SIGRAV SCHOOL
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# BINARY SYSTEMS AS TEST-BED OF GRAVITY THEORIES

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- Motion of Binary Pulsars in General Relativity
- Timing of Binary Pulsars in General Relativity
- Testing Relativistic Gravity with Binary Pulsars (I and II)

A BIT TECHNICAL

## LECTURE N° 1

MOTION OF BINARY PULSARS IN GENERAL RELATIVITY

# MOTION OF BINARY PULSARS IN GENERAL RELATIVITY

## TRADITIONAL APPROACH TO PROBLEM OF MOTION FOR WEAKLY SELF-GRAVITATING BODIES

Einstein's field equations (signature $-+++$; MTW conventions)

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

stress-energy tensor of matter

e.g. perfect fluid

$$T^{\mu\nu} = (\varepsilon + p)\, u^\mu u^\nu + p\, g^{\mu\nu} \qquad g_{\mu\nu} u^\mu u^\nu = -1$$

proper energy density  proper pressure

If isentropic equation of state: $p = p(\varepsilon)$

Define $\quad r \equiv \exp \int \frac{d\varepsilon}{\varepsilon + p(\varepsilon)}$

Bianchi: $\nabla_\nu (R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}) = 0 \Rightarrow \nabla_\nu T^{\mu\nu} = 0$

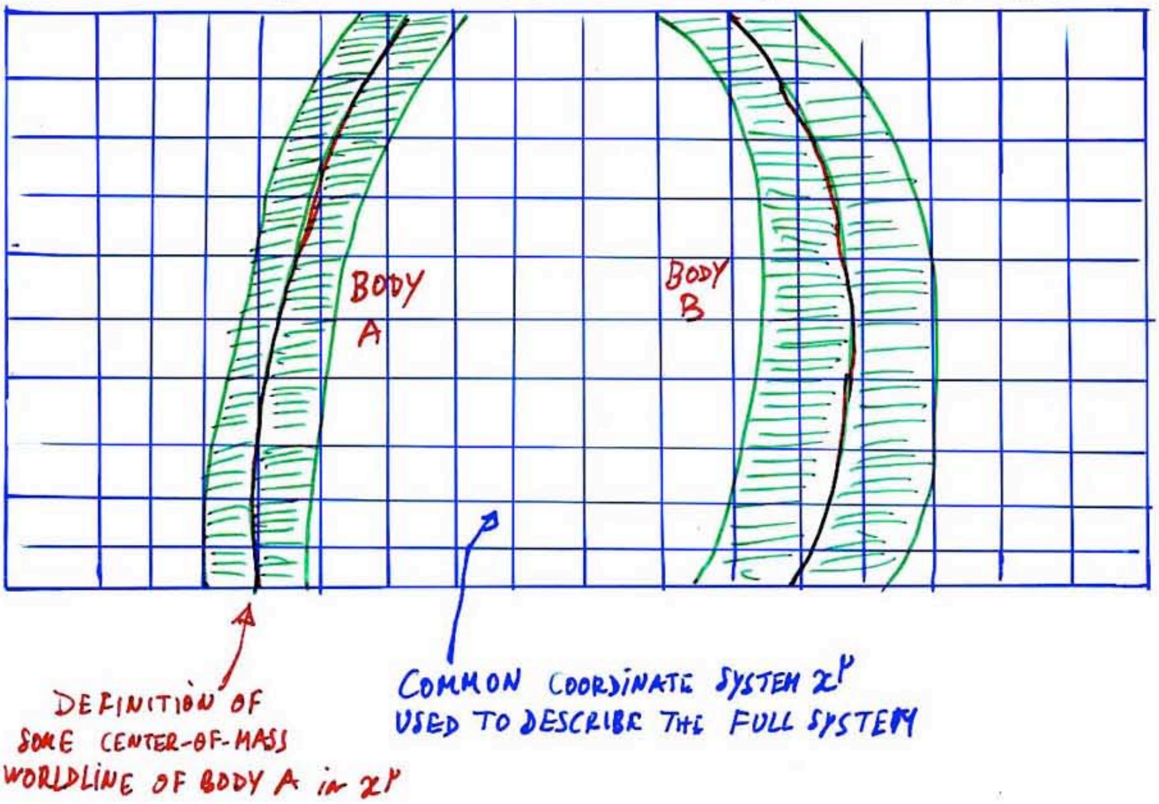
$\Rightarrow \quad \nabla_\mu (r u^\mu) = 0$ "conservation of rest mass"

AND $\quad (\varepsilon + p)\, u^\nu \nabla_\nu u^\mu = -(g^{\mu\nu} + u^\mu u^\nu)\, \partial_\nu p$

Equation of motion of fluid 4-velocity $u^\mu$

# ONE-CHART APPROACH TO PROBLEM OF MOTION

Droste 1916, De Sitter 1916, Levi-Civita '37, Fock '39, Papapetrou '51, ...




- EXPAND $g_{\mu\nu}(x^\lambda) = \eta_{\mu\nu} + G h^{(1)}_{\mu\nu} + G^2 h^{(2)}_{\mu\nu} + \cdots$
- SOLVE EINSTEIN'S EQS BY SUCCESSIVE APPROXIMATIONS (FIX COORDINATES)
- USE 'POST-NEWTONIAN' SIMPLIFICATIONS : $\partial_0 h = \frac{1}{c}\partial_t h \ll \partial_i h$; $\frac{v}{c} \ll 1$

$$p \ll \varepsilon \sim \rho c^2$$

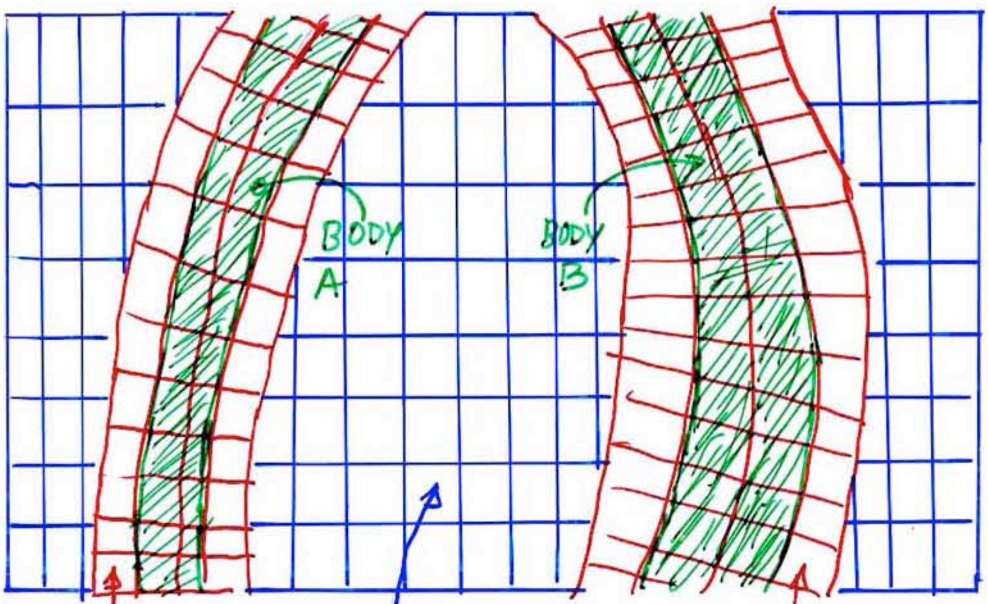
- INTEGRATE $\nabla_\nu T^{\mu\nu} = 0$ OVER VOLUME OF EACH BODY TO DEDUCE SOME EVOLUTION EQ. FOR SOME 'CENTER-OF-MASS'

