ON REGULAR SUBGROUPS OF $\mathsf{SL}_3(\mathbb{R})$

SAMI DOUBA AND KONSTANTINOS TSOUVALAS

Abstract. Motivated by a question of M. Kapovich, we show that the $\mathbb{Z}^2$ subgroups of $\mathsf{SL}_3(\mathbb{R})$ that are *regular* in the language of Kapovich–Leeb–Porti, or *divergent* in the sense of Guichard–Wienhard, are precisely the lattices in minimal horospherical subgroups. This rules out any relative Anosov subgroups of $\mathsf{SL}_3(\mathbb{R})$ that are not in fact Gromov-hyperbolic. By work of Oh, it also follows that a Zariski-dense discrete subgroup Γ of $\mathsf{SL}_3(\mathbb{R})$ contains a regular $\mathbb{Z}^2$ if and only if Γ is commensurable to a conjugate of $\mathsf{SL}_3(\mathbb{Z})$. In particular, a Zariski-dense regular subgroup of $\mathsf{SL}_3(\mathbb{R})$ contains no $\mathbb{Z}^2$ subgroups.

1. Introduction

Our discussion is motivated by the following question of M. Kapovich, also considered by D. Long and A. Reid.

Question 1.1. ([8, Prob. 3.3]) *Is there a subgroup of $\mathsf{SL}_3(\mathbb{Z})$ isomorphic to $\mathbb{Z}^2 * \mathbb{Z}$?*

We remark that $\mathsf{SO}_{3,1}(\mathbb{Z})$ contains copies of $\mathbb{Z}^2 * \mathbb{Z}$, and hence so does $\mathsf{SL}_n(\mathbb{Z})$ for each $n \geqslant 4$. While we do not resolve Question 1.1, we establish the following.

Theorem 1.2. *There is no regular subgroup of $\mathsf{SL}_3(\mathbb{R})$ isomorphic to $\mathbb{Z}^2 * \mathbb{Z}$.*

Regularity (defined with respect to a parabolic subgroup) is a form of discreteness for subgroups of, or representations into, noncompact semisimple Lie groups that coincides with discreteness in the rank-one setting, but is strictly stronger in higher rank. These subgroups already appear in work of Benoist [1], and are the *divergent* subgroups of Guichard and Wienhard [7]; see Section 2 for the precise definition. For instance, the aforementioned copies of $\mathbb{Z}^2 * \mathbb{Z}$ in $\mathsf{SL}_n(\mathbb{Z})$ for $n \geqslant 4$ are regular, as are Anosov representations of Gromov-hyperbolic groups [12, 7] and, more generally, relative Anosov representations of relatively hyperbolic groups [9, 17]. On the other hand, a lattice in a Cartan subgroup of $\mathsf{SL}_3(\mathbb{R})$ is not regular. Indeed, we show that regular $\mathbb{Z}^2$ subgroups of $\mathsf{SL}_3(\mathbb{R})$ are of a very particular form. Recall that a (resp., *minimal, maximal*) *horospherical* subgroup of $\mathsf{SL}_3(\mathbb{R})$ is by definition the unipotent radical of a (resp., maximal, minimal) proper parabolic subgroup of the latter.

Theorem 1.3. *A representation $\rho : \mathbb{Z}^2 \to \mathsf{SL}_3(\mathbb{R})$ is regular if and only if $\rho(\mathbb{Z}^2)$ is a lattice in a minimal horospherical subgroup of $\mathsf{SL}_3(\mathbb{R})$.*

It follows from Theorem 1.3 and results[1] of Oh [13, 14] that any Zariski-dense discrete subgroup of $\mathsf{SL}_3(\mathbb{R})$ containing a regular $\mathbb{Z}^2$ is in fact commensurable to an $\mathsf{SL}_3(\mathbb{R})$-conjugate of $\mathsf{SL}_3(\mathbb{Z})$. Theorem 1.2 now follows since any discrete $\mathbb{Z}^2 * \mathbb{Z}$ in $\mathsf{SL}_3(\mathbb{R})$ is necessarily Zariski-dense, while $\mathbb{Z}^2 * \mathbb{Z}$ cannot be realized as a lattice in $\mathsf{SL}_3(\mathbb{R})$, for instance, because groups of the latter form

