THIN RIGHT-ANGLED COXETER GROUPS IN SOME UNIFORM ARITHMETIC LATTICES

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Abstract. Using a variant of an unpublished argument due to Agol, we show that an irreducible right-angled Coxeter group on $n \geq 3$ vertices embeds as a thin subgroup of a uniform arithmetic lattice in an indefinite orthogonal group $\mathrm{O}(p,q)$ for some $p, q \geq 1$ satisfying $p+q=n$.

Let $\mathbf{G}$ be a semisimple algebraic $\mathbb{R}$-group and Γ a lattice in $G := \mathbf{G}(\mathbb{R})$. A subgroup $\Delta \subset \Gamma$ is said to be *thin* if Δ is Zariski-dense in G but of infinite index in Γ. It follows from the Borel density theorem [Bor60, Corollary 4.3] and a classical result of Tits [Tit72, Theorem 3] that if $\mathbf{G}$ as above is nontrivial, connected, and without compact factors, then any lattice in G contains a thin nonabelian free subgroup. A famous construction of Kahn–Markovic [KM12] produces thin surface subgroups of all uniform lattices in $\mathrm{SO}(3,1)$ (see [Ham15], [LR16], [CF19], [KLM18] for some other manifestations of surface groups as thin groups). In [BL20], Ballas–Long show that many arithmetic lattices in $\mathrm{SO}(n,1)$ virtually embed as thin subgroups of lattices in $\mathrm{SL}_{n+1}(\mathbb{R})$, and raise the question as to which groups arise as thin groups. In this note, we observe the following.

Theorem 1. *An irreducible right-angled Coxeter group on $n \geq 3$ vertices embeds as a thin subgroup of a uniform arithmetic lattice in $\mathrm{O}(p,q)$ for some $p, q \geq 1$ satisfying $p+q=n$.*

To that end, let Σ_1 be a connected simplicial graph on $n \geq 3$ vertices; we think of Σ_1 as a Coxeter scheme in the sense of [VS93, pg. 201, Def. 1.7] all of whose edges are bold. Fix an order $v_1, \ldots, v_n$ on the vertices of Σ_1, and let W be the group given by the presentation with generators $\gamma_1, \ldots, \gamma_n$ subject to the relations $\gamma_i^2 = 1$ for $i = 1, \ldots, n$, and $[\gamma_i, \gamma_j] = 1$ for each distinct $i, j \in \{1, \ldots n\}$ such that v_i and v_j are not adjacent in Σ_1. The group W is the *(right-angled) Coxeter group* associated to the graph Σ_1. (This convention will be convenient for our purposes; however, in the literature, the right-angled Coxeter group associated to a graph Σ is often defined as the right-angled Coxeter group associated to the *complement graph* of Σ in our sense.) Let W^+ be the index-2 subgroup of W consisting of all elements that can be expressed as a product of an even number of the γ_i; that W^+ indeed constitutes an index-2 subgroup of W follows, for instance, from faithfulness of the representation σ_1 of W to be defined in the sequel.

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For $d \in \mathbb{R}$, let $M_d = (m_{ij}) \in \mathrm{M}_n(\mathbb{Z}[d])$ be the symmetric matrix given by

$$m_{ij} = \begin{cases} 1 & \text{if } i = j \\ -d & \text{if } i \neq j \text{ and } v_i, v_j \text{ are joined by an edge in } \Sigma_1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\epsilon > 0$ be such that M_d is positive-definite for $d \in [-\epsilon, \epsilon]$, and let $D \geq 1$ be such that M_d is nondegenerate and its signature constant as d varies within $[D, \infty)$. Note that M_1 is the Gram matrix of the Coxeter scheme Σ_1 (and the given order on the vertices of Σ_1). In particular, we have that $\epsilon < 1$. For $d > 1$, the matrix M_d is the Gram matrix of the Coxeter scheme Σ_d obtained from Σ_1 by replacing each edge with a dotted edge labeled by d. (Here, we are again using the conventions employed by [VS93].)

