GENERALIZED VIRASORO CONSTRAINTS IN KAC-MOODY COSET MODELS OF M-THEORY

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IHES

work with Axel Kleinschmidt and Hermann Nicolai arXiv: 0912.3491

(CLOSED) BOSONIC STRING IN FLAT SPACETIME

$$S_{\text{NAMBU-GOTO}}[X^\mu(\tau,\sigma)] = -T \iint d\tau\, d\sigma \sqrt{\left(\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma}\right)^2 - \left(\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau}\right)\left(\eta_{\kappa\lambda} \frac{\partial X^\kappa}{\partial \sigma} \frac{\partial X^\lambda}{\partial \sigma}\right)} \Rightarrow \text{NON-LINEAR EOM}$$

GAUGE SYMMETRY: DIFFEOMORPHISMS ON WORLDSHEET: $\tau' = f(\tau,\sigma)$, $\sigma' = g(\tau,\sigma)$

\+ RIGID POINCARÉ SYMMETRY: $X'^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu$

PERIODICITY IN σ

GAUGE-FIXED ACTION : CONFORMAL GAUGE

$$S_{GF}[X^\mu(\tau,\sigma)] = \frac{T}{2} \iint d\tau\, d\sigma\ \eta_{\mu\nu}\left(\frac{\partial X^\mu}{\partial \tau}\frac{\partial X^\nu}{\partial \tau} - \frac{\partial X^\mu}{\partial \sigma}\frac{\partial X^\nu}{\partial \sigma}\right) = \frac{T}{2}\iint d\tau\, d\sigma \left(\dot{X}^2 - X'^2\right)$$

$\Rightarrow$ LINEAR EOM : $\left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2}\right) X^\mu = 0$

REDUCED GAUGE SYMMETRY: $\tau' + \sigma' = f_+(\tau+\sigma)$, $\tau' - \sigma' = f_-(\tau-\sigma)$

NON-LINEAR CONSTRAINTS
FUBINI-VENEZIANO-VIRASORO

$$[(\partial_\tau + \partial_\sigma) X^\mu]^2 = 0$$
$$[(\partial_\tau - \partial_\sigma) X^\mu]^2 = 0$$

CONSERVED CURRENTS AND SUGAWARA STRUCTURE

S_{GF} IS INVARIANT UNDER

$$\delta_{\epsilon_m} X^\mu = \epsilon^\mu_n \, e^{+im(\tau-\sigma)} \quad \text{AND} \quad \delta_{\tilde{\epsilon}_m} X^\mu = \tilde{\epsilon}^\mu_m \, e^{+im(\tau+\sigma)}$$

$\Rightarrow$ TWO INFINITE SEQUENCES OF NOETHER CONSERVED QUANTITIES

$$J^\mu_n \propto \int d\sigma \, \frac{\partial X^\mu}{\partial \tau} \, e^{+in(\tau-\sigma)} \quad \text{AND} \quad \tilde{J}^\mu_n \; : \; \tau+\sigma$$

$\Rightarrow$ INTEGRABILITY OF GF EOM

$$: \; X^\mu(\tau,\sigma) = x^\mu_0 + \ell^2 p^\mu_0 \tau + \frac{i\ell}{2} \sum_{n\neq 0} \left(\frac{J^\mu_n}{n} e^{-in(\tau-\sigma)} + \frac{\tilde{J}^\mu_n}{n} e^{-in(\tau+\sigma)} \right)$$

VIRASORO CONSTRAINTS : FOURIER MODES

$$L_m \equiv T \int d\sigma \, e^{-im\sigma} \left[(\partial_\tau - \partial_\sigma) X^\mu \right]^2 \; ; \; \tilde{L}_m$$

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \eta_{\mu\nu} \, J^\mu_{m-n} \, J^\nu_n$$

$$L_m \approx 0$$
$$\tilde{L}_m \approx 0$$

CONSTRAINTS LINKED TO RESIDUAL GAUGE SYMMETRY OF S_{GF}

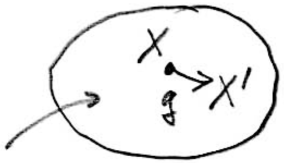
$$\tau' - \sigma' = f_-(\tau - \sigma)$$

CONSERVED "CHARGES" LINKED TO 'STRING' EXTENSION OF ORIGINAL TRANSLATIONAL SYMMETRY

$$\delta X^\mu = a^\mu$$

AFFINE KAC-MOODY ALGEBRA, SUGAWARA CONSTRUCTION

STRING ON GROUP MANIFOLD: WZW → G



RIGID SYMMETRY $X' = gX$

↓

'STRING' EXTENSION $g(\tau \pm \sigma)$

⇒ 2 INFINITE SEQUENCES OF NOETHER CONSERVED QUANTITIES

J^a_n ← a: Lie algebra, $n \in \mathbb{Z}$ $\qquad \tilde{J}^a_n$

ASSOCIATED VIRASORO GENERATORS (Sugawara)

$$\boxed{L_m = \frac{1}{2(k+h^\vee)} \sum_{n=-\infty}^{+\infty} \kappa_{ab} : J^a_{m-n} J^b_n :}$$

Affine Kac-Moody alg.

$$[J^a_m , J^b_n] = f^{ab}{}_c J^c_{m+n} + k^{ab} m \delta_{m,-n} K$$

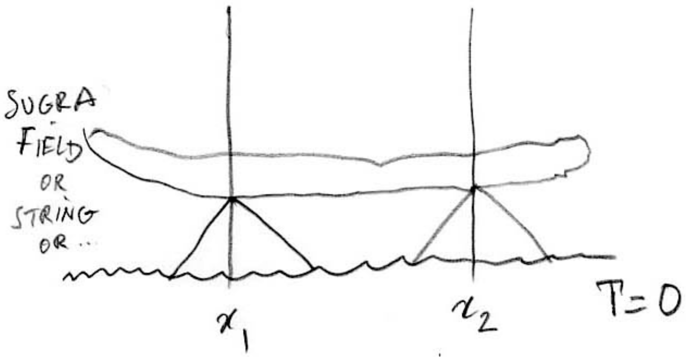
$$[d , J^a_m] = - m J^a_m$$

Associated Virasoro alg.