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[1] In greater detail, suppose that $\Gamma < \mathsf{SL}_3(\mathbb{R})$ is discrete and Zariski-dense, and that some minimal horospherical subgroup U of $\mathsf{SL}_3(\mathbb{R})$ is Γ-compact, where, following Oh [14], we say that a closed subgroup H of $\mathsf{SL}_3(\mathbb{R})$ is Γ-*compact* if $H/(H \cap \Gamma)$ is compact. Then Oh exhibits in [14, Prop. 4.1] a (Γ-compact) maximal horospherical subgroup V of $\mathsf{SL}_3(\mathbb{R})$ containing U such that the other minimal horospherical subgroup U' of $\mathsf{SL}_3(\mathbb{R})$ contained in V is also Γ-compact. By Zariski-density of Γ, there is some $\gamma \in \Gamma$ such that the Γ-compact minimal horospherical subgroups U and $\gamma U' \gamma^{-1}$ are opposite to one another [14, Prop. 3.6]. One now applies the main theorem of [13] to conclude that Γ is commensurable to a conjugate of $\mathsf{SL}_3(\mathbb{Z})$. See also Benoist's survey [2, Prop. 4.1].

enjoy Kazhdan's property (T) [11] (see also Furstenberg [6]). Moreover, as regularity is inherited by subgroups, and since $\mathsf{SL}_3(\mathbb{Z})$ contains a lattice in a Cartan subgroup of $\mathsf{SL}_3(\mathbb{R})$, we deduce the following.

Corollary 1.4. *A Zariski-dense regular subgroup of* $\mathsf{SL}_3(\mathbb{R})$ *contains no* $\mathbb{Z}^2$ *subgroups.*

We remark that if F is a lattice in a minimal horospherical subgroup of $\mathsf{SL}_3(\mathbb{R})$, then the limit set of F in the Furstenberg boundary of $\mathsf{SL}_3(\mathbb{R})$ is the set of all projective flags of the form (z,ℓ), where either the point $z \in \mathbb{P}(\mathbb{R}^3)$ is fixed and $\ell \subset \mathbb{P}(\mathbb{R}^3)$ varies among all projective lines in $\mathbb{P}(\mathbb{R}^3)$ containing z, or the projective line ℓ is fixed and z varies among all points of ℓ; for the precise notion of limit set used here, see Section 2. Thus, another consequence of Theorem 1.3 is that a relative Anosov subgroup Γ of $\mathsf{SL}_3(\mathbb{R})$ contains no $\mathbb{Z}^2$ subgroups, since the limit set of such Γ in the Furstenberg boundary of $\mathsf{SL}_3(\mathbb{R})$ is antipodal in the language of Kapovich–Leeb–Porti; see [9, 17].

It is known that a group admitting a relative Anosov representation is relatively hyperbolic with respect to a family of virtually nilpotent subgroups; see [9, 17]. Since polycyclic groups that lack $\mathbb{Z}^2$ subgroups are virtually cyclic, and since groups that are hyperbolic relative to virtually cyclic (more generally, hyperbolic) subgroups are themselves hyperbolic, we conclude the following from the previous paragraph.

Corollary 1.5. *Relative Anosov subgroups of* $\mathsf{SL}_3(\mathbb{R})$ *are Gromov-hyperbolic.*

In fact, in forthcoming work [16] of the second-named author with F. Zhu, Corollary 1.5 is used to prove the stronger statement that a relative Anosov subgroup of $\mathsf{SL}_3(\mathbb{R})$ is virtually a free group or a surface group.

The relevance of Theorem 1.3 to Question 1.1 is further explained by the following proposition.

Proposition 1.6. *Let* Γ *be a lattice in a real linear algebraic semisimple Lie group* G *of noncompact type and* P *be a proper parabolic subgroup of* G*. Assume that* P *is conjugate to its opposite. If* $\Delta < \Gamma$ *is* P*-regular in* G *and there is a point in* G/P *that is opposite to each point in the limit set of* Δ *in* G/P*, then for some infinite-order* $\gamma \in \Gamma$*, the subgroup* $\langle \Delta, \gamma \rangle < \Gamma$ *decomposes as* $\Delta * \langle \gamma \rangle$*.*

Thus, if there had been a regular $\mathbb{Z}^2$ in $\mathsf{SL}_3(\mathbb{Z})$ with "small" limit set in $\mathbb{P}(\mathbb{R}^3)$—a scenario that is ruled out by Theorem 1.3—then Proposition 1.6 would have furnished a $\mathbb{Z}^2 * \mathbb{Z}$ subgroup of $\mathsf{SL}_3(\mathbb{Z})$, and even a regular such subgroup by work of Dey and Kapovich [4, Thm. 3.2].

In light of Corollary 1.4, a result [3, Thm. 1.1] of the second-named author with R. Canary asserting that Anosov subgroups of $\mathsf{SL}_3(\mathbb{R})$ are virtually isomorphic to Fuchsian groups, and aforementioned forthcoming work of the second-named author with Zhu, the following question seems natural.

