GEOMETRIC AND ARITHMETIC PROPERTIES OF LÖBELL POLYHEDRA

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Abstract. The Löbell polyhedra form an infinite family of compact right-angled hyperbolic polyhedra in dimension 3. We observe, through both elementary and more conceptual means, that the "systoles" of the Löbell polyhedra approach 0, so that these polyhedra give rise to particularly straightforward examples of closed hyperbolic 3-manifolds with arbitrarily small systole, and constitute an infinite family even up to commensurability. By computing number theoretic invariants of these polyhedra, we refine the latter result, and also determine precisely which of the Löbell polyhedra are quasi-arithmetic.

1. Introduction

For each $n \geqslant 5$, consider a combinatorial 3-polyhedron whose "top" and "bottom" faces are n-gons and whose "lateral" surface consists of $2n$ pentagons (see Figure 1 and Figure 3, left, for illustrations of the case $n = 6$). By Andreev's theorem [3, 26], this abstract polyhedron can be realized as a compact right-angled polyhedron L_n in $\mathbb{H}^3$ for any $n \geqslant 5$. For instance, the polyhedron L_5 is the right-angled hyperbolic dodecahedron.

The polyhedra L_n are called *Löbell polyhedra*, after F. R. Löbell, who constructed the first example [18] of a closed oriented hyperbolic 3-manifold by gluing eight copies of the polyhedron L_6. The subgroup $\Gamma_n < \mathrm{Isom}(\mathbb{H}^3)$ generated by the reflections in the faces of L_n is a cocompact right-angled reflection group; we refer to the quotients $\mathbb{H}^3/\Gamma_n$ as *Löbell orbifolds.* Such an orbifold $\mathbb{H}^3/\Gamma_n$ can be visualized as the polyhedron L_n itself with reflective singularities in its faces. The example of Löbell is a particular instance of a general construction of degree-8 manifold covers of right-angled reflection 3-orbifolds; see [29] or [32, Section 3.1].

Much is understood about Löbell polyhedra. In particular, Vesnin [31] computed their volumes $\mathrm{vol}(L_n)$, and showed that $\mathrm{vol}(L_n)$ is an increasing function of n. Moreover, Mednykh and Vesnin [21] showed that the distance δ_n between the top and bottom faces of the Löbell polyhedron L_n satisfies

$$\cosh \delta_n = \frac{\cos\left(\frac{\pi}{n}\right)}{\cos\left(\frac{2\pi}{n}\right)}.$$

The following is an elementary consequence of the above identity.

Observation 1.1. *The distance δ_n between the top and bottom faces of the Löbell polyhedron L_n approaches 0 as $n \to \infty$.*

Since the systole of a compact right-angled reflection orbifold is twice the minimal distance between two nonadjacent walls (see Section 3.4 for the definition of the systole of a hyperbolic orbifold), and by an application of a collar lemma as in [6]

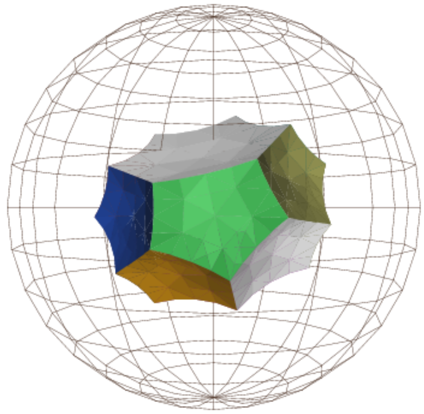

FIGURE 1. The Löbell polyhedron L_6, displayed in the Poincaré ball model of $\mathbb{H}^3$ using Geomview.

to the lateral faces of L_n, whose areas are fixed, the following is immediate from Observation 1.1.

Corollary 1.1.1. *For sufficiently large n, the systole of the Löbell orbifold L_n is precisely $2\delta_n$.*

Observation 1.1 also yields the following.

Corollary 1.1.2. *Let M_n be manifold covers of the Löbell orbifolds L_n of uniformly bounded degree (for instance, the degree-8 manifold covers discussed above). Then the systole of M_n approaches 0 as $n \to \infty$.*

Indeed, for n a multiple of 3, we provide an explicit degree-8 oriented manifold cover of L_n whose systole is precisely $2\delta_n$ for sufficiently large n; see Figure 2. We thus obtain straightforward examples of closed hyperbolic 3-manifolds with arbitrarily small systole. Such manifolds were known to exist in dimension 3 by Thurston's hyperbolic Dehn filling theorem (see, for instance, [8, Sections E.5 and E.6]). In fact, the Löbell polyhedron L_n decomposes into $2n$ copies of a polyhedron T_n (see Section 3.1) which may be viewed as a Dehn filling of a cusped polyhedron T_∞, so that Observation 1.1 ultimately also follows from a Dehn filling argument; see Section 4.

Agol [1] provided another construction that, given an input arithmetic lattice $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$ and $\varepsilon > 0$, outputs a closed hyperbolic 3-manifold with systole at most ε. Agol originally suggested this construction in dimension 4 (where the problem had theretofore been open), but it evidently also applies in lower dimensions, and in fact applies in all dimensions by work of Belolipetsky and Thomson [7] (alternatively, by a result of Bergeron, Haglund, and Wise [9, Theorem 1.4]). While the output lattices are nonarithmetic for fixed[1] Γ and sufficiently small ϵ, they are nevertheless all quasi-arithmetic, as observed by Thomson [28]. On the other hand, the reflection lattices Γ_n are eventually not quasi-arithmetic.

