CUBULATED HYPERBOLIC GROUPS ADMIT ANOSOV REPRESENTATIONS

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Abstract. We prove that any hyperbolic group acting properly discontinuously and cocompactly on a CAT(0) cube complex admits a projective Anosov representation into $\mathrm{SL}(d, \mathbb{R})$ for some d. More specifically, we show that if Γ is a hyperbolic quasiconvex subgroup of a right-angled Coxeter group C, then a generic representation of C by reflections restricts to a projective Anosov representation of Γ.

Contents

1. Introduction

The prototypical examples of Gromov-hyperbolic groups are the convex cocompact subgroups of $\mathrm{PO}(n, 1) = \mathrm{Isom}(\mathbb{H}^n)$, where $\mathbb{H}^n$ denotes real hyperbolic n-space, and more generally, of $\mathrm{Isom}(X)$, where X is a rank-one symmetric space of noncompact type. It is known that not all hyperbolic groups can arise in this fashion: there are hyperbolic groups which do not admit faithful representations into *any* matrix group [36, 9, 57, 58], and even among linear hyperbolic groups there are examples which fail to admit discrete faithful representations in any rank-one Lie group [57, 24].

Apart from the convex cocompact subgroups of linear rank-one Lie groups, a significant source of linear hyperbolic groups is the class of hyperbolic groups that are *virtually special* in the sense of Haglund and Wise [32]. A theorem of Agol [1] established that the virtually compact special hyperbolic groups are precisely those hyperbolic groups that are *cubulated*, that is, those that act properly discontinuously and cocompactly on CAT(0) cube complexes. Many diverse examples of cubulated hyperbolic groups can be found "in nature" and in the literature: see e.g. [52, 61, 53, 2, 63, 25]. In fact, there are currently no known examples of noncubulated convex cocompact subgroups of $\mathrm{PO}(n, 1)$, and indeed, Wise conjectured that no such subgroups exist [62, Conj. 13.52]. The following question is also due to Wise [62, Prob. 13.53].

Question 1.1. Does every cubulated hyperbolic group embed as a convex cocompact subgroup of $\mathrm{PO}(n,1)$ for some $n \geq 2$?

There are positive results towards Question 1.1 for some low-dimensional cubulated hyperbolic groups; see e.g. [36, 48, 33]. However, the question remains open even for some of the most standard examples of cubulated hyperbolic groups. For example, it is not known if every hyperbolic right-angled Coxeter group embeds convex cocompactly in $\mathrm{PO}(n,1)$ for some n, or even if the latter holds for such Coxeter groups of arbitrarily large virtual cohomological dimension. On the other hand, Danciger–Guéritaud–Kassel [13] and Lee–Marquis [47] (see also [14]) have shown that every hyperbolic Coxeter group admits an *Anosov* representation.

1.1. **Anosov representations.** Anosov representations were introduced by Labourie [45], and their theory subsequently developed by Guichard–Wienhard [29], Kapovich–Leeb–Porti [40, 41, 42], Guéritaud–Guichard–Kassel–Wienhard [28], Bochi–Potrie–Sambarino [4], and many others. They have emerged as a successful generalization of convex cocompact representations in arbitrary rank. An Anosov representation $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ gives a quasi-isometric embedding of Γ into the target group, and the class of Anosov representations is stable under small perturbations: the set of Anosov representations from Γ into $\mathrm{GL}(d,\mathbb{R})$ is an open subset of $\mathrm{Hom}(\Gamma, \mathrm{GL}(d,\mathbb{R}))$ [45]. Anosov representations may also be characterized in terms of the existence of limit maps relating the dynamics of Γ acting on its boundary to the dynamics of the image subgroup acting on a flag variety [40], and are closely related to convex cocompact actions in real projective space [12].

The class of groups admitting an Anosov representation is known to be strictly larger than that of groups which admit a convex cocompact embedding in rank one [24]. However, little is currently understood about the possible isomorphism types of groups admitting Anosov representations. Any such group is necessarily finite-by-linear, and is Gromov-hyperbolic [40, 4], but as observed by Canary [7, Q. 50.2], there are no other known restrictions:

Question 1.2. Does every linear hyperbolic group admit an Anosov representation?

Even though Question 1.2 is open, there are still surprisingly few constructions available for producing concrete examples of groups admitting Anosov representations, beyond the groups that are already known to admit convex cocompact representations in rank one. Apart from hyperbolic Coxeter groups, Kapovich [37] has shown that fundamental groups of certain Gromov–Thurston manifolds admit Anosov representations (note that in high dimensions, groups of the latter form are not commensurable to Coxeter groups [35]). In another direction, some combination theorems for Anosov subgroups have been established in recent years [11, 23, 21, 22], which can be used to prove that the class of groups admitting Anosov embeddings into $\mathrm{SL}(d,\mathbb{R})$ for some d is closed under free products [12, 24].

In this paper, we prove:

Theorem 1.3. *Every cubulated hyperbolic group admits an Anosov representation.*

Theorem 1.3 significantly enlarges the family of groups known to admit Anosov representations; see Section 1.4 below for some example applications. However, our proof also provides new examples of Anosov representations even for groups already known to admit them, since our main theorem (Theorem 1.5 below) can provide different Anosov representations for different cubulations of the same group.

Before stating the result, we briefly recall the definition of an Anosov representation. For a matrix $g \in \mathrm{GL}(d, \mathbb{R})$ and $1 \le k \le d$, we let $\sigma_k(g)$ denote the k^{th} largest *singular value* of g, i.e. the k^{th} largest eigenvalue of the matrix $\sqrt{gg^T}$, counted with multiplicity.

Definition 1.4. Let Γ be a finitely generated group, equipped with the word metric $|\cdot|$ induced by a finite generating set. A representation $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is *k-Anosov* for some $1 \le k < d$ if there are constants $A, B > 0$ so that for all $\gamma \in \Gamma$,

$$\log\left(\frac{\sigma_k(\rho(\gamma))}{\sigma_{k+1}(\rho(\gamma))}\right) \ge A|\gamma| - B.$$

A 1-Anosov representation is also called a *projective* Anosov representation. We will discuss singular values and Anosov representations in further detail in Section 6. For now, we remark that Definition 1.4 bears little resemblance to Labourie's original definiton of an Anosov representation, which was stated in terms of a certain flow on a bundle associated to the representation; the equivalence of the definitions is a theorem due to Kapovich–Leeb–Porti [40] and Bochi–Potrie–Sambarino [4].

Our main theorem is as follows:

Theorem 1.5. *Let (C, S) be a right-angled Coxeter system, let $\Gamma \hookrightarrow C$ be a quasiconvex embedding of a hyperbolic group Γ into C, and let $\rho\colon C \to \mathrm{SL}^{\pm}(|S|, \mathbb{R})$ be a simplicial representation of C whose Cartan matrix is* fully nondegenerate*. Then the restriction $\rho|_\Gamma$ is 1-Anosov.*

In this paper, a *simplicial representation* of a right-angled Coxeter group C is a deformation of the well-known *geometric representation* $C \to \mathrm{SL}^{\pm}(|S|, \mathbb{Z})$ studied by Tits [5]; see Section 4.1 for more detail. The geometric representation itself essentially never satisfies the technical condition demanded by Theorem 1.5, but representations with this property do exist for any right-angled Coxeter group. In fact, these representations form an open dense subset of the space of simplicial representations, and can even be arranged to have image lying in $\mathrm{O}(p, q)$ or $\mathrm{SL}^{\pm}(|S|, \mathbb{Z})$ (see Remark 4.9).

Thus, our proof of Theorem 1.3 does not reprove the result of Haglund–Wise and Agol that cubulated hyperbolic groups are linear. Rather, we use their characterization of cubulated hyperbolic groups as hyperbolic virtual quasiconvex subgroups of right-angled Coxeter groups, and apply Theorem 1.5. Theorem 1.3 then follows once we know that admitting an Anosov representation is a commensurability invariant (see [24, Lemma 2.1]).

1.2. **Actions of reflection groups on projective space.**

In the special case where the ambient right-angled Coxeter group C in Theorem 1.5 is Gromov-hyperbolic, then work of Danciger–Guéritaud–Kassel–Lee–Marquis [14] (see also [13]) implies that any representation $\rho\colon C \to \mathrm{SL}^{\pm}(|S|, \mathbb{R})$ as in the theorem is already 1-Anosov; in that case it follows easily that the restriction of ρ to any quasiconvex subgroup of C is 1-Anosov also.

However, the Haglund–Wise construction typically yields a quasiconvex embedding of a compact special hyperbolic group Γ into a *non-hyperbolic* right-angled Coxeter group C, and when this occurs we cannot invoke the results in [13] or [14]. Further, we know of no procedure that replaces C with a hyperbolic Coxeter group (although we do not know of any reason such a procedure cannot exist).

Our proof of Theorem 1.5 ultimately differs significantly from the approach in [13] and [14], and provides a new proof of the fact that hyperbolic right-angled Coxeter groups admit Anosov representations. Indeed, our methods rely on a completely different characterization of Anosov representations. However, the starting point for both proofs is the same: we consider representations $C \to \mathrm{SL}^{\pm}(|S|, \mathbb{R})$ which are *generated by reflections*. The general theory of such representations was developed by Vinberg [60], generalizing Tits' study of the geometric representation, and centers around the action of C on a *convex domain* in real projective space $\mathbb{P}(\mathbb{R}^{|S|})$.

As an illustrative example, the geometric representation of the free product $\Gamma = \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$ realizes Γ as a reflection lattice in $\mathrm{O}(2,1)$, acting on the *projective* or *Klein* model of the hyperbolic plane embedded in $\mathbb{P}(\mathbb{R}^3)$ as a convex ball. Although Γ is a hyperbolic group in this case, the geometric representation fails to be Anosov, as the product of any pair of distinct generators is a nontrivial unipotent element in $\mathrm{O}(2,1)$. However, Γ also admits *convex cocompact* representations into $\mathrm{O}(2,1)$, where the product of any pair of distinct generators is instead loxodromic. Such representations also preserve another convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^3)$, called the *Vinberg domain*, which is *not* projectively equivalent to the Klein model for $\mathbb{H}^2$.

1.3. **Proof idea.** Our proof of Theorem 1.5 heavily exploits the relationship between the projective geometry of the Vinberg domain Ω for a right-angled Coxeter system (C, S) acting by reflections, and the combinatorial geometry of the *Davis complex* $\mathrm{D}(C, S)$, a natural CAT(0) cube complex with a properly discontinuous and cocompact C-action. Specifically, we relate *half-spaces* in $\mathrm{D}(C, S)$ to certain convex subsets of projective space, which we call *half-cones*. By examining the nesting properties of half-cones, we are able to prove that if a geodesic γ_n in an irreducible right-angled Coxeter group C does not get "stuck" inside of a proper *standard* subgroup in C (i.e. a subgroup generated by a subset of the generating set S) then the singular value gaps of the sequence $\rho(\gamma_n)$ grow at a uniform rate. This directly verifies the condition in Definition 1.4.

Our proof also needs to handle the case where the geodesic γ_n spends arbitrarily long amounts of time inside of standard subgroups of C, and this is where the majority of the work takes place. The strategy is to induct on the size of the generating set S, and assume that the geodesic γ_n experiences uniform singular value gap growth on each sub-geodesic lying in a proper standard subgroup of C. The challenge is then to "glue together" the singular value gap growth on each of these sub-geodesics.

This "gluing" process is somewhat involved, but there are essentially only two different techniques at play. One of them is a *uniform transversality* argument for stable and unstable subspaces of elements in $\mathrm{SL}^{\pm}(|S|, \mathbb{R})$, and relies on an understanding of the convex projective geometry of the Vinberg domain for ρ. The other technique is to apply the *higher-rank Morse lemma* and *local-to-global principle* of Kapovich–Leeb–Porti [42, 43] (see also [49]), a pair of deep theorems about the geometry of certain quasi-geodesic sequences in higher-rank symmetric spaces.

Remark 1.6. An interesting feature of our proof is that it uses the hyperbolicity of the quasiconvex subgroup Γ only indirectly, in the form of a condition on the walls in $\mathrm{D}(C, S)$ crossed by an arbitrary geodesic in Γ (see Definition 3.1). As a consequence, our proof actually shows that this condition implies hyperbolicity of Γ, since any group admitting an Anosov representation is necessarily hyperbolic; see Remark 3.3. In the special case where $\Gamma = C$, this gives an alternative (albeit inefficient) proof of Gromov's "no empty square" characterization of hyperbolic right-angled Coxeter groups relying on the theory of Anosov representations (this was also accomplished in [13] and [47]).

1.4. **Examples and applications.** Theorem 1.3 provides evidence that groups admitting Anosov representations are in some sense "abundant." One way to make this precise is Gromov's density model for random finitely presented groups: at density $< \frac{1}{12}$, a random group satisfies the $C'(\frac{1}{6})$ small cancellation condition, and such groups are hyperbolic and cubulated [61]; more generally, at density $< \frac{1}{6}$, random groups are hyperbolic and cubulated (see [27, 9.B], [53]). Hence, by Theorem 1.3, random finitely presented groups at density $< \frac{1}{6}$ admit Anosov representations.

These results tell us that we cannot control the dimension of the Anosov representations provided by Theorem 1.3 even for torsion-free cubulated hyperbolic groups of bounded cohomological dimension: for fixed n, a random group at any positive density has no n-dimensional linear representations with finite kernel [44], while random groups at density $< \frac{1}{2}$ have cohomological dimension 2.

We mention, however, that not all groups that admit 1-Anosov representations into $\mathrm{SL}(d,\mathbb{R})$—indeed, not all convex cocompact subgroups of rank-one Lie groups—are cubulated. For example, any action of a discrete group with Kazhdan's property (T) on a finite-dimensional CAT(0) cube complex has a global fixed point [51], ruling out cubulability for uniform lattices in $\mathrm{Sp}(n,1)$, $n \geq 2$, and $F_{4(-20)}$. For subtler reasons, it is also true that uniform lattices in $\mathrm{PU}(n,1)$, $n \geq 2$, fail to be cubulated [19, 56].

1.4.1. *Strict hyperbolization.* Recent work of Lafont and Ruffoni [46] has established that the Charney–Davis *strict hyperbolization* process [10] also yields cubulated hyperbolic groups, which allows us to produce examples of Anosov subgroups with various "exotic" properties. For instance, Ontaneda [54] has used strict hyperbolization to construct new examples of closed negatively curved Riemannian manifolds in any dimension $n \geq 4$ which are not homeomorphic to any locally symmetric space of rank one. Work of Januszkiewicz–Świątkowski [35] implies that the fundamental groups of these manifolds are not commensurable to any Coxeter group when $n > 61$, meaning that Theorem 1.3 gives the first proof that these groups admit Anosov representations.

For another sample application of Theorem 1.3 and strict hyperbolization, recall that if M is a closed negatively curved Riemannian manifold, then the Gromov boundary of $\pi_1(M)$ is a topological sphere. However, Davis–Januszkiewicz [17] showed that the latter may fail if M is merely a closed aspherical manifold with hyperbolic fundamental group. The Davis–Januszkiewicz examples are constructed via strict hyperbolization, so their fundamental groups are cubulated by the work of Lafont–Ruffoni. Combining these results with a theorem of Bestvina [3, Thm. 2.8] and Theorem 1.3 yields the following:

Theorem 1.7. *For every $n \geq 4$, there is some $d \in \mathbb{N}$ and an Anosov subgroup of* $\mathrm{SL}(d,\mathbb{R})$ *whose Gromov boundary is not homeomorphic to an n-sphere, but is nevertheless a homology n-manifold with the homology of an n-sphere.*

This theorem may be viewed as a positive answer for each $n \geq 4$ to a variant of a question of Kapovich [38, Q. 9.4], within the broader realm of Anosov groups. It in fact follows from known results that Theorem 1.7 also holds for $n = 3$. Indeed, for an example with $n = 3$, it suffices to take one of the Anosov representations guaranteed by Danciger–Guéritaud–Kassel [13] of a right-angled Coxeter group given by a flag no-square triangulation of a nontrivial homology 3-sphere (see [15]); for the existence of such triangulations, see [55]. That the latter approach fails as soon as $n \geq 4$ follows from aforementioned work of Januszkiewicz–Świątkowski [35, Sect. 2.2].

Remark 1.8. The situation for groups admitting 1-Anosov representations into $\mathrm{SL}(d,\mathbb{R})$ appears to be strikingly different from the situation for groups admitting *Borel* Anosov representations into $\mathrm{SL}(d,\mathbb{R})$ (a representation $\rho\colon \Gamma \to \mathrm{SL}(d,\mathbb{R})$ is Borel Anosov if it is k-Anosov for every $1 \leq k < d$). Indeed, Sambarino conjectured that every group admitting a

Borel Anosov representation into $\mathrm{SL}(d,\mathbb{R})$ is virtually either a free group or the fundamental group of a closed surface, and this conjecture has been verified for infinitely many d (see [8], [59], [20]).

1.5. **Further questions.** We have already observed that Theorem 1.5 would no longer hold if we removed the hypothesis regarding fully nondegenerate Cartan matrices (see the end of Section 1.2). However, it seems plausible that a version of Theorem 1.5 could hold with a considerably relaxed version of this hypothesis, as long as we impose some assumptions on the quasiconvex embedding $\Gamma \hookrightarrow C$. For instance, it might be sufficient to ask for Γ to have finite intersection with every standard virtually unipotent subgroup of C. For appropriate quasiconvex embeddings, this could allow the conclusion of Theorem 1.5 to hold for an *arbitrary* representation of C by reflections (in particular, for the geometric representation).

Even in its current form, Theorem 1.5 still gives us a great deal of freedom to pick the simplicial representation $\rho\colon C \to \mathrm{SL}^{\pm}(d,\mathbb{R})$. In particular, for a fixed infinite right-angled Coxeter group C, we can pick ρ from a positive-dimensional submanifold M of the representation variety $\mathrm{Hom}(C,\mathrm{SL}^{\pm}(d,\mathbb{R}))$. We might want to consider properties of the restriction map $R\colon M \to \mathrm{Hom}(\Gamma,\mathrm{SL}^{\pm}(d,\mathbb{R}))$—for instance, it would be interesting to know the dimension of $R(M)$, or whether $R(M)$ contains any representations with Zariski-dense image.

1.6. **Acknowledgments.** The authors would like to thank Gye-Seon Lee, Jason Manning, Lorenzo Ruffoni, and Abdul Zalloum for productive discussion. We are also grateful to the Mathematisches Forschungsinstitut Oberwolfach and the organizers of the Arbeitsgemeinschaft: Higher Rank Teichmüller Theory (2241), where this work was initiated. S.D. was supported by the Huawei Young Talents Program. B.F. was partially supported by the grant NRF-2022R1I1A1A01072169 and received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (ERC starting grant DiGGeS, grant agreement No 715982, and ERC consolidator grant GeometricStructures, grant agreement No 614733). T.W. was partially supported by NSF grant DMS-2202770. F.Z. was partially supported by ISF grant 737/20 and an AMS-Simons Travel Grant.

2. Cube complexes and right-angled Coxeter groups

In this section we briefly review some essential background on the topic of nonpositively curved cube complexes, right-angled Coxeter groups, and the Davis complex. We refer to [32], [16] for further detail. Afterwards, we introduce a useful combinatorial framework for working with geodesics in the Davis complex, in the form of *itineraries*.

2.1. **CAT(0) cube complexes.** For our purposes, a *cube complex* X is a finite-dimensional cell complex in which each cell is a cube and attaching maps are combinatorial isomorphisms onto their images. A d-cell of X is a *d-cube*. A 0-cube is a *vertex*, a 1-cube is an *edge*, and a 2-cube is a *square*. Under the identification of a d-cube c of X with $[-1,1]^d$, a *midcube* of c is an intersection of c with a coordinate hyperplane of $\mathbb{R}^d$. By gluing midcubes of adjacent cubes of X whenever they meet, one obtains immersed subspaces of X, called *hyperplanes*, each of which also carries a natural cube complex structure. Note that a compact cube complex possesses only finitely many hyperplanes.

Two edges of a cube complex X are *elementary parallel* if they appear as opposite edges of a square in X. A *wall* of X is a class of the equivalence relation on the edge set of X generated by elementary parallelisms. Two edges of X are *parallel* if they belong to the same wall of X (in other words, if they are dual to the same hyperplane of X). Frequently, we will abuse terminology and refer to properties of "walls" when we really mean properties of the corresponding hyperplanes. In particular, when we say that an intersection of walls $W_1 \cap W_2$ is empty or nonempty, we mean to refer to an intersection of hyperplanes.

One says a cube complex X is *nonpositively curved* if the link of each vertex of X is a flag simplicial complex; recall that a simplicial complex L is *flag* if each clique $\mathcal{V}$ in L spans a $(|\mathcal{V}|-1)$-simplex. If X is moreover simply connected, then X is said to be CAT(0). Implicit in this terminology is a theorem of Gromov [26] that the path metric on a cube complex X induced by the Euclidean metric on each of its cubes is CAT(0) if and only if X is CAT(0) in the previous combinatorial sense. A key feature of a CAT(0) cube complex is that each of its hyperplanes is separating.

2.2. **Right-angled Coxeter groups.** Let Σ be a finite simplicial graph with vertex set S. The *right-angled Coxeter group* C_Σ with *nerve* Σ is the group given by the presentation with generating set S subject to the relations that two generators $s, t \in S$ commute if and only if s and t are adjacent as vertices in Σ. The pair (C_Σ, S) is a *right-angled Coxeter system*. A conjugate within C_Σ of an element of S is a *reflection*. If the complement of Σ is connected, we say (C_Σ, S) is *irreducible*; this is equivalent to saying that C_Σ does not decompose as a nontrivial direct product.

If (C, S) is a right-angled Coxeter system and $T \subset S$, we denote by $C(T)$ the subgroup of C generated by T. We refer to such subgroups of C as *standard subgroups*. For any $T \subset S$, the pair $(C(T), T)$ is again a right-angled Coxeter system.

2.2.1. *The Davis complex.* Given a finite simplicial graph Σ, there is a CAT(0) cube complex $\mathrm{D}(C_\Sigma, S)$, called the *Davis complex* of (C_Σ, S), on which the group C_Σ acts by combinatorial automorphisms, that may be constructed as follows. Let $\mathrm{D}'(C_\Sigma, S)$ be the square complex (i.e. 2-dimensional cube complex) obtained by attaching a square to each labeled 4-cycle of the form $stst$ in the Cayley graph $\mathrm{Cay}(C_\Sigma, S)$ of C_Σ with respect to the generating set S, where $s, t \in S$. Then $\mathrm{D}(C_\Sigma, S)$ is the unique nonpositively curved cube complex with 2-skeleton $\mathrm{D}'(C_\Sigma, S)$.

Each wall W in $\mathrm{D}(C_\Sigma, S)$ is fixed by a unique reflection r in C_Σ, and conversely each reflection r fixes a unique wall. As any reflection r is a conjugate of a unique $s \in S$, the edges comprising any given wall W of $\mathrm{D}(C_\Sigma, S)$ are all labeled with a single generator $s(W) \in S$. We call $s(W)$ the *type* of W. For each $s \in S$, we also let $W(s)$ denote the unique wall fixed by the reflection s.

2.2.2. *Quasiconvex subgroups.* Given a right-angled Coxeter system (C, S), a subgroup $\Gamma < C$ is *quasiconvex* (with respect to the standard generating set S) if, viewing Γ as a subset of the vertices of $\mathrm{D}(C, S)$, there is some $K > 0$ such that every combinatorial geodesic in $\mathrm{D}(C, S)$ with endpoints in Γ lies in the combinatorial K-neighborhood of Γ. (For example, if s_1, t_1, s_2, t_2 are distinct elements of S such that each element of $\{s_1, t_1\}$ commutes with each element of $\{s_2, t_2\}$, but s_i and t_i do not commute for $i = 1, 2$, then the cyclic subgroup $\langle s_1 t_1 s_2 t_2 \rangle < C$ is *not* quasiconvex.) A subgroup $\Gamma < C$ is quasiconvex if and only if there is a Γ-invariant convex subcomplex $\widetilde{Y}$ of $\mathrm{D}(C, S)$ on which Γ acts cocompactly; see [32, Cor. 7.8] or [31, Thm. H]. In particular, standard subgroups of C are quasiconvex. This second characterization of quasiconvexity is the one that will be relevant to us.

It turns out that the hyperbolic groups that are cubulated are precisely those that virtually embed as quasiconvex subgroups of right-angled Coxeter groups. Indeed, it follows from seminal work of Haglund–Wise [32] and Agol [1] that if a hyperbolic group Γ acts properly discontinuously and cocompactly on a CAT(0) cube complex $\widetilde{X}$, then there is a finite-index subgroup $\Gamma' < \Gamma$, a right-angled Coxeter system (C, S), an embedding $\iota\colon \Gamma' \to C$, and an ι-equivariant embedding of $\widetilde{X}$ as a convex subcomplex of $\mathrm{D}(C, S)$.

In more detail, Agol [1] showed that there is a finite-index torsion-free subgroup $\Lambda < \Gamma$ such that the cube complex $\widetilde{X}/\Lambda$ is *special* in the sense of Haglund and Wise [32], answering a question of the latter two authors [32, Prob. 11.7]. One can then pass to a deeper finite-index subgroup $\Gamma' < \Lambda$ such that $X := \widetilde{X}/\Gamma'$ is moreover *C-special*; see [32, Prop. 3.10]. The latter condition is designed to ensure the existence of a *local isometry* from the cube complex X to the orbicomplex $\mathrm{D}(C_X, S_X)/C_X$ for some right-angled Coxeter system (C_X, S_X) associated to X, inducing an embedding $\iota\colon \Gamma' \to C_X$ between (orbicomplex) fundamental groups, and lifting to an ι-equivariant embedding of $\widetilde{X}$ as a convex subcomplex of $\mathrm{D}(C_X, S_X)$. (Alternatively, one can find a finite-index subgroup $\Gamma'' < \Lambda$ so that $\widetilde{X}/\Gamma''$ admits a local isometry into the Salvetti complex for a right-angled Artin group, and then apply [18].)

We remark that, even in the case that the action of Γ on $\widetilde{X}$ is the action of a hyperbolic right-angled Coxeter group on its Davis complex, the right-angled Coxeter group C_X that one obtains via the process above may *not* be hyperbolic.

2.3. **Itineraries.** Throughout this paper, we will need a good understanding of the combinatorial behavior of geodesics in right-angled Coxeter groups, and especially geodesics lying in quasiconvex hyperbolic subgroups of right-angled Coxeter groups. It is often convenient to work not with the geodesics themselves, but rather some related combinatorial data in the form of an *itinerary*.

For the following, we fix a right-angled Coxeter system (C, S).

Definition 2.1. Recall that a wall W in the Davis complex $\mathrm{D}(C, S)$ can be viewed as an equivalence class of edges in the Cayley graph $\mathrm{Cay}(C, S)$. An *itinerary* is a sequence of walls $W_1, \ldots, W_n$, such that for some sequence of edges $e_i \in W_i$, the sequence $e_1 \cdots e_n$ is a geodesic edge path in $\mathrm{Cay}(C, S)$. We say that this edge path *follows* the itinerary $\mathcal{W}$.

A geodesic edge path $e_1 \cdots e_n$ in the Cayley graph $\mathrm{Cay}(C, S)$ always determines a geodesic word in S, by reading off the generators $s_i \in S$ labeling each edge e_i; conversely, a geodesic word in S gives rise to infinitely many different geodesic edge paths, one for each possible base point of the path in $\mathrm{Cay}(C, S)$. So a geodesic word records strictly less information than a geodesic edge path in $\mathrm{Cay}(C, S)$.

On the other hand an itinerary records strictly *more* information than a geodesic word but *less* information than a geodesic edge path: there may be many different edge paths which follow the same itinerary $W_1, \ldots, W_n$, but each of these edge paths determines the same geodesic word, namely the word

$$s(W_1) \cdots s(W_n).$$

Definition 2.2. If $\mathcal{W}$ is an itinerary $W_1, \ldots, W_n$, we say the geodesic word $s(W_1) \cdots s(W_n)$ is *traversed* by $\mathcal{W}$. We also say that $\mathcal{W}$ *traverses* the unique element $\gamma \in C$ represented by this geodesic word.

If $\alpha \in C$ is the initial vertex of some edge path following $\mathcal{W}$, then we say that $\mathcal{W}$ *departs from* α. Similarly, if $\beta \in C$ is the last vertex of an edge path following $\mathcal{W}$, then $\mathcal{W}$ *arrives at* β. If $\alpha, \beta \in C$, and $\mathcal{W}$ departs from α and traverses $\alpha^{-1}\beta$, then we say that $\mathcal{W}$ *joins* α to β. Note that this is stronger than saying that $\mathcal{W}$ departs from α and arrives at β.

The proposition below follows directly from the fact that walls are defined to be equivalence classes of edges in the Cayley graph of the Coxeter system (C, S):

Proposition 2.3. *Let $W_1, \ldots, W_n$ be an itinerary departing from the identity, and let $s_i = s(W_i)$ for all $1 \leq i \leq n$. Then $W_n = s_1 \cdots s_{n-1} W(s_n)$.*

In general a single itinerary can depart from different elements in C (and likewise can arrive at different elements in C). However, an itinerary always traverses a unique element.

Definition 2.4. If $\mathcal{U} = W_1, \ldots, W_n$ is an itinerary, we let $\gamma(\mathcal{U}) = \gamma(W_1, \ldots, W_n)$ denote the group element traversed by $\mathcal{U}$.

If W_i, W_j are walls in $\mathcal{U}$, with W_i appearing before W_j, we let $\gamma_{\mathcal{U}}(W_i, W_j)$ denote the group element traversed by the sub-itinerary of $\mathcal{U}$ beginning with W_i and ending with W_j.

2.3.1. *Partial order on walls.* As any wall in $\mathrm{D}(C, S)$ separates $\mathrm{D}(C, S)$ into two convex components, it follows that for any $\alpha, \beta \in C$, the walls appearing in an itinerary $W_1 \ldots, W_n$ joining α to β are precisely the walls in $\mathrm{D}(C, S)$ separating α from β. Motivated by this, we introduce the following notation.

Definition 2.5. For $\alpha, \beta \in C$, we let $\mathbf{W}(\alpha, \beta)$ denote the set of walls in $\mathrm{D}(C, S)$ separating α from β. We write $\mathbf{W}(\alpha)$ for $\mathbf{W}(\mathrm{id}, \alpha)$.

The set $\mathbf{W}(\alpha, \beta)$ is endowed with a partial order $<$, defined as follows: if $W_i, W_j \in \mathbf{W}(\alpha, \beta)$, then $W_i < W_j$ if W_i separates α from W_j in $\mathrm{D}(C, S)$ (equivalently, if W_j separates W_i from β).

Recall that two elements a, b in a poset are *incomparable* if neither $a < b$ nor $b < a$ holds. We say that two disjoint subsets A, B of a poset are *completely incomparable* if every element of A is incomparable to every element in B.

It is immediate that if $\alpha, \beta \in C$ and W_i, W_j are two walls in $\mathbf{W}(\alpha, \beta)$, then W_i and W_j are incomparable with respect to $<$ if and only if $W_i \cap W_j$ is nonempty. When this occurs, the generators $s(W_i)$ and $s(W_j)$ must commute.

Every itinerary joining α to β determines a total ordering of the set $\mathbf{W}(\alpha, \beta)$ which is compatible with the partial ordering $<$. The proposition below says that all compatible total orderings of $\mathbf{W}(\alpha, \beta)$ arise in precisely this way.

Proposition 2.6. *Let $\alpha, \beta \in C$. There is a one-to-one correspondence between the following three sets:*

(1) Itineraries joining α to β,
(2) Geodesic words in S representing $\alpha^{-1}\beta$,
(3) Total orderings of $\mathbf{W}(\alpha, \beta)$ which are compatible with $<$.