Question 1.7. *Is every regular Zariski-dense subgroup of* $\mathsf{SL}_3(\mathbb{R})$ *virtually isomorphic to a Fuchsian group?*

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2. Preliminaries

For two sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ of positive real numbers, we write $a_k \asymp b_k$ (resp., $a_k = O(b_k)$) if there is a constant $C > 1$ such that $C^{-1}a_k \leqslant b_k \leqslant Ca_k$ (resp., $a_k \leqslant Cb_k$) for every k.

Throughout this section, let G be a finite-center real semisimple Lie group with finitely many connected components and maximal compact subgroup $K < G$, and let $X = G/K$ be the associated symmetric space. Let P be a proper parabolic subgroup of G, so that P is the stabilizer in G of a point $z \in \partial_\infty X$, where $\partial_\infty X$ denotes the visual boundary of X. Pick a point $o \in X$, and let ξ be the geodesic ray in X emanating from o in the class of z. Fix also a Weyl chamber $\overline{\mathfrak{a}}^+ \subset X$ for G

in X with origin o containing the ray ξ. A sequence $(g_n)_{n\in\mathbb{N}}$ in G is *P-regular* if the vector-valued distances $d_{\overline{\mathfrak{a}}^+}(o, g_n o)$ diverge from each wall of $\overline{\mathfrak{a}}^+$ not containing ξ. This notion is independent of all the choices made after specifying the parabolic subgroup P. If Γ is a discrete group, a representation $\rho : \Gamma \to G$ is called *P-regular* if for every sequence $(\gamma_n)_{n\in\mathbb{N}}$ in Γ with $\gamma_n \to \infty$, the sequence $(\rho(\gamma_n))_{n\in\mathbb{N}}$ is P-regular. We remark that such a representation has finite kernel and discrete image, and is moreover P^{opp}-regular, where P^{opp} denotes a parabolic subgroup opposite to P. A subgroup Δ of G is called *P-regular* if the inclusion $\Delta \hookrightarrow G$ is P-regular. Notice that if a subgroup Γ of G is P-regular then so are all subgroups of Γ. For more background, see Kapovich–Leeb–Porti [10], as well as earlier work of Guichard–Wienhard [7] where the notion of P-regularity appears instead as *P-divergence*.

Example 2.1. (The case $G = \mathsf{SL}_3(\mathbb{K})$). Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and set $K_\mathbb{R} = \mathsf{SO}(3)$ and $K_\mathbb{C} = \mathsf{SU}(3)$. Any element $g \in \mathsf{SL}_3(\mathbb{K})$ can be written in the form

$$g = k_g \mathrm{diag}\big(\sigma_1(g), \sigma_2(g), \sigma_3(g)\big) k'_g \quad k_g, k'_g \in K_\mathbb{K},$$

where $\sigma_1(g) \geqslant \sigma_2(g) \geqslant \sigma_3(g)$ are uniquely determined and are called the *singular values* of g. The *Cartan projection*[2] of g is $\mu(g) = (\log\sigma_1(g), \log\sigma_2(g), \log\sigma_3(g)) \in \overline{\mathfrak{a}}^+$.

We will simply say a sequence in (resp., a representation into, subgroup of) $\mathsf{SL}_3(\mathbb{K})$ is *regular* if it is P-regular with respect to the stabilizer $P < \mathsf{SL}_3(\mathbb{K})$ of a line in $\mathbb{K}^3$. This language is unambiguous for representations into (hence subgroups of) $\mathsf{SL}_3(\mathbb{K})$; indeed, if P and Q are any two proper parabolic subgroups of $\mathsf{SL}_3(\mathbb{K})$, then a representation $\rho : \Gamma \to \mathsf{SL}_3(\mathbb{K})$ is P-regular if and only ρ is Q-regular. A sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathsf{SL}_3(\mathbb{K})$ is regular if and only if

$$\lim_{n\to\infty} \frac{\sigma_1(\rho(g_n))}{\sigma_2(\rho(g_n))} = \infty.$$

Note that, in this case, the sequence $\Big(\frac{1}{\sigma_1(\rho(g_n))}\rho(g_n)\Big)_{n\in\mathbb{N}}$ subconverges to a rank-1 matrix.

We will also use the following characterization of P-regularity in terms of the dynamics on the flag manifold G/P. A sequence $(g_n)_{n\in\mathbb{N}}$ is called *P-contracting* if there are points $z^+ \in G/P$ and $z^- \in G/P^{\mathrm{opp}}$ such that g_n converges uniformly on compact subsets of $C(z^-)$ to the constant function z^+, where $C(z^-)$ denotes the set of all points in G/P opposite to z^-. In this case, we write $(g_n)_n^+ := z^+$.