[1]Conjecturally, there is no dependence on Γ; see the discussion following Theorem 4.3 in [1], and [7, Section 5.1].

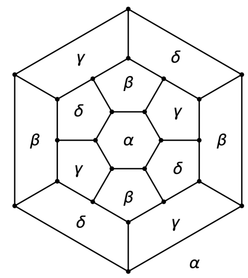

FIGURE 2. The above 4-coloring of the Löbell polyhedron L_6 and its analogues for the polyhedra L_{3k}, $k \geq 2$, determine (orientable) degree-8 manifold covers M_{3k} as in [32, Section 3.1]. Since the top and bottom faces of L_{3k} receive the same color, the manifold M_{3k} contains a closed geodesic of length $2\delta_{3k}$. For sufficiently large k, the systole of M_{3k} is in fact precisely $2\delta_{3k}$ by Corollary 1.1.1.

Theorem 1.2. *The Löbell polyhedron L_n is quasi-arithmetic if and only if $n = 5, 6, 8, 12$, and is properly quasi-arithmetic only when $n = 12$.*

Recall that a lattice is said to be *properly quasi-arithmetic* if it is quasi-arithmetic but not arithmetic. See Section 2.2 for definitions. We remark that it is unclear to us whether Theorem 1.2 exemplifies a more general phenomenon. More precisely, it appears that the following question is open.

Question 1.3. *Are only finitely many Dehn fillings of a complete finite-volume noncompact hyperbolic 3-orbifold quasi-arithmetic?*

The answer to Question 1.3 is known to be affirmative if one replaces "quasi-arithmetic" with "arithmetic"; see Maclachlan and Reid [19, Cor. 11.2.2].

Returning to our discussion on Löbell polyhedra, Vesnin [30] observed that the Löbell polyhedron L_n is nonarithmetic for $n \neq 5, 6, 7, 8, 10, 12, 18$. Antolín-Camarena, Maloney, and Roeder [5] later showed that L_n is arithmetic if and only if $n = 5, 6, 8$.

Our proof of Theorem 1.2 is straightforward and uses only classical tools from Vinberg's theory of hyperbolic reflection groups. Along the way, we compute the adjoint trace fields of the lattices Γ_n.

Theorem 1.4. *The adjoint trace field k_n of Γ_n is $\mathbb{Q}\left(\cos \frac{2\pi}{n}\right)$. In particular, if $p, q \geq 5$ are distinct primes, then the Löbell polyhedra L_p and L_q are incommensurable.*

It is shown in [19, Section 4.7.3] that there are infinitely many pairwise incommensurable compact Coxeter polyhedra in $\mathbb{H}^3$. However, we could not find in the literature a justification of the existence of infinitely many pairwise incommensurable *right-angled* such polyhedra. Indeed, this was our initial motivation for considering the Löbell polyhedra. Since $\deg(k_n) = \frac{\phi(n)}{2} \to \infty$ as $n \to \infty$, where ϕ is Euler's totient function, one can in fact conclude from Theorem 1.4 that there is no infinite subsequence of Γ_n consisting entirely of pairwise commensurable lattices. This fact is indeed already implied by Observation 1.1; see Remark 3.3.

It is worth mentioning that the existence of infinitely many pairwise incommensurable *noncompact* finite-volume right-angled polyhedra in $\mathbb{H}^3$ was already known.

For instance, Meyer–Millichap–Trapp [22] and Kellerhals [16] showed that the *ideal* right-angled antiprisms A_n provide a sequence of pairwise incommensurable reflection groups with the same trace fields as those of the Γ_n. There is in fact more to be said about the relationship between these two families of right-angled polyhedra: indeed, as observed by Kolpakov [17, Section 5.1], the Löbell polyhedra L_n can be viewed as having been obtained from the antiprisms A_n via Dehn filling; see Section 4. Another way of phrasing this is that the antiprisms are obtained from Löbell polyhedra by "contracting" certain edges to ideal vertices. We explain in Section 4 how this trick of contracting edges of a finite-volume right-angled polyhedron is a general method for constructing infinitely many commensurability classes of such polyhedra in dimension 3, and also why an analogous trick is not available in dimension 4 (Theorem 4.2), where the existence of such an infinite family appears to be open.

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2. Preliminaries

2.1. **Hyperbolic lattices.** Let $\mathbb{R}^{d,1}$ be the real vector space $\mathbb{R}^{d+1}$ equipped with the standard quadratic form f of signature $(d, 1)$, namely,

$$f(x) = -x_0^2 + x_1^2 + \cdots + x_d^2.$$

The hyperboloid $\mathcal{H} = \{x \in \mathbb{R}^{d,1} \mid f(x) = -1\}$ has two connected components

$$\mathcal{H}^+ = \{x \in \mathcal{H} \mid x_0 > 0\} \text{ and } \mathcal{H}^- = \{x \in \mathcal{H} \mid x_0 < 0\}.$$

The d-dimensional hyperbolic space $\mathbb{H}^d$ is the manifold $\mathcal{H}^+$ with the Riemannian metric ρ induced by restricting f to each tangent space $T_p(\mathcal{H}^+)$, $p \in \mathcal{H}^+$. This hyperbolic metric ρ satisfies $\cosh \rho(x, y) = -(x, y)$, where (x, y) is the scalar product in $\mathbb{R}^{d,1}$ associated to f. The hyperbolic d-space $\mathbb{H}^d$ is known to be the unique simply connected complete Riemannian d-manifold with constant sectional curvature -1. *Hyperplanes* of $\mathbb{H}^d$ are intersections of linear hyperplanes of $\mathbb{R}^{d,1}$ with $\mathcal{H}^+$, and are totally geodesic submanifolds of codimension 1 in $\mathbb{H}^d$.

