CONVEX COCOMPACT GROUPS IN REAL HYPERBOLIC SPACES WITH LIMIT SET A PONTRYAGIN SPHERE

SAMI DOUBA, GYE-SEON LEE, LUDOVIC MARQUIS, AND LORENZO RUFFONI

ABSTRACT. We exhibit two examples of convex cocompact subgroups of the isometry groups of real hyperbolic spaces with limit set a Pontryagin sphere: one generated by 50 reflections of $\mathbb{H}^4$, and the other by a rotation of order 21 and a reflection of $\mathbb{H}^6$. For each of them, we also locate convex cocompact subgroups with limit set a Menger curve.

1. INTRODUCTION

The *Sierpiński carpet* is the fractal curve obtained from a square by removing smaller and smaller squares, similarly to the way one obtains the Cantor set from an interval. In particular, the Sierpiński carpet has a well-defined collection of boundary squares. The *Pontryagin sphere* is obtained from the Sierpiński carpet by identifying the opposite sides of each boundary square as one would glue the edges of a square to form a torus. This results in a 2-dimensional compact metrizable space without non-empty open subsets embeddable in $\mathbb{R}^2$; see Figure 1. The Pontryagin sphere can also be described as the limit of a certain inverse system of closed connected orientable surfaces of positive genus; see [Jak91; Fis03; Zaw10; Ś20a; Ś20b] and §2.2. While this space is known to appear as the Gromov boundary of many Gromov hyperbolic groups, here we consider the problem of realizing the Pontryagin sphere as the limit set of concrete reflection groups in the isometry group $\mathrm{Isom}(\mathbb{H}^d)$ of a real hyperbolic space $\mathbb{H}^d$.

Theorem 1.1. *There is a convex cocompact right-angled reflection subgroup Γ^4 of $\mathrm{Isom}(\mathbb{H}^4)$ whose limit set is homeomorphic to the Pontryagin sphere.*

Theorem 1.2. *There is a 2-generated convex cocompact subgroup Γ^6_2 of $\mathrm{Isom}(\mathbb{H}^6)$ whose limit set is homeomorphic to the Pontryagin sphere. Moreover, Γ^6_2 contains a normal subgroup Γ^6 of index 21 which is a right-angled reflection group.*

The starting point of our work is the following. If W is a Gromov hyperbolic right-angled Coxeter group (RACG) whose nerve is a closed connected orientable surface of positive genus then the Gromov boundary $\partial_\infty W$ of W is a Pontryagin sphere; see [Fis03]. Hence, the up-shot of this paper is to find convex cocompact actions of two such Coxeter groups on $\mathbb{H}^4$ and $\mathbb{H}^6$, respectively.

The groups constructed in this paper are optimal in the following sense. The group in dimension $d = 4$ appears in the smallest possible dimension, as the Pontryagin sphere does not embed in $\mathbb{S}^2 = \partial_\infty\mathbb{H}^3$. In particular, Γ^4 is Zariski dense in $\mathrm{Isom}(\mathbb{H}^4)$. On the other hand, the group Γ^6_2 in dimension $d = 6$ has the smallest possible number of generators (as the limit set of a cyclic group consists of at most 2 points). Moreover, the groups in dimension $d = 6$ are Zariski dense, so their limit set is not contained in any round $\mathbb{S}^k \subseteq \mathbb{S}^5 = \partial_\infty\mathbb{H}^6$ for $k \leqslant 4$; see Remark 4.5.

The *Menger curve* is a fractal curve that is obtained from a cube by removing smaller and smaller cubes, similarly to the way one obtains the Sierpiński carpet from a square and the Cantor set from an interval. It can be characterized as the unique compact, metrizable, connected, locally connected, 1-dimensional space without

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local cut points and without non-empty open subsets embeddable in $\mathbb{R}^2$; see [And58a; And58b]. According to [KP96], the Pontryagin sphere has a dense collection of Menger curves, so it is natural to look for Menger curves that are limit sets of quasiconvex subgroups of Γ^4 or Γ^6. In Theorem 5.3 we show that both Γ^4 and Γ^6 have convex cocompact subgroups with limit set homeomorphic to the Menger curve. These subgroups are obtained by forgetting one (or more, suitably chosen) reflections. Previous examples of convex cocompact subgroups of $\mathrm{Isom}(\mathbb{H}^4)$ with Menger curve limit set were constructed by Bourdon [Bou97], see Remark 5.4.

We remark that indeed the answer to the following question does not appear to be known.

Question 1.3. *Does every Gromov hyperbolic group with boundary a Pontryagin sphere possess a quasiconvex subgroup with boundary a Menger curve?*

Note that the analogous statement for Gromov hyperbolic groups with boundary a 2-sphere, namely, that every group of the latter form possesses a quasiconvex subgroup with boundary a Sierpiński carpet, would follow from the Cannon conjecture, but even this implication requires non-trivial results about hyperbolic 3-manifolds, including several of the ingredients that went into the solution to the virtual Haken conjecture [Ber15].

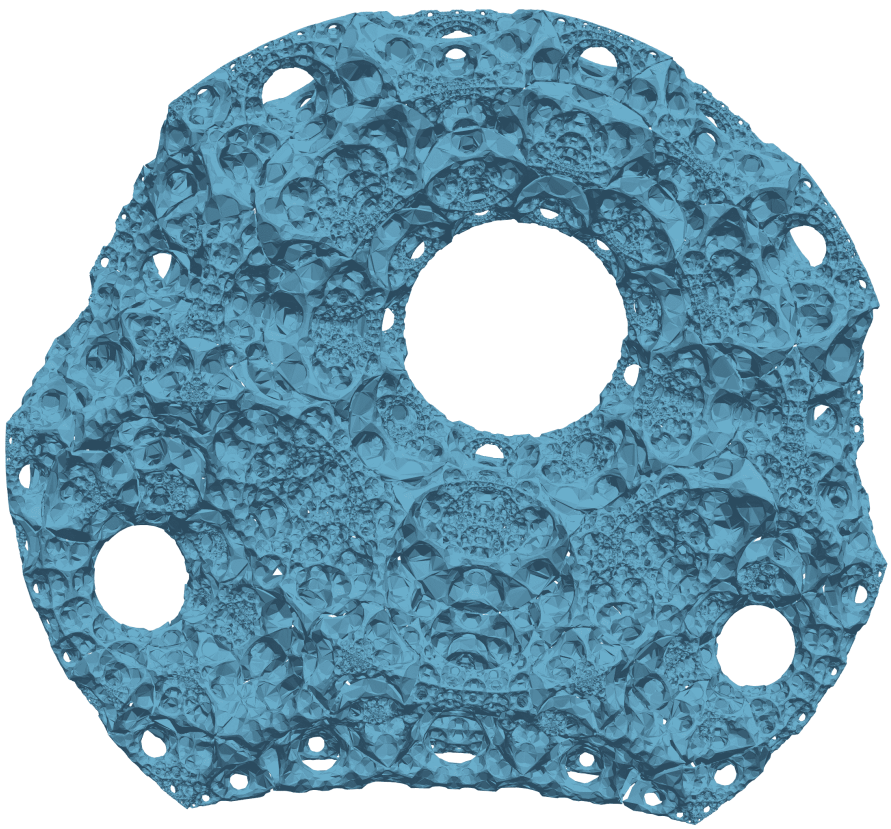

FIGURE 1. A rendering of the Pontryagin sphere, courtesy of Theodore Weisman. This is in fact a plot of the limit set of the group Γ^4 in Theorem 1.1.

