A NOVEL WITNESS TO INCOHERENCE OF $\mathrm{SL}_5(\mathbb{Z})$

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Abstract. Motivated by a question of Stover, we discuss an example of a Zariski-dense finitely generated subgroup of $\mathrm{SL}_5(\mathbb{Z})$ that is not finitely presented.

A group Γ is *coherent* if all its finitely generated subgroups are finitely presented. Otherwise, we say Γ is *incoherent*. In this note, we will concern ourselves with the case that Γ is a discrete subgroup of a noncompact simple Lie group. For a broader discussion on coherence, we refer the reader to Wise's survey [29].

Since finitely generated fundamental groups of (possibly open) surfaces are finitely presented, all Fuchsian groups are coherent. That the former statement holds also for 3-manifolds is due independently to Scott [23, 22] and Shalen (unpublished); in particular, all Kleinian groups are in fact coherent. On the other hand, the product $F_2 \times F_2$, where F_2 is a free group of rank two, admits a map onto $\mathbb{Z}$ whose kernel is finitely generated but (necessarily) not finitely presented [29, Example 9.22]. It follows that $\mathrm{SL}_n(\mathbb{Z})$ is incoherent for $n \geq 4$, since the latter possesses a block-diagonal copy of $F_2 \times F_2$. Whether or not $\mathrm{SL}_3(\mathbb{Z})$ is coherent is a question of Serre [28, Problem F14] and remains open.

While a simple Lie group G of real rank one possesses no discrete copies of $F_2 \times F_2$, many (conjecturally, all) lattices in G are incoherent as soon as G is not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$; see Kapovich–Potyagailo–Vinberg [16] and Kapovich [14]. We remark that, when $n \geq 5$, another way of certifying that $\mathrm{SL}_n(\mathbb{Z})$ is incoherent is to observe that $\mathrm{SL}_n(\mathbb{Z})$ contains a copy of the incoherent group $\mathrm{SO}(4,1;\mathbb{Z})$; indeed, as observed in [16], the first construction of a geometrically finite incoherent subgroup of $\mathrm{SO}(4,1)$, due to Kapovich and Potyagailo [15], can be carried out within $\mathrm{SO}(4,1;\mathbb{Z})$.

We call a finitely generated but not finitely presented subgroup of a group Γ a *witness to incoherence of* Γ. A question posed by Stover [24] that is not addressed by the above discussion is whether there are *Zariski-dense* witnesses to incoherence of $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 4$. Recall that a subgroup $\Delta < \mathrm{SL}_n(\mathbb{R})$ is *Zariski-dense* if every real polynomial in the matrix entries that vanishes on Δ in fact vanishes on all of $\mathrm{SL}_n(\mathbb{R})$. Since the subgroup $\mathrm{SO}(n-1,1)$ of $\mathrm{SL}_n(\mathbb{R})$ is defined by polynomial equations in the matrix entries, subgroups of $\mathrm{SO}(n-1,1)$ are not Zariski-dense in $\mathrm{SL}_n(\mathbb{R})$. Moreover, it follows from simplicity of $\mathrm{SL}_n(\mathbb{R})$ that no copy whatsoever of $F_2 \times F_2$ in $\mathrm{SL}_n(\mathbb{R})$ is Zariski-dense (note that the standard block-diagonal copies of $F_2 \times F_2$ in $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 4$, fail to be irreducible, let alone Zariski-dense).

It was suggested to the author by K. Tsouvalas that, if one allows freely decomposable examples, then Stover's question has an affirmative answer at least for $n \geq 5$. Indeed, a ping-pong argument in the projective space $\mathbb{P}(\mathbb{R}^n)$ demonstrates that $\mathrm{SL}_n(\mathbb{Z})$ contains for each $n \geq 5$ a subgroup decomposing as $\Gamma_0 * F$, where $\Gamma_0 < \mathrm{SO}(4,1;\mathbb{Z})$ is a geometrically finite incoherent group arising from the

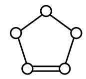

FIGURE 1. The diagram Σ.

Kapovich–Potyagailo construction and F is a Zariski-dense Anosov free subgroup of $\mathrm{SL}_n(\mathbb{Z})$ [11]. For any witness $\Delta_0 < \Gamma_0$ to incoherence of Γ_0, the subgroup $\langle \Delta_0, F\rangle < \mathrm{SL}_n(\mathbb{Z})$ is then a Zariski-dense witness to incoherence of $\mathrm{SL}_n(\mathbb{Z})$. The purpose of this note is to describe a Zariski-dense witness to incoherence of $\mathrm{SL}_5(\mathbb{Z})$ of a different nature.

