SAINT THOMAS 0
16-21 March 06
"CONFRONTING GRAVITY"

THEORETICAL CHALLENGES IN COALESCING BINARY BLACK HOLES

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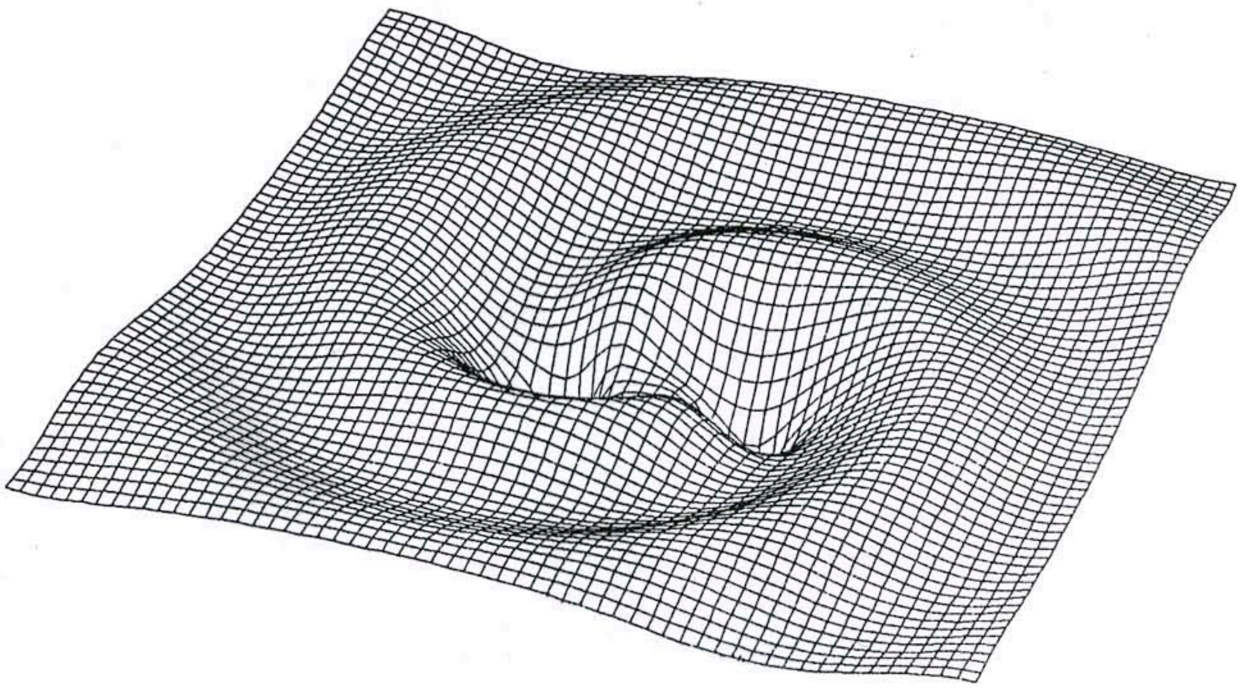
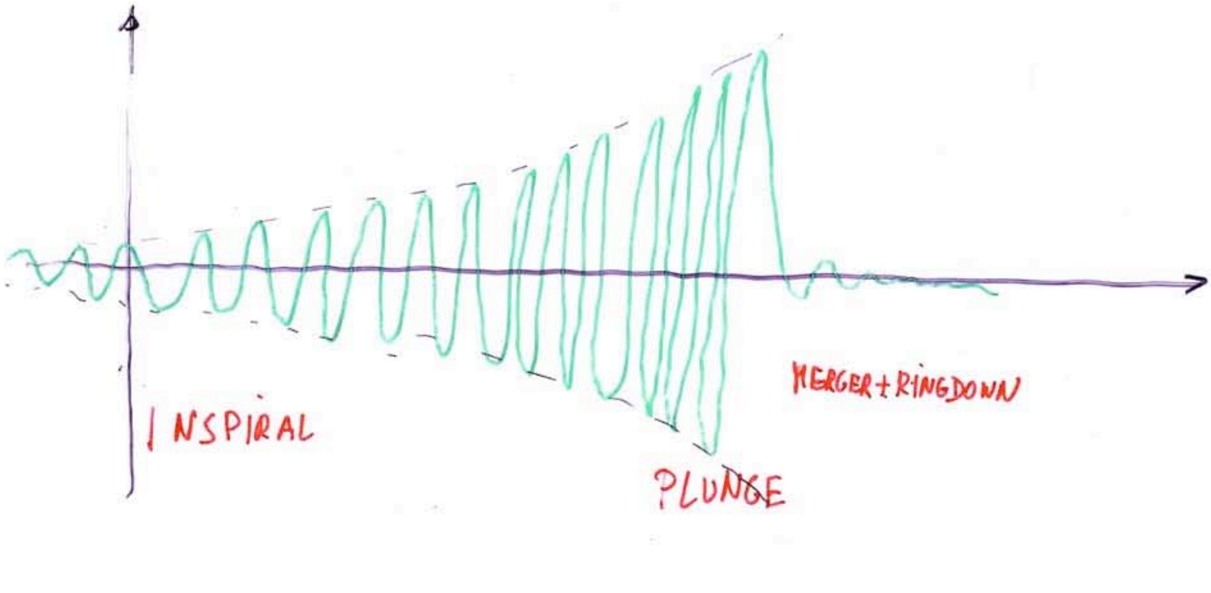
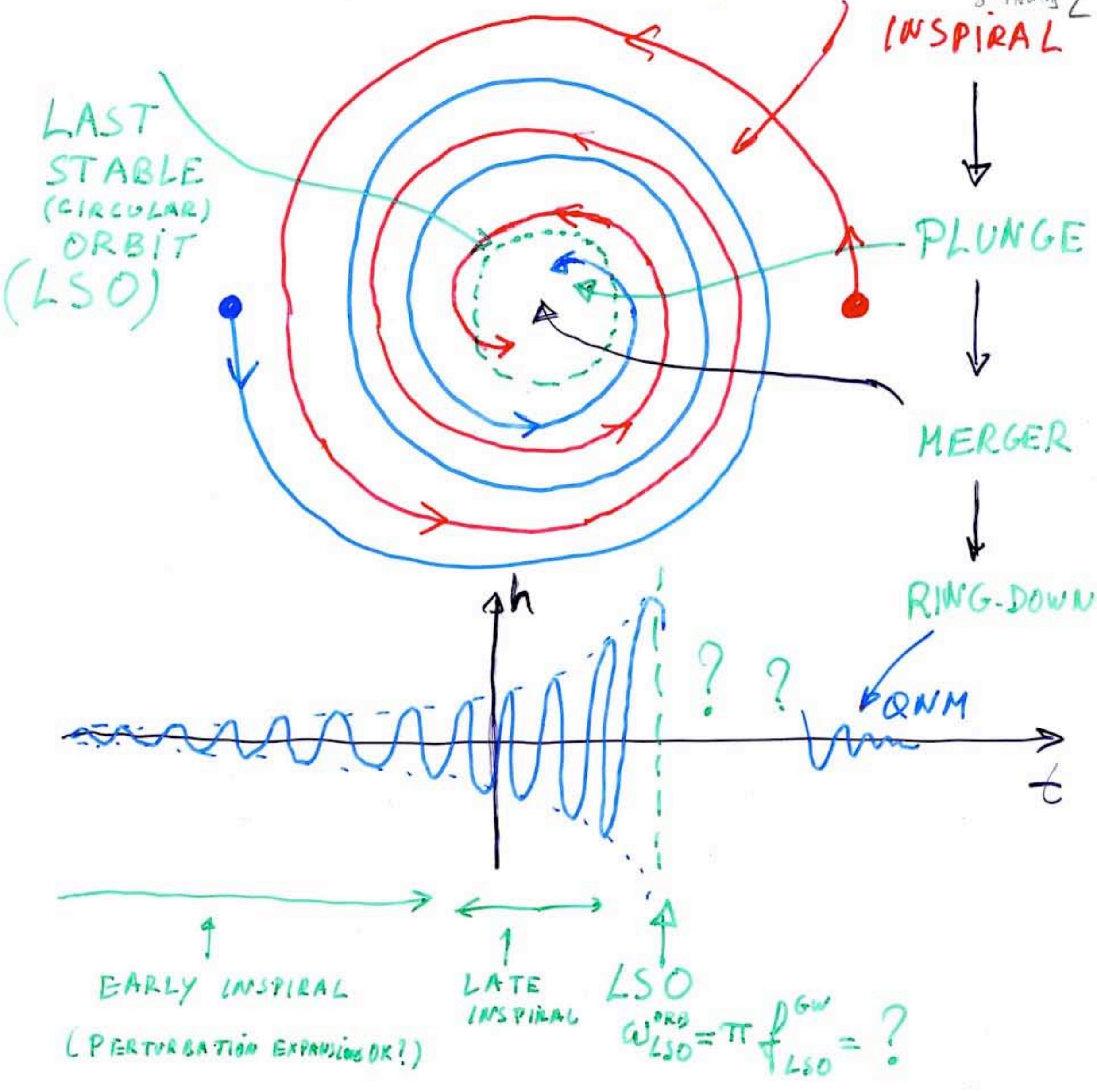
T 13/2 22
BLOG
StThomas 1

COALESCING BINARY





THE PROBLEM



USUAL LORE:

CAN USE ANALYTICAL RESULTS (PERTURBATIVE) → ← MUST USE NUMERICAL RESULTS FOR LATE INSPIRAL + PLUNGE + MERGER + RING-DOWN →

OUR CLAIM

CAN USE ANALYTICAL RESULTS → + SOME ROUGH ESTIMATE OF MERGER

Nia 4.4

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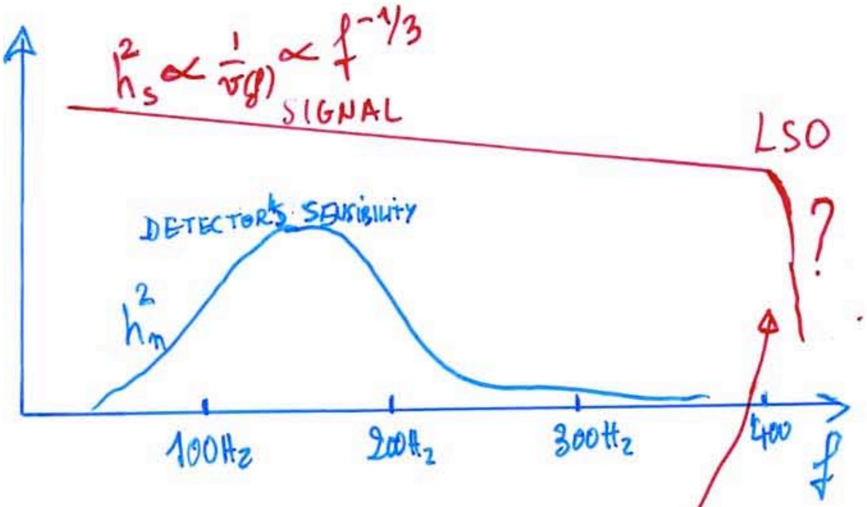
BH NS

