# A HYPERBOLIC LATTICE IN EACH DIMENSION WITH ZARISKI-DENSE SURFACE SUBGROUPS

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Abstract. For each integer $n \geq 3$, we exhibit a nonuniform arithmetic lattice in $\mathrm{SO}(n,1)$ containing Zariski-dense surface subgroups.

It follows from a straightforward ping pong argument that any lattice in $\mathrm{SO}(n,1)$, $n \geq 2$, contains a Zariski-dense copy of a noncocompact discrete subgroup of $\mathrm{SO}(2,1)$, namely, a Zariski-dense free subgroup (this may also be seen as a consequence of the Borel density theorem [Bor60, Corollary 4.3] and [Tit72, Theorem 3]). It is thus natural to look for Zariski-dense subgroups of lattices in $\mathrm{SO}(n,1)$ that are isomorphic to cocompact discrete subgroups of, that is, uniform lattices in, $\mathrm{SO}(2,1)$; the latter are virtually fundamental groups of closed hyperbolic surfaces. We refer to fundamental groups of such surfaces as *surface groups*.

While nonuniform lattices in $\mathrm{SO}(2,1)$ are virtually free and hence do not possess surface subgroups, it is expected that any lattice in $\mathrm{SO}(n,1)$ for $n \geq 3$ contains surface subgroups, and even Zariski-dense such subgroups. This has been established for $n = 3$ by Cooper, Long, and Reid [CLR97] in the nonuniform case (see also [CF19, KW21]) and by Kahn and Markovic [KM12] in the uniform case; and for odd $n \geq 3$ by Hamenstädt [Ham15] (see also [KLM18]) in the uniform case. However, while some standard constructions of closed arithmetic hyperbolic manifolds of arbitrary dimension contain immersed totally geodesic surfaces (see, for instance, [Ben04, Example 8]), the author was not aware of an example in the literature of a lattice in $\mathrm{SO}(n,1)$ for each $n \geq 3$ containing *Zariski-dense* surface subgroups. The purpose of this note is to present such an example in each dimension.

**Theorem 1.** *For each $n \geq 3$, there is a nonuniform arithmetic lattice in $\mathrm{SO}(n,1)$ containing Zariski-dense surface subgroups.*

Before we proceed, we fix some notation. Given an integer $n \geq 1$, a subdomain $D \subset \mathbb{R}$, and a symmetric matrix $Q \in \mathrm{M}_{n+1}(D)$ of signature $(n,1)$, we denote by $\mathrm{O}(Q;D)$ (resp., $\mathrm{SO}(Q;D)$) the set of all matrices $g \in \mathrm{GL}_{n+1}(D)$ (resp., $g \in \mathrm{SL}_{n+1}(D)$) satisfying $g^T Q g = Q$. The hypersurface of $\mathbb{R}^{n+1}$ consisting of all $x \in \mathbb{R}^{n+1}$ satisfying $x^T Q x = -1$ is a two-sheeted hyperboloid; we denote by $\mathrm{O}'(Q;D)$ the subgroup of $\mathrm{O}(Q;D)$ preserving each sheet. When $Q$ is the standard form $\mathrm{diag}(1,\ldots,1,-1)$, we write $\mathrm{O}(n,1)$ (resp., $\mathrm{SO}(n,1)$, $\mathrm{O}'(n,1)$) in the place of $\mathrm{O}(Q;\mathbb{R})$ (resp., $\mathrm{SO}(Q;\mathbb{R})$, $\mathrm{O}'(Q;\mathbb{R})$). We view $\mathrm{O}'(n,1)$ as the isometry group of $n$-dimensional real hyperbolic space $\mathbb{H}^n$ via the hyperboloid model of the latter.

We now proceed to the examples. For $m \geq 3$, let $K_m \in \mathrm{M}_m(\mathbb{Z})$ be the matrix all of whose entries are equal to 1; let $B_m \in \mathrm{M}_m(\mathbb{Z})$ be the matrix with 2's on the

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diagonal, 1's on the superdiagonal and subdiagonal, and 0's everywhere else; and let $Q_m = B_m - K_m$. What follows is the key observation of this note.

