Planar percolation with a glimpse of Schramm–Loewner Evolution

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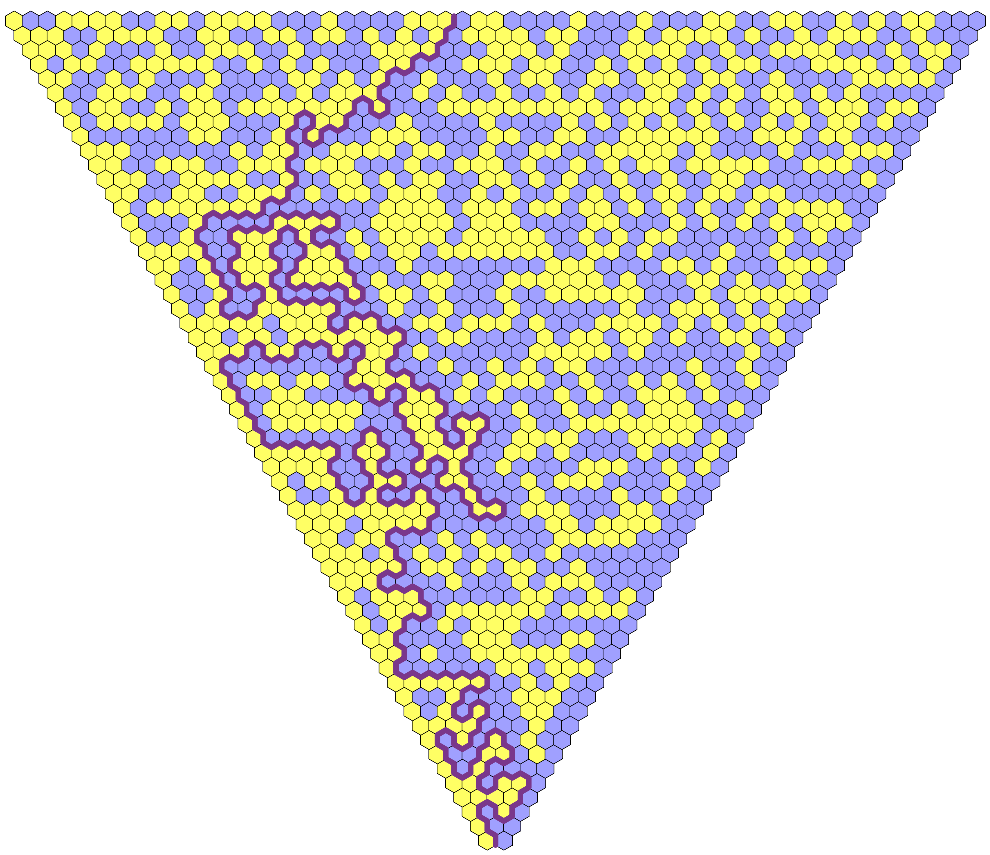

Abstract

In recent years, important progress has been made in the field of two-dimensional statistical physics. One of the most striking achievements is the proof of the Cardy-Smirnov formula: this theorem, together with the introduction of Schramm-Loewner Evolution and techniques developed over the years in percolation, allow precise descriptions of the critical and near-critical regimes of the model. This survey aims to describe the different steps leading to the proof that the infinite-cluster density $\theta(p)$ for site percolation on the triangular lattice behaves like $(p-1/2)_+^{5/36+o(1)}$ when p approaches its critical value $p_c = 1/2$.

1 Introduction

Percolation as a physical model was introduced by Broadbent and Hammersley in 1950 [BH57]. For $p \in (0,1)$, *(site) percolation* on the triangular lattice $\mathbb{T}$ is a random configuration supported on the vertices (or *sites*), each one being *open* with probability p and *closed* otherwise, independently of the others. By duality, this can also be seen as a random coloring of the faces of the hexagonal lattice; we will use this representation extensively, if only because it makes for prettier pictures... Denote the measure on configurations by $\mathbb{P}_p$. For general background on percolation, we refer the reader to the books of Grimmett [Gri99] and Kesten [Kes82].

We will be interested in the connectivity properties of the model. Two sites a and b of the triangular lattice are *connected* (which will be denoted by $a \leftrightarrow b$) if there exists a path of neighboring open sites starting at a and ending at b. If there exists a path of neighboring closed sites starting at a and ending at b, we will write $a \overset{\star}{\leftrightarrow} b$. A *cluster* is a connected component of open sites.

It is classical that there exists $p_c \in (0,1)$ — called the *critical point* — such that for $p < p_c$, there exists almost surely no infinite cluster, while for $p > p_c$, there exists almost surely a unique such cluster.

Theorem 1.1. *The critical of site-percolation on the triangular lattice point equals* $1/2$.

This theorem was first proved in the case of bond percolation on the square lattice by Kesten in [Kes80].

The proof can be summarized as follow. First, crossing probabilities at $p = 1/2$ for rectangles $[0,n] \times [0,\rho n]$ are proved to remain bounded away from 0 and 1 uniformly in n. The main ingredients are the self-duality of the model at $p = 1/2$ and what is known as Russo-Seymour-Welsh theory. Second, crossing probabilities are proved to converge to 0 and 1 as n goes to infinity when $p < 1/2$ and $p > 1/2$ respectively. The proof of this statement is based on a sharp threshold argument.

Once the critical point has been determined, it is natural to study the *phase transition* of the model, *i.e.* the behavior when p crosses p_c. Physicists are interested in the thermodynamical properties of the model, such as the *infinite cluster density*

$$\theta(p) := \mathbb{P}_p(0 \leftrightarrow \infty)$$

when $p > p_c$, the *susceptibility* (or *mean cluster-size*)

$$\xi(p) := \sum_{x \in \mathbb{T}} \mathbb{P}_p(0 \leftrightarrow x)$$

or the *correlation length* L_p (see Definition 3.5) when $p < p_c$. The behavior of these quantities near p_c is governed by power laws, more precisely one expects:

$$\begin{aligned} \theta(p) &= (p - p_c)^{\beta + o(1)} && \text{when } p \searrow p_c, \\ \xi(p) &= (p - p_c)^{-\nu' + o(1)} && \text{when } p \nearrow p_c, \\ L_p &= (p - p_c)^{-\nu + o(1)} && \text{when } p \nearrow p_c. \end{aligned}$$

These *critical exponents* β, ν and ν' are not independent of each other but satisfy equations: Kesten's *scaling relations* relate β and ν to the so-called monochromatic one-arm and polychromatic four arm exponents at criticality. The important feature of these relations is that they relate quantities defined away from criticality to fractal properties of the critical regime. In other words, the behavior of percolation through its phase transition (when p varies from slightly below to slightly above p_c) is intimately related to its behavior at p_c. The scaling relations enable mathematicians to focus on the critical phase: if connectivity properties of the critical phase can be understood, then critical exponents for thermodynamical quantities will follow.

We now turn to the study of critical planar percolation and briefly recall the historic on the subject. In the seminal papers [BPZ84a] and [BPZ84b], Belavin, Polyakov and Zamolodchikov postulated *conformal invariance* in the scaling limit (under all conformal transformations of sub-regions) of critical two-dimensional statistical models: the renormalization group formalism suggests that the scaling limit of critical models is a fixed point for the renormalization transformation. The fixed point being unique, the scaling limit should be invariant under translation, rotation and scaling; since it can be described by quantum local fields, it was natural to expect that the field describing the scaling limit of critical regime would itself be invariant under all transformations which are locally compositions of translations, rotations and scalings — *i.e.*, conformal maps.

From a mathematical perspective, the notion of conformal invariance of a model is ill-posed, since the meaning of scaling limit is not even clear in general. The following solution to this problem can be implemented: the scaling limit of the model could simply be less rich and retain the information given by "interfaces" only. There is no reason why all the information of a model should be encoded into information on interfaces, but one can hope that most of the relevant quantities can be recovered from it. The advantage of this approach is that there exists a mathematical setting for families of continuous curves.