$(10\,M_\odot, 1{,}4\,M_\odot)$

SYSTEM

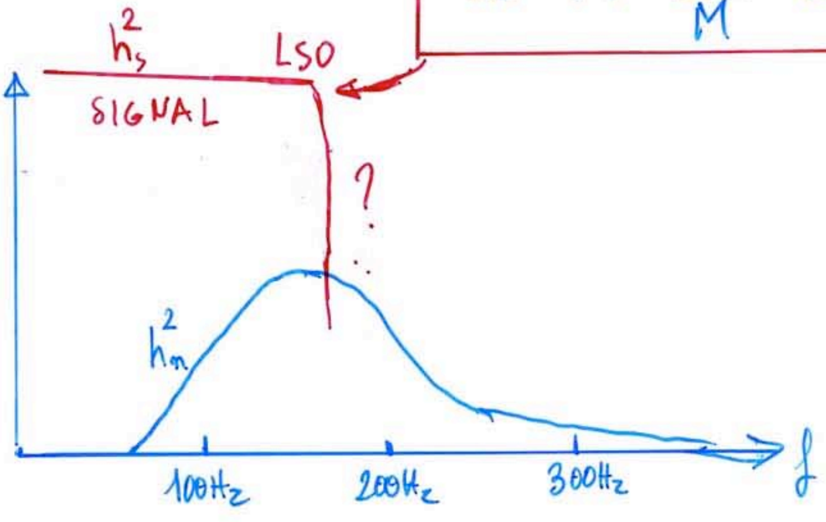


LSO CUTOFF EXPECTED AROUND $F_{LSO} \sim 4400 \frac{M_\odot}{M}$ Hz

BH BH

$(15\,M_\odot, 15\,M_\odot)$

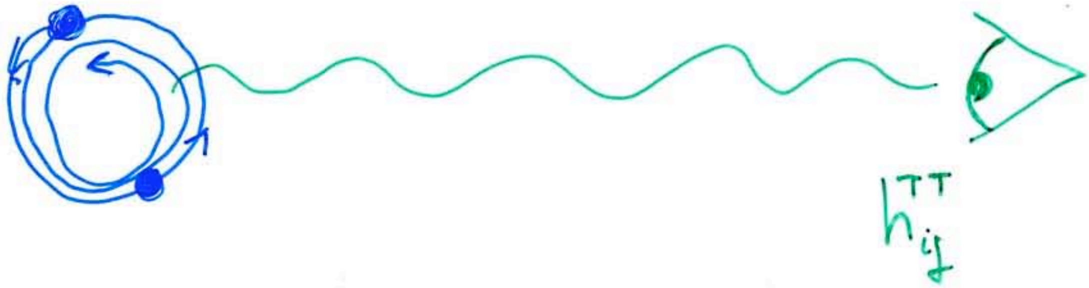
SYSTEM



(1,4, 1.4)
(10, 1.4) SYSTEMS : STANDARD PN TEMPLATE WAVEFORMS OK

(10, 10)
(15, 15) SYSTEMS : NEED GOOD TEMPLATES AROUND LSO

THEORETICAL CHALLENGES



- NEED EQUATIONS OF MOTION TO VERY HIGH PERTURBATION ORDER
- NEED GW GENERATION FORMALISM TO HIGH PERTURBATION ORDER
- NEED RESUMMATION METHODS BECAUSE
  - PERTURBATION EXPANSION OF PHASING BOTH SLOWLY AND ERRATICALLY CONVERGENT
  - TRANSITION INSPIRAL → "PLUNGE" TRIGGERED BY NON-PERTURBATIVE RELATIVISTIC EFFECTS
- NEED (QUASI-) ANALYTICAL DESCRIPTION OF INSPIRAL + PLUNGE + MERGER SIGNAL BECAUSE OF NEED OF (NEARLY) CONTINUOUS FAMILY OF ACCURATE GW TEMPLATES DEPENDING ON $\gtrsim 6$ (INTRINSIC) REAL PARAMETERS

ADVOCATED PHILOSOPHY

- TAKE ADVANTAGE OF GREAT FLEXIBILITY OF ANALYTICAL APPROACHES
- CAN EASILY VARY CONTINUOUS PARAMETERS: $m_1, m_2, \vec{S}_1, \vec{S}_2$
- CAN INCORPORATE PERTURBATIVE AND NON-PERTURBATIVE INFORMATION FROM NUMERICAL RELATIVITY

"FLEXIBILITY PARAMETERS" IN ANALYTICAL DESCRIPTION CAN BE FITTED TO NUMERICAL EXPERIMENTS

BOTH IN HAMILTONIAN AND WAVE FORM OF THE

EFFECTIVE ONE-BODY APPROACH

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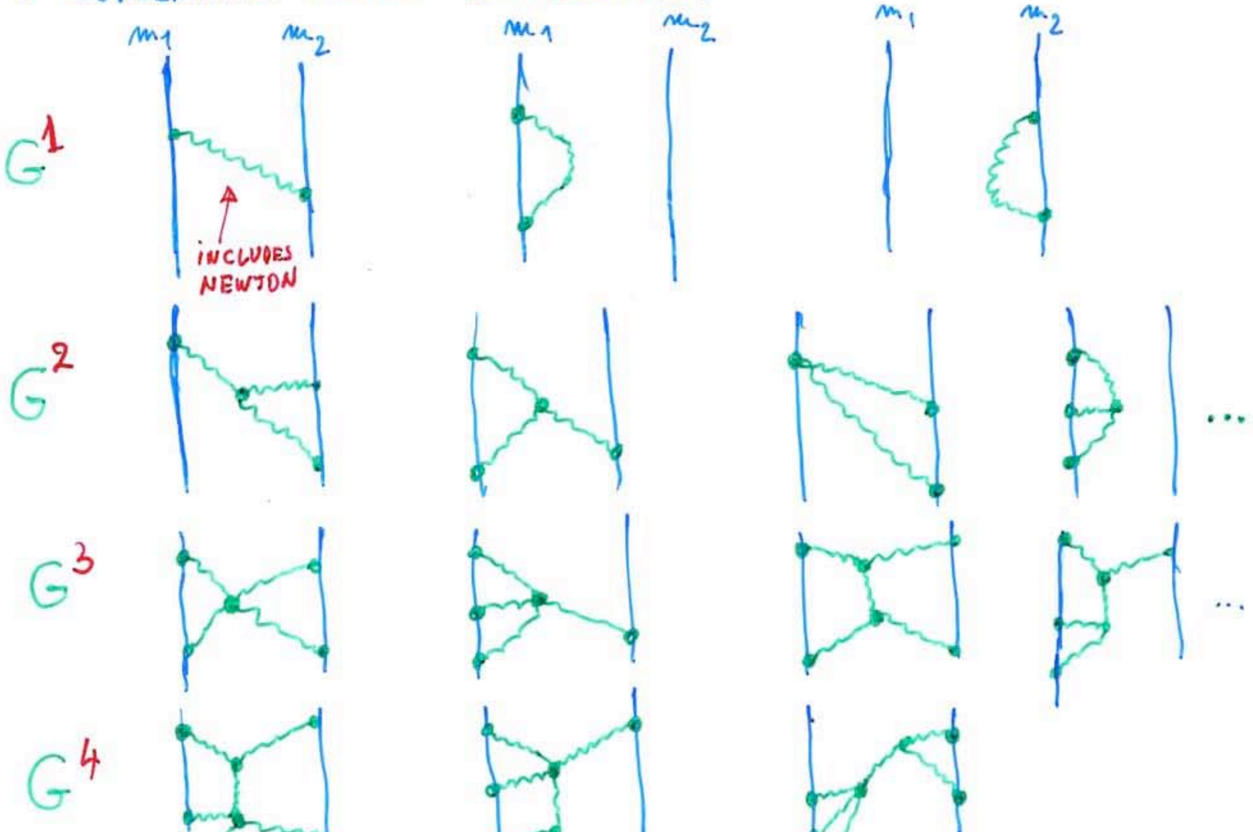
HIGH-PERTURBATION ORDER CALCULATIONS OF EQS. OF MOTION 6

- REPLACE

BECAUSE OF **EFFACEMENT OF INTERNAL STRUCTURE** UP TO $\left(\frac{v}{c}\right)^{10} \sim$ 5PN ORDER
(Damour, '82)

- CONSERVATIVE PART OF HAMILTONIAN :



- NEED **DIMENSIONAL REGULARIZATION** ('t Hooft Veltman '72) TO COMPUTE G^4 (3PN ~ 3 LOOP) (Damour Jaranowski Schäfer '00, Blanchet Damour Esposito F. '04)

3 PN HAMILTONIAN (Damour, Jaranowski, Schäfer '00) 17

$$H(\mathbf{x}_a,\mathbf{p}_a)=\sum_a m_a c^2 + H_{\rm N}(\mathbf{x}_a,\mathbf{p}_a)+\frac{1}{c^2}H_{\rm 1PN}(\mathbf{x}_a,\mathbf{p}_a)+\frac{1}{c^4}H_{\rm 2PN}(\mathbf{x}_a,\mathbf{p}_a)+\frac{1}{c^6}H_{\rm 3PN}(\mathbf{x}_a,\mathbf{p}_a)+\mathcal{O}\left(\frac{1}{c^8}\right). \tag{5}$$

At the Newtonian order, i.e. when keeping the rest-mass term $\Sigma_a m_a c^2$ and the Newtonian-level Hamiltonian,

$$H_{\rm N}(\mathbf{x}_a,\mathbf{p}_a)=\sum_a \frac{\mathbf{p}_a^2}{2m_a}-\frac{1}{2}\sum_a\sum_{b\neq a}\frac{G\,m_a\,m_b}{r_{ab}}, \tag{6}$$

$$H_{\rm 1PN}(\mathbf{x}_a,\mathbf{p}_a)=-\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left[-12\frac{\mathbf{p}_1^2}{m_1^2}+14\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right]$$
$$+\frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}}+(1\longleftrightarrow 2),$$

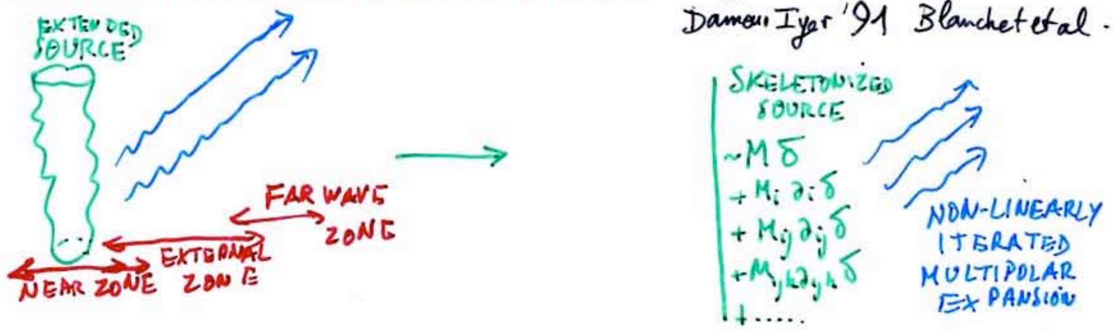
$$H_{\rm 2PN}(\mathbf{x}_a,\mathbf{p}_a)=\frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left[5\frac{(\mathbf{p}_1^2)^2}{m_1^4}-\frac{11}{2}\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2}-\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}+5\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right.$$
$$\left.-6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}-\frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right]$$
$$+\frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left[m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2}+19\frac{\mathbf{p}_2^2}{m_2^2}\right)-\frac{1}{2}(m_1+m_2)\frac{27(\mathbf{p}_1\cdot\mathbf{p}_2)+6(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right]$$
$$-\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2+5m_1m_2+m_2^2)}{r_{12}^2}+(1\longleftrightarrow 2).$$