## PROBLEMS:

- ? DEFINITION OF C. OF MASS, MASS, SPIN, MULTIPOLE MOMENTS ?
- UNPHYSICAL DEFORMATIONS OF BODY ; COUPLES EXTERNAL AND INTERNAL PROBLEMS
- 2 COORDINATE DEFORMATIONS IMPORTANT IN VLBI AND OTHER HIGH-PRECISION MEASUREMENTS : LORENTZ + EINSTEIN+.. VARIABLE 'CONTRACTIONS'

# MULTI-CHART APPROACH TO PROBLEM OF MOTION

GENERAL THEORY IN WEAKLY SELF-GRAVITATING CASES
Brumberg, Kopeikin, 1988-90; Damour, Soffel, Xu, PRD 1991-1994



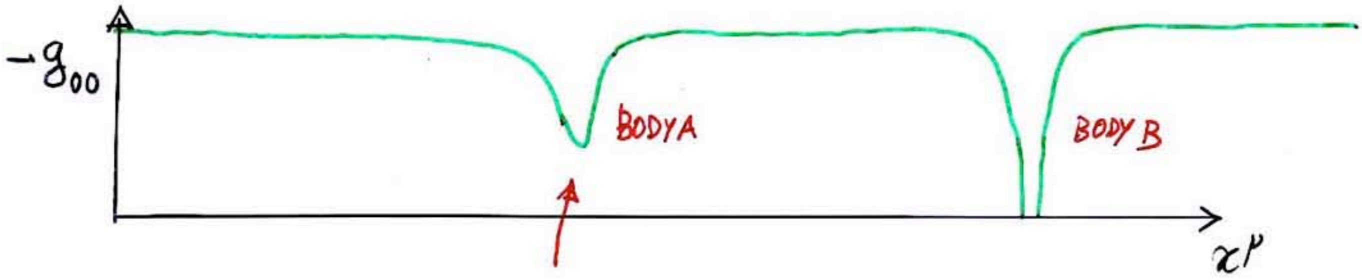

GLOBAL CHART $x^\mu$

LOCAL CHART $X^\alpha_A$ USED IN AND AROUND BODY A

LOCAL CHART $X^\alpha_B$ USED IN AND AROUND BODY B

- WORK BOTH WITH $g^{GLOBAL}_{\mu\nu}(x^\lambda)$ AND $G^{LOCAL}_{\alpha\beta}(X^\gamma_A)$
- USE $x^\mu = f^\mu(X^\alpha)$ TO SIMPLIFY $G^{LOCAL}_{\alpha\beta} = G^{BODY\,A}_{\alpha\beta}$ + 'SMALL TIDAL EFFECTS'
- USE LOCAL FRAME TO DEFINE C OF MASS WORLDLINE, AS WELL AS MASS AND MULTIPOLE MOMENTS OF EACH BODY
- USEFUL FOR RELATIVISTIC DESCRIPTION OF SOLAR-SYSTEM AS MEASURED BY MODERN TECHNOLOGIES.

# MOTION OF STRONGLY SELF-GRAVITATING BODIES 




$$-g_{00} = 1 - \frac{2Gm_A}{R}$$

FOR NEUTRON STAR, $m_A \simeq 1.4\, M_\odot$

$Gm_A \simeq 2\ \mathrm{km}$

$$\frac{2Gm_A}{R_A} \simeq \frac{4\,\mathrm{km}}{10\,\mathrm{km}} = 0.4$$

$$-g_{00}(R_A) = 1 - 0.4$$

FOR BLACK HOLE:

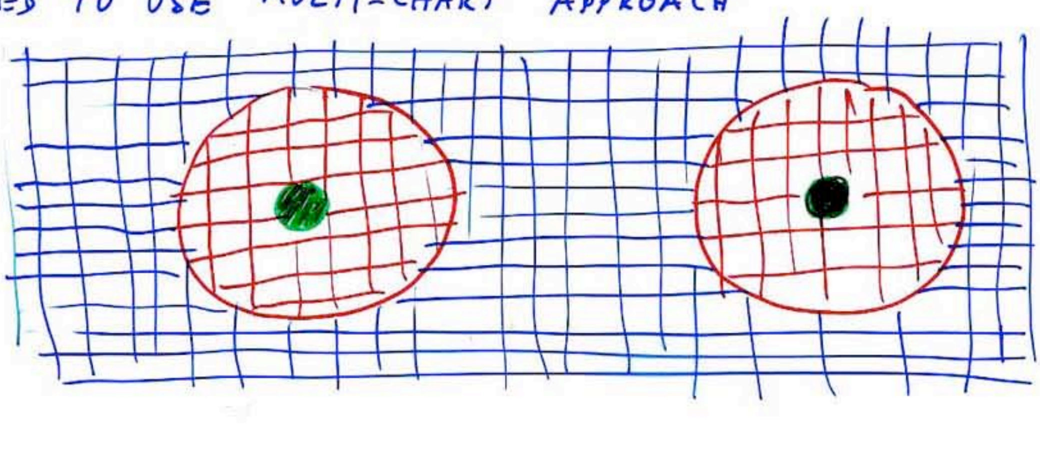
$$\frac{2Gm_B}{R_B} \equiv 1$$

- CANNOT (A PRIORI) USE THE TRADITIONAL ONE-CHART APPROACH

BECAUSE COUPLING BETWEEN INTERNAL $\cap$ EXTERNAL EFFECTS IS ILL-TREATED AND MIGHT LEAD TO FICTITIOUS EFFECTS

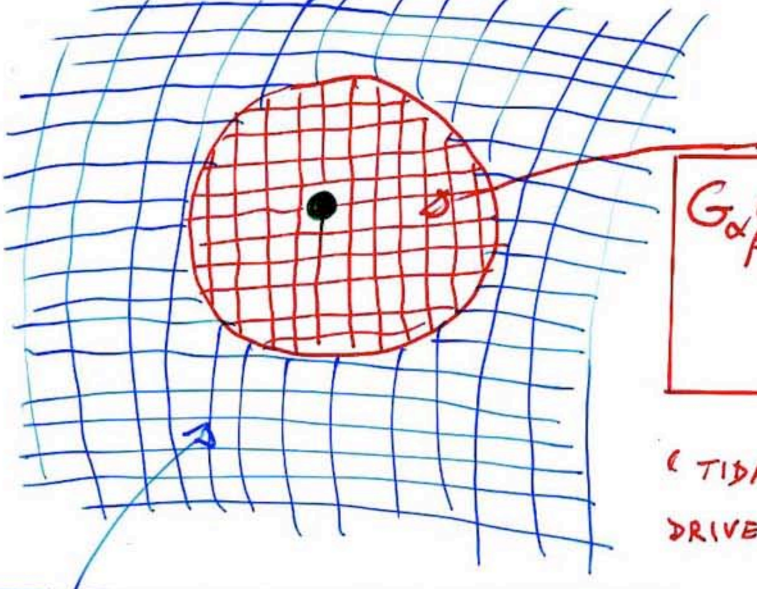
$$\propto \left(\frac{Gm_A}{c^2 R_A}\right)^n \times \frac{Gm_B}{|x_A - x_B|} \sim (0.4)^n \frac{Gm_B}{|x_A - x_B|}$$ : NUMERICALLY LARGE

