Geometry and Physics after 100 Years
of Einstein's Relativity.
Albert Einstein Institute April 5-8, 2005

SYMMETRY AND CHAOS IN GRAVITY AND SUPERGRAVITY

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SYMMETRY IN GRAVITY

Buchdal 1954 $\quad$ static solutions of GR_4 $\quad g_{0i} = 0$

discrete symmetry $\quad g_{00} \to \frac{1}{g_{00}} \quad h_{ij} = -g_{00} g_{ij} = \text{fixed}$

Ehlers 1959 $\quad$ stationary solutions of GR_4 or $\exists$ 1 Killing vector e.g. $k = \frac{\partial}{\partial x^3}$

continuous symmetry $\sim SL(2,\mathbb{R})_E$

$$Z' = \frac{aZ+b}{cZ+d} \qquad Z = B + i\,\Delta$$

$ad - bc = 1$
$a,b,c,d \in \mathbb{R}$

NON-LOCAL DUAL TO $g_{3\mu}/g_{33} \equiv A_\mu$ $\qquad \Delta = g_{33}$

$$\varepsilon_{\mu\nu\lambda}\partial^\lambda B = \Delta^2(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

Matzner, Misner 1967 $\quad$ if $\exists$ 2 commuting Killing vectors k_1, k_2

continuous symmetry $SL(2,\mathbb{R})_{MM}$

linked to freedom in linear combination of Killing v's

$$k'_a = \Lambda_a^{\ b}\, k_b$$

LOCAL IN GRAVITY VARIABLES

Geroch 1972 $\quad$ 2 commuting Killing vectors $+\ \varepsilon_{\mu\nu\rho\sigma}\, \overset{1}{k}{}^\mu \overset{2}{k}{}^\nu \nabla^\rho \overset{a}{k}{}^\sigma = 0$

interplay of $SL(2,\mathbb{R})_E \times SL(2,\mathbb{R})_{MM}$

$\Rightarrow$ infinite dimensional Lie group

Julia 1981 $\quad$ Geroch group $= \widehat{SL(2,\mathbb{R})} = A_1^{(1)}$ o═o

Breitenlohner, Maison '84, '87 $\quad$ AFFINE KAC-MOODY GROUP: EXTENSION OF $SL(2,\mathbb{R})$

Belinski, Zakharov ...

SYMMETRY IN SUPERGRAVITY

Cremmer Julia 1979 $\quad$ $SUGRA_{11}$ WITH 7 Kommuting Killing Vectors

$D=11 \longrightarrow D=4$

E_7 SYMMETRY 

$D=11 \downarrow$

$D=3$ (8 KV's) E_8 

Marcus, Schwarz '83

$D=2$ Nicolai '87 $E_9 = E_8^{(1)}$ 

?? $D=1 \overset{?}{\Rightarrow}$ E_{10} 

Julia,
Nicolai,
Mizoguchi

NB1: We consider here continuous symmetries: $SL(2,\mathbb{R}), \dots, E_{10}(\mathbb{R})$

Acting on solutions: solution $\rightarrow$ solution'

We consider un-compactified $SUGRA_{11}$

For compactified, quantum theories classical continuous symmetry groups are broken down to discrete groups: $SL(2,\mathbb{Z}), \dots, E_{10}(\mathbb{Z})$?

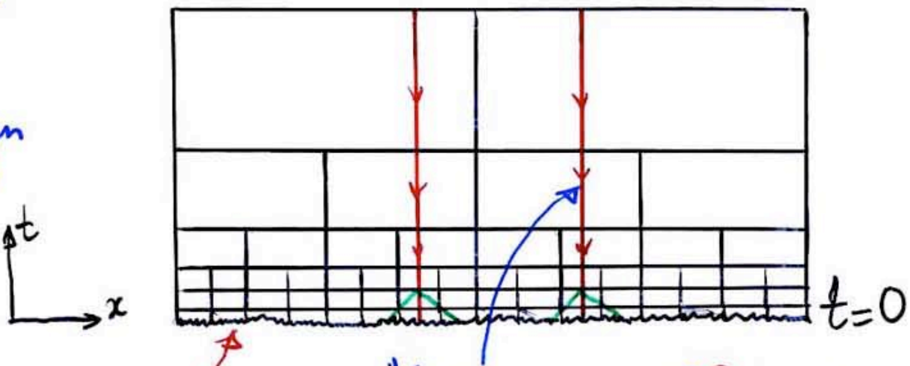
NB2: Other structures of E_{10} showed up in string / M-thy

- spectrum of BPS states: root lattice (Harvey, Moore '96)
- U-duality of toroidally compact. M-thy (Obers, Pioline '98, Banks, Fischler, Motl '99)

CHAOS IN GRAVITY

- Belinsky, Khalatnikov, Lifshitz 1969 (BKL)

infinitely "oscillatory" behaviour of generic inhomogeneous solution of $R_{\mu\nu}=0$ near $t=0$ in $D=4$



$ds^2 = -dt^2 + t^{2p_1}\,{}'dx^2{}' + t^{2p_2}\,{}'dy^2{}' + t^{2p_3}\,{}'dz^2{}'$

$p_a(t,x)$ depend both on t and x

- asymptotically $(t\to 0)$ similar to behaviour in homogeneous cosmological models, e.g. Bianchi IX (BKL '69, Misner '69)

- roughly speaking: BKL limit: $\frac{\partial}{\partial x} g_{\mu\nu}(t,x) \ll \frac{\partial}{\partial t} g_{\mu\nu}(t,x)$

$$PDE(t,x) \to \{ODE(t)\}_x$$

- Demaret, Henneaux, Spindel 1985

BKL CHAOS DISAPPEAR IN $D \geqslant 11$ $\Rightarrow$ MONOTONIC KASNER-LIKE BEHAVIOUR AS $t\to 0$

CHAOS IN SUPERGRAVITY$_{11}$

(AND STRING THIES$_{10}$)

- Damour, Henneaux 2000 THE BOSONIC SECTORS OF

$D=11$ SUPERGRAVITY (Cremmer Julia Scherk) $S=\int d^{11}x \sqrt{-G}\left[R(G)-(dA_3)^2+c\,A_3 \wedge dA_3 \wedge dA_3\right]$

AS WELL AS THE $D=10$ STRING THEORIES (I, IIA, IIB, HO, HE)

ARE ALL CHAOTIC À LA BKL

CHAOS AND SYMMETRY

- Damour, Henneaux 2001

$\exists$ A HIDDEN SYMMETRY IN THE COSMOLOGICAL CHAOS OF SUGRA$_{11}$, AS WELL AS THE VARIOUS STRING THEORIES

WEYL GROUP OF HYPERBOLIC KAC-MOODY ALGEBRA E_{10}

BKL-LIKE CHAOTIC OSCILLATIONS = CHAOTIC BILLIARD MOTION WITHIN WEYL CHAMBER OF E_{10}

- Damour, Henneaux, Nicolai 2002

$\exists$ APPROXIMATE EQUIVALENCE BETWEEN

$D=11$ SUGRA $\Longleftrightarrow$ $D=1$ COSET MODEL $E_{10}/K(E_{10})$

NULL GEODESIC ON COSET SPACE

- Damour, Henneaux, Julia, Nicolai 2001

WEYL GROUP OF AE_n LIES BEYOND THE BKL GRAVITY CHAOS IN $D=n+1$

USUAL GR_4 { BILLIARD MOTION WITHIN WEYL CHAMBER OF HYPERBOLIC KAC-MOODY ALG $\widehat{AE_3}=\widehat{\widehat{A_1}}=\widehat{SL(2,\mathbb{R})}$ = GEROCH