$$H^{\rm reg}_{\rm 3PN}(\mathbf{x}_a,\mathbf{p}_a)=-\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7}+\frac{1}{32}\frac{Gm_1m_2}{r_{12}}\left[-14\frac{(\mathbf{p}_1^2)^3}{m_1^6}+4\frac{\big((\mathbf{p}_1\cdot\mathbf{p}_2)^2+4\mathbf{p}_1^2\mathbf{p}_2^2\big)\mathbf{p}_1^2}{m_1^4m_2^2}+\frac{\big(\mathbf{p}_1^2\mathbf{p}_2^2-2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\big)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3}\right.$$
$$-10\frac{\big(\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2+\mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\big)\mathbf{p}_1^2}{m_1^4m_2^2}+24\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2}+2\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}$$
$$+\frac{\big(7\mathbf{p}_1^2\mathbf{p}_2^2-10(\mathbf{p}_1\cdot\mathbf{p}_2)^2\big)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3}+6\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2}$$
$$\left.+15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}-18\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}+5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\right]$$
$$+\frac{G^2m_1m_2}{r_{12}^2}\left[\frac{1}{16}(m_1-27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4}-\frac{115}{16}m_1\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2}+\frac{1}{48}m_2\frac{25(\mathbf{p}_1\cdot\mathbf{p}_2)^2+371\,\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2}\right.$$
$$+\frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3}-\frac{1}{8}m_1\frac{\big(15\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)+11(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)\big)(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2}+\frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3}$$
$$-\frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2}+\frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}+\frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}$$
$$\left.-\frac{1}{48}(220m_1+193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right]+\frac{G^3m_1m_2}{r_{12}^3}\left[-\frac{1}{48}\left(466\,m_1^2+\left(473-\frac{3}{4}\pi^2\right)m_1m_2+150\,m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\right.$$
$$+\frac{1}{16}\left(77(m_1^2+m_2^2)+\left(143-\frac{1}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+\frac{1}{16}\left(61\,m_1^2-\left(43+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}$$
$$\left.+\frac{1}{16}\left(21(m_1^2+m_2^2)+\left(119+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right]$$
$$+\frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left[\left(\frac{227}{3}-\frac{21}{4}\pi^2\right)m_1+m_2\right]+(1\longleftrightarrow 2). \tag{12}$$

HIGH-PERTURBATION ORDER CALC. OF GRAVIT WAVEFORMS St THOMAS 7

MULTIPOLAR POST-MINKOWSKIAN FORMALISM

Bonnor '59 Thorne '80 Blanchet Damour '86
Damour Iyer '91 Blanchet et al ...



+ MATCHING

RELATIVISTIC MULTIPOLE MOMENTS OF SOURCE

- IN EXTERNAL-ZONE ALGORITHM — EXTERNAL ('SOURCE') MOMENTS

$$\left\{ I(t),\ I_i(t),\ I_{\langle ij \rangle}(t),\ I_{\langle ijk \rangle}(t),\ \cdots \right\}$$

$$\left\{ J_i(t),\ J_{\langle ij \rangle}(t),\ J_{\langle ijk \rangle}(t),\ \cdots \right\}$$

+ "GAUGE MULTIPOLES" $W_{\dots}$ ....

- IN FAR WAVE-ZONE — RADIATIVE MOMENTS

$$h_{ij}^{TT} = \frac{4G}{c^2 R}\left[U_{ij}\left(T - \frac{R}{c}\right) + U_{ijk}(T-R)\, N^k + \dots - \frac{4}{3c} N_a \epsilon_{ab(i} V_{j)b}\left(T - \frac{R}{c}\right) - \dots \right]^{TT}$$

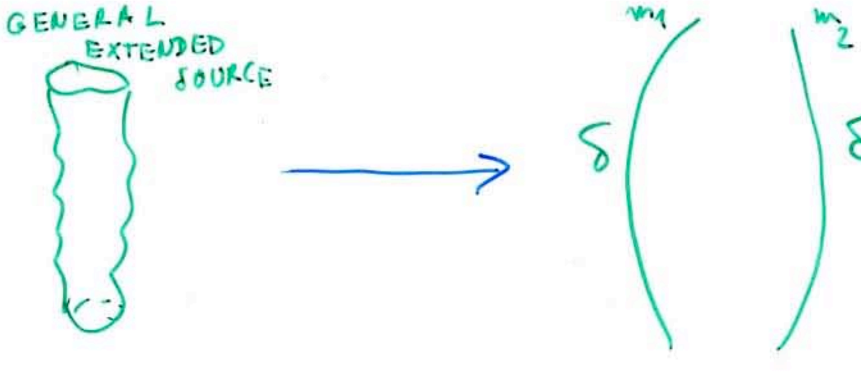
- LINKS: SOURCE ⟶ EXTERNAL MOMENTS ⟶ RADIATIVE MOMENTS

$$U_{ij}(T) = \overset{(2)}{I}_{ij}(T) + \frac{2GM}{c^3}\int_0^{+\infty} d\tau\, \overset{(4)}{I}_{ij}(T-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12}\right] + \cdots$$

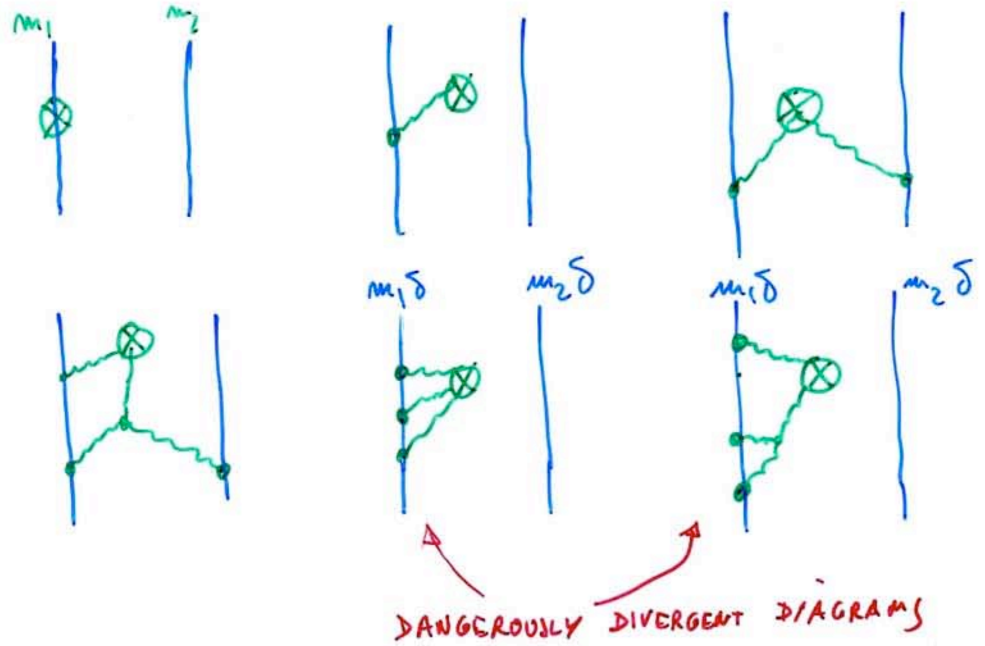
$$I_{ij}(t) = \int d^3x\, \hat{x}^{ij}\, \frac{T^{00} + T^{ss}}{c^2} + \frac{1}{14}\int d^3x\, \hat{x}^{ij}\, \frac{\vec{x}^2}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{T^{00} + T^{ss}}{c^2}\right) + \cdots$$

3 PN (~3 LOOP) MULTIPOLE MOMENTS OF BINARY BLACK HOLE

- REPLACE



- MULTIPOLE MOMENTS:



- NEED DIMENSIONAL REGULARIZATION TO COMPLETE THE CALCULATION

(Blanchet, Damour, Esposito-Farèse, Iyer '05)

Nice 4.9
ST THOMAS 9

"TAYLOR" PN-EXPANDED RESULTS: RADIATION FROM BINARY SYSTEM

Blanchet et al...

$\gamma \equiv \frac{GM}{r_{harmonic}}$, $\nu \equiv m_1 m_2 / M^2$

- $\frac{I_{ij}}{\mu} = \left(1 + \gamma\left(-\frac{1}{42} - \frac{13}{4}\nu\right) + \gamma^2\left(-\frac{461}{1512} - \frac{18395}{1512}\nu \ldots\right) + \gamma^3\left(\ldots + \ldots\nu + \ldots\nu^2 + \nu^3 + \ln\right)\right)\hat{x}^{ij} + \cdots$

- $U_{ij} = I^{(2)}_{ij} + \text{TAILS} + \cdots$

- $\frac{dE}{dt} = \frac{1}{5}\left(U^{(1)}_{ij}\right)^2 + \cdots\left(U^{(2)}_{ijk}\right)^2 + \cdots\left(V^{(1)}_{ij}\right)^2 + \cdots$

- $\frac{dE}{dt} = \frac{32}{5}\frac{c^5}{G}\nu^2\gamma^5\left\{1 + \left(-\frac{2927}{336} - \frac{5}{4}\nu\right)\gamma + \left(\cdots + \cdots\nu\right)\gamma^2 + \left(\cdots + \cdots\nu + \cdots\nu^2 + \ln\right)\gamma^3\right\}$

$\gamma = \frac{GM}{c^2 r}$, harmonic coords

IN TERMS OF

$$v_\omega \equiv \left(\frac{GM\omega}{c^3}\right)^{1/3}$$

- $\frac{dE}{dt} = \frac{32}{5}\nu^2 v_\omega^{10}\left\{1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)v_\omega^2 + 4\pi v_\omega^3 + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \ldots\right)v_\omega^4\right.$
$\left. + \left(\cdots + \cdots\nu\right)v_\omega^5 + \left(\cdots + \cdots\nu + \cdots\nu^2\right)v_\omega^6 + \left(\cdots + \cdots\nu\right)v_\omega^7\right.$

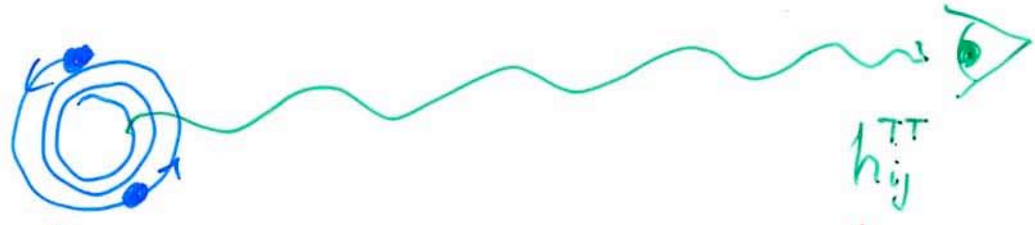
ALL KNOWN FOR $\nu \neq 0$

$+ A^0_8 v_\omega^8 + A^0_9 v_\omega^9 + A^0_{10} v_\omega^{10} + A^0_{11} v_\omega^{11}$

KNOWN ONLY FOR $\nu = 0$: TEST-MASS LIMIT

Poisson ; Tanaka Tagoshi Sasaki '96

WHAT IS NEEDED FOR DESCRIBING COALESCING BINARIES



EoM ($a=1,2$) $\quad \dfrac{d^2\vec{x}_a}{dt^2} = \vec{a}_a^{\,CONS} + \vec{a}_a^{\,RR}$

$$r h_{ij}^{TT} = \left[U_{ij} + U_{ijk}\frac{N^k}{c} + \dots \right]^{TT}$$

NEED RADIATION REACTION TO HIGHEST POSSIBLE ACCURACY

$$a^{RR} = \frac{GM}{r^2}\left[\frac{v^5}{c^5} + \frac{v^7}{c^7} + \underbrace{\frac{v^8}{c^8} + \frac{v^9}{c^9} + \frac{v^{10}}{c^{10}} + \frac{v^{11}}{c^{11}} + \frac{v^{12}}{c^{12}}}_{\text{HEURISTICALLY OBTAINED BY ASSUMING BALANCE OF } E,\ \vec{J}} \right]$$

