# DENSITY OF SYSTOLES OF HYPERBOLIC MANIFOLDS

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Abstract. We show that for each $n \geq 2$, the systoles of closed hyperbolic $n$-manifolds form a dense subset of $(0, +\infty)$. We also show that for any $n \geq 2$ and any Salem number $\lambda$, there is a closed arithmetic hyperbolic $n$-manifold of systole $\log(\lambda)$. In particular, the Salem conjecture holds if and only if the systoles of closed arithmetic hyperbolic manifolds in some (any) dimension fail to be dense in $(0, +\infty)$.

The *systole* $\mathrm{sys}(M)$ of a closed nonpositively curved Riemannian manifold $M$ is the length of a shortest closed geodesic in $M$. It follows for instance from residual finiteness of surface groups that there is a sequence of closed hyperbolic surfaces with systole going to infinity. Since for any $\epsilon > 0$ and any closed surface $M$ of negative Euler characteristic there is a hyperbolic metric on $M$ of systole $< \epsilon$, continuity of the systole function on moduli space then yields for any $L > 0$ a closed hyperbolic surface of systole precisely $L$.

By Mostow rigidity [16], the set of systoles of closed hyperbolic $n$-manifolds is countable for $n \geq 3$. Nevertheless, we show the following.

**Theorem 1.** *For any $n \geq 2$, the set of systoles of closed hyperbolic $n$-manifolds is dense in* $(0, +\infty)$.

That 0 is an accumulation point of systoles of closed hyperbolic 3-manifolds follows for instance from Thurston's hyperbolic Dehn filling theory (see [4, Sections E.5 and E.6]). The analogous fact for 4-manifolds was established by Agol [1], whose strategy was to "inbreed" arithmetic manifolds along totally geodesic hypersurfaces. This strategy was successfully extended to arbitrary dimensions independently by Belolipetsky–Thomson [3] and Bergeron–Haglund–Wise [6]. The present note is an elaboration on the utility of this inbreeding technique.

We also establish a variant of Theorem 1 for arithmetic hyperbolic manifolds. A *Salem number* is a real algebraic integer $\lambda > 1$ of degree $\geq 4$ that is Galois-conjugate to $\lambda^{-1}$ and all of whose remaining Galois conjugates lie on the unit circle.

**Theorem 2.** *Let $\lambda > 1$ be a Salem number. Then for any $n \geq 2$, there is a closed arithmetic hyperbolic $n$-manifold of systole precisely* $\log(\lambda)$.

The manifolds we exhibit in the proof of Theorem 2 are arithmetic of simplest type (the latter being the only arithmetic construction that applies in every dimension) and are moreover *classical* in the language of Emery–Ratcliffe–Tschantz [10]. For $n \geq 4$, it follows from Meyer's theorem on quadratic forms and [10, Thm. 5.2] that the exponential lengths of closed geodesics in any classical simplest-type closed arithmetic hyperbolic $n$-manifold are Salem numbers.[1]

[1] Exponential geodesic lengths in closed arithmetic hyperbolic surfaces or 3-manifolds can be *quadratic* "Salem numbers"; the latter are sometimes included in the definition of a Salem number. Exponential geodesic lengths in odd-dimensional closed arithmetic hyperbolic orbifolds need not

Lehmer's conjecture for Salem numbers, or the *Salem conjecture* for short, asserts that the Salem numbers are bounded away from 1; see [13, Section 13] and [18, p. 31]. In light of the relationship between Salem numbers and lengths of geodesics in arithmetic hyperbolic manifolds, Theorem 2 yields the following reformulation of the Salem conjecture.

**Corollary 2.1.** *The Salem conjecture holds if and only if, for some, equivalently, any, $n \geq 2$, the set of systoles of closed arithmetic hyperbolic $n$-manifolds fails to be dense in* $(0, +\infty)$.

*Proof.* It is known generally that, given any irreducible symmetric space $X$ of noncompact type, the Salem conjecture implies a uniform lower bound on systoles of closed arithmetic locally symmetric manifolds modeled on $X$; see Frączyk–Pham [11]. On the other hand, since positive powers of Salem numbers remain Salem numbers, failure of the Salem conjecture would imply density of Salem numbers in $(1, +\infty)$, and hence density of systoles of closed arithmetic hyperbolic $n$-manifolds in $(0, +\infty)$ for each $n \geq 2$ by Theorem 2. □

We motivate the proofs of Theorems 1 and 2 with the following remark.