Fact 2.2. ([10, Prop. 4.15]). *A sequence in G that is P-contracting is also P-regular. A sequence in G that is P-regular possesses a subsequence that is P-contracting.*

In particular, a sequence $(g_n)_{n\in\mathbb{N}}$ in G is P-regular if and only if every subsequence of $(g_n)_{n\in\mathbb{N}}$ possesses a P-contracting subsequence.

The *limit set* of a subgroup $\Gamma < G$ in the flag manifold G/P, denoted by Λ_Γ^P, is by definition the set of $(\gamma_n)_n^+ \in G/P$ for all P-contracting sequences $(\gamma_n)_{n\in\mathbb{N}}$ in Γ. If P is conjugate to P^{opp}, two subsets $\Lambda_1, \Lambda_2 \subset G/P$ are *antipodal* if each element of Λ_1 is opposite to each element of Λ_2.

The proof of the following lemma uses the fact that, for a matrix $g = (g_{ij})_{i,j=1}^3$ in $\mathsf{SL}_3(\mathbb{C})$, one has $\frac{1}{\sqrt{3}}||g||_2 \leqslant \sigma_1(g) \leqslant ||g||_2$, where $||g||_2 := (\sum_{i,j=1}^3 |g_{ij}|^2)^{1/2}$ is the ℓ^2-matrix norm of g.

Lemma 2.3. *Let $(g_n)_{n\in\mathbb{N}}$ be an infinite unbounded sequence of matrices in $\mathsf{SL}_3(\mathbb{C})$ with*

$$g_n = \begin{pmatrix} 1 & x_n & y_n \\ 0 & 1 & z_n \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $(g_n)_{n\in\mathbb{N}}$ is regular if and only if

$$\lim_{n\to\infty} \frac{x_n^2 + y_n^2 + z_n^2}{|x_n| + |z_n| + |x_n z_n - y_n|} = \infty.$$

[2] This is the vector-valued distance $d_{\overline{\mathfrak{a}}^+}(o, go)$ with respect to a particular choice of point $o \in X := \mathsf{SL}_3(\mathbb{K})/K_\mathbb{K}$ and Weyl chamber for $\mathsf{SL}_3(\mathbb{K})$ in X with origin o.

Proof. A straightforward calculation shows that for every $n \in \mathbb{N}$ we have that

$$g_n^{-1} = \begin{pmatrix} 1 & -x_n & x_n z_n - y_n \\ 0 & 1 & -z_n \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $g_n \in \mathsf{SL}_3(\mathbb{C})$, we have $\sigma_1(g_n)\sigma_2(g_n)\sigma_3(g_n) = 1$ and $\sigma_1(g_n^{-1}) = \sigma_3(g_n)^{-1}$ for every n, and hence we obtain

$$\frac{\sigma_1(g_n)}{\sigma_2(g_n)} = \frac{\sigma_1(g_n)^2\sigma_3(g_n)}{\sigma_1(g_n)\sigma_2(g_n)\sigma_3(g_n)} = \frac{\sigma_1(g_n)^2}{\sigma_1(g_n^{-1})}.$$

Now since

$$\sigma_1(g_n) \asymp |x_n| + |y_n| + |z_n|, \quad \sigma_1(g_n^{-1}) \asymp |x_n| + |x_n z_n - y_n| + |z_n|,$$

the conclusion follows. □

3. Proof of Theorem 1.3

Proof of Theorem 1.3. Suppose that $\rho : \mathbb{Z}^2 \to \mathsf{SL}_3(\mathbb{R})$ is a regular representation. We first prove that the image of ρ is unipotent. Fix a $\mathbb{Z}$-basis $x, y \in \mathbb{Z}^2$ for $\mathbb{Z}^2$.

Claim 1. The image $\rho(\mathbb{Z}^2)$ is a unipotent subgroup of $\mathsf{SL}_3(\mathbb{R})$.

Proof of Claim 1. Suppose otherwise. Assume first that all the eigenvalues of $\rho(x)$ are distinct. Then, up to conjugation within $\mathsf{SL}_3(\mathbb{C})$, the image of ρ is a diagonal subgroup of $\mathsf{SL}_3(\mathbb{C})$. Since ρ is discrete, we have that $\mu(\rho(\mathbb{Z}^2))$ contains the intersection of $\overline{\mathfrak{a}}^+$ with a lattice in $\mathfrak{a}$. It follows that ρ is not regular in this case.