WAVE FORM :

$$U_{ij} = \mu\, x^{\langle i} x^{j\rangle} \left\{ 1 + \frac{v^2}{c^2} + \frac{v^3}{c^3} + \frac{v^4}{c^4} + \frac{v^5}{c^5} + \frac{v^6}{c^6} + \frac{v^7}{c^7} \right\}$$

\+ NEED CONSERVATIVE DYNAMICS TO HIGHEST POSSIBLE ACCURACY

$$a^{CONS} = \frac{GM}{r^2}\left[1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} + \frac{v^6}{c^6} \right]$$

OR

$$H[x,p] = H_0 + \frac{1}{c^2}H_2 + \frac{1}{c^4}H_4 + \frac{1}{c^6}H_6$$

RESUMMATION TECHNIQUES

B5[16] Nice 4.10

E.G. GW ENERGY FLUX AT THE LSO (FOR TEST PARTICLE)

IN HARMONIC COORDS $\gamma = \frac{GM}{c^2 r_h}$

$$F_{TM}(\gamma) = 1 - 1.74(5\gamma) + 1.12(5\gamma)^{3/2} + 1.29(5\gamma)^2 + \dots$$

5γ = 1 AT SCHWARZ. LSO "$R = 6GM$", $r_h = 5GM$

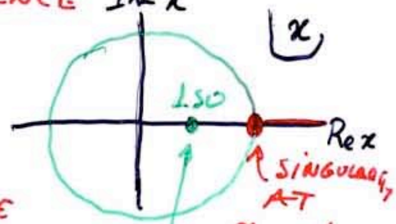
NO CONVERGENCE AT ALL

INVARIANT FUNCTION $F(x)$ $\quad x \equiv \left(\frac{GM\omega^{orb}}{c^3}\right)^{2/3}$ IN SCHW: $x = \frac{GM}{c^2 R} = \frac{1}{6}$ AT LSO

$$F_{TM}(x) = 1 - 0.619(6x) + 0.855(6x)^{3/2} - 0.137(6x)^2 + \dots$$

SLOW CONVERGENCE

BUT CONVERGENCE BECAUSE



NEED AN ACCELERATOR OF CONVERGENCE

USE PN INFORMATION IN OPTIMAL WAY

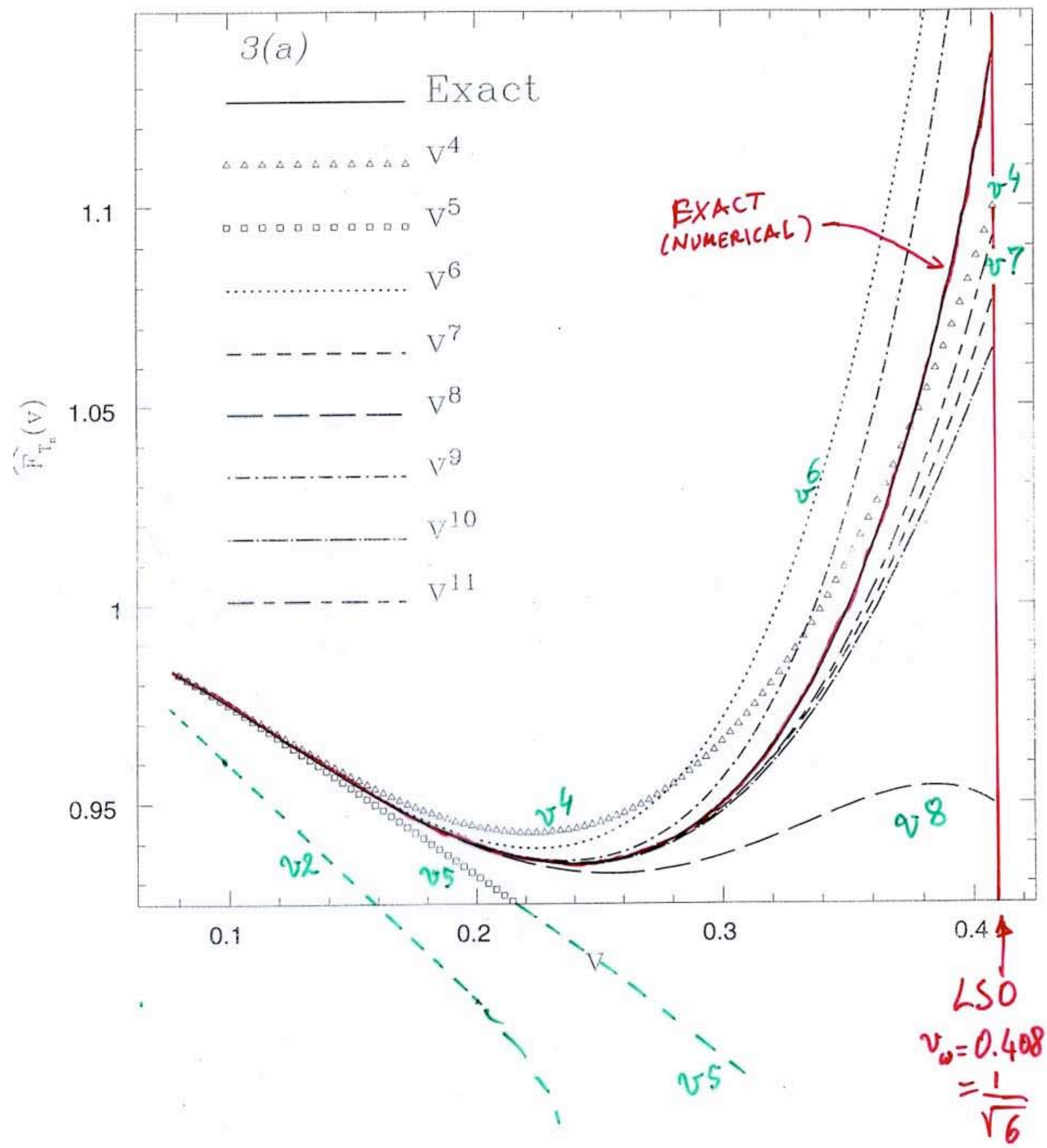
TECHNIQUES (ALL DETERMINED AT 3PN DJS'99)

- WORK ONLY WITH INVARIANT FUNCTIONS
- USE KNOWLEDGE OF TEST-MASS LIMIT, TAKING $\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$ AS DEFORMATION PARAMETER (NOT ASSUMED SMALL)
- INTRODUCE NEW FUNCTIONS
- USE PADÉ RESUMMATION

TAYLOR APPROXIMANTS TO $\dot{E}/\dot{E}_N$

$\nu = 0$ TEST-MASS

Damour Iyer Sathyaprakash '98
Exact Numerical from Poisson

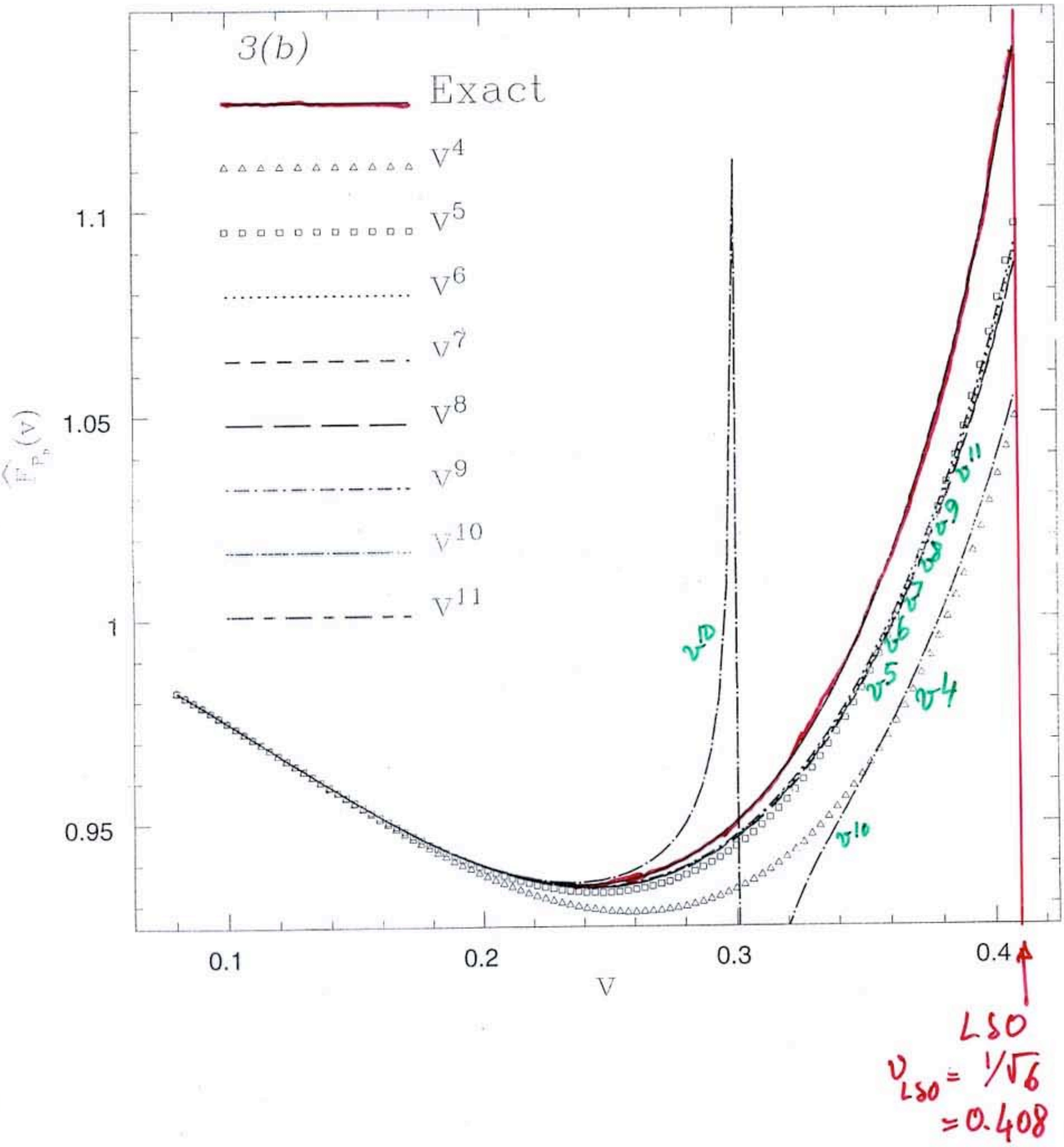


Mia 4.13

PADÉ APPROXIMANTS TO $\dot{E}/\dot{E}_N$

$\nu = 0$ TEST-MASS

Damour Iyer Sathyaprakash '98
Exact Numerical from Poisson



RESUMMATION OF THE (CONSERVATIVE) DYNAMICS

- SEVERAL POSSIBILITIES EXPLORED & COMPARED (DIS98, DJS00)
- MOST ROBUST, EFFICIENT AND VERSATILE (AND ELEGANT!)