- NEED TO USE MULTI-CHART APPROACH



# MULTI-CHART AND MATCHING

Einstein, Infeld, Hoffmann '38, Manasse '63, Demianski, Grischchuk '74, D'Eath '75, Kates '80, Damour '83, Thorne, Hartle '85



STRONG-FIELD METRIC OF ISOLATED BODY A

$$\boxed{G_{\alpha\beta}(X^\gamma) = G^{(0)}_{\alpha\beta}(X^\gamma, m_A) + H^{(1)}_{\alpha\beta}(X^\gamma) + H^{(2)}_{\alpha\beta} + \cdots}$$

'TIDAL' PERTURBATION DRIVEN BY FAR-AWAY COMPANION

$$\boxed{g_{\mu\nu}(x^\lambda) = \eta_{\mu\nu} + G h^{(1)}_{\mu\nu}(x) + G^2 h^{(2)}_{\mu\nu}(x) + \cdots}$$

PERTURBATION OF FLAT SPACE DRIVEN BY THE TWO MASSES $m_A$, $m_B$

THE TWO EXPANSIONS MUST MATCH IN SOME INTERMEDIATE DOMAIN: $\frac{G m_A}{c^2} \lesssim R_A \ll R \ll d = |x_A - x_B|$

$\exists$ NON-LINEAR TRANSFORMATION BETWEEN THE TWO COORDINATE SYSTEMS

$$x^\mu = (ct, x^i) \longleftrightarrow X^\alpha = (cT, X^a)$$

$$\boxed{x^\mu = z^\mu(T) + e^\mu_a(T) X^a + \frac{1}{2} f^\mu_{ab}(T) X^a X^b + \cdots}$$

IMAGE OF 'CENTER' $X^a = 0$ IN EXTERNAL CHART

$\Rightarrow$ DEFINES 'WORLDLINE OF A'

LINEAR DEFORMATION

$e^0_a \sim v^a_A / c$

$e^i_a \sim \delta^i_a + \frac{v_A^2}{c^2} + \frac{G m_B}{d c^2}$

NON-LINEAR DEFORMATIONS

# SIMPLEST CASE: NON-SPINNING BODIES

- EFFACEMENT OF INTERNAL STRUCTURE (Damour '83)

$$\delta G^{\text{STRUCTURE-DEPENDENT}}_{\alpha\beta}(X^\gamma) \equiv H^{\text{STRUC-DEP}}_{\alpha\beta}(X) \sim \frac{G^6}{c^{12}}\, k\, \frac{m_A^5\ m_B}{R^3\ \ d^3}$$

"Love number" ($k$) — tidal field of companion ($d^3$)

BY MATCHING $\Rightarrow$ EFFECT IN EXTERNAL METRIC

$$\delta g^{\text{STRUCT-DEP}}_{\mu\nu}(x) \sim \frac{G^6}{c^{12}}\, k\, \frac{m_A^5}{|x-z_A|^3}\, \frac{m_B}{|z_A-z_B|^3}$$

5PN EFFECT, i.e. $\left(\frac{v}{c}\right)^{10}$ SMALLER THAN NEWTON

- USEFUL TECHNICAL TOOL:
  REPRESENT COMPACT BODIES BY DELTA-FUNCTION SOURCES
  \+ USE ANALYTIC CONTINUATION TO DEAL WITH SELF-GRAVITY EFFECTS

EITHER RIESZ-KERNELS OR DIMENSIONAL-CONTINUATION (SIMPLER)

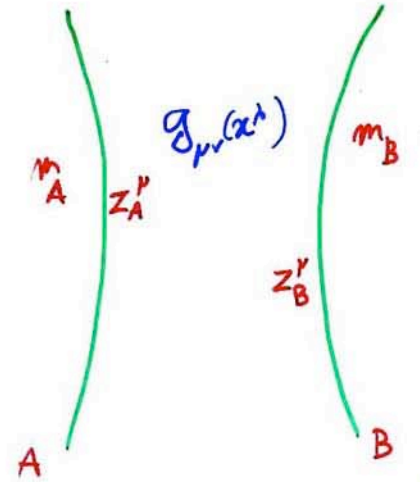
$$\boxed{S = \int \frac{d^{d+1}x}{c} \sqrt{g}\, \frac{c^4}{16\pi G} R(g) - \sum_a m_a c \int \sqrt{-g_{\mu\nu}(z_a^\lambda)\, dz_a^\mu\, dz_a^\nu}}$$

WHERE $d = 3+\varepsilon$, $\varepsilon \in \mathbb{C}$

POINT-PARTICLE ACTION

$g_{\mu\nu}(x^\lambda = z_a^\lambda)$ A PRIORI SINGULAR, BUT $d \in \mathbb{C} \Rightarrow$ WELL-DEFINED

# RELATIVISTIC GRAVITATIONAL INTERACTION OF TWO POINT MASSES




E.G. HARMONIC COORDINATES $g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0$ (BACK TO ONE CHART)

EXPLICIT FORM OF EINSTEIN'S EQS $\qquad \partial_\lambda g_{\mu\nu} \equiv g_{\mu\nu,\lambda}$

$$-g^{\alpha\beta}\partial_{\alpha\beta}g_{\mu\nu} + g^{\alpha\beta}g^{\gamma\delta}\Big(g_{\mu\alpha,\gamma}\,g_{\nu\beta,\delta} - g_{\mu\alpha,\gamma}\,g_{\nu\delta,\beta} + g_{\mu\alpha,\gamma}\,g_{\beta\delta,\nu} + g_{\nu\alpha,\gamma}\,g_{\beta\delta,\mu} - \frac{1}{2}g_{\alpha\gamma,\mu}\,g_{\beta\delta,\nu}\Big)$$

$$= \frac{16\pi G}{c^4}\Big(T_{\mu\nu} - \frac{1}{d-1}\,g_{\mu\nu}\,T_\lambda^{\ \lambda}\Big)$$

$$T_{\mu\nu}(x) = \sum_a m_a c^2 \int ds_a\, \frac{g_{\mu\alpha}(z_a)\,g_{\nu\beta}(z_a)}{\sqrt{g(z_a)}}\,\frac{dz_a^\alpha}{ds_a}\,\frac{dz_a^\beta}{ds_a}\,\delta^{(d+1)}\big(x^\lambda - z_a^\lambda(s_a)\big)$$

