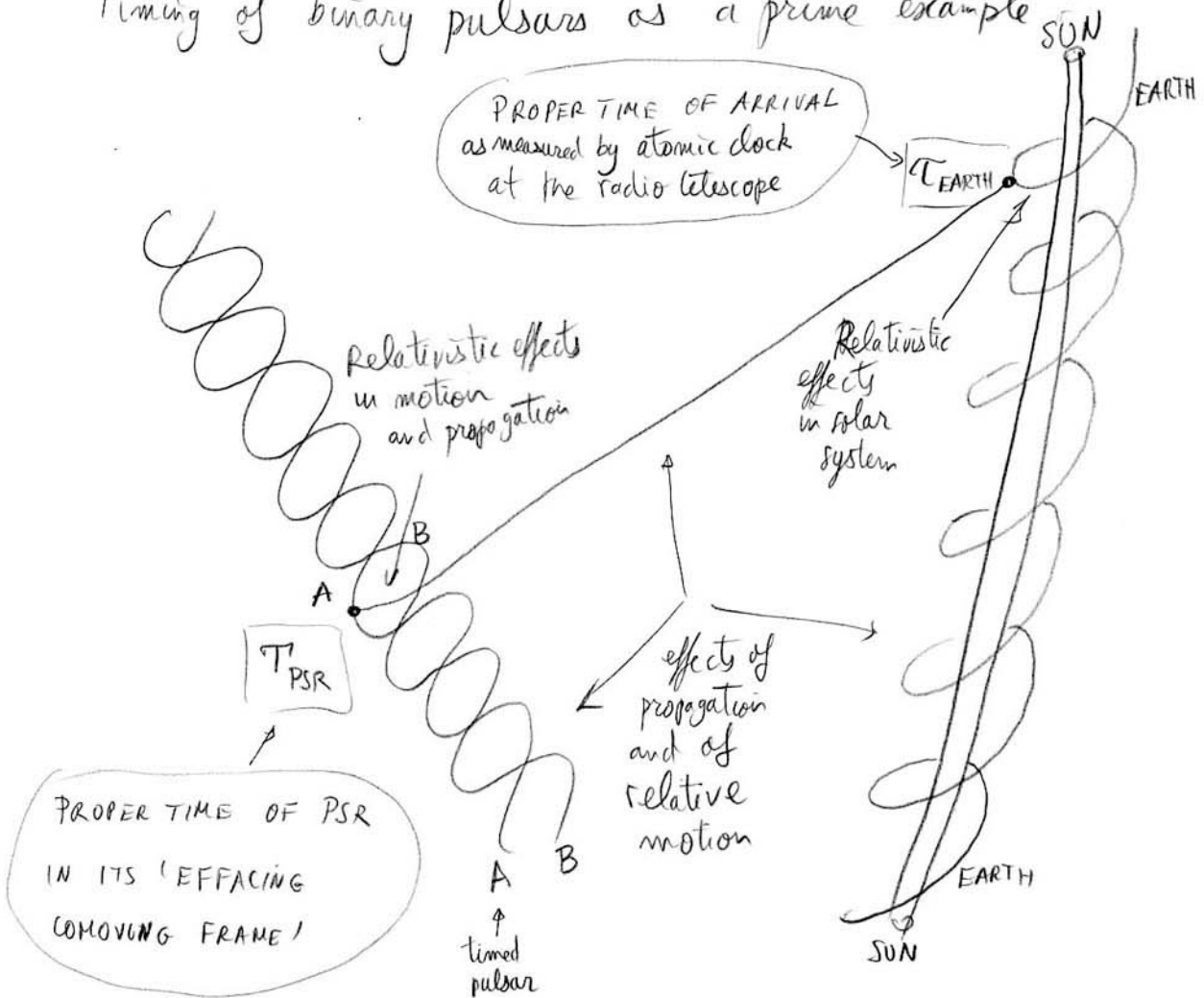
4.2 Binary pulsars and the physical meaning of General Relativity

? Physical meaning of GR with its 'general covariance', 'coordinates', ...

- Einstein confused for several years
- 'Problem of motion' marred by 'coordinate ambiguities' for years

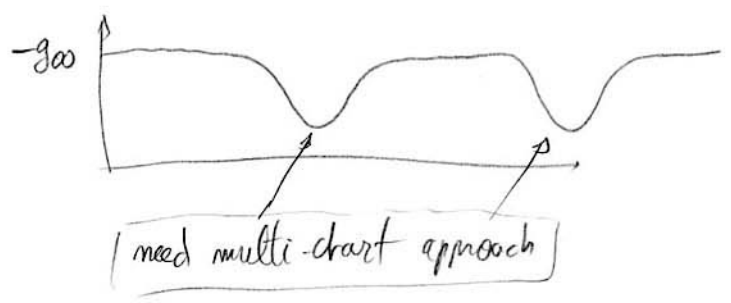
GR defines its own interpretation

Timing of binary pulsars as a prime example



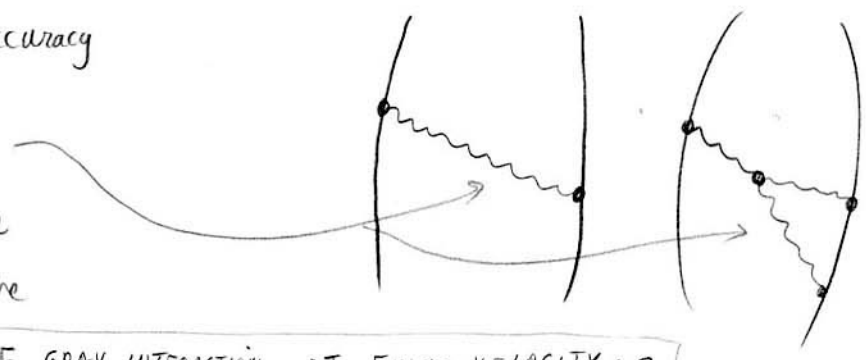
4.3 Motion of binary pulsars: strong fields, grav. waves

- Binary Pulsar
 - strongly self-gravitating



- high timing accuracy

need to carefully take into account the



PROPAGATION OF GRAV. INTERACTION AT FINITE VELOCITY: c

- Text book 'quadrupole formula' approach is unsatisfactory
- Often said that binary pulsars are only an indirect proof of \exists of G waves.

Actually, when done more correctly (μ , τ of Milon)

binary pulsars are a direct proof of the propagation of the gravitational interaction between A and B at velocity c and with spin 2 propagator.

+ confirmations of the fully nonlinear structure (strong-field effects)

4.4 Relativistic dynamics of binary systems

AGR4.4

Matched Multi-chart (+ Analytically Continued Skeletonized) approach

⇒ Eqs of Motion (say in harmonic coordinates) : Damour-Deruelle '81; Damour '82)

$$\frac{d^2 \vec{z}_A^i}{dt^2} = \underbrace{A_0^{Ai}}_{\text{Newton}}(\vec{z}_A - \vec{z}_B) + \frac{1}{c^2} A_2^{Ai}(\vec{z}_A - \vec{z}_B, \vec{v}_A, \vec{v}_B) + \frac{1}{c^4} A_4^{Ai}(z_A - z_B, v_A, v_B) + \dots$$

+ (Spin-orbit terms)

2.5 PN

needed to explain PSR observations

? How to solve this complicated dynamics

① Consider the time-symmetric 2PN dynamics

$$\frac{d^2 \vec{z}_A}{dt^2} = \vec{A}_0^A + \frac{1}{c^2} \vec{A}_2^A + \frac{1}{c^4} \vec{A}_4^A$$

② Exploit the conserved quantities at 2PN to solve 2PN dynamics

③ Use Lagrange method of Variation of Constants of 2PN dynamics to compute the effect of \vec{A}_5 , and spin-orbit terms.

4.5 Action, Symmetries and Conserved Quantities
at 2PN

From EoM \rightarrow brute force construction of ^{2PN} Lagrangian (Damour, Deruelle '81)

$$L^{2PN}(\vec{z}_A, \vec{z}_B, \vec{v}_A, \vec{v}_B, \vec{a}_A, \vec{a}_B) = L_0(z, v) + \frac{1}{c^2} L_2(z, v) + \frac{1}{c^4} L_4(z, v, a)$$

↑ velocities
 ↑ accelerations $\vec{a}_A = \frac{d^2 \vec{z}_A}{dt^2}$
 ↑ Newton $\sum \frac{1}{2} m_A \vec{v}_A^2 + \frac{1}{2} \sum_{A \neq B} \frac{G m_A m_B}{|\vec{z}_A - \vec{z}_B|}$
 ↑ Lorentz-Droste '17, Fichtenholz '50

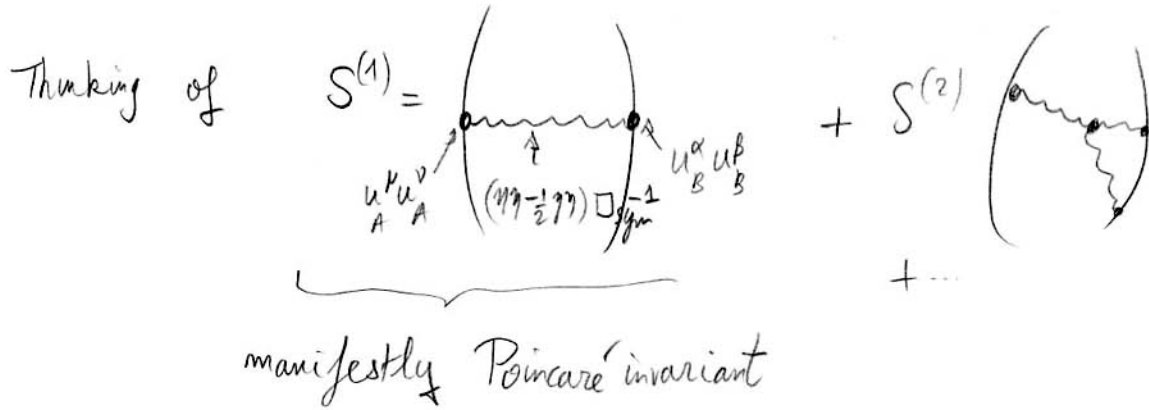
$$L_4 \sim v^6 + v^4 \frac{G}{r} + v^2 \frac{G^2}{r^2} + \frac{G^3}{r^3} + G a v^2$$

numerically similar if $a \sim \frac{G}{r^2}$
 but not allowed to replace EoM in L