"EFFECTIVE ONE-BODY" APPROACH

BASIC IDEA STARTED IN QED (Brézin Itzykson Zinn-Justin 70, Todorov 70)

NEW FORM ADAPTED TO GR (Buonanno Damour 99, 00; Damour 01; Damour Jaranowski Schäfer 00)

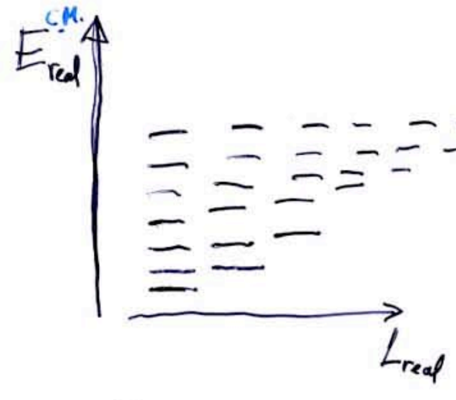
QUANTUM ENERGY STATES OF TWO DIFFERENT SYSTEMS

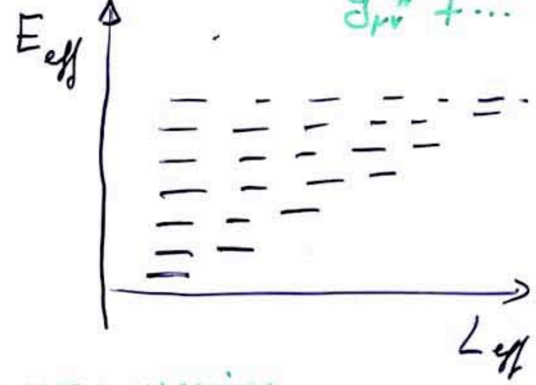
REAL SYSTEM
IN THE CENTER OF MASS FRAME

EFFECTIVE SYSTEM

$\mu = \frac{m_1 m_2}{m_1 + m_2}$

$g^{eff}_{\mu\nu} + \cdots$





i.e. DELAUNAY HAMILTONIAN

ADIABATIC INVARIANTS $I_i = \oint p_i \, dq^i$

$$E^{C.M.}_{real} = E_{real}(L_{real}, N_{real}) \qquad E_{eff} = E_{eff}(L_{eff}, N_{eff})$$

MAPPING:

$$\begin{cases} L_{real} = L_{eff} & \text{BEING } L_{eff}/\hbar,\ L_{real}/\hbar,\ N_{real}/\hbar,\ N_{eff}/\hbar \in \mathbb{N} \\ N_{real} = N_{eff} \\ E_{real} = f(E_{eff}) \end{cases}$$

DETERMINES f AND $g^{eff}_{\mu\nu}(x^{eff}) + \cdots$

Nice 4.15

EOB: ENORMOUS SIMPLIFICATION OF DYNAMICS

f ALWAYS UNIVERSAL:

$$\frac{E_{eff}}{\mu} = \frac{E_{real}^2 - m_1^2 - m_2^2}{2 m_1 m_2}$$

$$\left[H_{real}^{CM}\right]^{2PN} = \frac{1}{2\mu}\vec{P}^2 - \frac{GM}{r} + \frac{1}{c^2}\left(p^4 + \cdots \frac{(GM)^2}{r^2}\right) + \frac{1}{c^4}\left(p^6 + p^4\frac{GM}{r} + \cdots + \left(\frac{GM}{r}\right)^3\right)$$

13 TERMS

$\longleftrightarrow$ GEODESIC MOTION OF PARTICLE OF MASS μ IN

$$ds^2 = -A(r)\,dt^2 + \frac{D(r)}{A(r)}dr^2 + r^2 d\Omega^2$$

$$A^{2PN}(r) = 1 - \frac{2GM}{r} + 2\nu\left(\frac{GM}{r}\right)^3$$

$$D^{2PN}(r) = 1 - 6\nu\left(\frac{GM}{r}\right)^2$$

Buonanno Damour '00

NB: $\nexists$ 1PN CORRECTIONS!

◎ 3PN NEED TO ADD p^4 TERMS TO GEODESIC HJ: $p^2 + \mu^2 + O(\vec{P}^4) = 0$

24 TERMS IN C.M.

$$A^{3PN}(r) = 1 - \frac{2GM}{r} + 2\nu\left(\frac{GM}{r}\right)^3 + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\nu\left(\frac{GM}{r}\right)^4$$

$$D^{3PN}(r) = 1 - 6\nu\left(\frac{GM}{r}\right)^2 + 2(3\nu - 26)\nu\left(\frac{GM}{r}\right)^3$$

Damour Jaranowski Schäfer 00

EFFECTIVE ONE BODY METRIC @ 2PN, 3PN (1PN)

ALWAYS : $\boxed{\dfrac{E_{eff}}{\mu} = \dfrac{E_{real}^2 - m_1^2 - m_2^2}{2 m_1 m_2}}$

1PN $H^{C.M.}_{1PN} = 6$ TERMS $\longrightarrow$ GEODESIC MOTION OF PARTICLE μ IN SCHWARZSCHILD:

REDUCED TO C.M.

$$ds^2_{EFF} = -\left(1 - \frac{2GM}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega^2$$

2PN $H^{C.M.}_{2PN} = 6 + 7$ TERMS $\longrightarrow$ GEODESIC MOTION ON

$$ds^2_{EFF} = -A(r)\,dt^2 + \frac{D(r)}{A(r)}\,dr^2 + r^2 d\Omega^2$$

condensed (in 2)

$\boxed{\nu \equiv \dfrac{m_1 m_2}{(m_1+m_2)^2} \quad 0 \leq \nu \leq 1/4}$

$$A^{2PN}(r) = 1 - \frac{2GM}{r} + 2\nu\left(\frac{GM}{r}\right)^3$$

$$D^{2PN}(r) = 1 - 6\nu\left(\frac{GM}{r}\right)^2$$

ν = DEFORMATION PARAMETER

3PN $H^{C.M.}_{3PN} = 6 + 7 + 11$ $\longrightarrow$ $\boxed{A^{3PN}(r) = 1 - \dfrac{2GM}{r} + 2\nu\left(\dfrac{GM}{r}\right)^3 + \left(\dfrac{94}{3} - \dfrac{41}{32}\pi^2 + 2\omega_s\right)\nu\left(\dfrac{GM}{r}\right)^4}$

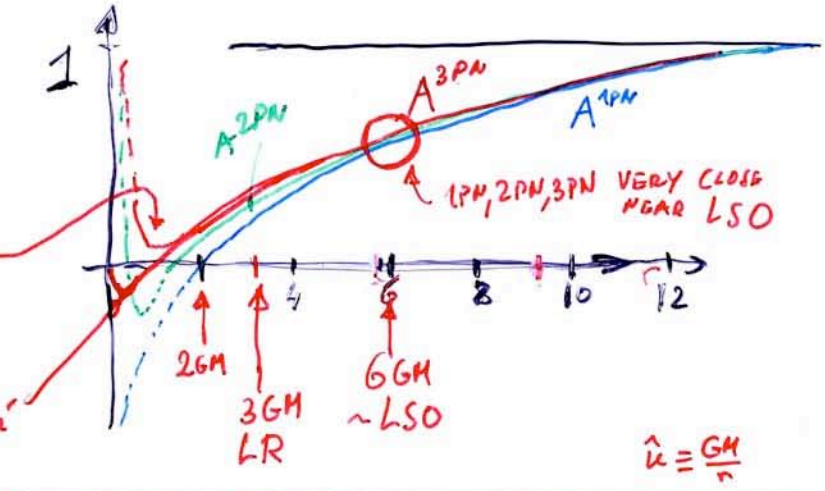
condensed in 3 coefficients OF WHICH ONLY ONE IS REALLY IMPORTANT

$$D^{3PN}(r) = 1 - 6\nu\left(\frac{GM}{r}\right)^2 + 2(3\nu - 26)\,\nu\left(\frac{GM}{r}\right)^3$$

$$+ : \ p^2 \to p^2\left[1 + 2(4 - 3\nu)\,\nu\left(\frac{GM}{r}\right)^2\left(\frac{\vec{n}\cdot\vec{p}}{\mu}\right)^4\right]$$

RADIAL FUNCTION

$$-g_{00}(r) \equiv A(r)$$



"UNPHYSICAL" WHEN $r \lesssim 2$

$\downarrow$

IMPROVED PADÉ "RESUMMED"

$\hat{u} \equiv \frac{GM}{r}$

$$\boxed{A(r) \equiv P^1_3\left[A^{3PN}(r)\right] = \frac{1 - a_1\hat{u}}{1 + b_1\hat{u} + b_2\hat{u}^2 + b_3\hat{u}^3}}$$

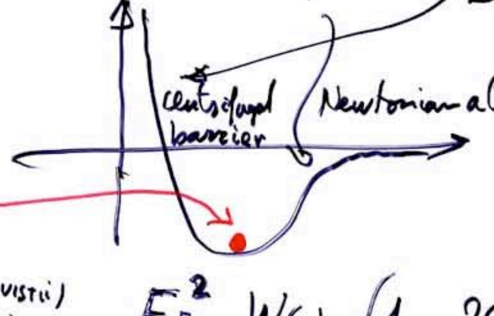
+ RECENT INCLUSION OF SPIN EFFECTS (Damour 01)

LAST STABLE ORBIT.

Nice 4.17

NON RELATIVISTIC CIRCULAR ORBIT: RADIAL POTENTIAL: $E_{NR} = -\frac{GM\mu}{r} + \frac{L^2}{2r^2}$

∃ STABLE CIRCULAR ORBITS ∀L, i.e. ∀r→0

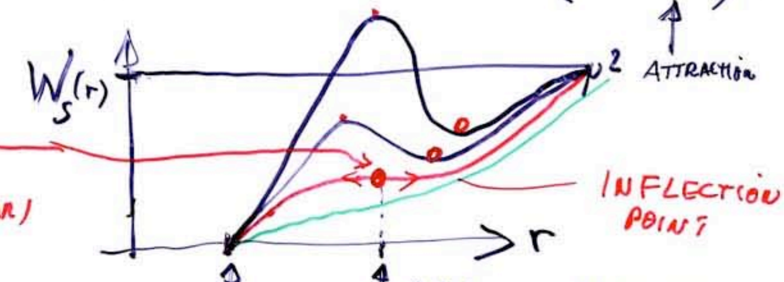


TEST PARTICLE AROUND SCHWARZSCHILD BLACK HOLE (RELATIVISTIC) RADIAL POTENTIAL

$$E_R^2 = W_S(r) = \left(1 - \frac{2GM}{r}\right)\left(\mu^2 + \frac{L^2}{r^2}\right)$$

ATTRACTION; CENTRIFUGAL BARRIER

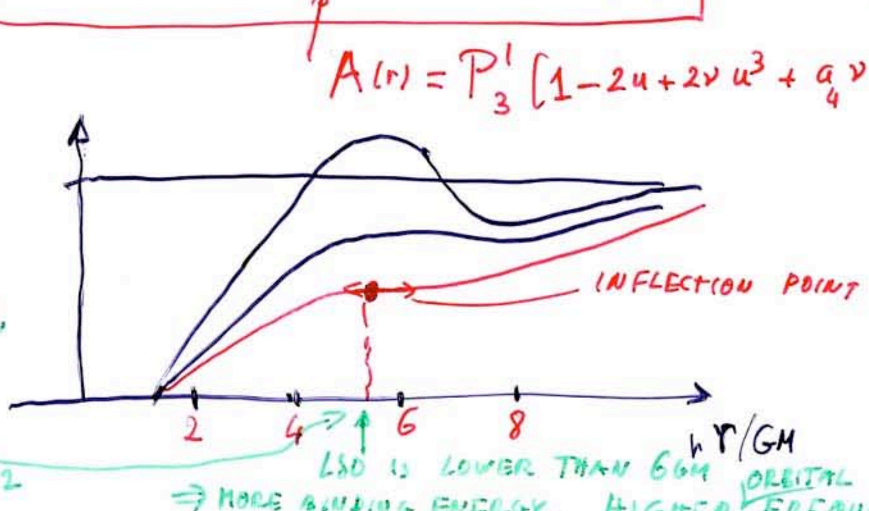
∃ LAST STABLE (CIRCULAR) ORBIT !