**Lemma 2.** *The symmetric matrix $Q_{n+1}$ has signature $(n, 1)$ for $n \geq 3$.*

*Proof.* The author is grateful to Yves Benoist for the following efficient argument. For $m > 0$, let $v_m := (1, \dots, 1)^T \in \mathbb{R}^m$ and let $H_{m-1}$ be the orthogonal complement of $\langle v_m \rangle$ in $\mathbb{R}^m$ with respect to the standard inner product on $\mathbb{R}^m$. Now let $n \geq 3$. Then $v_{n+1}^T Q_{n+1} v_{n+1} = -n^2 + 2n + 1 < 0$, so it suffices to show that the restriction of $Q_{n+1}$ to $H_n$ is positive-definite. Indeed, since the restriction of the form $K_{n+1}$ to $H_n$ is trivial, we have that the forms $Q_{n+1}$ and $B_{n+1}$ have the same restriction to $H_n$, so that it is enough to show that $B_{n+1}$ is positive-definite. This is true since we may view $B_{n+1}$ as the matrix representation of the standard inner product on $\mathbb{R}^{n+2}$ restricted to $H_{n+1}$ with respect to the basis $((-1)^{i+1}(e_i - e_{i+1}))_{i=1}^{n+1}$ of $H_{n+1}$, where $(e_1, \dots, e_{n+2})$ is the standard basis for $\mathbb{R}^{n+2}$. $\square$

*Proof of Theorem 1.* Let $n \geq 3$. By the Borel–Harish-Chandra theorem [BHC62] (and by Lemma 2), we have that $\Lambda_n := \mathrm{O}'(Q_{n+1}; \mathbb{Z})$ is a lattice in $\mathrm{O}(Q_{n+1}; \mathbb{R})$. As will become clear in the course of the proof, the lattice $\Lambda_n$ is nonuniform. We show that $\Lambda_n$ contains a Zariski-dense subgroup isomorphic to a cocompact lattice in $\mathrm{O}'(2, 1)$ generated by the reflections in the sides of a hyperbolic right-angled $2n$-gon. This will complete the proof since $Q_{n+1}$ has signature $(n, 1)$ by Lemma 2.

To that end, let $(W_n, (s_1, \dots, s_{n+1}))$ be the right-angled Coxeter system associated to the matrix $Q_{n+1}$; that is, let $W_n$ be the right-angled Coxeter group given by the presentation with generators $s_1, \dots, s_{n+1}$ subject to the relations $s_i^2 = 1$ for $i = 1, \dots, n+1$ and $[s_i, s_j] = 1$ whenever the $(i, j)^{\text{th}}$ entry of $Q_{n+1}$ is 0 (in our case, the latter happens exactly when $|i - j| = 1$). The image $\Gamma_n$ of the Tits representation $\rho_n : W_n \to \mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$ associated to the Coxeter system $(W_n, (s_1, \dots, s_{n+1}))$ lies in $\Lambda_n$ and is Zariski-dense in $\mathrm{O}(Q_{n+1}; \mathbb{R})$ [BdlH04]. Interpreting $\mathrm{O}'(Q_{n+1}; \mathbb{R})$ as the group of conformal diffeomorphisms of $\mathbb{S}^{n-1} = \partial \mathbb{H}^n$ by conjugating $\mathrm{O}(Q_{n+1}; \mathbb{R})$ to $\mathrm{O}(n, 1)$ within $\mathrm{GL}_{n+1}(\mathbb{R})$, we have that $\gamma_i := \rho_n(s_i)$ is an inversion in a (round) hypersphere $S_i$ of $\mathbb{S}^{n-1}$ for $i = 1, \dots, n+1$. Moreover, we have that $S_i$ is orthogonal to $S_{i+1}$ for $i = 1, \dots, n$, and that $S_i$ and $S_j$ are tangent for $1 \leq i < j - 1 \leq n$. The latter follows from the fact that $\gamma_i \gamma_j$ is nontrivial and unipotent, hence parabolic, for such $i$ and $j$.