Proof. The correspondence between the first two sets is immediate, once we recognize that itineraries joining α to β are in one-to-one correspondence with geodesic edge paths in the Cayley graph $\mathrm{Cay}(C, S)$ joining α to β. We have already observed that any itinerary joining α to β gives rise to an ordering on $\mathbf{W}(\alpha, \beta)$ compatible with $<$, so we just need to check that any such ordering determines an itinerary.

First observe that for any $\gamma \in C$ and any wall W in $\mathrm{D}(C, S)$, if no walls separate γ from W, then W contains a unique edge incident to γ. To see this, let $H_\pm$ denote the half-spaces in $\mathrm{D}(C, S)$ bounded by W, chosen so that $\gamma \in H_-$. We consider a minimal-length edge path p in $\mathrm{Cay}(C, S)$ joining γ to H_+. The last edge e_2 in p must belong to W, since it crosses from H_- to H_+. If there is more than one edge in p and e_1 is the next-to-last edge, then, by minimality of p, the edges e_1 and e_2 do not lie in a common square of $\mathrm{D}(C, S)$. It follows that the edge path $e_1 e_2$ is a geodesic segment for the CAT(0) metric on $\mathrm{D}(C, S)$ (indeed, it is more generally true that a local isometry between CAT(0) cube complexes is an isometric embedding with respect to their CAT(0) metrics; see [32, Lem. 2.11], [6, Prop. II.4.14]). The nearest point projection to e_i of the hyperplane Π_i dual to e_i is the midpoint of e_i, so that the Π_i are disjoint. The wall corresponding to Π_1 thus separates γ from W, a contradiction.

Now, fix an ordering $W_1, \ldots, W_n$ on $\mathbf{W}(\alpha, \beta)$ which is compatible with $<$. The previous claim tells us that W_1 contains a unique edge in $\mathrm{Cay}(C, S)$ incident to α. If α' is the other endpoint of this edge, then $W_2, \ldots, W_n$ is an ordering on $\mathbf{W}(\alpha', \beta)$, compatible with the partial ordering $<$ on this set. Proceeding iteratively, we then construct an edge path in $\mathrm{Cay}(C, S)$ from α to β which crosses exactly the sequence of walls $W_1, \ldots, W_n$, meaning this sequence is an itinerary. □

Definition 2.7. We say two itineraries $\mathcal{U}, \mathcal{U}'$ are *equivalent* if there are elements $\alpha, \beta \in C$ so that both $\mathcal{U}$ and $\mathcal{U}'$ join α to β.

Proposition 2.6 means that equivalent itineraries $\mathcal{U}, \mathcal{U}'$ always consist of the same set of walls. And, if $\mathcal{U}$ joins α to β for some $\alpha, \beta \in C$, then so does any equivalent itinerary $\mathcal{U}'$. Thus the third condition of Proposition 2.6 ensures that the definition of "equivalence" actually describes an equivalence relation.

2.3.2. *Efficient itineraries.* Whenever W and W' are walls in $\mathrm{D}(C, S)$, then there is always some itinerary $\mathcal{U}$ whose first wall is W and whose last wall is W'. Every wall W_i in $\mathcal{U}$ must either separate W from W', or intersect at least one of W, W'. Proposition 2.6 tells us that we can always find another itinerary equivalent to $\mathcal{U}$ by putting all of the walls in $\mathcal{U}$ intersecting W or W' either first or last. That is, if there are any walls in $\mathcal{U}$ which intersect either W or W', we can reorder the walls and restrict to a strictly shorter sub-itinerary to get a new (non-equivalent) itinerary whose first wall is W and whose last wall is W'. On the other hand, if $\mathcal{U}$ is *any* itinerary with initial wall W and final wall W', then $\mathcal{U}$ must contain every wall separating W from W'.

Definition 2.8. We say that an itinerary $\mathcal{U} = W_1, \ldots, W_n$ is *efficient* if every wall W_i with $1 < i < n$ is disjoint from both W_1 and W_n.

Given any two distinct walls W, W', the argument above shows that there is always an efficient itinerary with initial wall W and final wall W', and that each efficient itinerary between W and W' must consist of the same set of walls. The itinerary orders these walls in a way which is compatible with the partial ordering: $W_i < W_j$ if W_i separates W from W_j. Thus Proposition 2.6 means that any pair of efficient itineraries between W, W' are equivalent, and we can define the following.

Definition 2.9. Let W_1, W_2 be distinct walls in $\mathrm{D}(C, S)$. We let $\gamma(W_1, W_2)$ denote the unique element in C traversed by any efficient itinerary whose first wall is W_1 and whose last wall is W_2.

In general, not every itinerary is equivalent to an efficient itinerary, although this is "almost" true in the special case where $\mathrm{D}(C, S)$ is hyperbolic; see Lemma 3.6 below.

3. Bounded product projections

In this section we fix a right-angled Coxeter system (C, S), and use the setup from the previous section to prove some combinatorial results about *hyperbolic subcomplexes* of the Davis complex $\mathrm{D}(C, S)$. Our main aim is to prove Proposition 3.4, which implies that every geodesic in a hyperbolic subcomplex of $\mathrm{D}(C, S)$ is traversed by an itinerary consisting almost entirely of "regularly-spaced" pairwise disjoint walls. In later sections, we will be able to work with geodesics in C by only considering this set of disjoint walls.

Definition 3.1. Suppose that (C, S) is a right-angled Coxeter system. Let $\gamma \in C$, and let $D > 0$. We say a group element $\gamma \in C$ has *D-bounded product projections* if every pair of disjoint completely incomparable subsets $A, B \subset \mathbf{W}(\gamma)$ satisfies $\min(|A|, |B|) \le D$. We say that a subgroup $\Gamma \le C$ has *D-bounded product projections* if every $\gamma \in \Gamma$ has D-bounded product projections. We just say Γ has *bounded product projections* if there exists some $D > 0$ so that Γ has D-bounded product projections.

More intuitively, the elements in C with bounded product projections are precisely those elements γ whose geodesic representatives do not "travel diagonally" in a combinatorially embedded Euclidean 2-plane $E \hookrightarrow \mathrm{D}(C, S)$; "diagonally" is in reference to the product structure $E = \mathbb{R} \times \mathbb{R}$ induced by the cubulation of E. A geodesic representing an element with D-bounded product projections *may* spend an arbitrary amount of time in a 2-flat, but it must spend all but D of its length traveling parallel to one of the $\mathbb{R}$ factors.

The following is immediate from [30, Thm. 4.1.3].

Lemma 3.2. *Let Γ be a quasiconvex subgroup of C. If Γ is hyperbolic, then Γ has bounded product projections.*

Remark 3.3. It is shown in [30] that the converse of Lemma 3.2 also holds, that is, that if Γ has bounded product projections, then Γ is hyperbolic. In fact, this direction also follows from the proof of Theorem 1.5; our proof shows that if Γ has bounded product projections, then certain representations of C restrict to Anosov representations of Γ, and it follows from [42], [4] that any group admitting an Anosov representation is hyperbolic.

Whenever $\mathcal{W}$ is an itinerary, we let $|\mathcal{W}|$ denote the number of walls appearing in $\mathcal{W}$. If $\mathcal{W} = W_1, \ldots, W_n$ and $\mathcal{U} = U_1, \ldots U_m$ are itineraries, we write $\mathcal{W}, \mathcal{U}$ for the concatenation

$$W_1, \ldots, W_n, U_1, \ldots, U_m,$$

as long as this sequence of walls is also an itinerary.

Proposition 3.4. *Given $D > 0$, there exists $R > 0$ (depending only on D) satisfying the following. Suppose that $\gamma \in C$ has D-bounded product projections. Then any itinerary traversing γ is equivalent to an itinerary $\mathcal{U}$ of the form*

$$\{W_1\}, \mathcal{V}_1, \{W_2\}, \mathcal{V}_2, \ldots, \{W_n\}, \mathcal{V}_n,$$

such that:

(a) *Every $\mathcal{V}_i$ satisfies $|\mathcal{V}_i| \le R$,*
(b) *Every wall in $\mathcal{V}_i$ intersects W_i,*
(c) *For every $i \ne j$ we have $W_i \cap W_j = \emptyset$,*
(d) *Every wall W_i intersects at most R other walls in $\mathcal{U}$.*

Remark 3.5. Conditions (b) and (d) in the proposition together imply (a), but we still list (a) above because later we will use (b), (c) and a stronger form of (a) to prove (d).

Proposition 3.4 tells us in particular that the length of any geodesic in $\mathrm{D}(C, S)$ representing some $\gamma \in C$ with D-bounded product projections can be estimated (up to a multiplicative constant) as the maximal length of a *chain* $W_1 < W_2 < \ldots < W_n$ in $\mathbf{W}(\gamma)$. In fact, this weaker statement holds for *arbitrary* $\gamma \in C$; the point of the proposition is that when γ has D-bounded product projections, every wall in the chain is disjoint from almost every other wall in $\mathbf{W}(\gamma)$.

The proof of Proposition 3.4 is purely combinatorial. In fact, it relies only on the poset structure of the set of walls separating a pair of elements in C.

We first prove a useful lemma:

Lemma 3.6. *Suppose that $\gamma = \alpha^{-1}\beta$ is a nontrivial element in C with D-bounded product projections. Then there exists a minimal wall W in $\mathbf{W}(\alpha, \beta)$ such that $W \cap W' \neq \emptyset$ for at most $(2D+1) \cdot 4^D$ walls W' in $\mathbf{W}(\alpha, \beta)$.*

In particular, the lemma tells us that any group element γ with D-bounded product projections is traversed by an itinerary which is "nearly" efficient.

Proof. Without loss of generality we may assume $\alpha = \mathrm{id}$ and consider the set of walls $\mathbf{W}(\gamma)$. We let $\mathbf{W}_{\min}(\gamma)$ denote the minimal walls in $\mathbf{W}(\gamma)$.

Every minimal element in a poset is incomparable with every other minimal element. So, the bounded product projections property implies that the number of walls in $\mathbf{W}_{\min}(\gamma)$ is at most $2D+1$, since otherwise we could partition $\mathbf{W}_{\min}(\gamma)$ into disjoint completely incomparable subsets, both containing at least $D+1$ elements.

We can think of every wall $W \in \mathbf{W}(\gamma)$ as having one of finitely many "intersection types," determined by the walls in $\mathbf{W}_{\min}(\gamma)$ which W intersects. Precisely, we define an "intersection mapping" $I_{\min} : \mathbf{W}(\gamma) \to 2^{\mathbf{W}_{\min}(\gamma)}$, by:

$$I_{\min}(W) = \{V \in \mathbf{W}_{\min}(\gamma) : V \cap W \neq \emptyset\}.$$

We claim the following:

Claim. Let U_1, U_2 be subsets of $\mathbf{W}_{\min}(\gamma)$, such that $U_1 \not\subseteq U_2$ and $U_2 \not\subseteq U_1$. Then the sets of walls $A = I_{\min}^{-1}(U_1)$ and $B = I_{\min}^{-1}(U_2)$ are completely incomparable.

To prove the claim, observe that if the hypothesis holds, then there is a wall $V_1 \in U_1 \setminus U_2$ and a wall $V_2 \in U_2 \setminus U_1$. Since V_1 does not intersect any wall in B, and V_1 is minimal, we have $V_1 < W_B$ for every $W_B \in B$. Similarly, we have $V_2 < W_A$ for every $W_A \in A$. And, since every wall $W_A \in A$ intersects V_1 nontrivially, each W_A is incomparable with V_1. So, there is a total ordering of $\mathbf{W}(\gamma)$ (compatible with $<$) where each $W_A \in A$ precedes V_1, hence precedes every $W_B \in B$. Arguing symmetrically, there is also a compatible ordering of the walls where every $W_B \in B$ precedes every $W_A \in A$. Thus, A and B are completely incomparable, proving the claim.

Next, consider the collection of subsets $T \subset 2^{\mathbf{W}_{\min}(\gamma)}$ given by

$$T = \{U \subset \mathbf{W}_{\min}(\gamma) : |I_{\min}^{-1}(U)| > 2D+1\}.$$

The previous claim, together with the bounded product projections axiom, implies that if U_1 and U_2 are subsets of $\mathbf{W}_{\min}(\gamma)$, neither of which is a subset of the other, then either $|I_{\min}^{-1}(U_1)| \leq D$ or $|I_{\min}^{-1}(U_2)| \leq D$—so in particular at most one of U_1, U_2 can lie in T. This means that there is at most one maximal element in T, with respect to the partial ordering on $2^{\mathbf{W}_{\min}(\gamma)}$ given by inclusion. However, the element $\mathbf{W}_{\min}(\gamma) \in 2^{\mathbf{W}_{\min}(\gamma)}$ cannot itself lie in T: by definition, every element in $I_{\min}^{-1}(\mathbf{W}_{\min}(\gamma))$ is incomparable with every minimal wall, so every element in $I_{\min}^{-1}(\mathbf{W}_{\min}(\gamma))$ is itself minimal in $\mathbf{W}(\gamma)$ and therefore $|I_{\min}^{-1}(\mathbf{W}_{\min}(\gamma))| \leq |\mathbf{W}_{\min}(\gamma)| \leq 2D+1$.

We conclude that T is either empty, or it has a unique maximal element which is not all of $\mathbf{W}_{\min}(\gamma)$. In either case, there must be some wall $W_- \in \mathbf{W}_{\min}(\gamma)$ such that $W_- \notin U$ for any $U \in T$.

In other words, for any set $U \subset \mathbf{W}_{\min}(\gamma)$ containing W_-, we have $|I_{\min}^{-1}(U)| \leq 2D+1$. And by definition, the set of walls in $\mathbf{W}(\gamma)$ intersecting W_- is the union of the sets $I_{\min}^{-1}(U)$ over all subsets $U \subset \mathbf{W}_{\min}(\gamma)$ with $W_- \in U$. Since the number of such sets U is at most $2^{|\mathbf{W}_{\min}(\gamma)-1|} \leq 2^{2D} = 4^D$, we conclude that the number of walls in $\mathbf{W}(\gamma)$ intersecting W_- is at most $(2D+1) \cdot 4^D$. $\square$

Proof of Proposition 3.4. Consider an itinerary traversing γ. As in the proof of the previous lemma, without loss of generality we may assume that this itinerary joins id to γ, and contains exactly the walls in $\mathbf{W}(\gamma)$. Because of Proposition 2.6, our goal is to find a total ordering on $\mathbf{W}(\gamma)$, compatible with $<$, which gives an itinerary $\mathcal{U}$ of the desired form.

We will find the desired itinerary $\mathcal{U}$ iteratively. We let $R' = (2D + 1) \cdot 4^D$. Using Lemma 3.6, we choose a minimal wall $W_1 \in \mathbf{W}(\gamma)$ which intersects at most R' other walls in $\mathbf{W}(\gamma)$. Then, we let $\mathcal{V}_1$ be an itinerary consisting of the set of walls in $\mathbf{W}(\gamma)$ which intersect W_1, arranged in an arbitrary order compatible with $<$. We obtain an itinerary traversing γ of the form

$$\{W_1\}, \mathcal{V}_1, \mathcal{V}_1',$$

where $|\mathcal{V}_1| \leq R'$, and every wall in $\mathcal{V}_1'$ is disjoint from W_1.

Using Lemma 3.6 again, we pick a minimal wall W_2 in $\mathcal{V}_1'$ which intersects at most R' walls in $\mathcal{V}_1'$. We let $\mathcal{V}_2$ be the walls in $\mathcal{V}_1' - \{W_2\}$ which intersect W_2 (again arranged in an arbitrary compatible order), and obtain another equivalent itinerary

$$\{W_1\}, \mathcal{V}_1, \{W_2\}, \mathcal{V}_2, \mathcal{V}_2'.$$

We proceeed iteratively in this fashion until we have eventually obtained an itinerary $\mathcal{U}$ of the form

$$\{W_1\}, \mathcal{V}_1, \{W_2\}, \mathcal{V}_2, \ldots, \{W_n\}, V_n,$$

such that each $\mathcal{V}_k$ satisfies $|\mathcal{V}_k| \leq R'$, and for any k, ℓ with $k < \ell$, the wall W_k is disjoint from both W_ℓ and every wall in $\mathcal{V}_\ell$.

So, as long as we take $R \geq R' = (2D+1) \cdot 4^D$, the itinerary $\mathcal{U}$ satisfies conditions (a), (b), and (c) in the statement of the proposition. It remains to show that for some choice of R, this itinerary also satisfies condition (d). We claim that taking $R = R'D + D$ is sufficient.

To see this, fix k, and consider the set $I(W_k)$ of walls in $\mathbf{W}(\gamma)$ which intersect W_k. We wish to show that $|I(W_k)| \leq R'D + D$. We know that W_k is disjoint from every W_i for $i \neq k$ and from every wall in $\mathcal{V}_\ell$ for every $\ell > k$. So, every wall in $I(W_k)$ is contained in some $\mathcal{V}_i$ for $i \leq k$. Since each $\mathcal{V}_i$ contains at most R' walls, there are at most $R'D$ walls contained in the union

$$\bigcup_{j=k-D+1}^{k} I(W_k) \cap \mathcal{V}_j.$$

So, if I_{k-D} denotes the set of walls

$$I_{k-D} = \bigcup_{i=1}^{k-D} I(W_k) \cap \mathcal{V}_i,$$

we must have $|I(W_k)| \leq |I_{k-D}| + R'D$.

For any i, j with $i < j < k$, we have $W_i < W_j < W_k$. This means that if some wall V is incomparable with both W_i and W_k, it is also incomparable with W_j. Now, if $V \in I_{k-D}$, we know that $V \cap W_i$ is nonempty for some $i \leq k - D$, and $V \cap W_k$ is nonempty by assumption, so necessarily V intersects W_j for every j with $i \leq j \leq k$. In particular, V intersects W_j for every j with $k - D \leq j \leq k$.

That is, every wall in I_{k-D} intersects each of the $D + 1$ walls $W_{k-D}, \ldots, W_k$. But then the fact that γ has D-bounded product projections implies that $|I_{k-D}| \leq D$, hence $|I(W_k)| \leq R'D + D$ as required. □

Corollary 3.7. *The itinerary $\mathcal{U}$ coming from Proposition 3.4 satisfies the following properties:*

(1) For every $i < j$, the group element $\gamma_{\mathcal{U}}(W_i, W_j)$ satisfies

$$\gamma_{\mathcal{U}}(W_i, W_j) = \eta_i \cdot \gamma(W_i, W_j) \cdot \eta_j,$$

where $|\eta_i|, |\eta_j| < R$.

(2) Suppose $\mathcal{U}'$ is equivalent to $\mathcal{U}$. Then any sub-itinerary of $\mathcal{U}'$ with length greater than $2R$ is equivalent to an itinerary of the form

$$\mathcal{Y}_i, \mathcal{W}_{ij}, \mathcal{Y}_j,$$

for some $i \leq j$, where $\mathcal{W}_{ij}$ is an efficient itinerary between W_i and W_j, and $|\mathcal{Y}_i|, |\mathcal{Y}_j| < R$.

Proof. (1) Fix $i < j$, and consider the sub-itinerary $\mathcal{U}_{ij} = \{W_i\}, \mathcal{V}_i, \ldots \{W_j\}$. We know that at most R walls in this sub-itinerary intersect W_i, and at most R walls in this sub-itinerary intersect W_j, so there is an equivalent itinerary of the form $\mathcal{Y}_i, \{W_i\}, \mathcal{Y}, \{W_j\}, \mathcal{Y}_j$, where $|\mathcal{Y}_i|, |\mathcal{Y}_j| < R$, and the walls in $\mathcal{Y}_i$ and $\mathcal{Y}_j$ are precisely the walls in $\mathcal{U}_{ij}$ which respectively intersect W_i and W_j. Then the walls in $\mathcal{Y}$ are precisely the walls separating W_i from W_j, so

$$\gamma(\mathcal{U}_{ij}) = \gamma(\mathcal{Y}_i)\gamma(W_i, W_j)\gamma(\mathcal{Y}_j).$$

(2) Let $\mathcal{U}'$ be equivalent to $\mathcal{U}$, and let $\mathcal{Y}$ be a sub-itinerary of $\mathcal{U}'$. Let $\mathbf{W}$ be the ordered set of walls $W_1 < \ldots < W_n$. First, suppose that $\mathcal{Y}$ contains at least one wall $W_i \in \mathbf{W}$. We can then choose $W_i \leq W_j$ to be (respectively) minimal and maximal walls in $\mathbf{W} \cap \mathcal{Y}$; then the argument from the previous case shows that $\mathcal{Y}$ is equivalent to an itinerary $\mathcal{Y}_i, \mathcal{W}_{ij}, \mathcal{Y}_j$ as required. So, we just need to show that if $|\mathcal{Y}| > 2R$ then $\mathcal{Y}$ contains at least one wall in $\mathbf{W}$.

To see this, we consider the total orderings of the walls in $\mathcal{U}$ given by the equivalent itineraries $\mathcal{U}$ and $\mathcal{U}'$; both of these total orderings must be compatible with the partial order $<$ on all the walls in $\mathcal{U}$. For each $W_i \in \mathbf{W}$, we let $n(W_i)$ denote the index of W_i in $\mathcal{U}$, and let $n'(W_i)$ denote the index of W_i in $\mathcal{U}'$. Now, since each W_i is independent from at most R other walls in $\mathcal{U}$ with respect to the partial order $<$, we must have $|n(W_i) - n'(W_i)| \leq R$ for every i. Then, since every sub-itinerary of $\mathcal{U}$ with length at least R contains at least one wall in $\mathbf{W}$, the same is true for every sub-itinerary of $\mathcal{U}'$ with length at least $2R$. □

4. Reflection groups acting on convex projective domains

Let C be a right-angled Coxeter group with generating set S. In this section, we discuss the theory of representations $\rho\colon C \to \mathrm{SL}^{\pm}(|S|, \mathbb{R})$ which are *generated by linear reflections*, as studied by Vinberg [60]. Such representations give rise to a "projective model" for the Davis complex of (C, S), in the form of a *convex projective domain* Ω preserved by ρ. We refer to [60], [14] for further background.

4.1. **Cartan matrices and simplicial representations.** Let V be a real vector space with dimension d.

Definition 4.1. A *linear reflection* is an element in $\mathrm{SL}^{\pm}(V)$ with a $(d-1)$-dimensional eigenspace with eigenvalue 1, and a 1-dimensional eigenspace with eigenvalue -1. Equivalently, a linear reflection is any element $r \in \mathrm{SL}^{\pm}(V)$ which can be written $r = \mathrm{id} - v \otimes \alpha$, where $\alpha \in V^*$, $v \in V$, and $\alpha(v) = 2$.

We refer to the -1-eigenspace of r (which is spanned by the vector v) as the *polar* of r. The 1-eigenspace of r, given by $\ker(\alpha)$, is the *reflection hyperplane* of r.

The vector v and dual vector α are determined up to a choice of scale: we can replace v with λv and α with $\lambda^{-1}\alpha$ for any nonzero $\lambda \in \mathbb{R}$ to obtain the same reflection.

We say that a representation $\rho\colon C \to \mathrm{SL}^{\pm}(V)$ is *generated by reflections* if ρ maps each $s \in S$ to some linear reflection in $\mathrm{SL}^{\pm}(V)$. There is always at least one discrete faithful representation $\rho\colon C \to \mathrm{SL}^{\pm}(|S|, \mathbb{R})$ which is generated by reflections, called the *geometric representation*. In fact, whenever C is an infinite right-angled Coxeter group, there is an uncountable (continuous) family of conjugacy classes of such representations.

Below, we explain how to construct the representations in this family.

Definition 4.2. Let (C, S) be a right-angled Coxeter system, and write $S = \{s_1, \ldots, s_n\}$. We say that an $|S| \times |S|$ real matrix A is a *Cartan matrix* for (C, S) if it satisfies the following three criteria:

(1) For every i, we have $A_{ii} = 2$.
(2) For all $i \neq j$ such that s_i and s_j commute, we have $A_{ij} = 0$.
(3) For all $i \neq j$ such that s_i and s_j do *not* commute, we have $A_{ij} < 0$ and $A_{ij}A_{ji} \geq 4$.

Remark 4.3. It also makes sense to consider Cartan matrices for an *arbitrary* (i.e. not necessarily right-angled) Coxeter group C. The definition is slightly more complicated in this case. We omit it as it is not relevant for the present paper.

We can use any Cartan matrix A for a Coxeter group C to define a representation $\rho_A\colon C \to \mathrm{SL}^{\pm}(|S|, \mathbb{R})$: we let $n = |S|$, let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$, and let $\{e^1, \ldots, e^n\}$ be the corresponding dual basis. For each i, we set $\alpha_i = e^i$, and let v_i be the vector

$$v_i = \sum_{k=1}^{n} A_{ki} e_k.$$

Then the group element $\mathrm{id} - v_i \otimes \alpha_i$ is a linear reflection and for every i, j we have $\alpha_i(v_j) = A_{ij}$. One can check directly that the assignment $s_i \mapsto (\mathrm{id} - v_i \otimes \alpha_i)$ determines a representation of the Coxeter group C.

Definition 4.4. Given a Cartan matrix A for a Coxeter group C, we let $\rho_A\colon C \to \mathrm{SL}^{\pm}(n, \mathbb{R})$ denote the representation determined by the assignment

$$s_i \mapsto (\mathrm{id} - v_i \otimes \alpha_i)$$

described above.

We will refer to ρ_A as the *simplicial representation* associated to A. The terminology is motivated by the fact that ρ_A induces a discrete and faithful action of C on a convex domain in projective space $\mathbb{P}(\mathbb{R}^n)$, with fundamental domain a simplex (see Theorem 4.11 below).

Definition 4.5. Fix a right-angled Coxeter system (C, S), and let A be the unique Cartan matrix for (C, S) which satisfies $A_{ij} = A_{ji} = -2$ for every $i \neq j$ such that s_i and s_j do not commute. The simplicial representation ρ_A associated to this Cartan matrix is called the *geometric* (or *Tits*) representation of C.

The geometric representation was first studied by Tits [5], who proved that it is always faithful with discrete image in $\mathrm{SL}^{\pm}(|S|, \mathbb{R})$. In [60], Vinberg proved that the same is true for every simplicial representation ρ.

Remark 4.6. If the Cartan matrix A is symmetric, then one can define a (possibly degenerate) bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{|S|}$ by setting $\langle e_i, e_j \rangle = A_{ij}$. In this case, the simplicial representation ρ_A defined above is isomorphic to the *dual* of the representation

$$s_i \mapsto (\mathrm{id} - \langle \cdot, e_i \rangle e_i),$$

which preserves the bilinear form $\langle\cdot,\cdot\rangle$. If the form $\langle\cdot,\cdot\rangle$ is nondegenerate, then it induces an isomorphism from $\mathbb{R}^{|S|}$ to its dual, and ρ_A preserves the corresponding (nondegenerate) bilinear form (meaning $\rho_A(C)$ has image lying in some $O(p,q)$ with $p+q=|S|$).

4.1.1. *Nondegeneracy conditions.* Several times in this paper, we will want to consider the restriction of a simplicial representation ρ of a right-angled Coxeter group C to some standard subgroup $C(T) \leq C$. When $C(T)$ is a proper standard subgroup, the restricted representation cannot be irreducible, as it has an invariant subspace $V_T = \operatorname{span}\{v_s : s \in T\}$. So, for each subset $T \subseteq S$, we let $\rho_T \colon C(T) \to \mathrm{SL}^{\pm}(V_T)$ denote the representation induced by the restriction of ρ to $C(T)$.

To facilitate inductive arguments, we would like to have a condition which guarantees that the representation ρ_T is isomorphic to a simplicial representation of $C(T)$, and inherits some nondegeneracy properties of the original representation ρ.

We know that the representation ρ_T will be isomorphic to a simplicial representation precisely when the set of restrictions $\{\alpha_t|_{V_T} : t \in T\}$ is a basis for V_T^*. The Cartan matrix associated to ρ_T is then the principal submatrix of the Cartan matrix for ρ corresponding to the subset T. With this in mind, we define the following:

Definition 4.7. We will say that a matrix A is *fully nondegenerate* if all of its principal minors are nonzero.

The lemma below is completely elementary, but key to several of our later arguments.

Lemma 4.8. *Let $\rho\colon C \to \mathrm{SL}_{\pm}(|S|,\mathbb{R})$ be a simplicial representation of a right-angled Coxeter group C with fully nondegenerate Cartan matrix. For any subset $T \subset S$, define $V_T = \operatorname{span}\{v_s : s \in T\}$ and $V_T^{\perp} = \bigcap_{s\in T}\ker(\alpha_s)$. Then there is a $C(T)$-invariant decomposition $V = V_T \oplus V_T^{\perp}$, and the representation $\rho_T\colon C(T) \to \mathrm{SL}^{\pm}(V_T)$ is isomorphic to a simplicial representation.*

In the terminology of [14], Lemma 4.8 says that the simplicial representation ρ associated to a fully nondegenerate Cartan matrix is both *reduced* and *dual-reduced*, and that the same is true for the representations $\rho_T \colon C(T) \to \mathrm{SL}^{\pm}(V_T)$ for every $T \subseteq S$.

Proof. The nondegeneracy of the Cartan matrix implies that the subsets $\{v_s : s \in T\}$ and $\{\alpha_s : s \in T\}$ are both linearly independent in $\mathbb{R}^{|S|}$ and its dual, respectively. This implies that the subspaces V_T and $V_T^{\perp}$ have complementary dimension. Since the principal minor of the Cartan matrix corresponding to the subset T is nonzero, the set of restrictions $\{\alpha_s|_{V_T} : s \in T\}$ is also linearly independent in the dual V_T^*. This implies that V_T and $V_T^{\perp}$ are transverse, and that ρ_T is isomorphic to a simplicial representation. $\square$

Remark 4.9. Given a right-angled Coxeter group C, one can always find a fully nondegenerate Cartan matrix A for C. In fact, the space of fully nondegenerate matrices is open and dense in the space of Cartan matrices for C, since each principal minor of A is a polynomial in the parameters determining A which is not identically zero. If desired, one can also arrange for this fully nondegenerate matrix to be symmetric, or to have integer entries.

4.2. **Invariant convex projective domains.** Let V be a real vector space. Recall that a subset $\tilde{\Omega} \subset V$ is a *convex cone* if it is convex and invariant under multiplication by positive real numbers. A *convex domain* in $\mathbb{P}(V)$ is the image of an open convex cone under the projectivization map $V - \{0\} \to \mathbb{P}(V)$. A convex domain is *properly convex* if its closure is a convex subset of some affine chart in $\mathbb{P}(V)$ (equivalently, if its closure does not contain any projective line).

Tits and Vinberg proved that simplicial representations are discrete and faithful by showing that there is a certain ρ-invariant convex domain in $\mathbb{P}(\mathbb{R}^{|S|})$ with nonempty interior, and that the action of C on this domain is faithful and properly discontinuous. The structure of this domain is essential to this paper, so we describe it below.

Definition 4.10. Let A be a Cartan matrix for a Coxeter group C, which determines vectors $v_s \in \mathbb{R}^{|S|}$ and dual vectors $\alpha_s \in (\mathbb{R}^{|S|})^*$ for each $s \in S$.

- The *fundamental simplicial cone* for the representation ρ_A is the set
$$\tilde{\Delta} = \bigcap_{s \in S} \{v \in \mathbb{R}^{|S|} : \alpha_s(v) \leq 0 \text{ for all } s \in S\}.$$
The *fundamental simplex* Δ is the projectivization of $\tilde{\Delta}$ in $\mathbb{P}(\mathbb{R}^{|S|})$.
- The *Tits cone* is the interior of the set
$$\bigcup_{\gamma \in C} \rho_A(\gamma)\tilde{\Delta}.$$
The *Vinberg domain* is the projectivization of the Tits cone in $\mathbb{P}(\mathbb{R}^{|S|})$. We usually denote the Tits cone by $\tilde{\Omega}_{\mathrm{Vin}}$, and the Vinberg domain by Ω_{Vin}.