$$[L_m , L_n] = (m-n) L_{m+n} + \frac{k \dim \mathfrak{g}}{12(k+h^\vee)} m(m^2-1) \delta_{m,-n}$$

$$[L_m , J^a_n] = - n J^a_{m+n}$$

NEAR SPACELIKE SINGULARITY LIMIT AND HYPERBOLIC KAC MOODY COSET MODELS



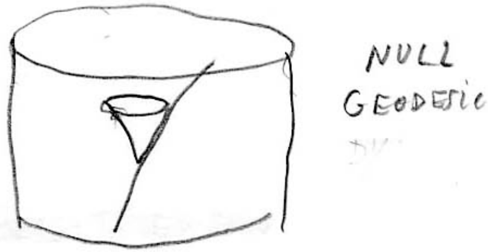
GRADIENT EXPANSION (BKL)

(~ SMALL TENSION EXPANSION $\alpha' \to \infty$)

$$\partial_{x_1}^{m_1} \partial_{x_2}^{m_2} \cdots \partial_{x_{10}}^{m_{10}} \ll \partial_T^{m_1+m_2+\cdots+m_{10}}$$

∞ dim coset, e.g. $E_{10}/K(E_{10})$

$AE_d/K(AE_d)$

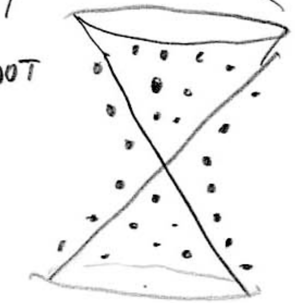


HEIGHT EXPANSION IN KAC-MOODY ALGEBRA

$$\alpha = n_0\alpha_0 + n_1\alpha_1 + \cdots + n_9\alpha_9$$

ROOT

SIMPLE ROOTS



KAC-MOODY ALGEBRAS

generalization of $[J_z, J_+] = +J_+$; $[J_z, J_-] = -J_-$

(J_z: DIAGONALIZABLE; J_+: RAISING GENERATOR; J_-: LOWERING GENERATOR)

TRIANGULAR decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$

- $\mathfrak{n}^-$: LOWERING gators $F_\alpha \equiv E_{-\alpha}$
- $\mathfrak{h}$: CARTAN SA, simultaneously diagonalizable gators $\mathfrak{h} = \{ h = \sum_{a=1}^{r} \beta^a h_a \}$
- $\mathfrak{n}^+$: RAISING gators E_α

$$[h, E_\alpha^{(s)}] = \alpha(h) E_\alpha^{(s)}$$

$$[h, F_\alpha^{(s)}] = -\alpha(h) F_\alpha^{(s)}$$

$$[E_\alpha^{(s)}, E_\beta^{(t)}] = C_{\alpha+\beta}^{(stu)} E_{\alpha+\beta}^{(u)}$$

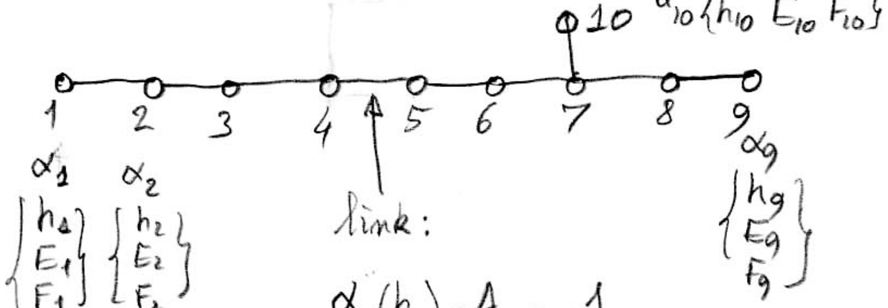
ROOT: eigenvalue of ad_h:
linear form on $h = \beta^a h_a$: $\alpha(h) = \sum_a \alpha_a \beta^a$

$\exists$ basis h_i, $i=1\dots r$ of CSA, and associated simple roots $\alpha_{(i)}$: $E_{\alpha_{(i)}}$, $F_{\alpha_{(i)}}$ s.t.

$$\alpha_{(i)}(h_j) = A_{ji}$$

(CARTAN MATRIX) $A_{ii} = +2$, $-A_{ij} \in \mathbb{N}$

E_{10}



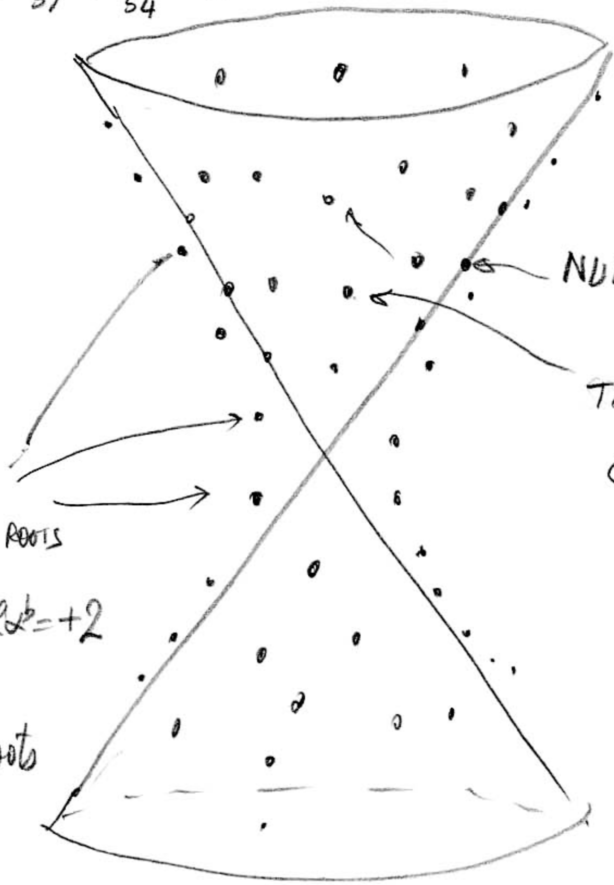
$\alpha_4(h_5) = A_{54} = -1$

10-dim Lorentzian

β^a CSA space : $G_{ab}\beta^a\beta^b$

$\simeq$ ROOT SPACE $\quad \alpha^a \equiv G^{ab}\alpha_b$

ROOT DIAGRAM



$\alpha^2 \equiv G^{ab}\alpha_a\alpha_b = G_{ab}\alpha^a\alpha^b = +2$

including 10 simple roots

$\alpha_{(i)}$; $i = 1 \dots 10$

DECOMPOSING E_{10} w.r.t. $GL(10)$ subalgebra

⟶ $GL(10)$ generators $K^a{}_b$ $\quad [K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b$