We conclude with a few comments on realizing the Pontryagin sphere as a limit set in other spaces. By embedding $\mathbb{H}^4$ as a totally geodesic subspace in higher-dimensional hyperbolic spaces, one can realize Γ^4 as a convex cocompact subgroup of $\mathrm{Isom}(\mathbb{H}^d)$ for each $d \geqslant 4$. It is then not difficult to show that, for each $d \geqslant 4$, there is a finite-index reflection subgroup of Γ^4 that admits arbitrarily small deformations that are Zariski dense in $\mathrm{Isom}(\mathbb{H}^d)$ (note that sufficiently small deformations will still be convex cocompact reflection groups). As for the complex hyperbolic spaces $\mathbb{H}^d_{\mathbb{C}}$, it does not appear to be known whether there is a convex cocompact subgroup of $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$ with limit set a Pontryagin sphere, though Granier [Gra15] constructed convex cocompact subgroups of $\mathrm{Isom}(\mathbb{H}^2_{\mathbb{C}})$ with limit set a Menger curve (see also recent related work of Ma–Xie [MX24]). Jeffrey Danciger and Theodore Weisman have communicated to us that they can show that the deformations of certain cusped hyperbolic 3-orbifold groups inside $\mathrm{Isom}(\mathbb{H}^3_{\mathbb{C}})$ constructed in [PT23] descend to convex cocompact representations of suitable quotients with limit set a Pontryagin sphere. Outside of the world of symmetric spaces, one can use Ontaneda's Riemannian hyperbolization [Ont20] to construct many pinched Hadamard manifolds in any dimension $d \geqslant 4$, whose isometry groups admit convex cocompact subgroups with limit set a Pontryagin sphere.

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2. Preliminaries

2.1. **Simplicial complexes.** Let K be a simplicial complex. In this paper K will always be finite. We will use the following terminology and notation:

- The *d-skeleton* of K is denoted $K^{(d)}$.
- K is *flag* if any collection of $d+1$ pairwise adjacent vertices spans a d-simplex. If K is any simplicial complex, then the *flag complex* on K is the complex obtained by adding all simplices whose 1-skeleton appears in K. In particular, a flag complex is completely determined by its 1-skeleton.
- If $L \subseteq K$ is a subcomplex, we say that L is *full* (or *induced*) if whenever $d+1$ vertices span a d-simplex in K, they also span a d-simplex in L. In particular, note that a full subcomplex of a flag complex is flag, and that if H is a full subcomplex of L and L is a full subcomplex of K, then H is a full subcomplex of K.
- K is *flag-no-square* if it is flag and has no induced squares (i.e., every square has a chord).
- The *puncture-respecting cohomological dimension* of K is defined to be:
$$\mathrm{pcd}(K) = \max_{\sigma} \max\left\{ n \;\middle|\; \overline{H}^n(K \smallsetminus \sigma) \neq 0 \right\},$$
where $\overline{H}^n$ denotes reduced cohomology and σ is a possibly empty subcomplex of K; see [DŚ21].

2.2. **The Pontryagin sphere.** The *Pontryagin sphere* is a 2-dimensional continuum (i.e., connected compact metrizable space), which arises as the limit of a certain inverse system of closed surfaces of positive genus; see [Jak91; Fis03; Zaw10; Ś20a; Ś20b] and Figure 1. The name "sphere" can be misleading here. Indeed, not only is this space not a manifold, but also none of its open subsets can be embedded in $\mathbb{R}^2$, except the empty one. Note that Pontryagin also defined an infinite family of 2-dimensional continua called *Pontryagin surfaces*, which are also obtained as inverse limits of 2-dimensional complexes but are not manifolds. However, the Pontryagin sphere is not one of the Pontryagin surfaces: while the former embeds in $\mathbb{R}^3$ and has rational cohomological dimension 2, the latter do not, see [Dra05, Example 1.9].

The notion of dimension we consider in this paper is the *Lebesgue covering dimension* (or *topological dimension*), i.e., the minimum integer d for which every open cover has an open refinement such that every point is contained in at most $d+1$ open sets. This is a well defined topological invariant for all metric spaces, and agrees with the standard notion of dimension for manifolds and CW-complexes.

Here are some more details about the construction of the Pontryagin sphere. Let $\{f_n : S_n \to S_{n-1}\}_{n\in\mathbb{N}}$ be an inverse system of closed connected orientable surfaces and continuous maps between them. Its inverse limit is called a *tree of surfaces* if the bonding maps f_n satisfy some technical conditions, which roughly speaking say that S_n is obtained from S_{n-1} by performing a connected sum with some closed orientable surfaces, in such a way that the portion of S_{n-1} which is engaged in the connected sum becomes dense eventually. The reader should imagine starting with a surface S_0 and performing an infinite sequence of connected sums with surfaces along smaller and smaller disks that become more and more densely packed. If all surfaces involved are 2-spheres, then the limit is a 2-sphere. If finitely many surfaces have positive genus, then the limit is a closed surfaces of positive genus. If infinitely many surfaces have positive genus, then the limit is the Pontryagin sphere. This space feels like a 2-sphere with infinitely many handles attached, but since we are attaching handles all over the place, the result is a fractal space rather than a surface of infinite type, see Figure 1.

In our context, an inverse system as above arises as follows. Let X be a CAT(0) cube complex in which the link of every vertex is a flag triangulation of a closed connected orientable surface of positive genus. Choose a base point p in the interior of a cube and consider a family of closed metric balls $B(p,r_n)$ of radius $r_n > 0$ centered at p. Assume for simplicity that p and r_n are chosen generically, so that no ball contains a vertex on its boundary. If $B(p,r_n)$ does not contain any vertex, then $\partial B(p,r_n)$ is just the space of directions at p and is homeomorphic to a 2-sphere. More generally, as observed in [DJ91, §3.d], $\partial B(p,r_n)$ is homeomorphic to the surface obtained from $\partial B(p,r_{n-1})$ by taking a connected sum with the surfaces $\mathrm{lk}(v,X)$, where v ranges over the vertices contained in $B(p,r_n) \setminus B(p,r_{n-1})$. We obtain an inverse system of surfaces, in which the bonding maps are induced by the nearest point projections $B(p,r_n) \to B(p,r_{n-1})$. Note that the collection of geodesic rays issuing from the base point p and hitting a vertex is dense in the space of directions at p, so the connected sums involved in the inverse limit engage a dense portion of the space of directions at p. The *visual boundary* $\partial_\infty X$ of X is the limit of this inverse system, and is homeomorphic to the Pontryagin sphere.