Let us first start with the study of one curve. Fix a simply connected domain (Ω, a, b) with two points on the boundary and consider discretizations $(\Omega_\delta, a_\delta, b_\delta)$ of (Ω, a, b) by a triangular lattice of mesh size δ. Assume now that the sites along the boundary arc ∂_{ab} are open and that those along the arc ∂_{ba} are closed. There exists a unique interface (which consists in a chain of bonds of the dual hexagonal lattice) between open and closed sites going from a to b. In order to see this, the correspondence between face percolation on the hexagonal lattice and site percolation on the triangular one is useful. We call this interface the *exploration path* and denote it by γ_δ.

Conformal field theory predicts that γ_δ converges when δ goes to 0 to a random continuous non-self-crossing curve between a and b in Ω which must be conformally invariant, where conformal invariance has now a precise meaning:

Definition 1.2. *A family of random continuous curves $\gamma_{(\Omega,a,b)}$ indexed by simply connected domains with two marked points on the boundary (Ω, a, b) is* conformally invariant *if for any (Ω, a, b) and any conformal map $\psi : \Omega \to \mathbb{C}$,*

$$\psi(\gamma_{(\Omega,a,b)}) \text{ has the same law as } \gamma_{(\psi(\Omega),\psi(a),\psi(b))}.$$

In 1999, Schramm proposed a natural candidate for the possible conformally invariant families of continuous non-self-crossing curves. He noticed that the interfaces of various models further satisfy the *domain Markov property* (see Section 2.4) which, together with the assumption of conformal invariance, determines the possible families of curves. In [Sch00], he introduced the *Schramm-Loewner evolution* — or SLE for short. SLE(κ), for $\kappa > 0$, is the random Loewner evolution with driving process $\sqrt{\kappa}B_t$, where (B_t) is a standard Brownian motion. By construction, the process is conformally invariant, random and fractal. The prediction of conformal field theory then translates into the following predictions for percolation: the limit of $(\gamma_\delta)_{\delta>0}$ in (Ω, a, b) is SLE(6).

For completeness, let us mention that families of interfaces in a percolation model are also expected to converge in the scaling limit to a conformally invariant family of non-intersecting loops. By consistency, each loop should look like an SLE(6) process. Sheffield and Werner [SW10a, SW10b] introduced a one-parameter family of processes of non-intersecting loops which are conformally invariant — called the Conformal Loop Ensembles CLE(κ) for $\kappa > 8/3$. Non-surprisingly, the loops of CLE(κ) are locally similar to SLE(κ).

Even though we now have a mathematical frame for conformal invariance, it remains an extremely hard task to prove convergence of the interfaces to SLE curves. Observe that working with interfaces offers a further simplification: properties of these interfaces should also be conformally invariant. Therefore, we could simply look at an *observable* of the model, *i.e.* something that one can measure by looking at the configuration. Of course, it is not clear that this observable would tell us anything about critical exponents, but it already represents a significant step toward conformal invariance.

In 1994, Langlands, Poulliot and Saint-Aubin [LPSA94] published numerical values in agreement with the conformal invariance (in the scaling limit) of crossing probabilities in the percolation model. More precisely, they checked that taking different topological rectangles, the probability $\mathcal{C}_\delta(\Omega, A, B, C, D)$ of having a path of adjacent open sites between the boundary arcs AB and CD converges when δ goes to 0 towards a limit which is the same for (Ω, A, B, C, D) and $(\Omega', A', B', C', D')$ if they are image of each other by a conformal map. Notice that the existence of such a crossing property can be expressed in terms of properties of an interface, thus keeping this discussion in the frame proposed earlier.

The paper [LPSA94], while only numerical, attracted many mathematicians to the domain. The same year, Cardy [Car92] proposed an explicit formula for the limit of crossing probabilities. In 2001, Smirnov proved Cardy's formula rigorously for critical site percolation on the triangular lattice, hence rigorously providing a concrete example of a conformally invariant property of the model:

Theorem 1.3 (Smirnov [Smi01])**.** *The probability of the event $\mathcal{C}_\delta(\Omega, A, B, C, D)$ has a limit $f(\Omega, A, B, C, D)$ as δ goes to 0. Furthermore, the limit satisfies the two following properties:*

- *It is equal to AB/AC if Ω is a equilateral triangle with vertices A, C and D;*

- *It is conformally invariant, in the following sense: if Φ is a conformal map from Ω to another simply connected domain $\Omega' = \Phi(\Omega)$, and extends continuously to $\partial\Omega$, then*

$$f(\Omega, A, B, C, D) = f(\Phi(\Omega), \Phi(A), \Phi(B), \Phi(C), \Phi(D)).$$

The fact that Cardy's formula takes such a simple form for equilateral triangles was first observed by Carleson. Notice that the Riemann mapping theorem and conformal invariance then give the value of f for every conformal rectangle.

A remarkable consequence of this theorem is that the mechanism can be reversed: even though Cardy's formula seems to be much weaker than convergence to SLE(6), they are actually equivalent. In other words, conformal invariance of one well-chosen observable of the model can be sufficient to prove conformal invariance of interfaces, and in particular convergence of the exploration path to an SLE curve:

Theorem 1.4 (Smirnov, see also [CN07])**.** *Let Ω be a simply connected domain with two marked points a and b on the boundary. Let γ_δ be the exploration path of the critical percolation as described in the previous paragraphs. Then the law of γ_δ converges weakly, as $\delta \to 0$, to that of chordal Schramm-Loewner evolution with $\kappa = 6$.*

Let us mention that convergence to CLE(6) was proved in [CN06], thus providing a proof of the full conformal invariance of percolation interfaces.

Convergence to SLE(6) is important for many reasons. Since SLE itself is very well understood (in particular, its fractal properties are known), it enables the computation of several critical exponents describing the critical phase. We will introduce these exponents during the study, but let us now state the result non-formally (see Theorem 4.1 or [SW01]):

- the probability that there exists an open path from the origin to the boundary of the box of size n decays equals $n^{-5/48+o(1)}$.

- the probability that there exist four arms, two open and two closed going from the origin to the boundary of the box of size n equals $n^{-5/4+o(1)}$.

Convergence to SLE(6) is the main step in the derivation of critical exponents describing fractal properties of the critical regime (for instance the arm exponents). Together with Kesten's scaling relations (Theorem 4.3 or [Kes87]), the previous asymptotics imply the following result, which is the main focus of this survey:

Theorem 1.5. *For site percolation on the triangular lattice, $p_c = 1/2$ and as $p \to 1/2$,*

$$\theta(p) = (p - 1/2)_+^{5/36+o(1)}.$$

Organization of the survey

The first section is devoted to the geometry of the $p = 1/2$ percolation. Uniform bounds away from 0 and 1 are proved for crossing probabilities. In a second part, these crossing probabilities are proved to converge in the scaling limit (Cardy-Smirnov's formula). The

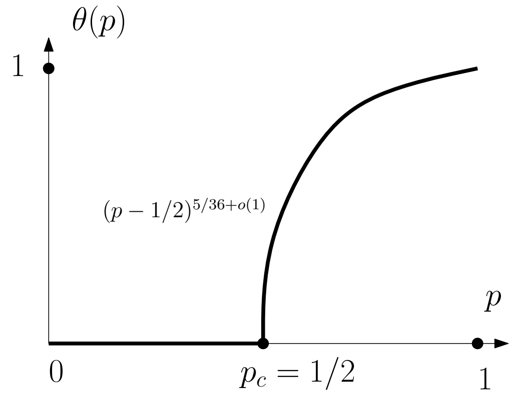

Figure 1: Cluster density with respect to p. Non-trivial facts in this picture include $p_c = 1/2$, $\theta(p_c) = 0$ and the behavior near the critical point. We do not investigate properties such as continuity of the cluster density away from p_c.

last part of this section presents a sketched proof of convergence of the exploration path to SLE(6).

The second section studies the percolation away from $p = 1/2$. First, we prove that $p_c = 1/2$. Then, we introduce the notion of correlation length and study properties of percolation below the correlation length.

The third section deals with critical exponents at criticality (in the form of some of the *arm-exponents*) and then goes into some details about Kesten's scaling relations.

The last part gathers a few open questions which we found relevant to the topic.

