ANOSOV GROUPS THAT ARE INDISCRETE IN RANK ONE

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Abstract. We exhibit Anosov subgroups of $\mathsf{SL}_d(\mathbb{R})$ that do not embed discretely in any rank-1 simple Lie group of noncompact type, or indeed, in any finite product of such Lie groups. These subgroups are isomorphic to free products $\Gamma * \Delta$, where Γ is a uniform lattice in $\mathsf{F}_4^{(-20)}$ and Δ is a uniform lattice in $\mathsf{Sp}(m,1)$, $m \geqslant 51$.

1. Introduction

Throughout this note, a *rank*-1 *Lie group* is a Lie group isogenous to the isometry group of an irreducible symmetric space of noncompact type and real rank 1. Such a symmetric space is isometric (up to scaling) to one of $\mathbb{R}\mathbf{H}^n, \mathbb{C}\mathbf{H}^n, \mathbb{H}\mathbf{H}^n$, $n \geqslant 2$, or $\mathbb{O}\mathbf{H}^2$; correspondingly, each rank-1 Lie group is isogenous to one of $\mathsf{SO}(n,1)$, $\mathsf{SU}(n,1)$, $\mathsf{Sp}(n,1)$, $n \geqslant 2$, or $\mathsf{F}_4^{(-20)}$; see, for instance, [Wol11, Thm. 8.12.2].

Since their introduction in Labourie's seminal paper on the Hitchin component [Lab06] and the further development of their theory by Guichard–Wienhard [GW12], Kapovich–Leeb–Porti [KLP17, KLP18a, KLP18b], Guéritaud–Guichard–Kassel–Wienhard [GGKW17], Bochi–Potrie–Sambarino [BPS19], and others, Anosov representations have emerged as successful higher-rank generalizations of convex cocompact representations into rank-1 Lie groups. For a survey on Anosov representations and their strong dynamical, geometric, and topological properties, see [Kas18].

That being said, the authors were not aware of an example in the literature of a Gromov-hyperbolic group that admits an Anosov embedding into a higher-rank Lie group but does not already embed as a convex cocompact subgroup of a rank-1 Lie group. The purpose of this note is to furnish such examples. Indeed, we observe the following.

Theorem 1.1. *Let Γ_1 and Γ_2 be uniform lattices in $\mathsf{F}_4^{(-20)}$. Then the free product $\Gamma_1 * \Gamma_2$ admits no discrete and faithful representation into any rank-1 Lie group.*

One can replace Γ_2 in the statement of Theorem 1.1 with any nontrivial group (indeed, if Γ_2 is nontrivial then $\Gamma_1 * \Gamma_1$ embeds as a subgroup of $\Gamma_1 * \Gamma_2$). If we replace Γ_2 with a uniform quaternionic hyperbolic lattice of large dimension, we obtain a stronger conclusion.

Theorem 1.2. *Let Γ be a uniform lattice in $\mathsf{F}_4^{(-20)}$ and Δ a uniform lattice in $\mathsf{Sp}(m,1)$, where $m \geqslant 51$. Then the free product $\Gamma * \Delta$ admits no discrete and faithful representation into any Lie group isogenous to a product of rank-1 Lie groups.*

On the other hand, it follows from a recent combination theorem of Dey–Kapovich–Leeb [DKL19], as well as a result announced by Danciger–Guéritaud–Kassel [DGK17, Prop. 12.5] and proved in their forthcoming work [DGK], that the free products in Theorems 1.1 and 1.2 admit Anosov embeddings; see Section 4. The proofs of Theorems 1.1 and 1.2 make crucial use of the rank-1 superrigidity theorems of Corlette [Cor92] and Gromov–Schoen [GS92].

The first examples of linear Gromov-hyperbolic groups that do not admit discrete and faithful representations into any rank-1 Lie group were exhibited in [TT21, Thm. 1.2 & 1.7] as amalgamated products of two copies of a torsion-free uniform lattice $\Delta < \mathsf{Sp}(m,1)$, $m \geqslant 2$, along a maximal cyclic subgroup of Δ. It was suggested to the second author by Beatrice

Pozzetti that such amalgams also admit Anosov embeddings, though we do not pursue this here.