$a, b = 1, 2, \dots 10$

Lie algebra element E_{10} :

$$\gamma = h^b{}_a K^a{}_b + A_{abc} E^{(abc)} + A_{a_1 \dots a_6} E^{[a_1 \dots a_6]} + A_{a_0|a_1 \dots a_8} E^{a_0|a_1 \dots a_8} + \cdots$$

$$+ \bar{A}^{abc} F_{[abc]} + \bar{A}^{a_1 \dots a_6} F_{[a_1 \dots a_6]} + \bar{A}^{a_0|a_1 \dots a_8} F_{a_0|a_1 \dots a_8} + \cdots$$

"level 0" wrt $GL(10)$ ($h^b{}_a K^a{}_b$); level +1 ($A_{abc} E^{(abc)}$); level +2 ($A_{a_1 \dots a_6} E^{[a_1 \dots a_6]}$); level +3 ($A_{a_0|a_1 \dots a_8} E^{a_0|a_1 \dots a_8}$); level −1 ($\bar{A}^{abc} F_{[abc]}$)

with, e.g.,

$$[E^{a_1 a_2 a_3}, E^{a_4 a_5 a_6}] = E^{a_1 a_2 a_3 a_4 a_5 a_6}$$

level $\quad +1 \quad +1 \quad\quad +2$

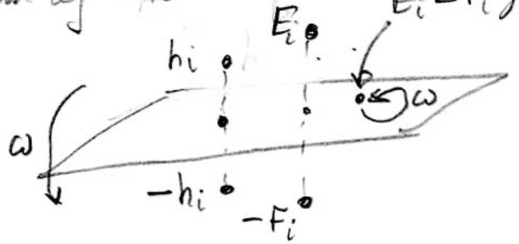
$$[E^{a_1 a_2 a_3}, E^{b_1 \dots b_6}] \sim E^{[a_1|a_2 a_3] b_1 \dots b_6}$$

$+1 \quad +2 \quad\quad +3$

$K(E_{10})$ AND $E_{10}/K(E_{10})$

Maximal compact subgroup of canonical real form of E_{10}

$K(E_{10})$ = fixed set of Chevalley involution ω



~ "antisymmetric" gators

$$J_\alpha \equiv E_\alpha - E_\alpha^T \qquad \gamma^T \equiv -\omega(\gamma)$$

$$J_\alpha^T = -J_\alpha$$

$$\rho = k_{[ab]} J^{ab} + k_{[abc]} J^{abc} + k_{a_1 \cdots a_6} J^{a_1 \cdots a_6} + k_{a_0 | a_1 \cdots a_8} J^{a_0 | a_1 \cdots a_8}$$

$$\rho^T = -\rho$$

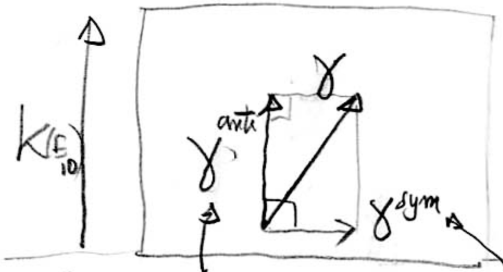
$J^{ab} = K^a{}_b - K^b{}_a$ — $SO(10)$

$J^{abc} \equiv E^{abc} - F_{abc}$

$J^{a_1 \cdots a_6} \equiv E^{a_1 \cdots a_6} - F_{a_1 \cdots a_6}$ ……

similarly to $M_{ab} \equiv M_{(ab)} + M_{[ab]}$

M_{ab}: $gl(n)$; $M_{(ab)}$: coset $gl(n) \ominus so(n)$; $M_{[ab]}$: $so(n)$



$$\gamma \equiv \frac{1}{2}(\gamma - \gamma^T) + \frac{1}{2}(\gamma + \gamma^T)$$

KAC-MOODY COSET MODEL (BOSONIC PART)

(null) geodesic on G/K

$$S^{COSET}_{Bos}[g(t)] = \int dt \, \frac{1}{4n(t)} \, \| v^{sym} \|^2$$

$g \in G/K$

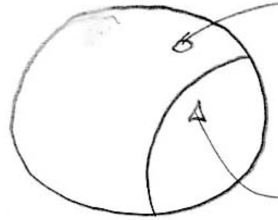
invariant Cartan-Kac metric on G

$$v^{sym} \equiv \frac{1}{2}(v + v^T) \; ; \qquad v \equiv \partial_t g \, g^{-1}$$

velocity in $Lie(G)$

worldline "lapse" $n(t)$ — velocity on coset G/K

big generalization of, e.g., $SL(2)/SO(2)$



Lobatchewski plane

geodesic $S = \int dt \, \frac{1}{2}(\dot{\theta}^2 + \sinh^2\theta \, \dot{\varphi}^2)$

SYMMETRIES:

GLOBAL G SYMMETRY: $g(t) \to k(t) \, g(t) \, g_0$ (local K, global G)

GAUGE worldline reparametrizations: $t' = f(t)$, $n(t)\,dt = n'(t')\,dt'$

$n(t)$: Lagrange xier → associated constraint $\| v^{sym} \|^2 = 0$ — OK with Lorentzian signature on G/K

CORRESPONDENCE $E_{10}/K(E_{10})$ AND $SUGRA_{11}$ (OR M-THY)

(Damour Henneaux Nicolai '02)

Iwasawa: $g(t) = e^{h^a{}_b(t) K^b{}_a} e^{\frac{1}{3!} A_{a_1 a_2 a_3}(t) E^{a_1 a_2 a_3} + \frac{1}{6!} A_{a_1 \dots a_6}(t) E^{a_1 \dots a_6} + \frac{1}{9!} A_{a_0 | a_1 \dots a_8}(t) E^{a_0 | a_1 \dots a_8} + \dots}$

↓

$$S^{coset}_{BOS} = \int \frac{dt}{n(t)} \Big[\frac{1}{4} \left(g^{ac} g^{bd} - g^{ab} g^{cd} \right) \dot g_{ab} \dot g_{cd} + \frac{1}{2} \frac{1}{3!} \dot A_{a_1 a_2 a_3} \dot A^{a_1 a_2 a_3} + \frac{1}{2} \frac{1}{6!} DA_{a_1 \dots a_6} DA^{a_1 \dots a_6} + \frac{1}{2} \frac{1}{9!} DA_{a_0 | a_1 \dots a_8} DA^{a_0 | a_1 \dots a_8} + \dots \Big]$$