*Remark* 3. We provide an argument along the lines of [1] that, given any $L > 0$, there is a closed hyperbolic surface of systole precisely $L$. Indeed, it suffices to find a closed hyperbolic surface $M_L$ of systole $\geq L$ possessing two disjoint 2-sided simple closed geodesics $\Sigma_1$ and $\Sigma_2$ and an orthogeodesic segment $\omega$ joining the $\Sigma_i$ of length $L/2$ and whose length is minimal among all orthogeodesic segments with endpoints on $\Sigma_1 \cup \Sigma_2$. Cutting $M_L$ along the $\Sigma_i$ and then doubling the resulting surface along its boundary yields a (possibly disconnected) closed hyperbolic surface of systole precisely $L$.

One can construct such a surface $M_L$ as follows. Let $P \subset \mathbb{H}^2$ be a right-angled pentagon with an edge $\widetilde{\omega}$ of length precisely $L/2$, and let $H_1, H_2 \subset \mathbb{H}^2$ be the walls of $P$ adjacent to $\widetilde{\omega}$. Let $\Gamma_P < \mathrm{Isom}(\mathbb{H}^2)$ be the group generated by the reflections in the walls of $P$. We can find a larger right-angled convex polygon $Q \subset \mathbb{H}^2$ that is a union of finitely many $\Gamma_P$-translates of $P$ such that the walls of $Q$ that enter the $\frac{L}{2}$-neighborhood of $H_i$ are orthogonal to $H_i$ for $i = 1, 2$ (this idea originates in work of Scott [19]; see also [2, Section 3.1]). By residual finiteness and virtual torsion-freeness of the group $\Gamma_Q < \mathrm{Isom}(\mathbb{H}^2)$ generated by the reflections in the walls of $Q$, there is a finite-index subgroup $\Lambda < \Gamma_Q$ such that $\Lambda \backslash \mathbb{H}^2$ is a surface of systole $\geq L$. We may now take $M_L = \Lambda \backslash \mathbb{H}^2$, the $\Sigma_i$ to be the projections to $M_L$ of the $H_i$, and $\omega$ to be the projection of $\widetilde{\omega}$.

*Proof of Theorem 1.* Given $L, \epsilon > 0$, we exhibit a closed hyperbolic $n$-manifold $M$ with $|\mathrm{sys}(M) - L| < \epsilon$. As in Remark 3, it suffices to find a closed hyperbolic $n$-manifold $M_{L,\epsilon}$ of systole $\geq L + \epsilon$ possessing two disjoint 2-sided properly embedded totally geodesic hypersurfaces $\Sigma_1$ and $\Sigma_2$, and an orthogeodesic segment $\omega$ joining the $\Sigma_i$ of length within $\epsilon/2$ from $L/2$ and whose length is minimal among all orthogeodesic segments with endpoints on $\Sigma_1 \cup \Sigma_2$. We can then cut $M_{L,\epsilon}$ along $\Sigma_1 \cup \Sigma_2$, select the component $N$ containing $\omega$ of the resulting manifold with boundary, and define $M$ to be the double of $N$. The double of $\omega$ is indeed a shortest closed geodesic in $M$ of length within $\epsilon$ from $L$.

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be Salem numbers even in this more general sense; see Lemma 4.10 and Theorem 4.11 in [17], as well as [10, Thm. 7.7].