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2. Preliminaries

Let G be a finite-center real semisimple Lie group with finitely many connected components and K a maximal compact subgroup of G. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{g} = \mathrm{Lie}(\mathsf{G})$ and $\overline{\mathfrak{a}}^+ \subset \mathfrak{a}$ a dominant Weyl chamber, so that there exists a Cartan decomposition $\mathsf{G} = \mathsf{K}\exp(\overline{\mathfrak{a}}^+)\mathsf{K}$. Let $\mu : \mathsf{G} \to \overline{\mathfrak{a}}^+$ be the associated Cartan projection. Given a non-empty set Θ of simple restricted roots of $\mathfrak{g}$, a representation $\rho : \Gamma \to \mathsf{G}$ of a finitely generated group Γ is Θ-*Anosov* if there exist $c, C > 0$ such that for every $\gamma \in \Gamma$ and $\theta \in \Theta$,

$$\theta\big(\mu(\rho(\gamma))\big) \geqslant c\big|\gamma\big|_\Gamma - C, \tag{1}$$

where $|\cdot|_\Gamma$ denotes word length with respect to some fixed finite generating set of Γ. That this characterization is equivalent to Labourie's original dynamical definition was established independently by Kapovich–Leeb–Porti [KLP18b] and Bochi–Potrie–Sambarino [BPS19]. It is visible from this definition that if $\rho|_{\Gamma_0}$ is Θ-Anosov for some finite-index subgroup $\Gamma_0 < \Gamma$ then ρ is itself Θ-Anosov.

Observe that when G is a rank-1 Lie group then (1) simply says that ρ is a quasi-isometric embedding, i.e., that ρ is convex cocompact. We say a subgroup $\Gamma < \mathsf{G}$ is Θ-*Anosov* if Γ is the image of a Θ-Anosov representation into G.

We now clarify condition (1) for the specific case $\mathsf{G} = \mathsf{SL}_d(\mathbb{R})$. Given $g \in \mathsf{SL}_d(\mathbb{R})$, denote by $\mu_1(g) \geqslant \mu_2(g) \geqslant \ldots \geqslant \mu_d(g)$ the logarithms of the singular values of g in non-increasing order (counting multiplicity). A representation $\rho : \Gamma \to \mathsf{SL}_d(\mathbb{R})$ is P_i-*Anosov*, $1 \leqslant i \leqslant d-1$, if there exist $c, C > 0$ such that

$$\mu_i\big(\rho(\gamma)\big) - \mu_{i+1}\big(\rho(\gamma)\big) \geqslant c|\gamma|_\Gamma - C$$

for every $\gamma \in \Gamma$. We will use repeatedly the following fact.

Lemma 2.1. *Suppose a finite-index normal subgroup $\Gamma_0 < \Gamma$ embeds as a P_1-Anosov subgroup of $\mathsf{SL}_d(\mathbb{R})$. Then Γ embeds as a P_1-Anosov subgroup of $\mathsf{SL}_r(\mathbb{R})$ for some $r \in \mathbb{N}$.*

Proof. Let $\rho : \Gamma_0 \hookrightarrow \mathsf{SL}_d(\mathbb{R})$ be the inclusion and $\rho^{\mathrm{ind}} : \Gamma \to \mathsf{SL}^{\pm}_{dm}(\mathbb{R})$ the induced representation (see, for instance, [FH91, Section 3.3]), where $m = [\Gamma : \Gamma_0]$. Since ρ is faithful, the same is true for ρ^{ind}. Set $\ell = dm + 1$ and let $\hat{\rho} : \Gamma \to \mathsf{SL}_\ell(\mathbb{R})$ be the composition of ρ^{ind} with the embedding $\mathsf{SL}^{\pm}_{dm}(\mathbb{R}) \to \mathsf{SL}_\ell(\mathbb{R})$ given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & \det(A) \end{pmatrix}.$$

Since the restriction $\rho^{\mathrm{ind}}|_{\Gamma_0}$ is an m-fold direct sum of P_1-Anosov representations, and since $\hat{\rho}|_{\Gamma_0}$ is obtained from $\rho^{\mathrm{ind}}|_{\Gamma_0}$ by inserting a 1 on the diagonal, we have that $\hat{\rho}|_{\Gamma_0}$ is P_m-Anosov in $\mathsf{SL}_\ell(\mathbb{R})$, and so the latter is also true for $\hat{\rho}$. If ℓ is even, we replace $\hat{\rho}$ with the representation obtained from $\hat{\rho}$ by inserting a 1 on the diagonal (we still call the latter representation $\hat{\rho}$), and increase ℓ by 1. If ℓ is odd, we keep $\hat{\rho}$ and ℓ as is. Note that in any case $\hat{\rho}$ remains P_m-Anosov.