(\Rightarrow change in coordinate system: Schäfer '84, Damour-Schäfer '85...)

however $+ Q(a - A(z, v))$ do not contribute
 \Rightarrow quadratic ('double zero')
 and can be used to reduce a -dependence
 to be linear in accelerations

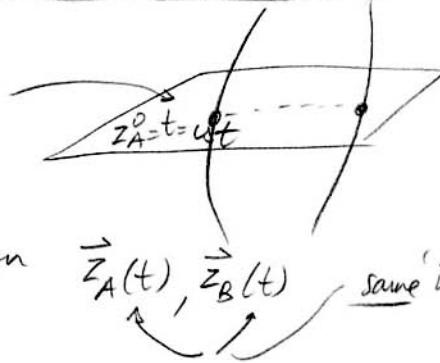
Symmetries



→ manifest invariance under

$$z_A^{\mu'} = \Lambda^{\mu'}_{\nu} z_A^{\nu} + b^{\mu'}$$

Z does not respect



→ more complicated action on $\vec{z}_A(t), \vec{z}_B(t)$ same 'instant'

N.B.: Poincaré symmetry = Residual symmetry of harmonic coords gauge-fixed Einstein action

$$S = \int R - \sum m_A \int ds_A + k \frac{g_{\mu\nu}}{\sqrt{g}} \partial_\alpha (\sqrt{g} g^{\mu\alpha}) \partial_\beta (\sqrt{g} g^{\nu\beta})$$

locally diff-invariant $x' = f(x)$

gauge-fixing term: only invariant under $x'^{\mu} = A^{\mu}_{\nu} x^{\nu} + B^{\mu}$

+ boundary condition: asymptotic flatness $g_{\mu\nu}^{(x)} \rightarrow \eta_{\mu\nu}$



restricts $A^{\mu}_{\nu} = \Lambda^{\mu}_{\nu}$

In other gauges, eg ADM, asymptotic flatness \rightarrow Poincaré gp acts as global symmetry group. However, not realized as $z = Az + b$ on worldlines

Noether

Poincaré group: $\overbrace{J_{ij}}^{J_{\mu\nu}}$ J_{0i} $P_\mu = (P_0, P_i)$
 3 rotations + 3 boosts + 4 translations
 → leaves $L(z, v, a) \equiv$ invariant modulo some $\frac{d}{dt}(\dots)$
 → 10 Noetherian conserved quantities (see Damour-Dernelbe '81)

(total energy)	$E^{2PN}(z, v, a) \equiv \sum_A (\dot{p}_{Ai} v_A^i + q_{Ai} a_A^i) - L_{2PN}(z, v, a)$
(lin. momentum)	$P_i^{2PN}(z, v, a) \equiv \sum_A P_{Ai}$
(angul. momentum)	$J_{ij}^{2PN}(z, v, a) \equiv \sum_A (z_A^i P_{Aj} - z_A^j P_{Ai} + v_A^i q_{Aj} - v_A^j q_{Ai})$
(center of mass conserved quantity)	$K_{2PN}^i(z, v, a, t) \equiv G_{2PN}^i(z, v) - t \cdot P_i$

$$G_{2PN}^i = \sum_A \left(m_A + \frac{1}{c^2} (m_A \vec{v}_A^2 - \frac{1}{2} \frac{G m_A m_B}{r_{AB}}) + \frac{1}{c^4} (v_A^4 \frac{v_A^2 G}{r} + \frac{G^2}{r^2}) \right) z_A^i - \frac{7}{4c^4} G m_A m_B (\vec{m}_{AB} \cdot (\vec{v}_A + \vec{v}_B)) (v_A^i - v_B^i)$$

4.6 Variation of 2PN conserved quantities under ^{dynamics} 2.5PN

after reduction

$$\begin{aligned} \frac{d}{dt} C_{2PN}^{2PN}(z, v) &= \sum_A \frac{\partial C_{2PN}}{\partial z_A^i} v_A^i + \frac{\partial C_{2PN}}{\partial v_A^i} \left(A_0^i + \frac{1}{c^2} A_2^i + \frac{1}{c^4} A_4^i + \frac{1}{c^5} A_5^i \right) \\ &= \sum_A \frac{\partial C_{2PN}}{\partial v_A^i} \frac{1}{c^5} A_5^i + O(\frac{1}{c^6}) \\ &= \sum_A \frac{\partial C_{Newt}}{\partial v_A^i} \frac{1}{c^5} A_5^i + O(\frac{1}{c^6}) \end{aligned}$$

Insert explicit expansion of

$$A_{5}^{Ai} = \frac{4}{5} \frac{G^2 m_A m_B}{r_{AB}^3} \left[v_{AB}^i \left(-v_{AB}^2 + \frac{2Gm_A}{r_{AB}} - \frac{8Gm_B}{r_{AB}} \right) + m_{AB}^i \left(\frac{m_A}{m_B} \frac{v_{AB}^i}{r_{AB}} \right) \right] \left[\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \right]$$

$$\left[3v_{AB}^2 - 6\frac{Gm_A}{r_{AB}} + \frac{52Gm_B}{3r_{AB}} \right]$$

after reduction

$$\frac{d}{dt} P_i^{2PN} \downarrow (z, v) = -\frac{1}{c^5} \frac{d}{dt} P_i^5(z, v) + 0 + O(c^{-6})$$

$$\frac{d}{dt} (G_i^{2PN} - t P_i^{2PN}) = -\frac{1}{c^5} \frac{d}{dt} (G_i^5(z, v) - t P_i^5(z, v)) + 0 + O(c^{-6})$$

→ Allows one to define 6 2.5 PN conserved quantities

$$P_i^{2.5PN} \equiv P_i^{2PN} + \frac{P_i^5}{c^5}; \quad K_i^{2.5PN} \equiv \left(G_i^{2PN} + \frac{G_i^5}{c^5} \right) - t P_i^{2.5PN}$$

→ Definition of 2.5 PN-accurate center of mass frame

$$0 = P_i^{2.5PN} = K_i^{2.5PN} = G_i^{2.5PN}$$

Explicit calculations with A_5^i harmonic \rightarrow

AGR 4.9

$$\frac{d}{dt} E_{(z,0)}^{2PN} = -\frac{1}{c^5} \frac{d}{dt} E_{(z,0)}^5 - \frac{G}{5c^5} I_{ij}^{(3)} I_{ij}^{(3)} + \mathcal{O}(1/c^6)$$

$$\frac{d}{dt} J_{ij}^{2PN}(z,0) = -\frac{1}{c^5} \frac{d}{dt} J_{ij}^5(z,0) - \frac{2G}{5c^5} (I_{is}^{(2)} I_{js}^{(3)} - I_{js}^{(2)} I_{is}^{(3)}) + \mathcal{O}(1/c^6)$$

where

$$I_{ij} = \sum_A m_A \langle z_A^i z_A^j \rangle$$

Schwarzschild masses of A, B

coincides with expected
'quadrupolar' losses of
energy and ang. mom. in GWaves at ∞

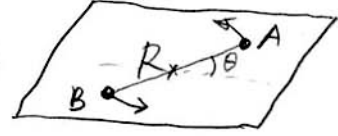
Actually, this agreement is comforting for showing that A_5^i drains out of the binary system the expected losses (so that $A_5^i = \text{Radiation Damping} = \text{Radiation Reaction}$) but is useless as far as computing the observable effects in the binary dynamics.

4.7 Lagrange method of 'variation of constants'

In 2.5 PN center of mass frame

$$\frac{d}{dt} J_{ij}^{2PN} \propto J_{ij}^{2PN} \Rightarrow$$

Fixed Plane of Motion



In fixed plane \rightarrow Polar coordinates for relative motion

$$\vec{z} \equiv \vec{z}_A - \vec{z}_B = (R \cos \theta, R \sin \theta, 0)$$

Considers first 2PN relative motion

\rightarrow conserved quantities

$$c_1 \equiv [E^{2PN}]_{\text{com}}$$

$$c_2 \equiv [\vec{J}^{2PN}]_{\text{com}}$$

$$\left(\frac{dR}{dt}\right)^2 = A + \frac{2B}{R} + \frac{C}{R^2} + \frac{D_1}{R^3} + \frac{D_2}{R^4} + \frac{D_3}{R^5} \equiv \mathcal{R}(R, c_1, c_2)$$

$$\frac{d\theta}{dt} = \frac{H}{R^2} + \frac{I_1}{R^3} + \frac{I_2}{R^4} + \frac{I_3}{R^5} \equiv \mathcal{G}(R, c_1, c_2)$$

where, e.g.,

$$\begin{cases} A = 2 E^{2PN} \left(1 + \frac{3}{2} (3\nu - 1) \frac{E^{2PN}}{c^2} + \dots \frac{\dots}{c^4} \right) = 2 c_1 \left(1 + \frac{3}{2} (3\nu - 1) \frac{c_1}{c^2} + \frac{g(c_1, c_2)}{c^4} \right) \\ \vdots \\ H = J^{2PN} \left(1 + (3\nu - 1) \frac{E^{2PN}}{c^2} + \dots \frac{\dots}{c^4} \right) = c_2 \left(1 + (3\nu - 1) \frac{c_1}{c^2} + \frac{g(c_1, c_2)}{c^4} \right) \end{cases}$$

with

$$\nu \equiv \frac{m_A m_B}{(m_A + m_B)^2} = \frac{\mu}{M} \quad 0 \leq \nu \leq \frac{1}{4}$$