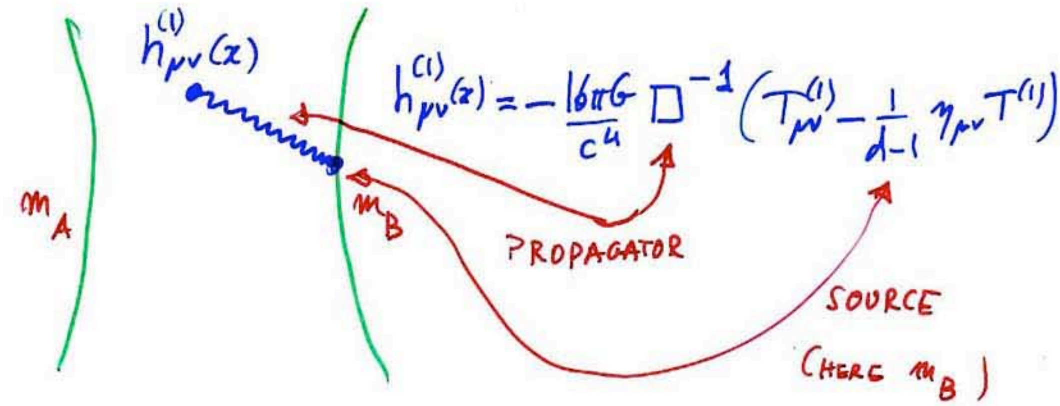
$$ds_a = \sqrt{-g_{\mu\nu}(z_a^\lambda)\,dz_a^\mu\,dz_a^\nu}$$

SOLVE EINSTEIN'S EQS BY SUCCESSIVE APPROXIMATION

$$g_{\mu\nu}(x^\lambda) = \eta_{\mu\nu} + h^{(1)}_{\mu\nu}(x^\lambda) + h^{(2)}_{\mu\nu}(x^\lambda) + \cdots$$

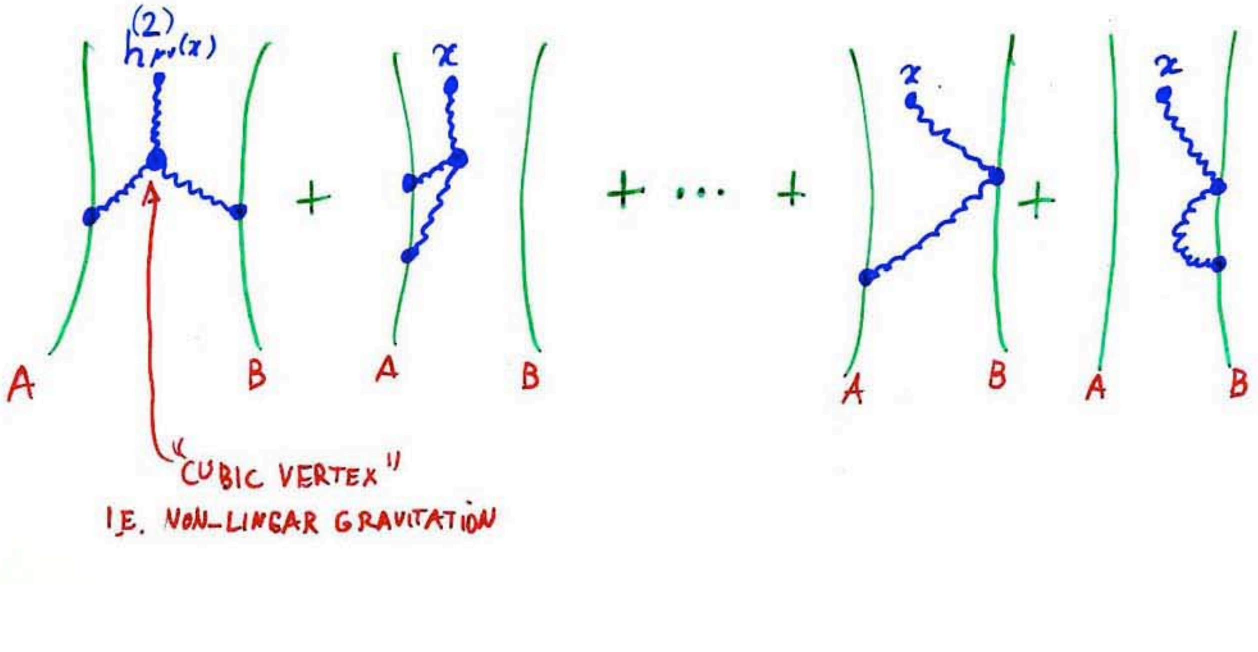
$$\Box\, h^{(1)}_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T^{(1)}_{\mu\nu} - \frac{1}{d-1}\,\eta_{\mu\nu}\, T^{(1)}\right)$$

$$T^{(1)}_{\mu\nu} = \sum_a m_a c^2 \int ds_a\, \eta_{\mu\alpha}\,\eta_{\nu\beta}\,\frac{dz_a^\alpha}{ds_a}\frac{dz_a^\beta}{ds_a}\,\delta^{(d+1)}(x - z_a)$$




AT NEXT APPROXIMATION

$$\Box\, h^{(2)} \sim h^{(1)}\partial^2 h^{(1)} + \partial h^{(1)} \partial h^{(1)} + h^{(1)}\, T^{(1)}$$



# DYNAMICS OF THE COMPACT BODIES



$$S[z_a^\lambda, g_{\mu\nu}(x)] = -\sum_a m_a c \int \sqrt{-g_{\mu\nu}(z_a) dz_a^\mu dz_a^\nu} + \int \frac{d^{d+1}x}{c} \sqrt{g} \frac{c^4}{16\pi G} R(g)$$

+ surface term

REPLACE $g_{\mu\nu}(x)$ BY SOLUTION: $g_{\mu\nu}[x; z_a^\lambda]$ (IN A CERTAIN GAUGE)

$$= \eta_{\mu\nu} + h^{(1)}_{\mu\nu}[z_a^\lambda] + \cdots$$

('FOKKER ACTION')

$$S^{\text{REDUCED}}[z_a^\lambda] = S[z_a^\lambda; g_{\mu\nu}[x; z_b^\lambda]]$$

$$= -\sum_a m_a c \int \sqrt{-\eta_{\mu\nu} dz_a^\mu dz_a^\nu} + \frac{1}{4} \int \frac{d^{d+1}x}{c} h^{(1)}_{\mu\nu}(x) T^{(1)}_{\mu\nu}(x) + \cdots$$

ACTION FOR FREE PARTICLES

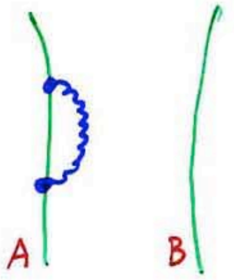
FIRST-ORDER INTERACTION BETWEEN THE PARTICLES

$$\boxed{S^{(0)}_{\text{FREE}} \equiv -\int dt\, m_a c^2 \sqrt{1 - \vec{v}_a^{\,2}/c^2}}$$


$$\boxed{S^{(1)}_{\text{INTERACTION}} \equiv -\frac{4\pi G}{c^4} \int \frac{d^{d+1}x}{c} \underset{\text{SOURCE}}{T^{\mu\nu}} \underset{\text{PROPAGATOR}}{\Box^{-1}} \underset{\text{SOURCE}}{\left(T_{\mu\nu} - \frac{1}{d-1}\eta_{\mu\nu} T\right)}}$$

# SELF-GRAVITY EFFECTS, DIMENSIONAL CONTINUATION, EFFACEMENT

MEANING OF :




?