← indices raised by g^{ab}

with $g^{ab} = (e^h)^a{}_c (e^h)^b{}_c$; $DA_{a_1 \dots a_6} = \dot A_{a_1 \dots a_6} + 10 A_{[a_1 \dots a_3} \dot A_{a_4 \dots a_6]}$

$$DA_{a_0 | a_1 \dots a_8} = \dot A_{a_0 | a_1 \dots a_8} + 42 A_{\langle 3} \dot A_{6 \rangle} - 42 \dot A_{\langle 3} A_{6 \rangle} + 280 A_{\langle 3} A_3 \dot A_{3 \rangle}$$

↓ coupled ODE's in worldline $n(t) = 1$

$$\begin{cases} \ddot g_{ab} = \dot g^2 + \dot A_3^2 + \dot A_6^2 + \dot A_9^2 \\ \ddot A_{a_1 a_2 a_3} = \dot g \dot A_3 + \dot A_3 A_6 + A_3 \dot A_6 \\ \ddot A_{a_1 \dots a_6} = \dot g \dot A_6 + \dot A_3 A_9 + A \dot A \dots \\ \partial_t \left(DA^{a_0 | a_1 \dots a_8} \right) = 0 \end{cases}$$

$G_{\mu\nu}(t, \vec x)$; $A_{\mu\nu\lambda}(t, \vec x)$; $\psi_\mu(t, \vec x)$

$$S^{Cremmer\ Julia\ Scherk} = \int d^{11}x \Big\{ \frac{E R(G)}{4} - \frac{E}{48} (dA_3)^2 + \frac{2}{(12)^4} F_4 \wedge F_4 \wedge A_3 - \frac{i}{2} \bar\psi_\mu \Gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{i}{96} \left(\bar\psi_\mu \Gamma^{\mu\alpha\beta\gamma\delta\nu} \psi_\nu + 12 \bar\psi^\alpha \Gamma^{\beta\gamma} \psi^\delta \right) F_{\alpha\beta\gamma\delta} + \dots \Big\}$$

↓

coupled PDE's in spacetime

gauge $N(t) = \sqrt{\det G_{ab}}$, $N^a = 0$

$$\begin{cases} \partial_0 \left(G^{ac} \partial_0 G_{cb} \right) = \frac{G}{6} F^{a\beta\gamma\delta} F_{b\beta\gamma\delta} - \frac{\delta^a_b}{72} G F^2 - 2 G R^a{}_b(\Gamma, C) \\ \partial_0 \left(G F^{0abc} \right) = \frac{\varepsilon^{abca_1 a_2 a_3 b_1 \dots b_4}}{144} F_{0a_1 a_2 a_3} F_{b_1 \dots b_4} + F\, C - \partial_d \left(G F^{dabc} \right) \\ \partial_0 F_{abcd} = 6 F_{0e[ab} C^e_{cd]} + 4 \partial_{[a} F_{0bcd]} \end{cases}$$

$C^a{}_{bc}(\vec x)$; $d\theta^a = \frac{1}{2} C^a{}_{bc}\, \theta^b \wedge \theta^c$; $\theta^a = E^a_i(\vec x) dx^i$

CORRESPONDENCE OF DYNAMICAL (EVOLUTION) EQS UP TO HEIGHT 29

DICTIONARY $g^{ab}(t) = G^{ab}(t,\vec{x}_0)$; $\mathring{A}_{a_1a_2a_3}(t) = \mathcal{F}_{0a_1a_2a_3}(t,\vec{x}_0)$ (in frame $E^a_i(x)dx^i$) ; $DA^{a_1\cdots a_6} = -\frac{1}{4!}\varepsilon^{a_1\cdots a_6 b_1\cdots b_4}\mathcal{F}_{b_1\cdots b_4}$; $DA^{a_0|a_1\cdots a_8} = \frac{3}{2}\varepsilon^{a_1\cdots a_8 b_1 b_2}C^{b}_{b_1b_2}$

+ FERMIONIC extension of coset model $S^{coset}_{FERMI} = \int dt \{ -\frac{i}{2}(\Psi(t)|\mathcal{D}^{vs}\Psi(t))_{vs} \}$ (Damour, Kleinschmidt Nicolai 06, de Buyl Henneaux Paulot '06)

+ STRUCTURE of higher-order M-theory corrections (Damour Nicolai 2005)

$$S_M \ni \int \frac{d^{11}x}{\ell_P^3}\sqrt{G}\left[t_8t_8R^4 + \frac{1}{4}\varepsilon_8\varepsilon_8R^4 - 4\,\varepsilon_{11}A_3[\mathrm{tr}R^4 - \frac{1}{4}(\mathrm{tr}R^2)^2] + R^3(DF)^2 + \cdots\right]$$

Green Schwarz '82; Sakai Tanii '87; Gross Witten '86, Deser Seminara, '99 '00; Green Vanhove '97; Green Gutperle Vanhove '97, ..., Tseytlin ....

BUT WHAT ABOUT THE CONSTRAINT EQS OF SUGRA$_{11}$?

gauge-fixed : $N(t) = \sqrt{\det G_{ab}}$; $N^a = 0$; $A_{0ijk} = 0$

among $\mathcal{G}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{3}\mathcal{F}_{\mu\cdots}\mathcal{F}_{\nu}^{\cdots} + \frac{g_{\mu\nu}}{36}\mathcal{F}^2 \approx 0$ $\rightarrow$ $\mathcal{G}_{00} \approx 0$ Hamiltonian ; $\mathcal{G}_{0i} \approx 0$ Momentum of Diffeomorphism

$\mathcal{M}^{\mu\nu\lambda} \equiv \nabla_\sigma \mathcal{F}^{\sigma\mu\nu\lambda} + \epsilon\,\mathcal{F}_4\mathcal{F}_4$ $\longrightarrow$ $\mathcal{M}^{0ij}$ Gauss