2.3. **Right-angled Coxeter groups.** Given a flag complex K, the *right-angled Coxeter group* (RACG) defined by K is the group W_K generated by one involution for each vertex of K, with two generators commuting exactly when the corresponding vertices are adjacent. In other words, it is defined by the following presentation

$$W_K = \langle s \in K^{(0)} \mid s^2 = 1, [s,t] = 1 \text{ iff } (s,t) \in K^{(1)} \rangle.$$

The RACGs considered in this paper arise concretely as groups generated by reflections in the codimension-1 faces of certain polytopes in real hyperbolic d-space $\mathbb{H}^d$. More generally, a RACG W_K can be regarded as a group generated by reflections in the hyperplanes of the associated *Davis complex*. This is a CAT(0) cube complex whose 1-skeleton may be identified with the Cayley graph of W_K and where the link of each vertex is isomorphic to the flag complex K. For background on Coxeter groups the reader is referred to [Dav08].

The flag complex K is called the *nerve* of the group W_K. Note that W_K is completely determined by the 1-skeleton of K. So, if $\mathscr{G}$ is a finite simplicial graph, we also use the notation $W_{\mathscr{G}}$ for the RACG W_K, where K is the flag complex on $\mathscr{G}$.

We now collect some well-known properties of W_K that can be read off of its nerve for future reference. Given a full subcomplex $L \subseteq K$, the subgroup $\langle L^{(0)} \rangle$ is called the *special subgroup* defined by L.

Lemma 2.1. *Let K be a flag complex. Then the following hold.*

1. *If $L \subseteq K$ is a full subcomplex, then $\langle L^{(0)} \rangle$ is naturally isomorphic to W_L and is quasiconvex (with respect to the generating set $K^{(0)}$ for W_K).*
2. *W_K is Gromov hyperbolic if and only if K is flag-no-square.*
3. *If K is flag-no-square, then $\dim(\partial_\infty W_K) = \mathrm{pcd}(K)$, where $\partial_\infty W_K$ is the Gromov boundary of W_K.*

Proof. The properties of special subgroups in (**1.**) are classical, see e.g. [Dav08, §4.1]. Item (**2.**) is the right-angled case of Moussong's theorem [Mou88], see also [Dav08, Corollary 12.6.3]. The equality in (**3.**) follows from [BM91, Corollary 1.4.(e)] and [Dav08, Corollary 8.5.5]. ☐

In many cases of interest, the topology of the nerve determines more than just the dimension of the Gromov boundary. For instance, we will use the following result from [Fis03].

Lemma 2.2. *Let K be a flag-no-square triangulation of a closed connected orientable surface of genus $g \geqslant 1$. Then the Gromov boundary of W_K is homeomorphic to the Pontryagin sphere.*

2.4. **Convex cocompactness.** Let Γ be a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^d)$. The *limit set* of Γ is the closed subset of $\mathbb{S}^{d-1} = \partial_\infty \mathbb{H}^d$ consisting of all accumulation points of an orbit of Γ (this does not depend on the orbit). We say that Γ is *convex cocompact* if Γ acts cocompactly on the convex hull of its limit set.

Lemma 2.3. *Let Γ be a convex cocompact subgroup of* $\mathrm{Isom}(\mathbb{H}^d)$. *Then the following hold.*

1. *Γ is Gromov hyperbolic, and the limit set of Γ in $\mathbb{S}^{d-1}$ is homeomorphic to the Gromov boundary of Γ.*
2. *If $H \leqslant \Gamma$ is quasiconvex, then H is convex cocompact.*

Proof. Part (1.) is [Swe01, Theorem 12]. Part (2.) follows from the fact that an H-orbit is quasiconvex in the corresponding Γ-orbit, together with the characterization of convex cocompactness in terms of quasiconvexity of orbits from [Swe01, Main Theorem]. ☐

3. Construction in $\mathbb{H}^4$

3.1. **The 600-cell as a union of two solid tori.** Among the six convex regular 4-polytopes, there is precisely one with 600 facets, namely, the hexacosichoron. From now on, we will refer to the simplicial complex given by the boundary of the hexacosichoron as the 600-cell, denoted C_{600}. As such, C_{600} is a triangulation of the 3-dimensional sphere $\mathbb{S}^3$ consisting of 600 tetrahedra, 1200 triangles, 720 edges, and 120 vertices. Each vertex is incident to 20 tetrahedra and 12 edges. We will now review a certain decomposition of C_{600} that is relevant for the purposes of this paper. For additional details about the 600-cell see [Cox70, pp. 19-23, §9].

We may decompose C_{600} as a union of two solid tori T_0 (the *bottom torus*), T_1 (the *top torus*), and an *interface* I homeomorphic to $T^2 \times [0,1]$. Each solid torus can be constructed as follows:

1. A *flying saucer* is the 3-dimensional simplicial complex obtained by gluing 5 tetrahedra along a common edge. Equivalently, this is obtained by taking the suspension of the boundary of a pentagon and connecting the suspension vertices with an additional *internal* edge, and then filling with tetrahedra. The boundary is a pentagonal bipyramid. See Figure 2.
2. Consider the complex obtained by stacking 10 flying saucers vertex to vertex in such a way that the internal edges form a cycle of length 10. Now, fill in the gaps between two consecutive flying saucers with 10 *interstitial* tetrahedra so that the union of two consecutive flying saucers with the interstitial tetrahedra is an icosahedral pyramid. See Figure 2. The resulting complex is a triangulation of a solid torus T_i consisting of 150 tetrahedra.
3. The surface ∂T_i of each of T_i is the same as the outer surface of the complex consisting of 10 pentagonal antiprisms—so-called *drums*—stacked together to form a torus with 100 exposed[1] triangles (with a contribution of 10 from each drum), 150 exposed edges, and 50 exposed vertices. In particular the triangulation of ∂T_i is that of a flat torus by 100 equilateral triangles, in which every vertex has 6 neighbors. See Figure 3.

The interface I has the following structure:

1. To each of the 100 exposed triangles of T_0, we glue a *raised* tetrahedron, in the following way: the raised tetrahedra come in pairs, identified along the triangular faces sharing an edge that is horizontal in Figure 3. The unique common vertex among such a pair of tetrahedra which is not a vertex of T_0 will be called the *raised vertex* of that pair. There are 50 raised vertices. We do the same for T_1.

[1] A simplex is *exposed* if it is contained in ∂T_i.

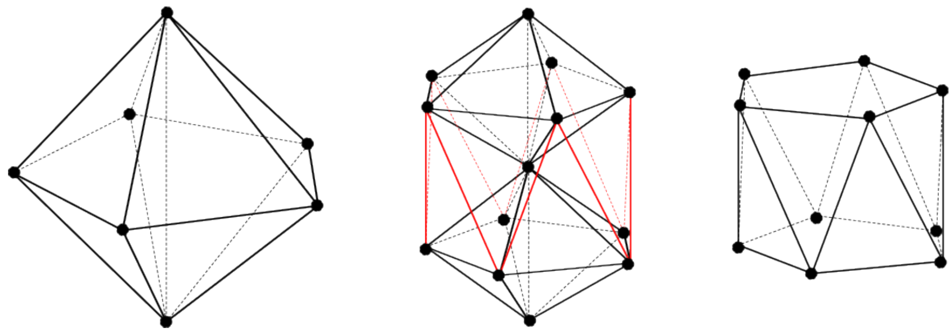


FIGURE 2. From left to right: a flying saucer, two flying saucers stacked vertex-to-vertex with ten interstitial tetrahedra (indicated in red) around their vertex of intersection, and a drum.