Notation and basic properties

Except otherwise stated, $\mathbb{T}$ will denote the triangular lattice with mesh size 1 embedded in the complex plane $\mathbb{C}$, containing a vertex at the origin and a vertex at 1 — complex coordinates will be used frequently to specify the position of a point. Let $d_{\mathbb{T}}(\cdot,\cdot)$ be the graph distance in $\mathbb{T}$. Define the ball $\Lambda_n := \{x \in \mathbb{T} : d_{\mathbb{T}}(x,0) \le n\}$ (balls have hexagonal shapes). Let $\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1}$ be the internal boundary of Λ_n.

We write $u_p \asymp v_p$ if there exist two constants $0 < A, B < \infty$ not depending on p such that $Au_p \le v_p \le Bu_p$ for all p in a neighborhood of p_c. We also write $u_p \lesssim v_p$ if there exists $0 < B < \infty$ such that $u_p \le Bv_p$ for all p.

The Harris inequality and monotonicity of percolation will be used a few times. Let us recall these two facts now. An event is called *increasing* if it is preserved by addition of open sites, see Section 2.2 of [Gri99] (a typical example is the existence of an open path from one set to another). The inequality $p < p'$ implies that $\mathbb{P}_p(A) \le \mathbb{P}_{p'}(A)$ for any increasing event A. Moreover, for every $p \in [0,1]$ and A, B two increasing events,

$$\mathbb{P}_p(A \cap B) \ge \mathbb{P}_p(A)\mathbb{P}_p(B) \qquad \text{(Harris inequality)}.$$

The van den Berg-Kesten inequality [vdBK85] will also be used extensively. For two increasing events A and B, let $A \circ B$ be the event that A and B occur disjointly, meaning that $\omega \in A \circ B$ if there exist two sets of sites E and F (possibly depending on ω) such that one can verify that $\omega \in A$ (resp. $\omega \in B$) by looking at sites in E (resp. F) only. Then, for every $p \in [0,1]$ and A, B two increasing events,

$$\mathbb{P}_p(A \circ B) \le \mathbb{P}_p(A)\mathbb{P}_p(B) \qquad \text{(BK inequality)}.$$

This inequality was improved by Reimer [Rei00], who proved that one can relax the condition on A and B being increasing: let $p \in [0,1]$ and A, B two events,

$$\mathbb{P}_p(A \Box B) \le \mathbb{P}_p(A)\mathbb{P}_p(B) \qquad \text{(Reimer inequality)},$$

where $A \Box B$ denotes the disjoint occurrence of A and B.

2 Crossing probabilities and conformal invariance at the critical point

2.1 Circuits in annuli

In this whole section, we let $p = 1/2$. Let $\mathcal{E}_n$ be the event that there exists a circuit of adjacent open sites in $\Lambda_{3n} \setminus \Lambda_n$ that surrounds the origin:

Theorem 2.1. *There exists $C > 0$ such that for every $n > 0$, $\mathbb{P}_{1/2}(\mathcal{E}_n) \ge C$.*

This theorem was first proved in a corresponding form in the case of bond percolation on the square lattice by Russo [Rus78] and by Seymour and Welsh [SW78]. It led to many applications, several of which will be discussed in this survey.

Such a bound (and its proof) is typical of the behavior of percolation at the self-dual point $p = 1/2$: it is indeed natural to expect that the probability of $\mathcal{E}_n$ event goes to 0 (resp. 1) below (resp. above) 1/2. Making this vague statement rigorous is not elementary and is indeed the whole point of Theorem 1.1.

Proof. We present one of the many proofs of Theorem 2.1, inspired by a argument due to Smirnov and available in [Wer09] and in [Gri10].

Step 1: Let $n > 0$ and index the sides of Λ_n as in Fig. 2. Consider the event that ℓ_1 is connected by an open path to $\ell_3 \cup \ell_4$. The complement of this event is that ℓ_2 is connected by a closed path to $\ell_5 \cup \ell_6$. Using the symmetry between closed and open sites and the invariance of the model under rotations of angle $\pi/3$ preserving the lattice, $\mathbb{P}_{1/2}(\ell_1 \leftrightarrow \ell_3 \cup \ell_4)$ is equal to 1/2.

In fact, we also have that $\mathbb{P}_{1/2}(\ell_1 \leftrightarrow \ell_4) \ge 1/8$. Indeed, either this is true or, going to the complement, $\mathbb{P}_{1/2}(\ell_1 \leftrightarrow \ell_3) \ge 1/2 - 1/8$. But in this case, using the Harris inequality,

$$\mathbb{P}_{1/2}(\ell_1 \leftrightarrow \ell_4) \;\ge\; \mathbb{P}_{1/2}(\ell_1 \leftrightarrow \ell_3)\mathbb{P}_{1/2}(\ell_2 \leftrightarrow \ell_4) \;\ge (3/8)^2 \ge 1/8.$$

Step 2: Now consider $R_n = \Lambda_n \cup (\Lambda_n - 2n\mathrm{i})$ and index the sides of R_n as in Fig. 2. For a path γ from ℓ_1 to ℓ_4, define the domain Ω_γ to consist in the sites of R_n strictly to the right of $\gamma \cup \sigma(\gamma)$, where σ is the reflection with respect to ℓ_1. Once again, the complement of $\{\ell_4 \cup \gamma \leftrightarrow \ell_{10} \cup \ell_{11} \text{ in } \Omega_\gamma\}$ is $\{\ell_9 \cup \sigma(\gamma) \overset{\star}{\leftrightarrow} \ell_2 \cup \ell_3 \text{ in } \Omega_\gamma\}$. The switching of colors and the symmetry with respect to ℓ_1 imply that the probability of the former is larger than $1/2$ (it is not exactly equal to $1/2$, since the site on $\gamma \cap \ell_1$ is assumed to be open).

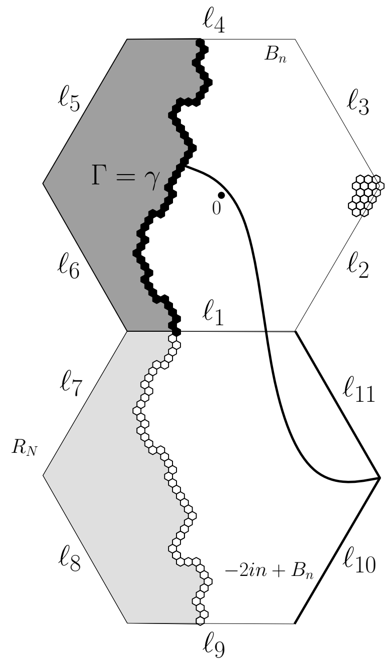

Figure 2: The dark gray area is the set of sites which are discovered after conditioning on $\{\Gamma = \gamma\}$. The white area is Ω_γ.

If $E := \{\ell_1 \leftrightarrow \ell_4\}$ occurs, set Γ to be the left-most crossing between ℓ_1 and ℓ_4. For a given path γ from ℓ_1 to ℓ_4, the event $\{\Gamma = \gamma\}$ is measurable only in terms of sites to the left or in γ. In particular, conditioning on $\{\Gamma = \gamma\}$, the configuration in Ω_γ is a percolation configuration, so that

$$\mathbb{P}_{1/2}\big((\ell_4 \cup \gamma) \leftrightarrow (\ell_{10} \cup \ell_{11}) \text{ in } \Omega_\gamma \mid \Gamma = \gamma\big) \;\geq\; 1/2.$$

Therefore,

$$\begin{aligned}
\mathbb{P}_{1/2}\big(\ell_4 \leftrightarrow (\ell_{10} \cup \ell_{11})\big) &= \mathbb{P}_{1/2}\big(\ell_4 \leftrightarrow (\ell_{10} \cup \ell_{11})\ ,\ E\big) \\
&= \sum_\gamma \mathbb{P}_{1/2}\big(\ell_4 \leftrightarrow (\ell_{10} \cup \ell_{11})\ ,\ \Gamma = \gamma\big) \\
&\geq \sum_\gamma \mathbb{P}_{1/2}\big((\ell_4 \cup \gamma) \leftrightarrow (\ell_{10} \cup \ell_{11}) \text{ in } \Omega_\gamma\ ,\ \Gamma = \gamma\big) \\
&\geq \sum_\gamma \frac{1}{2}\mathbb{P}_{1/2}(\Gamma = \gamma) \;=\; \frac{1}{2}\mathbb{P}_{1/2}(E) \;=\; \frac{1}{16}.
\end{aligned}$$

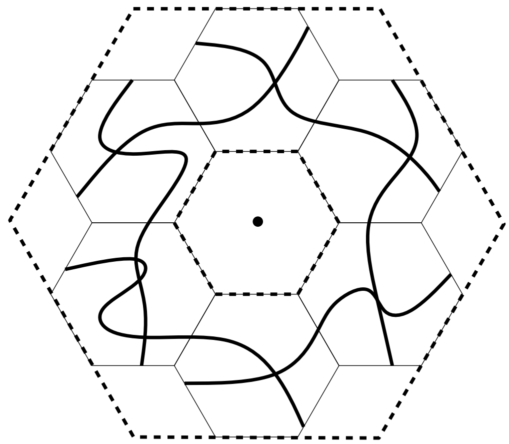

Figure 3: Six "rectangles" which, when crossed, ensure the existence of a circuit in the annulus.