Let $\mathrm{O}_{d,1} = \mathbf{O}(f, \mathbb{R})$ be the orthogonal group of the form f, and $\mathrm{O}'_{d,1} < \mathrm{O}_{d,1}$ be the subgroup (of index 2) preserving $\mathcal{H}^+$. The group $\mathrm{O}'_{d,1}$ preserves the metric ρ on $\mathbb{H}^d$, and is in fact the full group $\mathrm{Isom}(\mathbb{H}^d)$ of isometries of the latter.

If $\Gamma < \mathrm{O}'_{d,1}$ is a lattice, i.e., if Γ is a discrete subgroup of $\mathrm{O}'_{d,1}$ with a finite-volume fundamental domain in $\mathbb{H}^d$, then the quotient $M = \mathbb{H}^d/\Gamma$ is a complete finite-volume *hyperbolic orbifold*. If Γ is torsion-free, then M is a complete finite-volume Riemannian manifold, and is called a *hyperbolic manifold*.

Now set $G = \mathrm{O}'_{d,1}$, and suppose $\mathbf{G}$ is an admissible (for G) algebraic k-group, i.e. $\mathbf{G}(\mathbb{R})^o$ is isomorphic to G^o and $\mathbf{G}^\sigma(\mathbb{R})$ is a compact group for any non-identity embedding $\sigma\colon k \hookrightarrow \mathbb{R}$. Then any subgroup $\Gamma < G$ commensurable with the image in G of $\mathbf{G}(\mathcal{O}_k)$ is an *arithmetic lattice* (in G) with *ground field* k.

Since G also admits non-arithmetic lattices, we discuss some weaker notions of arithmeticity for lattices in G. Following Vinberg [33], a lattice $\Gamma < G$ is called *quasi-arithmetic* with *ground field* k if some finite-index subgroup of Γ is contained in the image in G of $\mathbf{G}(k)$, where $\mathbf{G}$ is some admissible algebraic k-group, and is called *properly quasi-arithmetic* if Γ is quasi-arithmetic, but not arithmetic on the nose.

It is worth stressing that the notion of quasi-arithmeticity is indeed broader than that of arithmeticity; as was mentioned in the introduction, the nonarithmetic closed hyperbolic manifolds constructed by Agol [1] and Belolipetsky–Thomson [7] exist in all dimensions and, as observed by Thomson [28], are quasi-arithmetic. The first examples of properly quasi-arithmetic lattices in dimensions 3, 4, and 5 were constructed by Vinberg [33] via reflection groups.

2.2. **Convex polyhedra and arithmetic properties of hyperbolic reflection groups.** A *(hyperbolic) reflection group* is a discrete subgroup of $\mathrm{O}'_{d,1}$ generated by reflections in hyperplanes. The fixed hyperplanes of the reflections in a finite-covolume reflection group $\Gamma < \mathrm{O}'_{d,1}$ divide $\mathbb{H}^d$ into isometric copies of a single finite-volume convex polyhedron $P \subset \mathbb{H}^d$. The polyhedron P is a *Coxeter polyhedron*, that is, a finite-sided convex polyhedron in which the dihedral angle between any two adjacent facets is an integral submultiple of π. We say P is a *fundamental chamber* for Γ. Conversely, given a finite-volume Coxeter polyhedron $P \subset \mathbb{H}^d$, the group generated by the reflections in all the supporting hyperplanes, or *walls*, of P is a finite-covolume reflection group $\Gamma < \mathrm{O}'_{d,1}$ with fundamental chamber P. We thus frequently conflate finite-volume Coxeter polyhedra in $\mathbb{H}^d$ with their corresponding lattices in $\mathrm{O}'_{d,1}$ (or their corresponding hyperbolic orbifolds).

Let $H_e = \{x \in \mathbb{H}^d \mid (x, e) = 0\}$ be a hyperplane in $\mathbb{H}^d \subset \mathbb{R}^{d,1}$ whose linear span in $\mathbb{R}^{d,1}$ has normal vector $e \in \mathbb{R}^{d,1}$ with $(e, e) = 1$, and $H_e^- = \{x \in \mathbb{H}^d \mid (x, e) \leq 0\}$ be the half-space associated with it. If

$$P = \bigcap_{j=1}^{N} H_{e_j}^-$$

is a Coxeter polyhedron in $\mathbb{H}^d$, then the matrix $G(P) = \{g_{ij}\}_{i,j=1}^N = \{(e_i, e_j)\}_{i,j=1}^N$ is its *Gram matrix*. We write $K(P) = \mathbb{Q}\left(\{g_{ij}\}_{i,j=1}^N\right)$ and denote by $k(P)$ the field generated by all possible cyclic products of the entries of $G(P)$; we call the field $k(P)$ the *ground field* of P. For convenience, the set of all cyclic products of entries of a given matrix $A = (a_{ij})_{i,j=1}^N$, i.e., the set of all possible products of the form $a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k i_1}$, will be denoted by $\mathrm{Cyc}(A)$. Thus, we have $k(P) = \mathbb{Q}(\mathrm{Cyc}(G(P))) \subset K(P)$.

The following criterion allows us to determine if a given finite-covolume hyperbolic reflection group Γ with fundamental chamber P is arithmetic, quasi-arithmetic, or neither.