AEI 5

GENERIC, INHOMOGENEOUS COSMOLOGICAL SINGULARITIES OF

$$S^{(D)} = \int d^D x \sqrt{-G}\left[R(G) - \frac{1}{2\times 4!} F_{\mu\nu\lambda\rho}F^{\mu\nu\lambda\rho}\right] + \frac{1}{(12)^4}\int A_3 \wedge F_4 \wedge F_4$$

$D=11$

$F_4 = dA_3$

HAMILTONIAN FORMALISM

RESCALED LAPSE $\tilde{N} \equiv \frac{N}{\sqrt{g}}$

$$S = \int dx^0 \int d^{10}x \left[\pi^{ij}\dot{g}_{ij} + \frac{1}{3!}\pi^{ijk}\dot{A}_{ijk} - \tilde{N}\mathcal{H} + \cdots \right]$$

MOMENTUM CONSTRAINT
GAUSS CONSTRAINT
VANISH ON SHELL IN
GAUGE $G_{0i} = 0$
$A_{0ij} = 0$

HAMILTONIAN DENSITY (OF WEIGHT 2)

$\frac{1}{D-2}$

CS SHIFT

$$\boxed{\mathcal{H}[g_{ij},\pi^{ij},A_{ijk},\pi^{ijk}] = \pi^{ij}\pi_{ij} - \frac{1}{9}\pi^i_i\pi^j_j + \frac{1}{2\times 3!}\left(\pi^{ijk} - c\,\varepsilon^{10}A_3F_4\right)^2 + \frac{g}{2\times 3!}\left(F_{ijkl}\right)^2 - gR(g)}$$

KINETIC GRAVITATIONAL ENERGY

POTENTIAL GRAVITATIONAL ENERGY

(ELECTRIC ENERGY)

(MAGNETIC ENERGY)

IWASAWA DECOMPOSITION OF SPATIAL METRIC

$$ds^2 = -N^2 dt^2 + g_{ij}(t,x)\, dx^i dx^j$$

OR MORE GENERALLY $\omega^i_j(x)\, dx^j$

$$= -N^2 dt^2 + \sum_{a=1}^{10} e^{-2\beta^a} (\theta^a)^2$$

$\theta^a = \mathcal{N}^a{}_i \, dx^i$

"DIAGONAL COMPONENTS" OF g_{ij}

UPPER TRIANGULAR "OFF DIAGONAL" COMPONENTS

$$\mathcal{N}^a{}_i = \begin{pmatrix} 1 & * & * & * & * \\ & 1 & * & * & * \\ & & 1 & * & * \\ & 0 & & 1 & * \\ & & & & \ddots \; 1 \end{pmatrix}$$

$$(g_{ij}, \pi^{ij}) \longrightarrow \Big(\underbrace{\beta^a, \pi_a}_{\text{DIAGONAL DOF}} ; \underbrace{\mathcal{N}^a{}_i , \mathcal{P}^i{}_a}_{\text{OFF DIAGONAL DOF}} \Big)$$

KINETIC GRAVITATIONAL TERMS

$$\pi^{ij}\pi_{ij} - \frac{1}{9}\pi^i{}_i \pi^j{}_j = \frac{1}{4} G^{ab} \pi_a \pi_b + \frac{1}{2}\sum_{a<b} e^{-2(\beta^b - \beta^a)} \left(\mathcal{N}^b{}_i \, \mathcal{P}^i{}_a \right)^2$$

Lorentzian quadratic form

"SYMMETRY WALL" linked to kinetic terms of off-diagonal metric

$$G^{ab}\pi_a\pi_b \equiv \sum_{a=1}^{10} \pi_a^2 - \frac{1}{9}\left(\sum_{a=1}^{10} \pi_a\right)^2$$

STRUCTURE OF SUPERGRAVITY$_{11}$ HAMILTONIAN

$$\boxed{\mathcal{H} = \frac{1}{4} G^{ab}\pi_a \pi_b + \sum_A c_A(Q, P, \partial_x\beta, \partial_x Q, \dots)\, e^{-2 w_A(\beta)}}$$

KINETIC TERMS OF CONJUGATE TO β^a's

Q_{11}: N^a_i, A_{ijk}

$P_=$: P^i_a, π^{ijk}

"WALL FORMS" LINEAR COMBINATIONS OF β^a's

SYMMETRY WALLS: $w^{(S)}_{ab}(\beta) \equiv \beta^b - \beta^a \quad ; \quad a<b$

'ELECTRIC' WALLS: $e_{abc}(\beta) \equiv \beta^a + \beta^b + \beta^c \quad ; \quad a \neq b \neq c \neq a$

'MAGNETIC' WALLS: $m_{a_1a_2a_3a_4}(\beta) = \sum_{e=1}^{10} \beta^e - \beta^{a_1} - \beta^{a_2} - \beta^{a_3} - \beta^{a_4}$ $\quad a_1 \neq a_2 \neq a_3 \neq \dots \neq b_6$

$= \beta^{b_1} + \beta^{b_2} + \beta^{b_3} + \beta^{b_4} + \beta^{b_5} + \beta^{b_6}$

GRAVITATIONAL WALLS: $\alpha_{abc}(\beta) = \sum_e \beta^e + \beta^a - \beta^b - \beta^c \qquad a \neq b \quad b \neq c \quad c \neq a$

$\mu_a(\beta) = \sum_e \beta^e - \beta^a$

ASYMPTOTIC SUGRA-DYNAMICS NEAR $T = 0$

$$\mathcal{H}(\beta,\pi;Q,P) = \frac{1}{4} G^{ab}\pi_a\pi_b + \sum_A c_A(Q,P,\dots)\, e^{-2 w_A(\beta)}$$

BIG BANG SINGULARITY

$T \to 0$

$\sqrt{g} = e^{-\Sigma\beta} \to 0$

$\sum_{a=1}^{10} \beta^a \to +\infty$

$$Q = Q^\infty + \mathcal{O}\Big(\sum_B e^{-2w_B(\beta)}\Big)$$

$$P = P^\infty + \mathcal{O}\Big(\sum_B e^{-2w_B(\beta)}\Big)$$

"FREEZING" OF Q AND P NEAR SINGULARITY

(Damour, Henneaux, Nicolai 2003)