FOR TWO COMPARABLE MASSES

$$\boxed{E_{eff}^2 = W(r) = A(r)\left(\mu^2 + \frac{L^2}{r^2}\right)} \qquad u \equiv \frac{GM}{r}$$

$$A(r) = P^1_3\left[1 - 2u + 2\nu u^3 + a_4 \nu u^4\right]$$



DEFORMATION AS A FUNCTION OF $\nu = m_1 m_2 / M^2$

LSO is LOWER THAN 6GM

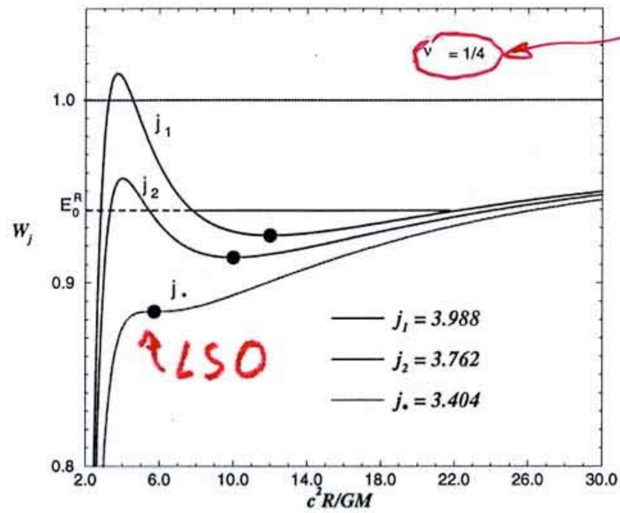
⇒ MORE BINDING ENERGY, HIGHER ORBITAL FREQUENCY

T18/3
Nice 4.18

EFFECTIVE RADIAL POTENTIAL

2PN
EFFECTIVE
ONE-BODY

CASE $m_1 = m_2$



(Buonanno Damour 99)

FIG. 1. *The effective radial potential $W_j(R)$ (at the 2PN level and for $\nu = 1/4$) versus the dimensionless radial variable $c^2R/(GM)$ for three different values of the dimensionless angular momentum $j = c\mathcal{J}_{\rm real}/(GM\mu)$. Note that the effective radial potential tends to one for $R \to \infty$. The stable circular orbits are located at the minima of the effective potential and are indicated by heavy black circles. The innermost stable circular orbit corresponds to the critical value j_*. In the case of the j_1 curve the orbit of a particle with energy $E_0^R = \widehat{\mathcal{E}}_0$ is an elliptical rosette.*

quantity coincides (in view of our rules) with the corresponding real two-body dimensionless angular momentum j:

$$j_0 \equiv \frac{c\,\mathcal{J}_0}{G\,M_0\,m_0} = \frac{c\,\mathcal{J}_{\rm real}}{G\,M\,\mu} \equiv j\,. \tag{5.17}$$

[Note that our definition of the j's differs by a factor c from the one used in the previous section.]

As usual, because of the inequality (5.10), when j and $\widehat{\mathcal{E}}_0 \equiv \mathcal{E}_0/(m_0\,c^2)$ are fixed, the trajectory of a particle following a geodesic in the effective metric (5.2) can be qualitatively read on Fig. 1. For instance, in the case illustrated for the j_1 curve ($E_0^R \equiv \widehat{\mathcal{E}}_0$ line), the orbit will be an elliptical rosette, with the radial variable oscillating between a minimum and a maximum (solid line in Fig. 1). The stable circular orbits are located at the minima of the effective potential (the maxima being unstable circular orbits). The innermost stable

LSO : RESULTS (DJS 00)

DEFINE $\frac{|E^{Schw}_{LSO} - M|}{M} = \frac{1}{4}\left(1 - \frac{1}{3}\sqrt{8}\right) = 1.43\%$ $\quad \omega^{Sch} = \frac{1}{GM}\frac{1}{6^{3/2}}$

$\frac{E^{LSO}-M}{M} / 1.43\%$				ω/ω^{Schw}			
1PN	2PN	3PN	ω_s	1PN	2PN	3PN	ω_s
		−1.042	−10			1.064	−10
−1.007	−1.048	−1.168	0	1.015	1.075	1.297	0
		−1.212	+10			1.383	+10

i.e. $\frac{E^{LSO} - M}{M} = -1.67\%$ → 1.67% M EMITTED AS GW

WEAK BUT SIGNIFICANT DEPENDENCE ON THE REG. AMBIGUITY PARAMETER ω_s !

IN FACT: EVEN LESS IMPORTANT THAN IT SEEMS (BD 00)

BECAUSE WHEN: 3PN CONSERVATIVE (EOB) DYNAMICS + PADÉ-RESUMMED RADIATION REACTION:

$$\frac{dr}{dt} = \frac{\partial H_{real}}{\partial p_r} \,, \quad \frac{d\varphi}{dt} = \frac{\partial H_{real}}{\partial p_\varphi}$$

$$\frac{dp_r}{dt} = -\frac{\partial H_{real}}{\partial r} \,, \quad \frac{dp_\varphi}{dt} = \mathcal{F}^{RAD\,REAC}_\varphi(\omega) = -\frac{32}{5}\mu\nu v_\omega^7 \frac{f_{DIS}(v_\omega;\nu)}{1 - v_\omega/v^{DIS}_{pole}}$$

$\omega \equiv \frac{d\varphi}{dt}$, $\quad v_\omega \equiv \left(\frac{GM\omega}{c^3}\right)^{1/3}$

WITH $\frac{H_{real}}{M} = \sqrt{1 + 2\nu\left(\frac{H_{eff}}{\mu} - 1\right)}$

⇒ **SMOOTH TRANSITION FROM INSPIRAL TO PLUNGE AROUND THE LSO**

Nio 4-20

INCLUSION OF SPIN EFFECTS

(Damour '01)

- NON-SPINNING EFFECTIVE ONE-BODY

⇒ HAMILTONIAN FOR RELATIVE MOTION

$$H(\vec{x}, \vec{p}) \longleftrightarrow \approx \text{GEODESIC MOTION IN SOME } g_{\mu\nu}^{EFF} + \mathcal{O}(p^4) \text{ CORRECTIONS}$$

C.M. VARIABLES $\vec{x} = \vec{x}_1 - \vec{x}_2$; $\vec{p} = \vec{p}_1 = -\vec{p}_2$

ν-DEFORMATION OF SCHWARZSCHILD METRIC

- EFFECTIVE ONE-BODY WITH SPINS

CONVENIENCE OF HAMILTONIAN APPROACH:

$$\{x^i, p_j\} = \delta^i_j \; ; \{S_1^i, S_1^j\} = \varepsilon^{ijk} S_1^k \; ; \{S_2^i, S_2^j\} = \varepsilon^{ijk} S_{2k}$$

$$H(\vec{x}, \vec{p}, \vec{S}_1, \vec{S}_2) \longleftrightarrow \approx \text{GEODESIC MOTION IN SOME } g_{\mu\nu}^{EFF}(M^{eff}, \vec{S}^{eff}) + \text{CORRECTIONS}$$

ν-DEFORMATION OF KERR METRIC

UNIQUELY DETERMINED AT LOWEST ORDER IN $\vec{S}_1, \vec{S}_2$

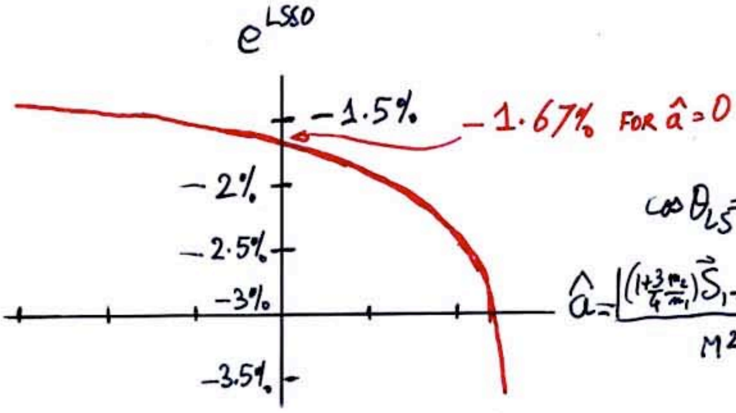
- A PRIORI RELIABLE ONLY FOR SMALL SPINS
- BUT VERY FLEXIBLE: NEED NEW INPUTS (EITHER ANALYTICAL OR NUMERICAL) TO DETERMINE IT BETTER

LONG-TERM: <u>NEED</u> ANALYTIC (BUT NUMERIC-FITTED) $H(x\, p\, \vec{S}_1 \vec{S}_2)$ TO COVER ALL PHASE SPACE

BINDING ENERGY OF THE LAST STABLE SPHERICAL ORBIT

2 SPINNING BLACK HOLES (3 PN SPINNING EOB)

(Damour '01)



$$\cos\theta_{LS} = \pm \frac{1}{\sqrt{3}}$$

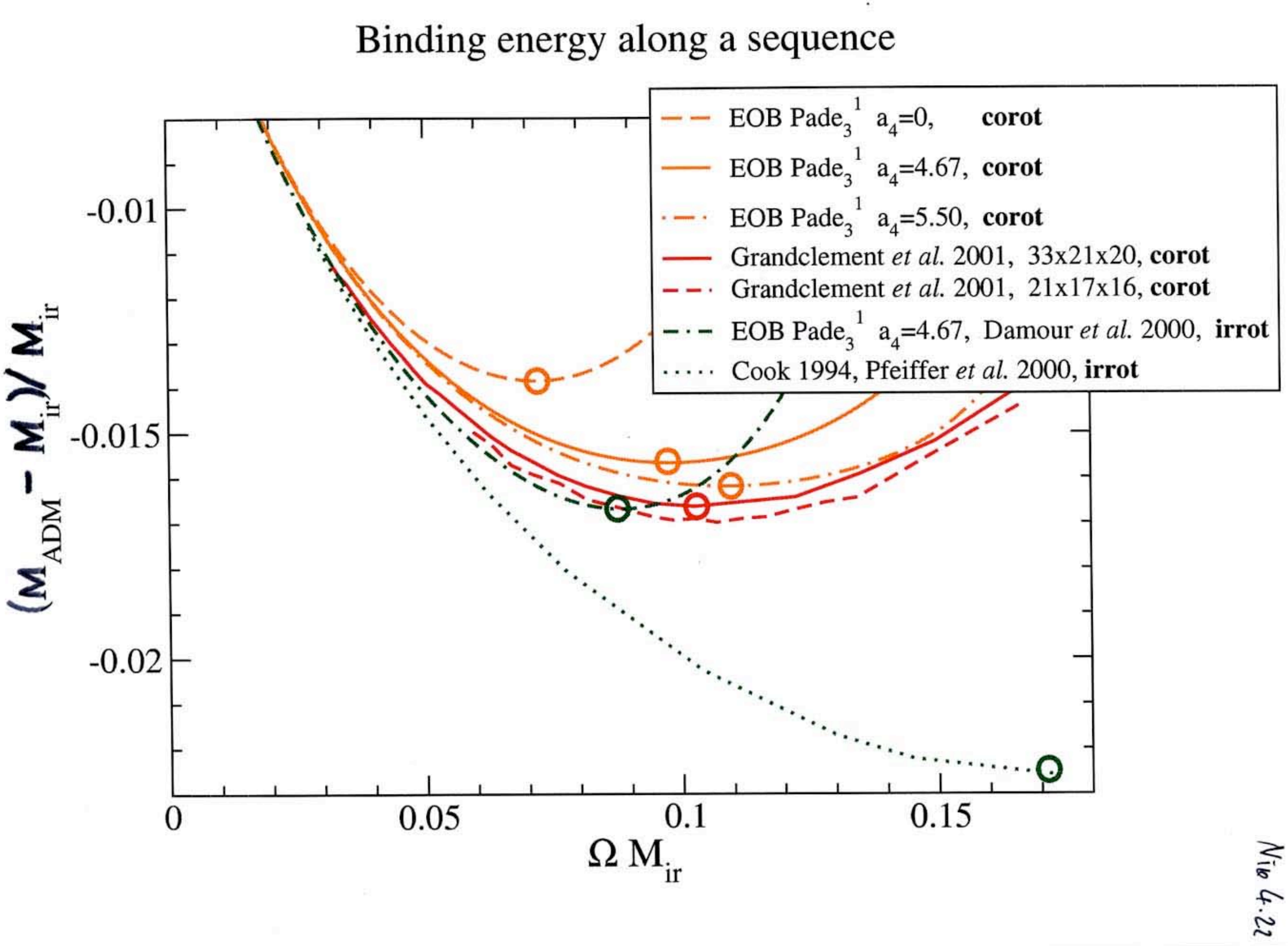
$$\hat{a} = \frac{\left|\left(1+\frac{3}{4}\frac{m_2}{m_1}\right)\vec{S}_1 + \left(1+\frac{3}{4}\frac{m_1}{m_2}\right)\vec{S}_2\right|}{M^2}$$