Theorem 2.1 (Vinberg's arithmeticity criterion [33])**.** *Let $\Gamma < \mathrm{O}'_{d,1}$ be a reflection group with finite-volume fundamental chamber $P \subset \mathbb{H}^d$. Then Γ is arithmetic if and only if each of the following conditions holds:*

- **(V1)** *$K(P)$ is a totally real algebraic number field;*
- **(V2)** *for any embedding $\sigma\colon K(P) \to \mathbb{R}$, such that $\sigma|_{k(P)} \neq \mathrm{Id}$, the matrix $G^\sigma(P)$ is positive semi-definite;*
- **(V3)** $\mathrm{Cyc}(2 \cdot G(P)) \subset \mathcal{O}_{k(P)}$,

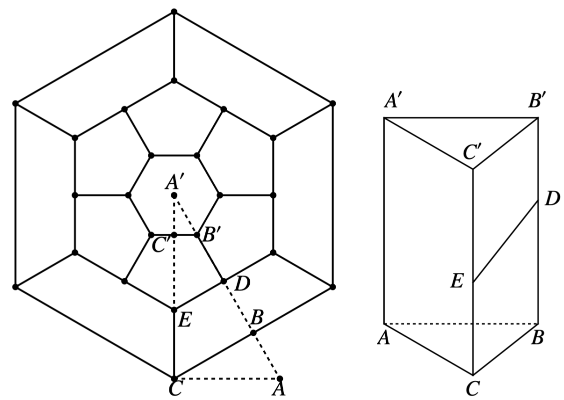

FIGURE 3. The Löbell polyhedron L_6 and its slice, the truncated tetrahedron T_6. We use the analogous edge labeling for all the truncated tetrahedra T_n.

and, in this case, the ground field of Γ *is* $k(P)$. *The group* Γ *is quasi-arithmetic if and only if it satisfies conditions* **(V1)**–**(V2)**, *but not necessarily* **(V3)**, *and, in this case, the ground field of* Γ *is again* $k(P)$.

Remark 2.2. Note that $2\cos\frac{\pi}{n}$ is always an algebraic integer. Thus, if there are no dashed edges in the Coxeter–Vinberg diagram of a finite-volume Coxeter polyhedron P, then condition **(V3)** above automatically holds, and there is no distinction between arithmeticity and quasi-arithmeticity for the associated reflection group Γ. In particular, a triangle group acting on $\mathbb{H}^2$ is quasi-arithmetic precisely when it is arithmetic.

Remark 2.3. Work of Vinberg [34, Section 4] implies that the ground field of a finite-covolume hyperbolic reflection group coincides with its adjoint trace field, and is thus a commensurability invariant.

3. Geometry and arithmetic of Löbell orbifolds

3.1. **A decomposition of L_n.** For any $n \geqslant 5$ the Löbell polyhedron L_n admits a decomposition into $2n$ isometric "slices" T_n, each of which may be regarded as a twice-truncated tetrahedron; this decomposition is illustrated in Figure 3 for $n = 6$, where the hyperbolic triangles ABC and $A'B'C'$ are exactly the results of these truncations.

The polyhedron T_n is a Coxeter polyhedron whose edges are labeled in Figure 3, right. The Coxeter–Vinberg diagram for T_n is given in Figure 4. The weight d_n in this diagram is equal to $\cosh\delta_n$ since the distance between the top and bottom faces of L_n is the same as that for T_n; note that δ_n is also equal to the length of the edge AA'.

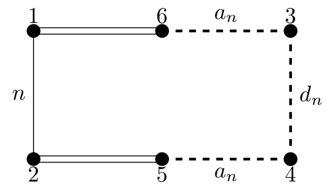

FIGURE 4. The Coxeter–Vinberg diagram of the polytope T_n

The Gram matrix of T_n is

$$G_n := G(T_n) = \begin{pmatrix} 1 & -\cos\left(\frac{\pi}{n}\right) & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2} \\ -\cos\left(\frac{\pi}{n}\right) & 1 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 & -d_n & 0 & -a_n \\ 0 & 0 & -d_n & 1 & -a_n & 0 \\ 0 & -\frac{1}{2}\sqrt{2} & 0 & -a_n & 1 & 0 \\ -\frac{1}{2}\sqrt{2} & 0 & -a_n & 0 & 0 & 1 \end{pmatrix}$$

We know that the signature[2] of G_n is $(3, 1, 2)$, since $T_n \subset \mathbb{H}^3$ is compact. Therefore, using the fact $\det G_n$ and all 5×5 principal minors of G_n vanish, we obtain that $d_n^2 = (2a_n^2 - 1)(a_n^2 - 1)$. This allows us to compute d_n and a_n:

$$d_n = \cosh \delta_n = \frac{\cos \frac{\pi}{n}}{\cos \frac{2\pi}{n}}; \quad a_n = \sqrt{1 + \frac{1}{2\cos \frac{2\pi}{n}}}.$$

3.2. **Proof of Theorem 1.4.** By Remark 2.3, we have that the adjoint trace field k_n of the lattice Γ_n coincides with the ground field of the polyhedron T_n for $n \geq 5$. Our computations in Section 3.1 allow us to determine the cyclic products $C_n := \mathrm{Cyc}(G_n)$ using the Coxeter–Vinberg diagram shown in Figure 4. We have

$$\mathbb{Q}(C_n) = \mathbb{Q}\left(\left\{\cos\frac{2\pi}{n}; \quad 1 + \frac{1}{2\cos\frac{2\pi}{n}}; \quad \frac{\cos^2\frac{\pi}{n}}{\cos^2\frac{2\pi}{n}}; \quad \frac{\cos^2\frac{\pi}{n}}{\cos\frac{2\pi}{n}}\left(1 + \frac{1}{2\cos\frac{2\pi}{n}}\right)\right\}\right).$$