2. The interface is obtained by interlocking the raised tetrahedra for the top and bottom tori, so that the raised vertex of a raised tetrahedron on T_0 gets identified with an exposed vertex in T_1, and vice versa. After doing so, some gaps are left, that can be filled with 100 *filling* tetrahedra, each of which has an edge on T_0 and an edge on T_1.

Let v be a vertex on the surface of one of the T_i, say, the bottom torus T_0. We now describe the 12 neighbors of v in C_{600} with respect to the above decomposition of C_{600}:

1. 2 neighbors in the interior of T_0.

2. 6 neighbors w_i on ∂T_0. The edges $[v, w_i]$ are on ∂T_0.

3. 4 neighbors on ∂T_1; indeed v is a raised vertex for T_1, and is hence the raised vertex for a pair of raised tetrahedra of T_1.

3.2. **Proof of Theorem 1.1.** Among the six convex regular 4-polytopes, there is precisely one with 120 facets, namely, the dodecacontachoron. The right-angled dodecacontachoron can be realized as a compact polytope $\mathscr{P}_{120}$ in $\mathbb{H}^4$. By Poincaré's polyhedron theorem, the group $\mathrm{Refl}(\mathscr{P}_{120})$ generated by the reflections in the facets of $\mathscr{P}_{120}$ is a uniform lattice in $\mathbb{H}^4$ and $\mathrm{Refl}(\mathscr{P}_{120})$ is the RACG defined by the 1-skeleton of the dual of $\partial\mathscr{P}_{120}$, i.e., the 1-skeleton of C_{600}, so that $\mathrm{Refl}(\mathscr{P}_{120}) \cong W_{\mathsf{C}_{600}}$.

Being isomorphic to a uniform lattice of $\mathrm{Isom}(\mathbb{H}^4)$, the group $W_{\mathsf{C}_{600}}$ is a Gromov hyperbolic group with Gromov boundary $\mathbb{S}^3$. In particular, C_{600} is a flag-no-square complex, by Lemma 2.1.(**2.**). We will show that the subgroup Γ^4 of $W_{\mathsf{C}_{600}}$ generated by the vertices in ∂T_0 (or, equivalently, ∂T_1) has limit set a Pontryagin sphere. The triangulation of ∂T_0 is as in Figure 3, with opposite edges are identified.

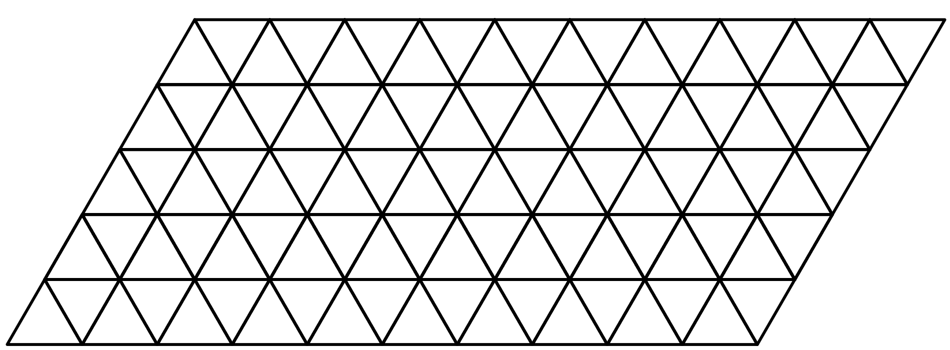


FIGURE 3. The triangulation of ∂T_0 with 50 vertices, 150 edges, and 100 triangles. Opposite edges are identified. This defines a RACG generated by 50 reflections inside $W_{\mathsf{C}_{600}}$.

Proposition 3.1. *The subcomplex ∂T_0 of C_{600} is a flag-no-square triangulation of a torus and a full subcomplex of C_{600}.*

Proof. The triangulation is clearly flag-no-square; indeed this is a piece of the equilateral triangulation of $\mathbb{R}^2$, with sides longer than 4, see Figure 3. To see that ∂T_0 is a full subcomplex we argue as follows. Let v be a vertex of ∂T_0. The above description of the neighbors of v shows that all the edges of C_{600} connecting v with another vertex on ∂T_0 are already contained in ∂T_0, namely they are the 6 edges to the 6 neighbors on ∂T_0. So, the 1-skeleton of ∂T_0 is an induced subgraph of the 1-skeleton of C_{600}. Since ∂T_0 is a flag 2-complex, it follows that ∂T_0 is a full subcomplex. □

With this proposition, we can see that the limit set of Γ^4 is a Pontryagin sphere, as follows. Since ∂T_0 is a full subgraph of C_{600}, it follows from (**1.**) in Lemma 2.1 that the special subgroup Γ^4 is quasiconvex and isomorphic to the abstract RACG $W_{\partial T_0}$. Since $\mathrm{Refl}(\mathscr{P}_{120})$ acts cocompactly on $\mathbb{H}^4$, Lemma 2.3 implies that Γ^4 is convex cocompact in $\mathrm{Isom}(\mathbb{H}^4)$ and in particular that the limit set of Γ^4 is homeomorphic to the Gromov boundary of Γ^4. Finally, since the nerve ∂T_0 of $W_{\partial T_0}$ is a flag-no-square triangulation of a closed orientable surface of positive genus, the Gromov boundary of $\Gamma^4 \cong W_{\partial T_0}$ is a Pontryagin sphere by Lemma 2.2.

Remark 3.2. By construction, Γ^4 is a subgroup of a uniform lattice $\mathrm{Refl}(\mathscr{P}_{120})$ in $\mathbb{H}^4$. It follows from work of Bugaenko [Bug84] that $\mathrm{Refl}(\mathscr{P}_{120})$ has finite index in the group $O(f_\varphi, \mathbb{Z}[\varphi])$, where $\varphi = \frac{1+\sqrt{5}}{2}$, and f_φ is the bilinear form on $\mathbb{R}^5$ given by the diagonal matrix $(1,1,1,1,-\varphi)$, so that $\mathrm{Refl}(\mathscr{P}_{120})$ is an arithmetic reflection lattice.

Remark 3.3. Figure 1 was obtained by Theodore Weisman by plotting the limit set of the action of Γ^4 on $\mathbb{H}^4$ given by restricting the action of $\mathrm{Refl}(\mathscr{P}_{120})$.

4. Construction in $\mathbb{H}^6$

4.1. **An abstract Coxeter group with 21 generators.** Let T_{21} be the flag-no-square simplicial complex depicted in Figure 4 (vertices sharing the same label should be identified and the boundary edge should be identified accordingly), it is a simplicial complex whose geometric realization is a topological surface S with 21 vertices, 63 edges and 42 faces, so that $\chi(S) = 0$. Hence, T_{21} is a flag-no-square triangulation of a 2-torus. The goal is to build a convex cocompact action of $W_{T_{21}}$ on $\mathbb{H}^6$ as a reflection group.