BINDING INCREASES WHEN SPINS ARE ~ PARALLEL

USES SPIN-MODIFIED, PADÉED A-FUNCTION:

$$\bar{A}(\hat{u},\hat{a}^2) = P_3^1\left[1 - 2\hat{u} + \hat{a}^2\hat{u}^2 + 2\nu\hat{u}^3 + a_4(\nu)\hat{u}^4\right]$$

SUGGESTS THAT FIRST DETECTIONS WILL INVOLVE ~ PARALLEL SPINS


Binding energy along a sequence
EOB Pade$_3^1$ a_4=0, **corot**
EOB Pade$_3^1$ a_4=4.67, **corot**
EOB Pade$_3^1$ a_4=5.50, **corot**
Grandclement *et al.* 2001, 33x21x20, **corot**
Grandclement *et al.* 2001, 21x17x16, **corot**
EOB Pade$_3^1$ a_4=4.67, Damour *et al.* 2000, **irrot**
Cook 1994, Pfeiffer *et al.* 2000, **irrot**
-0.01
-0.015
-0.02
0
0.05
0.1
0.15
ΩM_{ir}
$(M_{ADM} - M_{ir})/M_{ir}$
Nio 4.22

Nice 4.24

INSPIRAL, PLUNGE AND MERGER VIA EOB FORMALISM

Buonanno Damour 00

- EQS OF MOTION (QUASI-CIRCULAR ORBITS)

$$\begin{cases} \dfrac{dr}{dt} = \dfrac{\partial \mathcal{H}(r, p_r, p_\varphi)}{\partial p_r} \\[2ex] \dfrac{d\varphi}{dt} \equiv \omega = \dfrac{\partial \mathcal{H}(r, p_r, p_\varphi)}{\partial p_\varphi} \\[2ex] \dfrac{dp_r}{dt} = - \dfrac{\partial \mathcal{H}(r, p_r, p_\varphi)}{\partial r} \\[2ex] \dfrac{dp_\varphi}{dt} = \mathcal{F}_\varphi \end{cases}$$

EOB (← RESUMMED) HAMILTONIAN

$$\mathcal{H} = \frac{1}{\nu}\sqrt{1 + 2\eta\left[\sqrt{A(r)\left(1 + \frac{p_r^2}{B(r)} + \frac{p_\varphi^2}{r^2} + O(p^4)\right)} - 1\right]}$$

EFFECTIVE METRIC COEFF

$\mathcal{F}_\varphi$ ← PADÉ-RESUMMED RADIATION REACTION = ANG. MOM. LOSS

$$\mathcal{F}_\varphi = -\frac{32}{5}\eta\,\omega^5 (r\psi)^4 \; \frac{\hat{f}^{\text{Padé}}_{DIS}(\omega r \psi^{1/3}, \nu)}{1 - \omega r \psi^{1/3}/v_{pole}}$$

- USE SOME $h^{TT}_{ij} = \mathcal{F}[\vec{x}, \vec{v}]$ TO ESTIMATE GW SIGNAL
- AFTER CROSSING $r = 3$ (LIGHT RING)

⟶ MATCH TO RING-DOWN GW SIGNAL MADE OF QUASI-NORMAL MODES

(cf Davis Ruffini Press Price 1971)

RESULTS AT 2.5 PN

RESUMMED RADIATION REACTION FOR CIRCULAR ORBITS

LET $v \equiv \left(\frac{GM\omega}{c^3}\right)^{\frac{1}{3}} \equiv x^{1/2}$

ANG. MOMENTUM RR: $\mathcal{F}_\varphi^{PN} = -\frac{32}{5G} \nu^2 \frac{v^{10}}{\omega} \left[1 + a_2 v^2 + a_3 v^3 + a_4 v^4 + a_5 v^5 \right]$

COEFFICIENTS DEPEND ON ν

e.g. $a_2(\nu) = -\frac{1247}{336} - \frac{35}{12}\nu$

INVARIANT FUNCTION BUT BAD BECAUSE $\exists$ NO $a_1 v^1$

$\Rightarrow$ INJECT INFO THAT $\mathcal{F}_\varphi^{TM}$ HAS A SIMPLE POLE AT $v = v_{LR}^{TM} = \frac{1}{\sqrt{3}}$

$\Rightarrow$ NEW FUNCTION: $\hat{\mathcal{F}} = \left(1 - \frac{v}{v_{pole}(\nu)}\right) * \mathcal{F} = \cdots \left[1 + v + v^2 + \cdots \right]$

PADE THIS EXPANSION

$$\mathcal{F}_\varphi\left(\omega = \frac{d\varphi}{dt}\right) = -\frac{32}{5G} \nu^2 \frac{v^{10}}{\omega} \frac{1}{1 - \frac{v}{v_{pole}(\nu)}} \cfrac{1}{1 + \cfrac{c_1(\nu) v}{1 + \cfrac{c_2(\nu) v}{1 + \cfrac{c_3(\nu) v}{1 + \cfrac{c_4(\nu) v}{1 + c_5(\nu) v}}}}}$$

Damour, Iyer, Sathyaprakash '98

Nic 4.2.07 T815b

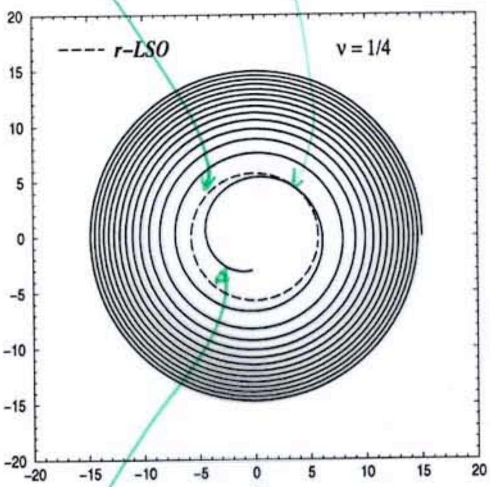
(RESUMMED) EFFECTIVE ONE BODY DYNAMICS

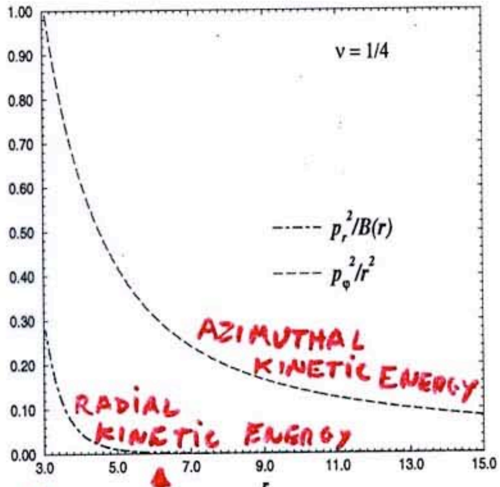
+ RESUMMED RADIATION REACTION (QUASI-CIRCULAR ORBITS)

(1) YIELDS INITIAL DYNAMICAL DATA ($q_1 q_2 P_1 P_2$) AT BEGINNING OF PLUNGE: 0.6 ORBIT LEFT

GOOD IN VIEW OF STATE OF THE ART NUMERICAL SIMULATIONS (Pretorius05)

TRANSITION INSPIRAL → PLUNGE WITH ARBITRARY MASS RATIO





LIGHT-RING

LSO

REMAINS QUASI-CIRCULAR DURING THE WHOLE PLUNGE

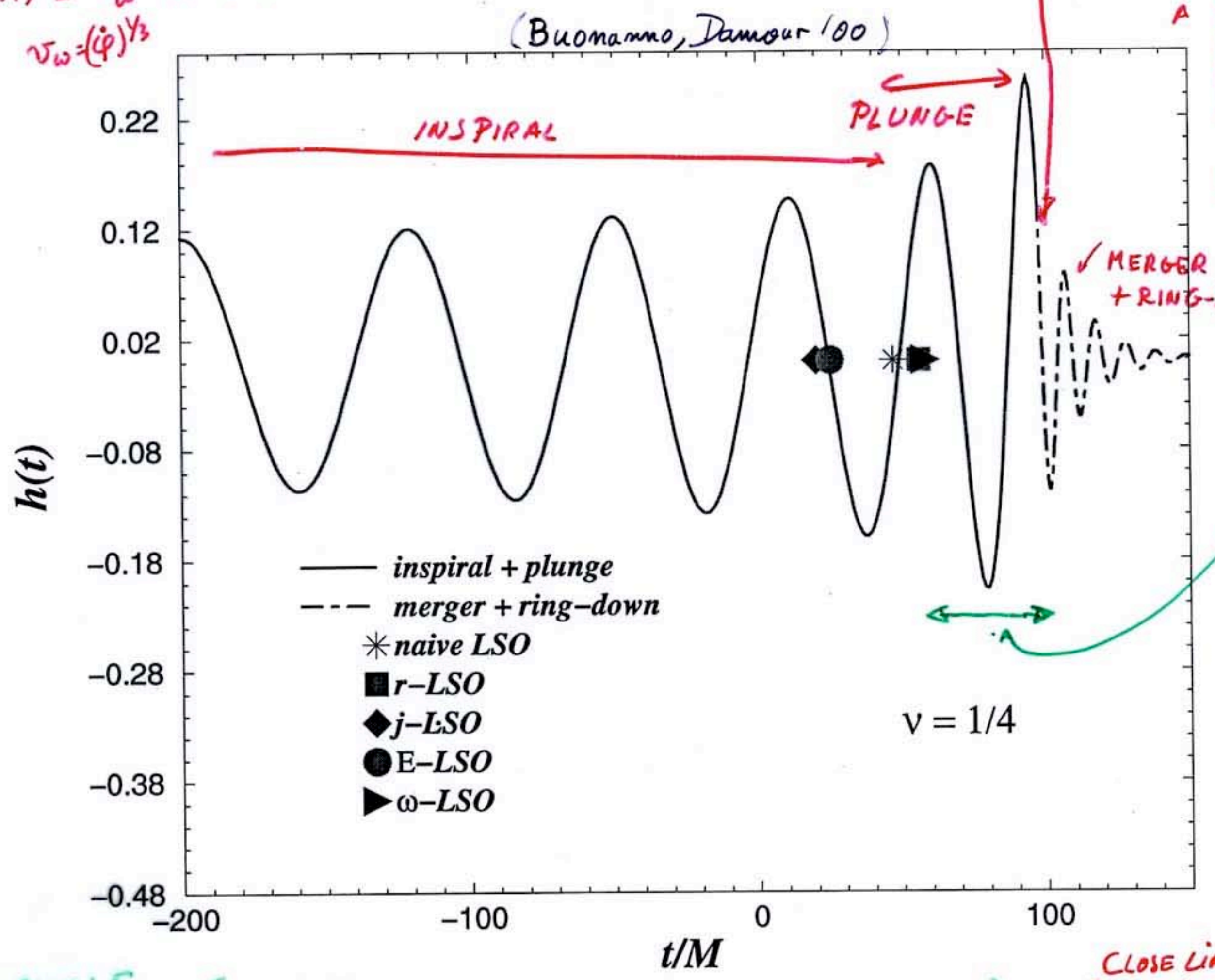
(2) FIRST ESTIMATE OF FULL WAVEFORM: "6M" → "3M" ≃ MERGER