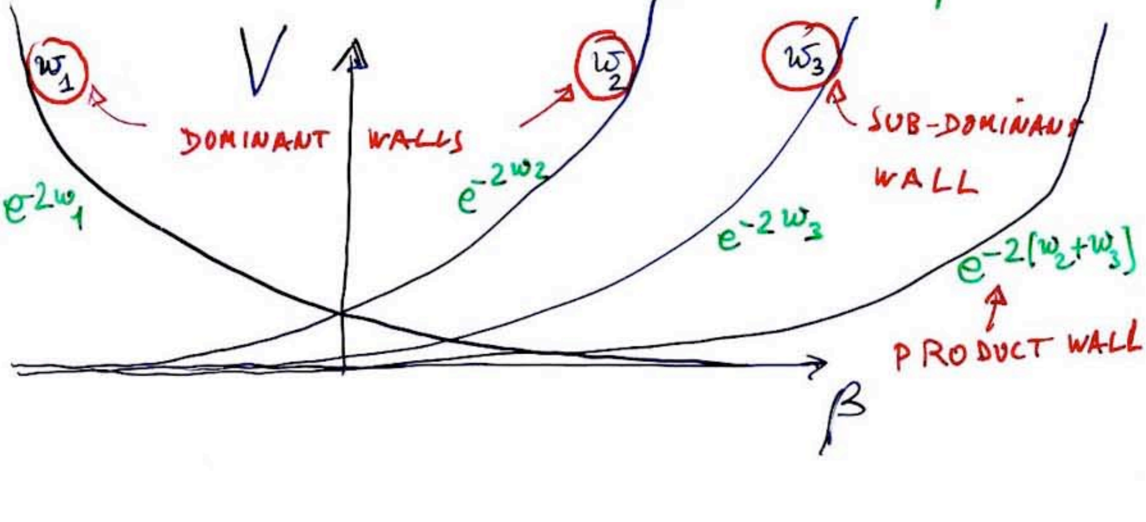
$$\mathcal{H}(\beta,\pi) \simeq \frac{1}{4} G^{ab}\pi_a\pi_b + \sum_A c_A(Q^\infty,P^\infty)\, e^{-2w_A(\beta)} + \sum_{A,B} c_{A,B}(Q^\infty,P^\infty)\, e^{-2[w_A+w_B]} + \dots$$

REDUCED DYNAMICS FOR 'DIAGONAL' DOF

FROZEN VALUES

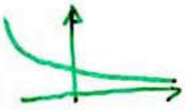
LINEAR COMBINATIONS OF ORIGINAL WALL FORMS

EXPONENTIAL ('TODA') WALLS IN β-SPACE

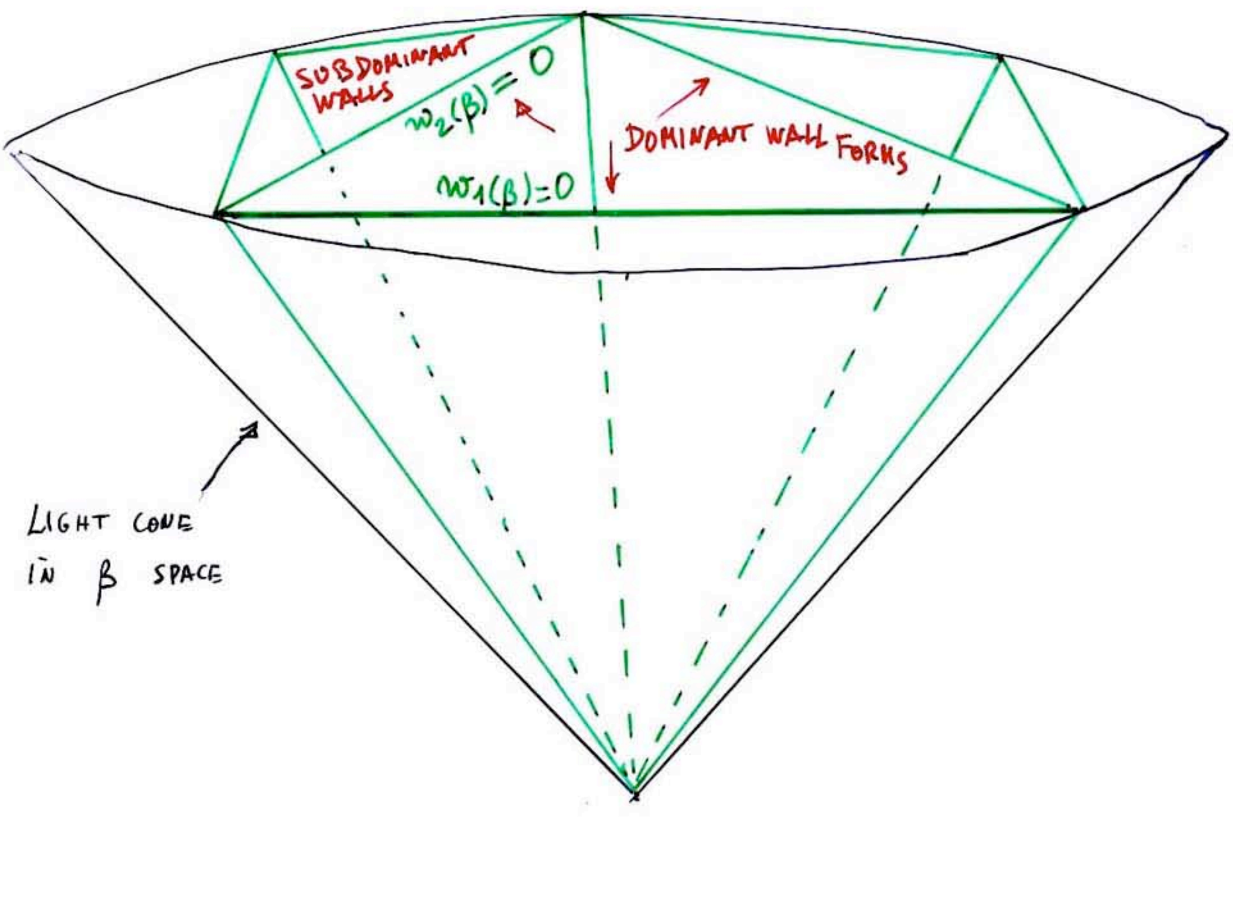


BILLIARD IN β SPACE

LORENTZIAN-SIGNATURE METRIC: $G^{ab}\pi_a\pi_b \leftrightarrow G_{ab}\,d\beta^a d\beta^b$

$e^{-2w(\beta)} =$  $\simeq$ SHARP WALL 

$$G_{ab}\,d\beta^a d\beta^b = \sum_{a=1}^{10}(d\beta^a)^2 - \left(\sum_{a=1}^{10} d\beta^a\right)^2$$