In fact, the presentation of $W_{T_{21}}$ can be expressed rather simply, and usefully, as follows. Let $M = (m_{i,j})$ be the symmetric 21×21 matrix with entries in $\{1,2,\infty\}$ given by

$$m_{i,j} = \begin{cases} 1 & \text{if } i = j; \\ 2 & \text{if } |i-j| = \pm 1, \pm 4 \text{ or } \pm 5 \,(\mathrm{mod}\ 21); \\ \infty & \text{otherwise.} \end{cases}$$

Then

$$W_{T_{21}} = \langle s_1, \dots, s_{21} \mid (s_i s_j)^{m_{i,j}} = 1,\ \forall 1 \leqslant i,j \leqslant 21 \rangle.$$

Since T_{21} is flag-no-square, we have by Lemma 2.1.(**2.**) that $W_{T_{21}}$ is Gromov hyperbolic.

4.2. **Construction of the reflection group Γ^6 of Theorem 1.2.** In this section, we construct an action $W_{T_{21}}$ on $\mathbb{H}^6$ as a reflection group.

Proposition 4.1. *There exists a convex cocompact right-angled reflection group Γ^6 of $\mathrm{Isom}(\mathbb{H}^6)$ whose limit set is homeomorphic to the Pontryagin sphere.*

Proof. Let $A = (A_{i,j})$ be the symmetric 21×21 matrix given by

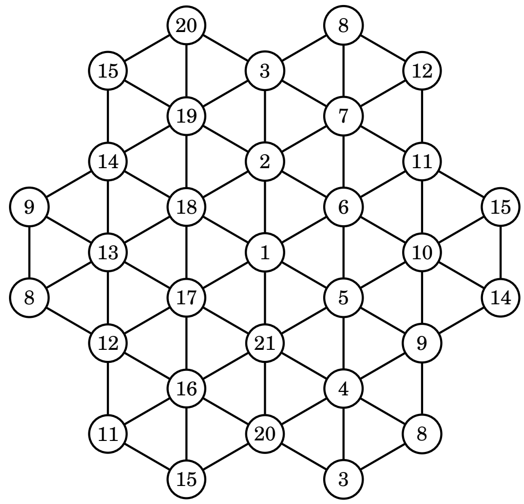


FIGURE 4. The nerve of the Coxeter group $W_{T_{21}}$

$$A_{i,j} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } |i-j| = \pm 1, \pm 4 \text{ or } \pm 5 \,(\bmod 21); \\ -u & \text{if } |i-j| = \pm 3, \pm 6 \text{ or } \pm 9 \,(\bmod 21); \\ -v & \text{if } |i-j| = \pm 2, \pm 8 \text{ or } \pm 10 \,(\bmod 21); \\ -w & \text{if } |i-j| = \pm 7 \,(\bmod 21), \end{cases}$$

where $u = \frac{1}{50}\left(27 + 7\sqrt{21}\right)$, $v = \frac{1}{50}\left(21 + 11\sqrt{21}\right)$ and $w = \frac{1}{50}\left(49 + 9\sqrt{21}\right)$. A computation shows that the matrix A has rank 7 and signature $(6,1,14)$.

So, [Vin85, Theorem 2.1] shows that there is a right-angled Coxeter polytope $\mathscr{P}_{21}$ with 21 facets in $\mathbb{H}^6$ whose Gram matrix is A. The group $\Gamma^6 = \mathrm{Refl}(\mathscr{P}_{21})$ generated by the 21 reflections along the boundary hyperplanes of $\mathscr{P}_{21}$ is isomorphic to $W_{T_{21}}$, by Poincaré's polyhedron theorem.

The group Γ^6 is convex cocompact, since $u, v, w > 1$ (see [DH13, Theorem 4.12]). Lemma 2.3.(**1.**) shows that the Gromov boundary of $W_{T_{21}}$ is homeomorphic to the limit set of Γ^6. As the nerve of $W_{T_{21}}$ is a flag-no-square triangulation of a torus, the Gromov boundary of $W_{T_{21}}$ is a Pontryagin sphere, by Lemma 2.2. Hence, the limit set of the convex cocompact subgroup Γ^6 is a Pontryagin sphere. □

4.3. A two generator group that contains Γ^6 as a finite-index subgroup.

Proposition 4.2. *There exists a 2-generated convex cocompact subgroup Γ^6_2 of* $\mathrm{Isom}(\mathbb{H}^6)$ *containing Γ^6 as a normal subgroup of index* 21.

Proof. Let

$$R_m = \begin{pmatrix} \cos\left(\frac{2m\pi}{21}\right) & -\sin\left(\frac{2m\pi}{21}\right) \\ \sin\left(\frac{2m\pi}{21}\right) & \cos\left(\frac{2m\pi}{21}\right) \end{pmatrix} \quad \text{and} \quad \sigma = \begin{pmatrix} R_1 & 0 & 0 & 0 \\ 0 & R_4 & 0 & 0 \\ 0 & 0 & R_5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}(6,1).$$

Let $y = \left(\alpha, 0, \alpha, 0, \alpha, 0, \beta\right)$, where $\alpha = \frac{1}{5}\sqrt{11 + \sqrt{21}}$ and $\beta = \frac{1}{5}\sqrt{8 + 3\sqrt{21}}$. Note that y is a space-like vector of $\mathbb{R}^{6,1}$ with length 1, that is, $\langle y, y \rangle = 1$, where $\langle\ ,\ \rangle$ is the standard Minkowski metric on $\mathbb{R}^{6,1}$. Therefore, for each integer $k \in \{1, 2, \ldots, 21\}$, a reflection $s_k \in \mathrm{O}(6,1)$ can be defined by

$$s_k(x) = x - 2\langle \sigma^{k-1}(y), x \rangle \sigma^{k-1}(y) \quad \text{for all } x \in \mathbb{R}^{6,1}.$$

Note that $s_k = \sigma^{k-1} s_1 \sigma^{-k+1}$ for all $k \in \{1,2,\dots,21\}$ and we can compute that

$$\langle \sigma^{i-1}(y), \sigma^{j-1}(y) \rangle = A_{i,j} \quad \text{for all } i,j \in \{1,2,\dots,21\}.$$

Since the group Γ^6 and the group generated by $s_1,\dots,s_{21}$ have the same dimension and the same Gram matrix, they are conjugate by [Vin85, Theorem 2.1]. Let Γ^6_2 be the group generated by σ and s_1. From the relations, for $k \in \{1,2,\dots,21\}$, $\sigma^k s_1 \sigma^{-k} = s_k$, we get that Γ^6 is a normal subgroup of Γ^6_2. The index of Γ^6 in Γ^6_2 is equal to the order of σ in the quotient Γ^6_2/Γ^6, hence this index is 1,3,7 or 21. If the order of σ in Γ^6_2/Γ^6 is not 21 then Γ^6 contains σ, σ^3, or σ^7, and hence contains an element of order 21, 7, or 3. But, in a Coxeter group, a finite-order element is conjugate into a spherical special subgroup (see [Dav08, Theorem. 12.3.4.(i)]). In particular, in a right-angled Coxeter group, finite-order elements are of order a power of 2, and hence σ must be of order 21 in Γ^6_2/Γ^6. □

Proof of Theorem 1.2. A finite-index subgroup of a convex cocompact group is convex cocompact as well, and has the same limit set. So, Propositions 4.1 and 4.2 together imply Theorem 1.2. □

4.4. **A few remarks.**

Proposition 4.3. *The group Γ^6 is not contained, even virtually, in an arithmetic lattice in* $\mathrm{Isom}(\mathbb{H}^6)$.