+ some $\mathcal{F}$ and R Bianchi constraints

ANALOGY: STRING — KAC-MOODY COSET

	STRING	KAC-MOODY COSET
GAUGE-FIXED ACTION	$S=\iint d\tau d\sigma\left(\dot{X}^2-X'^2\right)$	$S=\int \frac{dt}{n(t)}\left[(\dot{g}g^{-1})^{sym}\right]^2$
INFINITE SYMMETRY	$\delta_{\epsilon_m} X^\mu = \epsilon_m^\mu e^{im(\tau\mp\sigma)}$	$g(t)\to k(t)\,g(t)\,g_0$ ← E_{10} global symmetry
INFINITE # CONSERVED Noether CHARGES	J_m^μ	$J=g^{-1}\frac{v^{sym}}{n}g=\sum_{\alpha,s} J_\alpha^{(s)} E_\alpha^{(s)} \in Lie(E_{10})$
INTEGRABLE	$X_L^\mu=\frac{i}{2}\sum_{m\neq 0}\frac{J_m^\mu}{m}e^{-im(\tau-\sigma)}$	$g(t)=e^{tJ}g(0)$ in suitable gauge
INFINITE # CONSTRAINTS	$L_m\sim\int d\sigma\, e^{-im\sigma}(\partial X)^2$	know SUGRA constraints $\mathcal{G}_{00}\approx 0$, $\mathcal{G}_{0i}\approx 0$, $M^{\alpha\beta}\approx 0$ together with infinite # of spatial gradients
sugawara structure	$L_m=\frac{1}{2}\sum_{n=-\infty}^{+\infty}\eta_{\mu\nu}J^\mu_{m-n}J^\nu_n$	← YES for first known constraints DKN '07
Affine KM alg.	$[J_m^a, J_n^b]=f^{ab}{}_c J^c_{m+n}+k^{ab}m\,\delta_{m+n}K$	Hyperbolic KM alg. $\{J_\alpha, J_\beta\}=f_{\alpha\beta}{}^{\alpha+\beta}J_{\alpha+\beta}$
Constraint alg	$[L_m, L_n]=(m-n)L_{m+n}+c(m^3-m)\delta_{m+n}$	? what is the E_{10} analog of L_m and of the Virasoro alg. ?

SUGAWARA STRUCTURE OF KNOWN CONSTRAINTS

Hamiltonian constraint $\quad g_{00} \approx 0 \longleftrightarrow \frac{\delta S^{coset}}{\delta n(t)} \approx 0$, i.e. $\mathcal{L}^{coset} = \|P\|^2 \approx 0$

denoting $P \equiv v^{sym}$
$Q \equiv v^{antisym}$
$v \equiv \dot{g} g^{-1} \equiv P + Q$

$$\boxed{\mathcal{J} = g^{-1} P g} \longrightarrow \boxed{\|\mathcal{J}\|^2 = 0}$$

$$\mathcal{J} = \frac{\overset{(-3)}{J}{}^{m_0|m_1\cdots m_8}}{8!} F_{m_0|m_1\cdots m_8} + \frac{\overset{(-2)}{J}{}^{m_1\cdots m_6}}{6!} F_{m_1\cdots m_6} + \overset{(-1)}{J}{}^{m_1 m_2 m_3} F_{m_1 m_2 m_3} + \overset{(0)}{J}{}^{n}{}_{m} K^{m}{}_{n} + \frac{1}{3!} \overset{(1)}{J}_{mnp} E^{mnp} + \cdots$$

Other SUGRA constraints in GL(10) decomp. (under truncation $\overset{(4)}{P} = 0$ etc..)

Momentum: $\overset{(3)}{C}_{a_1\cdots a_9} = 3 \overset{(0)}{P}_{c[a_1} \overset{(3)}{P}_{c|a_2\cdots a_9]} + 28 \overset{(1)}{P}_{[a_1 a_2 a_3} \overset{(2)}{P}_{a_4\cdots a_9]} \longrightarrow \overset{(-3)}{\mathcal{L}}{}^{n_1\cdots n_9} = 3 \overset{(0)}{J}{}^{n_1}{}_{p} \overset{(-3)}{J}{}^{p|n_2\cdots n_9} + 28 \overset{(-1)}{J}{}^{n_1 n_2 n_3} \overset{(-2)}{J}{}^{n_4\cdots n_9}$

Gauss: $\overset{(4)}{C}_{b_1\cdots b_{10}\|a_1 a_2} = 3 \overset{(1)}{P}_{a_1 b_1 b_2} \overset{(3)}{P}_{a_2|b_3\cdots b_{10}} + \frac{21}{5} \overset{(2)}{P}_{a_1 b_1\cdots b_5} \overset{(2)}{P}_{a_2 b_6\cdots b_{10}} \to \overset{(-4)}{\mathcal{L}} = 3 \overset{(-1)}{J}{}^{m_1 n_1 n_2} \overset{(-3)}{J}{}^{n_2|m_3\cdots m_{10}} + \frac{21}{5} \overset{(-2)}{J}{}^{n_1 m_1\cdots m_5} \overset{(-2)}{J}{}^{n_2 m_6\cdots m_{10}}$

Bianchi-F: $\overset{(5)}{C}_{b_1\cdots b_{10}\|a_1\cdots a_5} = 3 \overset{(2)}{P}_{a_1\cdots a_4 b_1 b_2} \overset{(3)}{P}_{a_5|b_3\cdots b_{10}} \longrightarrow \overset{(-5)}{\mathcal{L}}{}^{m_1\cdots m_{10}\|n_1\cdots n_5} = 3 \overset{(-2)}{J}{}^{n_1\cdots n_4 m_1 m_2} \overset{(-3)}{J}{}^{n_5|m_3\cdots m_{10}}$

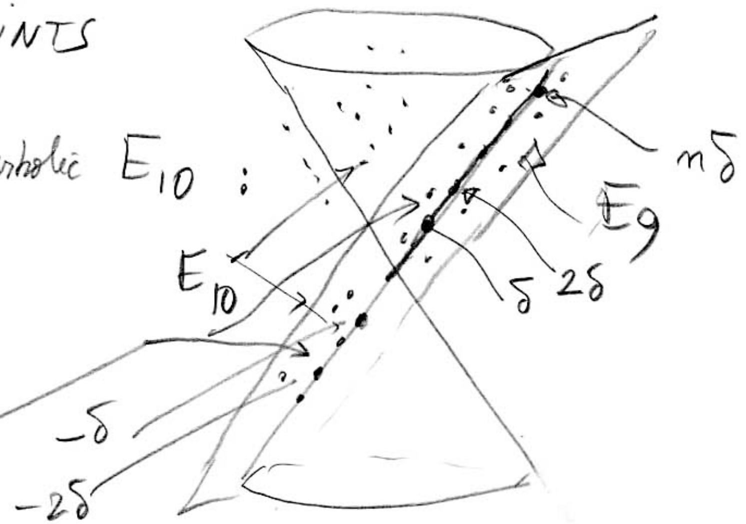
Bianchi-R: $\overset{(6)}{C}_{b_1\cdots b_{10}\|a_0|a_1\cdots a_7} = 9 \overset{(3)}{P}_{a_0|b_1\cdots b_8} \overset{(3)}{P}_{b_9|b_{10} a_1\cdots a_7} \to \overset{(-6)}{\mathcal{L}}{}^{m_1\cdots m_{10}\|n_0|n_1\cdots n_7} = 9 \overset{(-3)}{J}{}^{n_0|m_1\cdots m_8} \overset{(-3)}{J}{}^{m_9|m_{10} n_1\cdots n_7}$