Theorem 4.11 (See Tits [5], Vinberg [60])**.** *Let ρ be a simplicial representation of a Coxeter group C. Then:*

(i) The Tits cone $\tilde{\Omega}_{\mathrm{Vin}}$ is a nonempty convex open subset of $\mathbb{R}^{|S|}$;
(ii) the action of C on Ω_{Vin} is faithful and properly discontinuous;
(iii) the simplex $\Delta \cap \Omega_{\mathrm{Vin}}$ is a fundamental domain for the action;
(iv) the dual graph to the tiling of Ω_{Vin} by copies of $\Delta \cap \Omega_{\mathrm{Vin}}$ is equivariantly identified with the Cayley graph of C with generating set S.

Remark 4.12. We have only stated Theorem 4.11 for simplicial representations of the Coxeter group C. However, Vinberg's result also applies to a broad family of representations of C generated by linear reflections. In particular, a version of the theorem applies to representations $\rho\colon C \to \mathrm{SL}^{\pm}(V)$ where $\dim V$ is not necessarily equal to $|S|$.

While the Vinberg domain Ω_{Vin} associated to a simplicial representation ρ is always a convex domain, it is not necessarily *properly* convex. However, Vinberg also gave conditions which make it possible to guarantee that this holds in broad circumstances:

Proposition 4.13 (See [60, Lemma 15 and Proposition 22])**.** *Let A be a Cartan matrix for an irreducible Coxeter group C, and suppose that* $\det(A) \neq 0$. *Then the following are equivalent:*

(1) The matrix A is of negative type*, i.e. A has a negative eigenvalue.*
(2) The Vinberg domain Ω_{Vin} for ρ_A is properly convex.
(3) The Coxeter group C is infinite.

Sometimes, when working with representations of C generated by reflections, it will also be convenient to consider ρ-invariant domains $\Omega \subset \mathbb{P}(\mathbb{R}^{|S|})$ *other* than the Vinberg domain.

Definition 4.14. Let $\rho\colon C \to \mathrm{SL}^{\pm}(|S|, \mathbb{R})$ be a simplicial representation of an irreducible right-angled Coxeter group C. We will call any ρ-invariant nonempty convex domain $\Omega \subset \Omega_{\mathrm{Vin}}$ a *reflection domain* for ρ.

Typically, we lose nothing by requiring reflection domains to lie inside of the Vinberg domain Ω_{Vin}, instead of merely asking for them to be ρ-invariant convex open sets in $\mathbb{P}(\mathbb{R}^{|S|})$. The reason is the following fact, due to Danciger-Guéritaud-Kassel-Lee-Marquis:

Proposition 4.15 (See [14, Proposition 4.1])**.** *Let (C,S) be an irreducible infinite right-angled Coxeter system with $|S|>2$, and let ρ be a simplicial representation of C with nonsingular Cartan matrix. Then Ω_{Vin} contains every ρ-invariant properly convex domain in $\mathbb{P}(\mathbb{R}^{|S|})$.*

4.3. **Dual domains and the dual Vinberg domain.** Let V be a real vector space and let $\tilde{\Omega}$ be an open convex cone in V. The cone $\tilde{\Omega}$ determines a *dual cone* $\tilde{\Omega}^* \subset \mathbb{P}(V^*)$, defined by

$$\tilde{\Omega}^* = \{w \in V^* : w(v) < 0 \quad \forall x \in \overline{\tilde{\Omega}} - \{0\}\}.$$

Then if $\Omega \subset \mathbb{P}(V)$ is some convex domain, the dual domain $\Omega^* \subset \mathbb{P}(V^*)$ is the projectivization of $\tilde{\Omega}^*$, for some (any) convex cone $\tilde{\Omega}$ projecting to Ω.

It follows immediately that Ω^* is invariant under the dual action of any $g \in \mathrm{SL}^{\pm}(V)$ which preserves Ω. In addition, it is not hard to verify that if Ω is both properly convex and open, then so is Ω^*. We note further that duality reverses inclusions: if $\Omega_1 \subset \Omega_2$, then $\Omega_2^* \subset \Omega_1^*$.

4.3.1. *Dual simplicial representations.* We now let V be the vector space $\mathbb{R}^{|S|}$, and let $\rho\colon C \to \mathrm{SL}^{\pm}(V)$ be a simplicial representation for a right-angled Coxeter group C. If the associated Cartan matrix is nonsingular, then the dual representation $\rho^* : C \to \mathrm{SL}^{\pm}(V^*)$ is isomorphic to a simplicial representation of C whose Cartan matrix is the transpose of the Cartan matrix of ρ. Since ρ preserves the Vinberg domain Ω_{Vin}, the dual representation preserves the dual domain Ω^*_{Vin}.

When C is irreducible, infinite, and not virtually cyclic, Proposition 4.15 implies that Ω^*_{Vin} is contained in the Vinberg domain $\mathscr{O}_{\mathrm{Vin}}$ associated to the dual (simplicial) representation ρ^*. Explicitly, if we let $\tilde{\mathscr{D}}$ be the cone

$$\{w \in V^* : v_s(w) \leq 0 \quad \forall s \in S\},$$

and let $\mathscr{D}$ be the projectivization of $\tilde{\mathscr{D}}$, then $\mathscr{O}_{\mathrm{Vin}}$ is the projectivization of the interior of

$$\tilde{\mathscr{O}}_{\mathrm{Vin}} := \bigcup_{\gamma \in C} \rho^*(\gamma)\tilde{\mathscr{D}}.$$

Theorem 4.11 also applies to ρ^*, with $\mathscr{D}$ taking the place of Δ and $\mathscr{O}_{\mathrm{Vin}}$ taking the place of Ω_{Vin}.

We always have:

Proposition 4.16 (The dual of a reflection domain is a reflection domain)**.** *Let A be a nonsingular Cartan matrix for an infinite irreducible right-angled Coxeter group C, and let ρ_A be the associated simplicial representation. Let $\Omega_{\mathrm{Vin}}, \mathscr{O}_{\mathrm{Vin}}$ be the Vinberg domains for ρ_A, ρ_A^*, respectively. If $\Omega \subseteq \Omega_{\mathrm{Vin}}$ is a reflection domain for ρ_A, then $\Omega^* \subseteq \mathscr{O}_{\mathrm{Vin}}$.*

When C is not virtually cyclic this result follows from Proposition 4.15, and if C is an infinite dihedral group this can be checked directly.

5. Walls and half-cones in reflection domains

In this section we continue to explore some features of the projective geometry of reflection domains. Our main goal is to translate our combinatorial understanding of geodesics in a right-angled Coxeter group into information about the action of the corresponding sequence of group elements in $\mathrm{SL}^{\pm}(n,\mathbb{R})$.

Definition 5.1. Let C be an irreducible right-angled Coxeter group, and let Ω be a reflection domain for a simplicial representation ρ of C. A *wall* W in Ω is the fixed-point set in Ω of a reflection r in C. Whenever W is a wall, we let $\overline{W}$ denote the closure of W in $\overline{\Omega}$. Equivalently, $\overline{W}$ is the closure of W in projective space $\mathbb{P}(V)$.

The *polar* of a wall W is the polar of the reflection r fixing W.

Every wall in Ω is the intersection of some projective hyperplane with Ω, so if Ω is properly convex, then each wall separates Ω into two convex components. Further, every wall W is a union of codimension-1 faces of tiles in Ω, corresponding exactly to the set of edges in the Cayley graph of C fixed by the reflection that defines W. Consequently, we have the following:

Proposition 5.2. *Let Ω be a reflection domain for a simplicial representation of an infinite irreducible right-angled Coxeter group C. There is an equivariant one-to-one correspondence between walls in Ω and walls in the Davis complex $\mathrm{D}(C,S)$, which satisfies the following properties.*

(1) *Two walls in Ω intersect if and only if the corresponding walls in $\mathrm{D}(C,S)$ intersect (equivalently, if and only if the corresponding reflections in C commute).*
(2) *Fix a basepoint x_0 in the interior of $\Delta \cap \Omega$, and let $\gamma \in C$. The set of walls in Ω separating x_0 from $\rho(\gamma) \cdot x_0$ corresponds precisely to the set of walls in $\mathrm{D}(C,S)$ separating* id *from γ.*

Proposition 5.2 tells us that the walls in any reflection domain carry all of the same combinatorial information as the walls in the Davis complex. So, throughout the rest of this section, we will freely apply the combinatorial setup and results regarding walls in Sections 2 and 3 to the walls in a reflection domain Ω.

In particular, for any group element $\gamma \in C$, we can regard the poset of walls in $\mathrm{D}(C,S)$ separating id from γ as a poset of projective walls in Ω. Moreover, the notion of an *itinerary* traversing an element in C also makes sense in this context.

Definition 5.3. If $W_1, \dots, W_n$ is a sequence of walls in a reflection domain Ω corresponding to an itinerary of walls in the Davis complex $\mathrm{D}(C,S)$, we will say that $W_1, \dots, W_n$ is an *itinerary in* Ω or an Ω*-itinerary.* The element $\gamma \in C$ *traversed* by this itinerary is the element traversed by the corresponding itinerary in $\mathrm{D}(C,S)$.

5.1. Disjoint walls in Ω_{Vin} and $\overline{\Omega_{\mathrm{Vin}}}$.

The projective walls in the Vinberg domain Ω_{Vin} actually carry some additional combinatorial information beyond what is encoded in the corresponding walls in the Davis complex $\mathrm{D}(C,S)$: the intersection pattern of the *closures* of the set of walls in $\overline{\Omega_{\mathrm{Vin}}}$ is also informed by the structure of the Coxeter group C.

Specifically, we have the following:

Lemma 5.4. *Let C be a right-angled Coxeter group, and let Ω_{Vin} be the Vinberg domain for a simplicial representation of C. If W_1, W_2 are walls in Ω_{Vin}, then $\overline{W}_1 \cap \overline{W}_2 \neq \emptyset$ if and only if $\gamma(W_1, W_2)$ lies in a proper standard subgroup of C.*

Lemma 5.4 will follow from Lemma 5.5 and Lemma 5.6 below. For these two results, we assume that ρ is a simplicial representation for a right-angled Coxeter system (C,S).

Lemma 5.5. *If $W_1, \dots, W_n$ is an efficient Ω_{Vin}-itinerary between W_1 and W_n, then $\overline{W_1} \cap \overline{W_n} = \bigcap_{i=1}^n \overline{W_i}$.*

Proof. First, observe that if $W_1 \cap W_n \neq \emptyset$, then W_1, W_n is already an efficient itinerary between W_1 and W_n. So we can suppose that $W_1 \cap W_n = \emptyset$. It suffices to show that

$\overline{W_1} \cap \overline{W_n} \subset \overline{W_i}$ for every $1 < i < n$, so fix a wall W_i strictly between W_1 and W_n. Since $W_1, \dots, W_n$ is an efficient itinerary, W_i separates W_1 from W_n: that is, $\Omega_{\mathrm{Vin}} - W_i$ has exactly two connected components O_-, O_+, with $W_1 \subset O_-$ and $W_n \subset O_+$.

By convexity, $\overline{\Omega_{\mathrm{Vin}}} - \overline{W_i}$ also has exactly two components O'_-, O'_+, which satisfy $\overline{O_-} = \overline{O'_-}$ and $\overline{O_+} = \overline{O'_+}$ (as for walls, the closures are taken in projective space). We have $\overline{W_1} \subset \overline{O_-}$ and $\overline{W_n} \subset \overline{O_+}$, and therefore $\overline{W_1} \cap \overline{W_n} \subset \overline{O'_-} \cap \overline{O'_+} = \overline{W_i}$. □

Lemma 5.6. *Let $W_1, \dots, W_n$ be an Ω_{Vin}-itinerary departing from the identity, and let $s_i = s(W_i)$ for $1 \le i \le n$. Then $\bigcap_{i=1}^n \overline{W(s_i)} = \bigcap_{i=1}^n \overline{W_i}$.*

Proof. We induct on n. The case $n = 1$ is immediate. When $n > 1$, we let $\gamma_{n-1} = s_1 \cdots s_{n-1}$. By Proposition 2.3 we have $W_n = \rho(\gamma_{n-1})W(s_n)$, so inductively we have

$$\bigcap_{i=1}^{n} \overline{W_i} = \left(\bigcap_{i=1}^{n-1} \overline{W_i}\right) \cap \overline{W_n} = \left(\bigcap_{i=1}^{n-1} \overline{W(s_i)}\right) \cap \rho(\gamma_{n-1})\overline{W(s_n)}.$$

Since $\rho(s_i)$ fixes $\overline{W(s_i)}$ pointwise for each $1 \le i \le n$, both $\rho(\gamma_{n-1})$ and $\rho(\gamma_{n-1}^{-1})$ fix the intersection $\bigcap_{i=1}^{n-1} \overline{W(s_i)}$ pointwise. So, we have

$$\begin{aligned}\left(\bigcap_{i=1}^{n-1} \overline{W(s_i)}\right) \cap \rho(\gamma_{n-1})\overline{W(s_n)} &= \rho(\gamma_{n-1}) \left(\left(\rho(\gamma_{n-1}^{-1}) \bigcap_{i=1}^{n-1} \overline{W(s_i)}\right) \cap \overline{W(s_n)}\right) \\ &= \rho(\gamma_{n-1}) \bigcap_{i=1}^{n} \overline{W(s_i)} = \bigcap_{i=1}^{n} \overline{W(s_i)}.\end{aligned}$$

□

Proof of Lemma 5.4. Fix an efficient itinerary $\mathcal{V} = V_1, \dots, V_n$ whose first wall is W_1 and whose last wall is W_2. For any $h \in C$ we have $\gamma(W_1, W_2) = \gamma(hW_1, hW_2)$. So, after translating W_1, W_2 and $\mathcal{V}$ by an appropriate element h, we can assume that $\mathcal{V}$ departs from the identity.

Set $s_i = s(V_i)$ for $1 \le i \le n$, so that $\gamma(W_1, W_2) = s_1 \cdots s_n$. From Lemma 5.5 and Lemma 5.6, we know that $\overline{W_1} \cap \overline{W_2}$ is nonempty if and only if the intersection $\bigcap_{i=1}^n \overline{W(s_i)}$ is nonempty. But since the representation ρ is simplicial, this occurs if and only if the set S' of generators appearing in the geodesic word $s_1 \cdots s_n$ is a proper subset of S, i.e. if $\gamma(V_1, V_n) = s_1 \cdots s_n$ lies in a proper standard subgroup of C. □

5.2. **Half-cones.** Consider an infinite geodesic sequence γ_n in a right-angled Coxeter group C. If we fix a reflection domain Ω for some simplicial representation ρ of C, the geodesic sequence corresponds to an (infinite) Ω-itinerary $\mathcal{W} = W_1, W_2, \dots$

Each wall W_n in $\mathcal{W}$ cuts Ω into a pair of connected components, which we call *half-spaces*; we can fix a basepoint $x_0 \in \Omega$, and let H_n be the half-space which does *not* contain x_0.

By considering the infinite intersection of closed half-spaces $\bigcap_{n=1}^\infty \overline{H_n}$, we can try and find a well-defined "limit point" for the geodesic $\rho(\gamma_n)$ in projective space $\mathbb{P}(V)$. However, we will run into some difficulties: while it is often true that pairs of half-spaces are *nested*, (meaning that $H_{n+k} \subset H_n$ for some $n, k > 0$), they will never be *strongly* nested, i.e. they will *never* satisfy $\overline{H_{n+k}} \subset H_n$. This makes it hard to guarantee that the intersections $\bigcap_{i=1}^n \overline{H_n}$ decrease in size at a "uniform rate."

We solve this problem by enlarging the half-spaces H_n to subsets of projective space which we call *half-cones*. Half-cones satisfy the same nesting properties as half-spaces (see Lemma 5.13 below). They will often (but not always) additionally satisfy the *strong* nesting property as well, which allows us to employ them in asymptotic arguments later on. We will also be able to give a fairly complete description of when the strong nesting property fails (see Lemma 5.14), which will be key to later inductive arguments.

For the precise definitions, we fix an irreducible infinite right-angled Coxeter group C, let V denote the vector space $\mathbb{R}^{|S|}$, and let $\rho\colon C \to \mathrm{SL}^{\pm}(V)$ be a simplicial representation of C. We let $\Omega \subset \Omega_{\mathrm{Vin}}$ be a reflection domain for ρ, so that Ω is the projectivization of an invariant convex sub-cone $\tilde{\Omega} \subset \tilde{\Omega}_{\mathrm{Vin}}$ of the Tits cone.

We let $\tilde{\Delta}_\Omega$ denote the set $\tilde{\Delta} \cap \tilde{\Omega}$. Then the projectivization Δ_Ω of $\tilde{\Delta}_\Omega$ is a fundamental domain for the action of C on Ω.

Definition 5.7. Let W be a wall in Ω, so that W is the fixed-point set of a reflection $\mathrm{id} - v \otimes \alpha$ for some $v \in V$, $\alpha \in V^*$ with $\alpha(v) = 2$. Up to replacing α, v with $-\alpha, -v$, we may assume that $\alpha(x) \leq 0$ for every $x \in \tilde{\Delta}_\Omega$.

- We let $\widetilde{\mathbf{Hs}}_+(W)$ denote the interior of the convex cone $\{x \in \tilde{\Omega} : \alpha(x) \geq 0\}$. The *positive half-space* over W, denoted $\mathbf{Hs}_+(W)$, is the projectivization of $\widetilde{\mathbf{Hs}}_+(W)$. Similarly, $\widetilde{\mathbf{Hs}}_-(W)$ is the interior of the cone $\{x \in \tilde{\Omega} : \alpha(x) \leq 0\}$, and the *negative half-space* $\mathbf{Hs}_-(W)$ is the projectivization of $\widetilde{\mathbf{Hs}}_-(W)$.
- We let $\widetilde{\mathbf{Hc}}_+(W)$ denote the interior of the convex hull (in V) of $\ker(\alpha) \cap \tilde{\Omega}$ and v. The *positive half-cone* over W, denoted $\mathbf{Hc}_+(W)$, is the projectivization of $\widetilde{\mathbf{Hc}}_+(W)$. We define the set $\widetilde{\mathbf{Hc}}_-(W)$ and the *negative half-cone* $\mathbf{Hc}_-(W)$ in the same way, but with v replaced with $-v$. See Figure 1.

For $s \in S$, we let $\mathbf{Hs}_\pm(s)$ and $\mathbf{Hc}_\pm(s)$ denote the positive and negative half-spaces and half-cones over the wall $W(s)$ fixed by $\rho(s)$.

Note that the positive half-space $\mathbf{Hs}_+(W)$ over a wall W is precisely the connected component of $\Omega - W$ whose closure does *not* contain the fundamental domain Δ_Ω. This also gives a way to distinguish positive and negative half-cones (see Lemma 5.9 below).

Lemma 5.8. *Let $W_1, \ldots, W_n$ be an Ω-itinerary departing from the identity, traversing a geodesic word $s_1 \cdots s_n$ in C. Then $\mathbf{Hs}_\pm(W_n) = \rho(s_1 \cdots s_{n-1})\mathbf{Hs}_\pm(s_n)$ and $\mathbf{Hc}_\pm(W_n) = \rho(s_1 \cdots s_{n-1})\mathbf{Hc}_\pm(s_n)$.*

Proof. We know from Proposition 2.3 that W_n is the wall $\rho(s_1 \cdots s_{n-1})W(s_n)$. So, if we fix $\alpha \in V^*$ so that $W_n = [\ker \alpha \cap \tilde{\Omega}]$ and $\alpha(\tilde{\Delta}_\Omega) \leq 0$, we know that either $\alpha = \rho^*(s_1 \cdots s_{n-1})\alpha_{s_n}$ or $\alpha = -\rho^*(s_1 \cdots s_{n-1})\alpha_{s_n}$.

Now, since the walls $W_1, \ldots, W_n$ are precisely the walls separating $\tilde{\Delta}_\Omega$ from $\rho(s_1 \cdots s_n)\tilde{\Delta}_\Omega$, we must have

$$\alpha(\rho(s_1 \cdots s_n)\tilde{\Delta}_\Omega) \geq 0,$$

or equivalently $(\rho^*(s_n \cdots s_1)\alpha)(\tilde{\Delta}_\Omega) \geq 0$. So, since $(\rho^*(s_n)\alpha_{s_n})(\tilde{\Delta}_\Omega) = -\alpha_{s_n}(\tilde{\Delta}_\Omega) \geq 0$, we have $\alpha = \rho^*(s_1 \cdots s_{n-1})\alpha_{s_n}$. This proves the desired result for half-spaces.

A dual argument then shows that, if $v \in V$ is a lift of the polar of W with $\alpha(v) = 2$, then $\rho(s_1 \cdots s_{n-1})v = v_{s_n}$, which proves the result for half-cones. □

Lemma 5.9. *For every wall W in Ω, we have $\mathbf{Hs}_\pm(W) \subset \mathbf{Hc}_\pm(W)$.*

Proof. By Lemma 5.8, we just to check that $\mathbf{Hs}_\pm(s) \subset \mathbf{Hc}_\pm(s)$ for every $s \in S$. In fact, since the reflection $\rho(s)$ interchanges $\mathbf{Hs}_+(s)$ with $\mathbf{Hs}_-(s)$, and $\mathbf{Hc}_+(s)$ with $\mathbf{Hc}_-(s)$, we just need to verify that $\mathbf{Hs}_+(s) \subset \mathbf{Hc}_+(s)$.

We write $\rho(s) = \mathrm{id} - v_s \otimes \alpha_s$, and consider $x \in \widetilde{\mathbf{Hs}}_+(s) - \ker(\alpha_s)$. Since $\rho(s)$ interchanges $\widetilde{\mathbf{Hs}}_+(s)$ and $\widetilde{\mathbf{Hs}}_-(s)$, we know that $\rho(s)x \in \widetilde{\mathbf{Hs}}_-(s)$. Then since $\tilde{\Omega}$ is a convex cone, the line

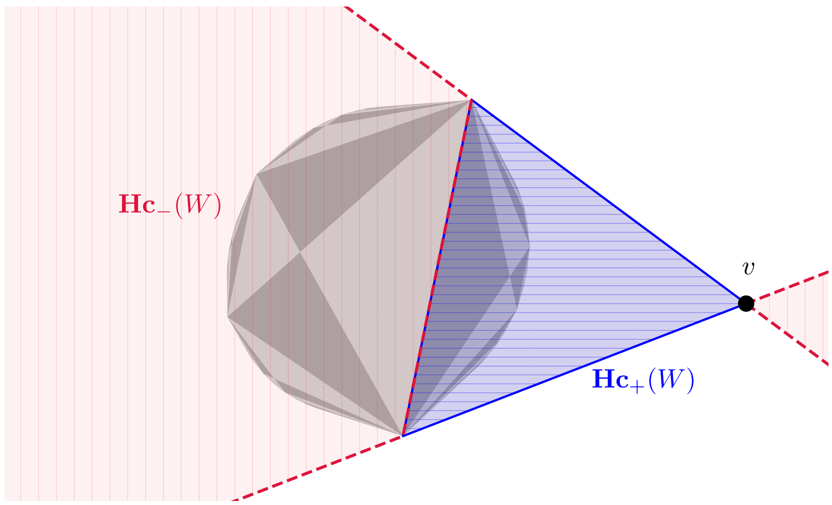

FIGURE 1. Positive and negative half-cones over a wall W in a Vinberg domain Ω_{Vin}. The polar of W is v. The negative half-cone over W is not contained in the depicted affine chart.

segment $\ell \subset V$ joining x to $\rho(s)x$ lies in $\tilde{\Omega}$. The segment ℓ is $\rho(s)$-invariant, and it must be transverse to $\ker(\alpha_s)$ at a point $x_0 \in \ker(\alpha_s) \cap \tilde{\Omega}$. So, the endpoints of ℓ have the form $x_0 \pm tv_s$ for some $t > 0$. But then since $\alpha_s(x) > 0$ by assumption we must have $x = x_0 + tv_s$, hence $x \in \widetilde{\mathbf{Hc}}_+(s)$. $\square$

5.3. **Dual walls and dual half-cones.**

Definition 5.10. Given Ω a reflection domain for a simplicial representation ρ, and W a wall in Ω, fixed by a reflection $\rho(\gamma)$ for $\gamma \in C$, we write W^* to denote the wall in the dual reflection domain Ω^* fixed by the reflection $\rho^*(\gamma)$.

We observe the following useful consequence of Lemma 5.4:

Lemma 5.11. *Let C be an irreducible right-angled Coxeter group, and let Ω_{Vin} be the Vinberg domain for a simplicial representation ρ of C with nonsingular Cartan matrix. Suppose that W_1, W_2 are two walls in Ω_{Vin} such that $\overline{W}_1 \cap \overline{W}_2 = \emptyset$. Then $\overline{W_1^*} \cap \overline{W_2^*} = \emptyset$.*

Proof. Let γ_1 and γ_2 be the reflections in C such that $\rho(\gamma_i)$ fixes W_i for $i = 1, 2$. By Lemma 5.4, the walls W_1 and W_2 have disjoint closures if and only if $\gamma(W_1, W_2)$ does not lie in a proper standard subgroup of C. But this condition depends only on γ_1 and γ_2, and not on the specific simplicial representation ρ.

Further, since the Cartan matrix of ρ is nonsingular, the dual representation ρ^* is also a simplicial representation. This means that W_1 and W_2 have disjoint closures if and only if the walls in $\mathscr{O}_{\mathrm{Vin}}$ fixed by $\rho^*(\gamma_1)$, $\rho^*(\gamma_2)$ have disjoint closures (see Section 4.3). However, $\mathscr{O}_{\mathrm{Vin}}$ contains Ω^*_{Vin} (Proposition 4.16), which means that these two walls contain W_1^* and W_2^*. $\square$

If $\tilde{\Omega} \subset \tilde{\Omega}_{\mathrm{Vin}}$ is the invariant cone over a reflection domain for ρ, then the dual cone $\tilde{\Omega}^*$ is an invariant sub-cone of $\tilde{\mathscr{O}}_{\mathrm{Vin}}$, projecting to the dual reflection domain Ω^*. So, the previous section tells us that we can define half-spaces $\mathbf{Hs}_\pm(W^*)$ and half-cones $\mathbf{Hc}_\pm(W^*)$ over the wall W^* in Ω^*.

The half-cones $\mathbf{Hc}_\pm(W)$ are themselves properly convex domains, which means that we can also define their duals $\mathbf{Hc}_\pm(W)^* \subset \mathbb{P}(V^*)$ (see the beginning of Section 4.3).

Lemma 5.12. *For any wall W in a reflection domain Ω, we have $\mathbf{Hc}_+(W)^* = \mathbf{Hc}_-(W^*)$.*

Proof. We will verify that this holds when W is the projective wall W_s fixed by a reflection $\rho(s)$ for $s \in S$. In this case W_s^* is the projective wall in Ω^* fixed by $\rho^*(s)$. We can use Lemma 5.8 (and its dual version) to see that the general case follows from this one: let $\mathcal{W}$ be an Ω-itinerary $W_1, \ldots, W_n$ departing from the identity, with $W_n = W$, and let $s_1 \cdots s_n$ be the geodesic word traversed by $\mathcal{W}$. Then:

$$\begin{aligned}\mathbf{Hc}_+(W)^* &= (\rho(s_1 \cdots s_{n-1})\mathbf{Hc}_+(W_{s_n}))^* \\ &= \rho^*(s_1 \cdots s_{n-1})\mathbf{Hc}_+(W_{s_n})^* \\ &= \rho^*(s_1 \cdots s_{n-1})\mathbf{Hc}_-(W_{s_n}^*) \\ &= \mathbf{Hc}_-(W^*).\end{aligned}$$

So, now fix $s \in S$. By definition, we have

$$\widetilde{\mathbf{Hc}}_+(W_s) = \{x + tv_s : x \in \ker(\alpha_s) \cap \tilde{\Omega}, t > 0\},$$

which means that $\widetilde{\mathbf{Hc}}_+(W_s)^*$ is the interior of the set

$$\{w \in V_S^* : w(x + tv_s) \leq 0 \text{ for all } x \in \ker(\alpha_s) \cap \tilde{\Omega}, t > 0\}.$$

Now let w be any point in $\widetilde{\mathbf{Hc}}_+(W_s)^*$. Since v_s does not lie in $\ker(\alpha_s)$, we can uniquely write $w = w_1 + w_2$ for $w_1 \in \mathbb{R}\alpha_s$ and $w_2 \in \ker(v_s)$ (viewing v_s as an element of V^{**}). Since v_s is a nonzero vector in the closure of $\widetilde{\mathbf{Hc}}_+(W_s)$, we know $w(v_s) < 0$. Then because $w(v_s) = w_1(v_s)$, and $\alpha_s(v_s) = 2$, we must have $w_1 = t\alpha_s$ for $t < 0$. And, any $x \in \ker(\alpha_s) \cap \tilde{\Omega}$ lies in the closure of $\widetilde{\mathbf{Hc}}_+(W_s)$ also, so we know $w(x) < 0$, hence $w_2(x) < 0$.

By Lemma 5.9, we know that for any $v \in \tilde{\Omega}$, we can write $v = x + tv_s$ for $x \in \ker(\alpha_s) \cap \tilde{\Omega}$ and $t \in \mathbb{R}$. Then $w_2(v) = w_2(x) < 0$, which means that $w_2 \in \ker(v_s) \cap \tilde{\Omega}^*$. This tells us that we can write $\widetilde{\mathbf{Hc}}_+(W_s)^*$ as the interior of

$$\{w + t\alpha_s : w \in \ker(v_s) \cap \tilde{\Omega}^*, t < 0\},$$

which is precisely the definition of $\widetilde{\mathbf{Hc}}_-(W_s^*)$. □

5.4. **Nested half-cones.** For the rest of the section, we will work exclusively with the Vinberg domain Ω_{Vin} for a simplicial representation ρ, rather than an arbitrary reflection domain. Our goal is to understand when the half-cones over a pair of walls W_1, W_2 in Ω_{Vin} are (strongly) nested inside of each other.

Lemma 5.13 (Half-cones nest)**.** *Let C be an irreducible infinite right-angled Coxeter group, and let $\rho\colon C \to \mathrm{SL}^\pm(V)$ be a simplicial representation. Suppose that W_1 and W_2 are walls in Ω_{Vin} such that $W_1 \cap W_2 = \emptyset$ and $\mathbf{Hs}_+(W_2) \subset \mathbf{Hs}_+(W_1)$. Then $\mathbf{Hc}_+(W_2) \subset \mathbf{Hc}_+(W_1)$. Moreover, if $\overline{W_1} \cap \overline{W_2} = \emptyset$, then $\partial\mathbf{Hc}_+(W_1) \cap \overline{\mathbf{Hc}_+(W_2)}$ is contained in $\partial\mathbf{Hc}_+(W_1) \cap \overline{W_2}$.*

Proof. We let γ_1, γ_2 be the elements of C such that $\rho(\gamma_i)$ fixes W_i for $i = 1, 2$. We fix $\alpha_1, \alpha_2 \in V^*$ so that $\widetilde{\mathbf{Hs}}_+(W_1)$ lies in the positive half-space of V defined by α_1, and similarly for $\alpha_2, \widetilde{\mathbf{Hs}}_+(W_2)$. Then we let $\tilde{W}_1 = \ker \alpha_1 \cap \tilde{\Omega}_{\mathrm{Vin}}$ and $\tilde{W}_2 = \ker \alpha_2 \cap \tilde{\Omega}_{\mathrm{Vin}}$, and fix vectors $v_1, v_2 \in V$ so that $\widetilde{\mathbf{Hc}}_+(W_i)$ is the interior of the convex hull of $\tilde{W}_i$ and v_i for $i = 1, 2$. Since

$\mathbf{Hs}_+(W_2) \subset \mathbf{Hs}_+(W_1)$, we know that $\tilde{W}_2 \subset \widetilde{\mathbf{Hc}}_+(W_1)$. We will show that $v_2 \in \overline{\widetilde{\mathbf{Hc}}_+(W_1)}$, and that if $\overline{W_2} \cap \overline{W_1} = \emptyset$, then $v_2 \in \widetilde{\mathbf{Hc}}_+(W_1)$. Then both parts of the lemma will follow from convexity of $\mathbf{Hc}_+(W_1)$.