To that end, let $k = \mathbb{Q}(\sqrt{2})$ and $\mathcal{O}_k = \mathbb{Z}[\sqrt{2}]$. Define $f$ to be the quadratic form on $\mathbb{R}^{n+1}$ given by

$$f(x_1, \cdots, x_{n+1}) = x_1^2 + \cdots + x_n^2 - \sqrt{2}x_{n+1}^2.$$

We identify the level set $\{x \in \mathbb{R}^{n+1} | f(x) = -1, x_{n+1} > 0\}$ with $n$-dimensional hyperbolic space $\mathbb{H}^n$ and $\mathrm{O}'(f;\mathbb{R})$ with $\mathrm{Isom}(\mathbb{H}^n)$, where $\mathrm{O}'(f;\mathbb{R})$ is the index-2 subgroup of $\mathrm{O}(f;\mathbb{R})$ preserving $\mathbb{H}^n$. By the Borel–Harish-Chandra theorem [7], we have that $\Gamma := \mathrm{O}'(f;\mathcal{O}_k)$ is a uniform arithmetic lattice of $\mathrm{Isom}(\mathbb{H}^n)$.

Denote by $H_1$ the hyperplane $\{x_1 = 0\}$ in $\mathbb{H}^n$. Since $\mathrm{O}'(f;k)$ is dense in $\mathrm{O}'(f;\mathbb{R})$, there is an element $g \in \mathrm{O}'(f;k)$ such that

$$\left| \mathrm{dist}_{\mathbb{H}^n}(H_1, gH_1) - \frac{L}{2} \right| < \frac{\epsilon}{2}.$$

Let $\widetilde{\omega}$ be the orthogeodesic segment in $\mathbb{H}^n$ connecting $H_1$ and $H_2 := gH_1$. By the proof of Lemma 3.1 in Belolipetsky–Thomson [3], we may now pass to a non-zero ideal $I \subset \mathcal{O}_k$ such that $\mathrm{dist}_{\mathbb{H}^n}(H_1, \gamma H_2) \geq \mathrm{dist}_{\mathbb{H}^n}(H_1, H_2)$ for each $\gamma$ in the principal congruence subgroup $\Gamma(I) < \Gamma$ of level $I$. Up to diminishing the ideal $I$, we can further assume that

(1) the principal congruence subgroup $\Gamma(I)$ is torsion-free, so that $M_{L,\epsilon} := \Gamma(I)\backslash\mathbb{H}^n$ is a manifold;
(2) we have $\mathrm{sys}(M_{L,\epsilon}) \geq L + \epsilon$;
(3) for $i = 1, 2$ and $\gamma \in \Gamma(I)$, either $\gamma H_i = H_i$, in which case $\gamma$ furthermore preserves each side of $H_i$, or $\mathrm{dist}_{\mathbb{H}^n}(H_i, \gamma H_i) \geq \mathrm{dist}_{\mathbb{H}^n}(H_1, H_2)$; in particular, the projections $\Sigma_i$ of the $H_i$ to $M_{L,\epsilon}$ are 2-sided and properly *embedded*.

Note that item (2) can be arranged by residual finiteness of the ring $\mathcal{O}_k$, whereas item (3) can be ensured by the fact that the stabilizer in $\Gamma$ of either side of $H_i$ is an intersection of congruence subgroups of $\Gamma$ for $i = 1, 2$; see [14] and [5, Lemme principal]. We now have that the $\Sigma_i$ are disjoint and that any orthogeodesic segment in $M_{L,\epsilon}$ with endpoints on $\Sigma_1 \cup \Sigma_2$ has length at least that of the projection $\omega$ of $\widetilde{\omega}$ to $M_{L,\epsilon}$. Thus, the manifold $M_{L,\epsilon}$, the hypersurfaces $\Sigma_i$, and the orthogeodesic segment $\omega$ are as desired. □

*Remark* 4. The proof of Theorem 1 demonstrates that we may take the closed hyperbolic manifolds giving rise to a dense set of systoles in $(0, +\infty)$ to be quasi-arithmetic in the sense of Vinberg and all share the same adjoint trace field (indeed, the same Vinberg ambient algebraic group); see [20] and compare [8, Thm. 1.3]. On the other hand, by varying the form $f$ in the proof, one easily produces a family of pairwise incommensurable closed hyperbolic manifolds in each dimension whose systoles remain dense in $(0, +\infty)$.