Theorem 1. *There is a Zariski-dense witness to incoherence of* $\mathrm{SL}_5(\mathbb{Z})$ *of cohomological dimension* 3 *and whose limit set in* $\mathbb{P}(\mathbb{R}^5)$ *in the sense of Guivarc'h* [12] *is connected.*

It also seems likely that the witness described below is one-ended, but we have not yet succeeded in demonstrating this.

Proof. Let W be the Coxeter group given by the diagram Σ shown in Figure 1. The Cartan matrix A of Σ is

$$A = \begin{pmatrix} 2 & -\sqrt{2} & 0 & 0 & -1 \\ -\sqrt{2} & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Since A has signature $(4,1)$, the Coxeter group W can be realized as a reflection group in $\mathrm{Isom}(\mathbb{H}^4)$, and this realization is unique up to conjugation, so that we may conflate W with its image in $\mathrm{Isom}(\mathbb{H}^4)$; see Vinberg [26, Thm. 2.1]. Moreover, since all proper induced subdiagrams of Σ are elliptic, we have that each fundamental chamber for W in $\mathbb{H}^4$ is a compact Coxeter simplex $P \subset \mathbb{H}^4$.

Remark 2. The compact hyperbolic Coxeter simplices were enumerated by Lannér [19]. The simplex P is one among only five such simplices in dimension 4, the highest dimension in which such simplices exist, and, crucially for us, is the only one among those with the property that, for each of its dihedral angles θ, the quantity $4\cos^2\theta$ is an integer. We remark also that the Galois conjugate A^σ of A is positive-definite, where $\sigma : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ is the unique nontrivial automorphism, so that W is in fact an arithmetic subgroup of $\mathrm{Isom}(\mathbb{H}^4)$.

While it is unclear to us whether the simplex P tiles a compact right-angled polyhedron in $\mathbb{H}^4$, it is nevertheless true that some finite-index subgroup of W abstractly embeds in a right-angled Coxeter group. Indeed, the latter remains true when W is replaced with an arbitrary finitely generated Coxeter group by work of Haglund and Wise [13]; alternatively, one can exploit arithmeticity of W (see Remark 2) and apply a result of Bergeron, Haglund, and Wise [7]. In particular, the group W is virtually residually finite rationally solvable (RFRS) in the sense of Agol [1], and hence, by a theorem of Kielak [18], possesses a torsion-free finite-index subgroup $\Lambda < W$, which we may assume preserves orientation, that maps onto $\mathbb{Z}$ with finitely generated kernel $\Delta \lhd \Lambda$. That Δ is not finitely presented is an application of the following lemma to $M = \Lambda\backslash\mathbb{H}^4$ and $\tilde{M} = \Delta\backslash\mathbb{H}^4$; for a more general statement, see Llosa Isenrich, Martelli, and Py [20, Prop. 14]. Note that $\chi(\Lambda\backslash\mathbb{H}^4) > 0$ by the Chern–Gauss–Bonnet theorem.

Lemma 3. *Let M be a closed connected oriented aspherical 4-manifold with $\chi(M) \neq 0$. Then $\pi_1(\tilde{M})$ is not finitely presented for any infinite cyclic cover $\tilde{M}$ of M.*

Proof. By a result of Milnor [21], there is some $i \in \mathbb{N}$ such that $H_i(\tilde{M};\mathbb{Q})$ has infinite dimension. Since $\tilde{M}$ is an open 4-manifold, we have $b_i(\tilde{M}) = 0$ for $i > 3$. Now if $\pi_1(\tilde{M})$ is not finitely generated, then we are done. Otherwise, the dimension of $H_1(\tilde{M};\mathbb{Q})$ is finite, and hence, by the partial duality theorem of Kawauchi [17], we have $H_3(\tilde{M};\mathbb{Q}) = \mathbb{Q}$. This forces the dimension of $H_2(\tilde{M};\mathbb{Q})$ to be infinite, so that $\pi_1(\tilde{M})$ cannot be finitely presented. □

Now let $\rho : W \to \mathrm{GL}_5(\mathbb{R})$ be the representation given by

$$\rho(s_i)(v) = v - (v^T A' e_i) e_i$$

for each $v \in \mathbb{R}^5$, where $s_1, \ldots, s_5 \in W$ are the standard generators of W, and