In the remaining case, up to conjugating ρ within $\mathsf{SL}_3(\mathbb{R})$, we have

$$\rho(x) = \begin{pmatrix} \lambda_x & 1 & 0 \\ 0 & \lambda_x & 0 \\ 0 & 0 & \lambda_x^{-2} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \lambda_y & \alpha_y & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_y^{-2} \end{pmatrix},$$

for some $\lambda_x, \lambda_y, \alpha_y \in \mathbb{R}$. Then we have

$$\rho(x^n y^m) = \lambda_x^n \lambda_y^m \begin{pmatrix} 1 & \lambda_x^{-1} n + \alpha_y \lambda_y^{-1} m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda_x^{-3n}\lambda_y^{-3m} \end{pmatrix}$$

for $n, m \in \mathbb{Z}$. Now there is an infinite sequence of distinct pairs of integers $(n_k, m_k)_{k\in\mathbb{N}}$ such that $\lim_k(\lambda_x^{-1} n_k + \alpha_y \lambda_y^{-1} m_k) = 0$ and $\lim_k(\lambda_x^{n_k}\lambda_y^{m_k}) = \infty$; note we can indeed ensure the latter, since otherwise discreteness of ρ would be violated. Observe that $\sigma_1(\rho(x^{n_k} y^{m_k})) \asymp \lambda_x^{n_k}\lambda_y^{m_k}$ as $k \to \infty$ and that the sequence of matrices $\left(\frac{1}{\lambda_x^{n_k}\lambda_y^{m_k}}\rho(x^{n_k} y^{m_k})\right)_{k\in\mathbb{N}}$ converges to a matrix of rank 2. In particular, the sequence $(\rho(x^{n_k} y^{m_k}))_{k\in\mathbb{N}}$ cannot be regular, so that ρ is not regular. □

Therefore, the image of the representation $\rho : \mathbb{Z}^2 \to \mathsf{SL}_3(\mathbb{R})$ has to be unipotent. We show that $\rho(\mathbb{Z}^2)$ lies in a minimal horospherical subgroup of $\mathsf{SL}_3(\mathbb{R})$. Up to conjugation, we may assume that

$$\rho(x) = \begin{pmatrix} 1 & a_x & b_x \\ 0 & 1 & c_x \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 1 & a_y & b_y \\ 0 & 1 & c_y \\ 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

where $a_x, b_x, a_y, b_y \in \mathbb{R}$. Since $\rho(x)$ commutes with $\rho(y)$, we have that $a_x c_y = a_y c_x$.

Claim 2. The identity $a_y c_x = a_x c_y = 0$ holds.

Proof of Claim 2. We prove the claim by contradiction. Assuming $a_y c_x \neq 0$, we will exhibit infinite sequences $(w_m)_{m\in\mathbb{Z}}$ in $\mathbb{Z}^2$ such that $\left(\frac{\sigma_1}{\sigma_2}(\rho(w_m))\right)_{m\in\mathbb{Z}}$ has an infinite bounded subsequence.

Set $\lambda := \frac{c_x}{a_x} = \frac{c_y}{a_y} \neq 0$. By conjugating the image of ρ with the diagonal matrix $\operatorname{diag}(1,1,\lambda) \in \mathsf{GL}_3(\mathbb{R})$, we may assume that $a_x = c_x$ and $a_y = c_y$, and hence

$$\rho(x) = \begin{pmatrix} 1 & a_x & b_x \\ 0 & 1 & a_x \\ 0 & 0 & 1 \end{pmatrix}, \ \rho(y) = \begin{pmatrix} 1 & a_y & b_y \\ 0 & 1 & a_y \\ 0 & 0 & 1 \end{pmatrix}.$$

A straightforward calculation shows that, for $m, n \in \mathbb{Z}$,

$$\rho(x^n) = \begin{pmatrix} 1 & na_x & n\big(b_x - \frac{a_x^2}{2}\big) + \frac{n^2 a_x^2}{2} \\ 0 & 1 & na_x \\ 0 & 0 & 1 \end{pmatrix}, \ \rho(y^m) = \begin{pmatrix} 1 & ma_y & n\big(b_y - \frac{a_y^2}{2}\big) + \frac{m^2 a_y^2}{2} \\ 0 & 1 & ma_y \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho(x^n y^m) = \begin{pmatrix} 1 & a(m,n) & b(m,n) \\ 0 & 1 & a(m,n) \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{aligned} a(m,n) &:= na_x + ma_y, \\ b(m,n) &:= mna_x a_y + \frac{n^2 a_x^2}{2} + \frac{m^2 a_y^2}{2} + n\Big(b_x - \frac{a_x^2}{2}\Big) + m\Big(b_y - \frac{a_y^2}{2}\Big) \\ &= \frac{1}{2}\big(na_x + ma_y\big)^2 + n\Big(b_x - \frac{a_x^2}{2}\Big) + m\Big(b_y - \frac{a_y^2}{2}\Big) \\ &= \frac{1}{2} a(m,n)^2 + \frac{B_x}{a_x} a(m,n) + m\Big(B_y - \frac{a_y}{a_x} B_x\Big) \\ &= \frac{1}{2}\bigg(\Big(a(m,n) + \frac{B_x}{a_x}\Big)^2 - \frac{2B_x^2}{a_x^2} + mZ_{x,y}\bigg), \end{aligned}$$