Remark 4.4. We do not know if Γ^6 is contained, virtually or not, in a non-arithmetic lattice in $\mathrm{Isom}(\mathbb{H}^6)$.

Proof. The Gram matrix of $\mathscr{P}_{21}$ contains three principal submatrices that are very symmetric. More precisely, for $j = 1,2,3$, let $S_j = \{k \in \{1,\dots,21\} \mid k \equiv j\}$. Then the principal submatrix A_j of the Gram matrix A obtained by deleting the rows and columns not in S_j is equal to :

$$A_1 = A_2 = A_3 = \begin{pmatrix} 1 & -u & -u & -u & -u & -u & -u \\ -u & 1 & -u & -u & -u & -u & -u \\ -u & -u & 1 & -u & -u & -u & -u \\ -u & -u & -u & 1 & -u & -u & -u \\ -u & -u & -u & -u & 1 & -u & -u \\ -u & -u & -u & -u & -u & 1 & -u \\ -u & -u & -u & -u & -u & -u & 1 \end{pmatrix}$$

Geometrically, this means that the Coxeter polytope $\mathscr{P}_{21}$ is the intersection of three isometric regular simplicies Δ_j, for $j = 1,2,3$.

We claim that $\mathrm{Refl}(\Delta_1)$ is not virtually contained in an arithmetic lattice, which trivially implies that $\mathrm{Refl}(\mathscr{P}_{21})$ is not virtually contained in an arithmetic lattice.

To that end, note first that $\mathrm{Refl}(\Delta_1)$ is Zariski dense in $\mathrm{O}(f)$ by [BH04], where f is the Tits bilinear form on $\mathbb{R}^7$ given by A_1. Now suppose $\mathrm{Refl}(\Delta_1)$ were contained in an arithmetic lattice of $\mathrm{O}(f,\mathbb{R})$. Then, by [Vin71, Thm. 3], the trace of $\mathrm{Ad}(\gamma)$ would be an algebraic integer for each $\gamma \in \mathrm{Refl}(\Delta_1)$, where Ad denotes the adjoint representation of $\mathrm{O}(f,\mathbb{R})$. On the other hand, since $\Gamma \subset \mathrm{O}(f,\mathbb{Q}(\sqrt{21}))$, this would imply that the trace of γ is an algebraic integer for each $\gamma \in \mathrm{Refl}(\Delta_1)$; see [Vin71, Theorems 1 and 2 and Lemma 2]. However, the trace of the product of two generators of $\mathrm{Refl}(\Delta_1)$ is equal to $2(u^2+1) = \frac{1}{625}(3008 + 378\sqrt{21})$, which is not an algebraic integer. □

Remark 4.5. Since the subgroup $\mathrm{Refl}(\Delta_1)$ is Zariski-dense in $\mathrm{O}(f)$, we have that the group Γ^6 is Zariski-dense in $\mathrm{O}(f)$ as well. Hence, the limit set of Γ^6 is not contained in any round k-sphere of $\mathbb{S}^k = \partial\mathbb{H}^{k+1}$, for $k \leqslant 4$.

5. Special subgroups with limit set a Menger curve

In the sequel, a simplical complex K is *non-planar* if K does not embed in $\mathbb{R}^2$. The purpose of this section is to prove the following proposition.

Proposition 5.1. *Let K be a flag-no-square triangulation of a closed connected orientable surface S of genus $g \geqslant 1$. Let $n \geqslant 1$, let $V = \{v_1, \dots, v_n\} \subseteq K^{(0)}$ be a collection of pairwise non-adjacent vertices, and let L be the full subcomplex spanned by $K^{(0)} \smallsetminus V$. Then the following hold.*

1. *L is a non-planar inseparable full subcomplex with* $\mathrm{pcd}(L) = 1$.
2. *The special subgroup W_L has Gromov boundary homeomorphic to the Menger curve.*

Our proof builds on the work in [Ś16; DŚ21; DHW23] and needs the following terminology. We will say that K is *inseparable* if it is connected and has no separating simplex, no separating pair of non-adjacent vertices, and no separating suspension of a simplex. In other words, K cannot be disconnected by removing any of the following *forbidden subcomplexes*: a vertex, an edge, a triangle, two non-adjacent vertices, two edges meeting at a vertex, or two triangles meeting along an edge.

Lemma 5.2. *Let K be a flag-no-square triangulation of a closed connected orientable surface S of genus $g \geqslant 1$. Let $M \subseteq K$ be a non-planar inseparable full subcomplex. Let $v \in M^{(0)}$ such that $\mathrm{lk}(v, M)$ is a circle, and let L be the full subcomplex of M spanned by $M^{(0)} \smallsetminus \{v\}$. Then L is a non-planar inseparable full subcomplex with* $\mathrm{pcd}(L) = 1$.

Proof. We start by noticing that both M and L are flag-no-squares complexes of dimension at most 2, because they are full subcomplexes of K. Since M is non-planar and not a sphere, and L is obtained from M by removing the open disk bounded by the circle $\mathrm{lk}(v, M)$, it follows that L is non-planar as well.

To compute the cohomological dimension, note that for any (possibly empty) simplex σ of L, the complex $L \smallsetminus \sigma$ deformation retracts to a graph (it can be thickened to a subsurface with non-empty boundary of S). In particular, the cohomology of $L \smallsetminus \sigma$ always vanishes in degree 2 and above. On the other hand, note that L does not deformation retract to a tree, because L is non-planar. So we conclude that $\mathrm{pcd}(L) = 1$.

To conclude, we show that L is inseparable. Since M is connected and $\mathrm{lk}(v, M)$ is connected, we have that L is connected. (This follows from Mayer–Vietoris applied to the decomposition $M = \mathrm{st}(v, M) \cup L$, for which $\mathrm{st}(v, M) \cap L = \mathrm{lk}(v, M)$.) Let $A \subseteq L$ be a subcomplex that disconnects L but not M. We need to check that A is not among the forbidden subcomplexes. Since $\mathrm{lk}(v, M)$ is connected, $A \cap \mathrm{lk}(v, M)$ must disconnect $\mathrm{lk}(v, M)$. Since $\mathrm{lk}(v, M)$ is a circle, $A \cap \mathrm{lk}(v, M)$ must have at least 2 non-adjacent vertices. In particular, A cannot be a simplex. For the remaining cases we argue by contradiction. Let $A = \{p, q\} * \sigma$ be the suspension of a simplex. Then $A \cap \mathrm{lk}(v, M)$ consists of the two suspension points $\{p, q\}$. But then for any vertex $r \in \sigma$ we have that p, q, r, v form an induced square in M, which cannot happen. Finally, let $A = \{u, w\} \subseteq \mathrm{lk}(v, M)$ be a pair of non-adjacent vertices. But then $\{u, v, w\}$ separates M. This is absurd since M has no separating suspension of a simplex. This concludes the proof that L is inseparable. □

Proof of Proposition 5.1. First of all, note that since K is a simplicial triangulation of a surface, the star of any vertex is an embedded closed disk, hence the link is a circle. Moreover, the open stars of the vertices in V are disjoint.