In fact, it is possible to achieve the latter without varying $f$; one can select from any family of closed hyperbolic manifolds whose systoles are dense in $(0, +\infty)$ pairwise incommensurable manifolds whose systoles remain dense in $(0, +\infty)$. Indeed, given a closed hyperbolic manifold $M$, the set of all geodesic lengths of manifolds commensurable to $M$ is closed and discrete in $[0, +\infty)$. When $M$ is nonarithmetic, the latter follows from Margulis's arithmeticity criterion [15, Thm. IX.6.5], and when $M$ is arithmetic, from the fact that, for fixed $D > 0$ and $\mu > 1$, there are only finitely many monic integer polynomials of degree $\leq D$ and Mahler measure $\leq \mu$; see the discussion immediately following Conjecture 10.2 in [12].

*Remark* 5. Instead of appealing to the work of Belolipetsky–Thomson (loc. cit.) to find the manifold $M_{L,\epsilon}$ in the proof of Theorem 1, we could have instead used [6, Cor. 1.12] as in the proof of Theorem 4 in [9]. An interesting feature of the former approach is that it afforded us a manifold $M_{L,\epsilon}$ that is *congruence* arithmetic.

*Proof of Theorem 2.* Let $\mu = \lambda + \lambda^{-1}$. Let $k = \mathbb{Q}(\mu)$, and $\mathcal{O}_k$ be the ring of integers of $k$. Following the proof of [10, Thm. 6.3], we define the form $f$ in $n+1$ variables over $k$ to be that given by the symmetric matrix

$$\begin{pmatrix} 1 & & \mu/2 \\ & I_{n-1} & \\ \mu/2 & & 1 \end{pmatrix}.$$

The form $f$ is of signature $(n,1)$, so that we may again identify $\mathbb{H}^n$ with one of the two sheets of the hyperboloid $\{x \in \mathbb{R}^{n+1} | f(x) = -1\}$ and $\mathrm{Isom}(\mathbb{H}^n)$ with $\mathrm{O}'(f;\mathbb{R})$, where $\mathrm{O}'(f;\mathbb{R})$ is the index-2 subgroup of $\mathrm{O}(f;\mathbb{R})$ preserving $\mathbb{H}^n$. Moreover, the field $k$ is totally real, and for any embedding $\sigma : k \to \mathbb{R}$ other than the identity (of which at least one exists), the form $f^\sigma$ is positive definite. It follows that $\Gamma := \mathrm{O}'(f;\mathcal{O}_k)$ is a uniform arithmetic lattice of $\mathrm{O}'(f;\mathbb{R}) = \mathrm{Isom}(\mathbb{H}^n)$, which contains the reflections

$$\tau_1 := \begin{pmatrix} -1 & & -\mu \\ & I_{n-1} & \\ & & 1 \end{pmatrix}, \quad \tau_2 := \begin{pmatrix} -\mu & & 1-\mu^2 \\ & I_{n-1} & \\ 1 & & \mu \end{pmatrix}$$

whose respective fixed hyperplanes $H_1, H_2 \subset \mathbb{H}^n$ satisfy

$$\mathrm{dist}_{\mathbb{H}^n}(H_1, H_2) = \log(\lambda)/2.$$

As in the proof of Theorem 1, by the aforementioned work of Belolipetsky–Thomson (loc. cit.), we may pass to a non-zero ideal $I \subset \mathcal{O}_k$ such that

- the principal congruence subgroup $\Gamma(I)$ is torsion-free, so that $M_{\log(\lambda)} := \Gamma(I)\backslash\mathbb{H}^n$ is a manifold;
- we have $\mathrm{sys}(M_{\log(\lambda)}) \geq \log(\lambda)$;
- the $H_i$ project to disjoint properly embedded 2-sided totally geodesic hypersurfaces $\Sigma_1$ and $\Sigma_2$ in $M_{\log(\lambda)}$;
- any orthogeodesic segment in $M_{\log(\lambda)}$ with endpoints on $\Sigma_1 \cup \Sigma_2$ has length $\geq \log(\lambda)/2$.

Now cutting $M_{\log(\lambda)}$ along the $\Sigma_i$ and then doubling the resulting manifold along its boundary yields a (possibly disconnected) closed manifold $M$ of systole precisely $\log(\lambda)$ each of whose components is arithmetic (since $\tau_i \in \Gamma$ for $i = 1, 2$). □

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