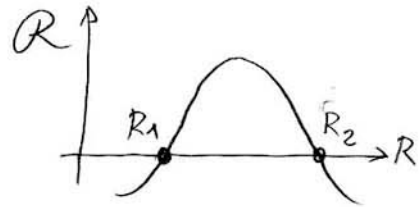
Structure of 2PN solution

Introduce TWO ANGULAR VARIABLES

$$l, m \in [0, 2\pi)$$

$$\begin{aligned} R &= S(l; c_1, c_2) \\ \theta &= m + W(l; c_1, c_2) \\ l &= 2\pi \frac{t}{P(c_1, c_2)} + c_0 \\ m &= 2\pi \frac{t}{Q(c_1, c_2)} + c_3 \end{aligned}$$

hyperelliptic functions of l ;
periodic on $[0, 2\pi]$



$$P(c_1, c_2) = 2 \int_{R_1}^{R_2} \frac{dR}{\sqrt{R(R, c_1, c_2)}} \quad ; \quad \frac{P(c_1, c_2)}{Q(c_1, c_2)} = \frac{1}{\pi} \int_{R_1}^{R_2} \frac{g(R, c_1, c_2)}{\sqrt{R(R, c_1, c_2)}} dR$$

Structure of 2.5 PN solution (Damour '83; Damour, Gopakumar, Iyer '04)

$$\begin{aligned} R &= S(l; c_1(t), c_2(t)) \\ \theta &= m + W(l; c_1(t), c_2(t)) \\ l &= 2\pi \int_0^t \frac{dt'}{P(c_1(t'), c_2(t'))} + c_0(t) \\ m &= 2\pi \int_0^t \frac{dt'}{Q(c_1(t'), c_2(t'))} + c_3(t) \end{aligned}$$

Same functions
 S, W, P, Q
as before,
but now

$$c_\alpha \equiv (c_1, c_2, c_0, c_3)$$

↑
 $\alpha = 1, 2, 0, 3$

vary with time
accordingly with,

$$\frac{dc_\alpha(t)}{dt} = \frac{1}{c_5} F_\alpha(l; c_\beta)$$

$$F_\alpha(l) \prec \begin{matrix} \text{periodic in } l \\ \propto \vec{A}_5 \end{matrix}$$

4.8 Explicit form of 'first post-Keplerian' solution AGR 4.12

1PN Lagrangian

(Damour Deruelle 1985)

$$L = \sum_A \left(\frac{1}{2} m_A \dot{\vec{v}}_A^2 + \frac{1}{8} m_A \frac{\dot{\vec{v}}_A^4}{c^2} \right) + \frac{G m_A m_B}{r_{AB}} + \frac{G m_A m_B}{2 c^2 r_{AB}} \left[3 \dot{\vec{v}}_A^2 + 3 \dot{\vec{v}}_B^2 - 7 (\dot{\vec{v}}_A \cdot \dot{\vec{v}}_B) - (\ddot{\vec{r}}_{AB} \cdot \dot{\vec{v}}_A) (\ddot{\vec{r}}_{AB} \cdot \dot{\vec{v}}_B) - \frac{G(m_A + m_B)}{r_{AB}} \right]$$

→ invariant under Poincaré group → 10 Noetherian quantities

$E^{1PN}, \vec{P}, \vec{J}, \vec{K}^{1PN}$ → 1PN center of mass frame

→ 1PN Lagrangian for relative motion in center of mass frame

$\hat{L}(\vec{R}, \vec{V}) = \frac{1}{2} \vec{V}^2 + \frac{GM}{R} + \frac{1}{8} (1-3\nu) \frac{\vec{V}^4}{c^2}$ <p style="margin-left: 20px;">↑ divided by</p> $+ \frac{GM}{2Rc^2} \left[(3+\nu) \vec{V}^2 + \nu (\vec{N} \cdot \vec{V})^2 - \frac{GM}{R} \right]$ <p style="margin-left: 20px;">$p \equiv \frac{m_A m_B}{M}$</p>	$\vec{R} \equiv \vec{z}_A - \vec{z}_B$ $\vec{V} \equiv \dot{\vec{v}}_A - \dot{\vec{v}}_B$ $M \equiv m_A + m_B$ $\nu \equiv \frac{m_A m_B}{M^2}$ $\equiv p/M$
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conserved quantities

$$\left\{ \begin{aligned} \hat{E} &= \vec{V} \cdot \frac{\partial \hat{L}}{\partial \vec{V}} - \hat{L} \\ \hat{J} &= \vec{R} \times \frac{\partial \hat{L}}{\partial \vec{V}} \end{aligned} \right.$$

In polar coordinates in plane of motion



$$\vec{R} = (R \cos \theta, R \sin \theta, 0)$$

$$\left(\frac{dR}{dt} \right)^2 = A + \frac{2B}{R} + \frac{C}{R^2} + \frac{D}{R^3} ; \quad \frac{d\theta}{dt} = \frac{H}{R^2} + \frac{I}{R^3}$$

as in Newtonian approx, but $A = 2\hat{E} \left(1 + \frac{\dots}{c^2} \right)$ etc.
 $D = \mathcal{O}\left(\frac{1}{c^2}\right)$ purely 1PN
 as Newton $H = \hat{J} \left(1 + \frac{\dots}{c^2} \right)$
 $I = \mathcal{O}\left(\frac{1}{c^2}\right)$ purely 1PN

NB: Contrary to usual Newtonian solution, one is not interested in eliminating t to get the orbit $\left(\frac{dR}{d\theta} \right)^2 = f(R)$.

One needs the motion: $(R(t), \theta(t))$

→ conchoidal transformation trick

absorb $D = \mathcal{O}(c^{-2})$

into a constant shift δR :

$$R = \bar{R} + \frac{D}{2C}$$

$$\delta R \frac{\partial}{\partial R} \left(A + \frac{2B}{R} + \frac{C}{R^2} \right) = -2\delta R \left(\frac{B}{R^2} + \frac{C}{R^3} \right)$$

↑ neglected $\delta R^2 = \mathcal{O}(c^{-4})$
 ↑ modifies C
 ↑ absorbs D

$$\left(\frac{d\bar{R}}{dt} \right)^2 = A + \frac{2B}{\bar{R}} + \frac{\bar{C}}{\bar{R}^2}$$

$$\bar{C} \equiv C - \frac{BD}{C}$$

as in Newtonian theory

→ can be solved by Kepler's parametrization

• Finally (after using another conchoidal trick for solving $d\theta/dt = \dots$) one can write explicit 1PN solution in 'QUASI-KEPLERIAN FORM'

$n \equiv \frac{2\pi}{P_b}$

$$n(t - t_0) = u - e_t \sin u$$

$$R \equiv r_{AB} = a_R (1 - e_R \cos u)$$

$$\theta - \theta_0 = (1+k) A_{e_\theta}(u) \rightarrow A_e(u) \equiv 2 \arctan \left[\sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2} \right]$$

angle
 $u \in [0, 2\pi]$
'relativistic eccentric anomaly'

• Differs from Newton only in

- eccentricity $e \left\{ \begin{array}{l} e_t \\ e_R \\ e_\theta \end{array} \right\}$ explicitly computed in terms of \hat{E}, \hat{J}

- θ motion multiplied by $1+k$

Periastron advance

$u : 0 \rightarrow 2\pi \Rightarrow R_{min} \rightarrow R_{min} = a_R (1 - e_R)$ "periastron to periastron"

$t \rightarrow t + \frac{2\pi}{n} \Leftrightarrow \left\{ n = \frac{2\pi}{P_b} \right\}$ periastron-to-periastron (binary) period

$\theta \rightarrow \theta + 2\pi(1+k) \Rightarrow \left\{ k = \left[\frac{\Delta\theta}{2\pi} \right]_{\text{orbital period}} \right\}$



$$k = \frac{3}{c^2} \left(\frac{GM}{\hat{J}} \right)^2 + O\left(\frac{1}{c^4}\right)$$

For binary pulsars: need higher accuracy on k AGR4.15

2PN + spin-orbit effects: need hyperelliptic (complete) integrals
 (evaluated à la Sommerfeld by contour integrals: Damour-Schiafer '88)

$$k = \frac{3}{c^2 \gamma^2} \left[1 + \left(\frac{5-v}{2}\right) \frac{\hat{E}}{c^2} + \left(\frac{35-5v}{4} \frac{1}{2}\right) \frac{1}{c^2 \gamma^2} \right] + \sum_A \frac{B_A}{s_A^2 i} \left[(1-3s_A^2 i) (\vec{k} \cdot \vec{s}_A) - \cos i \vec{k}_0 \cdot \vec{s}_A \right]$$

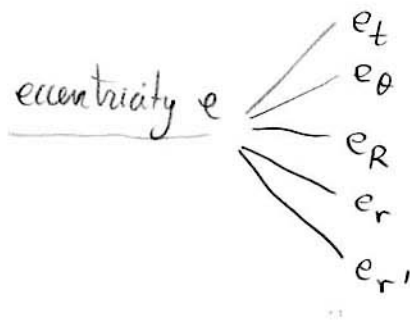
\uparrow $\hat{J} = \frac{\vec{J}}{GM}$ \uparrow 2PN correction \uparrow spin-orbit contributions