We now visualize $\mathbb{S}^{n-1}$ via stereographic projection onto $\mathbb{R}^{n-1}$ from the tangency point $\infty$ of $S_1$ and $S_{n+1}$. Under this projection, the hyperspheres $S_1$ and $S_{n+1}$ are parallel hyperplanes of $\mathbb{R}^{n-1}$, while the remaining hyperspheres are contained in some ball $B \subset \mathbb{R}^{n-1}$. Since the stabilizer $\mathrm{Stab}_{\Lambda_n}(\infty)$ of $\infty$ in $\Lambda_n$ contains the reflections $\gamma_1$ and $\gamma_{n+1}$ in the parallel Euclidean hyperplanes $S_1$ and $S_{n+1}$, respectively, and since $\Lambda_n$ is a lattice in $\mathrm{O}'(Q_{n+1}; \mathbb{R})$, we must have that $\mathrm{Stab}_{\Lambda_n}(\infty)$ is a lattice in $\mathrm{Isom}(\mathbb{R}^{n-1})$ by the Margulis lemma (see, for instance, [BP92, Prop. D.2.6]). In particular, there is some translation $\sigma \in \mathrm{Stab}_{\Lambda_n}(\infty)$ with nontrivial $S_1$-component, and so $\tau := (\gamma_1 \sigma \gamma_1)\sigma$ is a nontrivial translation parallel to $S_1$ (and $S_{n+1}$). We now replace $\tau$ with a sufficiently high power so that $B \cap \tau(B) = \emptyset$. Appropriately defined, the common exterior of the hyperspheres $S_1, \dots, S_{n+1}$ and their images under $\tau$ produce a Coxeter polytope in $\mathbb{H}^n$ with the correct dihedral angles so that $\langle \Gamma_n, \tau \Gamma_n \tau^{-1} \rangle < \Lambda_n$ is isomorphic to the right-angled $2n$-gon group (see, for example, the introduction of [Vin85]). $\square$

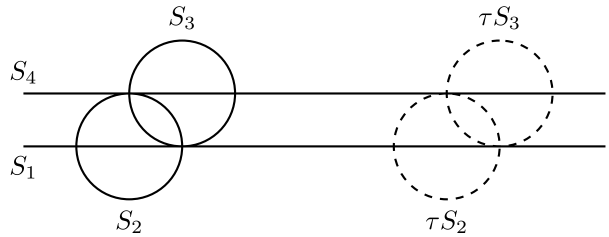


FIGURE 1. Visualizing the case $n = 3$. The sphere $\mathbb{S}^2$ is stereographically projected onto the plane from the tangency point of the circles $S_1$ and $S_4$, so that $S_1$ and $S_4$ project to parallel lines. Up to a Euclidean similarity, the circles $S_i$ are as in the figure. Zariski-density in $\mathrm{O}(Q_4; \mathbb{R})$ of the subgroup $\Gamma_3$ can be deduced from the fact that no vertical line is orthogonal to both $S_2$ and $S_3$. Any lattice in $\mathrm{O}'(3,1)$ containing the inversions in the $S_i$ also contains inversions in two circles resembling the dashed circles above; the latter represent the images of $S_2$ and $S_3$ under a horizontal translation $\tau$ of large magnitude. The subgroup of $\mathrm{O}'(3,1)$ generated by the inversions in the above six circles is (abstractly) a right-angled hexagon group. Extending each circle $S_i$ to a sphere $S_i'$ in $\mathbb{S}^3$ orthogonal to the page, and denoting by $S_5'$ the Euclidean plane parallel to the page and resting on top of $S_2'$ and $S_3'$, we have that the subgroup of $\mathrm{O}'(4,1)$ generated by the inversions in $S_1', \dots, S_5'$ is (conjugate to) the right-angled pentagon group $\Gamma_4'$ in Remark 7.

*Remark* 3. A ping pong argument following [Mas88, Section VII.E] demonstrates that in fact $\langle \Gamma_n, \tau \rangle < \Lambda_n$ decomposes as the HNN extension $\Gamma_n *_{\langle \gamma_1, \gamma_{n+1} \rangle}$ given by the identity map on $\langle \gamma_1, \gamma_{n+1} \rangle$.