For $d \geq 1$, let $\sigma_d : W \to \mathrm{GL}_n(\mathbb{R})$ be the Tits–Vinberg representation associated to the Coxeter scheme Σ_d and the given order on its vertices; this is the representation given by

$$\sigma_d(\gamma_i)(v) = v - 2(v^T M_d e_i)e_i$$

for $i = 1, \ldots, n$ and $v \in \mathbb{R}^n$, where $(e_1, \ldots, e_n)$ is the standard basis for $\mathbb{R}^n$. It follows from Vinberg's theory of reflection groups that the representations σ_d, $d \geq 1$, are faithful [Vin71, Theorem 5] (see Lecture 1 in [Ben04] for an exposition). This family of representations was studied in [DGK20].

If $M \in \mathrm{M}_n(\mathbb{R})$ is a symmetric matrix and A is a subdomain of $\mathbb{C}$, we write

$$\mathrm{O}(M; A) = \{g \in \mathrm{GL}_n(A) \, : \, g^T M g = M\},$$
$$\mathrm{SO}(M; A) = \{g \in \mathrm{SL}_n(A) \, : \, g^T M g = M\}.$$

Note that we have $W_d := \sigma_d(W) \subset \mathrm{O}(M_d; \mathbb{R})$ by design.

Lemma 2. *The group W_d is Zariski-dense in $\mathrm{O}(M_d; \mathbb{R})$ for $d \geq D$.*

Proof. The proof of the main theorem in [BdlH04] applies here, so we only sketch the argument provided there. Let $d \geq D$ and let G_d be the Zariski-closure of W_d in $\mathrm{O}(M_d; \mathbb{R})$. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of $\mathrm{O}(M_d; \mathbb{R})$ and G_d, respectively. It is enough to show that $\mathfrak{g} = \mathfrak{h}$, since the Zariski-closure of $\mathrm{SO}(M_d; \mathbb{R})^\circ$ is $\mathrm{SO}(M_d; \mathbb{R})$ and since $W_d \not\subset \mathrm{SO}(M_d; \mathbb{R})$.

For each distinct pair $i, j \in \{1, \ldots, n\}$, let $E_{i,j}$ be the orthogonal complement of $\langle e_i, e_j \rangle$ in $\mathbb{R}^n$ with respect to M_d. The subgroup of $\mathrm{O}(M_d; \mathbb{R})$ consisting of all elements that fix each vector in $E_{i,j}$ is a 1-dimensional closed subgroup of $\mathrm{O}(M_d; \mathbb{R})$ whose identity component $T_{i,j}$ corresponds to a subspace $\langle X_{i,j} \rangle$ of $\mathfrak{g}$ for some $X_{i,j} \in \mathfrak{g}$. Since M_d is nondegenerate, the elements $X_{i,j}$ form a basis for $\mathfrak{g}$ as a vector space [BdlH04, Lemme 7]. Thus, to show $\mathfrak{g} = \mathfrak{h}$, it suffices to show that $X_{i,j} \in \mathfrak{h}$ for each distinct pair $i, j \in \{1, \ldots, n\}$.

To that end, let $i, j \in \{1, \ldots, n\}$, $i \neq j$, and suppose first that v_i and v_j are adjacent in Σ_1. Then $\sigma_d(\gamma_i \gamma_j)$ generates an infinite cyclic subgroup of $T_{i,j}$, so that $T_{i,j} \subset G_d$. It follows that $X_{i,j} \in \mathfrak{h}$ in this case. One now verifies that, since Σ_1 is connected, any Lie subalgebra of $\mathfrak{g}$ that contains $X_{i,j}$ for all i, j such that v_i, v_j are adjacent in fact contains $X_{i,j}$ for each distinct pair $i, j \in \{1, \ldots, n\}$ [BdlH04, Preuve du Théorème, second cas]. □

Now let $K \subset \mathbb{R}$ be a real quadratic extension of $\mathbb{Q}$, let $\tau : K \to K$ be the nontrivial element of $\mathrm{Gal}(K/\mathbb{Q})$, and let $\mathcal{O}_K$ be the ring of integers of K. Then by