WEAKLY CONSERVED: $\partial_t \overset{(\ell)}{C} = \sum_{\ell' \geq \ell} \overset{(\ell' \ell)}{P} \overset{(\ell')}{C}$ (and triangular)

SUGAWARA structure + STRONGLY conserved

ROOT STRUCTURE OF CONSTRAINTS

Relation between Affine KM(E_9) and Hyperbolic E_{10} :



Affine Virasoro Constraints : $L_n \sim L_{n\delta}$

linked to $\mathbb{Z}$ line of null roots $\pm n\delta$

? Relation between GL(10)-decomposed $SUGRA_{11}$ constraints and E_{10} root diagram?

? Relation between $SUGRA_{10}$ IIA constraints and $SUGRA_{10}$ IIB ones (Kleinschmidt Nicolai 05)

? Relation between Affine Virasoro and Hyperbolic constraints ?

ROOT STRUCTURE OF $SUGRA_{11} \sim$ IIA constraints

Root: eigenvalue of ad_h

$[h, E_\alpha] = \alpha(h)\, E_\alpha$

e.g. $J = \cdots + \frac{1}{3!} \overset{(-1)}{J}{}^{mnp} F_{mnp} + \cdots$

$\overset{(-1)}{J}_\alpha \, E_\alpha$ ← root at level -1 associated to F_{mnp}

Constraints seen either in: enveloping alg. $\quad E_\alpha E_\beta$: root $\alpha+\beta$

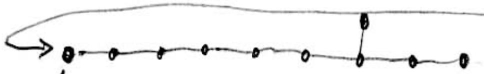
or Poisson algebra $\quad J_\alpha J_\beta$: root $\alpha+\beta$

e.g. $GL(10)$ multiplet $\overset{(-3)}{\mathcal{L}}{}^{[n_1 \cdots n_9]}$ = highest weight $\overset{(-3)}{\mathcal{L}}{}^{2345678910}$ + action of lowering $K^a{}_b$ generators

$\overset{(-3)}{\mathcal{L}}{}^{234678910}$ associated with root $\alpha = \Lambda_1 \equiv -\delta^{(3)} = -(\alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6 + 6\alpha_7 + 4\alpha_8 + 2\alpha_9 + 3\alpha_{10})$

Λ_1: fundamental weight associated to node 1

$-\delta^{(3)}$: negative, null root



1

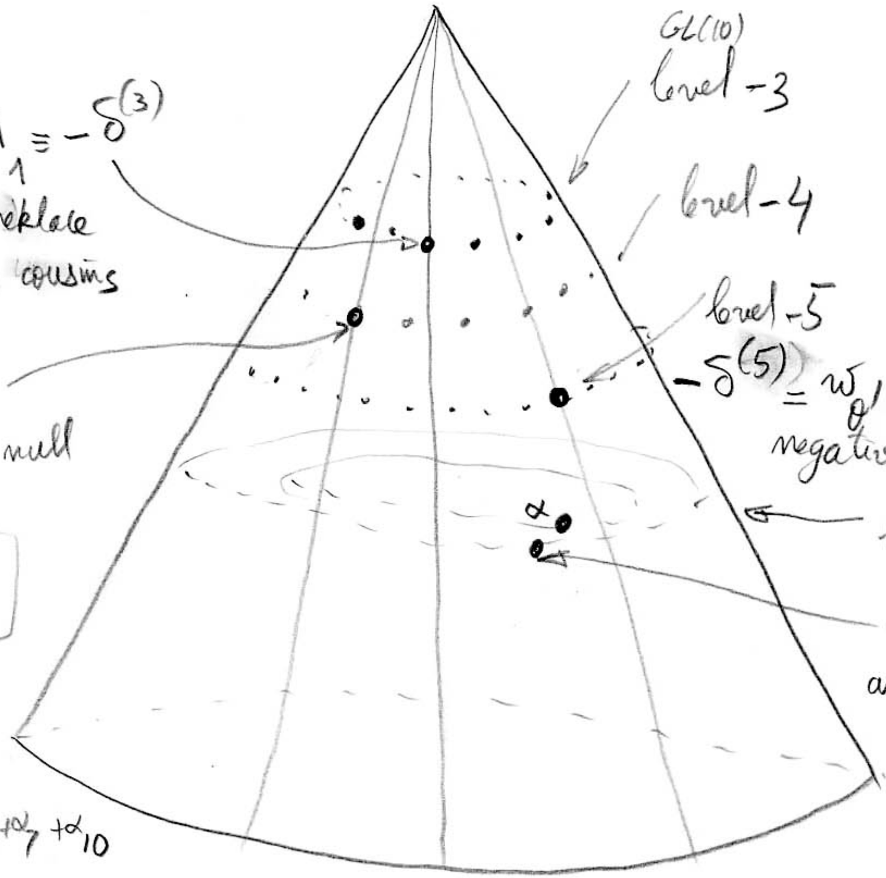
$\langle \Lambda^i \mid \alpha_j \rangle = \delta^i_j$

"ROOT SUPPORT" OF SUGRA CONSTRAINTS :

$\mathcal{L}_{\alpha}^{\text{CONSTRAINT}}$ (α ← ROOT)