Step 3: Invoking the Harris inequality,

$$\mathbb{P}_{1/2}(\ell_4 \leftrightarrow \ell_9) \;\geq\; \mathbb{P}_{1/2}\big(\ell_4 \leftrightarrow (\ell_{10} \cup \ell_{11})\big)\mathbb{P}_{1/2}\big((\ell_2 \cup \ell_3) \leftrightarrow \ell_9\big) \;\geq \frac{1}{16^2}.$$

Assuming that the six rectangles described in Fig. 3 are crossed (in the sense that there are paths between opposite short edges), the result follows from a last use of Harris inequality. □

The first corollary of Theorem 2.1 is the following lower bound on p_c (the result can also be proved using an elegant argument by Zhang which invokes the uniqueness of the infinite cluster when it exists, see Section 11 of [Gri99]).

Corollary 2.2 (Harris [Har60]). *For percolation on the triangular lattice, $\theta(1/2) = 0$; in particular, $p_c \geq 1/2$.*

Proof. Let us prove that when $p = 1/2$, 0 is almost surely not connected by a closed path to infinity (it is the same probability for an open path). Let $N > 0$. The origin being connected to $\partial\Lambda_{3^N}$ by a closed path implies that for every $n < N$, $\mathcal{E}_{3^n}^c$ occurs. Therefore,

$$\mathbb{P}_{1/2}(0 \overset{\star}{\leftrightarrow} \partial\Lambda_{3^N}) \;\leq\; \mathbb{P}_{1/2}\left(\bigcap_{n<N} \mathcal{E}_{3^n}^c\right) \;=\; \prod_{n<N} \mathbb{P}_{1/2}\left(\mathcal{E}_{3^n}^c\right) \;\leq\; (1-C)^N, \tag{2.1}$$

where C is the constant in Theorem 2.1. In the second inequality, the independence between percolation in different annuli is crucial. In particular, the left-hand term converges to 0 as $N \to \infty$, so that $\theta(1/2) = 0$. Hence, by the definition of p_c, $p_c \geq 1/2$. □

2.2 Discretization of domains and crossing probabilities

In general, we are interested in crossing probabilities for more general shapes. More precisely, we wish to let the size of the graph go to infinity, but keeping the same global shape. A natural way to do this is to shrink the lattice instead of looking at bigger and bigger scales.

Consider a topological rectangle (Ω, A, B, C, D), *i.e.* a bounded simply connected domain $\Omega \neq \mathbb{C}$ with four distinct points A, B, C and D on its boundary, indexed in counter-clockwise order. [For simplicity, we are only considering domains whose boundary is a Jordan curve, and we will also silently assume that their discretizations below satisfy a similar criterion. The eager reader might want to check that the argument still goes through in the most general case, where A, B, C and D are prime ends of Ω, but they will soon notice that the added notational weight makes the proof more obscure.]

For $\delta > 0$, we will be interested in percolation on $\Omega_\delta := \Omega \cap \delta\mathbb{T}$. Note that we can see the boundary of Ω_δ as a self-avoiding curve s on Ω_δ^* (which is a subgraph of the hexagonal lattice). The graph Ω_δ should be seen as a discretization of Ω at scale δ. Let A_δ, B_δ, C_δ and D_δ be the sites of s that are closest to A, B, C and D respectively. They divide s into four arcs denoted by $A_\delta B_\delta$, $B_\delta C_\delta$, etc.

In the percolation setting, let $\mathcal{C}_\delta(\Omega, A, B, C, D)$ be the event that there is a path of open sites in Ω_δ between the intervals $A_\delta B_\delta$ and $C_\delta D_\delta$ of its boundary (more precisely connecting two sites of Ω_δ adjacent to $A_\delta B_\delta$ and $C_\delta D_\delta$ respectively). We call such a path a *crossing*, and the event a *crossing event*. Sometimes, we will say that the rectangle is *crossed* if there exists a crossing.

With a slight abuse of notation, we will denote the percolation measure with $p = 1/2$ on $\delta\mathbb{T}$ by $\mathbb{P}_{1/2}$ (even though the measure is the push-forward of $\mathbb{P}_{1/2}$ by the scaling $x \mapsto \delta x$). We first state a direct consequence of Theorem 2.1:

Corollary 2.3. *Let (Ω, A, B, C, D) be a topological rectangle. There exist $0 < c_1, c_2 < 1$ such that for every $\delta > 0$,*

$$c_1 \;\leq\; \mathbb{P}_{1/2}\big[\mathcal{C}_\delta(\Omega, A, B, C, D)\big] \;\leq\; c_2.$$

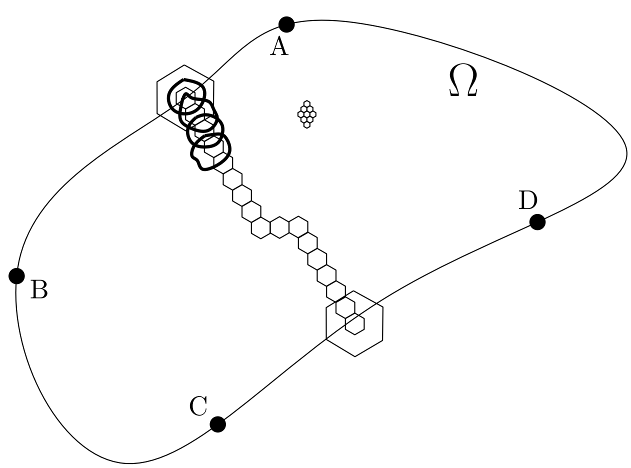

Figure 4: Circuits in annuli linking two edges of a topological rectangle. If each of these annuli contains an open circuit disconnecting the interior from the exterior boundary, we obtain an open path connecting the two sides.

Proof. It is sufficient to prove the lower bound, since the upper bound is a consequence of the following fact: the complement of $\mathcal{C}_\delta(\Omega, A, B, C, D)$ is the existence of a closed path from $B_\delta C_\delta$ to $D_\delta A_\delta$, it has same probability as $\mathcal{C}_\delta(\Omega, B, C, D, A)$. Therefore, if the latter probability is bounded from below, the probability of $\mathcal{C}_\delta(\Omega, A, B, C, D)$ will be bounded away from 1.

Fix $\varepsilon \in \delta\mathbb{N}$ positive. For a hexagon h of radius $\varepsilon > 0$, we set $\tilde{h}$ to be the hexagon with the same center and radius 3ε. Now, consider a collection $h_1, \ldots, h_k$ of hexagons "parallel" to $\mathbb{H}$ and of radius ε satisfying the following conditions:

- h_1 intersects AB and h_k intersects CD,
- $\tilde{h}_1, \ldots, \tilde{h}_k$ intersect neither BC nor DA,
- h_i are adjacent and create a "path" in Ω from AB to CD.