Dirichlet's unit theorem, there is a unit $\alpha \in \mathcal{O}_K^*$ such that $\alpha \geq \max\{\frac{1}{\epsilon}, D\}$. Thus, we have

$$\frac{|\tau(\alpha)|}{\epsilon} \leq \alpha|\tau(\alpha)| = |\alpha \cdot \tau(\alpha)| = 1,$$

where the final equality holds because $\alpha \in \mathcal{O}_K^*$. We conclude that $|\tau(\alpha)| \leq \epsilon$, and so $M_{\tau(\alpha)}$ is positive-definite. It follows that $\Gamma := \mathrm{O}(M_\alpha; \mathcal{O}_K)$ is a uniform arithmetic lattice in $\mathrm{O}(M_\alpha; \mathbb{R})$ (for an efficient survey of the relevant facts, see, for instance, Section 2 of [GPS87]). Moreover, we have $W_\alpha \subset \mathrm{O}(M_\alpha; \mathbb{Z}[\alpha]) \subset \Gamma$.

Remark 3. Note that Galois conjugation by τ transports Γ and hence W_α into the compact group $\mathrm{O}(M_{\tau(\alpha)}; \mathbb{R})$. That finitely generated right-angled Coxeter groups embed in compact Lie groups had already been observed by Agol [Ago18] using a similar trick to the one above. Indeed, Agol's argument was the inspiration for this note.

Proof of Theorem 1. We show that W_α is a thin subgroup of $\Gamma \subset \mathrm{O}(M_\alpha; \mathbb{R})$. By Lemma 2, it suffices to show that W_α is of infinite index in Γ. Indeed, suppose otherwise. Then W_α is a uniform lattice in $\mathrm{O}(M_\alpha; \mathbb{R})$. If $n = 3$, then immediately we obtain a contradiction, since in this case W_α is virtually a closed hyperbolic surface group, whereas W is virtually free. If M_α has signature $(2, 2)$ (the one case under consideration in which $\mathrm{SO}(M_\alpha; \mathbb{R})^\circ$ is not simple), then we again obtain a contradiction as W has virtual cohomological dimension at most 3 (for instance, since the latter is an upper bound for the dimension of the Davis complex associated to the infinite right-angled Coxeter group W; see [Dav08, Chapter 1]), while the symmetric space associated to $\mathrm{O}(M_\alpha; \mathbb{R})$ has dimension 4. Now suppose that $n > 3$ and that the signature of M_α is not $(2, 2)$. There is some $\beta > \alpha$ and a path $[\alpha, \beta] \to \mathrm{GL}_n(\mathbb{R}), d \mapsto h_d$ such that $h_d^T M_d h_d = M_\alpha$ for all $d \in [\alpha, \beta]$ (this follows, for example, from the fact that $\mathrm{GL}_n(\mathbb{R})$ acts continuously and transitively on the set $\Omega \subset M_n(\mathbb{R})$ of symmetric matrices with the same signature as M_α, and so the orbit map $\mathrm{GL}_n(\mathbb{R}) \to \Omega, g \mapsto g^T M_\alpha g$ is a fiber bundle). Setting $g_d = h_d h_\alpha^{-1}$ for $d \in [\alpha, \beta]$, we have that $g_\alpha = I_n$ and $g_d^T M_d g_d = M_\alpha$ for $d \in [\alpha, \beta]$. For $d \in [\alpha, \beta]$, let $\rho_d = g_d^{-1} \sigma_d g_d$, and note

$$\rho_d(W) \subset g_d^{-1} \mathrm{O}(M_d; \mathbb{R}) g_d = \mathrm{O}(g_d^T M_d g_d; \mathbb{R}) = \mathrm{O}(M_\alpha; \mathbb{R}).$$

Let $\rho_d^+ = \rho_d\big|_{W^+}$ and $\sigma_d^+ = \sigma_d\big|_{W^+}$ for $d \in [\alpha, \beta]$. Then $\rho_\alpha^+(W^+)$ is a uniform lattice in the connected non-compact simple Lie group $\mathrm{SO}(M_\alpha; \mathbb{R})^\circ$, and the latter is not locally isomorphic to $\mathrm{SO}(2, 1)^\circ$ by our assumption that $n > 3$. Thus, by Weil local rigidity [Wei60, Wei62], up to choosing β closer to α, we may assume that for each $d \in [\alpha, \beta]$ there is some $a_d \in \mathrm{SO}(M_\alpha; \mathbb{R})^\circ$ such that

$$\rho_d^+ = a_d \rho_\alpha^+ a_d^{-1} = a_d \sigma_\alpha^+ a_d^{-1}. \tag{1}$$

But $\rho_d^+ = g_d^{-1} \sigma_d^+ g_d$, so we obtain from (1) that the trace $\mathrm{tr}(\sigma_d(\gamma_i \gamma_j))$ remains constant as d varies within $[\alpha, \beta]$, where $i, j \in \{1, \ldots, n\}$ are chosen so that the vertices v_i, v_j are adjacent in Σ_1.