$$A' = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -2 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Note that since A' has integer entries, in fact $\rho(W) \subset \mathrm{GL}_5(\mathbb{Z})$. The representation ρ is the Vinberg representation of W with Cartan matrix A'; since A' is of negative type, we have in particular that ρ is faithful and preserves a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^5)$; see Vinberg [25]. As in the hyperbolic setting, since all proper induced subdiagrams of Σ are elliptic, we have that $\rho(W)$ acts cocompactly on Ω (in the language of Benzécri [6], the group $\rho(W)$ *divides* Ω). The same is then true of $\rho(\Lambda)$. Since the matrix A' is not symmetric, the domain Ω is not an ellipsoid.[1] However, since W is Gromov-hyperbolic, we nevertheless have that Ω is strictly convex by work of Benoist [5]. Another theorem of Benoist [3, Thm. 3.6] (see also [4]) then asserts that $\rho(\Lambda)$ is Zariski-dense in $\mathrm{SL}_5(\mathbb{R})$. By simplicity of $\mathrm{SL}_5(\mathbb{R})$, the infinite normal subgroup $\rho(\Delta) \lhd \rho(\Lambda)$ remains Zariski-dense in $\mathrm{SL}_5(\mathbb{R})$. We conclude that $\rho(\Delta)$ is a Zariski-dense witness to incoherence of $\mathrm{SL}_5(\mathbb{Z})$.

That Δ has cohomological dimension at least 3 follows from the fact that Δ is the kernel of an epimorphism $\Lambda \to \mathbb{Z}$. On the other hand, the cohomological dimension of Δ is at most 3 since Δ is not a uniform lattice in $\mathrm{Isom}(\mathbb{H}^4)$. A reference for both of these facts is Brown [8, Ch. VIII].

Finally, we claim that the limit set of $\rho(\Delta)$ in $\mathbb{P}(\mathbb{R}^5)$ is connected, indeed, is the boundary $\partial\Omega$ of the properly convex domain Ω. This is an immediate consequence of the following two facts, both due to Benoist; see [3, Lem. 3.8] and [2, Lem. 3.6.ii], respectively. On the one hand, the limit set in $\mathbb{P}(\mathbb{R}^n)$ of a discrete subgroup of $\mathrm{SL}_n(\mathbb{R})$ dividing a strictly convex domain in $\mathbb{P}(\mathbb{R}^n)$ is the boundary of that domain, so that $\partial\Omega$ is the limit set in $\mathbb{P}(\mathbb{R}^5)$ of the larger group $\rho(\Lambda)$. On the other, the limit set in $\mathbb{P}(\mathbb{R}^n)$ of a Zariski-dense subgroup $\Gamma < \mathrm{SL}_n(\mathbb{R})$ is the smallest

[1] In fact, no subgroup of $\mathrm{SL}_n(\mathbb{Z})$ divides an ellipsoid in $\mathbb{P}(\mathbb{R}^n)$ for $n \geq 5$. Indeed, such a subgroup $\Gamma < \mathrm{SL}_n(\mathbb{Z})$ would have to be of finite index in $\mathrm{SO}(Q;\mathbb{Z})$ for some rational quadratic form Q in $n \geq 5$ variables, but such Q is isotropic by Meyer's theorem, so that Γ cannot be cocompact in $\mathrm{SO}(Q;\mathbb{R})$.

nonempty closed Γ-invariant subset of $\mathbb{P}(\mathbb{R}^n)$, so that the limit sets in $\mathbb{P}(\mathbb{R}^5)$ of $\rho(\Lambda)$ and its normal subgroup $\rho(\Delta)$ coincide. □

Remark 4. The representation ρ of W also appears in Choi and Choi [9, Prop. 11], and is analogous to the triangle group representations into $\mathrm{GL}_3(\mathbb{Z})$ discovered by Kac and Vinberg [27].

Remark 5. The above strategy will not produce a Zariski-dense witness to incoherence of $\mathrm{SL}_4(\mathbb{Z})$, since any finitely generated subgroup of $\mathrm{GL}_4(\mathbb{R})$ preserving and acting properly on a domain in $\mathbb{P}(\mathbb{R}^4)$—for instance, any Vinberg reflection group in $\mathrm{GL}_4(\mathbb{R})$, hence any Coxeter group on at most 4 vertices[2]—is virtually the fundamental group of a 3-manifold and is thus coherent. However, it seems plausible that suitable Vinberg representations of incoherent Coxeter groups might yield one-ended Zariski-dense witnesses to incoherence of $\mathrm{SL}_n(\mathbb{Z})$ for arbitrarily large n. The approach that appears most promising in this regard is to consider Vinberg representations of finite-index reflection subgroups $W_n < W$ of increasing rank, where W is a fixed Coxeter group that virtually maps onto $\mathbb{Z}$ with finitely generated but not finitely presented kernel. However, such representations will not divide projective domains in high dimensions, so different methods of certifying Zariski-density would be required.

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[2]That a Coxeter group on at most 4 vertices is virtually a 3-manifold group can also be deduced, by an argument along the lines of [10, Example 1.2], from the fact that any non-elliptic Coxeter system on at most 4 vertices has planar nerve.

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