CONSIDER LOWEST (NEWTONIAN) APPROXIMATION :

$$T^{00}(x) \simeq m_A c^2 \delta^{(d)}(\vec{x} - \vec{z}_A(t)) + m_B c^2 \delta^{(d)}(\vec{x} - \vec{z}_B(t)) \gg T^{0i} \text{ AND } T^{ij}$$

$$T^{00}(x) \equiv \rho c^2 \quad \text{WITH} \quad \rho = \sum_a m_a \delta^{(d)}(\vec{x} - \vec{z}_a(t))$$

$$\Box = \Delta - \frac{1}{c^2}\partial_t^2 \simeq \Delta$$

DEFINE GRAVIT. POTENTIAL $\Delta U = -4\pi G \frac{2(d-2)}{d-1} \rho$

'POISSON EQUATION'

$$S^{(1)}_{NEWTON} = +\int dt \left[ \frac{1}{2} \int d^d x \; \rho(x) \, U(x) \right] \qquad \rho \overset{\Delta^{-1}}{\bullet\!\sim\!\sim\!\sim\!\bullet} \rho$$

$$\text{IN } d=3 \Rightarrow U = \sum_a \frac{G m_a}{|\vec{x} - \vec{z}_a|} \Rightarrow S^{(1)} = \int dt \left[ \frac{1}{2} \sum_{a,b} \frac{G m_a m_b}{|\vec{z}_a - \vec{z}_b|} \right]$$

$\exists$ INFINITE SELF-GRAVITY TERMS $a = b = A$ OR $B$

$$\text{IN } d = 3 + \varepsilon \Rightarrow U = \frac{2(d-2)}{d-1} \tilde{k} \frac{G m_a}{|\vec{x} - \vec{z}_a|^{d-2}} \Rightarrow \text{SELF-GRAVITY TERMS} \propto G m_A^2 \, |\vec{z}_A - \vec{z}_A|^{2-d}$$

VANISHES FOR $\mathrm{Re}[d]$ SMALL ENOUGH

DIMENSIONAL CONTINUATION SETS TO ZERO ALL DANGEROUS SELF-GRAVITY TERMS

$\Leftrightarrow$ OK WITH EFFACEMENT OF INTERNAL STRUCTURE AND MATCHING METHOD

# DYNAMICS OF N COMPACT BODIES

(IN $d=3$ DIMENSIONS)

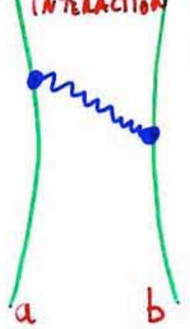
LAGRANGIAN $\quad S[\vec{z}_a] = \int dt\, L[\vec{z}_a]$

Lorentz, Droste 1917
Einstein Infeld Hoffmann 1938
Fock 1939, Landau Lifshitz

$$L[\vec{z}_a] = L^{(0)} + L^{(2\text{-BODY})} + L^{(3\text{-BODY})} + \cdots$$

FREE MOTION: $L^{(0)} = \sum - m_a c^2 \sqrt{1-\vec{v}_a^2/c^2}$

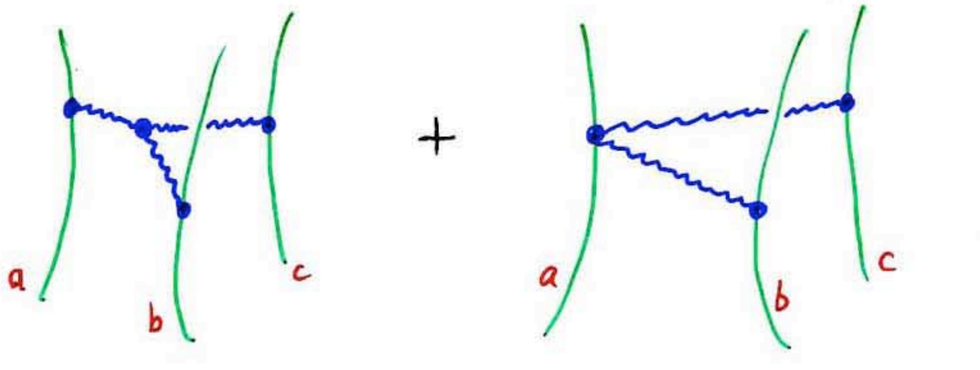
TWO-BODY INTERACTION




$$L^{2\text{-BODY}} = \frac{1}{2}\sum_{a\neq b} \frac{G m_a m_b}{r_{ab}}\left[1 + \frac{3}{2c^2}(\vec{v}_a^{\,2}+\vec{v}_b^{\,2}) - \frac{7}{2c^2}(\vec{v}_a\cdot\vec{v}_b) - \frac{1}{2c^2}(\vec{n}_{ab}\cdot\vec{v}_a)(\vec{n}_{ab}\cdot\vec{v}_b) + O\left(\frac{1}{c^4}\right)\right]$$

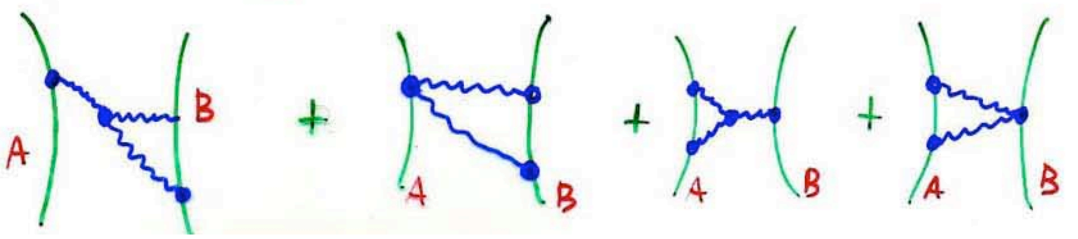
$r_{ab} \equiv |\vec{z}_a - \vec{z}_b| \quad ; \quad \vec{n}_{ab} \equiv (\vec{z}_a - \vec{z}_b)/r_{ab}$

THREE-BODY INTERACTIONS




$$L^{3\text{-BODY}} = -\frac{1}{2}\sum_{b\neq a\neq c} \frac{G^2 m_a m_b m_c}{r_{ab}\, r_{ac}\, c^2}$$

INCLUDES THE '2-BODY' CASE $b=c$



# SOLVING THE RELATIVISTIC TWO-BODY PROBLEM AT THE FIRST 'POST-KEPLERIAN' APPROXIMATION

(Damour, Deruelle 1985)

LAGRANGIAN $L = \sum_a -m_a c^2 \sqrt{1-\frac{\vec{v}_a^2}{c^2}} + L^{2\text{-BODY}} + L^{3\text{-BODY}}$

TAKE $a = A$ OR $B$ — EXPAND IN POWERS OF $\frac{\vec{v}_a}{c}$

$$L = L_{Kepler} + \frac{1}{c^2} L_{Post\text{-}Kepler} + \mathcal{O}\left(\frac{1}{c^4}\right)$$

$\mathcal{O}\left(\frac{1}{c^2}\right)$

$$L_K = \frac{1}{2} m_A \vec{v}_A^{\,2} + \frac{1}{2} m_B \vec{v}_B^{\,2} + \frac{G m_A m_B}{r_{AB}}$$

$$L_{PK} = \frac{1}{8} m_A \vec{v}_A^{\,4} + \frac{1}{8} m_B \vec{v}_B^{\,4} + \frac{G m_A m_B}{2 r_{AB}} \left[ 3\vec{v}_A^{\,2} + 3\vec{v}_B^{\,2} - 7\vec{v}_A \cdot \vec{v}_B - (\vec{n}_{AB} \cdot \vec{v}_A)(\vec{n}_{AB} \cdot \vec{v}_B) - G\frac{m_A + m_B}{r_{AB}} \right]$$