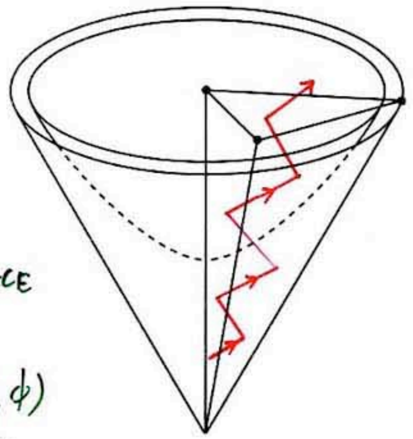


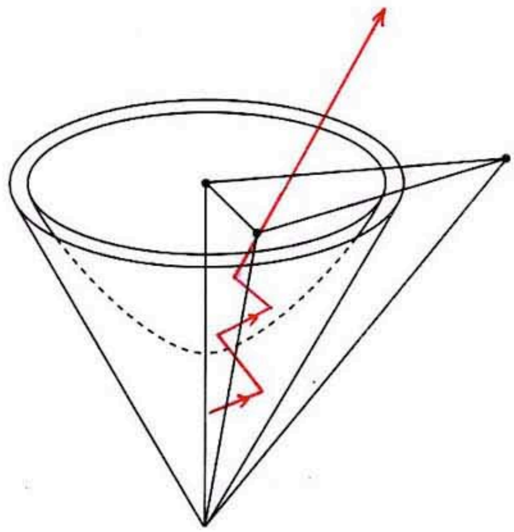
EINSTEIN BILLIARDS

CHAOTIC BILLIARD

E.G. PURE GRAVITY IN $D=3+1$
$d=3$

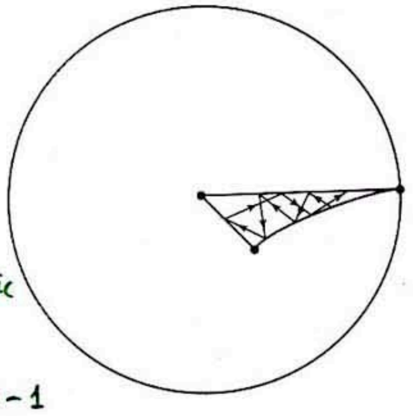
NON-CHAOTIC BILLIARD

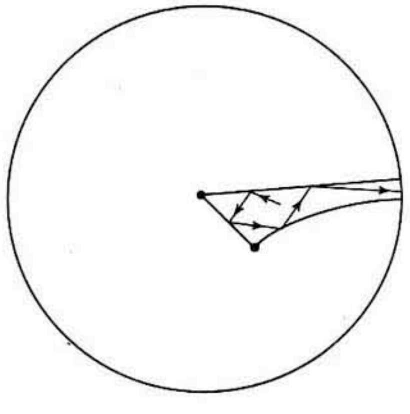




β-SPACE

$\beta^\mu = (\beta^a, \phi)$
$d+n$ dim

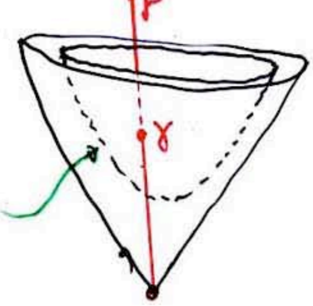




γ-SPACE
I.E.
HYPERBOLIC SPACE
H_{d+n-1}

$$\beta^\mu = \rho\, \gamma^\mu$$

$G_{\mu\nu}\gamma^\mu\gamma^\nu = -1$
ON UNIT FUTURE HYPERBOLOID



KAC-MOODY ALGEBRAS

SU(2) $\cong$ A_1 IS SL(2)

$$[J_z, J_+] = +J_+ \qquad [J_z, J_-] = -J_-$$

J_z: CARTAN GENERATOR (DIAGONAL); J_+: RAISING GENERATOR; J_-: LOWERING GENERATOR

CARTAN SUBALGEBRA : LINEAR SPACE $\mathbb{R}^r$ (r: RANK) $\quad \mathfrak{h} = \{\beta^a h_a ; a = 1, \dots, r\}$

β^a: coordinates in Cartan space: $h = \sum_a \beta^a h_a$; h_a: r independent Cartan gators

$$[h', h''] = 0$$

TRIANGULAR DECOMPOSITION: $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$

LOWERING GATORS F_α — CARTAN h — RAISING GATORS E_α

$$\boxed{[h, E_\alpha^{(s)}] = \alpha(h)\, E_\alpha^{(s)}}$$

(s): DEGENERACY INDEX

Cartan $h = \sum_a \beta^a h_a$; $E_\alpha^{(s)}$: RAISING GENERATOR(S)

ROOT $\equiv$ EIGENVALUE OF ad_h AS A LINEAR FORM OF $h \in$ CSA

$h = \beta^a h_a \to \alpha(h) = \alpha_a \beta^a \equiv \alpha(\beta)$

$$\boxed{\begin{aligned} [E_\alpha^{(s)}, E_\beta^{(t)}] &= C_{\alpha\beta}^{(stu)} E_{\alpha+\beta}^{(u)} \\ [h, F_\alpha^{(s)}] &= -\alpha(h)\, F_\alpha^{(s)} \end{aligned}}$$

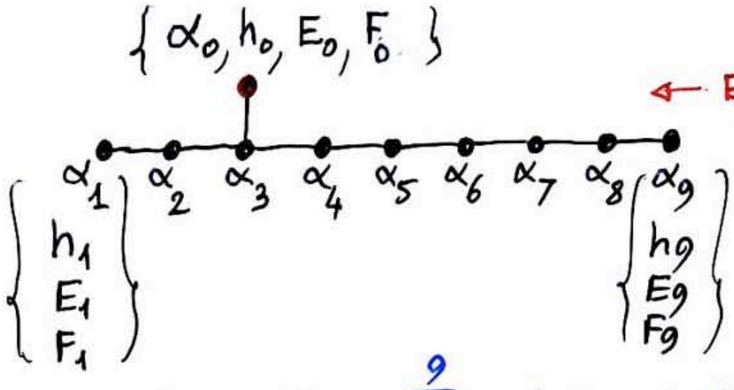
LOWERING GENERATORS : $F_\alpha^{(s)} \equiv E_{-\alpha}^{(s)}$