To prove (**1.**) we argue by induction on n. For $n = 1$, the statement follows from Lemma 5.2 with $M = K$ and $v \in K^{(0)}$ any vertex. Indeed, K is connected and non-planar. Moreover, since K triangulates a surface, it cannot be disconnected by any of the forbidden subcomplexes, so K is inseparable.

For the inductive step, let $n \geqslant 2$ and let $V = \{v_1, \dots, v_n\} \subseteq K^{(0)}$ be a collection of pairwise non-adjacent vertices. Let L be the full subcomplex spanned by $K^{(0)} \smallsetminus V$. By induction we know that the full subcomplex M spanned by $K^{(0)} \smallsetminus \{v_1, \dots, v_{n-1}\}$ is non-planar and inseparable. For $i = 1, \dots, n-1$ we have that v_i is not adjacent to v_n, so we have that $\mathrm{lk}(v_n, M) = \mathrm{lk}(v_n, K)$ is a circle. Since L is the full subcomplex spanned by $M^{(0)} \setminus \{v_n\}$, the desired properties of L follow from Lemma 5.2.

The statement in (**2.**) can be deduced from (**1.**) thanks to [DŚ21, Theorem 0.1.(2)]. We only need to check that L is not a join. By contradiction, say L is the join of two subcomplexes L_1, L_2. Since L has no induced squares and embeds in a surface, we have that (up to reversing the roles) L_1 is a single vertex and L_2 embeds in a circle. In particular, it follows that L embeds in a disk, contradicting the fact that L is non-planar. □

Since Γ^4 and Γ^6 are RACGs defined by flag-no-square triangulations of a torus, we can use Proposition 5.1 with $n = 1$ to obtain the following.

Theorem 5.3. *Both Γ^4 and Γ^6 contain special subgroups with limit set a Menger curve.*

Moreover, we can find smaller special subgroups with limit set a Menger curve, as follows.

The complex defining Γ^6 has a collection of 7 pairwise non-adjacent vertices, given by the 7 centers of the hexagonal tiling of the defining complex; see Figure 5. By Proposition 5.1, the special subgroup generated by the other 14 vertices has Gromov boundary homeomorphic to the Menger curve. Since the full subcomplex on these 14 vertices is 1-dimensional (i.e., it is a triangle-free graph) the Gromov boundary could also be computed with [DHW23, Corollary 1.7]. Note that this graph has $K_{3,3}$ as a minor, i.e., one can obtain $K_{3,3}$ from this graph by deleting edges, deleting vertices, and contracting edges. So, this graph is non–planar by Wagner's theorem.

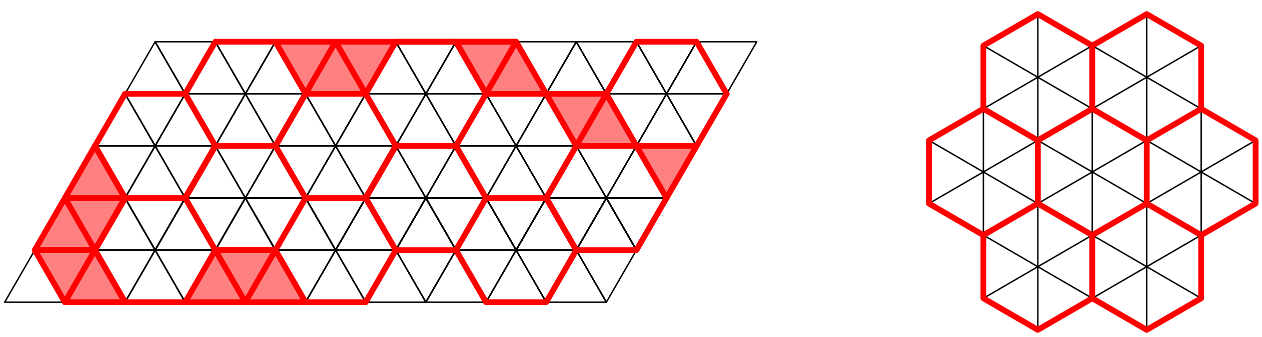


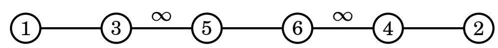
FIGURE 5. A special subgroup of Γ^d with limit set a Menger curve ($d = 4$ left, $d = 6$ right).

The complex defining Γ^4 has a collection of 14 pairwise non-adjacent vertices, which one can find by trying to fit a hexagonal grid in this complex; see Figure 5. By Proposition 5.1, the special subgroup generated by the other 36 vertices has Gromov boundary homeomorphic to the Menger curve.

We note that, in contrast to the case of Γ^6, one cannot directly apply [DHW23, Corollary 1.7] to find a special subgroup with limit set a Menger curve, because the complex K defining Γ^4 does not admit a triangle-free inseparable full subgraph. This can be seen as follows. Since every vertex in K has degree 6, in any triangle-free subgraph $L \subseteq K$, every vertex can have degree at most 3. On the other hand, if L has a vertex v of degree 1 or 2, then the link of v in L is a global cut vertex or a global cut pair respectively. It follows that every vertex of L must have degree exactly 3. In particular, every edge of L is contained in 2 edge-loops of length 6, so the complementary regions of L in K are given by regular hexagons. However, K is not tiled by hexagons, because the number of triangles is 100, which is not a multiple of 6.

Remark 5.4. Bourdon [Bou97] previously constructed examples of convex cocompact subgroups of $\mathrm{Isom}(\mathbb{H}^4)$ with limit set a Menger curve. It was explained to the first-named author by François Guéritaud that at least some of Bourdon's examples are (perhaps up to deformation) commensurable to hyperbolic reflection groups, as is illustrated by the following example.