'RESTRICTED' WAVE FORM

$h(t) \equiv v_\omega^2 \cos 2\varphi(t)$

$v_\omega = (\dot\varphi)^{1/3}$

(Buonanno, Damour '00)



MATCHING TO LEAST-DAMPED QUASI-NORMAL-MODE OF A KERR BH

(WITH $M = E^{\text{light-ring}}_{m_1 m_2}$, $J = J^{\text{light-ring}}_{m_1 m_2}$)

AT THE (ν-DEFORMED) LIGHT-RING

ONLY 1.2 GW PERIOD DURING PLUNGE ($\sim 6M \to \sim 3M$)

T18/6
Nie 4.25

POSSIBLE FUTURE IMPROVEMENTS

HIGHER PERTURBATIVE ACCURACY in H AND $\mathcal{F}_\varphi$

CLOSE LIMIT APPROXIMATION: Smarr, Price, Pullin...

NUMERICAL CALC.: recently: Baker et al '01

BH initial data: Cook; Baumgarte, Grandclément et al

EOB PREDICTION FOR THE FINAL SPIN

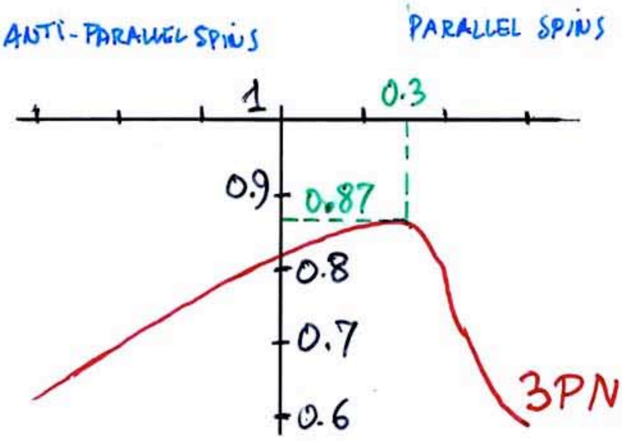
OF THE HOLE FORMED BY THE COALESCENCE OF TWO SPINNING HOLES

(Damour '01)

$\cos\theta_{LS} = \pm 1$

$S_1 = S_2$

$m_1 = m_2$



$\hat{a} = \frac{J_{BH}}{M_{BH}^2}$

- FOR NON-SPINNING HOLES: $\hat{a}^{final}_{EOB} \simeq 0.8$

  [CURRENT NUMERICAL RESULTS: $\hat{a}^{final}_{NUM} \simeq 0.7$ Pretorius '05, Baker et al '06]

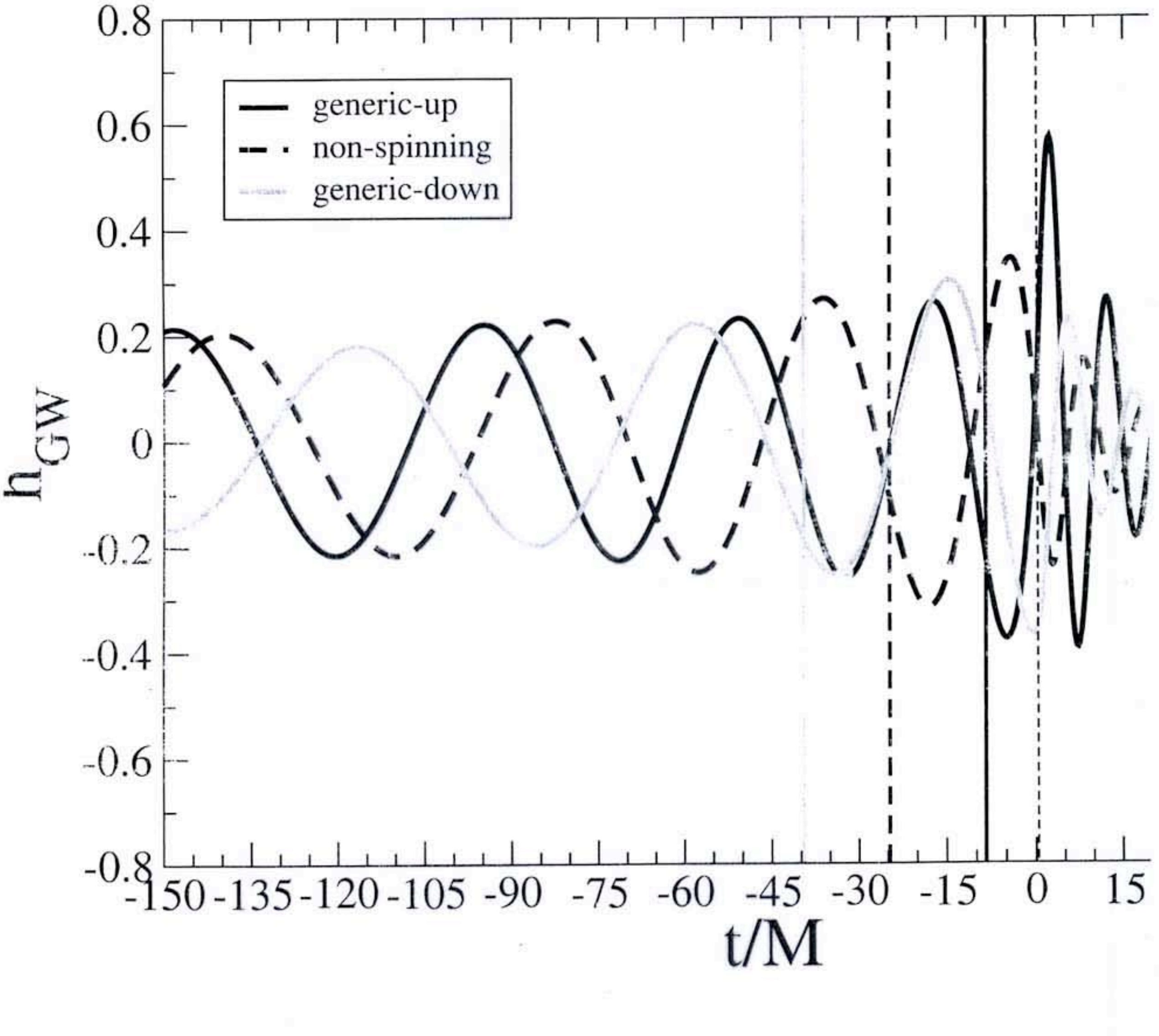
- MAXIMUM SPIN FOR $\frac{\vec{S}_1}{m_1^2} = \frac{\vec{S}_2}{m_2^2} = 0.3$: $\hat{a}^{final}_{EOB} \simeq 0.87 < 1$

- SENSITIVITY / FLEXIBILITY:
  ONE CAN FIT HIGHER-PN CONTRIBUTION TO EOB BY COMPARING TO NUMERICAL RESULTS

Nü 4.27

WAVEFORM FROM PRECESSING BINARIES OF SPINNING BLACK HOLES, USING EOB

Buonanno Chen Damour



POSSIBLE TESTS / IMPROVEMENTS OF EOB APPROACH

- FITS TO BINDING ENERGY OF 'WAVELESS' QUASI-STATIONARY BINARY BLACK HOLES
  - NEED 'WAVELESS' DATA (à la Gourgoulhon et al., following Isenberg + Wilson-Mathews)
  - NEED NON-CONFORMALLY FLAT METRIC
  - NEED ARBITRARY SPINS
  - CAN, IN PRINCIPLE, CHECK / LEAD TO MODIFICATION / DETERMINE NEW PARAMETERS IN SPINNING EOB HAMILTONIAN

- COMPARISON WITH EOB WAVEFORMS
  - COMPARISON FOR PLUNGING TEST-MASSES IN BHOLES

    HOPE TO LEARN BY COMPARING ANALYTICAL / NUMERICAL
  - COMPARISON FOR COMPARABLE-MASS COALESCING BHOLES

    PROBLEM OF INITIAL DATA

    CURRENT DATA ARE NEITHER 'WAVELESS', NOR CIRCULAR

    COMPARISON $\hat{a}^{final}_{NUM} \simeq 0.7 \sim \hat{a}^{final}_{EOB} \simeq 0.8$ IS ENCOURAGING

    CURRENT SIMULATIONS SUGGEST LARGER(?) $E^{NUM}_{rad} \approx 4\% \neq E^{EOB}_{PLUNGE + MERGER} \approx 1.4\%$

    BUT $E^{EOB}_{INSPIRAL + PLUNGE + MERGER} \approx 1.6\% + 1.4\% \approx 3\%$

- COMPARISON OF RECOIL ESTIMATES

LARGE VARIATION IN RESULTS: NUMERICAL: 30 – 300 km/s
Hermann '06 – Campanelli '05

ANALYTICAL: 50 – 250 km/s
Damour Gopakumar '06 Blanchet Q. Will '05

VERY SENSITIVE TO DETAILS OF GW EMISSION

CONCLUSIONS

- IF ONE CAN SUCCESSFULLY MATCH / FIT A SUITABLY EXTENDED EOB ANALYTICAL DESCRIPTION OF GW WAVEFORMS TO SUITABLY IMPROVED NUMERICAL RESULTS, ONE WILL HAVE AN EFFICIENT SEMI-PHENOMENOLOGICAL WAY OF BUILDING MULTI-PARAMETER BANKS OF 'TEMPLATES'

- EXISTING THEORETICAL TOOLS (HIGH-PERTURBATION-ORDER CALCULATIONS + RESUMMATION À LA PADÉ + EOB) ARE ≈ UNDER CONTROL FOR LATE INSPIRAL AND TRANSITION TO PLUNGE (CONVERGENCE ROUGHLY SIMILAR TO $\sum_n \left(\frac{1}{2}\right)^n$)

- WOULD BE USEFUL TO HAVE THEORETICAL TOOLS TO UNDERSTAND THE 'MERGER' + TRANSITION TO QNM RINGING OF DEFORMED BLACK HOLE ($R \sim 3M$) FOR COMPARABLE-MASS BH BINARIES