*Remark* 4. The surface subgroups produced above are geometrically finite but are not convex cocompact since they contain parabolics. They are also automatically *thin* in the sense of Sarnak [Sar14] since a surface group cannot be realized as a lattice in $\mathrm{O}'(n,1)$ for $n \geq 3$ (for instance, because the outer automorphism group of such a lattice is finite by Mostow–Prasad rigidity [Mos68, Pra73]).

*Remark* 5. In this remark, we use the language of Coxeter schemes following [Vin85, Section II.5]. To justify Zariski-density of $\Gamma_n$, and hence $\langle \Gamma_n, \tau \Gamma_n \tau^{-1} \rangle$, in $\mathrm{O}(Q_{n+1}; \mathbb{R})$, we appealed to the general result of Benoist and de la Harpe [BdlH04], which asserts in particular that if the Gram matrix $Q$ of a finite connected Coxeter scheme $\Sigma$ with no dotted edges is nondegenerate, then the Tits representation of the associated Coxeter group is Zariski-dense in $\mathrm{O}(Q; \mathbb{R})$. When $Q$ has a single negative eigenvalue (as is the case for $Q = Q_{n+1}$, $n \geq 3$, by Lemma 2), so that $Q$ is the Gram matrix of a hyperbolic Coxeter polytope [Vin85, Theorem 2.1], this also follows from the fact that, for $n \geq 2$, if $P \subset \mathbb{H}^n$ is an irreducible Coxeter polytope with finitely many bounding hyperplanes $\Pi_i$, then the subgroup of $\mathrm{O}'(n,1)$ generated by the reflections in the $\Pi_i$ is Zariski-dense in $\mathrm{O}(n,1)$ if and only if the $\Pi_i$ do not all share a point in $\mathbb{H}^n \cup \partial \mathbb{H}^n$ or a common orthogonal hyperplane in $\mathbb{H}^n$ (see, for instance, [DSO01, Theorem 1.3]), which holds if and only if the Gram matrix of $P$ has rank $n + 1$ [Vin85, Section I.1].

We remark that if all the edges of the Coxeter scheme $\Sigma$ are bold (in other words, if the entries of $Q$ are contained in $\{-1,0,1\}$, as is true for $Q = Q_{n+1}$), then the argument of Benoist and de la Harpe simplifies. For an outline of their argument in this case, see the proof of Lemma 2 in [Dou22].

*Remark* 6. Let $n \geq 4$. We have demonstrated that $\Lambda_n$ contains a Zariski-dense copy of the right-angled $2n$-gon group, but it is even true that $\Lambda_n$ contains a Zariski-dense copy of the right-angled $2(n-1)$-gon group. Indeed, by Lemma 2 and Remark 5, there is a unique hypersphere $S \subset \mathbb{S}^{n-1}$ that is simultaneously orthogonal to $S_1, \ldots, S_n$. We visualize $\mathbb{S}^{n-1}$ via stereographic projection onto $\mathbb{R}^{n-1}$ from the tangency point of $S_1$ and $S_n$. Under this projection, the hyperspheres $S$, $S_1$, and $S_n$ are hyperplanes of $\mathbb{R}^{n-1}$, while $S_2, \ldots, S_{n-1}$ are Euclidean $(n-2)$-spheres. As in the proof of Theorem 1, there is some Euclidean translation in $\Lambda_n$ that is not parallel to $S$, and hence some Euclidean translation in $\Lambda_n$ that is parallel to $S_1$ but not parallel to $S$. For a sufficiently high power $\tau'$ of the latter translation, we have that $\langle \gamma_1, \gamma_2, \ldots, \gamma_n, \tau'\gamma_{n-1}\tau'^{-1}, \tau'\gamma_{n-2}\tau'^{-1}, \ldots, \tau'\gamma_2\tau'^{-1} \rangle < \Lambda_n$ is a right-angled $2(n-1)$-gon group. Moreover, by Remark 5, this subgroup of $\Lambda_n$ is Zariski-dense in $\mathrm{O}(n,1)$ since there is no hypersphere in $\mathbb{S}^{n-1}$ that is simultaneously orthogonal to $S_1, S_2, \ldots, S_n, \tau' S_{n-1}, \tau' S_{n-2}, \ldots, \tau' S_2$.