(Such a chain exists if ε is chosen small enough.) Let E_i^δ be the event that there is an open circuit in $\Omega_\delta \cap (\tilde{h}_i \setminus h_i)$ surrounding $\Omega_\delta \cap h_i$. By construction, if each E_i^δ occurs, there is a path from AB to CD, see Fig. 4. Using Theorem 2.1, the probability of this is bounded from below by C^k uniformly in δ. Yet, one can choose $k = k(\Omega, A, B, C, D, \varepsilon)$ not depending on δ, which readily implies the claim. □

In particular, long rectangles are crossed in the long direction with probability bounded away from 0 as $\delta \to 0$. This result is the classical formulation of Theorem 2.1. We finish this section with a property of percolation with parameter 1/2:

Corollary 2.4. *There exist $\alpha, \beta > 0$ such that for every $n > 0$,*

$$n^{-\alpha} \ \leq \ \mathbb{P}_{1/2}(0 \leftrightarrow \partial\Lambda_n) \ \leq \ n^{-\beta}.$$

Proof. The existence of $\beta > 0$ is given as in (2.1). For the lower bound, we use the following construction: define $R_n := [0, 2^n] \times [0, 2^{n+1}]$ if n is odd, and $R_n := [0, 2^{n+1}] \times [0, 2^n]$ if it is even. Set F_n to be the event that R_n is crossed in the "long" direction. Corollary 2.3 applied to the topological rectangle of the form $[0,1] \times [0,2]$ implies the existence of $C_1 > 0$ such that $\mathbb{P}_{1/2}(F_n) \geq C_1$ for every $n > 0$. By the Harris inequality

$$\mathbb{P}_{1/2}(0 \leftrightarrow \partial\Lambda_{3^N}) \ \geq \mathbb{P}_{1/2}\left(\bigcap_{n<N} F_n\right) \ \geq \ \prod_{n<N} \mathbb{P}_{1/2}(F_n) \ \geq \ C_1^{N-1}.$$

This yields the existence of $\alpha > 0$. □

2.3 The Cardy-Smirnov formula

The subject of this section is the proof of Theorem 1.3. The proof of this theorem is very well (and very shortly) exposed in the original paper [Smi01]. It has been rewritten in a number of places including [BR06b, Gri10, Wer09]. We provide here a version of the proof that we used during the lecture in Florence (in particular with the same notations), which is mainly inspired by [Smi01] and [Bef07].

Proof. Fix (Ω, A, B, C) a topological triangle and $z \in \Omega$ (with the same caveat as in the previous proof that we will silently assume the boundary of Ω to be smooth, for notation's sake, but that the same proof applies to the general case of a simply connected domain). For $\delta > 0$, A_δ, B_δ, C_δ, z_δ are the points of Ω_δ^* closest to A, B, C and z, as before. Define $E_{A,\delta}(z)$ to be the event that there exists a non-self-intersecting path of open sites in Ω_δ, separating A_δ and z_δ from B_δ and C_δ — and $E_{B,\delta}(z)$, $E_{C,\delta}(z)$ similarly, with obvious circular permutations of the letters. Let $H_{A,\delta}(z)$ (resp. $H_{B,\delta}(z)$, $H_{C,\delta}(z)$) be the probability of $E_{A,\delta}(z)$ (resp. $E_{B,\delta}(z)$, $E_{C,\delta}(z)$).

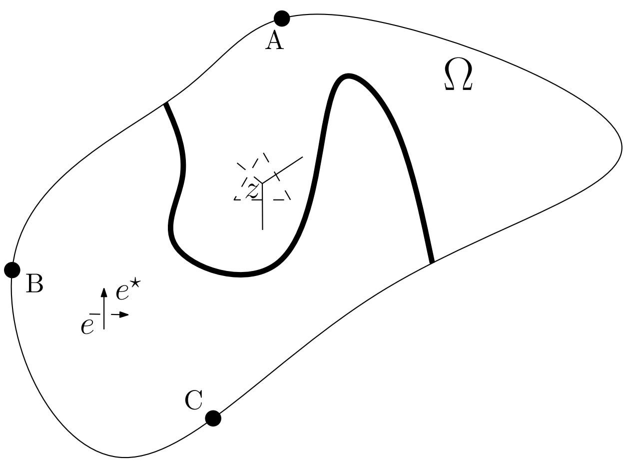

Figure 5: Picture of the event $E_{A,\delta}(z)$. Also depicted is one oriented edge e with its associated edge $e^\star$. The graph $\mathbb{T}_\delta$ is drawn with dotted lines while its dual $\mathbb{H}_\delta$ is drawn with solid lines.

The proof runs into three steps, the second one being the most important:

- First, prove that $(H_{A,\delta}, H_{B,\delta}, H_{C,\delta})_{\delta>0}$ is a precompact family of functions (with variable z).
- Second, let $\tau = e^{2i\pi/3}$ and introduce the following two sequences of functions defined by
$$H_\delta(z) := H_{A,\delta}(z) + \tau H_{B,\delta}(z) + \tau^2 H_{C,\delta}(z) \qquad S_\delta(z) = H_{A,\delta}(z) + H_{B,\delta}(z) + H_{C,\delta}(z),$$
and show that any subsequential limits h and s of these sequences are holomorphic. This statement is proved using Morera's theorem, based on the study of discrete integrals.
- Third, use boundary conditions to identify the possible h and s, and thus guarantee the possible subsequential limit of $(H_{A,\delta}, H_{B,\delta}, H_{C,\delta})_{\delta>0}$ to be unique. A byproduct of the proof is the exact computation of the limit of h and s, and thus of the limits of $(H_{A,\delta})$, $(H_{B,\delta})$ and $(H_{C,\delta})$.

Then, making the additional remark that $E_{C,\delta}(D_\delta)$ is the event $\mathcal{C}_\delta(\Omega, A, B, C, D)$, the limit of $H_{C,\delta}(D_\delta)$ as δ goes to 0 converge to the limiting crossing probability, thus concluding the proof of Theorem 1.3.

Precompactness The main remark is that if two points z, z' are surrounded by an open (or a closed) circuit, then the events $E_{A,\delta}(z')$ and $E_{A,\delta}(z)$ are realized simultaneously, so that

$$|H_{A,\delta}(z') - H_{A,\delta}(z)| \leq \mathbb{P}_p[z_\delta \text{ and } z'_\delta \text{ are not separated from } A_\delta \text{ and } B_\delta C_\delta \text{ in } \Omega_\delta]$$

For z and z' remaining at distance $\eta > 0$ away from A, B and C, Theorem 2.1 applied in roughly $\log(|z - z'|/\eta)/\log 3$ concentric annuli, there exist two positive constants K and ε depending only on η such that, for every $\delta > 0$,

$$|H_{A,\delta}(z') - H_{A,\delta}(z)| \leqslant K\,|z' - z|^\varepsilon \tag{2.2}$$

and a similar bound for $H_{B,\delta}$ and $H_{C,\delta}$. Hence, if we suitably extend these functions continuously to Ω, we obtain a family of uniformly Hölder maps from Ω to $[0, 1]$. The family is then relatively compact with respect to uniform convergence, and it is hence possible to extract a subsequence $(H_{A,\delta_n}, H_{B,\delta_n}, H_{C,\delta_n})_{n>0}$, with $\delta_n \to 0$, which converges uniformly to a triple of Hölder maps (h_A, h_B, h_C) from $\overline{\Omega}$ to $[0, 1]$. From now on, we set $h = h_A + \tau h_B + \tau^2 h_C$ and $s = h_A + h_B + h_C$ (they are the limits of $(H_{\delta_n})_{n>0}$ and $(S_{\delta_n})_{n>0}$ respectively).

Holomorphicity of h and s To prove that h is holomorphic, one can try to prove that H_{δ_n} is a sequence of (almost) *discrete holomorphic* functions, where one needs to specify what is meant by discrete holomorphic; in our case, we take it to mean that discrete contour integrals vanish. Indeed, Morera's theorem (see *e.g.* [Lan99]) yields that for any a simply connected domain Ω of the complex plane, and any continuous function $f : \Omega \to \mathbb{C}$, f is holomorphic if, and only if, for every simple, closed, smooth curve γ contained in Ω, the integral of f along γ vanishes: the previous definition is a natural discretization of this property. We refer to [Smi10] for more details on discrete holomorphicity, including other definitions of it, and its connections to statistical physics.

Consider a simple, closed, smooth curve γ contained in Ω. For every $\delta > 0$, let γ_δ be a discretization of γ contained in Ω_δ, *i.e.* a finite chain $(\gamma_\delta(k))_{0\leqslant k\leqslant N_\delta}$ of pairwise distinct sites of Ω_δ, ordered in the positive direction, such that for every index k, $\gamma_\delta(k)$ and $\gamma_\delta(k+1)$ are nearest neighbors, and chosen in such a way that the Hausdorff distance between γ_δ and γ goes to 0 with δ. Notice that N_δ can be taken of order δ^{-1}, which we shall assume from now on.