+ JACOBI + SERRE RELATIONS

E_{10}

rank 10 : $\dim \mathfrak{h} = 10$ AND $\exists$ 10 basic raising gators E_{α_i}

10 SIMPLE ROOTS



$\leftarrow E_0 = E^{123} \in E^{[abc]}$; $F_0 = F_{123} \in F_{[abc]}$

WITH h_0 DEFINES GL_{10} SUBALGEBRA

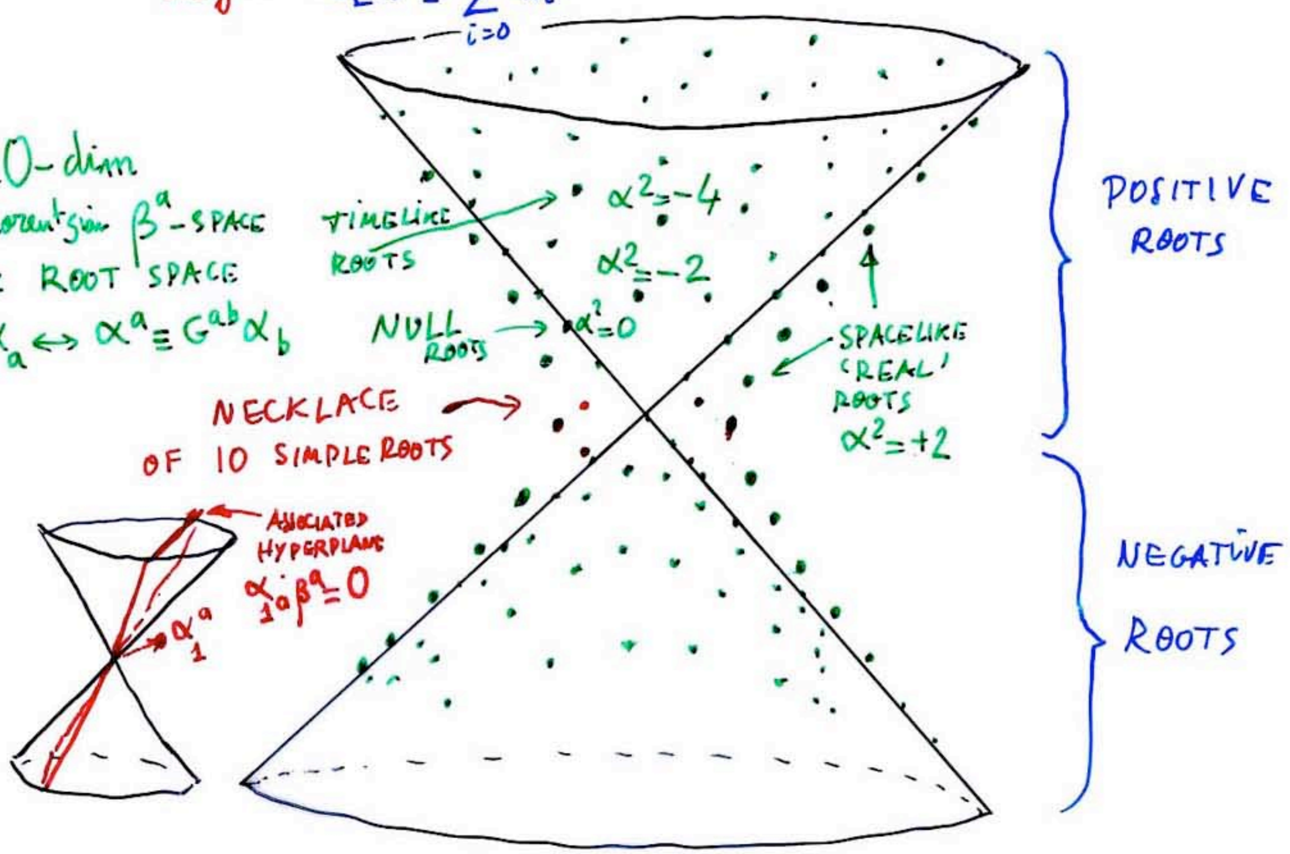
$$\{\text{ALL ROOTS}\} = \underbrace{\left\{\alpha = \sum_{i=0}^{9} n^i \alpha_i \, ; \, n^i \in \mathbb{N}\right\}}_{\text{POSITIVE ROOTS}} \cup \underbrace{\left\{\alpha = -\sum_{i=0}^{9} n^i \alpha_i \, ; \, n^i \in \mathbb{N}\right\}}_{\text{NEGATIVE ROOTS}}$$

height $ht[\alpha] = \sum_{i=0}^{9} n_i$

10-dim Lorentzian β^a-SPACE

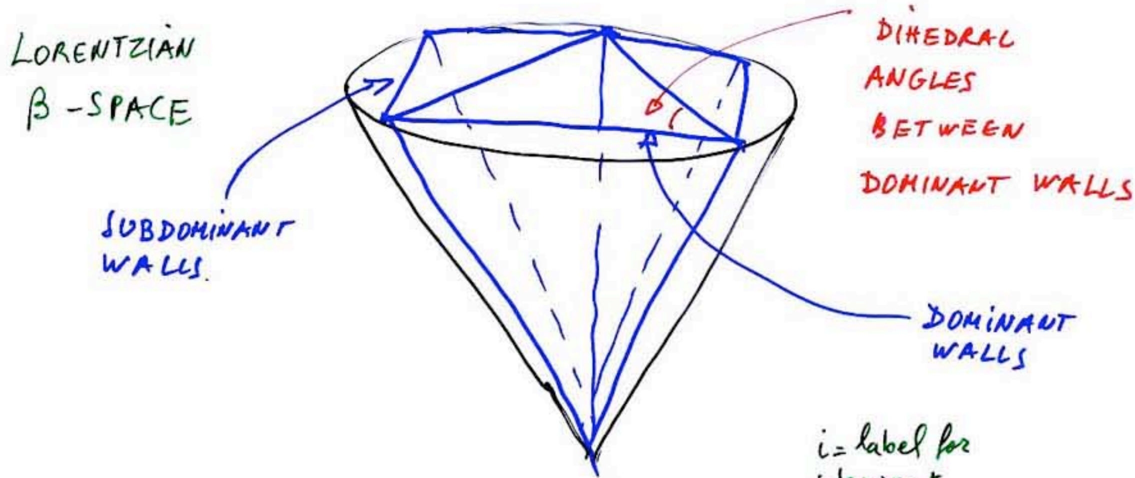
$\simeq$ ROOT SPACE

$\alpha_a \leftrightarrow \alpha^a \equiv G^{ab}\alpha_b$



COSMOLOGICAL SINGULARITIES AND KAC-MOODY ALGEBRAS

GEOMETRY OF HYPERBOLIC BILLIARD



DEFINE **CARTAN** MATRIX :

$$\boxed{A_{ij} \equiv 2\, \frac{\underset{\substack{i = \text{label for}\\ \text{dominant walls}}}{w_i} \cdot w_j}{w_i \cdot w_i}}$$

QUADRATIC FORM DEFINED BY $G_{\mu\nu}$

$$w_i(\beta) = w_{i\mu}\beta^\mu$$

$$w_i \cdot w_j \equiv G^{\mu\nu} w_{i\mu} w_{i\nu}$$

RESULT :

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 & \cdots & \\ -1 & 2 & \cdots & -1 & \\ 0 & \vdots & 2 & \cdots & -2 \\ \vdots & & \vdots & & \\ \vdots & -1 & & -1 & 2 \end{pmatrix}$$

ONLY NEGATIVE INTEGERS OFF THE DIAGONAL

E.G. PURE GRAVITY IN 3+1 DIM

$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

DYNKIN DIAGRAM

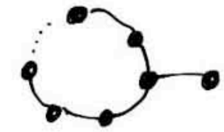
AE_3

OFF DIAGONAL
$-1, -1$
i.e. DIHEDRAL ANGLE
$120°$

OFF DIAGONAL
$-2, -2$

COSMOLOGICAL SINGULARITIES AND K-M ALGEBRAS

PURE GRAVITY
IN $D = d+1$ DIM $\rightarrow$

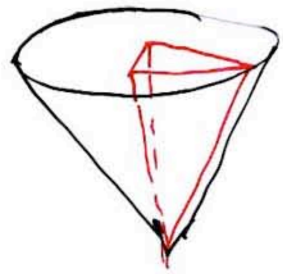


$AE_d \equiv A^{++}_{d-2} \equiv A^{H}_{d-2}$

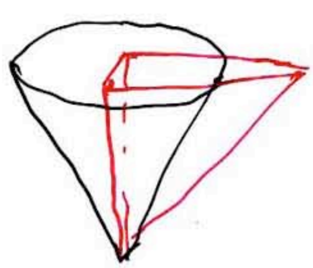
Damour, Henneaux, Julia, Nicolai '01 $\quad d=3$