*Remark* 7. There are more efficient examples in even dimensions. Indeed, let $n \geq 4$, and $Q'_{n+1} \in \mathrm{M}_{n+1}(\mathbb{Z})$ be the matrix obtained from $Q_{n+1}$ by replacing the top-right and bottom-left entries with 0's. Let $(W'_n, (t_1^{(n)}, \ldots, t_{n+1}^{(n)}))$ be the right-angled Coxeter system associated to $Q'_{n+1}$, so that $W'_n$ is a right-angled $(n+1)$-gon group. The associated Tits representation $\rho'_n : W'_n \to \mathrm{SL}^{\pm}_{n+1}(\mathbb{R})$ realizes $W'_n$ as a subgroup $\Gamma'_n$ of $\mathrm{O}(Q'_{n+1}; \mathbb{Z})$ in $\mathrm{O}(Q'_{n+1}; \mathbb{R})$. If $n$ is even, then $Q'_{n+1}$ has signature $(n,1)$ [Dou22, Example 4], and so again by [BdlH04] (or Remark 5), we have that $\Gamma'_n$ is Zariski-dense in $\mathrm{O}(Q'_{n+1}; \mathbb{R})$. In this manner (alternatively, via Remark 6), one for instance obtains a nonuniform arithmetic lattice in $\mathrm{SO}(4,1)$ containing a Zariski-dense copy of the fundamental group of a closed orientable genus-2 surface.

Now suppose instead that $n$ is odd. Then $Q'_{n+1}$ has signature $(n-1,1,1)$, with kernel spanned by the vector $u_{n+1} := ((-1)^i)_{i=0}^n \in \mathbb{R}^{n+1}$. Note that $\Gamma'_n$ is contained in the stabilizer $G_n$ of $u_{n+1}$ in $\mathrm{O}(Q'_{n+1}; \mathbb{R})$. Denoting by $V_n$ the quotient of $\mathbb{R}^{n+1}$ by the span of $u_{n+1}$, by $\overline{Q'_{n+1}}$ the form induced on $V_n$ by $Q'_{n+1}$, and by $\mathrm{O}(\overline{Q'_{n+1}})$ the group of linear automorphisms of $V_n$ preserving the form $\overline{Q'_{n+1}}$, we have a natural map $G_n \to \mathrm{O}(\overline{Q'_{n+1}})$. Since $Q_n$ is the matrix representation of the form $\overline{Q'_{n+1}}$ with respect to the basis $(\overline{e_1}, \ldots, \overline{e_n})$ for $V_n$, where $\overline{e_i}$ is the image in $V_n$ of the $i^{\text{th}}$ standard basis vector for $\mathbb{R}^{n+1}$, we may identify $\mathrm{O}(\overline{Q'_{n+1}})$ with $\mathrm{O}(Q_n; \mathbb{R})$ to obtain a map $\pi_n : G_n \to \mathrm{O}(Q_n; \mathbb{R})$; explicitly, this map sends a matrix $A = (a_{i,j})_{i,j} \in G_n$ to the matrix obtained from $A$ by first adding $a_{n+1,j}u_{n+1}$ to the $j^{\text{th}}$ column for $1 \leq j \leq n$ and then deleting the final row and column. In particular, we have that $\pi_n(\Gamma'_n) \subset \mathrm{O}(Q_n; \mathbb{Z})$ and that $\pi_n(\rho'_n(t_i^{(n)})) = \rho_{n-1}(s_i)$ for $i = 1, \ldots, n$. Moreover, the map $\pi_n$ is injective on $\Gamma'_n$; see [dC12] and the references therein. The conclusion is that $\Gamma_m$ is in fact contained in a right-angled $(m+2)$-gon subgroup of $\mathrm{O}(Q_{m+1}; \mathbb{Z})$, namely, $\pi_{m+1}(\Gamma'_{m+1})$, for $m \geq 4$ even.

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