For an edge $e \in \mathbb{H}_\delta$, define $e^\star$ to be the rotation by $\pi/2$ of e (it is an edge of the triangular lattice). For an edge e of the hexagonal lattice, let

$$H_\delta(e) := \frac{H_\delta(x) + H_\delta(y)}{2},$$

where $e = xy$.

The discrete curve γ_δ surrounds a finite family of edges of $\mathbb{T}_\delta$, which we shall denote by $Int(\gamma_\delta)$. An oriented edge $e^\star$ of $\mathbb{T}_\delta$ *belongs* to γ_δ if it is of the form $\gamma_\delta(k)\gamma_\delta(k+1)$ (we set $e \in \gamma_\delta$).

Define the discrete integral $I^\delta_\gamma(H)$ of H_δ (and similarly $I^\delta_\gamma(S)$ for S_δ) along γ_δ by

$$I^\delta_\gamma(H) := \sum_{e^\star\in\gamma} e^\star H_\delta(e).$$

Our goal is to prove that $I_\gamma^\delta(H)$ and $I_\gamma^\delta(S)$ converge to 0 when δ goes to 0. Since along the sequence (δ_n), they also converge to $\oint_\gamma h(z)\mathrm{d}z$ and $\oint_\gamma s(z)\mathrm{d}z$, it will imply that h and s are holomorphic via Morera's Theorem (notice that h and s are continuous as uniform limits of continuous functions).

For every oriented edge $e = xy \in \mathbb{H}_\delta$, set

$$P_{A,\delta}(e) \;=\; \mathbb{P}_{1/2}\big(E_{A,\delta}(y) \setminus E_{A,\delta}(x)\big),$$

and similarly P_B and P_C.

Lemma 2.5. *For any smooth γ, when δ goes to 0,*

$$I_\gamma^\delta(H) = \sum_{e^\star \subset Int(\gamma_\delta)} e^\star \left[P_A(e) + \tau P_B(e) + \tau^2 P_C(e)\right] + o(1), \tag{2.3}$$

$$I_\gamma^\delta(S) = \sum_{e^\star \subset Int(\gamma_\delta)} e^\star \left[P_A(e) + \tau^2 P_B(e) + \tau^4 P_C(e)\right] + o(1). \tag{2.4}$$

Proof. We treat the case of H_δ. For every oriented edge $e = xy$ in $\mathbb{H}_\delta$, define the following notation:

$$\partial_e H_\delta := H_\delta(y) - H_\delta(x),$$

where $e = xy$. If f is a face of $\mathbb{T}_\delta$, let ∂f be its oriented boundary, seen as a set of oriented edges. With these notations, we get the following identity:

$$I_\gamma^\delta(H) \;=\; \sum_{e^\star \in \gamma_\delta} e^\star H_\delta(e) \;=\; \sum_{f \in Int(\gamma_\delta)} \sum_{e^\star \in \partial f} e^\star H_\delta(e). \tag{2.5}$$

Indeed, in the last equality, each boundary term is obtained exactly once with the correct sign, and each interior term appears twice with opposite signs. The sum of $eH_\delta(e)$ around f can be rewritten in the following fashion:

$$\sum_{e^\star \in \partial f} e^\star H_\delta(e) = \sum_{e^\star = xy \in \partial f} \left(\frac{x+y}{2} - f\right) \partial_e H_\delta.$$

Putting this quantity in the sum (2.5), the term $\partial_e H_\delta = H_\delta(y) - H_\delta(x)$ appears twice notice for x, y nearest neighbors bordered by two triangles in γ_δ, and the factors $(x+y)/2$ cancel between the two occurrences, leaving only the difference between the centers of the faces, *i.e.* the dual edge of xy. Therefore,

$$I_\gamma^\delta(H) = \frac{1}{2} \sum_{e^\star \subset Int(\gamma_\delta)} e^\star \partial_e H_\delta + o(1). \tag{2.6}$$

In the previous equality, we used the fact that the total contribution of the boundary goes to 0 with δ. Indeed, $e^\star$ is of order δ, and

$$\partial_e H_\delta \;=\; P_{A,\delta}(e) - P_{A,\delta}(-e) \;+\; \tau(P_{B,\delta}(e) - P_{B,\delta}(-e)) \;+\; \tau^2(P_{C,\delta}(e) - P_{C,\delta}(-e)) \tag{2.7}$$

so that Theorem 2.1 gives a bound of $\delta^{1+\varepsilon}$ for $e^\star \partial_e H_\delta$. Since there are roughly δ^{-1} boundary terms, we obtain that the boundary contributes for at most δ^ε.

Replacing in (2.6) ∂H_δ by its expression (2.7), and re-indexing the sum to obtain each oriented edge in exactly one term, we get the announced equality. ☐

Lemma 2.6 (Smirnov [Smi01]). *For every edge e of $\Omega_\delta^\star$, we have the following identities:*

$$P_{A,\delta}(e_1) = P_{B,\delta}(e_2) = P_{C,\delta}(e_3),$$

where e_1, e_2, e_3 are three edges emanating from a site x.

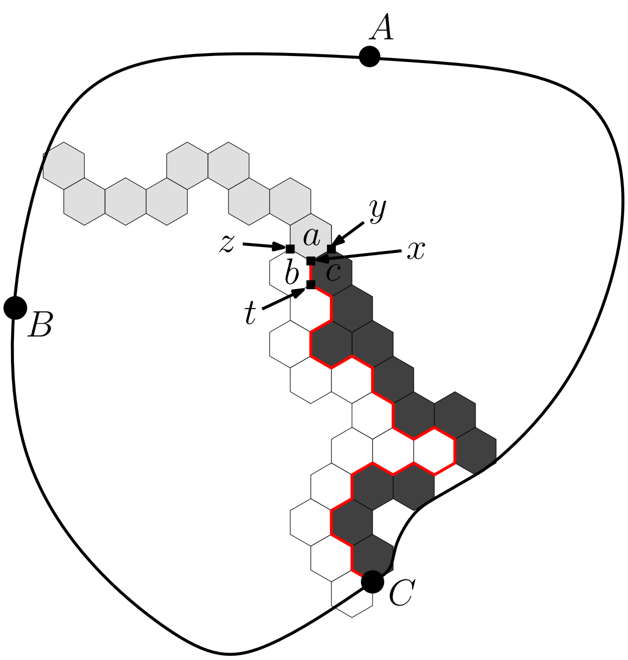

Figure 6: The dark gray and the white hexagons are the hexagons on $\overline{\Gamma}$, Γ being in black.

Even though we include the proof for completeness, we refer the reader to [Smi01] for the (elementary, but very clever) first proof of this result. The lemma extends to site-percolation with parameter 1/2 on any planar triangulation.

Proof. Index the three sites around x by a, b and c, and the sites by y, z and t as depicted in Fig. 6. We see events as subsets of $\{\text{Open,Closed}\}^{|\Omega_\delta|}$.

Let us prove that $P_{A,\delta}(e_1) = P_{B,\delta}(e_2)$. The event $E_{A,\delta}(y) \setminus E_{A,\delta}(x)$ occurs if and only if there are open paths from AB to a and from AC to c, and a closed path from BC to b.