where the constants $B_x, B_y, Z_{x,y} \in \mathbb{R}$ are defined as follows:

$$\begin{aligned} B_x := b_x - \frac{a_x^2}{2}, \ B_y := b_y - \frac{a_y^2}{2} \\ Z_{x,y} := 2\Big(B_y - \frac{a_y}{a_x} B_x\Big). \end{aligned}$$

Suppose first that $Z_{x,y} = 0$, and choose infinite sequences $(k_m)_{m \in \mathbb{N}}, (r_m)_{m \in \mathbb{N}}$ of integers such that

$$\big|a(k_m, r_m)\big| = \big|k_m a_x + r_m a_y\big| \leqslant 1$$

for every m. By our assumption that $Z_{x,y} = 0$, we have that $(b(k_m, r_m))_{m \in \mathbb{Z}}$ is also bounded, and hence so is $(\rho(x^{k_m} y^{r_m}))_{m \in \mathbb{N}}$, violating our assumption that ρ is discrete and faithful.

Now suppose that $Z_{x,y} \neq 0$. Let $m \in \mathbb{Z}$ with $mZ_{x,y} < 0$, and define

$$n_m := \bigg\lfloor -m\frac{a_y}{a_x} + \frac{1}{a_x}\sqrt{|mZ_{x,y}|} \bigg\rfloor$$

so that

$$\Big|a(m, n_m) - \sqrt{|mZ_{x,y}|}\Big| = |a_x| \Big| n_m + \frac{a_y}{a_x} m - \frac{1}{a_x}\sqrt{|mZ_{x,y}|}\Big| \leqslant |a_x|. \tag{2}$$

Note that $|a(m, n_m)| \asymp \sqrt{|m|}$, and hence

$$\begin{aligned} \big|b(m, n_m)\big| &\leqslant \frac{B_x^2}{a_x^2} + \frac{1}{2}\Big|a(m,n_m) + \frac{B_x}{a_x} - \sqrt{|mZ_{x,y}|}\Big| \cdot \Big|a(m,n_m) + \frac{B_x}{a_x} + \sqrt{|mZ_{x,y}|}\Big| \\ &\leqslant \frac{B_x^2}{a_x^2} + \bigg(|a_x| + \frac{|B_x|}{|a_x|}\bigg)\bigg(\big|a(m,n_m)\big| + \frac{|B_x|}{|a_x|} + \sqrt{|mZ_{x,y}|}\bigg) \\ &= O(\sqrt{|m|}), \quad mZ_{x,y} \to -\infty, \end{aligned}$$

where the second inequality follows from (2).

Finally, we claim that the sequence $\rho(w_m)_{m\in\mathbb{Z}}$, where $w_m := x^{n_m}y^m$, has an infinite subsequence that is not regular. Indeed, for $m \in \mathbb{Z}$ with $mZ_{x,y} < 0$, we have by Lemma 2.3 that

$$\frac{\sigma_1(\rho(w_m))}{\sigma_2(\rho(w_m))} \asymp \frac{2a(m,n_m)^2 + b(m,n_m)^2}{2\left|a(m,n_m)\right| + \left|a(m,n_m)^2 - b(m,n_m)\right|}$$

and the latter fraction remains bounded since $\left|a(m,n_m)\right| \asymp \sqrt{|m|}$ and $\left|b(m,n_m)\right| = O(\sqrt{|m|})$ as $mZ_{x,y} \to -\infty$.

We thus arrive at a contradiction, and so we conclude that $a_x c_y = a_y c_x = 0$. □

Completing the proof of Theorem 1.3. We have reduced to the case that ρ is as in (1) with $a_x c_y = a_y c_x = 0$.

Suppose first that $a_x = c_x = 0$ and $a_y c_y \neq 0$. In this case, we may define a new representation $\rho' : \mathbb{Z}^2 \to \mathsf{SL}_3(\mathbb{R})$ given by

$$\rho'(x) = \rho(xy), \ \rho'(y) = \rho(y).$$

Since ρ is assumed to be regular, the same holds for ρ'. Now note that the $(1,2)$ and $(2,3)$ entries of $\rho'(x)$ and $\rho'(y)$ are non-zero, so that the representation ρ' cannot be regular by Claim 1, a contradiction. By applying the same argument with x and y interchanged, we conclude that in fact $a_x = a_y = 0$ or $c_x = c_y = 0$ as desired.