Now consider the m^{th} exterior power $\bigwedge^m \hat{\rho} : \Gamma \to \mathsf{SL}\left(\bigwedge^m \mathbb{R}^\ell\right)$. Since ℓ is odd, we have that $\bigwedge^m \hat{\rho}$ is faithful. Moreover, since $\hat{\rho}$ is P_m-Anosov we have that $\bigwedge^m \hat{\rho}$ is P_1-Anosov. $\square$

Following [Mor15], we say that two Lie groups G_1 and G_2 are *isogenous* if there exist finite index subgroups $\mathsf{G}_i' < \mathsf{G}_i$ and finite normal subgroups $M_i < \mathsf{G}_i'$ such that $\mathsf{G}_1'/M_1 \cong \mathsf{G}_2'/M_2$.

The following proposition is a consequence of Archimedean superrigidity of lattices in $\mathsf{F}_4^{(-20)}$.

Proposition 2.2. *Let Γ be a lattice in $\mathsf{F}_4^{(-20)}$, and suppose that G is a rank-1 Lie group that is not isogenous to $\mathsf{F}_4^{(-20)}$. Then every representation $\rho : \Gamma \to \mathsf{G}$ has bounded image.*

Proof. Suppose that $\rho(\Gamma)$ has noncompact closure in G. Corlette's Archimedean superrigidity theorem [Cor92] provides a continuous ρ-equivariant totally geodesic embedding $\mathbb{O}\mathbf{H}^2 \hookrightarrow X_{\mathsf{G}}$, where X_{G} is the symmetric space associated to G. Since G is not isogenous to $\mathsf{F}_4^{(-20)}$, we may find a totally geodesic embedding $X_{\mathsf{G}} \hookrightarrow \mathbb{H}\mathbf{H}^m$ for $m \in \mathbb{N}$ large enough. In particular, we obtain a totally geodesic embedding of the Cayley hyperbolic plane $\mathbb{O}\mathbf{H}^2$ into $\mathbb{H}\mathbf{H}^m$, but this is impossible by the classification of totally geodesic subspaces of $\mathbb{H}\mathbf{H}^m$ [Mey15, Thm. 2.12]. □

3. Proofs of Theorems 1.1 & 1.2

Proof of Theorem 1.1. Assume for a contradiction that we have a discrete and faithful representation $\rho : \Gamma_1 * \Gamma_2 \to \mathsf{G}$, where Γ_1 and Γ_2 are uniform lattices in $\mathsf{F}_4^{(-20)}$, and G is a rank-1 Lie group. Since ρ is discrete and faithful on the factor Γ_1, we must have that G is isogenous to $\mathsf{F}_4^{(-20)}$ by Proposition 2.2. It follows that $\rho(\Gamma_1)$ is a uniform lattice of G since the virtual cohomological dimension of Γ_1 is equal to the dimension of $\mathbb{O}\mathbf{H}^2$. Since $\rho(\Gamma_1)$ is a lattice in G, and ρ is discrete and faithful, we deduce that Γ_1 is of finite index in $\Gamma_1 * \Gamma_2$. This is absurd since Γ_2 is nontrivial. □

To prove Theorem 1.2, we will make use of the following consequence of Gromov–Schoen superrigidity [GS92]. Similar arguments can be found in the proofs of [Kap05, Thm. 8.1] and [CST19, Thm. 3.1].

Proposition 3.1. *Let G be either $\mathsf{Sp}(m,1)$, $m \geqslant 2$, or $\mathsf{F}_4^{(-20)}$, and let $\Gamma < \mathsf{G}$ be a lattice. Suppose that $\rho : \Gamma \to \mathsf{GL}_d(\mathbb{R})$ is a representation with infinite image. Then there is a representation $\rho' : \Gamma \to \mathsf{GL}_d(\mathbb{C})$ with unbounded image.*