$AE_3 = A^{H}_1$

HYPERBOLIC ONLY
WHEN $d \leq 9$
$D \leq 10$



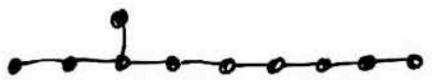
WHEN
$d \geq 10$
$D \geq 11$



SUPERSTRING MODIFIED
GRAVITY
$D = 10$ or 11

HYPERBOLIC

M, IIA, IIB :



E_{10}

I, HET

BE_{10}

BOSONIC STRING
$D = 26$

DE_{26}

DICTIONARY

$\beta^\mu \longleftrightarrow h = \sum_{\mu=1}^{r} \beta^\mu h_\mu$ Cartan element

DOMINANT WALLS $w_i \longleftrightarrow$ SIMPLE ROOTS α_i

GROUP OF REFLECTIONS IN COSMO. BILLIARD $\longleftrightarrow$ WEYL GROUP OF KM

BILLIARD TABLE $\longleftrightarrow$ WEYL CHAMBER

E_{10} AND A "HEIGHT EXPANSION" OF $SUGRA_{11}$
(OR "SMALL TENSION")

- CONSTRUCTION OF A (ONE-DIMENSIONAL) E_{10}-INVARIANT COSET MODEL

INFINITE DIMENSIONAL COSET SPACE $E_{10} / K(E_{10})$

KM GROUP

MAXIMAL COMPACT SUBGROUP
(CANONICAL REAL FORM OF E_{10})

$$\boxed{S_1^{E_{10}} = \frac{1}{4}\int \frac{dt}{n(t)} \left\langle \frac{d\mathcal{V}}{dt}\mathcal{V}^{-1} + \left(\frac{d\mathcal{V}}{dt}\mathcal{V}^{-1}\right)^T, \frac{d\mathcal{V}}{dt}\mathcal{V}^{-1} + \left(\frac{d\mathcal{V}}{dt}\mathcal{V}^{-1}\right)^T \right\rangle}$$

$\mathcal{V} \in E_{10}$

"TRANSPOSE" $E^T \equiv -\omega(E)$

Chevalley involution

$\omega(h_i) = -h_i$, $\omega(e_i) = -f_i$, $\omega(f_i) = -e_i$

E_{10} INVARIANT BILINEAR FORM

- ACTION INVARIANT UNDER: $\mathcal{V}(t) \rightarrow k(t)\, \mathcal{V}(t)\, g$

$k \in K(E_{10})$ $\qquad g \in E_{10}$

$k^T = k^{-1}$

- EXPLICIT (IWASAWA) PARAMETRIZATION OF E_{10} COSET ELEMENT

$$\mathcal{V} = \cancel{K} A N = \exp\left(h^a{}_b(t) K^b{}_a\right) \exp\left(\frac{A_{abc}(t)}{3!} E^{abc} + \frac{A_{a_1 \cdots a_6}}{6!} E^{a_1 \cdots a_6} + \frac{A_{a_0|a_1 \cdots a_8}}{9!} E^{a_0|a_1 \cdots a_8} + \cdots\right)$$

E^{abc}

$SU(10)$

- $$\boxed{n\mathcal{L}_1^{E_{10}} = \frac{1}{4}\left(g^{ac}g^{bd} - g^{ab}g^{cd}\right)\dot{g}_{ab}\dot{g}_{cd} + \frac{1}{2}\frac{1}{3!}\dot{A}_{a_1a_2a_3}\dot{A}^{a_1a_2a_3} + \frac{1}{2}\frac{1}{6!}DA_{a_1\cdots a_6}DA^{a_1\cdots a_6} + \frac{1}{2}\frac{1}{9!}DA_{a_0|a_1\cdots a_8}DA^{a_0|a_1\cdots a_8} + \cdots}$$

$g^{ab} = (e^h)^a{}_c (e^h)^b{}_c$; $DA_{a_1\cdots a_6} = \dot{A}_{a_1\cdots a_6} + 10\, A_{[a_1 \cdots a_3}\dot{A}_{a_4a_5a_6]}$; $DA_{a_1|a_2\cdots a_9} = \dot{A}_{a_1|a_2\cdots a_9} + 42 A_{\langle 3}\dot{A}_{6\rangle} - 42\dot{A}_{\langle 3}A_{6\rangle} + 280\, A_{\langle 3}A_3\dot{A}_{3\rangle}$

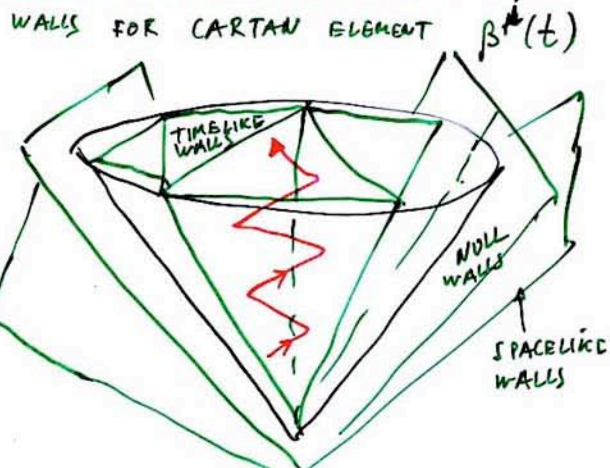
CORRESPONDENCE $E_{10}/K(E_{10})$ COSET $\Longleftrightarrow$ $SUGRA_{11}$

$$\mathcal{F}_4 = d\mathcal{A}_3$$

$$\mathcal{L}_1^{E_{10}} \sim (g^{-1}\dot{g})^2 + (\dot{A}_3)^2 + (\dot{A}_6 + A_3\dot{A}_3)^2 + (\dot{A}_9 + A_6\dot{A}_3 + A_3A_3\dot{A}_3)^2 + \dots$$

$$\Downarrow$$

BILLIARD WITH INFINITE NUMBER OF EXPONENTIAL WALLS FOR CARTAN ELEMENT $\beta^{\hat{a}}(t)$



$$\mathcal{H}_1 = \tfrac{1}{2}G^{\mu\nu}\pi_\mu\pi_\nu + \sum C_\alpha(q,p)\,e^{-2\alpha(\beta)}$$

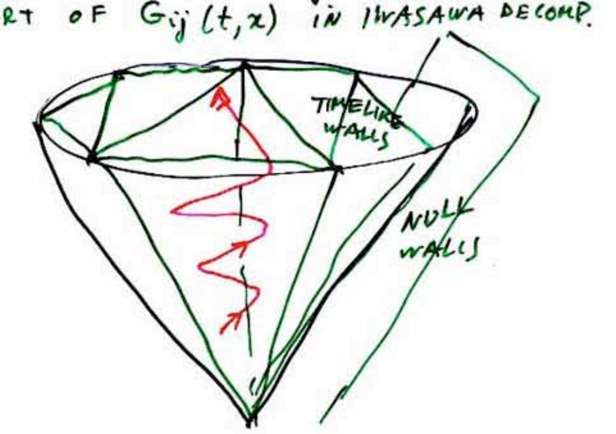
$$\alpha(\beta) = \sum_i m^i \alpha_i(\beta), \quad m^i \in \mathbb{N}$$