- Need also the separate motions of A and B:
 conchoidal transformation again

$$\begin{aligned} |\vec{z}_A - \vec{z}_B| &= R = a_R (1 - e_R \cos u) \\ |\vec{z}_A - \vec{z}_{cm}| &= r_A = a_r (1 - e_r \cos u) \\ |\vec{z}_B - \vec{z}_{cm}| &= r_B = a_{r'} (1 - e_{r'} \cos u) \end{aligned}$$

again like in Keplerian solution,

but with



all explicitly computed
 in terms of \hat{E}, \hat{J}

4.9 Radial motion at 2.5 PN approximation

At 2 PN, $R = r_{AB}$ is a periodic function of the 'true anomaly' l

$R = S(l; c_1, c_2)$ with $l = 2\pi \frac{t}{P(c_1, c_2)} + c_0$
 defined by \uparrow \uparrow \uparrow
 constant at 2PN

$R = a_R(c_1, c_2) (1 - e_R(c_1, c_2) \cos u)$
 $l = u - e_t(c_1, c_2) \sin u + \frac{f(c_1, c_2)}{c^4} \sin A_\theta(u) + \frac{g(c_1, c_2)}{c^4} (A_\theta(u) - u)$
 2PN corrections Jamou Schaefer '88
Schaefer Wex '93

Lagrange variation of constants: $\frac{dc_\alpha}{dt} = \frac{1}{c^5} F_\alpha(l; c_1, c_2)$

Radial motion at 2.5 PN:

$R = a_R(c_1(t), c_2(t)) (1 - e_R(c_1(t), c_2(t)) \cos u)$
 $l = u - e_t(c_1, c_2) \sin u + \frac{1}{c^4} c_1(t) \cdot c_2(t)$

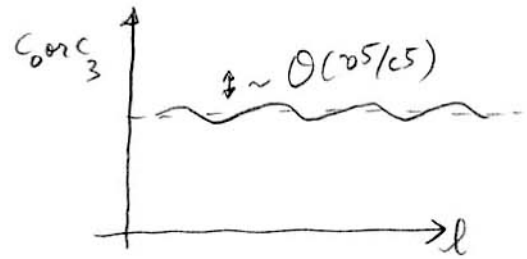
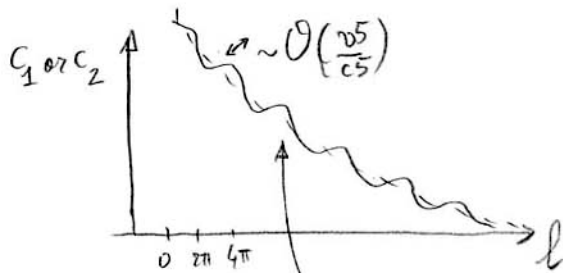
but with $l^{2.5PN} = 2\pi \int_0^t \frac{dt'}{P(c_1(t'), c_2(t'))} + c_0(t)$

CRUCIAL nontrivial time dependence, because periastrons are reached for $l = 0, 2\pi, 2 \times 2\pi, \dots, N \times 2\pi, \dots$

4.10 Evolution of the $c_\alpha(t)$ under \vec{A}_5

AGR 4.17

$$\frac{dc_\alpha}{dl} = \frac{1}{c^5} \tilde{F}_\alpha(l; c_1, c_2) \propto A_5^i \begin{cases} l\text{-even for } \alpha=1, 2 \\ l\text{-odd for } \alpha=0, 3 \end{cases}$$



'secular' evolution given by the averaged system

$$\frac{d}{dl} \bar{c}_\alpha = \frac{1}{c^5} \left\langle \tilde{F}_\alpha(l; c_1, c_2) \right\rangle_l \text{ with } c_1, c_2 \text{ fixed}$$

$$c_\alpha = \bar{c}_\alpha + \tilde{c}_\alpha \propto O\left(\frac{v^5}{c^5}\right)$$

↑ negligible in pulsar observables
(but possibly measurable in GW phasing of eccentric binaries, *Jayaram Gopakumar Iyer 104*)

Over times \ll time-scale of radiation reaction A_5 , i.e. $\frac{P_0}{(v/c)^5}$

$$\bar{c}_{\alpha=1,2} = \bar{c}_\alpha^0 + \dot{\bar{c}}_\alpha^0 t$$

$$\Rightarrow l \approx 2\pi \int_0^t \frac{dt'}{P(c_1^0 + \dot{c}_1^0 t', c_2^0 + \dot{c}_2^0 t')} + \bar{c}_0^0 + \cancel{\dot{\bar{c}}_0^0 t} + \frac{\text{periodic}}{c^5}$$

Simplification

orbital frequency " ω "

$$\frac{2\pi}{P(c_1, c_2)} \equiv n(c_1, c_2) = \frac{(-2\hat{E})^{3/2}}{GM} \left[1 + \frac{15-\nu}{4} \frac{\hat{E}}{c^2} + \frac{1}{c^4} f(\hat{E}, \hat{J}) \right]$$

↑
↑
↑

At N and 1PN only $c_1 = \hat{E}$ enters
 only at 2PN

→ $P(\dot{c}_1^0 + \dot{c}_1^0 t') \approx P_0 + \dot{P}_0 t'$ with $\dot{P}_0 = \frac{\partial P_0}{\partial c_1^0} \dot{c}_1^0$

$= \frac{\partial P}{\partial E} \left(\frac{dE}{dt} \right)_{\text{orbital period}}$

$$\dot{P}_0 = \frac{\partial P}{\partial E} \left\langle \vec{v}_A \cdot \frac{\vec{A}_5^A}{c^5} + \vec{v}_B \cdot \frac{\vec{A}_5^B}{c^5} \right\rangle$$

direct consequence of $\left(\frac{t-d/c}{c} \right) + \dots$

direct calculation from A_5

$$\dot{P}_0 = -\frac{192\pi}{5c^5} \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}} (GM\eta)^{5/3} \nu$$

$$l \approx 2\pi \int_0^t \frac{dt'}{P_0 + \dot{P}_0 t'} + \bar{c}_0^0 \approx 2\pi \left[\frac{t}{P_0} - \frac{1}{2} \frac{\dot{P}_0}{P_0} \left(\frac{t}{P_0} \right)^2 \right] + \bar{c}_0^0$$

→ date t_N of N^{th} periastron passage: $l_N = 2\pi N$

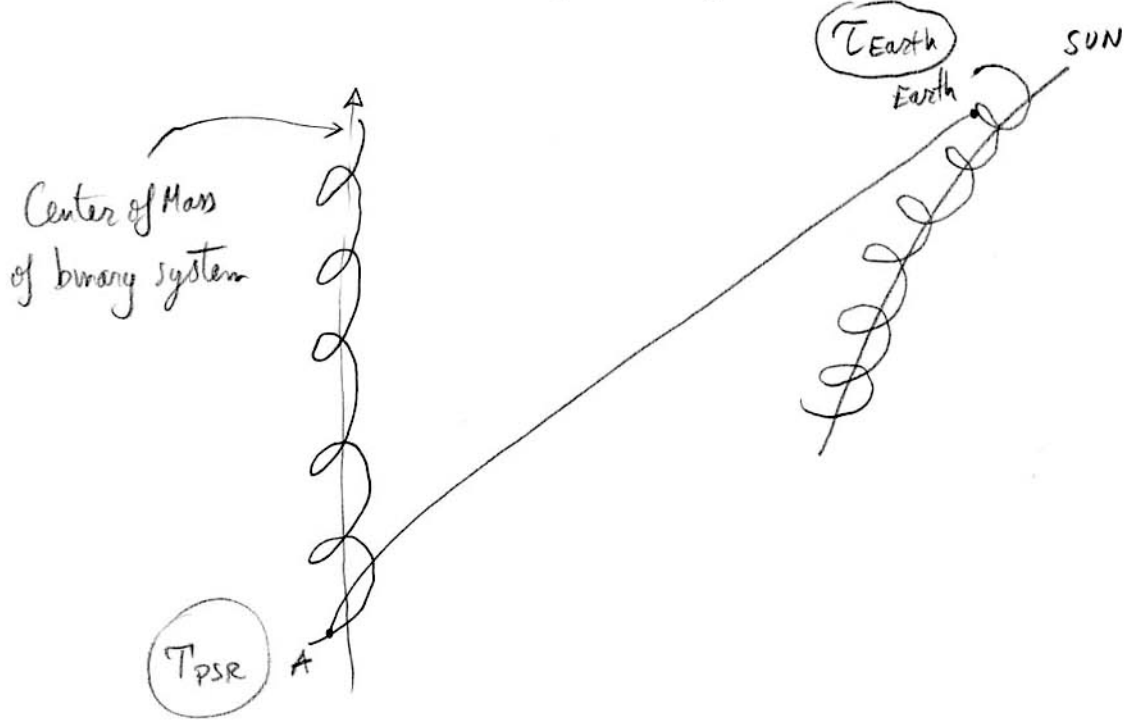
$$t_N \approx t_0 + P_0 N + \frac{1}{2} \frac{\dot{P}_0}{P_0} N^2$$

4.11

Relativistic timing of binary pulsars

AGR 4.19

Blandford Teukolsky '76 ... Damour Deruelle '86



CHAIN OF TRANSFORMATIONS: $T_{PSR} \longrightarrow T_{EARTH}$

• Within binary system

↙ focus on these

• Between binary system and solar system

↘ Dispersion of radio waves
 $\sim D/f^2$

↘ Variation of Doppler factor
between CoM worldline
and Barycenter of solar system