Proof. We may assume that ρ has bounded image, so that $\rho(\Gamma) \subset \mathsf{O}(n)$ up to postconjugation. Since Γ has Property (T) (see [BdlHV08] and the references therein), up to further postconjugation, we have that $\rho(\Gamma) \subset \mathsf{O}(n, \mathbb{K})$ for some number field $\mathbb{K} \subset \mathbb{R}$ [Rag72, Prop. 6.6]. Moreover, since Γ is finitely generated, we in fact have $\rho(\Gamma) \subset \mathsf{O}(n, A)$ for some finitely generated subdomain $A \subset \mathbb{K}$. We may now find embeddings $A \subset \mathbb{K}_i$, where $\mathbb{K}_1, \ldots, \mathbb{K}_r$ are local fields, with $\mathbb{K}_1, \ldots, \mathbb{K}_s$ Archimedean and $\mathbb{K}_{s+1}, \ldots, \mathbb{K}_r$ non-Archimedean, so that the diagonal embedding $A \hookrightarrow \prod_{i=1}^r \mathbb{K}_i$ is discrete. We thus obtain from ρ a discrete representation $\Gamma \to \prod_{i=1}^r \mathsf{GL}_n(\mathbb{K}_i)$. By the superrigidity result of Gromov–Schoen [GS92], we have that the projection $\Gamma \to \prod_{i=s+1}^r \mathsf{GL}_n(\mathbb{K}_i)$ is bounded, so that the projection $\Gamma \to \prod_{i=1}^s \mathsf{GL}_n(\mathbb{K}_i)$ is discrete. Since the latter representation has infinite image, we conclude that at least one of the projections $\Gamma \to \mathsf{GL}_n(\mathbb{K}_i)$, $1 \leqslant i \leqslant s$, has unbounded image. □

We deduce the following from Proposition 3.1.

Theorem 3.2. *Let Δ be a lattice in $\mathsf{Sp}(m,1)$, where $m \geqslant 51$. Suppose that H is a semisimple Lie group isogenous to $\mathsf{F}_4^{(-20)}$. Then every representation $\rho : \Delta \to \mathsf{H}$ has finite image.*

Proof. Let H_0 be a finite-index subgroup of H, and F_0 and F_1 finite normal subgroups of H_0 and $\mathsf{F}_4^{(-20)}$, respectively, such that $\mathsf{H}_0/F_0 \cong \mathsf{F}_4^{(-20)}/F_1$. Denote by $\mathfrak{g}$ the 52-dimensional real Lie algebra of $\mathsf{F}_4^{(-20)}$. Since F_1 is central in $\mathsf{F}_4^{(-20)}$, the adjoint representation $\mathrm{Ad} : \mathsf{F}_4^{(-20)} \to \mathsf{GL}(\mathfrak{g})$ induces a well-defined representation $\psi : \mathsf{H}_0/F_0 \to \mathsf{GL}(\mathfrak{g})$ with finite kernel.

We now pass to a finite-index subgroup Δ_0 of Δ such that $\rho(\Delta_0)$ is contained in H_0, and consider the composition $\phi := \psi \circ \pi \circ \rho : \Delta_0 \to \mathsf{GL}(\mathfrak{g})$, where π is the projection $\mathsf{H}_0 \to \mathsf{H}_0/F_0$. Observe that ρ has finite image if and only if ϕ does.

Now assume that ϕ has infinite image. In this case, Proposition 3.1 provides a representation $\phi' : \Delta_0 \to \mathsf{GL}_{52}(\mathbb{C})$ with unbounded image. In particular, by Corlette's Archimedean superrigidity theorem [Cor92] (see [FH12, Thm. 3.7]) there exists a continuous representation $\overline{\phi} : \mathsf{Sp}(m,1) \to \mathsf{GL}_{52}(\mathbb{C})$ and a representation $\phi_0 : \Delta \to \mathsf{GL}_{52}(\mathbb{C})$ with compact closure such that the images $\overline{\phi}(\Delta)$ and $\phi_0(\Delta)$ commute and $\phi(\gamma) = \overline{\phi}(\gamma)\phi_0(\gamma)$ for every $\gamma \in \Delta$. Since $\phi'(\Delta_0)$ has noncompact closure, the representation $\overline{\phi}$ is unbounded and hence has finite kernel. In particular, the Lie algebra of $\mathsf{Sp}(m,1)$ embeds into that of $\mathsf{GL}_{52}(\mathbb{C})$. However, this cannot happen since $m \geqslant 51$ and the dimension of the Lie algebra of $\mathsf{Sp}(m,1)$ is $2m^2+5m+3 > 2\cdot 52^2$. We obtain a contradiction, and hence the image of ϕ is finite. It follows that the image of ρ is finite. □

We are now ready to establish our main result.