Consider the 2-dimensional subspace spanned V_{12} by v_1 and v_2. If $V_{12} \cap \overline{\tilde{\Omega}_{\mathrm{Vin}}}$ is trivial, then since Ω_{Vin} is properly convex there is some $w \in \tilde{\Omega}^*_{\mathrm{Vin}}$ which vanishes on both v_1 and v_2. But then w is fixed by both $\rho^*(\gamma_1)$ and $\rho^*(\gamma_2)$, implying that $W_1^* \cap W_2^*$ is nonempty. But this occurs if and only if the reflections fixing W_1^* and W_2^* commute, which we know is not the case because $W_1 \cap W_2 = \emptyset$. We conclude that $V_{12} \cap \overline{\tilde{\Omega}_{\mathrm{Vin}}}$ is nontrivial, and let x be a nonzero point in this intersection.

Next, suppose that $V_{12} \cap \tilde{\Omega}_{\mathrm{Vin}}$ is trivial. Repeating the same argument from before, we see there is some nonzero $w \in \overline{\tilde{\Omega}^*_{\mathrm{Vin}}}$ which vanishes on v_1 and v_2 and therefore $\overline{W_1^*} \cap \overline{W_2^*} \neq \emptyset$. But by Lemma 5.11 this is impossible if $\overline{W_1} \cap \overline{W_2} = \emptyset$. We conclude that if $\overline{W_1} \cap \overline{W_2} = \emptyset$, then we can take our nonzero point $x \in V_{12} \cap \overline{\tilde{\Omega}_{\mathrm{Vin}}}$ to lie in $V_{12} \cap \tilde{\Omega}_{\mathrm{Vin}}$.

Now, we can view v_1 as a linear functional on V^*, whose kernel intersects the interior of the dual domain $\tilde{\Omega}^*_{\mathrm{Vin}}$ in the reflection wall for $\rho^*(\gamma_1)$. Equivalently, v_1 does not lie in the closure of the dual domain to $\tilde{\Omega}^*_{\mathrm{Vin}}$ in V^{**}. But this dual domain is canonically $\tilde{\Omega}_{\mathrm{Vin}}$, which means that $v_1 \notin \overline{\tilde{\Omega}_{\mathrm{Vin}}}$.

Because of this, Lemma 5.9 implies that $x = x_0 + tv_1$ for some $x_0 \in \overline{\tilde{W}_1}$ and $t \in \mathbb{R}$, hence $v_2 = x_0 + t'v_1$ for $t' \in \mathbb{R}$. Thus, we know that v_2 lies in the closure of the set $U = \widetilde{\mathbf{Hc}}_-(W_1) \cup \widetilde{\mathbf{Hc}}_+(W_1) \cup \tilde{W}_1$. Moreover, if $\overline{W_1} \cap \overline{W_2} = \emptyset$, we may take $x_0 \in \tilde{W}_1$, which implies that in fact $v_2 \in U$.

Now, since $\widetilde{\mathbf{Hs}}_+(W_1)$ contains $\widetilde{\mathbf{Hs}}_+(W_2)$, we must have $\tilde{W}_1 \subset \widetilde{\mathbf{Hs}}_-(W_2)$ and therefore

$$\rho(\gamma_2)\tilde{W}_1 \subset \widetilde{\mathbf{Hs}}_+(W_2) \subset \widetilde{\mathbf{Hs}}_+(W_1).$$

Letting $y \in \tilde{W}_1$, we see that $\alpha_1(\rho(\gamma_2)y) > 0$, i.e. $\alpha_1(y - \alpha_2(y)v_2) > 0$ and therefore $\alpha_1(v_2) > 0$. Thus, v_2 cannot lie in the closure of $\widetilde{\mathbf{Hc}}_-(W_1)$. This proves that $v_2 \in \overline{\widetilde{\mathbf{Hc}}_+(W_1)}$, and that $v_2 \in \widetilde{\mathbf{Hc}}_+(W_1)$ if $\overline{W_1} \cap \overline{W_2} = \emptyset$. □

5.4.1. *Strong nesting.* Now consider a pair of disjoint walls W_1, W_2 in Ω_{Vin}, such that $\mathbf{Hs}_+(W_2) \subset \mathbf{Hs}_+(W_1)$. We want to know precisely when the half-cone $\mathbf{Hc}_+(W_2)$ is *strongly* nested inside of $\mathbf{Hc}_+(W_1)$, i.e. when the closure of $\mathbf{Hc}_+(W_2)$ is contained in $\mathbf{Hc}_+(W_1)$.

It is clear that this strong nesting cannot occur if $\overline{W_1} \cap \overline{W_2}$ is nonempty (meaning, by Lemma 5.4, that the group element $\gamma(W_1, W_2)$ lies in a proper standard subgroup of C). One might hope that this is the only situation in which strong nesting fails, and indeed, if this were true, it would greatly simplify several inductive arguments in the final section of this paper. But it turns out that this is not the case; see Appendix A for an explicit counterexample.

However, in the special case where a geodesic representing the group element $\gamma = \gamma(W_1, W_2)$ lies in a hyperbolic subcomplex of $\mathrm{D}(C, S)$, the lemma below gives us some control over the situations where the half-cones over W_1, W_2 fail to strongly nest.

Lemma 5.14. *Let $\mathcal{W} = W_0, \ldots, W_{n+1}$ be an efficient itinerary in Ω_{Vin}, and suppose that $\mathbf{Hc}_+(W_{n+1}) \subset \mathbf{Hc}_+(W_0)$ but $\overline{\mathbf{Hc}_+(W_{n+1})} \not\subset \mathbf{Hc}_+(W_0)$. Suppose further that $\gamma(\mathcal{W})$ has D-bounded product projections. Then, for a constant R depending only on D, $\mathcal{W}$ is equivalent to an Ω_{Vin}-itinerary of the form $\mathcal{U}_-, \mathcal{U}_0, \mathcal{U}_+$, such that:*

(1) The itinerary $\mathcal{U}_0$ satisfies $|\mathcal{U}_0| < 2R$;

(2) The itineraries $\mathcal{U}_-, \mathcal{U}_+$ are both efficient;

(3) Both intersections $\bigcap_{U \in \mathcal{U}_+} \overline{U}$ *and* $\bigcap_{U \in \mathcal{U}_-} \overline{U}$ *are nonempty.*

In light of Lemma 5.4, the lemma above tells us that a pair of disjoint walls in the Vinberg domain will fail to have strictly nested half-cones only if the group element traversed by an efficient itinerary between them is (roughly) a product of two elements which each lie in a proper standard subgroup.

Proof. First, if $\overline{W_0} \cap \overline{W_{n+1}}$ is nonempty, then by Lemma 5.5 we can just take $\mathcal{U}_- = \mathcal{W}$ and $\mathcal{U}_0, \mathcal{U}_+$ to be empty. So, assume $\overline{W_0} \cap \overline{W_{n+1}} = \emptyset$.

From Lemma 5.13, if $\partial \mathbf{Hc}_+(W_0)$ intersects $\overline{\mathbf{Hc}_+(W_{n+1})}$, then ∂W_{n+1} contains a point x_{n+1} in the boundary of $\mathbf{Hc}_+(W_0)$. Then since $\overline{W_0} \cap \overline{W_{n+1}} = \emptyset$, there is a point $x_0 \in \partial W_0$ and a nontrivial projective segment $\ell \subset \partial \mathbf{Hc}_+(W_0)$ joining x_0 to x_{n+1}, such that the projective span L of ℓ contains the tip v_0 of $\mathbf{Hc}_+(W_0)$. By convexity, ℓ is contained in $\overline{\Omega_{\mathrm{Vin}}}$. And, since $\ell \subset \partial \mathbf{Hc}_+(W_0)$, Lemma 5.9 implies that $\ell \subset \partial \Omega_{\mathrm{Vin}}$. Further, the reflection R_0 fixing W_0 acts by a reflection on L fixing x_0, and preserves $\partial \Omega_{\mathrm{Vin}}$. So the union $\ell' = \ell \cup R_0 \ell$ is a projective segment in $\partial \Omega_{\mathrm{Vin}}$ containing x_0 in its interior.

Let v_{n+1} be the polar of W_{n+1}, and consider the projective subspace P spanned by L and v_{n+1}. First, suppose that P does not intersect the interior of Ω_{Vin}. Then, there is a *supporting hyperplane* H of Ω_{Vin} containing P, i.e. a hyperplane in $\mathbb{P}(V)$ which intersects $\overline{\Omega_{\mathrm{Vin}}}$ but not Ω_{Vin}. In particular, this supporting hyperplane contains both v_{n+1} and v_0, so it is preserved by the reflections fixing W_0 and W_{n+1}. So, H lies in $\overline{W_0^*} \cap \overline{W_{n+1}^*}$. But, Lemma 5.11 implies that this intersection is empty once $\overline{W_0} \cap \overline{W_{n+1}}$ is empty.

We conclude that P intersects the interior of Ω_{Vin}. In particular L is a proper subspace of P, so P is a projective 2-plane. Since each wall W_i for $1 < i < n$ is disjoint from both W_0 and W_{n+1}, the intersection $W_i \cap P$ cannot span P. So, $W_i \cap P$ is a projective segment ℓ_i. Each ℓ_i must separate $W_0 \cap P$ from $W_{n+1} \cap P$ in $\Omega_{\mathrm{Vin}} \cap P$, so the closure of ℓ_i intersects the (closed) segment ℓ at a point x_i. See Figure 2.

Then, by reordering the walls in $\mathcal{W}$, we can obtain an equivalent efficient itinerary where the corresponding ordering of the walls W_i is compatible with the (partial) ordering of the x_i along ℓ. We partition this itinerary into two pieces $\mathcal{U}'_-, \mathcal{U}'_+$, where $\mathcal{U}'_-$ consists of the walls W_i such that $x_i \in \ell - \{x_{n+1}\}$, and $\mathcal{U}'_+$ consists of the walls W_i such that $x_i = x_{n+1}$.

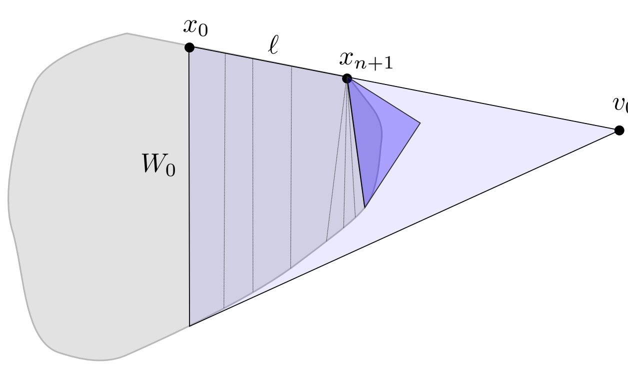

FIGURE 2. Illustration for the proof of Lemma 5.14. The walls separating W_0 from W_{n+1} all intersect either the interior of $\ell \cup R_0\ell$, or the endpoint x_{n+1}.

Lemma 3.6 then says that there is a uniform constant R so that $\mathcal{U}'_+$ is equivalent to an itinerary $\mathcal{U}''_0, \mathcal{U}_+$, where $|\mathcal{U}''_0| \leq R$ and $\mathcal{U}_+$ contains a unique minimal wall. Since $\mathcal{U}_+$ also contains the (unique) maximal wall W_{n+1} in $\mathcal{W}$, it must be efficient. Similarly, we can find an itinerary $\mathcal{U}_-, \mathcal{U}'_0$ equivalent to $\mathcal{U}'_-$, so that $\mathcal{U}_-$ is efficient, and $|\mathcal{U}'_0| \leq R$. We let $\mathcal{U}_0 = \mathcal{U}'_0, \mathcal{U}''_0$, so that $\mathcal{W}$ is equivalent to $\mathcal{U}_-, \mathcal{U}_0, \mathcal{U}_+$.

Since $x_{n+1} \in \overline{W_i}$ for each W_i in $\mathcal{U}_+$, we already know that the intersection $\bigcap_{U \in \mathcal{U}_+} \overline{U}$ is nonempty. To see that the intersection $\bigcap_{U \in \mathcal{U}_-} \overline{U}$ is also nonempty, let W_m be the (unique) maximal wall in $\mathcal{U}_-$, so that $\overline{W_m}$ intersects the interior of the projective line segment $\ell' \subset \partial\Omega_{\mathrm{Vin}}$ at the point x_m. Any supporting hyperplane of Ω_{Vin} at x_m must contain ℓ', which means that any such hyperplane contains the projective line L, and in particular contains the point v_0.

Let v_m be the polar of the wall W_m. By Lemma 5.9, the projective line spanned by x_m and v_m does not intersect Ω_{Vin}, so it is contained in some supporting hyperplane H of Ω_{Vin} at x_m. Then H is an element of $\overline{\Omega^*_{\mathrm{Vin}}}$ which contains both v_m and v_0, so it is preserved by the reflections fixing W_0 and W_m. That is, $H \in \overline{W_0^*} \cap \overline{W_m^*}$, so Lemma 5.11 implies that the intersection $\overline{W_0} \cap \overline{W_m}$ is also nonempty. Then we are done after applying Lemma 5.5. ☐

6. Singular values, stable and unstable subspaces, and regularity

The definition of an Anosov representation used in this paper (Definition 1.4) is rooted in linear algebra—specifically, in the *singular value decomposition* of matrices in $\mathrm{GL}(d, \mathbb{R})$. The main purpose of this section is to provide various methods for estimating singular values and singular value gaps, which we will use throughout the rest of the paper.

Much of the content of this section can also be interpreted in terms of the geometry and dynamics of semisimple Lie groups or their associated Riemannian symmetric spaces, and some of the cited results rely on this perspective (or other complicated tools) for their proofs. However, we will state all of the estimates we need in elementary terms, and refer to [4, Sect. 7] for an overview of the connection between the different viewpoints.

6.1. Singular values.

We equip $\mathbb{R}^d$ with its standard Euclidean inner product. Given a matrix $g \in \mathrm{GL}(d, \mathbb{R})$, recall that the i^{th} singular value $\sigma_i(g)$ is the i^{th} largest eigenvalue of the linear map $\sqrt{gg^T}$, where $\sqrt{gg^T}$ is the unique positive-definite matrix squaring to the positive-definite matrix gg^T.

The definition of $\sigma_i(g)$ depends on the choice of positive-definite inner product on $\mathbb{R}^d$. We obtain different singular values for g by choosing a different inner product $\langle \cdot, \cdot \rangle$, and using the adjoint of g with respect to $\langle \cdot, \cdot \rangle$ in place of the transpose matrix g^T. Geometrically, the i^{th} singular value is the length of the i^{th} longest axis of the image ellipsoid

$$\{gv : v \in \mathbb{R}^d, \|v\| = 1\},$$

measured with respect to the chosen inner product on $\mathbb{R}^d$.

If $\sigma_i(g) > \sigma_{i+1}(g)$, then we define the unstable subspace $E_i^+(g)$ as the subspace of $\mathbb{R}^d$ spanned by the i longest axes of this image ellipsoid. If $\sigma_{d-i}(g) > \sigma_{d-i+1}(g)$, we also have the stable space $E_i^-(g) = E_i^+(g^{-1})$, which is the subspace spanned by the i least expanded axes of the unit sphere $\{v \in \mathbb{R}^d : \|v\| = 1\}$.

All of this information is combined in the singular value decomposition (or KAK decomposition) of g: any element $g \in \mathrm{GL}(d, \mathbb{R})$ can be written as a product $g = ka\ell$, where a is a diagonal matrix with nonzero entries $\sigma_1, \ldots, \sigma_d$, and $k, \ell \in \mathrm{O}(d)$ are such that k takes the span of the first i standard basis vectors to $E_i^+(g)$, and ℓ takes $E_i^-(g)$ to the span of the last i standard basis vectors.

We let $\mu_i(g)$ denote $\log \sigma_i(g)$, and collect the logarithms together in the vector $\mu(g) := (\mu_1(g), \mu_2(g), \ldots, \mu_d(g))$. We will also write $\mu_{i,i+1}(g)$ as shorthand for $\mu_i(g) - \mu_{i+1}(g)$.

6.2. **Additivity estimates for singular value gaps.** As noted in the introduction, a representation $\rho\colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ of a finitely-generated group Γ is 1-Anosov if and only if there exist constants $A, B > 0$ such that $\mu_{1,2}(\rho(\gamma)) \geq A|\gamma| - B$ for all $\gamma \in \Gamma$.

This definition depends on a choice of word metric determined by a choice of finite generating set for the group Γ, but this does not matter since all such word metrics are quasi-isometrically equivalent. It also depends on the choice of inner product used to define singular values, but this also turns out not to matter. One justification is the following error estimate on logarithms of singular values:

Lemma 6.1 (Additivity estimate for μ; see [28, Fact 2.18])**.** *For any norm $||\cdot||$ on $\mathbb{R}^d$, there is a constant $K > 0$ such that for any $g, h_1, h_2 \in \mathrm{SL}^{\pm}(d, \mathbb{R})$, we have*

$$||\mu(h_1 g h_2) - \mu(g)|| \leq K(||\mu(h_1)|| + ||\mu(h_2)||).$$

In particular, for any $1 \leq i < d$, there is some $K' > 0$ such that

$$|\mu_{i,i+1}(h_1 g h_2) - \mu_{i,i+1}(g)| \leq K'(||\mu(h_1)|| + ||\mu(h_2)||).$$

As any pair of positive-definite inner products on $\mathbb{R}^d$ differ by composition with some fixed element $h \in \mathrm{GL}(d, \mathbb{R})$, the lemma implies in particular that choosing a different inner product on $\mathbb{R}^d$ to define singular values only changes each $\mu_k(g)$ by a uniformly bounded additive amount.

The lemma also tells us that if Γ is any finitely generated subgroup of $\mathrm{SL}^{\pm}(d, \mathbb{R})$, equipped with a word metric $|\cdot|$, then $||\mu(\gamma)||$ is at most $K|\gamma|$ for a uniform constant K. Thus, we can also apply the lemma to obtain the following useful result:

Lemma 6.2 ("Triangle inequality" for $\mu_{i,i+1}$)**.** *Let Γ be a finitely generated subgroup of $\mathrm{SL}^{\pm}(d, \mathbb{R})$, equipped with a word metric $|\cdot|$. There exists a constant $K > 0$ such that for any $\gamma, \eta_1, \eta_2 \in \Gamma$, we have*

$$|\mu_{i,i+1}(\eta_1 \gamma \eta_2) - \mu_{i,i+1}(\gamma)| \leq K(|\eta_1| + |\eta_2|).$$

We will frequently apply this lemma throughout the rest of the paper, as it allows us to estimate singular value gaps for the images of "nearby" elements in a finitely generated group Γ under some representation ρ.

We will also sometimes want to estimate singular value gaps for the product of a pair of *large* elements γ_1, γ_2 in Γ. This is possible as long as the stable and unstable subspaces of $\rho(\gamma_1), \rho(\gamma_2)$ exist and are *uniformly transverse*. In this situation, we can apply the following:

Lemma 6.3 (Uniform transversality estimate for $\mu_{1,2}$; see [4, Lemma A.7])**.** *Let g, h be two elements of $\mathrm{SL}^{\pm}(d, \mathbb{R})$, and suppose that $\mu_{1,2}(g) > 0$ and $\mu_{1,2}(h) > 0$. If $\theta = \angle(E^-_{d-1}(g), E^+_1(h))$, then we have*

$$\mu_{1,2}(gh) \geq \mu_{1,2}(g) + \mu_{1,2}(h) + 2\log(\sin\theta).$$

6.3. **Regularity, uniform gaps, and the local-to-global principle.** The next estimates in this section are less elementary in nature, but can still be stated in terms of linear algebra.

Definition 6.4 (Uniform regularity)**.** Let g_n be a (finite or infinite) sequence in $\mathrm{GL}(d, \mathbb{R})$. We say that the sequence g_n is *(A, B)-regular* for constants $A \geq 0, B \geq 0$ if for all n, m, we have

$$\mu_{1,2}(g_n^{-1} g_m) \geq A\mu_{1,d}(g_n^{-1} g_m) - B. \tag{1}$$

If an infinite sequence g_n is (A, B)-regular for some A, B, then we say that g_n is *uniformly regular*.

Sequences as in Definition 6.4 are a special case of the "coarsely uniformly regular" sequences defined by Kapovich-Leeb-Porti in [42]; in general a different type of "regularity" can be defined for each singular value gap $\mu_{i,i+1}$. For sequences lying in a finitely generated subgroup $\Gamma < \mathrm{GL}(d, \mathbb{R})$, we can also strengthen this notion of uniform regularity as follows:

Definition 6.5. Let Γ be a finitely generated subgroup of $\mathrm{GL}(d, \mathbb{R})$, and let γ_n be a sequence in Γ which is a geodesic with respect to a word metric $|\cdot|$. We say the geodesic γ_n *has (A, B)-gaps* if

$$\mu_{1,2}(\gamma_n^{-1}\gamma_m) \geq A|m-n| - B.$$

If a geodesic γ_n has (A, B)-gaps for some A, B, we also say it has *uniform gaps*.

A geodesic γ_n with uniform gaps is always uniformly regular: for any $g \in \mathrm{GL}(d, \mathbb{R})$, we have $\mu_{1,d}(g) = \log ||g|| + \log ||g^{-1}||$, where $||\cdot||$ is the operator norm on $\mathrm{GL}(d, \mathbb{R})$. This means that there is a constant K (depending only on Γ and on the choice of finite generating set determining $|\cdot|$) so that

$$\mu_{1,d}(\gamma_n^{-1}\gamma_m) \leq K|m-n|.$$

Geodesics with uniform gaps as in Definition 6.5 are a special case of the *uniformly regular undistorted* (or *URU*) sequences defined in the work of Kapovich–Leeb–Porti [40]; "undistorted" refers to the fact that these sequences map to quasi-geodesics in the Riemannian symmetric space associated to the semisimple Lie group $\mathrm{PGL}(d, \mathbb{R})$. We note that, in this language, 1-Anosov representations are precisely those which send sequences in the group which are geodesic with respect to a word metric to uniformly regular and undistorted sequences.

A key feature of geodesics with uniform gaps is that they have well-defined "limit points" in both projective space $\mathbb{P}(\mathbb{R}^d)$ and its dual (indeed, this is one way of defining limit maps for 1-Anosov representations). To state this result, we let $d_{\mathbb{P}}$ denote the metric on projective space obtained by viewing $\mathbb{P}(\mathbb{R}^d)$ as the quotient of the unit sphere in $\mathbb{R}^d$. Then we let $d_{\mathbb{P}}^*$ denote the metric on the Grassmannian $\mathrm{Gr}_{d-1}(\mathbb{R}^d)$ obtained by viewing the projectivization of each $(d-1)$-plane in $\mathrm{Gr}_{d-1}(\mathbb{R}^d)$ as a subset of $\mathbb{P}(\mathbb{R}^d)$, and taking Hausdorff distance with respect to $d_{\mathbb{P}}$.

Lemma 6.6 (Uniform convergence for geodesics with uniform gaps; see [4, Proposition 2.4])**.** *Let Γ be a finitely generated subgroup of $\mathrm{GL}(d, \mathbb{R})$. If γ_n is a geodesic in Γ with (A, B)-gaps, then the limits*

$$\lim_{n\to\infty} E_1^+(\gamma_n), \qquad \lim_{n\to\infty} E_{d-1}^-(\gamma_n)$$

exist in $\mathbb{P}(\mathbb{R}^d)$ and $\mathrm{Gr}_{d-1}(\mathbb{R}^d)$, respectively, and are uniform in A, B with respect to the metrics $d_{\mathbb{P}}$ and $d_{\mathbb{P}}^*$.

Remark 6.7. Lemma 6.6 also follows from the *higher-rank Morse lemma* of Kapovich–Leeb–Porti [42] (discussed briefly below), but we refer to the result in [4] since it is closer to the form that we actually need. Both of these results are stronger and more technical than what we have stated here—in particular, they imply a version of Lemma 6.6 which does not require the sequence γ_n to lie in a finitely generated subgroup of $\mathrm{GL}(d, \mathbb{R})$.

We refer also to [28, Sect. 5] for a related result that holds under the assumption that Γ is a word-hyperbolic group.

The next estimate in this section follows from several results of Kapovich–Leeb–Porti. The first, proved in [42], is a higher-rank version of the Morse lemma. It implies that uniformly regular undistorted sequences in a finitely generated subgroup $\Gamma < \mathrm{SL}^{\pm}(d,\mathbb{R})$ are essentially equivalent to uniformly *Morse* quasi-geodesic sequences in the associated Riemannian symmetric space X, i.e. quasi-geodesic sequences which stay uniformly close to certain "Morse subsets" in X. The next result, proved in [39] (or [43]), says that these Morse quasi-geodesics satisfy a *local-to-global* principle; a more precise version of this theorem was later proved by Riestenberg [49].

Combining these results gives us a local-to-global principle for geodesics with uniform gaps in a finitely generated group Γ, which we can state as follows:

Theorem 6.8 (Local-to-global principle; see [42, Theorem 1.3] and [43, Theorem 1.1] or [39, Theorem 7.18] or [49, Theorem 1.1])**.** *Let Γ be a finitely generated subgroup of $\mathrm{SL}^{\pm}(d,\mathbb{R})$. Given $A, B > 0$, there exist constants $A', B', \lambda > 0$ satisfying the following. Suppose that γ_n is a sequence in Γ which is a geodesic with respect to a word metric $|\cdot|$, and that for every $i < j$ with $|i-j| \leq \lambda$, the sub-geodesic $\gamma_i, \ldots, \gamma_j$ has (A, B)-gaps. Then γ_n has (A', B')-gaps.*

6.4. **Estimating singular value gaps using the Hilbert metric.** Our last estimate on singular values comes from convex projective geometry. Any properly convex domain Ω in real projective space $\mathbb{P}(\mathbb{R}^d)$ can be endowed with a natural metric d_Ω called the *Hilbert metric*. The Hilbert metric on Ω is always proper, geodesic, and invariant under any projective transformations preserving Ω.

We will use the Hilbert metric as a tool to study the behavior of *nested sequences* of properly convex domains in $\mathbb{P}(\mathbb{R}^d)$. Roughly, we will use the Hilbert metric to measure how much a projective transformation g *fails* to preserve a given properly convex domain Ω, and then use this to estimate the singular value gaps of g.

We provide this estimate in Lemma 6.11 below, but first we recall some basic definitions.

Definition 6.9. Let Ω be a properly convex domain. For $x, y \in \Omega$, we define the Hilbert distance $d_\Omega(x, y)$ by

$$d_\Omega(x,y) = \frac{1}{2}\log([a,b;x,y]),$$

where a, b are the unique points in $\partial\Omega$ such that a, x, y, b lie on a projective line in that order, and $[\cdot,\cdot;\cdot,\cdot]$ is the projective cross-ratio with formula

$$[a,b;c,d] = \frac{\|a-c\|}{\|a-d\|} \cdot \frac{\|b-d\|}{\|b-c\|}.$$

The distances appearing in the cross-ratio formula can be measured using any Euclidean metric on any affine chart containing a, b, c, d; the value of the cross-ratio does not depend on the choice of chart or metric. From this, it follows immediately that for any properly convex domain $\Omega \subset \mathbb{P}(V)$, any $g \in \mathrm{GL}(V)$, and any $x, y \in \Omega$, we have

$$d_\Omega(x,y) = d_{g\Omega}(gx, gy). \tag{2}$$

For our purposes, the most important property of the Hilbert metric is the following standard result, which can be verified by a straightforward computation:

Lemma 6.10 (Expansion of the Hilbert metric on nested domains)**.** *Let Ω_1, Ω_2 be properly convex domains in $\mathbb{P}(\mathbb{R}^d)$, and suppose $\Omega_1 \subseteq \Omega_2$. Then for all $x, y \in \Omega_1$, we have*

$$d_{\Omega_1}(x,y) \geq d_{\Omega_2}(x,y).$$

Further, if the strong inclusion $\overline{\Omega_1} \subset \Omega_2$ also holds, then there is a constant $\lambda > 1$ (depending only on Ω_1 and Ω_2) so that for all $x, y \in \Omega_1$, we have

$$d_{\Omega_1}(x, y) \geq \lambda \cdot d_{\Omega_2}(x, y).$$

Our main application of the Hilbert metric in this paper is the following. Here, and elsewhere, if Ω is a properly convex domain and $X \subseteq \Omega$, then $\mathrm{diam}_\Omega(X)$ refers to the diameter of X with respect to the Hilbert metric d_Ω.

Lemma 6.11 (Hilbert metric estimates singular value gaps)**.** *Fix properly convex domains $\Omega_1, \Omega_2 \subset \mathbb{P}(\mathbb{R}^d)$. There exists a constant $D > 0$, depending only on Ω_1, Ω_2, such that for any $g \in \mathrm{GL}(d, \mathbb{R})$ with $g\overline{\Omega_1} \subset \Omega_2$, we have*

$$\mu_{1,2}(g) \geq -\log(\mathrm{diam}_{\Omega_2}(g\Omega_1)) - D. \tag{3}$$

Proof. Let A_d be the standard affine chart $\{[a_1 : \ldots : a_{d-1} : 1] \subset \mathbb{P}(\mathbb{R}^d) : a_i \in \mathbb{R}\}$, equipped with the Euclidean metric induced by the standard inner product on $\mathbb{R}^d$. We let B be the unit ball in A_d centered at $[e_d] = [0 : \ldots : 0 : 1]$, and first consider the special case of the lemma where $\Omega_1 = \Omega_2 = B$, and g fixes both e_d and its orthogonal complement $e_d^\perp = \mathrm{span}\{e_1, \ldots, e_{d-1}\}$. That is, we assume that g is block-diagonal, of the form

$$\begin{pmatrix} H & \\ & \lambda \end{pmatrix}$$

for $H \in \mathrm{GL}(e_d^\perp)$ and $\lambda \in \mathbb{R}$. Then gB is an ellipsoid in A_d, centered at $[e_d]$, whose semi-major axis has length $\sigma_1(H)/|\lambda|$. Then if $g\overline{B} \subset B$, we have $\sigma_1(H)/|\lambda| < 1$ and thus $\sigma_1(g)/\sigma_2(g) = |\lambda|/\sigma_1(H)$.

The diameter of gB with respect to the Hilbert metric d_B is therefore given by

$$\begin{aligned}\mathrm{diam}_B(gB) &= \log\left(\frac{1 + e^{-\mu_{1,2}(g)}}{1 - e^{-\mu_{1,2}(g)}}\right) \\ &\geq 2\log(1 + e^{-\mu_{1,2}(g)}) \\ &\geq 2e^{-\mu_{1,2}(g)}.\end{aligned}$$

This proves that (3) holds in this special case, when we take $D = \log 2$. The rest of the proof amounts to reducing the general case to this one.