SIMPLE ROOTS

$$\mathcal{L}_{11}^{SUGRA} = \int d^{11}x \left[\sqrt{-G}\left[R(G) - \frac{(d\mathcal{A}_3)^2}{48}\right] + \frac{1}{(12)^4}\mathcal{F}_4\wedge\mathcal{F}_4\wedge\mathcal{A}_3\right]$$

$$\Downarrow$$

BILLIARD WITH LARGE BUT FINITE # OF EXPONENTIAL WALLS FOR $\beta^a(t,x)$, DIAGONAL PART OF $G_{ij}(t,x)$ IN IWASAWA DECOMP.



$$\mathcal{H}_{11} = \tfrac{1}{2}G^{\mu\nu}\pi_\mu\pi_\nu + \sum_A C_A(Q,P,\partial_x\beta,\partial_x Q,\dots)\times e^{-2w_A(\beta)}$$

$$w_A(\beta) = \sum_i m^i w_i(\beta)$$

DOMINANT WALLS

DICTIONARY

$$g^{ab}(t) = (e^h)^a{}_c (e^h)^b{}_c = G^{ab}(t,\vec{x}_0)$$

WRT A SPECIAL FRAME

$$\dot{A}_{a_1a_2a_3}(t) = \mathcal{F}_{0a_1a_2a_3}(t,\vec{x}_0) \qquad \theta^a(x) = e^a{}_i(x)\,dx^i$$

$$DA^{a_1\dots a_6}(t) = g^{a_1a_1'}\cdots g^{a_6a_6'}\left[\dot{A}_{a_1'\dots a_6'} + 10\,A_{[3}\dot{A}_{3]}\right] = -\frac{1}{4!}\varepsilon^{a_1\dots a_6 b_1\dots b_4}\mathcal{F}_{b_1\dots b_4}(t,\vec{x}_0)$$

$$DA^{b|a_1\dots a_8}(t) = g^{\cdots}\left[\dot{A}_9 + 42A_3\dot{A}_6 + 280A_3A_3\dot{A}_3\right] = +\frac{3}{2}\varepsilon^{a_1\dots a_8 b_1b_2}C^b{}_{b_1b_2}(\vec{x}_0)$$

$$d\theta^a = \tfrac{1}{2}C^a{}_{bc}\,\theta^b\wedge\theta^c$$

THE CORRESPONDENCE WORKS FOR ALL TERMS OF HEIGHT $\leqslant 29$

$$\sum_i m^i \leqslant 29 \qquad\qquad \sum_i m^i \leqslant 29$$

HIGHER-ORDER M-THEORY CORRECTIONS AND E_{10}

(Damour, Nicolai, 2005)

$$S_M = \int \frac{d^{11}x}{l_P^9} \sqrt{-G} \Big[\underbrace{R - F^2 + AFF}_{\text{TWO DERIVATIVES } g\partial^2 g + \partial^2 A} \Big] + \text{HIGHER-ORDER CORRECTIONS}$$

FROM
- ONE STRING-LOOP CORRECTIONS TO $D=10$ IIA AMPLITUDES Green Schwarz '82, Sakai, Tanii '87
- DIVERGENCES IN $D=11$ SUGRA Deser, Seminara '99, '00
- M-THEORY LOOPS Green Vanhove '97, Green Gutperle Vanhove '97
- ANOMALY IN 5-BRANE Duff, Liu, Minasian '95

+ Tseytlin '00, Peeters, Vanhove, Westerberg '01

$\nu > 0$

$$S_1 = \frac{\nu}{l_P^3} \int d^{11}x \sqrt{-G} \Big[t_8 t_8 R^4 - \frac{1}{4} \varepsilon_8 \varepsilon_8 R^4$$ ← EXPECTED FROM ONE-LOOP (IIA) LIGHT-CONE 4-PT AMPLITUDES

$$+ \frac{2}{4} \varepsilon_8 \varepsilon_8 R^4$$ ← CHANGES THE SIGN OF $\bar{E}_8$

$$-4 \varepsilon_{11} A_3 \left[\mathrm{tr} R^4 - \frac{1}{4} (\mathrm{tr} R^2)^2 \right]$$ ← CHERN-SIMONS-TYPE

$$+ \underbrace{R^2 (DF)^2 + R(DF)^3 + (DF)^4}_{\text{NOT KNOWN IN NICE FORM}} + \underbrace{\cdots + F^8}_{\text{NOT KNOWN}} \Big]$$

$$t_8 M^4 \equiv t_8^{\mu_1 \cdots \mu_8} M_{\mu_1 \mu_2} M_{\mu_3 \mu_4} M_{\mu_5 \mu_6} M_{\mu_7 \mu_8} \equiv 24\, \mathrm{tr} M^4 - 6 (\mathrm{tr} M^2)^2$$

(ANTISYMMETRIC)

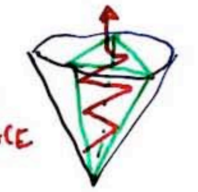
$$\varepsilon_8 \varepsilon_8 R^4 \equiv \delta^{\nu_1 \nu_2 \nu_3 \nu_4 \nu_5 \nu_6 \nu_7 \nu_8}_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7 \mu_8} R^{\mu_1 \mu_2}{}_{\nu_1 \nu_2} R^{\mu_3 \mu_4}{}_{\nu_3 \nu_4} R^{\mu_5 \mu_6}{}_{\nu_5 \nu_6} R^{\mu_7 \mu_8}{}_{\nu_7 \nu_8}$$

$$\equiv \bar{E}_8 = \text{EULER-LOVELOCK DENSITY}$$

LEADING-BEHAVIOUR OF $R^4 + \dots$ TERMS AS $T \to 0$

INTERMEDIATE ASYMPTOTICS : $T_P \equiv \frac{l_P}{c} \ll T \ll T_{\text{INITIAL}}$

SPATIAL ZEHNBEIN : $e^a_{\ p} = e^{-\beta^a} \mathcal{N}^a_{\ p}$ UPPER TRIANGULAR



β-SPACE

$\beta^a \to +\infty$ IN CHAOTIC E_{10} BILLIARD

"FREEZES" AS $T \to 0$

LEADING COMPONENTS OF R, F, DF

$$\boxed{\sigma \equiv \sum_{a=1}^{10} \beta^a}$$

$$\boxed{\begin{aligned} R_{0a0a} &\simeq e^{2\sigma}\, v_a\, \bar{v}_a \\ R_{abab} &\simeq e^{2\sigma}\, v_a\, v_b \\ D_0 F_{0abc} &\simeq e^{2\sigma}\, e^{-(\beta^a+\beta^b+\beta^c)}\, \varepsilon_{abc} \end{aligned}}$$