Proof of Theorem 1.2. Let G be a semisimple Lie group that is isogenous to a product of rank-1 Lie groups and $\rho : \Gamma * \Delta \to \mathsf{G}$ a representation. We prove that ρ cannot be discrete and faithful.

By our assumption on G, one can find a finite-index subgroup G_0 of G, rank-1 Lie groups $\mathsf{G}_1, \dots, \mathsf{G}_q$, and a continuous epimorphism $\pi : \mathsf{G}_0 \to \prod_{i=1}^q \mathsf{G}_i$ with finite kernel. Choose finite-index subgroups Γ_0 and Δ_0 of Γ and Δ, respectively, such that $\rho(\Gamma_0 * \Delta_0) \subset \mathsf{G}_0$. It suffices to prove that the composition $\phi := \pi \circ \rho : \Gamma_0 * \Delta_0 \to \prod_{i=1}^q \mathsf{G}_i$ cannot be discrete and faithful.

Let $\mathrm{pr}_i : \prod_{j=1}^q \mathsf{G}_j \to \mathsf{G}_i$ denote the projection onto the i^{th} factor, let $I_1 \subset \{1, \dots, q\}$ be the (possibly empty) set of indices i such that G_i is isogenous to $\mathsf{F}_4^{(-20)}$, and set $I_2 := \{1, \dots, q\} \smallsetminus I_1$. Since $m \geqslant 51$, by Theorem 3.2, the representation $\mathrm{pr}_i \circ \phi : \Delta_0 \to \mathsf{G}_i$ has finite image for every $i \in I_1$, so that the subgroup $\Delta_1 := \bigcap_{i \in I_1} \ker(\mathrm{pr}_i \circ \phi) < \Delta_0$ is of finite index. Moreover, by Theorem 2.2, for every $j \in I_2$, the image of the representation $\mathrm{pr}_j \circ \phi : \Gamma_0 \to \mathsf{G}_j$ is bounded since G_j is not isogenous to $\mathsf{F}_4^{(-20)}$. Now choose an arbitrary infinite sequence $(\gamma_n)_{n \in \mathbb{N}}$ of distinct elements of Γ and a non-trivial element $\delta \in \Delta_1$. Then the terms of the sequence

$$g_n = [\delta, \gamma_n] = \delta \gamma_n \delta^{-1} \gamma_n^{-1},$$

$n \in \mathbb{N}$, in $\Gamma_0 * \Delta_1$ are distinct. For every $n \in \mathbb{N}$ and $i \in I_1$, we have that $\mathrm{pr}_i(\phi(g_n)) = 1$ since $\mathrm{pr}_i(\phi(\delta)) = 1$. Moreover, for every $j \in I_2$, the sequence $(\mathrm{pr}_i(\phi(g_n))_{n \in \mathbb{N}}$ is bounded in G_j since $\mathrm{pr}_i(\phi(\Gamma_0))$ is bounded. It follows that $(\phi(g_n))_{n \in \mathbb{N}}$ is bounded in the product $\prod_{i=1}^q \mathsf{G}_i$ and hence $\pi \circ \rho$ cannot be discrete and faithful. □

Remark 3.3. If Γ_1 and Γ_2 are infinite-covolume convex cocompact subgroups of a rank-1 Lie group G, so that the Γ_i have nonempty domain of discontinuity on the visual boundary of the symmetric space of G, then classical arguments of Maskit [Mas88, Thm. VII.C.2] imply that the free product $\Gamma_1 * \Gamma_2$ also embeds as a convex cocompact subgroup of G. For any convex cocompact subgroup (in particular, any uniform lattice) Γ of a rank-1 Lie group not isogenous to $\mathsf{F}_4^{(-20)}$, there is some $m \in \mathbb{N}$ such that Γ acts convex cocompactly on $\mathbb{H}\mathbf{H}^m$ preserving a proper totally geodesic subspace of the latter; in particular, the free product of any two such Γ again admits a convex cocompact representation into a rank-1 Lie group.

4. Anosov subgroups and free products

In this section, we justify that the free products discussed in Section 3 admit Anosov embeddings. Using a combination theorem of Dey–Kapovich–Leeb [DKL19] (see also [DK22]), we show more generally that the property of admitting an Anosov embedding into some special linear group is preserved under taking finitely many free products.