First, we reduce the general situation to the case where $\overline{\Omega_2} \subset \Omega_1$, by choosing some $h_1 \in \mathrm{SL}(d, \mathbb{R})$ so that $\overline{\Omega_2} \subset h_1\Omega_1$. Then, if $\Omega_1' = h_1\Omega_1$, we have

$$\mathrm{diam}_{\Omega_2}(g\Omega_1) = \mathrm{diam}_{\Omega_2}(gh_1^{-1}\Omega_1'),$$

and by Lemma 6.2 the difference $|\mu_{1,2}(g) - \mu_{1,2}(gh_1^{-1})|$ is bounded by a constant only depending on h_1. So, we can replace Ω_1 with Ω_1' and g with gh_1^{-1}. This introduces uniformly bounded additive error to the left-hand side of (3), and does not affect the right-hand side.

Now, since $g\overline{\Omega_1} \subset \Omega_2 \subset \Omega_1$, the Brouwer fixed-point theorem implies that g fixes some $x \in \Omega_2$. Since duality reverses inclusions, we also know that $g^{-1}\overline{\Omega_2^*} \subset \Omega_1^* \subset \Omega_2^*$, and therefore g also fixes an element of Ω_1^*, which we can view as a hyperplane $U \subset \mathbb{R}^d$. Since $\overline{\Omega_2} \subset \Omega_1$, there is a uniform lower bound $\varepsilon > 0$ on the distance between x and U, in a fixed metric on $\mathbb{P}(\mathbb{R}^d)$. So, there is a compact set $K = K(\varepsilon)$ in $\mathrm{SL}(d, \mathbb{R})$ such that for some $h \in K$, the spaces hx, hU are orthogonal with respect to the standard inner product on $\mathbb{R}^d$.

By projective invariance of the Hilbert metric we know that

$$\mathrm{diam}_{\Omega_2}(g\Omega_1) = \mathrm{diam}_{h\Omega_2}(hg\Omega_1).$$

So, if we replace Ω_1, Ω_2 with $h\Omega_1, h\Omega_2$, and replace g with hgh^{-1}, the right-hand side of (3) again stays the same, and (again applying Lemma 6.2) the left-hand side only changes by an additive amount, bounded uniformly in terms of Ω_1, Ω_2. Then, after further conjugating by an orthogonal matrix, we can assume the fixed point of g in Ω_2 is $[e_d]$, and that $\overline{\Omega_1}$ lies in the g-invariant affine chart A_d.

For any $r > 0$, we let B_r denote the open ball of radius r centered at $[e_d]$ in the affine chart A_d. Since Ω_1, Ω_2 are both bounded open convex subsets in A_d, we can find radii $r_1, r_2 > 0$ such that $B_{r_1} \subset \Omega_1$, and $\overline{\Omega_2} \subset B_{r_2}$. We let h_1, h_2 denote the dilations in A_d by a factor of r_1, r_2, respectively, and we let $g' = h_2^{-1} g h_1$.

Using Lemma 6.10, we see that

$$\mathrm{diam}_{B_1}(g'B_1) = \mathrm{diam}_{B_{r_2}}(gB_{r_1}) < \mathrm{diam}_{\Omega_2}(g\Omega_1),$$

and one more application of Lemma 6.2 tells us that the difference between $\mu_{1,2}(g)$ and $\mu_{1,2}(g')$ is bounded by a constant depending only on Ω_1, Ω_2. This reduces the problem to the case that we originally proved. □

7. Stable and unstable subspaces in half-cones

Let (C, S) be an infinite irreducible right-angled Coxeter system with $|S| = d$, and let $\rho \colon C \to \mathrm{SL}^{\pm}(d, \mathbb{R})$ be a simplicial representation. As in the previous section, we assume $V = \mathbb{R}^d$ is endowed with its standard inner product, which allows us to define $\sigma_k(g)$ and $\mu_k(g)$ for every $g \in \mathrm{SL}^{\pm}(V)$; if $\mu_{k,k+1}(g) \neq 0$ we can also define $E_k^+(g)$ and $E_{d-k}^-(g)$.

In this section, we establish some estimates on the location of the subspaces $E_1^+(\rho(\gamma))$ and $E_{d-1}^-(\rho(\gamma))$ for certain elements $\gamma \in C$, in terms of the walls of an Ω_{Vin}-itinerary $\mathcal{W}$ traversing γ. Ultimately, we will use these estimates to argue that the stable and unstable subspaces of certain group elements $\rho(\gamma_1), \rho(\gamma_2)$ are *uniformly transverse*, which allows us to apply Lemma 6.3 to get an estimate on $\mu_{1,2}(\rho(\gamma_1\gamma_2))$.

The idea behind the first estimate is as follows. An unstable subspace $E_1^+(g)$ should be thought of as an "attracting" subspace for g in $\mathbb{P}(V)$. If γ is a group element separated from the identity by walls $W_1, \ldots, W_n$, and x_0 is a fixed basepoint in a reflection domain Ω, then $\rho(\gamma) \cdot x_0$ is separated from x_0 by this same set of walls, and thus an "attracting" subspace for $\rho(\gamma)$ should at least lie in the positive half-space and positive half-cone bounded by the wall W_1. In fact, this "attracting" subspace should lie in the positive half-cone bounded by the wall W_k, as long as k is much smaller than n.

To make this precise, we first need a general fact about group actions on convex projective domains:

Lemma 7.1. *Let Ω be a properly convex domain, and let $g_n \in \mathrm{SL}^{\pm}(d, \mathbb{R})$ be any sequence of elements such that $g_n\Omega = \Omega$, and $\mu_{1,2}(g_n) \to \infty$. Then for any point $x \in \Omega$, the set of accumulation points of the sequence $g_n x$ is the same as the set of accumulation points of the sequence $E_1^+(g_n)$.*

Proof. Fix $x \in \Omega$, and choose an arbitrary subsequence of g_n so that both of the sequences $g_n x$ and $E_1^+(g_n)$ converge. We will show that these sequences converge to the same point.

Let $\mathrm{Aut}(\Omega) < \mathrm{SL}^{\pm}(d, \mathbb{R})$ denote the subgroup $\{g \in \mathrm{SL}^{\pm}(d, \mathbb{R}) : g\Omega = \Omega\}$. The Hilbert metric on Ω (see the previous section) is always proper and $\mathrm{Aut}(\Omega)$-invariant, which implies that the action of $\mathrm{Aut}(\Omega)$ on Ω is proper. Using this, one can check (see e.g. [34, Proposition 5.6]) that any accumulation point of the sequence of stable subspaces $E_{d-1}^-(g_n)$ is a supporting hyperplane of Ω (recall from the proof of Lemma 5.14 that a supporting hyperplane is a projective hyperplane which intersects $\overline{\Omega}$, but not Ω itself). Up to subsequence, $E_{d-1}^-(g_n)$

converges to a fixed supporting hyperplane H. We can then appeal to the singular value decomposition (or KAK decomposition, see Section 6.1) of g_n to see that if z is any point in $\mathbb{P}(V) - H$, the sequence $g_n z$ converges to the limit of $E_1^+(g_n)$. In particular this holds for $z = x$ and we are done. $\square$

Before stating and proving our first estimate, we establish some more notation. As in the previous section, we let $d_{\mathbb{P}}$ denote a Riemannian metric on $\mathbb{P}(V)$ induced by the standard inner product on V.

Definition 7.2. For $x \in \mathbb{P}(V)$ and $\varepsilon > 0$, we let $B_\varepsilon(x)$ denote the open ε-ball about x, with respect to the metric $d_{\mathbb{P}}$. Similarly, if $Z \subset \mathbb{P}(V)$ is a subset, $N_\varepsilon(Z)$ denotes the ε-neighborhood of Z in $\mathbb{P}(V)$.

For $\varepsilon > 0$, the "$-\varepsilon$-neighborhood" of a set $Z \subset \mathbb{P}(V)$ is by convention the complement of the ε-neighborhood of the complement of Z, i.e. the set $N_{-\varepsilon}(Z) := \{x \in Z : B_\varepsilon(x) \subset Z\}$.

We now prove the first main estimate we need from this section:

Lemma 7.3 (Unstable subspaces lie close to half-cones)**.** *For any $\varepsilon > 0$ and any $N > 0$, there exists a constant $M > 0$ satisfying the following. Suppose that $\mathcal{W}$ is an Ω-itinerary $W_1, \ldots, W_n$ traversing a group element $\gamma \in C$, and suppose that $\mu_{1,2}(\rho(\gamma)) > M$. Then for any $k \le \min(n, N)$:*

(1) If $\mathcal{W}$ departs from the identity, then the unstable subspace $E_1^+(\rho(\gamma))$ is contained in $N_\varepsilon(\mathbf{Hc}_+(W_k))$.
(2) If $\mathcal{W}$ arrives at the identity, then the stable subspace $E_{d-1}^-(\rho(\gamma))$ (viewed as a subset of $\mathbb{P}(V)$) is disjoint from $N_{-\varepsilon}(\mathbf{Hc}_-(W_{n-k}))$.

Proof. Let ε, N be given, and fix $k \le N$. We will prove the first statement by contradiction: suppose that there is a sequence of group elements $\gamma_m \in C$ so that as $m \to \infty$, $\mu_{1,2}(\rho(\gamma_m))$ tends to infinity, but for some Ω-itinerary $W_{0,m}, \ldots W_{|\gamma_m|,m}$ joining the identity to γ_m, the point $E_1^+(\rho(\gamma_m))$ does *not* lie in the ε-neighborhood of $\mathbf{Hc}_+(W_{k,m})$. Since $W_{k,m}$ is the last wall in an itinerary of length k departing from the identity, after extracting a subsequence we can assume that $W_{k,m} = W_k$ for a wall W_k independent of m.

Fix a basepoint x in the fundamental domain $\Delta \cap \Omega$. Then by definition, W_k separates x from $\rho(\gamma_m)x$, so $\rho(\gamma_m)x$ lies in the halfspace $\mathbf{Hs}_+(W_k)$, and therefore $\rho(\gamma_m)x$ accumulates in $\overline{\mathbf{Hs}_+(W_k)}$. Then by Lemma 7.1, $E_1^+(\rho(\gamma_m))$ also accumulates in $\overline{\mathbf{Hs}_+(W_k)}$. But we know that $\overline{\mathbf{Hs}_+(W_k)} \subset \overline{\mathbf{Hc}_+(W_k)}$ by Lemma 5.9, so eventually $E_1^+(\rho(\gamma_m))$ lies in the ε-neighborhood of $\mathbf{Hc}_+(W_k)$, contradiction.

The second statement follows from the first via duality. Here are the details. Given $\gamma \in C$, we consider the group element $\rho^*(\gamma^{-1}) \in \mathrm{SL}^{\pm}(V^*)$. If $W_1, \ldots, W_n$ is an Ω-itinerary traversing γ and arriving at the identity, then $W_n, \ldots, W_1$ is an Ω-itinerary traversing γ^{-1} and departing from the identity. The sequence of dual walls $W_n^*, \ldots, W_1^*$ is an Ω^*-itinerary, also traversing γ^{-1} and departing from the identity.

Observe that the singular values of $\rho^*(\gamma^{-1})$ are precisely the same as the singular values of $\rho(\gamma)$, so in particular $\mu_{1,2}(\rho^*(\gamma^{-1})) = \mu_{1,2}(\rho(\gamma))$. Then, the first part of the lemma implies that if $\mu_{1,2}(\rho(\gamma))$ is sufficiently large, the unstable subspace $E_1^+(\rho^*(\gamma^{-1})) \in \mathbb{P}(V^*)$ is contained in an arbitrarily small neighborhood of $\mathbf{Hc}_+(W_{n-k}^*) = \mathbf{Hc}_-(W_{n-k})^*$, with respect to the metric $d_{\mathbb{P}}^*$ on $\mathbb{P}(V^*)$; recall that this is the metric on $\mathbb{P}(V^*)$ obtained by viewing $\mathbb{P}(V^*)$ as the space of projective hyperplanes in $\mathbb{P}(V)$, and then taking Hausdorff distance with respect to $d_{\mathbb{P}}$.

Therefore, if $K \subset \mathbb{P}(V)$ is the kernel of $E_1^+(\rho^*(\gamma^{-1}))$, then as long as $\mu_{1,2}(\rho(\gamma))$ is sufficiently large, there is a projective hyperplane H, within Hausdorff distance ε of K, so that every point in the closure of $\mathbf{Hc}_-(W_{n-k})$ is not contained in H.

The inner product on V induces an isomorphism $V^* \to V$, which identifies $\rho^*(\gamma^{-1}) \in \mathrm{SL}^\pm(V^*)$ with the transpose $\rho(\gamma)^T \in \mathrm{SL}^\pm(V)$. The kernel K of $E_1^+(\rho^*(\gamma^{-1}))$ is the orthogonal complement of the subspace $E_1^+(\rho(\gamma)^T)$, which is in turn the stable subspace $E_{d-1}^-(\rho(\gamma))$. We conclude that if $x \in \mathbf{Hc}_-(W_{n-k})$ is contained in $E_{d-1}^-(\rho(\gamma))$, then $d_{\mathbb{P}}(x, H) < \varepsilon$, hence x is distance at most ε from a point in the complement of $\mathbf{Hc}_-(W_{n-k})$. That is, no point in $N_{-\varepsilon}(\mathbf{Hc}_-(W_{n-k}))$ is contained in $E_{d-1}^-(\rho(\gamma))$. ☐

7.1. **Bounding unstable subspaces away from walls.** The estimate given by the previous lemma tells us that unstable subspaces for $\rho(\gamma)$ are located near half-cones over the walls $W_1, \ldots, W_n$ in Ω_{Vin} separating a basepoint x_0 from $\rho(\gamma)x_0$. We will want to apply this lemma to see that this unstable subspace is uniformly far from some hyperplane in the complement of $\mathbf{Hc}_+(W_1)$.

If the sequence of halfcones $\mathbf{Hc}_+(W_k)$ is *strongly nested* (see Section 5.4.1), then this poses little problem. However, if the closed walls separating x_0 from $\rho(\gamma)x_0$ have a common intersection in the boundary of Ω_{Vin}, then so do the boundaries of the half-cones over these walls. A priori, the unstable subspace $E_1^+(\rho(\gamma))$ could lie near this intersection of boundaries, and we would have no way to bound the distance between $E_1^+(\rho(\gamma))$ and the complement of $\mathbf{Hc}_+(W_1)$.

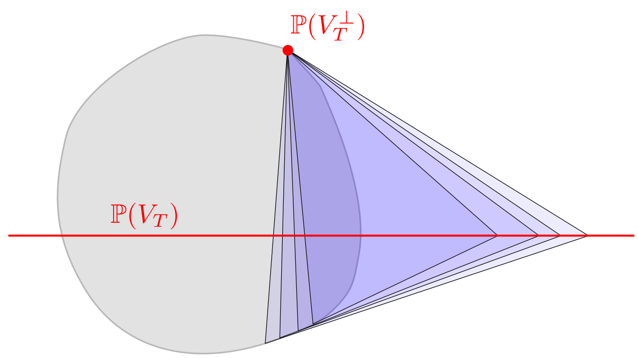

FIGURE 3. A sequence of nested half-cones over walls in an itinerary $W_1, \ldots, W_n$ where $\overline{W_1} \cap \overline{W_n} \neq \emptyset$. Lemma 7.3 and Lemma 7.4 show that if the element traversed by this itinerary has large singular value gaps, then its unstable subspace lies in $\mathbf{Hc}_+(W_k)$ for some large k, and is far from $\overline{W_1} \cap \overline{W_n}$. In particular, the subspace is far from the complement of $\mathbf{Hc}_+(W_1)$.

The lemma below tells us that this does not occur. The idea is that, when the closed walls $\overline{W_1}, \ldots, \overline{W_n}$ have a common intersection in $\partial\Omega_{\mathrm{Vin}}$, then γ lies "close" to a proper standard subgroup $C(T) < C$. The intersection of these walls lies in a "trivial subspace" for the action of this subgroup, which must be far from any attracting subspace for γ. See Figure 3.

Lemma 7.4. *Let Ω be a reflection domain for a simplicial representation ρ with fully nondegenerate Cartan matrix. Then there exist $\varepsilon_0, M > 0$ satisfying the following. Suppose that $W_1, \ldots, W_n$ is an Ω-itinerary departing from the identity, satisfying $W_1 \cap W_n = \emptyset$*

and $\overline{W_1} \cap \overline{W_n} \neq \emptyset$*, and let* $\gamma = \gamma(W_1, W_n)$*. If* $\mu_{1,2}(\rho(\gamma)) > M$*, then the distance between* $E_1^+(\rho(\gamma))$ *and* $\overline{W_1} \cap \overline{W_n}$ *is at least* ε_0*.*

Proof. We can replace the itinerary $W_1, \ldots, W_n$ with an efficient itinerary from W_1 to W_n also departing from the identity, since this does not affect the element $\gamma(W_1, W_n)$. For each i, we write $s_i \in S$ for the type $s(W_i)$ of the wall i. By Lemma 5.5 the intersection $\overline{W_1} \cap \overline{W_n}$ is equal to the intersection $\bigcap_{i=1}^n \overline{W_i}$, and by Lemma 5.6 this intersection is in turn contained in $\bigcap_{i=1}^n \mathbb{P}(\ker(\alpha_{s_i}))$.

Since this intersection is nonempty, the set of elements s_i for $1 \leq i \leq n$ lie in a proper subset $T \subsetneq S$. Then $\gamma \in C(T)$. Since the Cartan matrix for ρ is fully nondegenerate, Lemma 4.8 implies that $\rho(\gamma)$ acts block-diagonally on V with respect to a decomposition $V_T \oplus V_T^\perp$, where

$$V_T = \operatorname{span}\{v_s : s \in T\}, \qquad V_T^\perp = \bigcap_{s \in T} \ker(\alpha_s).$$

Since these subspaces are transverse, we can choose $\varepsilon_0 > 0$ so that the distance between the projective subspaces $\mathbb{P}(V_T), \mathbb{P}(V_T^\perp) \subset \mathbb{P}(V)$ is at least $2\varepsilon_0$. Since there are only finitely many possible choices for T, we may choose this ε_0 independently of γ.

Next, we can choose a linear map $h \in \mathrm{SL}(V)$, depending only on T, so that the decomposition $hV_T \oplus hV_T^\perp$ is orthogonal. Then the conjugate $g = h\rho(\gamma)h^{-1}$ acts block-diagonally on a pair of orthogonal subspaces, and Lemma 6.2 implies that there is a bound on the difference $|\mu_{1,2}(\rho(\gamma)) - \mu_{1,2}(g)|$, depending only on h (and therefore only on T). In particular, this bound is independent of γ, so if $\mu_{1,2}(\rho(\gamma))$ is sufficiently large, we know that $\mu_{1,2}(g) > 0$. Since g acts by the identity on $hV_T^\perp$, this means that $E_1^+(g)$ lies in the subspace hV_T.

Then, Lemma A.4 and A.5 in [4] imply that we have

$$\begin{aligned} d_{\mathbb{P}}\left(E_1^+(\rho(\gamma)), h^{-1}E_1^+(g)\right) &\leq d_{\mathbb{P}}\left(E_1^+(\rho(\gamma)), E_1^+(\rho(\gamma)h^{-1})\right) + d_{\mathbb{P}}\left(E_1^+(\rho(\gamma)h^{-1}), h^{-1}E_1^+(g)\right) \\ &\leq e^{\mu_{1,d}(h) - \mu_{1,2}(\rho(\gamma))} + e^{\mu_{1,d}(h) - \mu_{1,2}(\rho(\gamma)h^{-1})} \\ &\leq e^{-\mu_{1,2}(\rho(\gamma))}\left(e^{\mu_{1,d}(h)} + e^{2\mu_{1,d}(h)}\right) \end{aligned}$$

which is smaller than ε_0 once $\mu_{1,2}(\rho(\gamma))$ is sufficiently large. Since $h^{-1}E_1^+(g)$ lies in V_T, we conclude that the unstable subspace $E_1^+(\rho(\gamma))$ lies within distance ε_0 of $\mathbb{P}(V_T)$, hence it lies at least distance ε_0 from $\mathbb{P}(V_T^\perp)$. But we have ensured that the intersection $\overline{W_1} \cap \overline{W_n}$ lies in $\mathbb{P}(V_T^\perp)$, so we get the desired result. □

8. Proof of main theorem

In this section we finally set about proving Theorem 1.5. We assume that Γ is a hyperbolic quasiconvex subgroup of a right-angled Coxeter group C with generating set S. We let $d = |S|$, and fix a simplicial representation $\rho \colon C \to \mathrm{SL}^{\pm}(d, \mathbb{R})$ with fully nondegenerate Cartan matrix.

We want to prove the following uniform inequality: there exist constants $A, B > 0$ such that for any $\gamma \in \Gamma$,

$$\mu_{1,2}(\rho(\gamma)) \geq A|\gamma| - B.$$

To do so, it will be more convenient to work on a larger subset of C containing Γ:

Definition 8.1. Let $D > 0$. We let $\mathrm{BP}(D) \subset C$ denote the subset of group elements with D-bounded product projections (see Definition 3.1).

Remark 8.2. If C is not hyperbolic, then $\mathrm{BP}(D)$ is not a subgroup of C for any D.

Lemma 3.2 says precisely that there exists a uniform constant D such that $\Gamma \subseteq \mathrm{BP}(D)$. We choose to work with the set $\mathrm{BP}(D)$ rather than the subgroup Γ because $\mathrm{BP}(D)$ has the following useful property: if w is a geodesic word representing some $\gamma \in \mathrm{BP}(D)$, and w' is a subword of w, then w' also represents an element of $\mathrm{BP}(D)$. This means that we can prove statements about group elements in $\mathrm{BP}(D)$ by cutting those elements up into smaller pieces and applying inductive arguments.

Recall that for each subset $T \subseteq S$, the $C(T)$-invariant subspace $V_T \subseteq V$ is given by $\mathrm{span}\{v_s : s \in T\}$. We let $\rho_T \colon C(T) \to \mathrm{SL}^{\pm}(V_T)$ denote the representation of $C(T)$ obtained by restricting the ρ-action of $C(T)$ to V_T.

Below, we will show the following.

Proposition 8.3. *For any subset $T \subseteq S$ and any $D > 0$, there exist constants $A, B > 0$ such that, if $\gamma \in \mathrm{BP}(D) \cap C(T)$, then $\mu_{1,2}(\rho_T(\gamma)) \geq A|\gamma| - B$.*

This gives us the main theorem by taking $T = S$.

8.1. **Setting up the inductive statement.** Since we have assumed that the simplicial representation ρ has fully nondegenerate Cartan matrix, Lemma 4.8 implies that for any subset $T \subseteq S$, the restriction of the representation ρ to the subgroup $C(T)$ decomposes as a product $\rho_T \oplus \mathrm{id}$ on $V_T \oplus V_T^{\perp}$, where id denotes the trivial representation $C(T) \to \mathrm{SL}^{\pm}(V_T^{\perp})$. After a bounded change in inner product (which only changes singular values up to bounded multiplicative error), we can assume that the subspaces $V_T, V_T^{\perp}$ are orthogonal.

So, when $\gamma \in C(T)$, the inequality in Proposition 8.3 is satisfied with uniform constants for $\rho_T(\gamma)$ if and only if it is satisfied for $\rho(\gamma)$ (possibly with different constants). Further, the representation ρ_T is also a simplicial representation of the right-angled Coxeter group $C(T)$ with fully nondegenerate Cartan matrix.

So, to prove Proposition 8.3, we assume inductively that there are constants $A, B > 0$ so that for any proper subset $T \subsetneq S$, if $\gamma \in C(T) \cap \mathrm{BP}(D)$, then

$$\mu_{1,2}(\rho(\gamma)) \geq A|\gamma| - B. \tag{4}$$

We wish to show that, possibly after changing the constants A and B, this inequality also holds for any $\gamma \in \mathrm{BP}(D)$. If $|S| = 1$, the conclusion of Proposition 8.3 is obvious, since in this case $C \cong \mathbb{Z}/2$. So we just need to consider the inductive step. Before doing so, we make some simplifying arguments.

First, if C is finite, then so is $\mathrm{BP}(D)$, and the desired inequality holds trivially. So, we may assume C is infinite.

Next, for any $\gamma \in \mathrm{BP}(D)$, we can find a (possibly trivial) product decomposition of C into $C(S_1) \times C(S_2)$ so that $C(S_1)$ is irreducible and the image of γ under the projection $C \to C(S_2)$ has length at most D; this holds because every wall in $\mathrm{D}(C, S)$ whose type lies in S_1 intersects every wall whose type lies in S_2, and vice versa. Then, writing $\gamma = \gamma_1\gamma_2$ for $\gamma_1 \in C(S_1)$, $\gamma_2 \in C(S_2)$, we apply Lemma 6.2 to see that

$$\mu_{1,2}(\rho(\gamma)) \geq \mu_{1,2}(\rho(\gamma_1)) - KD$$

for a uniform constant K. So, by replacing γ with γ_1, for the purposes of proving Proposition 8.3 we can assume that $\gamma \in \mathrm{BP}(D)$ always lies in a subgroup $C(T)$ of C which is both infinite and irreducible.

If this subgroup is proper, then we can directly apply the inductive assumption (4) to obtain the desired lower bound on $\mu_{1,2}(\rho(\gamma))$. So we may assume that C is itself irreducible.

We have therefore reduced the inductive step of our proposition to the following statement:

Proposition 8.4. *Suppose that C is infinite and irreducible, and fix $D > 0$. If there exist uniform constants $A_0, B_0 > 0$ such that, for any proper subset $T \subsetneq S$ and any $\eta \in C(T) \cap \mathrm{BP}(D)$, we have*

$$\mu_{1,2}(\rho(\eta)) \geq A_0|\eta| - B_0, \tag{5}$$

then there are constants $A, B > 0$ such that for any $\gamma \in \mathrm{BP}(D)$, we have

$$\mu_{1,2}(\rho(\gamma)) \geq A|\gamma| - B.$$

The rest of the section is devoted to the proof of this proposition. So, from now on, we assume that the hypotheses of the proposition hold. Since we assume C is infinite and irreducible, we may assume that the Vinberg domain Ω_{Vin} is properly convex by Proposition 4.13.

8.2. **Cutting itineraries into pieces.** Fix an element $\gamma \in \mathrm{BP}(D)$. The first step in proving the estimate in Proposition 8.4 is to cut an itinerary traversing γ into several sub-itineraries, as follows. We let $\mathcal{U}$ be an itinerary traversing γ of the form

$$\{W_1\}, \mathcal{V}_1, \{W_2\}, \mathcal{V}_2, \ldots, \{W_n\}, \mathcal{V}_n,$$

which satisfies the conclusions of Proposition 3.4 (and of Corollary 3.7). In particular, in the partial order $<$ on walls in $\mathcal{U}$, we have $W_1 < \ldots < W_n$. We let $\mathbf{W}$ denote the set $\{W_i : 1 \leq i \leq n\}$, and for any $W_i, W_j \in \mathbf{W}$ with $W_i < W_j$, we let $\mathcal{U}(W_i, W_j)$ denote the sub-itinerary of $\mathcal{U}$ starting with W_i and ending with W_j. We also assume that the itinerary $\mathcal{U}$ departs from the identity; this means that $\mathbf{Hs}_+(W_j) \subset \mathbf{Hs}_+(W_i)$ whenever $i < j$.

We then choose a subset $\mathbf{Z} = \{Z_1, \ldots, Z_N\}$ of the walls W_i as follows: we take $Z_1 = W_1$. Then, for each $j > 1$, we take Z_j to be the first wall $W_i > Z_{j-1}$ such that $\overline{\mathbf{Hc}_+(W_i)} \subset \mathbf{Hc}_+(Z_{j-1})$. If there is no such wall, and Z_{j-1} is not already maximal in $\mathbf{W}$, we let $Z_j = Z_N$ be the maximal wall in $\mathbf{W}$. It follows that for every $i < j < N$, we have

$$\overline{\mathbf{Hc}_+(Z_j)} \subset \mathbf{Hc}_+(Z_i). \tag{6}$$

Each pair of walls Z_i, Z_{i+1} now determines a sub-itinerary $\mathcal{U}(Z_i, Z_{i+1})$ of $\mathcal{U}$.

8.3. **Outline of the rest of the proof.** We separate our sub-itineraries $\mathcal{U}(Z_i, Z_{i+1})$ into "short" and "long" ones, where the distinction depends on some uniform threshold on the length to be specified later.

If our itinerary spends a uniform proportion of its lifetime inside "short" sub-itineraries (i.e. a proportion above a uniform threshold τ, specified independent of γ), then the number of "short" sub-itineraries is at least a constant times $|\gamma|$. Using (6), we obtain a sequence of strictly nested half-cones. The Hilbert diameters of these half-cones are decreasing at a uniform exponential rate. Then we use Lemma 6.11 to get a corresponding uniform exponential estimate on the singular value gap.

Otherwise, we can assume the geodesic spends a large fraction of its lifetime inside of "long" sub-itineraries. For this case, we use the results of Section 5.4 to observe that each sub-itinerary traverses a group element which (up to uniformly bounded error) either lies in a proper standard subgroup of C, or else can be decomposed into into a product of two group elements, each of which lies in a proper standard subgroup of C. Using our inductive assumption, we can then show that each of our sub-itineraries traverses a group element whose singular value gap is uniformly exponential in its length.

We then need to show that when we "glue together" all of these sub-itineraries, we obtain an itinerary which also traverses a group element with uniformly exponential singular value gap. There are several different "gluing" steps in this process, but each of them essentially relies on just two techniques. First, if two adjacent "long" sub-itineraries are separated by a short sub-itinerary whose initial and final walls are well-separated in Ω_{Vin}, then we can use this separation to estimate positions of stable and unstable subspaces, and use uniform transversality (Lemma 6.3) to estimate singular value gaps.

On the other hand, if we *cannot* separate two adjacent "long" sub-itineraries $\mathcal{V}_-, \mathcal{V}_+$ by a pair of well-separated walls, then every sub-itinerary which "overlaps" both $\mathcal{V}_-$ and $\mathcal{V}_+$ must (up to some uniform error) traverse a group element lying in a proper standard subgroup of C. Once again, our inductive assumption implies that these overlapping sub-itineraries also have uniformly exponential singular value gaps. By applying the Kapovich-Leeb-Porti local-to-global principle (Theorem 6.8), we get a uniform gap estimate for the geodesic traversed by the concatenation $\mathcal{V}_-, \mathcal{V}_+$. The length constant in the local-to-global lemma is ultimately what determines the threshold for a "long" sub-itinerary.

By repeatedly using these two gluing arguments, we are eventually able to show that if the proportion of time spent in "long" sub-itineraries is sufficiently close to 1, then we obtain a uniform gap estimate for the entire itinerary. Combining this estimate with the previous case completes the proof.

8.4. **Estimating time spent in "short" sub-itineraries.** Given any itinerary $\mathcal{V}$, recall that the length $|\mathcal{V}|$ is the word-length of the element $\mathcal{V}$ traverses, or equivalently the number of walls appearing in $\mathcal{V}$. If $W_i < W_j < W_k$ are three walls in $\mathbf{W}$, then we have

$$|\mathcal{U}(W_i, W_k)| = |\mathcal{U}(W_i, W_j)| + |\mathcal{U}(W_j, W_k)| - 1,$$

since the wall W_j is counted twice on the right-hand side. It will therefore be convenient to introduce some additional notation, and let $d_{\mathcal{U}}(W_i, W_j)$ denote $|\mathcal{U}(W_i, W_j)| - 1$. With this notation, for any $W_i < W_j < W_k$, we have

$$d_{\mathcal{U}}(W_i, W_k) = d_{\mathcal{U}}(W_i, W_j) + d_{\mathcal{U}}(W_j, W_k). \tag{7}$$

Now suppose we have fixed a quantity $L > 0$. For any pair of walls $W_i < W_j$ in $\mathbf{W}$, we define the truncated length $\{d_{\mathcal{U}}(W_i, W_j)\}_L$ by

$$\{d_{\mathcal{U}}(W_i, W_j)\}_L := \begin{cases} d_{\mathcal{U}}(W_i, W_j), & d_{\mathcal{U}}(W_i, W_j) < L \\ 0, & d_{\mathcal{U}}(W_i, W_j) \geq L. \end{cases}$$

Similarly, we define $\{d_{\mathcal{U}}(W_i, W_j)\}^L$ to be $d_{\mathcal{U}}(W_i, W_j)$ if this is at least L, and 0 otherwise.