We claim, however, that $\mathrm{tr}(\sigma_d(\gamma_i \gamma_j)) = 4d^2 - 4 + n$ for $d \geq D$. Indeed, let $d \geq D$. Then M_d is nondegenerate, so that $\mathbb{R}^d$ splits as a direct sum of the 2-dimensional subspace $\langle e_i, e_j \rangle \subset \mathbb{R}^n$ and its orthogonal complement $E_{i,j}$ with respect to M_d. Each of γ_i and γ_j acts as the identity on $E_{i,j}$, so our claim is equivalent to the assertion that $\mathrm{tr}\left(\sigma_d(\gamma_i \gamma_j)\big|_{\langle e_i, e_j \rangle}\right) = 4d^2 - 2$, and the latter follows from the fact that, with respect to the basis (e_i, e_j) of $\langle e_i, e_j \rangle$, the matrices representing $\sigma_d(\gamma_i), \sigma_d(\gamma_j)$ are $\begin{pmatrix} -1 & 2d \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2d & -1 \end{pmatrix}$, respectively. □

Example 4. We consider the case that $n \geq 5$ and the complement graph of Σ_1 is the cycle $v_1 v_2 \dots v_n$. In this case, the group W may be realized as the subgroup of $\mathrm{Isom}(\mathbb{H}^2)$ generated by the reflections in the sides of a right-angled hyperbolic n-gon, so that W is virtually the fundamental group of a closed hyperbolic surface. We have

$$M_d = (1+d)I_n + d(J_n + J_n^{n-1}) - d(I_n + J_n + \dots + J_n^{n-1}) \tag{2}$$

where $J_n \in \mathrm{M}_n(\mathbb{C})$ is the matrix

$$J_n = \begin{pmatrix} e_2 & e_3 & \dots & e_n & e_1 \end{pmatrix}.$$

There is some $C \in \mathrm{GL}_n(\mathbb{C})$ such that

$$CJ_nC^{-1} = \mathrm{diag}(1, \zeta_n, \zeta_n^2, \dots, \zeta_n^{n-1})$$

where $\zeta_n = e^{2\pi i/n}$. Observe that

$$C(I_n + J_n + \dots + J_n^{n-1})C^{-1} = \mathrm{diag}(n, 0, \dots, 0)$$

$$C(J_n + J_n^{n-1})C^{-1} = \mathrm{diag}\left(2, 2\cos\frac{2\pi}{n}, 2\cos\frac{2\pi \cdot 2}{n}, \dots, 2\cos\frac{2\pi(n-1)}{n}\right).$$

It follows from (2) that, counted with multiplicity, the eigenvalues of M_d are $1 - d(n-3)$ and $1 + d\left(1 + 2\cos\frac{2\pi k}{n}\right)$, where $k = 1, \dots, n-1$. Note that for d sufficiently large, we have that $1 - d(n-3) < 0$, and that $1 + d\left(1 + 2\cos\frac{2\pi k}{n}\right) \geq 0$ if and only if $\cos\frac{2\pi k}{n} \geq -\frac{1}{2}$. We conclude that the signature of M_d is $(2\lfloor\frac{n}{3}\rfloor, n - 2\lfloor\frac{n}{3}\rfloor)$ for all d sufficiently large. In particular, if $n = 3m$, $m \geq 2$, then the signature of M_d is $(2m, m)$ for all d sufficiently large. The above discussion yields thin surface subgroups of uniform arithmetic lattices in $\mathrm{SO}(2\lfloor\frac{n}{3}\rfloor, n - 2\lfloor\frac{n}{3}\rfloor)$ for each $n \geq 5$.

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