Proposition 4.1. *Let Γ_1 and Γ_2 be P_1-Anosov subgroups of $\mathsf{SL}_n(\mathbb{R})$. Then $\Gamma_1 * \Gamma_2$ embeds as a P_1-Anosov subgroup of $\mathsf{SL}_N(\mathbb{R})$ for some $N \in \mathbb{N}$.*

We remark that Proposition 4.1 also follows from the theory of convex cocompactness in real projective spaces developed by Danciger–Guéritaud–Kassel [DGK17, Thm. 1.15], together with a result they have announced stating that a free product $\Gamma_1 * \Gamma_2$ of two discrete subgroups $\Gamma_1, \Gamma_2 < \mathsf{SL}_d(\mathbb{R})$ that are convex cocompact in, but do not divide a properly convex domain in, $\mathbb{P}(\mathbb{R}^d)$ embeds as a discrete subgroup of $\mathsf{SL}_d(\mathbb{R})$ that is again convex cocompact in $\mathbb{P}(\mathbb{R}^d)$ [DGK17, Prop. 12.5]. Their proof will appear in [DGK].

Remark 4.2. For every rank-1 Lie group G, one can find an integer $d = d(\mathsf{G})$ and a Lie group homomorphism $\psi : \mathsf{G} \to \mathsf{SL}_d(\mathbb{R})$ with the property that, for every convex cocompact subgroup $\Gamma < \mathsf{G}$ (for instance, every uniform lattice $\Gamma < \mathsf{G}$), the restriction $\psi|_\Gamma : \Gamma \to \mathsf{SL}_d(\mathbb{R})$ is P_1-Anosov; see, for instance, [GW12, Prop. 4.7]. One thus concludes from Proposition 4.1 that if G_1 and G_2 are semisimple linear algebraic $\mathbb{R}$-groups and $\Gamma_i < \mathsf{G}_i$, $i = 1, 2$, is a Θ_i-Anosov subgroup of G_i, then $\Gamma_1 * \Gamma_2$ embeds as a P_1-Anosov subgroup of $\mathsf{SL}_d(\mathbb{R})$ for some $d \in \mathbb{N}$.

Proof of Proposition 4.1. We assume throughout that the Γ_i are infinite. Indeed, if the Γ_i are both finite, so that they both embed discretely in $\mathrm{O}(M)$ for some $M \in \mathbb{N}$, then $\Gamma_1 * \Gamma_2$ embeds as a convex cocompact subgroup of $\mathsf{O}(M, 1)$ (see Remark 3.3), and hence as a P_1-Anosov subgroup of $\mathsf{SL}_{M+2}(\mathbb{R})$. Moreover, if Γ_i is infinite and Γ_j is finite, then the kernel of the projection $\Gamma_1 * \Gamma_2 \to \Gamma_j$ is of the form

$$\Gamma_i * \cdots * \Gamma_i * \mathbb{Z} * \cdots * \mathbb{Z},$$

so we have reduced to the case where the factors are all infinite, as the property of admitting a P_1-Anosov embedding into some special linear group passes to finite-index supergroups; see Lemma 2.1.

Since $\mathsf{SL}_n(\mathbb{R})$ acts on the space of symmetric $(n \times n)$ real matrices, preserving the positive-definite cone, we may assume up to replacing n with $\frac{n(n+1)}{2}$ that the Γ_i both preserve a (nonempty) properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^n)$. Now identify $\mathbb{R}^n$ with the linear hyperplane $\Pi := \{x_1 = 0\} \subset \mathbb{R}^{n+1}$ via the map $(x_2, \ldots, x_{n+1}) \mapsto (0, x_2, \ldots, x_{n+1})$, and view $\mathsf{SL}_n(\mathbb{R})$, and hence the Γ_i, as being included in $\mathsf{SL}_{n+1}(\mathbb{R})$ via the map

$$g \mapsto \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}.$$

Then the Γ_i are P_1-Anosov, and hence P_n-Anosov (see [GW12, Lem. 3.18 (i)]), in $\mathsf{SL}_{n+1}(\mathbb{R})$.