- USE CONSERVATION LAWS (LINKED TO POINCARÉ INVARIANCE OF DYNAMICS)
  - CENTER-OF-MASS INTEGRAL $\vec{K} = \vec{G} - t\vec{P}$
  - TOTAL LINEAR MOMENTUM $\vec{P}$
  - TOTAL ENERGY $E$
  - TOTAL ANGULAR MOMENTUM $\vec{J}$
- USE $\vec{P}, \vec{K}$ TO GO TO CENTER-OF-MASS FRAME: $\vec{P} = \vec{0} = \vec{K}$
- IN C.O.M. FRAME USE $E, \vec{J}$ TO REDUCE RELATIVE DYNAMICS TO RADIAL + ANGULAR MOTION:

$$\left(\frac{dR}{dt}\right)^2 = A + \frac{2B}{R} + \frac{C}{R^2} + \frac{D}{R^3}$$

$$\frac{d\theta}{dt} = \frac{H}{R^2} + \frac{I}{R^3}$$

USEFUL TRICK: USE SHIFTS $R = R' + \text{CST}$ TO ELIMINATE $D/R^3$ TERM

# EXPLICIT SOLUTION OF (1PK) TWO-BODY PROBLEM

(Damour Deruelle 85)

## QUASI-KEPLERIAN FORM :

Kepler equation for 'eccentric anomaly'

$$\boxed{n(t-t_0) = u - e_t \sin u}$$

ORBITAL FREQUENCY $n \equiv \frac{2\pi}{P_b}$ (PERIASTRON TO PERIASTRON)

TIME AT INFINITY IN CoM FRAME

eccentric anomaly

" t-eccentricity"

Angular motion

$$\boxed{\theta - \theta_0 = (1+k)\, 2 \arctan\left[\left(\frac{1+e_\theta}{1-e_\theta}\right)^{\frac{1}{2}} \tan\frac{u}{2}\right]}$$

fractional periastron advance per orbit

" θ- eccentricity"

$$k = \frac{\Delta\theta}{2\pi} = \frac{\langle\dot{\omega}\rangle}{n} = \frac{\langle\dot{\omega}\rangle P_b}{2\pi}$$

Radial motions

$$\boxed{\begin{aligned} R \equiv r_{AB} &= a_R\,(1 - e_R \cos u) \\ r_A \equiv |\vec{z}_A - \vec{z}_{CoM}| &= a_r\,(1 - e_r \cos u) \\ r_B \equiv |\vec{z}_B - \vec{z}_{CoM}| &= a_{r'}\,(1 - e_{r'} \cos u) \end{aligned}}$$

WITH DIFFERENT 'SEMI-MAJOR AXES' $a_R,\ a_r,\ a_{r'}$ AND 'RADIAL ECCENTRICITIES' $e_R,\ e_r,\ e_{r'}$

$\exists$ EXPLICIT FORMULAS RELATING $n, e_t, k, e_\theta, a_R, e_R, a_r, e_r, a_{r'}, e_{r'}$ TO $E$ AND $J$ AND THE MASSES, E.G.

$$n = \frac{(-2E)^{3/2}}{GM}\left[1 - \frac{1}{4}(\nu - 15)\frac{E}{c^2}\right]$$

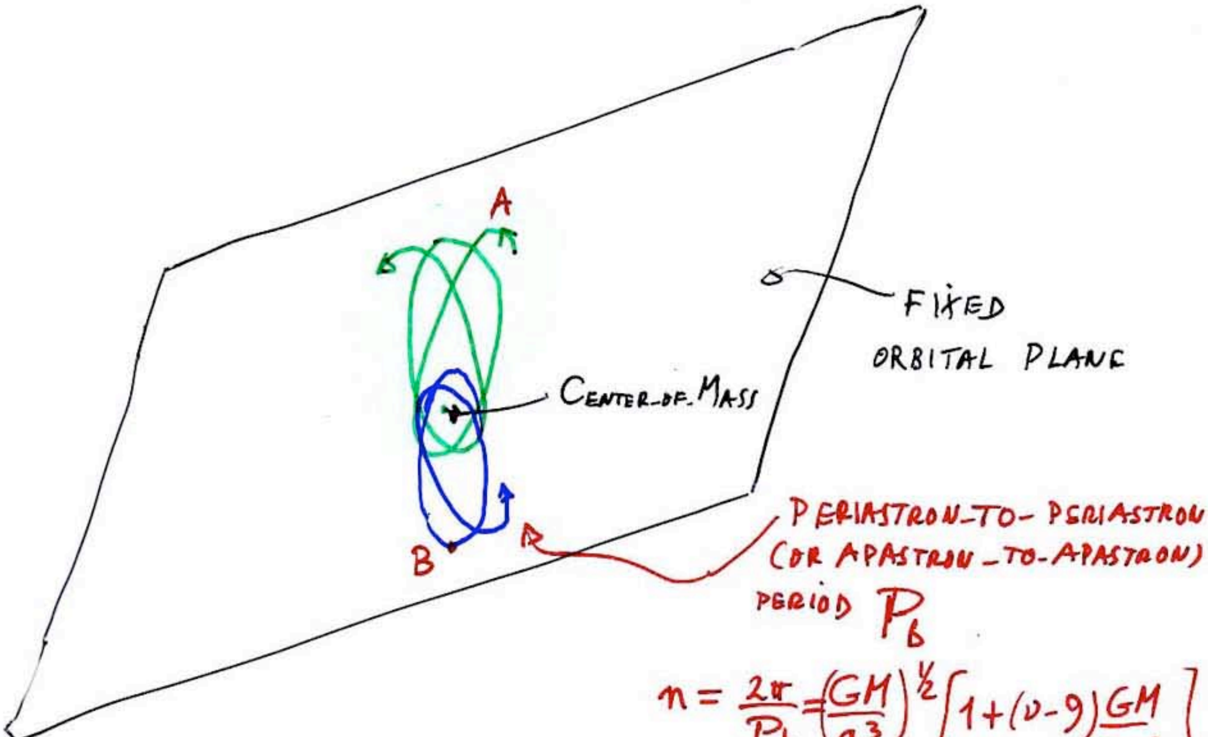
ORBITAL FREQUENCY

$M \equiv m_A + m_B$

$\nu \equiv m_A m_B/(m_A+m_B)^2$

# TWO-BODY RELATIVISTIC MOTION

## IN CENTER-OF-MASS FRAME




$$n = \frac{2\pi}{P_b} = \left(\frac{GM}{a_R^3}\right)^{1/2}\left[1 + (\nu - 9)\frac{GM}{a_R c^2}\right]$$

('MEANING' OF SEMI-MAJOR AXES AND ECCENTRICITIES

$$r_{AB}^{max} = R_{max} = a_R(1+e_R)\,; \quad r_{AB}^{min} = a_R(1-e_R)$$
$$r_A^{max} = a_r(1+e_r)\,; \quad r_A^{min} = a_r(1-e_r)$$
$$r_B^{max} = a_{r'}(1+e_{r'})\,; \quad r_B^{min} = a_{r'}(1-e_{r'})$$

$$Z \quad e_R \neq e_r \neq e_{r'}$$

HOWEVER ONE PROVES THAT $\dfrac{a_r}{\frac{1}{m_A}} = \dfrac{a_{r'}}{\frac{1}{m_B}} = \dfrac{a_R}{\frac{1}{m_A} + \frac{1}{m_B}}$ AS FOR NEWTON!