Thus

$$k_n = \mathbb{Q}(C_n) = \mathbb{Q}\left(\cos\frac{2\pi}{n}\right).$$

Since $\deg(k_n) = \frac{\phi(n)}{2}$, where ϕ is Euler's totient function, we have that for distinct primes p and q the fields k_p and k_q have different degrees, so that the polyhedra L_p and L_q are not commensurable. ■

3.3. **Proof of Theorem 1.2.** The reflection group Γ_n is commensurable with the group Λ_n generated by reflections in the walls of T_n. It thus remains to check quasi-arithmeticity of Λ_n using Vinberg's arithmeticity criterion (see Theorem 2.1).

The main result in [10] implies in particular that any face of a quasi-arithmetic hyperbolic Coxeter 3-polyhedron that is itself a Coxeter polygon is also quasi-arithmetic with the same ground field. Note that if some face F of a Coxeter 3-polyhedron meets its adjacent faces at even angles, i.e. angles of the form $\frac{\pi}{2m}$ for some $m \geq 1$, then F is a Coxeter polygon. It is shown in [10] that, in the latter case, if P is moreover arithmetic, then F is arithmetic as well.

[2]The *signature* of a real symmetric matrix A is the triple (p, q, r) of numbers of positive, negative, and zero eigenvalues of A, respectively.

$n=5$	$k_5=\mathbb{Q}\left(\cos\frac{2\pi}{5}\right)=\mathbb{Q}(\sqrt{5})$	$d_5=2$	$G_5^{\sigma_\ell}\geqslant 0$ for all $\ell\neq 1$
$n=6$	$k_6=\mathbb{Q}$	$d_6=1$	no nonidentity embeddings
$n=7$	$k_7=\mathbb{Q}\left(\cos\frac{2\pi}{7}\right)$	$d_7=3$	$G_7^{\sigma_2}$ has signature $(3,1)$
$n=8$	$k_8=\mathbb{Q}(\sqrt{2})$	$d_8=2$	$G_8^{\sigma_\ell}\geqslant 0$ for all $\ell\neq 1$
$n=10$	$k_{10}=\mathbb{Q}\left(\cos\frac{\pi}{5}\right)=\mathbb{Q}(\sqrt{5})$	$d_{10}=2$	$G_{10}^{\sigma_3}$ has signature $(3,1)$
$n=12$	$k_{12}=\mathbb{Q}\left(\cos\frac{\pi}{6}\right)=\mathbb{Q}(\sqrt{3})$	$d_{12}=2$	$G_{12}^{\sigma_\ell}\geqslant 0$ for all $\ell\neq 1$
$n=18$	$k_{18}=\mathbb{Q}\left(\cos\frac{2\pi}{18}\right)$	$d_{18}=3$	$G_{18}^{\sigma_5}$ has signature $(3,1)$

TABLE 1. Properties of the Gram matrices G_n under the nonidentity embeddings σ_ℓ of the fields k_n for $n=5,6,7,8,10,12,18$.

The polyhedron T_n has a $(2,4,n)$-triangular face orthogonal to all adjacent faces. For the $(2,4,n)$-triangle group, arithmeticity is equivalent to quasi-arithmeticity (see Remark 2.2). Takeuchi [27] showed that these triangle groups are arithmetic only for $n=5,6,7,8,10,12,18$. Thus, by the previous paragraph, we have that Λ_n is not quasi-arithmetic for n outside these values (and hence neither is Γ_n). It then suffices to check the conditions of Vinberg's criterion for the Gram matrix of the Coxeter polyhedron T_n for n within these values.

Denote by σ_ℓ the embeddings of the totally real number field $k_n=\mathbb{Q}\left(\cos\frac{2\pi}{n}\right)$, enumerated as follows:

$$\sigma_\ell\left(\cos\frac{2\pi}{n}\right)=\cos\frac{2\pi\ell}{n},\quad (\ell,n)=1,\quad 1\le\ell<n.$$

Note that $\sigma_\ell(k_n)=\mathbb{Q}\left(\cos\frac{2\pi\ell}{n}\right)$.

We see from Table 1, which is a result of computations made in Sage, that Λ_n is quasi-arithmetic only for $n=5,6,8,12$. In order to show that Λ_n is properly quasi-arithmetic if and only if $n=12$, one needs to check the condition **(V3)** of Theorem 2.1, that is, that $\mathrm{Cyc}(2G_n)\subset\mathcal{O}_{k_n}$ if and only if $n=5,6,8$. This can easily be done even without a computer. To avoid being redundant, we record here only the (most interesting) case $n=12$. Indeed, the other cases were already verified in [5].