Given the collection of walls $\mathbf{Z} = Z_1 < \ldots < Z_N$, we define the quantity

$$r_L(\mathbf{Z}) := \frac{1}{d_{\mathcal{U}}(Z_1, Z_N)} \sum_{i=1}^{N-1} \{d_{\mathcal{U}}(Z_i, Z_{i+1})\}_L.$$

Roughly, $r_L(\mathbf{Z})$ is the proportion of time $\mathcal{U}$ spends inside of "short" sub-itineraries bounded by elements of $\mathbf{Z}$, where an itinerary is "short" if its length is at most L.

Notation 8.5. To simplify notation, for the rest of the paper, if $\mathcal{W}$ is an itinerary, we abbreviate $\mu_{1,2}(\rho(\gamma(\mathcal{W})))$ to $\mu_{1,2}(\mathcal{W})$. Further, if W_1, W_2 are walls in Ω_{Vin}, we write $\mu_{1,2}(W_1, W_2)$ for $\mu_{1,2}(\rho(\gamma(W_1, W_2)))$.

Similarly, we write $E_i^{\pm}(\mathcal{W})$ for $E_i^{\pm}(\rho(\gamma(\mathcal{W})))$, and $E_i^{\pm}(W_1, W_2)$ for $E_i^{\pm}(\rho(\gamma(W_1, W_2)))$.

8.5. **Mostly short sub-itineraries.** We first want to prove the following:

Proposition 8.6. *For any $L > 0$ and $\tau \in (0, 1)$, there are constants $A \geq 0, B > 0$ (depending on L, τ) such that if $r_L(\mathbf{Z}) \geq \tau$, then $\mu_{1,2}(\mathcal{U}) \geq A|\mathcal{U}| - B$.*

Proof. We let $L > 0$ and $\tau \in (0,1)$ be given, and let m denote the number of indices i in $1, \dots, N$ such that $d_{\mathcal{U}}(Z_i, Z_{i+1}) < L$. Thus, we have

$$m \cdot L \geq \sum_{i=1}^{N-1} \{d_{\mathcal{U}}(Z_i, Z_{i+1})\}_L,$$

and since we assume $r_L(\mathbf{Z}) \geq \tau$, we have

$$m \geq \frac{\tau}{L} \cdot d_{\mathcal{U}}(Z_1, Z_N).$$

We now consider the sequence of half-cones $\mathbf{Hc}_+(Z_1), \dots, \mathbf{Hc}_+(Z_N)$. For every index n, let γ_n denote the group element traversed by the itinerary $\mathcal{U}(Z_1, \dots, Z_n)$, and let $s_n = s(Z_n)$. Then Lemma 5.8 implies that

$$\mathbf{Hc}_+(Z_n) = \rho(\gamma_n)\mathbf{Hc}_-(s_n),$$

where, for any $s \in S$, $\mathbf{Hc}_+(s)$ denotes the positive half-cone over the reflection wall for $\rho(s)$.

Let d_n denote the Hilbert metric on the half-cone $\mathbf{Hc}_+(Z_n)$. As the half-cones $\mathbf{Hc}_+(Z_{n-1})$ and $\mathbf{Hc}_+(Z_n)$ are always nested, Lemma 6.10 implies that $d_n \geq d_{n-1}$ for all $n \leq N$.

When $n < N$, then the half-cones $\mathbf{Hc}_+(Z_{n-1})$ and $\mathbf{Hc}(Z_n)$ are *strongly* nested. Equivalently,

$$\rho(\gamma_{n-1}^{-1}\gamma_n)\overline{\mathbf{Hc}_-(s_n)} \subset \mathbf{Hc}_-(s_{n-1}). \tag{8}$$

For any n, we have $\gamma_{n-1}^{-1}\gamma_n = s_{n-1}\gamma(Z_{n-1}, Z_n)$. So, if $d_{\mathcal{U}}(Z_{n-1}, Z_n) < L$, there only are finitely many possible choices for the group element $\rho(\gamma_{n-1}^{-1}\gamma_n)$ in the inclusion (8) above, as well as finitely many possible choices for the half-cones $\mathbf{Hc}_-(s_n)$ and $\mathbf{Hc}_-(s_{n-1})$.

We can then apply the sharper form of Lemma 6.10 to see that there is a uniform constant $\lambda > 1$ (depending only on ρ, L) so that, if $n < N$, $d_{\mathcal{U}}(Z_{n-1}, Z_n) < L$ and $x, y \in \mathbf{Hc}_+(Z_n)$, then

$$d_n(x,y) \geq \lambda \cdot d_{n-1}(x,y). \tag{9}$$

Now let ℓ be the last index smaller than N such that $d_{\mathcal{U}}(Z_{\ell-1}, Z_\ell) < L$, and let D_ℓ be the diameter of $\mathbf{Hc}_+(Z_\ell)$ with respect to the Hilbert metric on $\mathbf{Hc}_+(Z_{\ell-1})$, or equivalently the diameter of $\rho(\gamma_{\ell-1}^{-1}\gamma_\ell)\mathbf{Hc}_-(s_\ell)$ with respect to the Hilbert metric on $\mathbf{Hc}_-(s_{\ell-1})$. Since there are only finitely many possible values for these two strongly nested half-cones, we can bound D_ℓ beneath a uniform constant D, which depends only on L.

From (9), the inequality $d_n \geq d_{n-1}$, and (8) it follows that the diameter of the half-cone $\mathbf{Hc}_+(Z_N) = \rho(\gamma_N)\mathbf{Hc}_-(s_N)$ with respect to the Hilbert metric on $\mathbf{Hc}_+(Z_1)$ is at most $\lambda^{-(m-1)}D$.

As there are only finitely many possible values for $\mathbf{Hc}_+(Z_1)$ and $\mathbf{Hc}_-(s_N)$, we can then apply Lemma 6.11 to see that

$$\mu_{1,2}(\rho(\gamma_N)) \geq m \cdot \log(\lambda) - B$$

for some uniform constant B. By Proposition 3.4, the group element $\gamma_N = \gamma_{\mathcal{U}}(Z_1, Z_N)$ is within uniformly bounded distance of $\gamma(\mathcal{U})$ in the Coxeter group C, and m is uniformly linear in $d_{\mathcal{U}}(Z_1, Z_N)$ (which is within uniformly bounded additive error of $|\mathcal{U}|$). So we can apply Lemma 6.2 to complete the proof. □

8.6. **Mostly long sub-itineraries.** Our goal for the rest of the section is to prove the following:

Proposition 8.7. *There exist constants $A > 0$, $B, L \geq 0$ and $\tau \in (0, 1)$ so that if $r_L(\mathbf{Z}) \leq \tau$, then $\mu_{1,2}(\mathcal{U}) \geq A|\mathcal{U}| - B$.*

Since the constants A, B, L, τ above can all be chosen independently of $\mathcal{U}$ and γ, putting Proposition 8.7 together with Proposition 8.6 will imply Proposition 8.4, which in turn gives us the desired Proposition 8.3 (and thus our main theorem). Throughout the section, we will obtain many intermediate estimates for various sub-itineraries $\mathcal{U}'$ of $\mathcal{U}$, which all roughly have the form

$$\mu_{1,2}(\mathcal{U}') \geq A|\mathcal{U}'| - B$$

for some constants A, B. We will not keep track of the different constants A, B for all of these different estimates.

Most of this section involves obtaining and refining various estimates for $\mu_{1,2}(\mathcal{U}(W_i, W_j))$ and $\mu_{1,2}(W_i, W_j)$ for certain walls $W_i, W_j \in \mathbf{W}$. Due to Corollary 3.7.(1) and Lemma 6.2, we can go back and forth between $\gamma_{\mathcal{U}}(W_i, W_j)$ and $\gamma(W_i, W_j)$ when we make these estimates. Precisely, we can say the following:

Lemma 8.8. *There are uniform constants R_1, R_2 so that for any walls $W_i < W_j$ in $\mathbf{W}$, we have*

$$|\mu_{1,2}(\mathcal{U}(W_i, W_j)) - \mu_{1,2}(W_i, W_j)| < R_1$$

and

$$||\gamma(W_i, W_j)| - d_{\mathcal{U}}(W_i, W_j)| < R_2.$$

Using Lemma 6.2 and the above, we also obtain the following:

Lemma 8.9. *There is a uniform constant $K > 0$ so that for any walls $W_i < W_j < W_k$ in $\mathbf{W}$, we have*

$$|\mu_{1,2}(W_i, W_k) - \mu_{1,2}(W_i, W_j)| \leq K \cdot d_{\mathcal{U}}(W_j, W_k),$$
$$|\mu_{1,2}(W_i, W_k) - \mu_{1,2}(W_j, W_k)| \leq K \cdot d_{\mathcal{U}}(W_i, W_j).$$

Definition 8.10. Let W_i, W_ℓ be walls in $\mathbf{W}$ with $W_i < W_\ell$. For constants $A, B > 0$, we say that the pair of walls W_i, W_ℓ *has (A, B)-gaps* if for every pair of walls W_j, W_k with $W_i \leq W_j < W_k \leq W_\ell$, we have

$$\mu_{1,2}(W_j, W_k) \geq A|\gamma(W_j, W_k)| - B. \tag{10}$$

If $\mathcal{V}$ is any itinerary equivalent to a sub-itinerary of $\mathcal{U}$, we say that $\mathcal{V}$ *has (A, B)-gaps* if every pair of walls $W_i < W_\ell$ in $\mathcal{V} \cap \mathbf{W}$ has (A, B)-gaps.

Remark 8.11. Lemma 8.8 means that we could equivalently define pairs of walls with (A, B)-gaps by replacing $\mu_{1,2}(W_j, W_k)$ with $\mu_{1,2}(\mathcal{U}(W_j, W_k))$ and $|\gamma(W_j, W_k)|$ with $|\mathcal{U}(W_j, W_k)|$ (or with $d_{\mathcal{U}}(W_j, W_k)$) in (10) above—if we do, the constants A, B change, but by a controlled amount depending only on the group C and the representation ρ.

Using Lemma 6.2 and Corollary 3.7.(2), one can also see that, if $\mathcal{V}$ is a sub-itinerary of some itinerary equivalent to $\mathcal{U}$, then $\mathcal{V}$ has (A, B)-gaps if and only the geodesic following $\mathcal{V}$ has uniform gaps in the sense of Definition 6.5 (with constants depending on A, B).

8.6.1. *Finding sub-itineraries with uniform gaps.* We have used the set of walls $\mathbf{Z}$ to cut the itinerary $\mathcal{U}$ into pieces, so our first main task is to show that each piece corresponds to a geodesic segment in Γ with uniform gaps. This will follow from:

Proposition 8.12. *There are constants $A, B > 0$ so that every pair of walls $W < W'$ in $\mathbf{W}$ which satisfies $\overline{\mathbf{Hc}_+(W')} \not\subset \mathbf{Hc}_+(W)$ has (A, B)-gaps.*

We need several intermediate lemmas in order to prove Proposition 8.12. First, we observe that our inductive assumption (5) from Proposition 8.4 and the quasiconvexity of standard subgroups in C implies the following:

Lemma 8.13. *There are uniform constants A, B so that, if $\mathcal{V}$ is any itinerary equivalent to a sub-itinerary of $\mathcal{U}$, and $\gamma(\mathcal{V}) \in C(T)$ for some $T \subsetneq S$, then $\mathcal{V}$ has (A, B)-gaps.*

We next need two different "straightness" lemmas for sub-itineraries of $\mathcal{U}$. Both Lemma 8.14 and Lemma 8.15 below essentially say that if two adjacent itineraries $\mathcal{V}_-, \mathcal{V}_+$ in $\mathcal{U}$ have sufficiently large gaps, and there are walls between $\mathcal{V}_-$ and $\mathcal{V}_+$ which are well-separated in Ω_{Vin}, then $\mu_{1,2}(\mathcal{V})$ is approximately $\mu_{1,2}(\mathcal{V}_-) + \mu_{1,2}(\mathcal{V}_+)$. The conclusion of both lemmas is nearly the same, but in Lemma 8.15, we assume slightly weaker separation hypotheses and slightly stronger hypotheses on gaps than in Lemma 8.14 (we refer to Appendix A for a discussion of the difference between the separation hypotheses in these lemmas). We need both lemmas for the arguments throughout the rest of the section.

Lemma 8.14 (Additivity for well-separated itineraries, I)**.** *Given $\lambda > 0$, there are constants $M, D > 0$ so that the following holds. Suppose that $W_i < W_j < W_k < W_\ell$ are walls in $\mathbf{W}$ satisfying:*

(1) $\mu_{1,2}(W_i, W_j) > M$ and $\mu_{1,2}(W_k, W_\ell) > M$;
(2) $\overline{\mathbf{Hc}_+(W_k)} \subset \mathbf{Hc}_+(W_j)$.
(3) $d_{\mathcal{U}}(W_j, W_k) < \lambda$.

Then we have

$$\mu_{1,2}(W_i, W_\ell) \geq \mu_{1,2}(W_i, W_j) + \mu_{1,2}(W_k, W_\ell) - D.$$

Proof. Let $\mathcal{V}$ be an efficient itinerary from W_i to W_ℓ, and write $\mathcal{V}$ as a concatenation $\mathcal{V}_-, \mathcal{V}_+$ so that $\mathcal{V}_-$ contains W_i, W_j and $\mathcal{V}_+$ contains W_k, W_ℓ. We change basepoints so that $\mathcal{V}_-$ arrives at the identity and $\mathcal{V}_+$ departs from the identity. After doing so, the inclusion $\overline{\mathbf{Hs}_+(W_k)} \subset \mathbf{Hs}_+(W_j)$ becomes $\mathbf{Hs}_+(W_k) \subset \mathbf{Hs}_-(W_j)$, which means we can assume that $\overline{\mathbf{Hc}_+(W_k)} \subset \mathbf{Hc}_-(W_j)$. We write $H_- = \mathbf{Hc}_-(W_j)$ and $H_+ = \mathbf{Hc}_+(W_k)$.

We choose $\varepsilon > 0$ so that $N_{2\varepsilon}(H_+) \subset N_{-2\varepsilon}(H_-)$. Since the distance $|\gamma(W_j, W_k)| \leq |\mathcal{U}(W_j, W_k)|$ is bounded by λ, this ε can be chosen uniformly in λ. Next, we choose $M' > 0$ (depending only on ε and λ) as in Lemma 7.3, so that if $\mu_{1,2}(\mathcal{V}_+) > M'$, then $E_1^+(\rho(\gamma(\mathcal{V}_+))) \in N_\varepsilon(H_+)$, and if $\mu_{1,2}(\mathcal{V}_-) > M'$, then $E_{d-1}^-(\rho(\gamma(\mathcal{V}_-)))$ is disjoint from $N_{-\varepsilon}(H_-)$. Finally, since $d_{\mathcal{U}}(W_j, W_k) < \lambda$, we know that $\gamma(\mathcal{V}_-) = \gamma(W_i, W_j)\eta_-$ and $\gamma(\mathcal{V}_+) = \eta_+\gamma(W_k, W_\ell)$ for $\eta_\pm$ satisfying $|\eta_\pm| < \lambda$. So, by Lemma 6.2, we can find M so that if $\mu_{1,2}(W_i, W_j) > M$, then $\mu_{1,2}(\mathcal{V}_-) > M'$, and similarly for W_k, W_ℓ and $\mathcal{V}_+$.

Now, applying Lemma 7.3 and our hypothesis (1) above, we see that an ε-neighborhood of $E_1^+(\mathcal{V}_+)$ is disjoint from $E_{d-1}^-(\mathcal{V}_-)$. That is, this pair of subspaces is uniformly transverse (see Figure 4).

Thus, the additivity estimate (Lemma 6.3) together with Lemma 8.9 gives us the desired bound. ∎

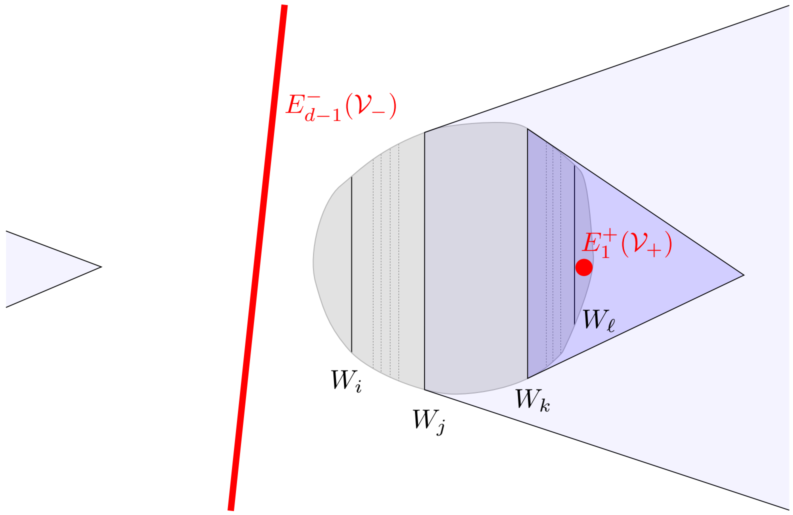

FIGURE 4. Illustration for the proof of Lemma 8.14. Since the half-cones over W_j, W_k strongly nest, the subspaces $E_1^+(\mathcal{V}_+)$ and $E_{d-1}^-(\mathcal{V}_-)$ are uniformly transverse.

Lemma 8.15 (Additivity for well-separated itineraries, II)**.** *Given $A, \lambda > 0$ and $B \geq 0$, there is a constant $D > 0$ so that the following holds. Suppose $W_i < W_j < W_k < W_\ell$ are walls in $\mathbf{W}$ satisfying:*

(1) The pairs of walls W_i, W_j and W_k, W_ℓ both have (A, B)-gaps.
(2) The intersection $\overline{W_j} \cap \overline{W_k}$ is empty.
(3) We have $d_{\mathcal{U}}(W_j, W_k) < \lambda$.

Then we have

$$\mu_{1,2}(W_i, W_\ell) \geq \mu_{1,2}(W_i, W_j) + \mu_{1,2}(W_k, W_\ell) - D. \tag{11}$$

Proof. First, observe that if either $|\gamma(W_i, W_j)|$ or $|\gamma(W_k, W_\ell)|$ is smaller than some constant N, then we can use Lemma 8.9 to find some constant D depending only on N so that the inequality (11) holds. So, throughout the proof, we will be able to assume that both $|\gamma(W_i, W_j)|$ and $|\gamma(W_k, W_\ell)|$ are larger than any given constant that depends only on A, B, λ. Since we have assumed that the pairs W_i, W_j and W_k, W_ℓ have (A, B)-gaps, this means we can also assume that $\mu_{1,2}(W_i, W_j)$ and $\mu_{1,2}(W_k, W_\ell)$ are larger than any given constant depending on A, B, λ.

Lemma 5.13 implies that we have a nesting of half-cones $\mathbf{Hc}_+(W_k) \subset \mathbf{Hc}_+(W_j)$. If this nesting is strict, the reasoning from the previous paragraph tells us that we can apply Lemma 8.14 and we will be done. So, we now assume that $\overline{\mathbf{Hc}_+(W_k)}$ is not contained in $\mathbf{Hc}_+(W_j)$.

As in the proof of Lemma 8.14, we let $\mathcal{V}$ be an efficient itinerary from W_i to W_ℓ, and write it as a concatenation $\mathcal{V}_-, \mathcal{V}_+$ for $\mathcal{V}_-$ containing W_i, W_j and $\mathcal{V}_+$ containing W_k, W_ℓ; in fact, we can ensure that the first wall of $\mathcal{V}_+$ is W_k. We can again assume that $\mathcal{V}_-$ arrives at the identity and $\mathcal{V}_+$ departs from the identity. As before, set $H_- = \mathbf{Hc}_-(W_j)$ and $H_+ = \mathbf{Hc}_+(W_k)$, so that H_+ (but not $\overline{H_+}$) is contained in H_-.

For each wall $W \in \mathcal{V}_+ \cap \mathbf{W}$, we let $E_1^+(W)$ denote the 1-dimensional unstable subspace of $\rho(\gamma_{\mathcal{V}_+}(W))$, where $\gamma_{\mathcal{V}_+}(W)$ is the group element traversed by a sub-itinerary of $\mathcal{V}_+$ starting with the initial wall of $\mathcal{V}_+$ and ending in W.

Let ε_0, M_0 be the fixed constants from Lemma 7.4. We know that $\mathcal{V}_+ = \mathcal{U}(W_k, W_\ell)$ has (A, B)-gaps. So, by applying the uniform convergence property for geodesics with uniform gaps in $\mathrm{SL}^{\pm}(d, \mathbb{R})$ (Lemma 6.6), we see that there is a fixed constant N (depending only on ε_0, A, B) so that if W is any wall in $\mathbf{W}$ separating W_k and W_ℓ, with $|\gamma(W_k, W)| \geq N$, we have

$$d_{\mathbb{P}}(E_1^+(W), E_1^+(\mathcal{V}_+)) < \varepsilon_0/2. \tag{12}$$

By increasing N if necessary, we can again apply the fact that $\mathcal{V}_+$ has uniform gaps to also ensure that as long as $|\gamma(W_k, W)| \geq N$, then $\mu_{1,2}(W_k, W) > M_0$.

Now, we may assume that $N > 2R$, and that $|\gamma(W_k, W_\ell)| \geq N$. Then, Corollary 3.7 (2) implies that there is a uniform $R > 0$ and some wall W in $\mathbf{W}$ with $W_k < W < W_\ell$ and $N < |\gamma(W_k, W)| \leq N + R$. We let H'_+ denote the half-cone $\mathbf{Hc}_+(W)$; from Lemma 5.13, we know that $H'_+ \subset H_+ \subset H_-$. This tells us that $\partial H'_+ \cap \partial H_- \subset \partial H_+ \cap \partial H_-$. But, Lemma 5.13 also implies that $\partial H'_+ \cap \partial H_-$ is a subset of ∂W, and that $\partial H_+ \cap \partial H_-$ is a subset of ∂W_k, which means we have

$$\partial H'_+ \cap \partial H_- \subset \overline{W} \cap \overline{W_k}. \tag{13}$$

We now consider two possibilities.

Case 1: $\overline{W_k} \cap \overline{W} = \emptyset$. In this case, the inclusion (13) means that the inclusion of half-cones $H'_+ \subset H_-$ is strict, i.e. that $\overline{H'_+} \subset H_-$. So, we can replace W_k with W and apply Lemma 8.14 (with $\lambda + N + R$ in place of λ); since the pairs W_i, W_j and W_k, W_ℓ have uniform gaps, as long as $|\gamma(W_i, W_j)|$ and $|\gamma(W_k, W_\ell)|$ are sufficiently large, the hypotheses of Lemma 8.14 are satisfied.

Case 2: $\overline{W_k} \cap \overline{W} \neq \emptyset$. In this case, we will show that the subspaces $E_1^+(\mathcal{V}_+)$ and $E_{d-1}^-(\mathcal{V}_-)$ are uniformly transverse by arguing that $E_1^+(\mathcal{V}_+)$ lies close to H'_+ and far from $\overline{W} \cap \overline{W_k}$, and $E_{d-1}^-(\mathcal{V}_-)$ does not lie too close to H_-; see Figure 5.

Since the closures $\overline{H'_+}$ and $\overline{H_-}$ are compact, we can choose some $\varepsilon > 0$ so that the intersection

$$N_{2\varepsilon}(H'_+) \cap N_\varepsilon(\mathbb{P}(V) - H_-)$$

is contained in the $\varepsilon_0/4$-neighborhood of $\partial H'_+ \cap \partial H_-$ (and thus, by (13), in the $\varepsilon_0/4$-neighborhood of $\overline{W} \cap \overline{W_k}$). Since $|\gamma(W_j, W)| < \lambda + N + R$, we can choose this ε uniformly in $\lambda, N, R, \varepsilon_0$. We can also ensure that $\varepsilon < \varepsilon_0/4$.

We now choose a constant M which is larger than the corresponding M from Lemma 7.3, taking the constant ε in the lemma to be our chosen ε, and N to be $\lambda + N + R$. We may assume that both $\mu_{1,2}(\mathcal{V}_+)$ and $\mu_{1,2}(\mathcal{V}_-)$ are at least M.

We claim that the distance between $E_1^+(\mathcal{V}_+)$ and $E_{d-1}^-(\mathcal{V}_-)$ must then be at least ε. To see this, suppose for a contradiction that there is some point $x \in E_{d-1}^-(\mathcal{V}_-)$ which is contained in the ε-neighborhood of $E_1^+(\mathcal{V}_+)$. Lemma 7.3 implies that $E_1^+(\mathcal{V}_+)$ is contained in $N_\varepsilon(H'_+)$ and $E_{d-1}^-(\mathcal{V}_-)$ is disjoint from $N_{-\varepsilon}(H_-)$. This means that x must lie in the intersection

$$N_{2\varepsilon}(H'_+) \cap (\mathbb{P}(V) - N_{-\varepsilon}(H_-)) = N_{2\varepsilon}(H'_+) \cap N_\varepsilon(\mathbb{P}(V) - H_-).$$

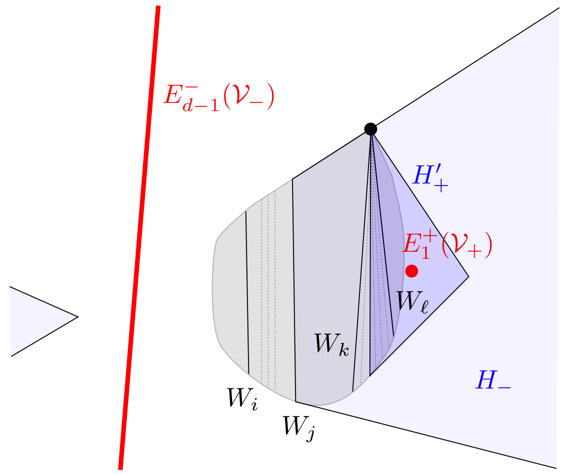

FIGURE 5. The subspaces $E^+_1(\mathcal{V}_+)$ and $E^-_{d-1}(\mathcal{V}_-)$ are uniformly transverse, since $E^+_1(\mathcal{V}_+)$ cannot lie close to the intersection of the boundaries of the half-cones H_-, H'_+.

In turn, this means that x is contained in the $\varepsilon_0/4$-neighborhood of $\overline{W_k} \cap \overline{W}$ and so $E^+_1(\mathcal{V}_+)$ is contained in the $\varepsilon_0/2$-neighborhood of $\overline{W_k} \cap \overline{W}$. Then, using (12), we see that $E^+_1(W)$ is contained in the ε_0-neighborhood of $\overline{W_k} \cap \overline{W}$. However, this contradicts Lemma 7.4.

We finally conclude that the subspaces $E^+_1(\mathcal{V}_+)$ and $E^-_{d-1}(\mathcal{V}_-)$ are uniformly transverse, which gives us the desired inequality by Lemma 6.3 and Lemma 8.9. □

The last intermediate lemma we need for the proof of Proposition 8.12 is a direct consequence of the Kapovich–Leeb–Porti local-to-global principle (Theorem 6.8):

Lemma 8.16 (Local-to-global for pairs with uniform gaps)**.** *For every $A, B > 0$, there exists $\lambda > 0$ and $A', B' > 0$ satisfying the following. Suppose that W_i, W_j are walls in $\mathbf{W}$ such that for every pair n, m with $d_{\mathcal{U}}(W_n, W_m) < \lambda$ and $W_i \leq W_n < W_m \leq W_j$, the pair W_n, W_m has (A, B)-gaps. Then W_i, W_j has (A', B')-gaps.*

Proof of Proposition 8.12. Fix walls $W_i < W_j$ in $\mathbf{W}$ so that $W \leq W_i < W_j \leq W'$. Since W_i and W_j separate W and W', Lemma 5.13 and the assumption $\overline{\mathbf{Hc}_+(W')} \not\subset \mathbf{Hc}_+(W)$ means that the strong inclusion $\overline{\mathbf{Hc}_+(W_j)} \subset \mathbf{Hc}_+(W_i)$ cannot hold. We want to find uniform constants A, B so that $\mu_{1,2}(W_i, W_j) \geq A|\gamma(W_i, W_j)| - B$.

If $\overline{W}_n \cap \overline{W}_m \neq \emptyset$, then Lemma 5.4 implies that $\gamma(W_i, W_j) \in C(T)$ for a proper standard subgroup $C(T) < C$, and we are done after applying Lemma 8.13. So, assume that $\overline{W_i} \cap \overline{W_j} = \emptyset$.

We consider an efficient itinerary $\mathcal{V}$ between W_i and W_j. By Lemma 5.14, $\mathcal{V}$ is equivalent to an itinerary of the form $\mathcal{W}_-, \mathcal{W}_0, \mathcal{W}_+$, where $|\mathcal{W}_0| < 2R$ for a uniform constant R, the itineraries $\mathcal{W}_-$ and $\mathcal{W}_+$ are efficient, and the intersections $\bigcap_{W \in \mathcal{W}_-} \overline{W}$ and $\bigcap_{W \in \mathcal{W}_+} \overline{W}$ are both nonempty. By Lemma 5.4, both $\gamma(\mathcal{W}_-)$ and $\gamma(\mathcal{W}_+)$ lie in (possibly different) proper standard subgroups of C.

We let $\mathcal{W}'_-$ be the concatenation $\mathcal{W}_-, \mathcal{W}_0$, so that $\mathcal{V} = \mathcal{W}'_-, \mathcal{W}_+$. By Lemma 8.13 and Lemma 8.9, we see that $\mathcal{W}'_-$ and $\mathcal{W}_+$ both have (A_0, B_0)-gaps, for uniform constants A_0, B_0.

We then choose constants A, B, λ as in the local-to-global lemma 8.16, depending on A_0, B_0.

Suppose that for every pair of walls $W_n, W_m \in \mathbf{W}$ such that $W_n \in \mathcal{W}'_-$, $W_m \in \mathcal{W}_+$, and $d_{\mathcal{U}}(W_n, W_m) < \lambda$, we have $\overline{W_n} \cap \overline{W_m} \neq \emptyset$. Lemma 5.4 then implies that each such pair satisfies $\gamma(W_n, W_m) \in C(T)$ for some proper subgroup $C(T) < C$, and then Lemma 8.13 implies that each such pair has (A_0, B_0)-gaps. Then Lemma 8.16 implies that the pair W_i, W_j has (A, B)-gaps, and we are done.

So, we may now suppose that there is some pair of walls $W_n, W_m \in \mathbf{W}$ such that $W_n \in \mathcal{W}'_-$, $W_m \in \mathcal{W}_+$, and $d_{\mathcal{U}}(W_n, W_m) < \lambda$, but $\overline{W_n} \cap \overline{W_m} = \emptyset$. Since $\mathcal{W}'_-, \mathcal{W}_+$ both have (A_0, B_0)-gaps, so do the pairs W_i, W_n and W_m, W_j. We can then apply Lemma 8.15 to the walls $W_i < W_n < W_m < W_j$ to complete the proof. $\square$

As a consequence of Proposition 8.12, we obtain:

Proposition 8.17. *There are uniform constants $A, B > 0$ so that every pair of consecutive walls $Z_i < Z_{i+1}$ in $\mathbf{Z}$ has (A, B)-gaps.*

Proof. Consider walls $W_n < W_m$ in $\mathbf{W}$ with $Z_i \leq W_n < W_m \leq Z_{i+1}$. We want to find constants A, B so that $\mu_{1,2}(W_n, W_m) \geq A|\gamma(W_n, W_m)| - B$. We may assume that $W_m > W_{n+1}$: otherwise, by Proposition 3.4 (b) we have $|\gamma(W_n, W_m)| = |\gamma(W_n, W_{n+1})| = 1$, and the desired inequality holds as long as we ensure $B > A$.