Consider the interface Γ between the open cluster connected to AC and the closed cluster connected to BC, starting at C up to the first time it hits x (it will do it if and only if there exist an open path from AB to a and a closed path from AC to c). Fix a deterministic path from C to x, the event $\{\Gamma = \gamma\}$ depends only on sites adjacent to γ (we denote the space of such sites $\overline{\gamma}$). Now, on $\{\Gamma = \gamma\}$, there exists a bijection between configurations with an open path from a to AB and configurations with a closed path from a to AB (by symmetry between open and closed edges in the domain $\Omega_\delta \setminus \overline{\gamma}$). This is true for any γ, hence there is a bijection between the event

$$E_{A,\delta}(y) \setminus E_{A,\delta}(x) = \bigcup_\gamma \{\Gamma = \gamma\} \cap \{a \leftrightarrow AB \text{ in } \Omega_\delta \setminus \overline{\gamma}\}$$

and

$$E := \bigcup_{\gamma} \{\Gamma = \gamma\} \cap \{a \overset{\star}{\leftrightarrow} AB \text{ in } \Omega_\delta \setminus \overline{\gamma}\}.$$

Note that $E_{B,\delta}(z) \setminus E_{B,\delta}(x)$ is the image of E after switching the colors, so that it is in bijection with it. This part is the key step of the lemma, and is sometimes called **color-switching trick**. Since $\mathbb{P}_{1/2}$ is simply the uniform measure on configurations, we obtain $P_{A,\delta}(e_1) = P(E) \;=\; P_{B,\delta}(e_2)$. □

We are now in a position to prove that $I_\gamma^\delta(H)$ and $I_\gamma^\delta(S)$ converge to 0. From Lemmas 2.5 and 2.6, we obtain by re-indexing the sum

$$I_\gamma^\delta(H) \;=\; \sum_{e^\star \subset Int(\gamma_\delta)} (e^\star + \tau(\tau.e)^\star + \tau^2(\tau^2.e)^\star) P_A(e) + o(1) = o(1)$$

using that

$$e^\star + \tau(\tau.e)^\star + \tau^2(\tau^2.e)^\star = 0. \tag{2.8}$$

Similarly, for s:

$$I_\gamma^\delta(S) \;=\; \sum_{e^\star \subset Int(\gamma_\delta)} (e^\star + (\tau.e)^\star + (\tau^2.e)^\star) P_A(e) + o(1) = o(1).$$

Here, we have used

$$e^\star + (\tau.e)^\star + (\tau^2.e)^\star = 0. \tag{2.9}$$

This concludes the proof of the holomorphicity of h and s.

Identification of s and h Let us start with s. Since it is holomorphic and real-valued, it must be constant. It is easy to see from the boundary conditions (near a corner for instance) that it is equal to 1. Now, consider h. Since h is holomorphic, it is sufficient to identify enough boundary conditions to specify it uniquely.

Let $z \in \Omega$. Since $h_A(z) + h_B(z) + h_C(z) = 1$, $h(z)$ is a barycenter of 1, τ and τ^2 and it belongs to the triangle with sites 1, τ and τ^2. Furthermore, if z is on the boundary of $\Omega_\delta^\star$ lying between B and C, $h_A(z) = 0$ (using Theorem 2.1), and thus $h_B(z) + h_C(z) = 1$ (since $s = 1$). Hence, $h(z)$ lies on the interval $[\tau, \tau^2]$ of the complex plane. Besides, $h(B) = \tau$ and $h(C) = \tau^2$, so h induces a continuous map from the boundary interval $[BC]$ of Ω onto $[\tau, \tau]$. By Theorem 2.1 yet again, h is one-to-one on this boundary interval. Similarly, h induces a bijection between the boundary interval $[AB]$ (resp. $[CA]$) of Ω and the complex interval $[1, \tau]$ (resp. $[\tau^2, 1]$), so putting the pieces together we see that h is a holomorphic map from Ω to the triangle with sites at 1, τ and τ^2 which extends continuously to $\bar{\Omega}$ and induces a continuous bijection between $\partial\Omega$ and the boundary of the triangle.

From standard results of complex analysis ("principle of corresponding boundaries", cf. for instance Theorem 4.3 in [Lan99]), this implies that h is actually a conformal map from Ω to the interior of the same triangle. But we know that h maps A (resp. B, C) to 1 (resp. τ, τ^2), and this determines it uniquely. In other words, there is only one possible limit for the triple (H_A, H_B, H_C) as δ goes to 0, which gives conformal invariance for free and concludes the proof of Theorem 1.3. □

As a corollary of the proof, we get a nice expression for h_A: if $\Phi_{\Omega,A,B,C}$ is the conformal map from Ω to the triangle mapping A, B and C as previously (which means of course that $\Phi_{\Omega,A,B,C} = h$) then

$$H_{A,\delta}(z) \to \frac{2\Re e(\Phi_{\Omega,A,B,C}(z)) + 1}{3}.$$

If Ω is the equilateral triangle itself, then h is the identity map and we obtain Cardy's formula in Carleson's form: if $D \in [CA]$ then

$$f(\Omega, A, B, C, D) \;=\; \frac{|CD|}{|AB|}.$$

It is also to be noted that (2.8) actually characterizes the triangular lattice (and therefore its dual the hexagonal one). So, it seems that the triangular lattice is the only one (apart from trivial modifications of it) in which a fully combinatorial proof of the holomorphicity of h is possible. On the other hand, the holomorphicity of s and therefore the fact that it equals 1 relies only on (2.9), which is true for any triangulation where Theorem 2.1 holds. This seems to be a fundamental property of critical two-dimensional percolation (and *might* be the key to understanding universality in this particular, limited case, though this is hardly even speculative). As of this time, no direct, combinatorial proof of this fact seems to be known.

2.4 Scaling limit of interfaces

A natural question at this point is the exact amount of information contained in Theorem 1.3. For instance, is it enough to derive precise results about the geometry of critical percolation clusters? It turns out that it is indeed the case, and in fact the full structure of the percolation scaling limit can be recovered from it through Schramm-Loewner Evolution; we now present the proof of Theorem 1.4. The strategy to prove that a family of parametrized curves converges to an SLE(κ) follows three steps:

- First, prove that the family of curves is tight.
- Then, show that any subsequential limit is a *time-changed Loewner chain* with a continuous driving process.
- Finally, show that the only possible driving process for the subsequential limits is $\sqrt{\kappa}B_t$ where B_t is a standard Brownian motion.

The main step is the third one. In order to identify Brownian motion as being the only possible driving process for the curve, we find computable martingales expressed in terms of the limiting curve. In our case, these martingales will be the limits of crossing probabilities; the fact that these (explicit) functions are martingales allows us to deduce martingale properties of the driving process. More precisely, we aim to use Lévy's theorem: a continuous real-valued process X such that X_t and $X_t^2 - at$ are martingales is necessarily of the form $\sqrt{a}B_t$.

2.4.1 A crash-course on Schramm-Loewner Evolution

In this paragraph, several non-trivial concepts about Loewner chains are used and we refer to the extensive literature for details. We briefly recall several useful facts in the next paragraph. We do not aim for completeness (see [Law05, Wer40, Wer05] for details). We simply introduce notions needed in the next sections. Recall that a *domain* is a simply connected open set not equal to $\mathbb{C}$.

Set $\mathbb{H}$ to be the upper half-plane. Fix a compact set $K \subset \overline{\mathbb{H}}$ such that $H = \mathbb{H} \setminus K$ is still simply connected. For such a domain H, Riemann's mapping theorem guarantees the existence of a conformal map from H onto $\mathbb{H}$. Moreover, there are *a priori* three real degrees of freedom in the choice of the conformal map, so that it is possible to fix its asymptotic behavior when z goes to ∞: let g_K be the unique conformal map from H onto $\mathbb{H}$ such that

$$g_K(z) \ := \ z + \frac{C}{z} + O\left(\frac{1}{z^2}\right).$$

The proof of the existence of this map is not completely obvious and requires the reflection principle. The constant C is called the *h-capacity* of H. It acts like a capacity: it is increasing in K and the h-capacity of λK is λ^2 times the h-capacity of K.

There is a natural way to parametrize continuous curves $\gamma : \mathbb{R}_+ \to \mathbb{H}$ with $\gamma(0) = 0$ and with γ going to ∞ when $t \to \infty$. For every s, let H_s be the connected component of $\mathbb{H}\setminus\gamma[0, s]$ containing ∞. We denote by K_s the *hull* created by $\gamma[0, s]$, *i.e.* the compact set $\overline{\mathbb{H}} \setminus H_s$. From the previous paragraph, K_s has a certain h-capacity C_s. The continuity of the curve guarantees that C_s grows continuously, so that it is possible to parametrize the curve in such a way that $C_s = 2t$ at time t. This parametrization is called the *h-capacity parametrization*. Note that in general, the previous operation is not a proper reparametrization, since any part of the curve "hidden from ∞" will not make the h-capacity grow, and thus will be mapped to the same point for the new curve.