$\overset{(-3)}{\mathcal{L}}$: $\Lambda_1 \equiv -\delta^{(3)}$

and its necklace of 10 sl_{10} cousins



$\overset{(-4)}{\mathcal{L}}{}^{a_1 \dots a_{10} \| m_1 m_2}$ — $\delta^{(4)}$

again negative, null

and, moreover,

$\delta^{(4)} = w_\theta(\delta^{(3)})$

↑ Weyl reflection of $\delta^{(3)}$ in

$\theta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_{10}$

$-\delta^{(5)} = w_{\theta'}(-\delta^{(4)})$

negative, null

both $-\delta^{(6)}$ null

and $\alpha^2 = -2$ time-like roots

FULL ROOT SUPPORT OF COSET CONSTRAINTS

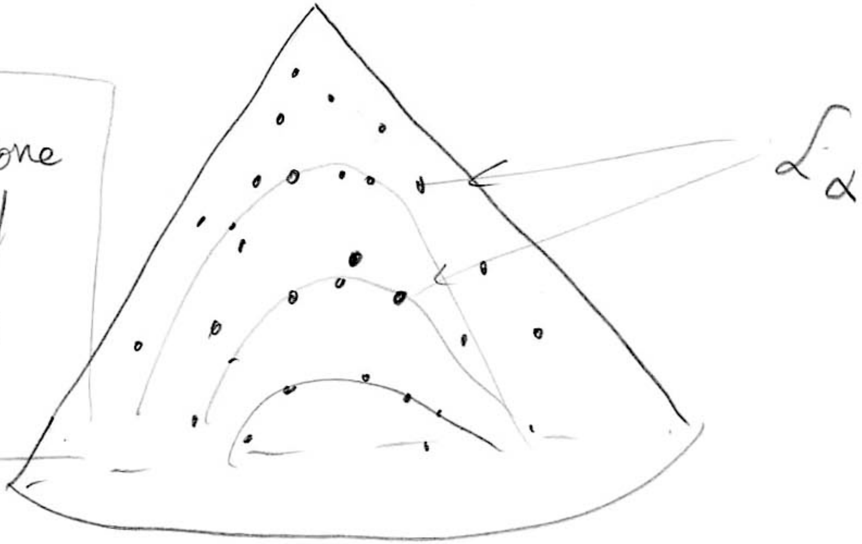
Expect: root support of fundamental rep. $L(\Lambda_1)$ 

+ spatial gradients e.g. $\partial_m \mathcal{H}_n \sim \mathcal{L}^{(-6)\,m_1 \cdots m_9 | n_1 \cdots n_9} \to$ root $-2\delta^{(3)}$

i.e. expect need to add multiples $-n\delta$

→ Using theorems of V. Kac

root support = cone of all past-directed null and time-like roots



'Sugawara content' of constraints

$$\mathcal{L}_\alpha = \sum_{\beta,\, s,s'} M_{s,s'}(\alpha,\beta)\, J^{(s)}_{\alpha-\beta}\, J^{(s')}_{\beta}$$

'constituent roots' $\beta_1 \equiv \alpha - \beta$, $\beta_2 = \beta$?

$\alpha = \beta_1 + \beta_2$

? numerical coefficients

SUGRA CONSTRAINTS AND CANONICAL NORMALIZATION

e.g. $\overset{(-3)}{\mathcal{L}}{}^{n_1\cdots n_9} = 28\, \overset{(-1)}{J}{}^{[n_1 n_2 n_3} \overset{(-2)}{J}{}^{n_4\cdots n_9]} + 3\, \overset{(-3)}{J}{}^{p\,|[n_1\cdots n_8} \overset{(0)}{J}{}^{n_9]}{}_{p}$

$\overset{(-4)}{\mathcal{L}}{}^{m_1\cdots m_{10}|n_1 n_2} = \frac{21}{5}\, \overset{(-2)}{J}{}^{n_1[m_1\cdots m_5} \overset{(-2)}{J}{}^{m_6\cdots m_{10}]n_2} + 3\, \overset{(-3)}{J}{}^{n_2|[m_1\cdots m_8} \overset{(-1)}{J}{}^{m_9 m_{10}]n_1}$

with normalization

$$\langle E_{\alpha_{real}}, \omega(E_{\alpha_{real}}) \rangle = 1$$

real roots among $GL(10)$ multiplet, $\alpha^2 = +2$

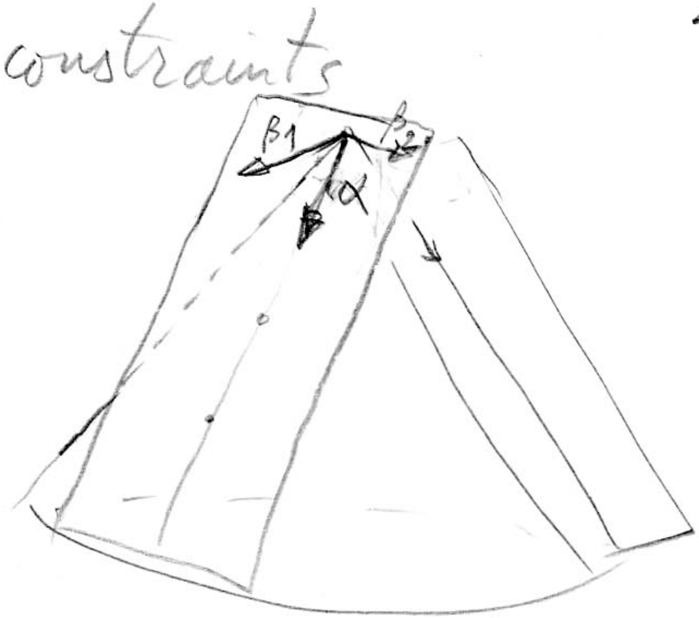
e.g. $$3\, \overset{(-3)}{\mathcal{L}}_{\alpha} = \sum_{\substack{\beta_1 \beta_1' \text{ real} \\ \beta_2 \beta_2' \text{ real} \\ \beta_1+\beta_2 = \alpha = \beta_1'+\beta_2'}} \overset{(-1)}{J}_{\beta_1} \overset{(-2)}{J}_{\beta_2} + \overset{(-3)}{J}_{\beta_1'} \overset{(0)}{J}_{\beta_2'}$$

(α: null)

SAME COEFFICIENT FOR ALL 'SPECIAL' CONFIGURATIONS

$\alpha_{null} = \beta_1^{real} + \beta_2^{real}$ decomp.