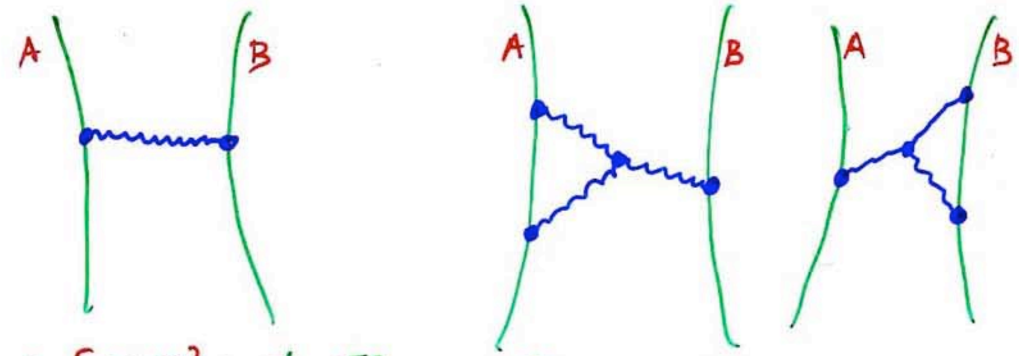
FRACTIONAL PERIASTRON ADVANCE PER ORBIT

$$k = \frac{\delta\theta}{2\pi} = \frac{\langle\dot{\omega}\rangle P_b}{2\pi} = \frac{3\,G(m_A+m_B)}{c^2 a_R (1-e^2)}$$

ANY eccentricity

$a_R, e_R, \ldots$ WILL ACQUIRE REAL OBSERVABLE MEANING ONLY AFTER ONE DEVELOPS THE 'TIMING FORMULA'

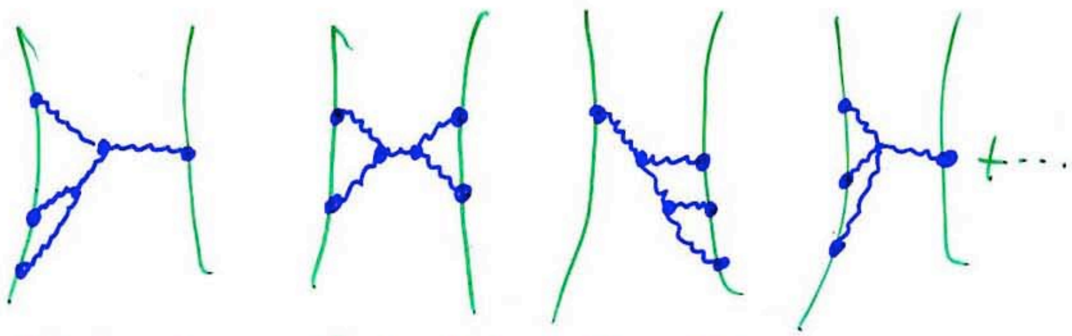
# HIGHER-ORDER RELATIVISTIC DYNAMICAL EFFECTS




$$G m_A m_B \left[1 + \frac{v^2}{c^2} + \underbrace{\frac{v^4}{c^4} + \frac{v^5}{c^5} + \dots}_{\text{HIGHER-ORDER}}\right] \qquad \frac{G^2}{c^2} m_A^2 m_B \left[1 + \underbrace{\frac{v^2}{c^2} + \frac{v^3}{c^3}}_{\text{HIGHER-ORDER}}\right]$$

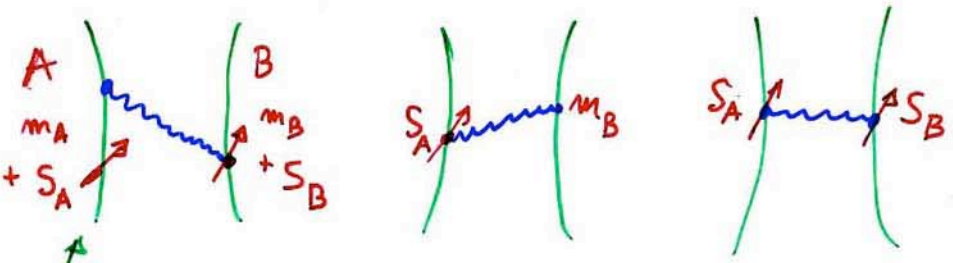
LINEAR IN G

QUADRATIC IN G



$$\underbrace{\frac{G^3}{c^4} m_A^3 m_B \left[1+\frac{v}{c}\right] \quad \frac{G^3}{c^4} m_A^2 m_B^2 \left[1+\frac{v}{c}\right] \quad \frac{G^3}{c^4} m_A m_B^3 \left[1+\frac{v}{c}\right]}_{\text{HIGHER-ORDER}}$$

\+ SPIN - DEPENDENT EFFECTS




SOURCE $\sim m_A \delta + S_A \nabla \delta$

G6
OL16

# $(V/c)^5$ EQUATIONS OF MOTION IN GENERAL RELATIVITY



accelerations. Then each body must satisfy the following equation of motion (Damour and Deruelle, 1981*a*; Damour, 1982):

$$a^i = A_0^i(\vec{z}-\vec{z}') + c^{-2}A_2^i(\vec{z}-\vec{z}', \vec{v}, \vec{v}') + c^{-4}A_4^i(\vec{z}-\vec{z}', \vec{v}, \vec{v}', \vec{S}, \vec{S}') + \underline{c^{-5}A_5^i(\vec{z}-\vec{z}', \vec{v}-\vec{v}')} + O(c^{-6}), \tag{154}$$

with

$$A_0^i = -Gm'R^{-2}N^i, \tag{155}$$

$$A_2^i = Gm'R^{-2}\{N^i[-v^2 - 2v'^2 + 4(vv') + \tfrac{3}{2}(Nv')^2 + 5(Gm/R) + 4(Gm'/R)] + (v^i - v'^i)[4(Nv) - 3(Nv')]\}, \tag{156}$$

"DRESSED MASSES" INCORPORATING STRONG-SELF-FIELD EFFECTS

$$A_4^i = B_4^i + C_4^i + D_4^i, \tag{157}$$

$$\begin{aligned} B_4^i = Gm'R^{-2}\{&N^i[-2v'^4 + 4v'^2(vv') - 2(vv')^2 + \tfrac{3}{2}v^2(Nv')^2 + \tfrac{9}{2}v'^2(Nv')^2 \\ &- 6(vv')(Nv')^2 - \tfrac{15}{8}(Nv')^4 + (Gm/R)(-\tfrac{15}{4}v^2 + \tfrac{5}{4}v'^2 - \tfrac{5}{2}(vv') + \tfrac{39}{2}(Nv)^2 \\ &- 39(Nv)(Nv') + \tfrac{17}{2}(Nv')^2) + (Gm'/R)(4v'^2 - 8(vv') + 2(Nv)^2 \\ &- 4(Nv)(Nv') - 6(Nv')^2)] \\ &+ (v^i - v'^i)[v^2(Nv') + 4v'^2(Nv) - 5v'^2(Nv') - 4(vv')(Nv) \\ &+ 4(vv')(Nv') - 6(Nv)(Nv')^2 + \tfrac{9}{2}(Nv')^3 \\ &+ (Gm/R)(-\tfrac{63}{4}(Nv) + \tfrac{55}{4}(Nv')) + (Gm'/R)(-2(Nv) - 2(Nv'))]\}, \end{aligned} \tag{158}$$