Now consider the pair of walls $W_n < W_{m-1}$. We know that $W_{m-1} < Z_{i+1}$, which means that the strong inclusion of half-cones $\overline{\mathbf{Hc}_+(W_{m-1})} \subset \mathbf{Hc}_+(W_n)$ cannot hold: in this case Lemma 5.13 would imply that also $\overline{\mathbf{Hc}_+(W_{m-1})} \subset \mathbf{Hc}_+(Z_i)$, and Z_{i+1} was chosen to be the minimal wall in $\mathbf{W}$ whose positive half-cone strongly nests into $\mathbf{Hc}_+(Z_i)$. Then Proposition 8.12 implies that $\mu_{1,2}(W_n, W_{m-1}) \geq A|\gamma(W_n, W_{m-1})| - B$ for some uniform constants A, B. So, we can apply Lemma 8.9 and the fact that $d_{\mathcal{U}}(W_{m-1}, W_m) < R$ to complete the proof. $\square$

8.6.2. *Combining sub-itineraries with uniform gaps using the local-to-global principle.* Proposition 8.12 shows that each pair of walls Z_i, Z_{i+1} defines a sub-itinerary $\mathcal{U}(Z_i, Z_{i+1})$ which has (A, B)-gaps, for uniform A, B. For the next step of the proof, we remove walls from the collection $\mathbf{Z}$ to "combine" adjacent sub-itineraries; we do so in a way which ensures that each sub-itinerary bounded by consecutive walls in $\mathbf{Z}$ has (A', B')-gaps, for uniform constants A', B' which do not depend on the itinerary $\mathcal{U}$.

More precisely, we modify $\mathbf{Z}$ via the following procedure. Using Proposition 8.12 and Proposition 8.17, we choose constants $A, B > 0$ so that every pair of consecutive walls Z_i, Z_{i+1} in $\mathbf{Z}$ has (A, B)-gaps, *and* so that every pair of walls $W < W'$ in $\mathbf{W}$ satisfying $\overline{\mathbf{Hc}_+(W')} \not\subset \mathbf{Hc}_+(W)$ also has (A, B)-gaps. Then, we fix local-to-global constants λ, A', B' from Lemma 8.16, depending on our chosen A, B. Now, suppose we have a sequence of consecutive elements $Z_j, Z_{j+1}, \ldots, Z_{j+n}$ in $\mathbf{Z}$ which satisfies the following property:

(†) For each wall Z_i in $\mathbf{Z}$ with $Z_j < Z_i < Z_{j+n}$, and every pair of walls $W, W' \in \mathbf{W}$ such that $W < Z_i < W'$ and $d_{\mathcal{U}}(W, W') < \lambda$, we have $\overline{\mathbf{Hc}_+(W')} \not\subset \mathbf{Hc}_+(W)$.

It immediately follows from Proposition 8.17, Proposition 8.12 and Lemma 8.16 that the pair of walls Z_j, Z_{j+n} has (A', B')-gaps. So, for each maximal subsequence $Z_j, \ldots, Z_{j+n}$ in $\mathbf{Z}$ satisfying (†), we delete all elements except Z_j and Z_{j+n} from $\mathbf{Z}$.

After this modification (and after reindexing $\mathbf{Z}$ appropriately), it is still true that each pair of consecutive walls Z_i, Z_{i+1} has (A, B)-gaps for uniform A, B. Further, since we have only deleted walls from $\mathbf{Z}$ (and we do not delete the walls Z_1 or Z_N), the quantity $r_L(\mathbf{Z})$

can only decrease for any fixed L. So we can still assume that $r_L(\mathbf{Z}) < \tau$ if this was true before we deleted walls from $\mathbf{Z}$.

After this modification, however, we gain the following. We now know that our $\mathbf{Z}$ satisfies the following additional property:

Proposition 8.18 (Long adjacent itineraries are well-separated)**.** *Let λ be the local-to-global constant defined above. Suppose that Z_{i-1}, Z_i, Z_{i+1} are three consecutive walls in $\mathbf{Z}$ such that $d_{\mathcal{U}}(Z_{i-1}, Z_i) \geq \lambda$ and $d_{\mathcal{U}}(Z_i, Z_{i+1}) \geq \lambda$. Then there is a pair of walls $W, W' \in \mathbf{W}$ satisfying:*

(1) $Z_{i-1} < W < Z_i < W' < Z_{i+1}$;
(2) $d_{\mathcal{U}}(W, W') < \lambda$;
(3) $\overline{\mathbf{Hc}_+(W')} \subset \mathbf{Hc}_+(W)$.

8.6.3. *Combining well-separated adjacent sub-itineraries.* For the next step, we again combine sequences of adjacent sub-itineraries by deleting elements from $\mathbf{Z}$.

Given $L_0 > 0$, we consider maximal subsequences $Z_j, Z_{j+1}, \ldots, Z_{j+n}$ of $\mathbf{Z}$ which satisfy:

($\star L_0$) for all indices i with $j \leq i < j+n$, we have $d_{\mathcal{U}}(Z_i, Z_{i+1}) \geq L_0$.

We claim the following:

Lemma 8.19. *There are uniform constants $A, L_0 > 0$ so that, if $Z_j, Z_{j+1}, \ldots, Z_{j+n}$ is a subsequence of $\mathbf{Z}$ satisfying ($\star L_0$), then*

$$\mu_{1,2}(Z_j, Z_{j+n}) \geq A \cdot d_{\mathcal{U}}(Z_j, Z_{j+n}).$$

Proof. As each pair of consecutive walls $Z_i < Z_{i+1}$ has uniform gaps, we know there are constants A', B' so that $\mu_{1,2}(Z_i, Z_{i+1}) \geq A' d_{\mathcal{U}}(Z_i, Z_{i+1}) - B'$. Let K be the constant from Lemma 8.9, let λ be the local-to-global constant from Proposition 8.18, and let M, D be the constants from Lemma 8.14 depending on λ. Then, choose L_0 large enough so that

$$A'L_0 - 3\lambda K - B' - D > M,$$

and define $A = (A'L_0 - 2\lambda K - B' - D)/L_0$, so that $AL_0 > M + \lambda K$.

To prove the proposition, we induct on the length of the sequence $Z_j, \ldots, Z_{j+n}$. In the base case $n = 1$, we have

$$\begin{aligned}\mu_{1,2}(Z_j, Z_{j+1}) &\geq A' d_{\mathcal{U}}(Z_j, Z_{j+1}) - B' \\ &= \left(A + \frac{2\lambda K + B' + D}{L_0}\right) d_{\mathcal{U}}(Z_j, Z_{j+1}) - B'.\end{aligned}$$

Since $d_{\mathcal{U}}(Z_j, Z_{j+1}) \geq L_0$ by assumption, we see that

$$\mu_{1,2}(Z_j, Z_{j+1}) \geq A \cdot d_{\mathcal{U}}(Z_j, Z_{j+1}) + 2\lambda K + D > A \cdot d_{\mathcal{U}}(Z_j, Z_{j+1}).$$

We now assume that $n > 1$, so that inductively we have

$$\mu_{1,2}(Z_j, Z_{j+n-1}) \geq A \cdot d_{\mathcal{U}}(Z_j, Z_{j+n-1}).$$

From Proposition 8.18, we can find walls W, W' with $W < Z_{j+n-1} < W'$ satisfying $\overline{\mathbf{Hc}_+(W')} \subset \mathbf{Hc}_+(W)$ and $d_{\mathcal{U}}(W, W') < \lambda$. In particular we know $d_{\mathcal{U}}(W, Z_{j+n-1}) < \lambda$, so by Lemma 8.9 we know that

$$\mu_{1,2}(Z_j, W) \geq A \cdot d_{\mathcal{U}}(Z_j, Z_{j+n-1}) - \lambda K. \tag{14}$$

In particular, since $d_{\mathcal{U}}(Z_j, Z_{j+n-1}) \geq L_0$, it follows that $\mu_{1,2}(Z_j, W) > M$.

We can also combine Lemma 8.9 with the argument from the base case and the fact that $d_{\mathcal{U}}(W', Z_{j+n}) < \lambda$ to see that

$$\mu_{1,2}(W', Z_{j+n}) \geq A \cdot d_{\mathcal{U}}(Z_{j+n-1}, Z_{j+n}) + \lambda K + D. \tag{15}$$

In particular, this tells us that $\mu_{1,2}(W', Z_{j+n}) > M + 2\lambda K + D > M$.

We now apply Lemma 8.14 to the walls $Z_j < W < W' < Z_{j+n}$ to see that

$$d_{\mathcal{U}}(Z_j, Z_{j+n}) \geq \mu_{1,2}(Z_j, W) + \mu_{1,2}(W', Z_{j+n}) - D.$$

Putting this inequality together with (14) and (15) and using the fact that $d_{\mathcal{U}}(Z_j, Z_{j+n}) = d_{\mathcal{U}}(Z_j, Z_{j+n-1}) + d_{\mathcal{U}}(Z_{j+n-1}, Z_{j+n})$ completes the proof. □

We keep the constant L_0 from Lemma 8.19 fixed for the rest of the paper. As in the previous step, we now modify the collection $\mathbf{Z}$ by replacing all maximal sequences $Z_j, Z_{j+1}, \ldots, Z_{j+n}$ satisfying ($\star L_0$) with Z_j, Z_{j+n}. Once again, we know that for any L, the quantity $r_L(\mathbf{Z})$ can only decrease after this modification.

After reindexing, the previous lemma now gives us the following:

Proposition 8.20. *For a uniform constant A_0, and every i with $1 \leq i < N$, if $d_{\mathcal{U}}(Z_i, Z_{i+1}) \geq L_0$, then $\mu_{1,2}(Z_i, Z_{i+1}) > A_0 \cdot d_{\mathcal{U}}(Z_i, Z_{i+1})$.*

In addition, $\mathbf{Z}$ now satisfies:

Proposition 8.21 (No adjacent long sub-itineraries)**.** *Let Z_{i-1}, Z_i, Z_{i+1} be consecutive walls in $\mathbf{Z}$. Then either $d_{\mathcal{U}}(Z_{i-1}, Z_i) < L_0$ or $d_{\mathcal{U}}(Z_i, Z_{i+1}) < L_0$.*

8.6.4. *Choosing the threshold for "long" sub-itineraries.* Let $N = |\mathbf{Z}|$. For any given $L > 0$, and for each $1 < i < j \leq N$, we now define

$$t_L^+(i, j) := \sum_{n=i}^{j} \{d_{\mathcal{U}}(Z_{n-1}, Z_n)\}^L.$$

We similarly define $t_L^-(i, j)$ via the truncated length:

$$t_L^-(i, j) := \sum_{n=i}^{j} \{d_{\mathcal{U}}(Z_{n-1}, Z_n)\}_L.$$

That is, $t_L^+(i, j)$ is the amount of time the itinerary $\mathcal{U}(Z_i, Z_j)$ spends inside of sub-itineraries $\mathcal{U}(Z_n, Z_{n+1})$ whose length is at least L, and $t_L^-(i, j)$ is the amount of time it spends inside of itineraries with length bounded strictly above by L.

It follows from (7) that for any L and any i, j we have

$$t_L^+(i, j) + t_L^-(i, j) = d_{\mathcal{U}}(Z_i, Z_j). \tag{16}$$

We also define the quantity

$$r_L(i, j) := t_L^-(i, j)/d_{\mathcal{U}}(Z_i, Z_j).$$

By definition, we have $r_L(\mathbf{Z}) = r_L(1, N)$.

Definition 8.22. We now let A_0, L_0 be the constants from Proposition 8.20. We then fix constants D_0, M_0 as in Lemma 8.14, where the constant λ in the lemma is chosen to be L_0. Then, we define

$$L_1 = \max\{M_0/A_0, D_0/A_0, L_0\}$$

and then choose constants D_1, M_1 as in Lemma 8.14 again, but this time taking $\lambda = L_1$.

Lemma 8.23. *Given $A, B, C > 0$, there exists $L > 0, \tau_1 \in (0,1)$ so that for any $i < j$, if $r_L(i,j) < \tau_1$, then*

$$A \cdot t_L^+(i,j) - B \cdot t_L^-(i,j) > C.$$

Proof. Since $t_L^-(i,j) = d_{\mathcal{U}}(Z_i, Z_j) r_L(i,j)$ and $t_L^+(i,j) = d_{\mathcal{U}}(Z_i, Z_j)(1 - r_L(i,j))$, the left-hand side of the inequality above is given by

$$d_{\mathcal{U}}(Z_i, Z_j) \cdot (A(1 - r_L(i,j)) - B r_L(i,j)).$$

If $r_L(i,j) < \tau_1$ for some $\tau_1 \in (0,1)$, the above expression is bounded below by

$$d_{\mathcal{U}}(Z_i, Z_j)(A(1 - \tau_1) - B\tau_1).$$

As $\tau_1 \to 0$, the quantity $(A(1-\tau_1) - B\tau_1)$ approaches A. In particular, if $\tau_1 < \frac{A}{2(A+B)}$, then the above expression is at least $d_{\mathcal{U}}(Z_i, Z_j) \cdot \frac{A}{2}$. Finally, the fact that $r_L(i,j) < \tau_1 < 1$ implies that $d_{\mathcal{U}}(Z_i, Z_j) > L$ since there has to be at least one long sub-itinerary, so the lemma follows if we take $L \geq 2C/A$. □

Definition 8.24. We define

$$B_0 := \max\{D_1, D_0, K\},$$

where K is the constant from Lemma 8.9, and then use the previous lemma to fix constants L, τ_1 so that

$$A_0 \cdot t_L^+(i,j) - B_0 \cdot t_L^-(i,j) > \max\{M_0, M_1\} \tag{17}$$

whenever $r_L(i,j) < \tau_1$. We also choose L large enough so that $L > L_1 \geq L_0$. The length L will be our threshold for "long" sub-itineraries.

8.6.5. *Combining all remaining sub-itineraries.*

Proposition 8.25. *For any $1 \leq i < j \leq N$, if $r_L(i,j) < \tau_1$, then*

$$\mu_{1,2}(Z_i, Z_j) \geq A_0 \cdot t_L^+(i,j) - B_0 \cdot t_L^-(i,j).$$

Proof. We will induct on $j - i$. In the base case $j = i+1$, then since $r_L(i,j) < \tau_1 < 1$ we must have $d_{\mathcal{U}}(Z_i, Z_{i+1}) \geq L$. Since $L \geq L_0$, Proposition 8.20 tells us that $\mu_{1,2}(Z_i, Z_j) \geq A_0 d_{\mathcal{U}}(Z_i, Z_j)$ and since $d_{\mathcal{U}}(Z_i, Z_j) = t_L^+(i,j)$ in this case, we are done.

So now suppose that $j - i > 1$. First, observe that if $d_{\mathcal{U}}(Z_i, Z_{i+1}) < L$, then $r_L(i+1, j) < r_L(i,j) < \tau_1$. So, by induction we have

$$\mu_{1,2}(Z_{i+1}, Z_j) \geq A_0 \cdot t_L^+(i+1,j) - B_0 \cdot t_L^-(i+1,j).$$

In this case, we also know that $t_L^+(i,j) = t_L^+(i+1,j)$. So, after applying Lemma 8.9, we have

$$\begin{aligned}\mu_{1,2}(Z_i, Z_j) &\geq A_0 \cdot t_L^+(i,j) - B_0 \cdot t_L^-(i+1,j) - K \cdot d_{\mathcal{U}}(Z_i, Z_{i+1}) \\ &\geq A_0 \cdot t_L^+(i,j) - B_0 \cdot t_L^-(i+1,j) - B_0 \cdot d_{\mathcal{U}}(Z_i, Z_{i+1}).\end{aligned}$$

But we also know that $t_L^-(i,j) = d_{\mathcal{U}}(Z_i, Z_{i+1}) + t_L^-(i+1,j)$ in this case, so this proves the desired inequality. A similar argument also shows that the inequality holds in the case where $d_{\mathcal{U}}(Z_{j-1}, Z_j) < L$.

So, we may now assume that we have $d_{\mathcal{U}}(Z_i, Z_{i+1}) \geq L$ and $d_{\mathcal{U}}(Z_{j-1}, Z_j) \geq L$. Our strategy in this case is to find a suitable pair of walls $Z_{n-1} < Z_n$ to break the itinerary $\mathcal{U}(Z_i, Z_j)$ into three pieces: a "short" piece in the middle, and two "mostly long" pieces on either side. Then we will apply induction and one of our additivity lemmas for well-separated walls (Lemma 8.14).

To find a suitable "short" sub-itinerary in the middle of $d_{\mathcal{U}}(Z_i, Z_j)$, we prove the following claim:

Claim. For some n with $i+1 < n < j$, we have

$$r_L(i, n-1) < \tau_1, \qquad r_L(n, j) < \tau_1, \tag{18}$$

and $d_{\mathcal{U}}(Z_{n-1}, Z_n) < L$.

To find such an n, first note that since $d_{\mathcal{U}}(Z_{j-1}, Z_j) \geq L > L_0$, Proposition 8.21 implies that $d_{\mathcal{U}}(Z_{j-2}, Z_{j-1}) < L_0 < L$ and thus $j-2 > i$. Then, if $r_L(i, j-2) < \tau_1$, we can satisfy the claim by taking $n = j-1$, since $r_L(j-1, j) = 0$. So, we can assume that $r_L(i, j-2) \geq \tau_1$, and let n be the minimal index such that $r_L(i, n) \geq \tau_1$.

Now, we have

$$\begin{aligned} r_L(i,j) &= \frac{1}{d_{\mathcal{U}}(Z_i, Z_j)}(t_L^-(i,n) + t_L^-(n,j)) \\ &= \frac{d_{\mathcal{U}}(Z_i, Z_n)}{d_{\mathcal{U}}(Z_i, Z_j)} r_L(i,n) + \frac{d_{\mathcal{U}}(Z_n, Z_j)}{d_{\mathcal{U}}(Z_i, Z_j)} r_L(n,j). \end{aligned}$$

From (7), we also know that

$$\frac{d_{\mathcal{U}}(Z_i, Z_n)}{d_{\mathcal{U}}(Z_i, Z_j)} + \frac{d_{\mathcal{U}}(Z_n, Z_j)}{d_{\mathcal{U}}(Z_i, Z_j)} = 1.$$

So, since $r_L(i,n) \geq \tau_1$ we must have $r_L(n,j) < \tau_1$, and by minimality of n we also know that $r_L(i, n-1) < \tau_1$. But in particular we also know $r_L(i, n-1) < r_L(i, n)$, implying $d_{\mathcal{U}}(Z_{n-1}, Z_n) < L$.

This proves the claim, so to finish the inductive step, we need to consider two cases:

Case 1: $d_{\mathcal{U}}(Z_{n-1}, Z_n) < L_1$: In this case, we consider the sub-itineraries $\mathcal{U}(Z_i, Z_{n-1})$ and $\mathcal{U}(Z_n, Z_j)$. Since $r_L(i, n-1) < \tau_1$ and $r_L(n, j) < \tau_1$, we may assume inductively that

$$\mu_{1,2}(i, n-1) \geq A_0 t_L^+(i, n-1) - B_0 t_L^-(i, n-1), \tag{19}$$

$$\mu_{1,2}(n, j) \geq A_0 t_L^+(n, j) - B_0 t_L^-(n, j). \tag{20}$$

It then follows directly from Definition 8.24 and (19), (20) above that

$$\mu_{1,2}(i, n-1) > M_1, \qquad \mu_{1,2}(n, j) > M_1.$$

So, since $d_{\mathcal{U}}(Z_{n-1}, Z_n) < L_1$, we can apply Lemma 8.14 to the walls Z_i, Z_{n-1}, Z_n, Z_j to obtain

$$\begin{aligned} \mu_{1,2}(Z_i, Z_j) &> \mu_{1,2}(Z_i, Z_{n-1}) + \mu_{1,2}(Z_n, Z_j) - D_1 \\ &\geq A_0(t_L^+(i, n-1) + t_L^+(n,j)) - B_0(t_L^-(i, n-1) + t_L^-(n,j)) - D_1 \\ &\geq A_0 t_L^+(i,j) - B_0(t_L^-(i, n-1) + t_L^-(n,j) + 1), \end{aligned}$$

where the last inequality holds because we have defined $B_0 \geq D_1$. But then since $1 \leq d_{\mathcal{U}}(Z_{n-1}, Z_n) < L$, we know that

$$t_L^-(i,j) = t_L^-(i, n-1) + t_L^-(n,j) + d_{\mathcal{U}}(Z_{n-1}, Z_n) \geq t_L^-(i, n-1) + t_L^-(n,j) + 1,$$

which proves the desired inequality in this case.

Case 2: $d_{\mathcal{U}}(Z_{n-1}, Z_n) \geq L_1$: In this case, since $L_1 \geq L_0$, Proposition 8.21 implies that the two sub-itineraries $\mathcal{U}(Z_{n-2}, Z_{n-1})$, $\mathcal{U}(Z_n, Z_{n+1})$ to the left and right of $\mathcal{U}(Z_{n-1}, Z_n)$ both have length less than L_0. Since we know $r_L(i, n-1) < \tau_1 < 1$ and $r_L(n, j) < \tau_1 < 1$, this also tells us that $i < n-2$ and $n+1 < j$, and that

$$r_L(i, n-2) < \tau_1, \qquad r_L(n+1, j) < \tau_1.$$

We then see directly from Definition 8.24 together with the induction hypothesis that $\mu_{1,2}(Z_i, Z_{n-2}) > M_0$ and $\mu_{1,2}(Z_{n+1}, Z_j) > M_0$. In addition, since $d_{\mathcal{U}}(Z_{n-1}, Z_n) \geq L_1 = \max\{L_0, M_0/A_0, D_0/A_0\}$, Proposition 8.20 implies that $\mu_{1,2}(Z_{n-1}, Z_n) \geq \max\{M_0, D_0\}$.

We now apply Lemma 8.14 twice. First, we apply the lemma to the walls $Z_i < Z_{n-2} < Z_{n-1} < Z_n$, which gives the bound

$$\mu_{1,2}(Z_i, Z_n) \geq \mu_{1,2}(Z_i, Z_{n-2}) + \mu_{1,2}(Z_{n-1}, Z_n) - D_0. \tag{21}$$

Since $\mu_{1,2}(Z_{n-1}, Z_n) \geq D_0$ we see that $\mu_{1,2}(Z_i, Z_n) > M_0$, which means we can then apply Lemma 8.14 to the walls $Z_i < Z_n < Z_{n+1} < Z_j$ to obtain

$$\mu_{1,2}(Z_i, Z_j) \geq \mu_{1,2}(Z_i, Z_n) + \mu_{1,2}(Z_{n+1}, Z_j) - D_0. \tag{22}$$

Putting (21) and (22) together we see that

$$\mu_{1,2}(Z_i, Z_j) \geq \mu_{1,2}(Z_i, Z_{n-2}) + \mu_{1,2}(Z_{n-1}, Z_n) + \mu_{1,2}(Z_{n+1}, Z_j) - 2D_0.$$

Since each of $d_{\mathcal{U}}(Z_{n-2}, Z_{n-1})$, $d_{\mathcal{U}}(Z_{n-1}, Z_n)$, and $d_{\mathcal{U}}(Z_n, Z_{n+1})$ is less than L, we know that $t_L^+(i, j) = t_L^+(i, n-2) + t_L^+(n+1, j)$. Thus, after applying induction to the terms $\mu_{1,2}(Z_i, Z_{n-2})$ and $\mu_{1,2}(Z_{n+1}, Z_j)$ in the inequality above, and discarding the (nonnegative) $\mu_{1,2}(Z_{n-1}, Z_n)$ term, we obtain

$$\begin{aligned}\mu_{1,2}(Z_i, Z_j) &\geq A_0 t_L^+(i, j) - B_0(t_L^-(i, n-2) + t_L^-(n+1, j)) - 2D_0 \\ &\geq A_0 t_L^+(i, j) - B_0(t_L^-(i, n-2) + t_L^-(n+1, j) + 2).\end{aligned}$$

For the last line we apply the fact that $B_0 \geq D_0$. Finally, since $t_L^-(n-2, n-1) \geq 1$ and $t_L^-(n, n+1) \geq 1$, we get

$$\begin{aligned}t_L^-(i, j) &= t_L^-(i, n-2) + t_L^-(n-2, n-1) + t_L^-(n-1, n) + t_L^-(n, n+1) + t_L^-(n+1, j) \\ &> t_L^-(i, n-2) + t_L^-(n+1, j) + 2,\end{aligned}$$

and we obtain the desired inequality in this case as well. □

Finally we obtain the estimate we originally wanted.

Proof of Proposition 8.7. We set

$$\tau = \min\left\{\tau_1, \frac{A_0}{2(A_0 + B_0)}\right\},$$

where A_0, B_0 are the constants from Proposition 8.25.

To simplify notation we write $r = r_L(\mathbf{Z})$. Since all of our modifications to $\mathbf{Z}$ have only decreased r, if we had $r < \tau$ before our modifications, this is still true for our current $\mathbf{Z}$.

By definition, we have $r \cdot d_{\mathcal{U}}(Z_1, Z_N) = t_L^-(1, N)$, and (16) implies that

$$t_L^+(1, N) = (1 - r) \cdot d_{\mathcal{U}}(Z_1, Z_N).$$

Then, since $r < \tau \leq \tau_1$, we can use Proposition 8.25 to obtain:

$$\begin{aligned}\mu_{1,2}(Z_1, Z_N) &\geq (A_0(1 - r) - B_0 r) \cdot d_{\mathcal{U}}(Z_1, Z_N) \\ &\geq (A_0(1 - \tau) - B_0\tau) \cdot d_{\mathcal{U}}(Z_1, Z_N) \\ &\geq \frac{A_0}{2} d_{\mathcal{U}}(Z_1, Z_N).\end{aligned}$$

Our construction ensures that $Z_1 = W_1$ and Z_N is always the maximal wall in $\mathbf{W}$. From Corollary 3.7 and the definition of $\mathcal{U}$, we know there is a uniform $R > 0$ so that $\gamma(\mathcal{U}) = \eta\,\gamma(Z_1, Z_N)\,\eta'$, for some $\eta, \eta' \in C$ with $|\eta|, |\eta'| < R$. Thus the desired estimate follows from Lemma 6.2. ☐

Appendix A. Failure of strong nesting for half-cones

The purpose of this appendix is to prove the following two claims:

Proposition A.1. *There exists a right-angled Coxeter group C, a simplicial representation ρ of C, and a pair of walls W, W' in the Vinberg domain Ω_{Vin} for ρ which satisfies the following properties:*

(1) $\gamma(W, W')$ does not lie in a proper standard subgroup of C (equivalently, by Lemma 5.4, $\overline{W} \cap \overline{W'} = \emptyset$);
(2) $\mathbf{Hs}_+(W') \subset \mathbf{Hs}_+(W)$;
(3) $\overline{\mathbf{Hc}_+(W')}$ is not *contained in $\mathbf{Hc}_+(W)$.*

Proposition A.2. *There exists a right-angled Coxeter group C and reflections R, R' in C such that, for any simplicial representation $\rho\colon C \to \mathrm{SL}^{\pm}(|S|, \mathbb{R})$ with fully nondegenerate Cartan matrix, and* any *reflection domain Ω for ρ, if W, W' are the walls in Ω preserved by R, R', then:*

(1) $\gamma(W, W')$ does not lie in a proper standard subgroup of C (so in particular $\overline{W} \cap \overline{W'} = \emptyset$);
(2) $\mathbf{Hs}_+(W') \subset \mathbf{Hs}_+(W)$;
(3) $\overline{\mathbf{Hc}_+(W')}$ is not *contained in $\mathbf{Hc}_+(W)$.*

Although the second proposition above implies the first, we will prove these results one at a time, since the construction for Proposition A.2 is a slightly more complicated variation of the construction for Proposition A.1.

Note that if Proposition A.1 were false, then the proofs in Section 8 of this paper would considerably simplify—in particular, the proof of Lemma 8.15 could be reduced to a direct application of Lemma 8.14. Proposition A.2 tells us that we cannot resolve the problem simply by replacing Ω_{Vin} in Section 8 with some other carefully chosen reflection domain.

Remark A.3. If C is a hyperbolic Coxeter group, it follows from [14, Corollary 1.11] that there is a simplicial representation $\rho\colon C \to \mathrm{SL}^{\pm}(V)$ and a reflection domain Ω for ρ so that half-cones over any two walls W, W' in Ω with disjoint closures will strongly nest. It seems likely that this holds even for some examples of simplicial representations of non-hyperbolic Coxeter groups, but we do not pursue this here.

A.1. **Proof of Proposition A.1.** Consider the right-angled Coxeter group C with generating set $S = \langle a, b, c, d, e \rangle$, and nerve given in Figure 6 below (recall that there is an edge between two vertices in the nerve precisely when the corresponding generators commute):

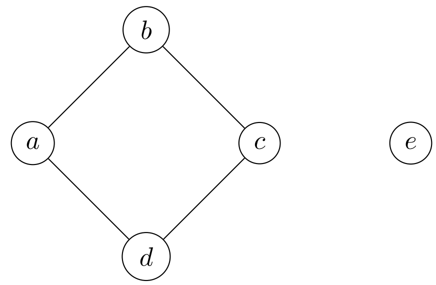

FIGURE 6. The nerve of the right-angled Coxeter group C in Proposition A.1.

That is, C is isomorphic to the group $(\mathbb{Z}/2*\mathbb{Z}/2)^2*(\mathbb{Z}/2)$, where the $\mathbb{Z}/2*\mathbb{Z}/2$ factors are generated by the pairs a, c and b, d, and e generates the remaining $\mathbb{Z}/2$ free factor. Observe that C is a minimal example of an irreducible non-hyperbolic right-angled Coxeter group: a theorem of Gromov (see [26, Section 4.2.C] or [16, Chapter 12]) implies that a right-angled Coxeter group fails to be hyperbolic precisely when the "empty square" on the vertices a, b, c, d appears as a full subgraph of the nerve. (The general version of this theorem for Coxeter groups which are not necessarily right-angled is due to Moussong [50].)

Let $V = \mathbb{R}^5$. We choose a nonsingular Cartan matrix for C and a simplicial representation determined by this Cartan matrix. We suppress the representation from the notation, and just let C act directly on $\mathbb{P}(V)$. For each $s \in \{a, b, c, d, e\}$, we let $H_s \subset \mathbb{P}(V)$ denote the (projective) reflection hyperplane for the reflection s, and we let F_s denote the closed face of the Tits simplex Δ fixed by s. Further, for any subset $S' \subset S$, we let $H_{S'}$ denote $\bigcap_{s\in S'} H_s$, and $F_{S'}$ denote $\bigcap_{s\in S'} F_s$.

Consider the 2-dimensional projective subspace $H_{\{a,c\}} = H_a \cap H_c$. This subspace contains the closed face $F_{\{a,c\}}$ of the fundamental simplex Δ fixed pointwise by the standard subgroup $C(a, c)$. Since $C(a, c)$ is infinite, $F_{\{a,c\}}$ must be contained in the boundary of Ω_{Vin} and hence so are all of its translates under the action of C.