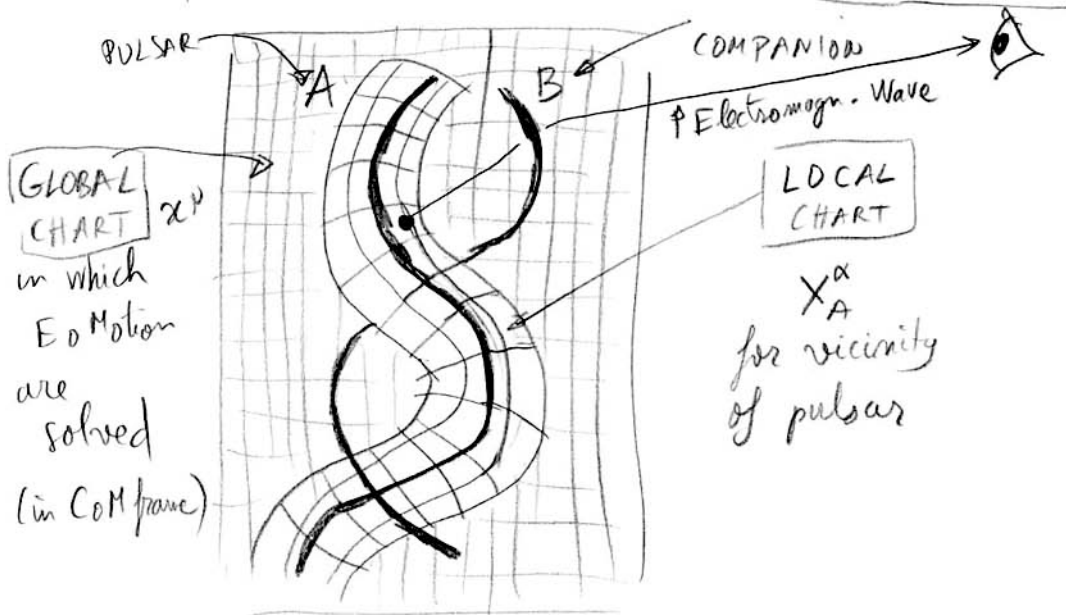
• Within solar system

↘ travel time in solar system

↘ gravitational delay (Shapiro)

↘ variable redshift $dT_{Earth}/dT_{Barycenter}$
(Einstein)

4.12 Various contributions to TIMING of PSR



need

$$x^\mu = z^\mu(\tau) + e_a^\mu(\tau) X^a + \frac{1}{2} f_{ab}^\mu(\tau) X^a X^b + \dots$$

↑
↑
 ('PROPER TIME' of PULSAR FRAME) ('ASYMPTOTIC FRAME')

$$ds^2 = -\left(1 - \frac{2GM}{c^2 R}\right) c^2 dT^2 + \left(1 + \frac{2GM}{c^2 R} + \dots\right) \delta_{ij} dx^i dx^j$$

Matching inner and outer problem →

$$\tau = \frac{1}{c} \int \sqrt{-g_{\mu\nu}(z_A^i) dz_A^\mu dz_A^\nu} \approx \int dt \sqrt{1 - \frac{2Gm_B}{c^2 r_{AB}} - \frac{\vec{v}_A^2}{c^2}} \approx \int dt \left(1 - \frac{Gm_B}{c^2 r_{AB}} - \frac{\vec{v}_A^2}{2c^2}\right)$$

'A-EXTERNAL METRIC' (see end of lecture 2)

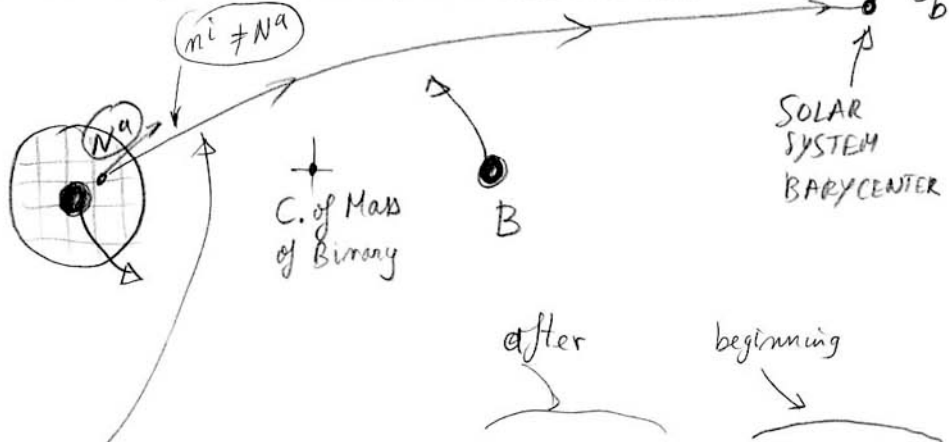
direction emitted radio wave

$$e_a^0 = \frac{v_a}{c} \left[1 + \frac{1}{2} \frac{\vec{v}^2}{c^2} + \frac{3Gm_B}{c^2 r_{AB}} + \dots\right] \rightarrow \text{ANGLE between } N^a|_{X^\alpha} \text{ and } n^i|_{x^\mu}$$

ABERRATION

$$\left[\delta_i^a + O\left(\frac{v^2}{c^2}\right) \right] n^i \approx N^a(\tau) + \frac{v_A^a}{c} - N^a \left(\frac{\vec{N} \cdot \vec{v}_A}{c} \right)$$

Propagation of electromagnetic signal



NULL GEODESIC: $0 = ds^2 = g_{\mu\nu}(x^\alpha) dx^\mu dx^\nu = G_{\alpha\beta}^A(x_A) dx_A^\alpha dx_A^\beta$

$$-(1 - \frac{2U}{c^2}) c^2 dt^2 + (1 + \frac{2U}{c^2}) d\vec{x}^2 = 0$$

$$dt \approx \frac{1}{c} (1 + \frac{2U}{c^2}) |d\vec{x}|$$

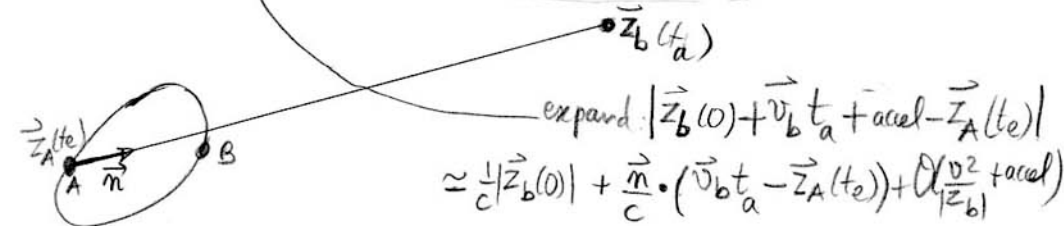
$$\int_{t_{match}}^{t_{arrival}} dt \approx \frac{1}{c} \int_{t_m}^{t_a} |d\vec{x}| + \frac{2}{c^3} \int_{t_m}^{t_a} \left(\frac{G m_A}{|\vec{x} - \vec{z}_A|} + \frac{G m_B}{|\vec{x} - \vec{z}_B|} \right) |d\vec{x}|$$

+ similar contributions in local frame with $\frac{G m_A}{|\vec{x}_A|}$ along straight line

expanded in tidal series, $G(t) \sim \frac{G m_B}{r_{AB}}$ absorbed in $x^\mu = f^\mu(x_A)$

Matching \rightarrow

$$t_{arrival} = t_{emission} + \frac{1}{c} |\vec{z}_B(t_a) - \vec{z}_A(t_e)| + \frac{2}{c^3} \int_{t_e}^{t_a} \frac{G m_B}{|\vec{x} - \vec{z}_B|} |d\vec{x}|$$



4.13 The DD Timing formula

(Damour & Deruelle '86) AGR 4.22

PULSAR
PROPER
TIME
 T



$$\frac{\Phi}{2\pi} \equiv N = N_0 + \nu_P T + \frac{1}{2} \dot{\nu}_P T^2 + \frac{1}{6} \ddot{\nu}_P T^3$$

Rotational Phase # of turns PSR frequency $\nu_P = \frac{1}{P}$

integer $N \iff T'_N =$ proper time of emission of N th pulse, corrected for aberration

then

$$D \cdot \tau_a = T + \Delta_R(T) + \Delta_E(T) + \Delta_S(T) + \Delta_A(T)$$

↑ Doppler factor ↑ Proper time of arrival at solar barycenter (corrected for dispersion + solar-system analogs of $\Delta_R, \Delta_E, \Delta_S$) ↑ ROEMER time delay ↑ EINSTEIN time delay ↑ SHAPIRO time delay ↑ ABERRATION time delay

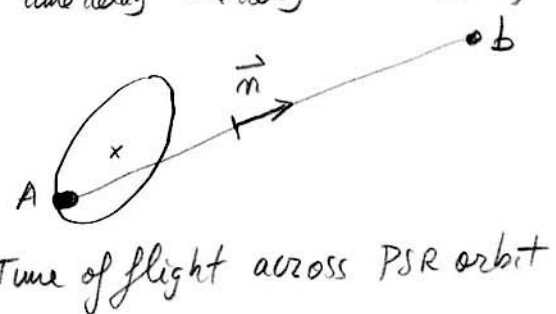
$$D = \frac{1 - \vec{m} \cdot \vec{v}_b / c}{\sqrt{1 - v_b^2 / c^2}}$$

$$\Delta_R = -\frac{1}{c} \vec{m} \cdot \vec{z}_A$$

$$\Delta_E = +\frac{1}{c^2} \int dt \left(\frac{G_{mB} + \frac{1}{2} \vec{v}_A^2}{r_{AB}} - \left\langle \frac{G_{mB} + \frac{1}{2} \vec{v}_A^2}{r_{AB}} \right\rangle \right)$$

$$\Delta_S = +\frac{1}{c^3} \int_{t_e}^{t_a} \frac{2 G_{mB}}{|\vec{x} - \vec{z}_B|} d\vec{x}$$

$$\Delta_A = \frac{1}{2\pi \nu_P} \frac{\vec{v}_A \cdot (\vec{m} \times \vec{e}_3)}{c (\vec{m} \times \vec{e}_3)^2}$$