Consider the Coxeter group W with the following Coxeter diagram.[2]



Denoting by $s_1,\dots,s_6$ the corresponding standard generators, consider the index-6 reflection subgroup W' of W generated by s_1, s_2, $t_1 := s_3$, $t_2 := s_4$, $t_3 := s_5s_3s_5$, $t_4 := (s_5s_6s_5)s_4(s_5s_6s_5)$, $t_5 := (s_6s_5s_6)s_3(s_6s_5s_6)$, and $t_6 := s_6s_4s_6$ (note that the dihedral subgroup of W generated by s_5 and s_6 is a retract of W, and W' is nothing but the kernel of the retraction map from W onto this dihedral subgroup). The subgroup $\Gamma < W'$ generated by s_1t_{2k+1} and s_2t_{2k+2} for $0 \leqslant k \leqslant 2$ is of index 4 in W' and is isomorphic to the group Γ_{63} in Bourdon's notation [Bou97]; the latter is, as demonstrated by Benakli [Ben92], a Gromov hyperbolic group with Gromov boundary a Menger curve. Now the group W can be realized as a convex cocompact reflection group in $\mathrm{Isom}(\mathbb{H}^4)$; indeed, by considering the signature of the associated Gram matrix, one sees that there is a unique way to do this (up to conjugation) so that the hyperbolic distance between the fixed hyperplane of s_3 and the fixed hyperplane of s_5 is equal to the distance between the fixed hyperplane of s_4 and the fixed hyperplane of s_6 (and in this case, this distance is $\operatorname{arccosh}(3/\sqrt{2})$). The resulting representation of the subgroup $\Gamma < W$ is of the form described in [Bou97].

References

[And58a] R. D. Anderson. "A characterization of the universal curve and a proof of its homogeneity". In: *Ann. of Math. (2)* 67 (1958), pp. 313–324.

[And58b] R. D. Anderson. "One-dimensional continuous curves and a homogeneity theorem". In: *Ann. of Math. (2)* 68 (1958), pp. 1–16.

[Ben92] N. Benakli. "Polyèdres et groupes hyperboliques : passage du local au global". PhD thesis. Université Paris-Sud, 1992.

[BH04] Y. Benoist and P. de la Harpe. "Adhérence de Zariski des groupes de Coxeter". In: *Compos. Math.* 140.5 (2004), pp. 1357–1366.

[Ber15] N. Bergeron. "Toute variété de dimension 3 compacte et asphérique est virtuellement de Haken (d'après Ian Agol et Daniel T. Wise)". In: *Astérisque* 367-368 (2015), Exp. No. 1078, vii–viii, 115–150.

[BM91] M. Bestvina and G. Mess. "The boundary of negatively curved groups". In: *J. Amer. Math. Soc.* 4.3 (1991), pp. 469–481.

[Bou97] M. Bourdon. "Sur la dimension de Hausdorff de l'ensemble limite d'une famille de sous-groupes convexes co-compacts". In: *C. R. Acad. Sci. Paris Sér. I Math.* 325.10 (1997), pp. 1097–1100.

[Bug84] V. O. Bugaenko. "Groups of automorphisms of unimodular hyperbolic quadratic forms over the ring $\mathbf{Z}[(\sqrt{5}+1)/2]$". In: *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* 5 (1984), pp. 6–12.

[Cox70] H. S. M. Coxeter. *Twisted honeycombs*. Vol. No. 4. Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics. American Mathematical Society, Providence, RI, 1970, pp. iv+47.

[DHW23] P. Dani, M. Haulmark, and G. Walsh. "Right-angled Coxeter groups with non-planar boundary". In: *Groups Geom. Dyn.* 17.1 (2023), pp. 127–155.

[DŚ21] D. Danielski and J. Świątkowski. "A note on the complete characterizations of hyperbolic Coxeter groups with Sierpiński curve boundary and with Menger curve boundary". In: *arXiv e-prints* (Oct. 2021). arXiv: 2110.01922 [math.GT].

[Dav08] M. W. Davis. *The geometry and topology of Coxeter groups*. Vol. 32. London Mathematical Society Monographs Series. Princeton University Press, Princeton, NJ, 2008, pp. xvi+584.

[DJ91] M. W. Davis and T. Januszkiewicz. "Hyperbolization of polyhedra". In: *J. Differential Geom.* 34.2 (1991), pp. 347–388.

[2]Here, we use the classical conventions for Coxeter diagrams, so that an edge between two vertices indicates that the product of the corresponding standard generators has order 3, an edge with label ∞ between two vertices indicates that the product of the corresponding standard generators has infinite order, and the lack of an edge between two vertices indicates that the corresponding standard generators commute.

[DH13] M. Desgroseilliers and F. Haglund. "On some convex cocompact groups in real hyperbolic space". In: *Geom. Topol.* 17.4 (2013), pp. 2431–2484.

[Dra05] A. N. Dranishnikov. "Cohomological dimension theory of compact metric spaces". In: *arXiv Mathematics e-prints* (Jan. 2005). arXiv: math/0501523 [math.GN].

[Fis03] H. Fischer. "Boundaries of right-angled Coxeter groups with manifold nerves". In: *Topology* 42.2 (2003), pp. 423–446.

[Gra15] J. Granier. *Groupes discrets en géométrie hyperbolique – aspects effectifs*. Thesis (Ph.D.)–Université de Fribourg. 2015.

[Jak91] W. Jakobsche. "Homogeneous cohomology manifolds which are inverse limits". eng. In: *Fundamenta Mathematicae* 137.2 (1991), pp. 81–95.

[KP96] P. Krupski and H. Patkowska. "Menger curves in Peano continua". In: *Colloq. Math.* 70.1 (1996), pp. 79–86.

[MX24] J. Ma and B. Xie. "The Menger curve and spherical CR uniformization of a closed hyperbolic 3-orbifold". In: *Geom. Dedicata* 218.4 (2024), Paper No. 84, 30.

[Mou88] G. Moussong. *Hyperbolic Coxeter groups*. Thesis (Ph.D.)–The Ohio State University. ProQuest LLC, Ann Arbor, MI, 1988, p. 55.

[Ont20] P. Ontaneda. "Riemannian hyperbolization". In: *Publ. Math. Inst. Hautes Études Sci.* 131 (2020), pp. 1–72.

[PT23] J. Paupert and M. Thistlethwaite. "Complex hyperbolic and projective deformations of small Bianchi groups". In: *Geom. Dedicata* 217.6 (2023), Paper No. 98, 18.

[Swe01] E. L. Swenson. "Quasi-convex groups of isometries of negatively curved spaces". In: *Topology Appl.* 110.1 (2001), pp. 119–129.

[Ś16] J. Świątkowski. "Hyperbolic Coxeter groups with Sierpiński carpet boundary". In: *Bull. Lond. Math. Soc.* 48.4 (2016), pp. 708–716.

[Ś20a] J. Świątkowski. "Trees of manifolds as boundaries of spaces and groups". In: *Geom. Topol.* 24.2 (2020), pp. 593–622.

[Ś20b] J. Świątkowski. "Trees of metric compacta and trees of manifolds". In: *Geom. Topol.* 24.2 (2020), pp. 533–592.

[Vin71] E. B. Vinberg. "Rings of definition of dense subgroups of semisimple linear groups". In: *Izv. Akad. Nauk SSSR Ser. Mat.* 35 (1971), pp. 45–55.

[Vin85] E. B. Vinberg. "Hyperbolic groups of reflections". In: *Uspekhi Mat. Nauk* 40.1(241) (1985), pp. 29–66, 255.

[Zaw10] P. Zawiślak. "Trees of manifolds and boundaries of systolic groups". In: *Fund. Math.* 207.1 (2010), pp. 71–99.

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