Lemma 3.1. *The reflection group Λ_{12} is properly quasi-arithmetic.*

Proof. Notice that $\cos(\frac{\pi}{12})=\frac{\sqrt{6}+\sqrt{2}}{4}$. The Gram matrix G_{12} of T_{12} is

$$\begin{pmatrix} 1 & -\frac{\sqrt{2}(1+\sqrt{3})}{4} & 0 & 0 & 0 & -\frac{1}{2}\sqrt{2}\\ -\frac{\sqrt{2}(1+\sqrt{3})}{4} & 1 & 0 & 0 & -\frac{1}{2}\sqrt{2} & 0\\ 0 & 0 & 1 & -\frac{\sqrt{2}(3+\sqrt{3})}{6} & 0 & -\frac{\sqrt{\sqrt{3}+3}}{\sqrt{3}}\\ 0 & 0 & -\frac{\sqrt{2}(3+\sqrt{3})}{6} & 1 & -\frac{\sqrt{\sqrt{3}+3}}{\sqrt{3}} & 0\\ 0 & -\frac{1}{2}\sqrt{2} & 0 & -\frac{\sqrt{\sqrt{3}+3}}{\sqrt{3}} & 1 & 0\\ -\frac{1}{2}\sqrt{2} & 0 & -\frac{\sqrt{\sqrt{3}+3}}{\sqrt{3}} & 0 & 0 & 1 \end{pmatrix}$$

We have that $k_{12}=\mathbb{Q}(\sqrt{3})$ and

$$K_{12}:=K(T_{12})=\mathbb{Q}\left(\sqrt{2},\sqrt{3},\sqrt{3+\sqrt{3}}\right)=\mathbb{Q}\left(\sqrt{2},\sqrt{3+\sqrt{3}}\right).$$

The latter field is of degree 8 with all embeddings given by

$$\sqrt{2} \to \pm\sqrt{2}, \quad \sqrt{3+\sqrt{3}} \to \pm\sqrt{3 \pm \sqrt{3}}.$$

We verified via Sage that for each[3] embedding $\sigma : K_{12} \to \mathbb{R}$ such that $\sigma \mid_{k_{12}} \neq 1$, the matrix G_{12}^{σ} has signature $(4, 0, 2)$.

Finally, we observe that not all cyclic products of $2 \cdot G_{12}$ are algebraic integers. For instance, we have that $\left(2\frac{\sqrt{\sqrt{3}+3}}{\sqrt{3}}\right)^2 = \frac{4(3+\sqrt{3})}{3} \notin \mathbb{Z}[\sqrt{3}] = \mathcal{O}_{k_{12}}$. ■

3.4. **Systoles of Löbell orbifolds.** For a lattice $\Gamma < \mathrm{Isom}(\mathbb{H}^d)$, the *systole* $\mathrm{sys}(\Gamma)$ of Γ is the minimal translation length of a loxodromic element of Γ. The *systole* $\mathrm{sys}(M)$ of the complete finite-volume hyperbolic orbifold $M = \mathbb{H}^d/\Gamma$ is simply the systole of the lattice Γ. We record in this section a couple of remarks about systoles of hyperbolic reflection orbifolds.

Remark 3.2. Suppose we have a sequence of reflection groups $\Gamma_n < \mathrm{Isom}(\mathbb{H}^d)$, $d \geq 2$, with the property that $\mathrm{sys}(\Gamma_n) \to 0$. Then Γ_n is arithmetic for at most finitely many n. Indeed, suppose otherwise, so that we may assume the Γ_n are all arithmetic. Since there are only finitely many maximal arithmetic reflection groups (see [24], [2], and [12]), up to further extraction, we may also assume the Γ_n are all contained in a single lattice $\Gamma < \mathrm{Isom}(\mathbb{H}^d)$, so that $\mathrm{sys}(\Gamma_n) \geq \mathrm{sys}(\Gamma)$, a contradiction.

Remark 3.3. It follows from Remark 3.2 that if $\Gamma_n < \mathrm{Isom}(\mathbb{H}^d)$, $d \geq 2$, is a sequence of finite-covolume reflection groups satisfying $\mathrm{sys}(\Gamma_n) \to 0$, then for each $m \in \mathbb{N}$, we have that Γ_m is commensurable to Γ_n for at most finitely many n. Indeed, suppose otherwise. Then, up to passing to a subsequence, we may assume that the Γ_n are all commensurable. Since the Γ_n are nonarithmetic by Remark 3.2, it follows from a result of Margulis [20, Theorem 1, page 2] that their commensurator $\Lambda < \mathrm{Isom}(\mathbb{H}^d)$ contains each Γ_n as a finite-index subgroup, so that $\mathrm{sys}(\Gamma_n) \geq \mathrm{sys}(\Lambda)$, a contradiction.

As observed in [7, Sections 5.2 and 5.3], the above conclusion in fact holds for any sequence of lattices $\Gamma_n < \mathrm{Isom}(\mathbb{H}^d)$ satisfying $\mathrm{sys}(\Gamma_n) \to 0$. Indeed, suppose one has such a sequence Γ_n where the Γ_n are all commensurable. If the Γ_n are nonarithmetic, then one obtains a contradiction as in the previous paragraph. If the Γ_n are arithmetic, then since they are commensurable, a uniform lower bound for $\mathrm{sys}(\Gamma_n)$ is provided by the fact that, given $d \in \mathbb{N}$ and $\mu > 1$, there are only finitely many monic integer polynomials of degree d and Mahler measure at most μ. We remark that the strongest form of Lehmer's conjecture would imply a uniform lower bound on the systole of any arithmetic locally symmetric orbifold; see the discussion immediately following Conjecture 10.2 in Gelander [13].

[3]In fact, since the semi-definiteness property can be checked via principal minors (and since these minors are computed using cyclic products contained in $k_{12} = \mathbb{Q}(\sqrt{3})$), it suffices to consider only a single embedding $\sigma\colon K_{12} \to \mathbb{R}$ mapping $\sqrt{3}$ to $-\sqrt{3}$.