after rescaling T by mean GR+SR effects in binary \rightarrow modifies D

Orders of magnitude of timing effects

$$D = 1 + \mathcal{O}\left(\frac{v_b^{\text{solar}} - v_{\text{com}}^{\text{binary}}}{c}\right) \approx 1 + \mathcal{O}(10^{-3})$$

orbital effects



$$\frac{GM}{a} \sim v^2$$

$$v \sim \sqrt{ma} \sim \frac{2\pi}{P_b} a$$

can be exactly absorbed in some rescalings, except for its time-dependent piece $\propto \frac{\text{accel.}}{d}$

typically $\left(\frac{v}{c} \sim (\text{few}) \times 10^{-3} \right) \left| \frac{P_b}{a} \sim \text{second} \right.$

$$\begin{aligned} \Delta_R &\sim \frac{a}{c} \left[1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} + \dots \right] \sim \frac{v}{c} \frac{P_b}{2\pi} \left[1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} + \dots \right] \\ &\sim \frac{a}{c} + \frac{GM}{c^3} + \frac{v^2}{c^2} \frac{GM}{c^3} \\ &\sim \underline{\underline{5}} + \underline{\underline{\text{few } \mu\text{s}}} + 10^{-12} \text{ s} \end{aligned}$$

$$\boxed{\frac{GM_{\odot}}{c^3} = 4.92549 \dots \mu\text{s}}$$

$$\Delta_E \sim \frac{v^2}{c^2} \frac{P_b}{2\pi} \sim \frac{v}{c} \Delta_R \left[1 + \frac{v^2}{c^2} + \dots \right] \sim \underline{\underline{\text{ms}}} + 10^{-9} \text{ s}$$

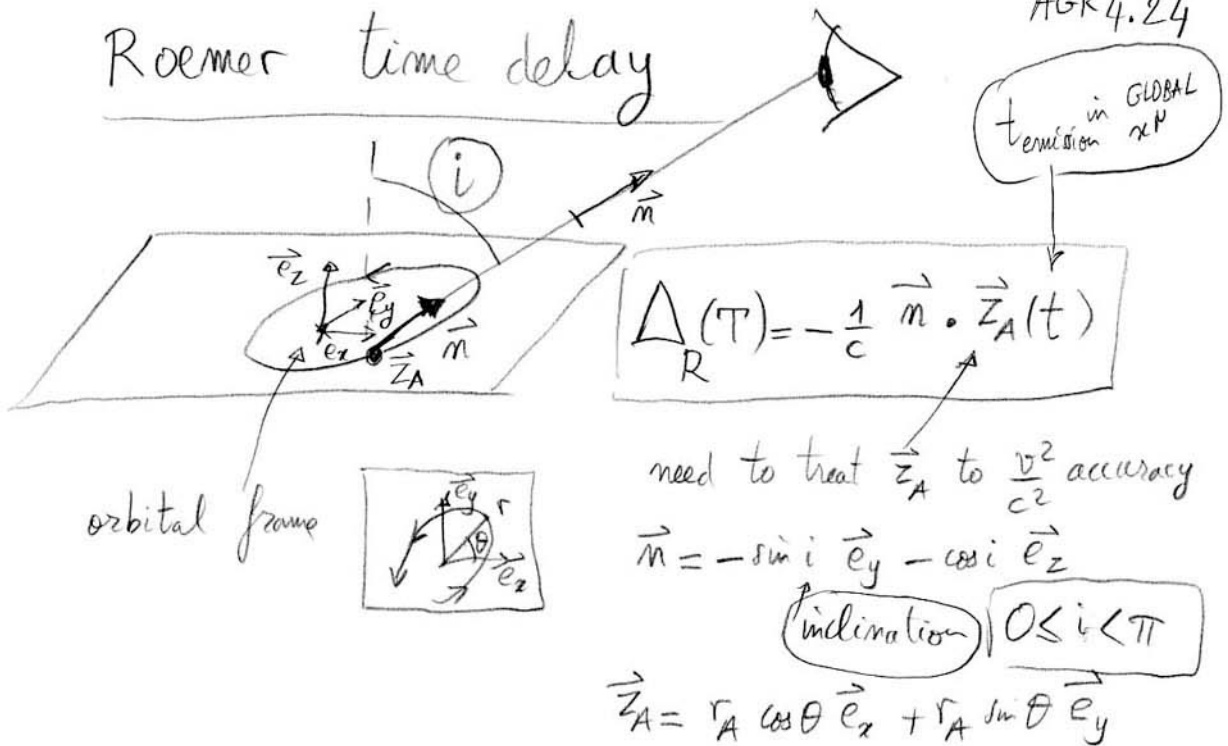
$$\Delta_S \sim \frac{v^3}{c^3} \frac{P_b}{2\pi} \sim \frac{v^2}{c^2} \Delta_R \sim \frac{GM}{c^3} \sim \underline{\underline{\mu\text{s}}} + 10^{-9} \text{ s} \leftarrow \text{from } \mathcal{O}\left(\frac{v}{c}\right) \text{ corrections}$$

$$\Delta_A \sim \frac{v}{c} \frac{P_b}{2\pi} \sim 10^{-3} \frac{P_p}{2\pi}$$

\Rightarrow 1PK TIMING FORMULA ACCURATE ENOUGH

Roemer time delay

AGR 4.24



$$\Delta_R = -\frac{1}{c} \vec{n} \cdot \vec{z}_A(t) = +\frac{1}{c} \sin i r_A \sin \theta$$

$$r_A = a_r (1 - e_r \cos u)$$

$$a_r = \left(\frac{m_B}{M} + \frac{D}{c^2} \right) a_R$$

$$\theta = \omega_0 + (1+k) 2 \arctan \left[\sqrt{\frac{1+e_r}{1-e_r}} \tan \frac{u}{2} \right]$$

$A_{e_r}(u)$

where $n(t-t_0) = u - e_f \sin u$

but $T = \int_{t_0}^{t_e} dt \left(1 - \frac{G m_B}{c^2 r_{AB}} - \frac{1}{2} \frac{v_A^2}{c^2} + \langle \rangle \right) = t - \Delta_E$

$$\Delta_E = \frac{G m_B}{c^2} \left(1 + \frac{m_B}{M} \right) \int \frac{dt}{R} - \langle \rangle dt = \frac{e}{n} \frac{G m_B (1 + m_B/M)}{c^2 a_R} \sin u$$

i.e. $t_e = T + \Delta_E = T + \gamma \sin u$

AGR 4.25

At this stage

$$\boxed{t \rightarrow T} \Rightarrow n(T - T_0) = u - \overset{0}{e}_T \sin u$$

$$\overset{0}{e}_T = e_t \left(1 + \frac{Gm_B(1+x_B)}{c^2 a_R} \right)$$

$x_B \equiv \frac{m_B}{M}$

Can further simplify unpleasant ratios $\frac{1 - e_r \cos u}{1 - e_\theta \cos u}$ by redefining the relativistic eccentric anomaly u

$$\boxed{u \rightarrow u^{\text{new}}} : n(T - T_0) = u^{\text{new}} - \overset{0}{e}_T \sin u^{\text{new}} \quad e_T = e_T^0 + e_\theta - e_r$$

Final DD Timing Formula

$$N = N_0 + \dot{\nu}_p T + \frac{1}{2} \ddot{\nu}_p T^2 + \frac{1}{6} \overset{\circ\circ}{\nu}_p T^3$$

$$D \overset{\text{arrival}}{\underset{\text{barycenter}}{T}} = T + \Delta_R(T) + \Delta_E(T) + \Delta_S(T) + \Delta_A(T)$$

$$\Delta_R(T) = \alpha \sin \omega [\cos u - e(1 + \delta_r)] + \alpha \sqrt{1 - e^2(1 + \delta_\theta)^2} \cos \omega \sin u$$

$$\Delta_E(T) = \gamma \sin u$$

$$\Delta_S(T) = -2r \ln \left\{ 1 - e \cos u - s [\sin \omega (\cos u - e) + \sqrt{1 - e^2} \cos \omega \sin u] \right\}$$

$$\Delta_A(T) = A \left\{ \sin[\omega + A_e(u)] + e \sin \omega \right\} + B \left\{ \cos[\omega + A_e(u)] + e \cos \omega \right\}$$

with

$$\alpha = \alpha_0 + \dot{\alpha} (T - T_0), \quad e = e_0 + \dot{e} (T - T_0)$$

$$\omega = \omega_0 + k A_e(u) \leftarrow$$

$$A_e(u) \equiv 2 \arctan \left[\sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2} \right]$$

$$u - e \sin u = 2\pi \left[\frac{T - T_0}{P_b} - \frac{1}{2} \dot{P}_b \left(\frac{T - T_0}{P_b} \right)^2 \right]$$

4.14

PARAMETRIZED POST-KEPLERIAN FORMALISM

(Damour Taylor '92)

- Fact: The mathematical structure of the DD timing formula

$$T_a = F [T_N ; \underbrace{P_b, T_0, e_0, \omega_0, x_0}_{\substack{\text{Keplerian parameters} \\ P_i^{PK}}}; \underbrace{k, \gamma, \dot{P}_b, r, s, \delta_\theta, \dot{e}, \dot{x}}_{\substack{\text{separately measurable} \\ \text{post-Keplerian param.} \\ P_i^{PK}}}; \underbrace{\delta_r, A, B, D}_{\substack{\text{not separately} \\ \text{measurable} \\ \text{post-Kepler. para.} \\ q_i^{PK}}}]$$

set to FIDUCIAL VALUES

holds in a large class of ^{alternative} relativistic theories of gravity: tensor-scalar

- the PK param. P_i^{PK} reflect various aspects of relativistic gravity, notably strong-field and/or radiative effects

k : periastron advance: depend on $\alpha_A \left(\overset{\text{spin}}{\bullet} \right) \alpha_B$