$v_a \equiv \partial_t \beta^a$

$\bar{v}_a \equiv \partial_t \sigma - \partial_t \beta^a$

LEADING CORRECTION TO HAMILTONIAN FOR β-DYNAMICS

$$\boxed{\delta\mathcal{H}(\beta,\pi) = c\ e^{-2w(\beta)}}$$

WALL FORM : $w(\beta) = -3\sigma = -3\sum_{a=1}^{10} \beta^a$

DOES CORRESPOND TO A ROOT OF E_{10}: $\ell = -10$, $ht = -115$

GENERATOR $E_{-3\sigma}$ = SINGLET OF GL_{10}

REMARKABLE COMPATIBILITY BETWEEN $[R^4 + \dots]$ AND E_{10}

E_{10} CONJECTURE MAKES PREDICTIONS ABOUT. $R^M (DF)^N$: $M + N = 3k + 1$

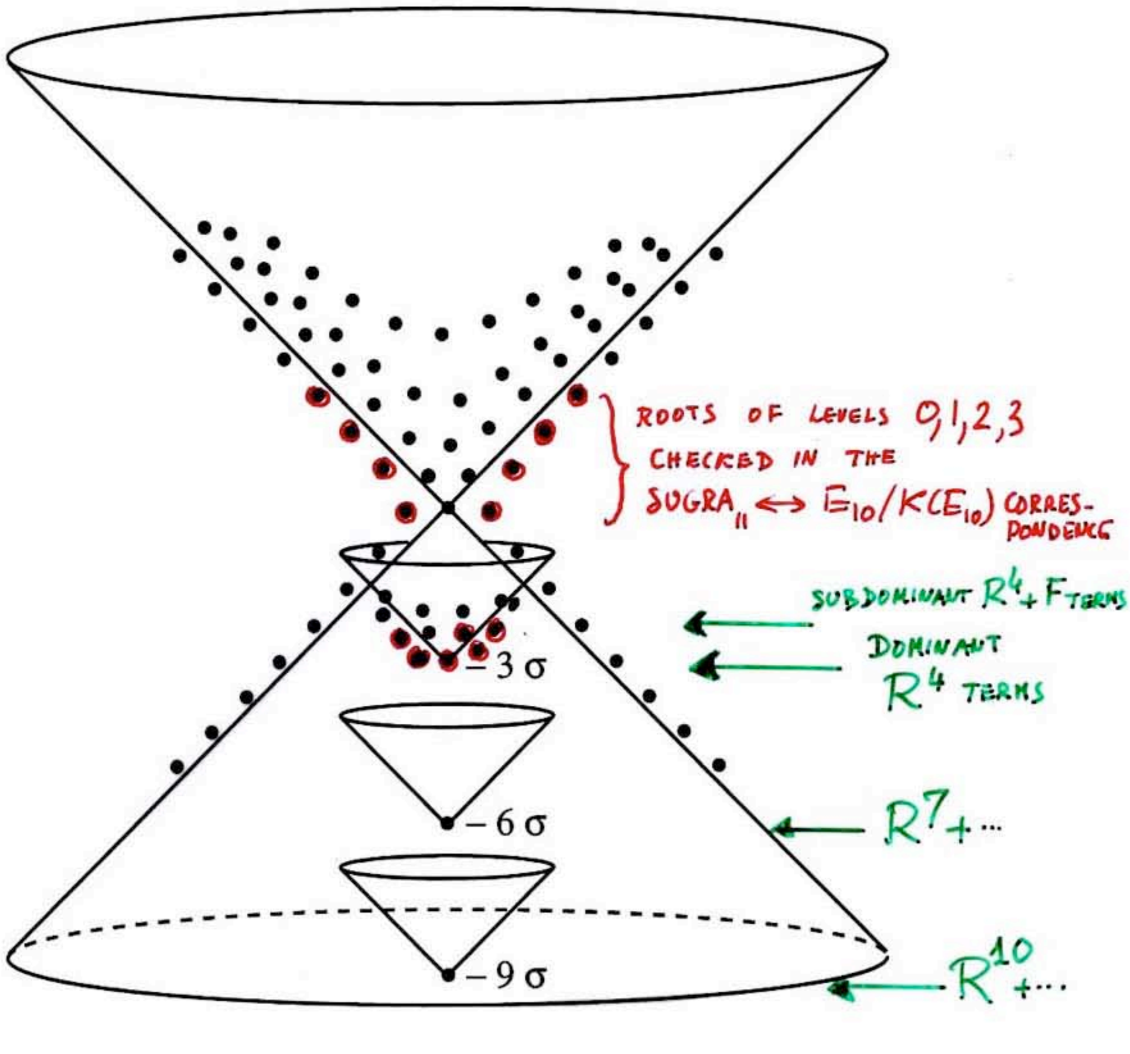
COEFFICIENT : $c = -\nu \left[O_{8[t_8 t_8 - \frac{\varepsilon_8 \varepsilon_8}{4}]}(v) + O_{8[\frac{\varepsilon_8 \varepsilon_8}{2}]}(v) \right]$

> 0

PROBABLY $c < 0$

OK NO BOUNCE

ROOTS OF E_{10}


ROOTS OF LEVELS 0,1,2,3
CHECKED IN THE
SUGRA$_{11}$ $\leftrightarrow$ $E_{10}/K(E_{10})$ CORRESPONDENCE
SUBDOMINANT R^4 + F TERMS
DOMINANT
R^4 TERMS
R^7 + ...
R^{10} + ...
-3σ
-6σ
-9σ

CONCLUSIONS

- $\exists$ TANTALIZING HINTS OF INFINITE-DIMENSIONAL HIDDEN SYMMETRY $E_{10}(\mathbb{R})$ OF $SUGRA_{11}$

  TRANSFORMING SOLUTION $\rightarrow$ SOLUTION'

  (SIMILAR TO $SL(2,\mathbb{R})_E$ FOR $GR^4_{3}{\downarrow}$, OR $\hat{A}_{1G}$ FOR $GR^4_{2}{\downarrow}$, OR E_9 FOR $SUGRA^{11}_{2}{\downarrow}$)

  PROBABLY BROKEN DOWN TO $E_{10}(\mathbb{Z})$ FOR M-THY

- BKL-LIKE EXPANSION $\frac{1}{c}\frac{\partial h}{\partial t} \gg \frac{\partial h}{\partial x}$ $\leftrightarrow$ "height expansion" $\sum_A c_A e^{-\alpha_A(\beta)}$

  WOULD BE A WAY TO REVEAL THIS SYMMETRY

  FORMALLY $\sim$ "SMALL (BULK) TENSION" LIMIT $T_b = \frac{c^4}{32\pi G} \rightarrow 0$

  ? ANY LINK WITH "SMALL STRING TENSION" $T_s = \frac{1}{2\pi\alpha'} \rightarrow 0$ LIMIT (Gross) ?

- DICTIONARY BETWEEN $SUGRA_{11} \longleftrightarrow E_{10}/K(E_{10})$ NULL GEODESIC

  HAS BEEN CONSTRUCTED ONLY FOR HEIGHTS ≤ 29

- HIGHER-ORDER M-THEORY CORRECTIONS $(R^4 + \dots)$ ARE

  COMPATIBLE WITH THIS CONJECTURED E_{10} SYMMETRY

  MAKES PREDICTIONS BOTH ABOUT M-THY CORRECTIONS AND ABOUT E_{10}