4. Löbell orbifolds and hyperbolic Dehn fillings of ideal right-angled antiprisms

Let $P \subset \mathbb{H}^3$ be a finite-volume Coxeter polyhedron with a compact edge e, and say the dihedral angle at e is $\frac{\pi}{m}$. Following Kolpakov [17], if $Q \subset \mathbb{H}^3$ is a finite-volume Coxeter polyhedron of the same combinatorial type and with the same dihedral angles as P except that the dihedral angle at e in Q is diminished to $\frac{\pi}{n}$ for some $n \geq m$, we say that Q is obtained from P via a *$\frac{\pi}{n}$-contraction* at e. For example, for $n \geq 5$ and $k \geq 2$, the polyhedron $P_{n,k} \subset \mathbb{H}^3$ whose Coxeter–Vinberg diagram is shown in Figure 5 is obtained from the truncated tetrahedron T_n via $\frac{\pi}{2k}$-contractions at the (analogues of the) edges $B'D$ and EC, while T_n is in turn obtained from T_5 via a $\frac{\pi}{n}$-contraction at the edge AA'; see Figures 3 and 4.

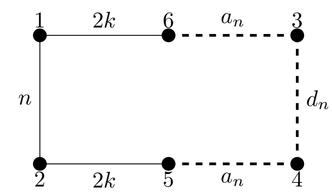

Figure 5. The Coxeter–Vinberg diagram of the polyhedron $P_{n,k}$

If instead Q and P differ (as labeled polyhedra) only in that, in Q, the edge e is replaced by an ideal vertex $v \in \partial\mathbb{H}^3$, then we say that Q is obtained from P by *contracting e to an ideal vertex*. Note that, if such a contraction exists, the dihedral angle at each edge adjacent to e in P (and each edge incident to v in Q) is $\frac{\pi}{2}$, and the faces sharing e in P are at least 4-sided.

In [16], Kellerhals studies a family of ideal right-angled polyhedra known as the antiprisms $A_n \subset \mathbb{H}^3$, $n \geq 3$, where she exploits a decomposition of each such polyhedron A_n into $2n$ copies of a polyhedron R_n, analogous to the decomposition of the Löbell polyhedron L_n into $2n$ copies of T_n; see Figure 6. In the above language, for $n \geq 5$, the polyhedron R_n is in fact obtained from T_n by contracting the edges $B'D$ and EC of T_n to ideal vertices. In particular, for such n, the antiprism A_n may be obtained from the Löbell polyhedron L_n by a sequence of edge contractions to ideal vertices. This was already observed by Kolpakov [17, Section 5.1]. Indeed,

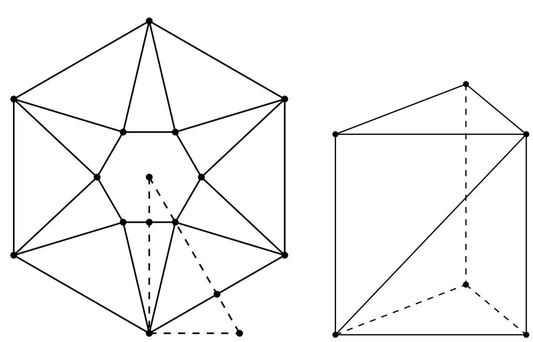

Figure 6. The antiprism A_6 and its slice R_6.

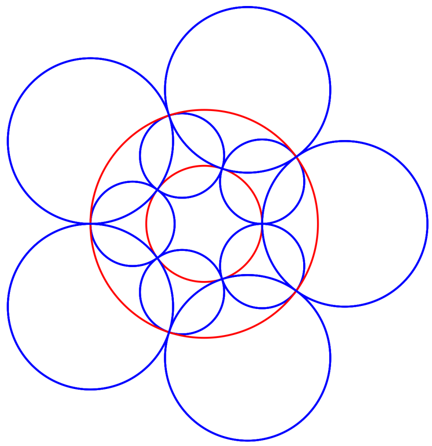

FIGURE 7. A visual proof that the distance between the "top" and "bottom" faces of the antiprism A_n approaches 0. Drawn above are the ideal boundaries of the walls of A_n for $n = 5$, 8, and 12 (from left to right), visualized via stereographic projection onto the page from the ideal boundary of $\mathbb{H}^3$ such that the circles corresponding to the top and bottom faces, indicated here in red, are concentric. Keeping fixed the Euclidean diameter of the inner red circle, we have that as n approaches ∞, the diameter of an inner blue circle approaches 0, so that the ratio between the diameters of the inner and outer red circles approaches 1. In fact, an elementary exercise in Euclidean geometry shows that this ratio is precisely $\frac{1+\sin(\frac{\pi}{n})}{\cos(\frac{\pi}{n})}$, corroborating a computation of Kellerhals [16, Examples 1 & 2].

the existence of the polyhedra $P_{n,k}$ and R_n, $n \geq 5$, $k \geq 2$, is predicted by the following result of Kolpakov, whose proof rests on Andreev's theorem [3, 26].