Finally, we verify that if $\rho(\mathbb{Z}^2)$ is a lattice in a minimal horospherical subgroup of $\mathsf{SL}_3(\mathbb{R})$, then ρ is indeed regular. This follows immediately from Lemma 2.3, but we present the following geometric argument that applies in any dimension. We first consider the case that $\rho(\mathbb{Z}^2)$ is a lattice in the unipotent radical of the stabilizer in $\mathsf{SL}_3(\mathbb{R})$ of a hyperplane in $\mathbb{R}^3$.

Claim 3. Let U be the unipotent radical of the stabilizer in $\mathsf{SL}_d(\mathbb{R})$ of a hyperplane $V \subset \mathbb{R}^d$. Then any lattice F in U is P-regular, where P is the stabilizer of a line in $\mathbb{R}^d$.

Proof of Claim 3. We identify the U-invariant affine chart $\mathbb{P}(\mathbb{R}^d) \smallsetminus \mathbb{P}(V)$ with $\mathbb{R}^{d-1}$, so that U acts on $\mathbb{R}^{d-1}$ via translations. For a point $z \in \mathbb{R}^{d-1}$ and $R > 0$, denote by $B(z,R)$ the Euclidean ball in $\mathbb{R}^{d-1}$ of radius R centered at z. Fix a point $z_0 \in \mathbb{R}^{d-1}$.

Now let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in F with $\gamma_n \to \infty$. Then, since $\mathbb{P}(\mathbb{R}^d)$ is compact, up to extraction, we have that $\gamma_n z_0 \to z^+$ for some $z^+ \in \mathbb{P}(\mathbb{R}^d)$. Moreover, since F acts properly on $\mathbb{R}^{d-1}$, we in fact have $z^+ \in \mathbb{P}(V)$.

We claim that $(\gamma_n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{R}^d$ to the constant function z^+. Indeed, let W_n be a metric $\frac{1}{n}$-neighborhood of z^+ in $\mathbb{P}(\mathbb{R}^d)$ with respect to the Fubini–Study metric on the latter; viewed in our chosen affine chart, the boundary of W_n is a two-sheeted hyperboloid for n sufficiently large. It suffices to show that for any $n \in \mathbb{N}$, there is some $N \in \mathbb{N}$ such that $W_n \supset \gamma_N B(z_0, n) = B(\gamma_N z_0, n)$. But this is true since, given any $n \in \mathbb{N}$, there is some $m \in \mathbb{N}$ such that $B(z,n) \subset W_n$ for each $z \in W_m$. □

In the remaining case, where $\rho(\mathbb{Z}^2)$ lies in the unipotent radical of the stabilizer of a *line* in $\mathbb{R}^3$, one argues as above with the dual representation ρ^* instead of ρ, as $\sigma_i(\rho^*(\gamma)) = \sigma_{4-i}(\rho(\gamma))^{-1}$ for $\gamma \in \mathbb{Z}^2$ and $1 \leqslant i \leqslant 3$. □

Remark 3.1. Following the above approach, it is not difficult to see that if $\langle a, b\rangle < \mathsf{SL}_3(\mathbb{R})$ is a discrete $\mathbb{Z}^2$ which is not contained in a minimal horospherical subgroup, then the limit set of $\langle a, b\rangle$ in $\mathbb{P}(\mathbb{R}^3)$ consists of at most three points.

4. Proof of Proposition 1.6

To prove Proposition 1.6, we use the following variant of the ping-pong lemma. Similar arguments appear in work of Dey and Kapovich [4], but we include them here for the convenience of the reader.

Lemma 4.1. *Let G be a Lie group acting continuously on a manifold $\mathcal{F}$. Suppose $\Gamma_1, \Gamma_2 < G$ are infinite*[3] *and that there are closed nonempty disjoint subsets $C_1, C_2 \subset \mathcal{F}$ such that $\gamma_i C_j \subset C_i$ for $\gamma_i \in \Gamma_i \smallsetminus \{1\}$ and $i \neq j$. Then $\langle \Gamma_1, \Gamma_2 \rangle < G$ is discrete and decomposes as $\Gamma_1 * \Gamma_2$.*

Proof. Let $\rho : \Gamma_1 * \Gamma_2 \to G$ be the map induced by the inclusions $\Gamma_i \subset G$ for $i = 1, 2$. Take a sequence $w_n \in \Gamma_1 * \Gamma_2$ and suppose for a contradiction that $w_n \neq 1$ for any $n \in \mathbb{N}$ but $\lim_n \rho(w_n) = 1 \in G$. Up to relabeling Γ_1 and Γ_2 and extracting a subsequence of $(w_n)_n$, we may assume that for some fixed $i \in \{1, 2\}$ and each $n \in \mathbb{N}$, the first letter (read from the left) in the canonical form of w_n belongs to $\Gamma_1 \smallsetminus \{1\}$ and the last belongs to $\Gamma_i \smallsetminus \{1\}$.