γ : Einstein delay: Einstein + SR redshift, depend on $\overset{A}{\bullet} \overset{G_{AB}}{\text{---}} \overset{B}{\bullet} \dots$

\dot{P}_b : radiation damping, spin + non-linearities

r, s : range and shape of Shapiro delay, depend light

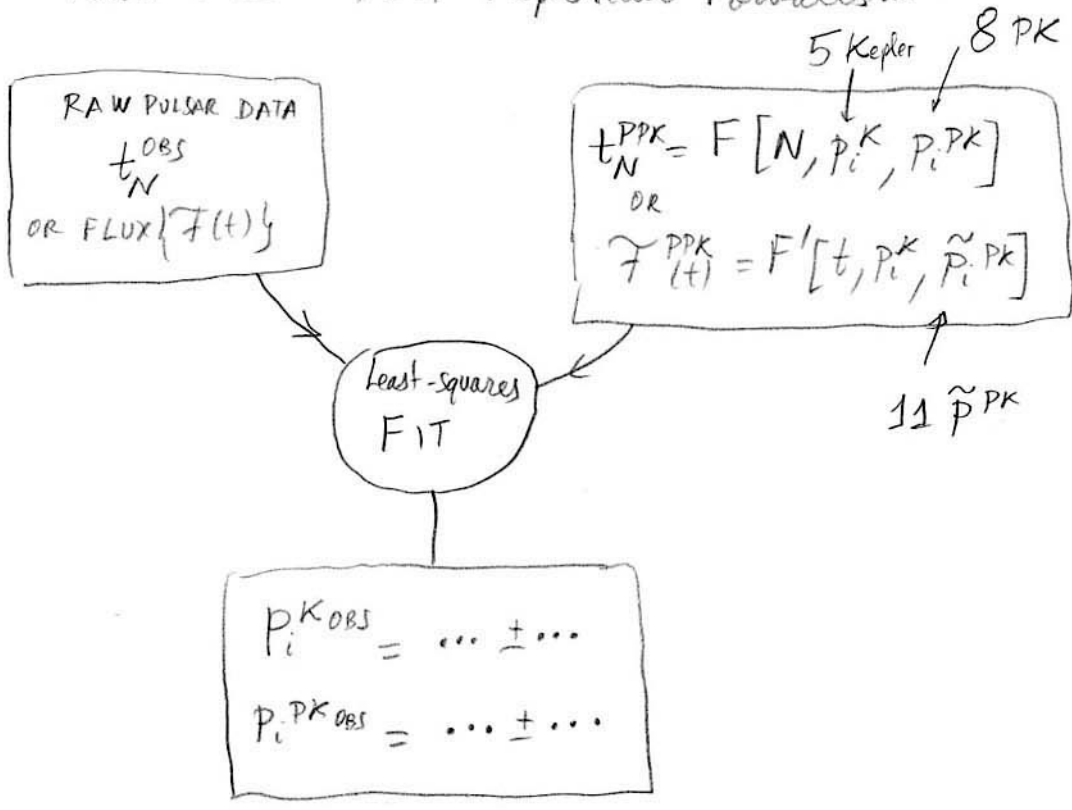
δ_θ : depend on many terms in 1PK Lagrangian

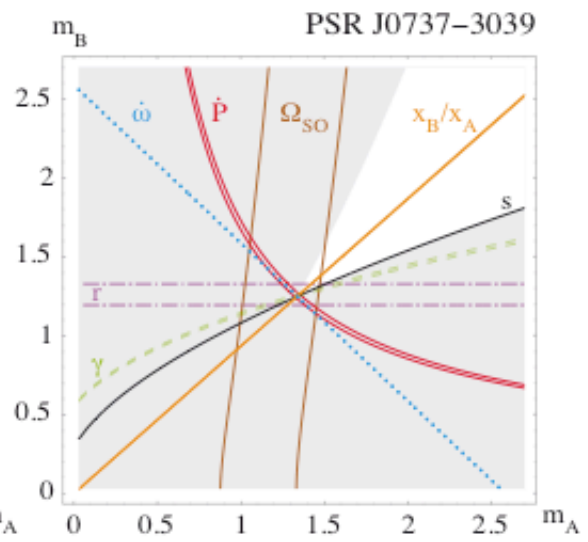
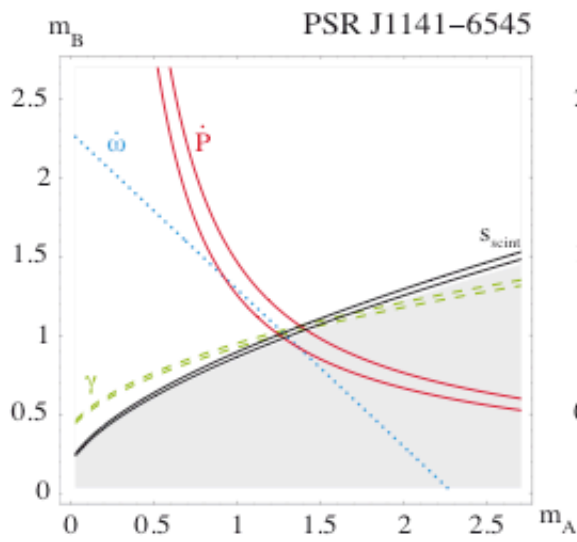
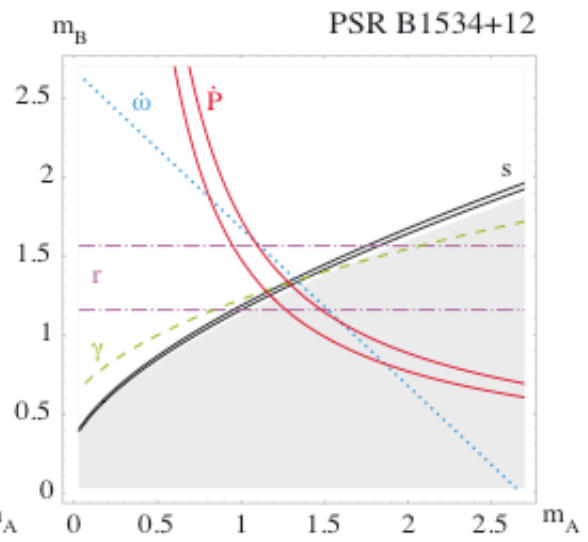
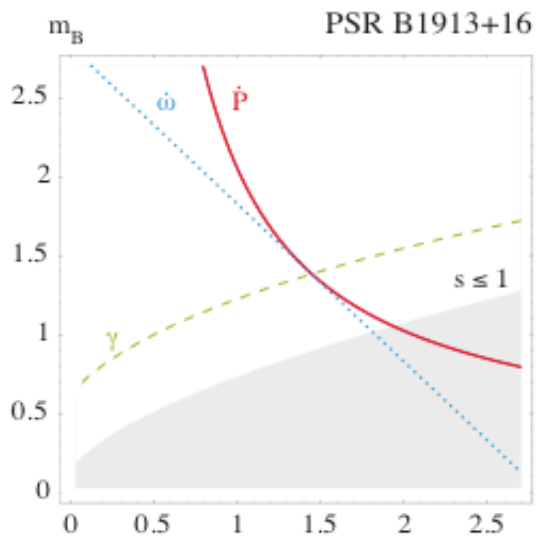
\dot{e}, \dot{x} : depend on radiation damping and spin-orbit effects
notably after absorbing A, B

- the relations $P_i^{PK} = f(P_i^K; m_A, m_B)$ depend on the theory of gravity and will be strongly modified wrt GR, if effacement is violated

PHENOMENOLOGICAL ANALYSIS OF BINARY PULSAR DATA

(Parametrized Post Keplerian Formalism)

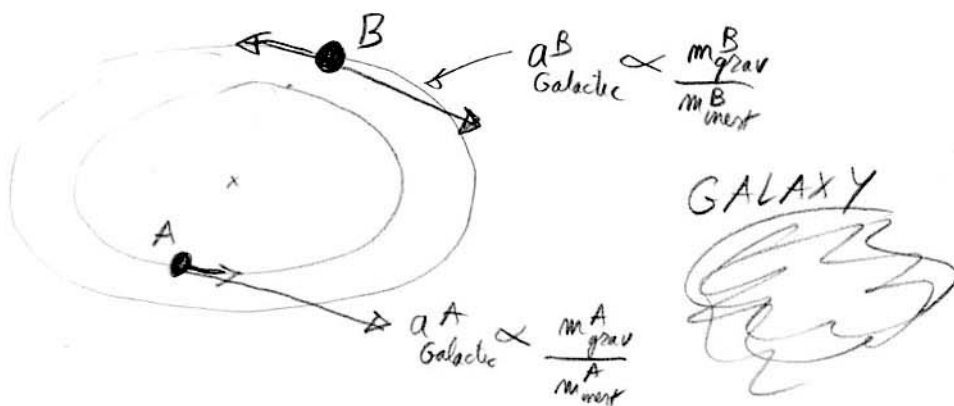




Other phenomenological tests:

Binary pulsar tests of violation of Strong Equivalence Principle
(Damour, Schäfer 1991)

Certain quasi-circular neutron-star white-dwarf systems



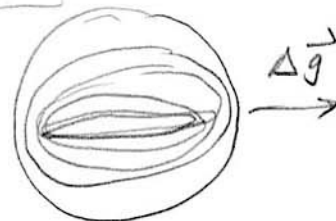
if $\Delta = \frac{m_{grav}^A}{m_{inert}^A} - \frac{m_{grav}^B}{m_{inert}^B} \neq 0 \Rightarrow$ 'gravitational Stark effect'

$$\frac{d^2 \vec{r}}{dt^2} + G_{AB} M \frac{\vec{r}}{r^3} + v^{3/2} \text{ terms} = \Delta \vec{g}_{galactic}$$