Theorem 4.1 (c.f. the proof of Prop. 1 in [17])**.** *Let $P \subset \mathbb{H}^3$ be a finite-volume Coxeter polyhedron with at least 5 faces and a compact edge e such that the dihedral angle at each edge adjacent to e is $\frac{\pi}{2}$, and the faces sharing e in P are at least 4-sided. If the dihedral angle at e in P is $\frac{\pi}{m}$, $m \geq 2$, then P admits a $\frac{\pi}{n}$-contraction P_n at e for each $n \geq m$, as well as a contraction P_∞ of e to an ideal vertex.*

Since we may instead view the P_n as having been obtained from P_∞ via Dehn filling[4], it follows from work of Dunbar and Meyerhoff [11] that the length of the (analogue of the) edge $e \subset P_n$ approaches 0 as $n \to \infty$. Taking P_n to be the truncated tetrahedron T_n and e the edge AA', we obtain another justification for Observation 1.1. Applying a similar argument to the R_n instead of the T_n, one concludes that the distance between the "top" and "bottom" faces of the antiprism

[4]More precisely, Dunbar and Meyerhoff adapt Thurston's theory of hyperbolic Dehn fillings to the setting of *oriented* hyperbolic 3-orbifolds. Applying this theory to the orientation covers of the P_n, where the P_n are viewed as reflection orbifolds, one concludes that the length of the edge e in P_n approaches 0 as $n \to \infty$.

A_n also approaches 0 as $n \to \infty$; for a justification of the latter that uses only elementary Euclidean geometry, see Figure 7.

We remark that the conditions of Theorem 4.1 are satisfied for any compact edge e of a finite-volume right-angled polyhedron $P \subset \mathbb{H}^3$. This suggests a method of constructing a family of new finite-volume right-angled polyhedra starting from the polyhedron P. Namely, for $n \geq 2$, let $P_n \subset \mathbb{H}^3$ be the polyhedron obtained from P by a $\frac{\pi}{n}$-contraction at e, let $\Lambda_n < \mathrm{Isom}(\mathbb{H}^3)$ be the associated reflection group, and let Δ_n be the stabilizer of e in Λ_n. Then the finite-volume polyhedron $Q_n = \bigcup_{\gamma \in \Delta_n} \gamma P_n$ is right-angled, and is compact if and only if P was. Moreover, for each $m \geq 2$, we have that Λ_m is commensurable to Λ_n for at most finitely many $n \geq 2$, for instance, because $\mathrm{sys}(\Lambda_n) \to 0$ as $n \to \infty$ (see Remark 3.3). We conclude our discussion by observing that this strategy for producing an infinite family of pairwise incommensurable finite-volume right-angled hyperbolic polyhedra fails in dimension 4, where the existence of such a family appears to be open. The observation is essentially that codimension-2 contractions in dimensions higher than 3 yield orbifolds à la Gromov–Thurston [15].

Theorem 4.2. *Let $n, m \geq 3$ be of the same parity and $d \geq 4$. Suppose $P_n \subset \mathbb{H}^d$ is a finite-volume Coxeter polyhedron all of whose dihedral angles are right angles except for a single dihedral angle of $\frac{\pi}{n}$. Suppose there is also a finite-volume Coxeter polyhedron $P_m \subset \mathbb{H}^d$ with the same combinatorics and dihedral angles as P_n except that the exceptional dihedral angle of P_m is $\frac{\pi}{m}$. Then $n = m$.*

We will make use of the following lemma.

Lemma 4.3. *Let P be a finite-volume Coxeter polyhedron in $\mathbb{H}^d$, $d \geq 4$, and suppose all dihedral angles of P are right angles except possibly for one dihedral angle formed by walls H_1 and H_2 of P. Then the group $\Gamma < \mathrm{Isom}(\mathbb{H}^d)$ generated by the reflections in all walls of P except H_1 and H_2 is Zariski-dense in $\mathrm{Isom}(\mathbb{H}^d)$.*

Proof of Lemma 4.3. Let $P' \subset \mathbb{H}^d$ be the (infinite-volume) polyhedron obtained from P by forgetting the walls H_1 and H_2, and let P_i be the intersection of P' with the hyperplane H_i of $\mathbb{H}^d$ for $i = 1, 2$. Then the $(d-1)$-dimensional right-angled hyperbolic polyhedron P_i is obtained from a finite-volume such polyhedron—namely, the polyhedron $P \cap H_i$—by forgetting a single wall—namely, the intersection $H_1 \cap H_2$. Since $d - 1 \geq 3$, it follows that the subgroup of $\mathrm{Isom}(H_i)$ generated by the reflections in the walls of P_i is Zariski-dense in $\mathrm{Isom}(H_i)$; see, for instance, [14, Section 1.7]. The lemma follows. ∎

Proof of Theorem 4.2. For $k = n, m$, let H_k and H_k' be the walls of P_k forming the exceptional dihedral angle of $\frac{\pi}{k}$, and let R_k be the union of the images of P_k under the reflection group D_k generated by the reflections in H_k and H_k'. Since n and m have the same parity, we may choose reflections $r_k \in D_k$ such that the $(d-1)$-dimensional polyhedra $\mathrm{Fix}(r_n) \cap R_n$ and $\mathrm{Fix}(r_m) \cap R_m$ have the same combinatorics and dihedral angles and are thus isometric by Mostow–Prasad rigidity [23, 25] (see also Andreev [3, 4]).

Now interbreed the R_k along $\mathrm{Fix}(r_k) \cap R_k$ and let R be the resulting finite-volume polyhedron in $\mathbb{H}^d$. Then there is an obvious dihedral group $D_{\frac{n+m}{2}}$ of combinatorial symmetries of R preserving dihedral angles. By Mostow–Prasad rigidity, each of these combinatorial symmetries is a hyperbolic isometry. However, by Lemma 4.3, there is (up to inverses) a unique hyperbolic isometry γ_k rotating the surface of any given slice of R_k two slices over, namely, the composition of the reflections in H_k and H'_k, so that γ_k has order k. Since γ_n and γ_m each generate the commutator subgroup of $D_{\frac{n+m}{2}}$, it follows that $n = m$. ■

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