Universality features of constraints



- 'Scaffold' of special sugawara configurations $\alpha_{null} = \beta_1^{real} + \beta_2^{real}$

→ unit coefficient, like Affine sugawara

$$L_{n\delta} \propto \sum_{\beta \in \Delta^{aff}} J_{n\delta - \beta} \, J_{\beta}$$

- II A versus II B

$A_9 = SL(10)$

↑

explicitly covariant under SL(10)

$A_8 = SL(9) \oplus A_1 = SL(2)$

↑ ↑

covariant under $SL(9) \times SL(2)$

$$\left[\mathcal{L}^{(-3)\,2345678910}_{IIA}\right]_{\text{real roots}} = \left[\mathcal{L}^{(-4)\,23456789}_{IIB}\right]_{\text{real roots}}$$

$$= \left(\frac{1}{3} L^{\text{affine}}_{-\delta}\right)_{\text{real roots}}$$

Non-universality features of constraints

- while $\left[\mathcal{L}_{IIA}\right]_{\text{real roots}} = \left[\mathcal{L}_{IIB}\right]_{\text{real roots}}$, $\left[\mathcal{L}_{IIA}\right]_{\text{imaginary roots}} \neq \left[\mathcal{L}_{IIB}\right]_{\text{imaginary roots}}$

  because of different manifest covariantization groups $SL(10) \neq SL(9)\times SL(2)$

first possibility: try to combine the different covariantizations to define 'more complete' coset constraints

second possibility: admit that the two different choices of parabolic subgroups

$$IIA = \text{Lie}(E_{10}) - \{f_{10}\} \neq IIB = \text{Lie}(E_{10}) - \{f_8\}$$

correspond to two different 'gauge fixings' of the underlying E_{10}-invariant coset action

$\Rightarrow$ two different additional constraints

$$S_{IIA} = \int dt \left[\frac{\|P\|^2}{n(t)} + \lambda^{\alpha}_{IIA}(t)\, \mathcal{L}^{IIA}_{\alpha} \right] \neq S_{IIB} = \int dt \left[\frac{\|P\|^2}{n(t)} + \lambda^{\alpha}_{IIB}(t)\, \mathcal{L}^{IIB}_{\alpha} \right]$$

Algebra of constraints : Generalized Virasoro alg.

it seems that, for each choice of **parabolic** subgroup (IIA or IIB)

$\Leftrightarrow$ associated **level** decomp. l_{IIA} counts e_{10}

l_{IIB} counts e_8

we have

$$\overset{-(l+1)}{\mathcal{L}} = \{ \overset{(-1)}{J} , \overset{(-l)}{\mathcal{L}} \}$$

$$\overset{(-l)}{\mathcal{L}} = \sum_{p+q=l} \overset{(-p)}{J} \cdot \overset{(-q)}{J}$$

HYPERBOLIC KM

$$\Rightarrow \quad \{ \mathcal{L}_\alpha , \mathcal{L}_\beta \} \sim \sum_\gamma J_{\alpha+\beta-\gamma} \mathcal{L}_\gamma$$

which generalizes

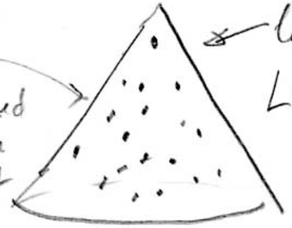
$$[L_{m\delta} , L_{n\delta}] = 2 (m-n) K \, L_{m\delta+n\delta}$$

Affine KM central element

CONCLUSIONS (1)

$\exists$ HINTS OF REMARKABLE STRUCTURE OF POSSIBLE CONSTRAINTS TO BE ADDED TO COSET ACTION

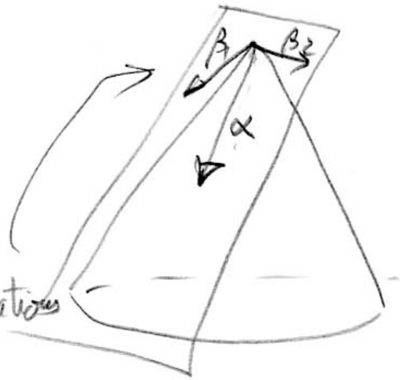
$\mathcal{L}_\alpha$ ↙ root on and within past light cone



← linked to $L(\Lambda_1) \oplus L(2\Lambda_1) \oplus \ldots$

$$\mathcal{L}_\alpha = \sum_{\beta_1+\beta_2=\alpha} J_{\beta_1} J_{\beta_2}$$

with especially simple and universal special configurations



- explicit Weyl invariance
- partial universality $[\mathcal{L}_{IIA}]_{real} = [\mathcal{L}_{IIB}]_{real}$

? However, $\mathcal{L}_{IIA} \neq \mathcal{L}_{IIB}$ because of different <u>parabolic</u> subgroups

but $\mathcal{L}_{IIA}$ invariant under $E_{10}^{(-\mathfrak{f}_{10})}$ and $\mathcal{L}_{IIB}$ invariant under $E_{10}^{(-\mathfrak{f}_8)}$

i.e. $\mathcal{L}_{IIA}$ is a rep. of $E_{10}^{(-\mathfrak{f}_{10})}$ (similarly for $\mathcal{L}_{IIA}^{massive}$ and $E_{10}^{(-\mathfrak{f}_9)}$)

? maybe analogous to light-cone constraints ≠ covariant ones, or light-cone′ ≠ light-cone″ ?

CONCLUSIONS (2)

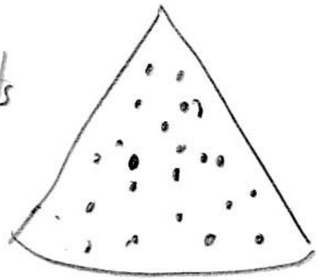
for better investigated case $IIA \sim SUGRA_{11}$ $E_{10}^{(-f_{10})}$ $\exists$ closed algebra of constraints

$$\{ \overset{(-p)}{J}, \overset{(-q)}{J} \} \sim \overset{(-p-q)}{J}$$

$$\{ \overset{(-p)}{J}, \overset{(-q)}{\mathcal{L}} \} \sim \overset{(-p-q)}{\mathcal{L}}$$ ⟵ very ≠ from <u>affine</u> KM case $\{J, \mathcal{L}\} \sim J$

$$\{ \overset{(-p)}{\mathcal{L}}, \overset{(-q)}{\mathcal{L}} \} \sim \sum \overset{(-p-q+r)}{J} \overset{(-r)}{\mathcal{L}}$$

Quantum coset model: ∞ # of constraints



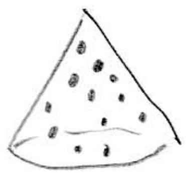
$$\hat{\mathcal{L}}_\alpha |\psi\rangle = 0$$

? welcome to reduce physical dof to vicinity of real roots, i.e. sugra fields + their gradients + ?

<u>Mathematically</u>:

~ spectral parameter, loop alg.

⟶



? generalization of loop alg.