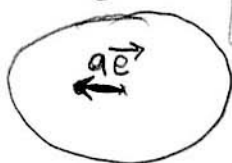
like $e \vec{E}_{external}$

Secular instability of Keplerian ellipse:

→ POLARIZATION after including $v^{3/2}$ terms



eccentricity vector



$$\vec{e}(t) = \vec{e}_{Relativistic(t) \text{ Precession}} + \vec{e}_{\Delta}$$

latest data
Stairs et al 05

$$e_{\Delta} = \frac{3}{2} \frac{\Delta g_{\perp}}{\dot{\omega}_R m a}$$

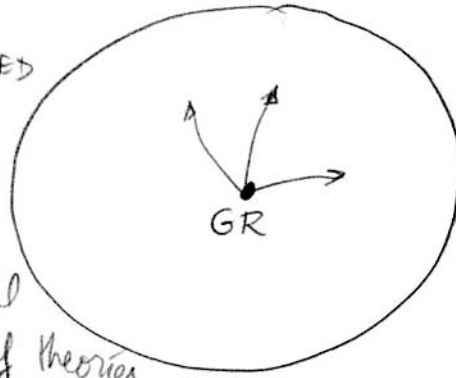
$$\Delta \lesssim 5 \times 10^{-3}$$

4.15 Theory-space approach to testing relativistic gravity with binary pulsar data

Instead of

YES or NO
GR or GR

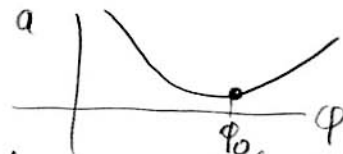
EMBED
GR
WITHIN
A
multi-
dimensional
Space of theories



Need to go beyond the weak-field PPN parametrization

"post-Einstein" para.: $\bar{\gamma} = \gamma^{PPN} - 1$; $\bar{\beta} = \beta^{PPN} - 1$; ...

E.g. TENSOR-SCALAR GRAVITY:



Arbitrary coupling function $a(\phi)$ if only one scalar ϕ

tensor-multi-scalar: $\gamma_{ab}(\phi^c) \nabla_\mu \phi^a \nabla^\mu \phi^b \oplus a(\phi^a)$
 $\phi^a = (\phi^1, \dots, \phi^n)$
(Damian Esposito-Farese 1992)
 \uparrow METRIC in ϕ -space \uparrow COUPLING FUNCTION

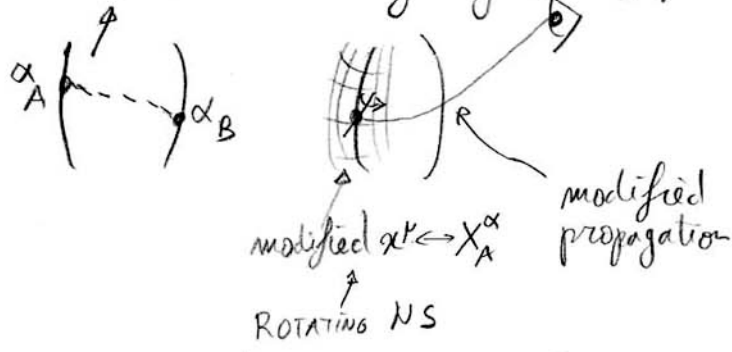
Various possibilities:

- Beyond PPN: $\bar{\gamma}$, $\bar{\beta}$, ϵ , ζ + expansion in $\frac{GM_A}{c^2 R_A}$
- Special classes of tensor-bi-scalar $T_2(\beta', \beta'')$

- One scalar and 2-parameter $T_1(\alpha_0, \beta_0)$: $a(\phi) = \alpha_0(\phi - \phi_0) + \frac{1}{2}\beta_0(\phi - \phi_0)^2$
 strong-field generalization of PPN

Modifications in $P_i^{PK} = f(P^K; m_A, m_B)$ AGR 4.30

Need to rederive Motion + Timing of binary pulsar



has $J = I(\varphi_{AA}) \Omega_A$
 adiabatic invariant \uparrow varies along orbit

E.g. $M = m_A + m_B$
 $\chi_A \equiv \frac{m_A}{M}$ $\chi_B \equiv \frac{m_B}{M}$

$$R^{GR}(m_A, m_B) = \frac{3}{1-e^2} \left(\frac{GMm}{c^3} \right)^{2/3} \rightarrow R^{TS}(m_A, m_B) = \frac{3}{1-e^2} \left(\frac{G_{AB} M m}{c^3} \right)^{2/3} \times \left[\frac{1 - \frac{1}{3} \chi_A \chi_B}{1 + \chi_A \chi_B} - \frac{\chi_A \chi_B \alpha_A^2 + \chi_B \chi_A \alpha_B^2}{6(1 + \chi_A \chi_B)^2} \right]$$

$$\gamma^{GR}(m_A, m_B) = \frac{e}{m} \left(\frac{GMm}{c^3} \right)^{2/3} \chi_B (\chi_B + 1) \rightarrow \gamma^{TS} = \frac{e}{m} \left(\frac{G_{AB} M m}{c^3} \right)^{2/3} \chi_B \left[\chi_B (1 + \chi_A \chi_B) + 1 + \chi_A \chi_B \right] \frac{\rightarrow \partial \ln I_A(\varphi_0) / \partial \varphi_0}{1 + \chi_A \chi_B}$$

$$\dot{P}_b^{GR}(m_A, m_B) = -\frac{192\pi}{5} \left(\frac{GMm}{c^3} \right)^{5/3} \frac{1 + \frac{73}{24} e^2 + \frac{37}{96} e^4}{(1-e^2)^{7/2}} \chi_A \chi_B \rightarrow \dot{P}_b^{TS} = \dot{P}_b^{quadrupole} + \dot{P}_b^{dipole} + \dots$$

\uparrow
 $\sim v^5/c^5$ $\sim v^3/c^3$!
 $\propto (\alpha_A - \alpha_B)^2$

Non-perturbative strong-field effects

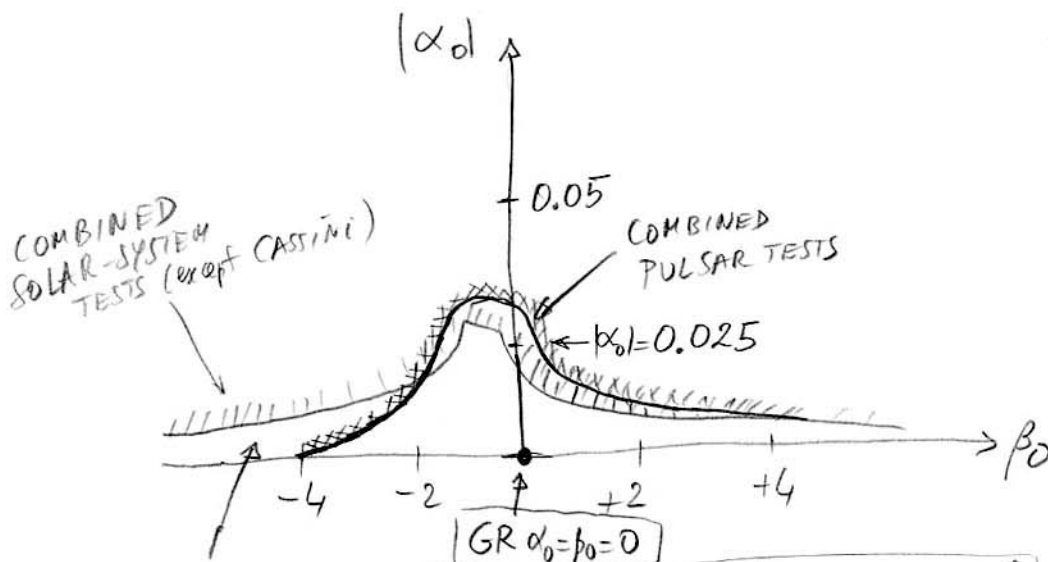
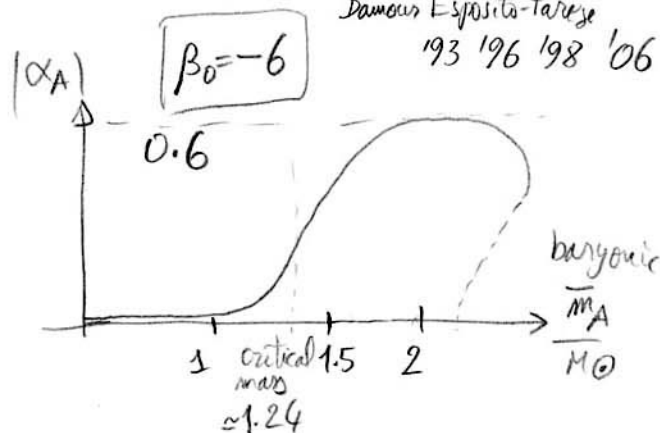
AGR 4.31

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Even if $\alpha_0 \ll 1$

$$\bar{\gamma} = -\frac{2\alpha_0^2}{1+\alpha_0^2} \ll 1$$

$$\bar{\beta} = \frac{1}{2} \frac{\beta_0 \alpha_0^2}{(1+\alpha_0^2)^2} \ll 1$$



BINARY PULSARS exclude a whole domain of theory space ($\beta_0 \lesssim -4$) which is allowed by all solar-system tests

(see course of Esposito-Farese and Bourbaphy 28 October: Kromer)

This shows that binary pulsar tests do go beyond stationary-weak-field tests and probe strong-field + radiative regime of relativistic gravity \rightarrow deep confirmations of GR in many of its essential aspects (non-linear, wave propagation, ...) [spin-orbit effects have also been confirmed]