Let $\mathcal{F}$ be the flag manifold of $\mathsf{SL}_{n+1}(\mathbb{R})$ consisting of all pairs (ℓ', π) where $\ell' \in \mathbb{P}(\mathbb{R}^{n+1})$ and π is a projective hyperplane in $\mathbb{P}(\mathbb{R}^{n+1})$ containing ℓ', and let $\Lambda_i \subset \mathcal{F}$ be the limit set of Γ_i in $\mathcal{F}$. For each pair $(\ell', \pi) \in \Lambda_i$, we have that $\ell' \in \partial\Omega$, that $\pi \cap \Omega = \varnothing$, and that π contains the point $[1 : 0 : \ldots : 0] \in \mathbb{P}(\mathbb{R}^{n+1})$. Choose a point $\ell \in \Omega$ and points $\ell^{\pm}$ on the projective line $L \subset \mathbb{P}(\mathbb{R}^{n+1})$ joining ℓ to the point $[1 : 0 : \ldots : 0]$ so that all four of the points just mentioned are distinct. Choose also projective hyperplanes $\pi^{\pm} \subset \mathbb{P}(\mathbb{R}^{n+1})$ containing $\ell^{\pm}$ and whose intersection with $\mathbb{P}(\Pi)$ is disjoint from $\overline{\Omega}$.

Under the above assumptions, the flags $(\ell^{\pm}, \pi^{\pm}) \in \mathcal{F}$ are transverse, and we can find an element $h \in \mathsf{SL}_{n+1}(\mathbb{R})$ that is simultaneously P_1- and P_n-proximal whose attracting and repelling fixed points in $\mathcal{F}$ are $(\ell^{\pm}, \pi^{\pm})$. Moreover, the sets $\{(\ell^{\pm}, \pi^{\pm})\}$ and $\Lambda_1 \cup \Lambda_2$ are antipodal in $\mathcal{F}$, in the sense that each flag in one set is transverse to each flag in the other. It follows from [DKL19, Lem. 4.24] that one can then find a neighborhood $U \subset \mathcal{F}$ of $\{(\ell^{\pm}, \pi^{\pm})\}$ so that the sets U and $\Lambda_1 \cup \Lambda_2$ remain antipodal in $\mathcal{F}$.

Up to replacing h with one of its powers, we have that $h\Lambda_2 \subset U$. Since $h\Lambda_2$ is the limit set of $h\Gamma_2 h^{-1}$ in $\mathcal{F}$, it follows from [DKL19, Thm. 1.3] that there are finite-index normal subgroups Γ_i' of Γ_i so that $\langle \Gamma_i', h\Gamma_2' h^{-1} \rangle < \mathsf{SL}_{n+1}(\mathbb{R})$ is naturally isomorphic to $\Gamma_1' * \Gamma_2'$ and is P_1-Anosov in $\mathsf{SL}_{n+1}(\mathbb{R})$.

Let $\Gamma_0 < \Gamma_1 * \Gamma_2$ be the intersection of the kernels of the compositions $\Gamma_1 * \Gamma_2 \to \Gamma_i \to \Gamma_i/\Gamma_i'$. Then Γ_0 is of finite index in $\Gamma_1 * \Gamma_2$ and is isomorphic to a group of the form

$$\Gamma_1' * \cdots * \Gamma_1' * \Gamma_2' * \cdots * \Gamma_2' * \mathbb{Z} * \cdots * \mathbb{Z}. \tag{2}$$

Since the Γ_i' are both infinite, any group of the above form embeds as a quasiconvex subgroup of the Gromov-hyperbolic group $\Gamma_1' * \Gamma_2'$; indeed, for any $\gamma_i \in \Gamma_i'$, $i = 1, 2$, of infinite order, the subgroup

$$\left\langle \gamma_2 \Gamma_1' \gamma_2^{-1}, \ldots, \gamma_2^r \Gamma_1' \gamma_2^{-r}, \gamma_1 \Gamma_2' \gamma_1, \ldots, \gamma_1^s \Gamma_2' \gamma_1^{-s}, \gamma_1^{s+1} \gamma_2 \gamma_1^{-(s+1)}, \ldots, \gamma_1^{s+q} \gamma_2 \gamma_1^{-(s+q)} \right\rangle$$

of $\Gamma_1' * \Gamma_2'$ is quasiconvex and is naturally isomorphic to a free product of the form (2) (that we may find γ_i of infinite order in Γ_i follows from the fact that the Γ_i are infinite finitely generated linear groups, for instance). Since we have already found a P_1-Anosov embedding of the latter into a special linear group, we conclude that Γ_0 also admits such a representation, and hence so does the finite-index supergroup $\Gamma_1 * \Gamma_2$ by Lemma 2.1. □

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