Suppose first that $i = 1$. Then $\rho(w_n) C_2 \subset C_1$ for each $n \in \mathbb{N}$. Selecting some $z \in C_2$, we then have $z = \lim_n \rho(w_n) z \in C_1$ since C_1 is closed, so that $z \in C_1 \cap C_2$, a contradiction.

Now suppose that $i = 2$. Pick an element $\gamma_1 \in \Gamma_1 \smallsetminus \{1\}$, and let $w'_n = \gamma_1 w_n \gamma_1^{-1}$ for $n \in \mathbb{N}$. Note that we still have $\lim_n \rho(w'_n) = 1$. If for some subsequence $(w'_{n_k})_k$ of $(w'_n)_n$ the canonical form of w'_{n_k} has odd length for each $k \in \mathbb{N}$, then one obtains a contradiction as in the previous paragraph. Otherwise, there is some $N \in \mathbb{N}$ such that the first letter (read from the left) in the canonical form of w_n is γ_1^{-1} for $n \geqslant N$. Now select $\gamma'_1 \in \Gamma_1 \smallsetminus \{1, \gamma_1\}$, and let $w''_n = \gamma'_1 w_n (\gamma'_1)^{-1}$ for $n \in \mathbb{N}$. Then again we have $\lim_n \rho(w''_n) = 1$, but now the canonical form of w''_n has odd length for $n \geqslant N$, so that we arrive at a contradiction as in the previous paragraph. □

Proof of Proposition 1.6. Since we have assumed that there is a point in G/P opposite to each point in Λ^P_Δ, we can find a compact neighborhood W_0 of Λ^P_Δ and a compact subset $U \subset G/P$ with nonempty interior such that U and W_0 are antipodal; see [5, Lem. 4.24]. As in [4, Rmk. 6.4], we have by P-regularity of Δ that $\delta U \subset W_0$ for each nontrivial element $\delta \in \Delta$ apart from a finite list $\delta_1, \dots, \delta_k \in \Delta \smallsetminus \{1\}$.

For $i = 1, \dots, k$, let Z_i be the set of all $z \in G/P$ such that z is not opposite to $\delta_i z$. Since each of the Z_i is a proper algebraic subset of G/P, we have that $U \smallsetminus \bigcup_{i=1}^k Z_i$ has nonempty interior. We can thus find a compact subset $V \subset U \smallsetminus \bigcup_{i=1}^k Z_i$ with nonempty interior such that V and $\delta_i V$ are antipodal for $i = 1, \dots, k$. Setting $W = W_0 \cup \bigcup_{i=1}^k \delta_i V$, we then have that V and W remain antipodal in G/P.

Now since Γ is a lattice in G, there is an element $g \in \Gamma$ generating a P-regular cyclic subgroup with $\Lambda^P_{\langle g \rangle} \subset \mathring{V}$ (one can always choose *P-proximal* such $g \in \Gamma$, the existence of which already follows, for instance, from [15, Lemma 1]). There is then some $N \in \mathbb{N}$ such that $g^n W \subset V$ for all $n \in \mathbb{Z}$ with $|n| \geqslant N$. Moreover, by design, we have $\delta V \subset W$ for each $\delta \in \Delta \smallsetminus \{1\}$. Setting $\gamma = g^N$, we conclude from Lemma 4.1 that $\langle \Delta, \gamma \rangle < \Gamma$ decomposes as $\Delta * \langle \gamma \rangle$. □

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[3] In fact, our argument requires only that $|\Gamma_i| > 2$ for $i = 1, 2$. The statement remains true if at least one of the Γ_i has size at least 3.

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Institut des Hautes Études Scientifiques, Université Paris-Saclay, 35 route de Chartres, 91440 Bures-sur-Yvette, France
E-mail address: douba@ihes.fr

CNRS, Laboratoire Alexander Grothendieck, Institut des Hautes Études Scientifiques, Université Paris-Saclay, 35 route de Chartres, 91440 Bures-sur-Yvette, France
E-mail address: tsouvkon@ihes.fr