$$C_4^i = G^3m'R^{-4}N^i[-\tfrac{57}{4}m^2 - 9m'^2 - \tfrac{69}{2}mm'], \tag{159}$$

$$D_4^i = \left(\frac{S^{ik}}{m} + 2\frac{S'^{ik}}{m'}\right)(v^l - v'^l)\left(\frac{Gm'}{R}\right)_{,kl} + \left(2\frac{S^{kl}}{m} + 2\frac{S'^{kl}}{m'}\right)(v^l - v'^l)\left(\frac{Gm'}{R}\right)_{,ik}, \tag{160}$$

and

$$\boxed{\begin{aligned} A_5^i = \tfrac{4}{5}G^2mm'R^{-3}\{&V^i[-V^2 + 2(Gm/R) - 8(Gm'/R)] \\ &+ N^i(NV)[3V^2 - 6(Gm/R) + \tfrac{52}{3}(Gm'/R)]\}. \end{aligned}} \tag{161}$$

GRAVITATIONAL RADIATION DAMPING

DIRECT

EFFECT OF PROPAGATION OF GRAVITY AT SPEED C

The two parameters $m$ and $m'$ appearing in eqs. (154)–(161) are the 'Schwarzschild masses' of the condensed bodies. They are two constants which appear in the external gravitational field, in which are hidden many internal structure effects (see the discussion of the 'effacement of internal structure' in Section 6.14). On the other hand, the spin tensors undergo a slow evolution (on the post-Newtonian time scale, i.e. $\beta_e^{-2}$ times the orbital period) which is also obtained in the Einstein–Infeld–Hoffmann–Kerr-type approach (Damour, 1982, and references therein). Introducing, *à la* Schiff, a suitable spin-vector, $\vec{S}$, associated with $S_{\mu\nu}$, the law of evolution ('spin precession') reads for the first body (see also references in Section 6.13.2)

$$\frac{d\vec{S}}{dt} = \left[\frac{Gm'}{c^2R^2}\vec{N} \times \left(\frac{3}{2}\vec{v} - 2\vec{v}'\right)\right] \times \vec{S} + O\left(\frac{1}{c^4}\right). \tag{162}$$

# PHYSICAL EFFECTS LINKED TO HIGHER-ORDER TERMS

EQ. OF MOTION

$$\frac{d^2\vec{z}}{dt^2} \sim \frac{Gm\vec{n}}{R^2}\left[\underbrace{1 + \frac{v^2}{c^2}}_{1PK} + \frac{v^4}{c^4} + \frac{v^5}{c^5} + \text{Spin-orbit terms}\right]$$

QUANTITATIVE MODIFICATIONS OF 1PK BUT NO NEW QUALITATIVE EFFECT (BECAUSE STILL CONSERVATIVE)

ACCELERATION OF ORBITAL MOTION + SECULAR VARIATION OF MANY QUANTITIES

WOBBLING OF ORBITAL PLANE + SPIN PRECESSION

MODIFICATION OF PERIASTRON ADVANCE

$x_1 \equiv \frac{m_A}{M} \equiv 1 - x_2$

$$k = \frac{3(GMn)^{2/3}}{c^2(1-e_T^2)}\left[1 + \frac{(GMn)^{2/3}}{c^2(1-e_T^2)}\left(\frac{39}{4}x_1^2 + \frac{27}{4}x_2^2 + 15x_1x_2\right) - \frac{(GMn)^{2/3}}{c^2}\left(\frac{13}{4}x_1^2 + \frac{1}{4}x_2^2 + \frac{13}{3}x_1x_2\right)\right.$$
$$\left. - \frac{cS_A}{Gm_A^2}\frac{(GMn)^{1/3}}{c}\frac{x_1(4x_1+3x_2)}{6(1-e_T^2)^{1/2}\sin^2 i}\left[(3\sin^2 i - 1)\vec{k}\cdot\vec{s}_A + \cos i\, \vec{K}_0\cdot\vec{s}_A\right] - A \leftrightarrow B\right]$$

FROM $v^4/c^4$ TERMS

FROM SPIN-ORBIT TERMS

EFFECTS OF $v^5/c^5$ TERMS :

ORBITAL PHASE $n(t-t_0) = 2\pi\frac{t-t_0}{P_b} \longrightarrow 2\pi\left[\frac{t-t_0}{P_b} - \frac{1}{2}\dot{P}_b\left(\frac{t-t_0}{P_b}\right)^2\right]$

$$\boxed{\dot{P}_b = -\frac{192\pi}{5c^5}\,\nu\,(GMn)^{5/3}\,\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}}}$$

# SUMMARIZING

- SPECIAL NATURE OF STRONG-SELF-GRAVITY EFFECTS IN GR

E.G. FOR NON-SPINNING COMPACT BODIES : EFFACEMENT OF INTERNAL STRUCTURE

UP TO $(v/c)^{10}$ $\Rightarrow$ EQS. MOTION DEPEND ONLY ON $m_A$ , $m_B$

" Schwarzschild-like masses" entering $G^{(0)\,\text{ISOLATED}}_{\alpha\beta}(x^\gamma, m_A)$

ALL STRONG-SELF-GRAVITY EFFECTS LUMPED IN $m_A$ , $m_B$

HOWEVER:
$$m_A \simeq \underbrace{\int d^3x\,\rho}_{+\text{internal energy}} - \frac{1}{2}\int\frac{d^3x\,d^3x'}{c^2}\frac{\rho\rho'}{|x-x'|} + \frac{1}{c^4}\iiint\cdots + \frac{1}{c^6}\iiint\cdots$$

MANY "Post Newtonian" CORRECTIONS

$$\simeq \underbrace{\bar{m}_A}_{\text{BARYON REST-MASS}}\Big[\underbrace{1}_{N} - \underbrace{0.1}_{1PN} + \underbrace{c_2(0.1)^2}_{2PN} + \underbrace{c_3(0.1)^3}_{3PN} + \underbrace{\cdots}_{4PN\cdots}\Big]$$

- $\dot{P}_b$ IS DUE TO TIME-ASYMMETRIC (AND NON-LINEAR) PROPAGATION OF GRAVITY BETWEEN A and B AT VELOCITY $c_g = c$

$\Box^{-1}_{\text{RETARDED}}$

THE USUALLY MENTIONNED 'QUADRUPOLE FORMULA' ARGUMENT IS
- NOT NEEDED
- NOT LINKED TO WHAT IS OBSERVED
- LESS CONCLUSIVE,
- INCORRECT IF BASED ON WEAK-FIELD $Q_{ij} \sim \int \rho x^{\langle i}x^{j\rangle}$

- $\exists$ QUITE SIMPLE QUASI-KEPLERIAN REPRESENTATION OF 1PK MOTION
(can be extended to 2PK : Damour Schäfer 88 ; Wex 95)

THIS FORM RESTS ON GENERAL STRUCTURE $E \sim A + 2B/R + C/R^2 + D/R^3$

$\Rightarrow$ PLAUSIBLY VALID IN MANY OTHER RELATIVISTIC THEORIES OF GRAVITY

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