Now, since the subgroup $C(b, d)$ centralizes $C(a, c)$, it preserves the subspace $H_{\{a,c\}}$, and in fact b and d act on this subspace by projective reflections fixing the lines $H_b \cap H_{\{a,c\}}$ and $H_d \cap H_{\{a,c\}}$. The relative interior of the orbit $C(b, d) \cdot F_{\{a,c\}}$ is an infinite-sided convex polygon P in $H_{\{a,c\}}$, which must be a subset of $\partial\Omega_{\mathrm{Vin}}$ (see Figure 7).

On the other hand, since the generator e does *not* commute with either a or c, the intersection $H_{\{a,c\}} \cap eH_{\{a,c\}}$ is the 1-dimensional subspace $H_{\{a,c\}} \cap H_e$. This tells us that the closed polygon $\overline{P}$ is exactly the intersection of $\overline{\Omega_{\mathrm{Vin}}}$ with $H_{\{a,c\}}$, i.e. it is a face of Ω_{Vin}.

Now, consider the geodesic word $w = bdeac$, and let $\mathcal{W}$ be an Ω_{Vin}-itinerary traversing w, departing from the identity. Since no pair of consecutive generators in w commutes, every pair of distinct walls in $\mathcal{W}$ is disjoint in Ω_{Vin}. This means that $\mathcal{W}$ must be efficient. So, if W, W' are respectively the first and last walls in $\mathcal{W}$, we must have $\gamma(W, W') = bdeac$.

As $W \cap W' = \emptyset$ and $\mathcal{W}$ departs from the identity, we know that $\mathbf{Hs}_+(W)$ contains $\mathbf{Hs}_+(W')$. By Proposition 2.3 we have $W' = bdea \cdot W(c)$, where $W(c)$ is the reflection wall in Ω_{Vin} for c. From this, it follows that:

Proposition A.4. *The intersection $\overline{W'} \cap \overline{P}$ is given by $bd \cdot F_{\{a,c,e\}}$, where $F_{\{a,c,e\}}$ is the edge of $F_{\{a,c\}}$ fixed by e.*

Proof. Since $\overline{W(c)} = H_c \cap \overline{\Omega_{\mathrm{Vin}}}$, and Ω_{Vin} is C-invariant, we have $\overline{P} \cap \overline{W'} = \overline{P} \cap bdea \cdot H_c$. Since P is invariant under b, d, this intersection is the same as $bd(\overline{P} \cap ea \cdot H_c)$. Then

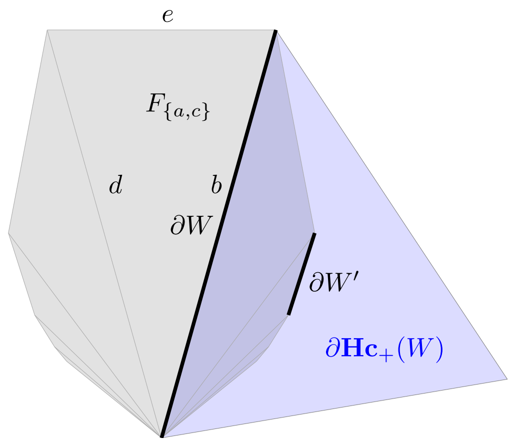

FIGURE 7. The polygon P giving a face of the Vinberg domain Ω_{Vin} for a simplicial representation of C. The polygon is tiled by copies of the triangle $F_{\{a,c\}}$; the walls of this tiling are translates of $H_{\{a,c\}} \cap \partial W(b)$ and $H_{\{a,c\}} \cap \partial W(d)$. The shaded triangle has a vertex at the polar of b, so it lies in the boundary of the half-cone $\mathbf{Hc}_+(W)$.

since e does not commute with c or a, we have $(ea \cdot H_c) \cap H_{\{a,c\}} = H_{\{a,c,e\}}$ and thus $\overline{P} \cap ea \cdot H_c = F_{\{a,c,e\}}$. □

We know that the polar of b lies in the projective subspace $H_{a,c}$ since b commutes with both a and c. So, we can use the argument from Lemma 5.9 to see that the boundary of the half-cone $\mathbf{Hc}_+(W)$ contains the connected component of $\overline{P} - \overline{W}$ which does *not* contain $F_{\{a,c\}}$. Thus, the boundary of $\mathbf{Hc}_+(W)$ contains $\overline{W'} \cap H_{a,c}$ and so the half-cones over these walls cannot strongly nest.

Remark A.5. For any given $k \geq 1$, we can also consider the word $w = (bd)^k e(ac)^k$, and an itinerary $\mathcal{W}$ traversing w departing from the identity. A nearly identical argument to the above shows that the initial and terminal walls W, W' of $\mathcal{W}$ also satisfy the conclusions of Proposition A.1. This proves that, in Proposition A.1, the group element $\gamma(W, W')$ cannot even be made to lie "close" to a proper standard subgroup: we cannot find a uniform constant R so that $\gamma(W, W') = \eta_1 \gamma \eta_2$ for η_1, η_2 satisfying $|\eta_i| < R$ and γ lying in a proper standard subgroup of C.

A.1.1. *Modifying the example.* The argument above also shows that the corresponding half-cones in the Vinberg domain $\mathscr{O}_{\mathrm{Vin}} \subset \mathbb{P}((\mathbb{R}^5)^*)$ for the dual representation ρ^* (see Section 4.3) do not strongly nest. By Lemma 5.12, the corresponding half-cones in the dual $\mathscr{O}^*_{\mathrm{Vin}}$ cannot strongly nest either. By Proposition 4.15, the reflection domain $\mathscr{O}^*_{\mathrm{Vin}}$ is contained inside of every reflection domain for ρ, so we denote it $\Omega_{\min}$.

One could still hope to find some reflection domain Ω lying between $\Omega_{\min}$ and Ω_{Vin} where the half-cones over the walls $W \cap \Omega$ and $W' \cap \Omega$ strongly nest. In fact, for the example above, it is possible to find such a domain, by taking Ω to be a small neighborhood of $\Omega_{\min}$ with respect to the Hilbert metric on Ω_{Vin}. This strategy works because the segments in $\partial\Omega_{\mathrm{Vin}}$ joining ∂W to $\partial W'$ are not contained in $\partial\Omega_{\min}$; see Figure 8. But this can fail if the intersection between $\partial\Omega_{\mathrm{Vin}}$ and $\partial\Omega_{\min}$ contains large-dimensional faces, which is what occurs in the next counterexample.

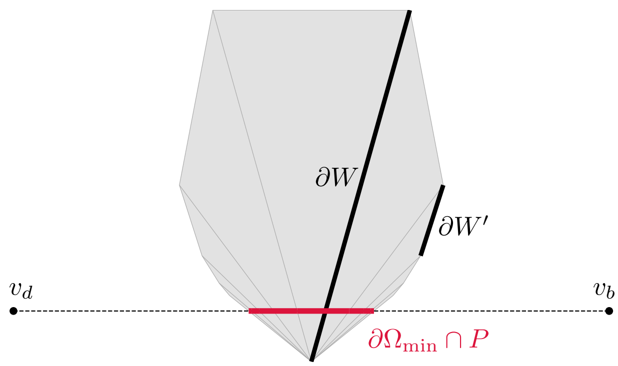

FIGURE 8. The boundary of the minimal reflection domain $\Omega_{\min}$ does not contain a segment joining W with W'. See Theorem A.7 below for a construction of $\Omega_{\min}$.

A.2. **Proof of Proposition A.2.** We consider a right-angled Coxeter group C whose generating set S splits into three disjoint subsets $D = \{d_1, d_2\}$, $T = \{t_1, t_2, t_3\}$, and $E = \{e\}$, with the following relations:

- Each $t_i \in T$ commutes with each $d_j \in D$;
- The generator e commutes with t_1 and t_3.

There are no other relations among the generators, meaning the system (C, S) has the nerve depicted in Figure 9 below. Observe that the subgroup $C(T)$ is an (∞, ∞, ∞) triangle group, which commutes with the infinite dihedral subgroup $C(D)$.

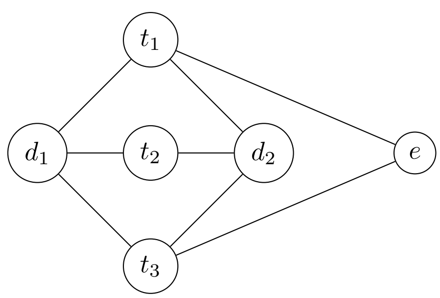

FIGURE 9. The nerve of the right-angled Coxeter group C in Proposition A.2.

As for the previous example, we let $V = \mathbb{R}^6$, and fix an arbitrary simplicial representation $C \to \mathrm{SL}^{\pm}(6, \mathbb{R})$ with fully nondegenerate Cartan matrix. We again omit the representation from the notation and allow C to act directly on V and $\mathbb{P}(V)$.

Consider the 3-dimensional projective subspace $H_D \subset \mathbb{P}(V)$, containing the closed face F_D of Δ. This face is a tetrahedron whose faces span fixed subspaces for the reflections in $T \cup E$. As every point in F_D has infinite stabilizer, it lies in the boundary of the Vinberg domain Ω_{Vin}.

Since the centralizer of the subgroup $C(D)$ is precisely $C(T)$, the interior of the orbit $C(T) \cdot F_D$ is a subset of H_D. In fact, one may apply the more general form of Theorem 4.11 given in [60] to see that $C(T) \cdot F_D$ is an infinite-sided convex polytope $P \subset H_D$, whose closure $\overline{P}$ is a face of the Vinberg domain tiled by copies of F_D. See Figure 10.

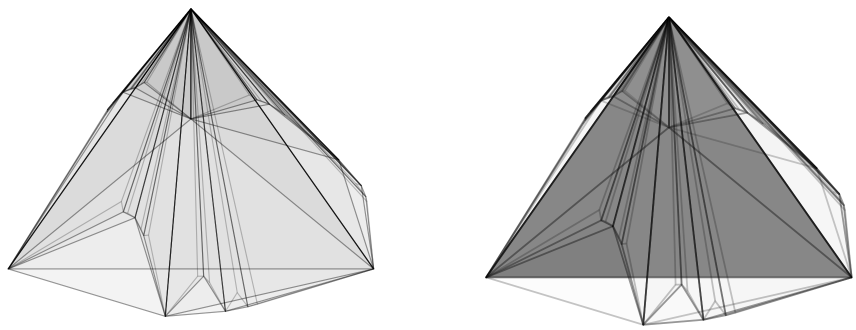

FIGURE 10. The convex polytope P (left and right). On the right, the tetrahedral fundamental domain F_D has also been highlighted. The reflections t_1, t_2, t_3 in the three faces of F_D meeting at the topmost vertex preserve the polytope P, while the reflection e in the bottom face $F_{D \cup E}$ does not.

It is possible to argue as in the proof of Proposition A.1 to see that there is a pair of walls W, W' in Ω_{Vin} with disjoint closures, such that the boundary of $\mathbf{Hc}_+(W)$ contains a component of $P - \overline{W}$ whose closure intersects $\overline{W'}$. We want to see that something similar occurs not just for walls in the Vinberg domain, but in *any* reflection domain for this simplicial representation.

For this, we consider the unique minimal reflection domain $\Omega_{\min}$ for the simplicial representation ρ, whose existence is guaranteed by the discussion in Section A.1.1. We can get a precise description of $\Omega_{\min}$ in our situation using a theorem of Danciger-Guéritaud-Kassel-Lee-Marquis.

Definition A.6. Let (C, S) be an infinite irreducible right-angled Coxeter system, and suppose C acts via a simplicial representation on $V = \mathbb{R}^{|S|}$ with nonsingular Cartan matrix. For any subset $S' \subseteq S$, we let $\tilde{\Sigma}_{S'} \subset \overline{\tilde{\Delta}}$ denote the set

$$\tilde{\Sigma}_{S'} = \left\{ x = \sum_{t \in S'} \lambda_t v_t : \alpha_s(x) \leq 0 \text{ and } \lambda_t \geq 0 \quad \forall s \in S, t \in S' \right\}.$$

We let $\Sigma_{S'}$ denote the projectivization of $\tilde{\Sigma}_{S'}$ in $\mathbb{P}(V)$, and write $\tilde{\Sigma} = \tilde{\Sigma}_S$ and $\Sigma = \Sigma_S$.

Theorem A.7 (See [14, Theorem 5.2])**.** *Let (C, S) be an infinite irreducible right-angled Coxeter group with $|S| > 2$, acting on $V = \mathbb{R}^{|S|}$ by a simplicial representation with nonsingular Cartan matrix. Then the set $\overline{\Omega_{\min}} = \overline{\bigcup_{\gamma \in C} \gamma \Sigma}$ is the closure of the unique reflection domain $\Omega_{\min} \subset \Omega_{\text{Vin}}$ which is contained in every reflection domain for ρ.*

Remark A.8. The version of Theorem A.7 proved in [14] holds under considerably weaker hypotheses than what we have stated here. In particular, the result in [14] can be applied to non-right-angled Coxeter groups and non-simplicial representations.

We now consider the intersection of the sets Σ and $\overline{\Omega_{\min}}$ with the projective subspace H_D, for the specific right-angled Coxeter group C we described above. Observe that the subgroup $C(T)$ acts on the subspace $V_T = \mathrm{span}\{v_{t_1}, v_{t_2}, v_{t_3}\}$, via a simplicial representation whose Cartan matrix is a (fully nondegenerate) principal submatrix of our original Cartan matrix. Thus $\Sigma_T \subset \Sigma \cap \mathbb{P}(V_T)$ is a hexagon, with alternating sides contained in the projective lines $H_{t_i} \cap \mathbb{P}(V_T)$ (see Figure 11).

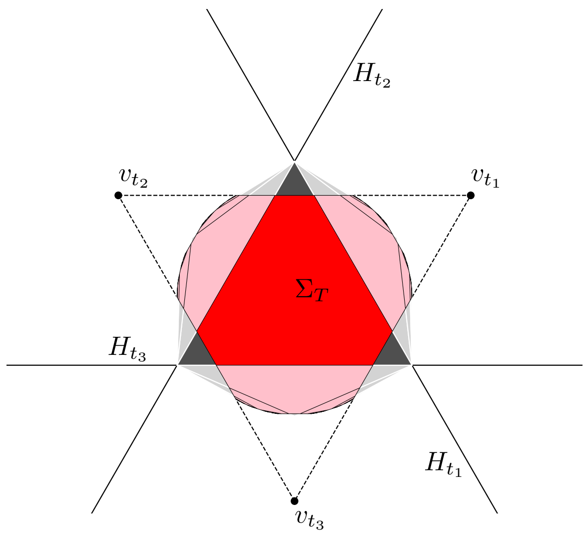

FIGURE 11. The action of the standard subgroup $C(T)$ via a simplicial representation on $\mathbb{P}(V_T)$. Copies of the red hexagon Σ_T tile the minimal domain $P_{\min}$ (in pink) for this simplicial representation.

In particular, since $C(D)$ and $C(T)$ commute, we know that $\mathbb{P}(V_T) \subset H_D$, and therefore $\Sigma_T \subset F_D$. In addition, since t_1 and t_3 commute with all of the generators in $D \cup E$, the subspace $\mathbb{P}(V_{t_1,t_3})$ is contained in the 2-dimensional projective subspace $H_{D\cup E} = H_D \cap H_E$, which means that the edge Σ_{t_1,t_3} of the hexagon Σ_T is contained in $H_{D\cup E}$ (see Figure 12, left).

Since $\Sigma_T \subset F_D$, the relative interior of the orbit $C(T) \cdot \Sigma_T$ is a $C(T)$-invariant convex subset $P_{\min}$ of the polytope P, contained in the $C(T)$-invariant subspace $\mathbb{P}(V_T)$; it is a copy of the minimal $C(T)$-invariant domain for the simplicial representation of $C(T)$ on V_T (see Figure 12, right).

We now consider the geodesic word $w = t_1 t_3 t_2 e d_1 d_2$, and let $\mathcal{W}$ be an Ω_{Vin}-itinerary traversing w, departing from the identity. As in the previous example, since no pair of consecutive generators in w commutes, the itinerary $\mathcal{W}$ is efficient. Thus, writing W_{Vin} and W'_{Vin} for the first and last walls of $\mathcal{W}$, $\gamma(W_{\mathrm{Vin}}, W'_{\mathrm{Vin}}) = t_1 t_3 t_2 e d_1 d_2$.

We let R, R' denote the reflections in C fixing $W_{\mathrm{Vin}}, W'_{\mathrm{Vin}}$, and let Ω be an arbitrary reflection domain for C. We let W, W' denote the walls in Ω fixed by R, R', and let $W_{\min}, W'_{\min}$ denote the walls in $\Omega_{\min}$ fixed by the same pair of reflections. By Proposition 4.15 and

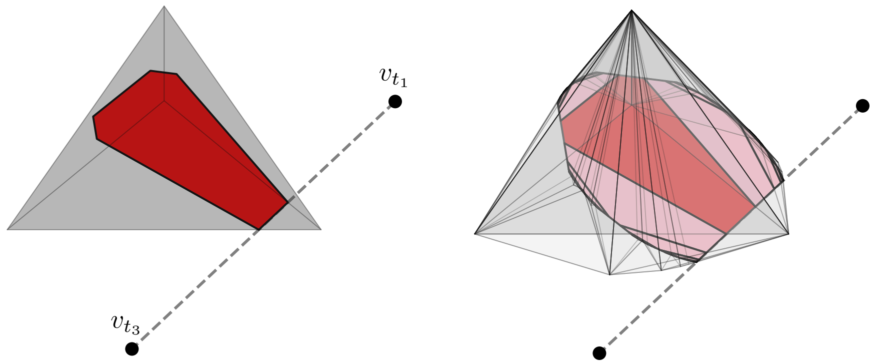

FIGURE 12. Left: the hexagon Σ_T, embedded in the tetrahedron F_D. The subset Σ_{t_1,t_3} is an edge of the hexagon contained in the bottom face $F_{D\cup E}$ of F_D. Right: the convex subset $P_{\min}$ embedded in the polytope P.

Theorem A.7, we have

$$\Omega_{\min} \subseteq \Omega \subseteq \Omega_{\mathrm{Vin}},$$

and

$$W_{\min} \subseteq W \subseteq W_{\mathrm{Vin}}, \qquad W'_{\min} \subseteq W' \subseteq W'_{\mathrm{Vin}}.$$

By Proposition 2.3, we have $W'_{\min} = t_1t_3t_2ed_1 \cdot (H_{d_2} \cap \Omega_{\min})$. Using this, we show:

Proposition A.9. *The intersection $\overline{W'_{\min}} \cap \overline{P_{\min}}$ contains $t_1t_3t_2 \cdot \Sigma_{t_1,t_3}$.*

Proof. Since $P_{\min}$ is $C(T)$-invariant and $\Omega_{\min}$ is C-invariant, we just need to check that $\overline{P_{\min}} \cap ed_1 \cdot H_{d_2}$ contains Σ_{t_1,t_3}. Since e does not commute with d_1 or d_2, we know that $ed_1 \cdot H_{d_2} \cap H_D = H_{D\cup E}$, and as $P_{\min} \subset H_D$ we therefore have

$$\overline{P_{\min}} \cap ed_1 \cdot H_{d_2} = \overline{P_{\min}} \cap H_{D\cup E}.$$

We have already seen that the edge Σ_{t_1,t_3} of the hexagon Σ_T is contained in $H_{D\cup E}$, which gives us the desired intersection since $C(T)$-translates of Σ_T tile $P_{\min}$. ☐

Now, since the Ω_{Vin}-itinerary $\mathcal{W}$ departs from the identity, we know that $\mathbf{Hs}_+(W_{\mathrm{Vin}})$ contains $\mathbf{Hs}_+(W'_{\mathrm{Vin}})$ and thus $\mathbf{Hs}_+(W') \subset \mathbf{Hs}_+(W)$. We consider the boundary of the half-cone $\mathbf{Hc}_+(W)$. Note that, since the entire tetrahedron F_D has infinite stabilizer in C, any point in $\overline{W} \cap F_D$ must lie in ∂W. In particular, since $W_{\min} \subset W$, the boundary ∂W must contain $\overline{W_{\min}} \cap F_D$, which is a reflection wall in the 2-dimensional domain $P_{\min}$.

As $P_{\min}$ is a reflection domain for $C(T)$ in the $C(T)$-invariant subspace $\mathbb{P}(V_T)$, Lemma 5.9 tells us that the closure of the half-cone $\mathbf{Hc}_+(\overline{W_{\min}} \cap P_{\min})$ on the domain $P_{\min}$ contains the component of $P_{\min} - \overline{W_{\min}}$ which does not contain Σ_T (see Figure 13).

In particular, as $W_{\min}$ is a reflection wall for t_1, this half-cone contains $t_1t_3t_2 \cdot \Sigma_{t_1,t_3}$. As $\overline{W_{\min}} \cap P_{\min} \subset \partial W$, we have

$$\mathbf{Hc}_+(\overline{W_{\min}} \cap P_{\min}) \subset \partial \mathbf{Hc}_+(W),$$

meaning that $\partial\mathbf{Hc}_+(W)$ contains $t_1t_3t_2 \cdot \Sigma_{t_1,t_3}$. Then, since $W'_{\min} \subset W'$ we see from Proposition A.9 that $\overline{W'} \cap \partial\mathbf{Hc}_+(W)$ is nonempty, meaning that the halfcones $\mathbf{Hc}_+(W)$ and $\mathbf{Hc}_+(W')$ cannot strongly nest.

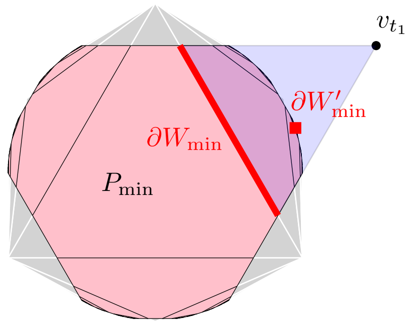

FIGURE 13. The boundary of the half-cone over $W_{\min}$ intersects the boundary of the wall $W'_{\min}$.

Remark A.10. As in Remark A.5, we can apply a nearly identical argument to the word $(t_1t_3t_2)^k e(d_1d_2)^k$ for any given $k > 0$ to see that the group element $\gamma(W, W')$ can also be made to lie arbitrarily far from any standard subgroup of C.

References

[1] I. Agol, *The virtual Haken conjecture*, Doc. Math., 18 (2013), pp. 1045–1087. With an appendix by Agol, Daniel Groves, and Jason Manning.

[2] N. Bergeron and D. T. Wise, *A boundary criterion for cubulation*, Amer. J. Math., 134 (2012), pp. 843–859.

[3] M. Bestvina, *Local homology properties of boundaries of groups*, Michigan Math. J., 43 (1996), pp. 123–139.

[4] J. Bochi, R. Potrie, and A. Sambarino, *Anosov representations and dominated splittings*, J. Eur. Math. Soc. (JEMS), 21 (2019), pp. 3343–3414.

[5] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], No. 1337, Hermann, Paris, 1968.

[6] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, vol. 319 of Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1999.

[7] R. Canary, *Anosov representations: Informal lecture notes*, 2021.

[8] R. Canary and K. Tsouvalas, *Topological restrictions on Anosov representations*, J. Topol., 13 (2020), pp. 1497–1520.

[9] R. D. Canary, M. Stover, and K. Tsouvalas, *New nonlinear hyperbolic groups*, Bulletin of the London Mathematical Society, 51 (2019), pp. 547–553.

[10] R. M. Charney and M. W. Davis, *Strict hyperbolization*, Topology, 34 (1995), pp. 329–350.

[11] J. Danciger, F. Guéritaud, and F. Kassel, *Examples and non-examples of convex cocompact groups in projective space*. In preparation.

[12] ———, *Convex cocompact actions in real projective geometry*, arXiv e-prints, (2017), p. arXiv:1704.08711.

[13] ———, *Convex cocompactness in pseudo-Riemannian hyperbolic spaces*, Geom. Dedicata, 192 (2018), pp. 87–126.

[14] J. Danciger, F. Guéritaud, F. Kassel, G.-S. Lee, and L. Marquis, *Convex cocompactness for Coxeter groups*, arXiv e-prints, (2021), p. arXiv:2102.02757.

[15] M. W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. (2), 117 (1983), pp. 293–324.

[16] ———, *The geometry and topology of Coxeter groups*, in Introduction to modern mathematics, vol. 33 of Adv. Lect. Math. (ALM), Int. Press, Somerville, MA, 2015, pp. 129–142.
[17] M. W. Davis and T. Januszkiewicz, *Hyperbolization of polyhedra*, J. Differential Geom., 34 (1991), pp. 347–388.
[18] ———, *Right-angled Artin groups are commensurable with right-angled Coxeter groups*, J. Pure Appl. Algebra, 153 (2000), pp. 229–235.
[19] T. Delzant and M. Gromov, *Cuts in Kähler groups*, in Infinite groups: geometric, combinatorial and dynamical aspects, vol. 248 of Progr. Math., Birkhäuser, Basel, 2005, pp. 31–55.
[20] S. Dey, *On Borel Anosov subgroups of* $\mathrm{SL}(d,\mathbb{R})$, arXiv e-prints, (2022), p. arXiv:2208.02109.
[21] S. Dey and M. Kapovich, *Klein-Maskit combination theorem for Anosov subgroups: Free products*, arXiv e-prints, (2022), p. arXiv:2205.03919.
[22] ———, *Klein-Maskit combination theorem for Anosov subgroups: Amalgams*, arXiv e-prints, (2023), p. arXiv:2301.02354.
[23] S. Dey, M. Kapovich, and B. Leeb, *A combination theorem for Anosov subgroups*, Math. Z., 293 (2019), pp. 551–578.
[24] S. Douba and K. Tsouvalas, *Anosov groups that are indiscrete in rank one*, arXiv preprint arXiv:2210.17549, (2022).
[25] A. Giralt, *Cubulation of Gromov-Thurston manifolds*, Groups Geom. Dyn., 11 (2017), pp. 393–414.
[26] M. Gromov, *Hyperbolic groups*, in Essays in group theory, vol. 8 of Math. Sci. Res. Inst. Publ., Springer, New York, 1987, pp. 75–263.
[27] ———, *Asymptotic invariants of infinite groups*, in Geometric group theory, Vol. 2 (Sussex, 1991), vol. 182 of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 1993, pp. 1–295.
[28] F. Guéritaud, O. Guichard, F. Kassel, and A. Wienhard, *Anosov representations and proper actions*, Geom. Topol., 21 (2017), pp. 485–584.
[29] O. Guichard and A. Wienhard, *Anosov representations: domains of discontinuity and applications*, Invent. Math., 190 (2012), pp. 357–438.
[30] M. F. Hagen, *Geometry and combinatorics of cube complexes*, McGill University (Canada), 2012.
[31] F. Haglund, *Finite index subgroups of graph products*, Geom. Dedicata, 135 (2008), pp. 167–209.
[32] F. Haglund and D. T. Wise, *Special cube complexes*, Geom. Funct. Anal., 17 (2008), pp. 1551–1620.
[33] P. Haïssinsky, *Hyperbolic groups with planar boundaries*, Invent. Math., 201 (2015), pp. 239–307.
[34] M. Islam and A. Zimmer, *A flat torus theorem for convex co-compact actions of projective linear groups*, J. Lond. Math. Soc. (2), 103 (2021), pp. 470–489.
[35] T. Januszkiewicz and J. Świątkowski, *Hyperbolic Coxeter groups of large dimension*, Comment. Math. Helv., 78 (2003), pp. 555–583.
[36] M. Kapovich, *Representations of polygons of finite groups*, Geom. Topol., 9 (2005), pp. 1915–1951.
[37] ———, *Convex projective structures on Gromov-Thurston manifolds*, Geom. Topol., 11 (2007), pp. 1777–1830.
[38] ———, *Kleinian groups in higher dimensions*, in Geometry and dynamics of groups and spaces, vol. 265 of Progr. Math., Birkhäuser, Basel, 2008, pp. 487–564.
[39] M. Kapovich, B. Leeb, and J. Porti, *Morse actions of discrete groups on symmetric space*, arXiv e-prints, (2014), p. arXiv:1403.7671.
[40] ———, *Anosov subgroups: dynamical and geometric characterizations*, Eur. J. Math., 3 (2017), pp. 808–898.
[41] ———, *Dynamics on flag manifolds: domains of proper discontinuity and cocompactness*, Geom. Topol., 22 (2018), pp. 157–234.
[42] ———, *A Morse lemma for quasigeodesics in symmetric spaces and Euclidean buildings*, Geom. Topol., 22 (2018), pp. 3827–3923.
[43] ———, *Morse actions of discrete groups on symmetric spaces: Local-to-global principle*, arXiv e-prints, (2023), p. arXiv:2301.04562.
[44] G. Kozma and A. Lubotzky, *Linear representations of random groups*, Bull. Math. Sci., 9 (2019), pp. 1950016, 12.
[45] F. Labourie, *Anosov flows, surface groups and curves in projective space*, Invent. Math., 165 (2006), pp. 51–114.
[46] J.-F. Lafont and L. Ruffoni, *Special cubulation of strict hyperbolization*, arXiv preprint arXiv:2206.03620, (2022).
[47] G.-S. Lee and L. Marquis, *Anti–de Sitter strictly GHC-regular groups which are not lattices*, Trans. Amer. Math. Soc., 372 (2019), pp. 153–186.

[48] V. Markovic, *Criterion for Cannon's conjecture*, Geom. Funct. Anal., 23 (2013), pp. 1035–1061.

[49] J. Maxwell Riestenberg, *A quantified local-to-global principle for Morse quasigeodesics*, arXiv e-prints, (2021), p. arXiv:2101.07162.

[50] G. Moussong, *Hyperbolic Coxeter groups*, ProQuest LLC, Ann Arbor, MI, 1988. Thesis (Ph.D.)–The Ohio State University.

[51] G. Niblo and L. Reeves, *Groups acting on* CAT(0) *cube complexes*, Geom. Topol., 1 (1997), p. approx. 7 pp.

[52] ———, *Coxeter groups act on* CAT(0) *cube complexes*, J. Group Theory, 6 (2003), pp. 399–413.

[53] Y. Ollivier and D. T. Wise, *Cubulating random groups at density less than* 1/6, Trans. Amer. Math. Soc., 363 (2011), pp. 4701–4733.

[54] P. Ontaneda, *Riemannian hyperbolization*, Publ. Math. Inst. Hautes Études Sci., 131 (2020), pp. 1–72.

[55] P. Przytycki and J. Świątkowski, *Flag-no-square triangulations and Gromov boundaries in dimension 3*, Groups Geom. Dyn., 3 (2009), pp. 453–468.

[56] P. Py, *Coxeter groups and Kähler groups*, Math. Proc. Cambridge Philos. Soc., 155 (2013), pp. 557–566.

[57] N. Tholozan and K. Tsouvalas, *Linearity and indiscreteness of amalgamated products of hyperbolic groups*, arXiv preprint arXiv:2112.05574, (2021).

[58] ———, *Residually finite non linear hyperbolic groups*, arXiv preprint arXiv:2207.14356, (2022).

[59] K. Tsouvalas, *On Borel Anosov representations in even dimensions*, Comment. Math. Helv., 95 (2020), pp. 749–763.

[60] E. B. Vinberg, *Discrete linear groups that are generated by reflections*, Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya, 35 (1971), pp. 1072–1112.

[61] D. T. Wise, *Cubulating small cancellation groups*, Geom. Funct. Anal., 14 (2004), pp. 150–214.

[62] ———, *The cubical route to understanding groups*, in Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, Kyung Moon Sa, Seoul, 2014, pp. 1075–1099.

[63] ———, *The structure of groups with a quasiconvex hierarchy*, vol. 209 of Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 2021.

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