From now on, assume the curve is parametrized via h-capacity. In particular, the curve can be encoded via the family of conformal maps g_t from H_t to $\mathbb{H}$, in such a way that

$$g_t(z) \ := \ z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right).$$

Under mild conditions, the infinitesimal evolution of the family (g_t) implies the existence of a continuous real valued process W_t such that for every t and $z \in H_t$,

$$\partial_t g_t(z) \ := \ \frac{2}{g_t(z) - W_t}.$$

The process W_t is called the *driving process* of γ. The typical required hypothesis in order to do so is the following "local growth" condition:

Local Growth Condition: for any $t \geq 0$ and for any ε, there exists $\delta > 0$ such that for any $s \leq t$, the diameter of $g_s(K_{s+\delta} \setminus K_s)$ is smaller than ε, where $K_s = \mathbb{H} \setminus H_s$ is the hull created by γ_s.

It is important to notice that the procedure is reversible under mild assumptions. If a continuous function W_t is given, it is possible to reconstruct the hull K_t as the set of points z for which the previous differential equation already blew up.

We are now in a position to define Schramm-Loewner Evolutions:

Definition 2.7 (SLE in the upper half-plane)**.** *The chordal Schramm-Loewner Evolution in* $\mathbb{H}$ *with parameter* $\kappa > 0$ *is the (random) Loewner chain with driving process* $W_t := \sqrt{\kappa} B_t$*, where* B_t *is a standard Brownian motion.*

Loewner chains in other domains are easy to define via conformal maps:

Definition 2.8 (SLE in a general domain)**.** *Fix a domain* Ω *with two points on the boundary* a *and* b *and assume it has a nice boundary (for instance a Jordan curve). The chordal Schramm-Loewner evolution with parameter* $\kappa > 0$ *in* (Ω, a, b) *is the image of the Schramm-Loewner evolution in the upper half-plane by a conformal map from* $(\mathbb{H}, 0, \infty)$ *onto* (Ω, a, b)*.*

To conclude this paragraph, let us justify the fact that these curves are natural scaling limits for interfaces of conformally invariant models. In order to explain this fact, we need the notion of *domain Markov property* for a family of random growing curves:

Definition 2.9. *A family of random continuous curves* $\gamma_{(\Omega,a,b)}$ *(parametrized via h-capacity) in simply connected domains is said to satisfy the domain Markov property if for every* (Ω, a, b)*, and every* $t > 0$*, the law of the curve* $\gamma[t, \infty)$ *conditionally on* $\gamma[0, t]$ *is the same as the law of* $\gamma_{(\Omega_t,\gamma_t,b)}$*, where* Ω_t *is the connected component of* $\Omega \setminus \gamma_t$ *containing* b*.*

Discrete interfaces in many models of statistical physics naturally satisfy this property (which can be seen as a variant of the DLR conditions), and therefore their scaling limits, provided that they exist, also do. Schramm proved the following result in [Sch00], which in some way justifies SLE processes as natural candidates for such scaling limits:

Theorem 2.10 (Schramm [Sch00])**.** *Every family of Loewner chains* $\gamma_{(\Omega,a,b)}$ *which*

- *is conformally invariant,*
- *satisfies the domain Markov property,*
- *satisfies that* $\gamma_{(\mathbb{H},0,\infty)}$ *is scale invariant,*

is a chordal Schramm-Loewner evolution with parameter $\kappa \in [0, \infty)$*.*

2.4.2 Tightness of the interfaces

Convergence of random parametrized curves (say with time-parameter in $[0, 1]$) is in the sense of the **weak topology** inherited from the following distance on curves:

$$d(\gamma_1, \gamma_2) \;=\; \inf_{\phi} \sup_{u \in [0,1]} |\gamma_1(u) - \gamma_2(\phi(u))|, \tag{2.10}$$

where the infimum is taken over all reparametrizations (*i.e.* strictly increasing continuous functions $\phi\colon [0, 1] \to [0, 1]$ with $\phi(0) = 0$ and $\phi(1) = 1$).

In this section, the following theorem is proved:

Theorem 2.11. *Fix a domain* (Ω, a, b)*, the family* $(\gamma_\delta)_{\delta>0}$ *of exploration paths for critical percolation in* (Ω, a, b) *is tight for the topology associated to the curve distance.*

The question of tightness for curves in the plane has been studied in the milestone paper [AB99]. In this paper, it is proved that a sufficient condition for tightness is the absence, at every scale, of annuli crossed back and forth an unbounded number of times. More precisely, for $x \in \Omega$ and $r < R$, let $\Lambda_r(x) = x + \Lambda_r$ and $S_{r,R}(x) = \Lambda_R(x) \setminus \Lambda_r(x)$ and define $\mathcal{A}_k(x; r, R)$ to be the existence of k crossing of the curve γ_δ between outer and inner boundaries of $S_{r,R}(x)$.

Theorem 2.12 (Aizenman-Burchard [AB99])**.** *Let Ω be a simply connected domain and let a and b be two marked points on its boundary. Denote by μ_δ the law of a random curve $\tilde{\gamma}_\delta$ on Ω_δ from a_δ to b_δ. If there exist $k \in \mathbb{N}$, $C_k < \infty$ and $\Delta_k > 2$ such that for all $\delta < r < R$ and $x \in \Omega$,*

$$\mu_\delta(\mathcal{A}_k(x; r, R)) \leq C_k \Big(\frac{r}{R}\Big)^{\Delta_k},$$

then the family of curves $(\tilde{\gamma}_\delta)$ is tight.

We now show how to exploit this theorem in order to prove Theorem 2.11. The main tool is Theorem 2.1.

Proof of Theorem 2.11. Fix $x \in \Omega$, $\delta < r < R$ and recall that the lattice has mesh size δ. Let k to be fixed later. The Reimer inequality implies

$$\mathbb{P}_p\big(\mathcal{A}_k(x; r, 3r)\big) \leq \big[\mathbb{P}_p\big(\mathcal{A}_1(x; r, 3r)\big)\big]^k.$$

Using Theorem 2.1, $\mathbb{P}_p\big(\mathcal{A}_1(x; r, 3r)\big) \leq 1 - \mathbb{P}_p(\mathcal{E}_n) < 1 - C$. Let us fix k large enough so that $(1-C)^k < 1/27$. Now, one can decompose the annulus $S_{r,R}(x)$ into roughly $\ln_3(R/r)$ annuli of the form $S_{r,3r}(x)$, so that for the previous k,

$$\mathbb{P}_p(\mathcal{A}_k(x; r, R)) \leq \Big(\frac{r}{R}\Big)^3. \tag{2.11}$$

Hence, Theorem 2.12 implies that the family (γ_δ) is tight. $\square$

2.4.3 Subsequential limits are Loewner chains

In the previous paragraph, traces of exploration paths were shown to be tight. We would now like to parametrize any subsequential limit curve as a Loewner chain, *i.e.* via its h-capacity. Furthermore, in order to be able to reconstruct the curve from the driving process associated to it, we require the Loewner chain to be generated by a curve. In this case, we say that the curve is a *time-changed Loewner chain*.

Theorem 2.13. *Any subsequential limit of the family $(\gamma_\delta)_{\delta>0}$ of exploration paths is a time-changed Loewner chain.*

As emphasized before, not every continuous curve is a time-changed Loewner chain, therefore an additional argument is needed, especially since the limiting curve is fractal-like and has many double points. A general characterization for a parametrized non-self-crossing curve in (Ω, a, b) to be a time-changed Loewner chain generated by a curve is the following:

- its h-capacity must be continuous,

- its h-capacity must be strictly increasing
- the curve grows locally seen from infinity in the following sense: for any $t \geq 0$ and for any ε, there exists $\delta > 0$ such that for any $s \leq t$, the diameter of $g_s(K_{s+\delta} \setminus K_s)$ is smaller than ε, where $K_s = \mathbb{H} \setminus H_s$ is the hull created by $\gamma[0, s]$.

The first condition is automatically satisfied by continuous curves. The third one follows from the two others when the curve is continuous, so that the only condition to check is the second one. This condition can be understood as being the fact that the tip of the curve is visible from b at every time. In other words, the family of hulls created by the curve (*i.e.* the complement of the connected component of $\Omega \setminus \gamma_t$ containing b) is strictly increasing. This is the case if the curve does not enter long fjords created by its past at every scale, see Fig. 7.

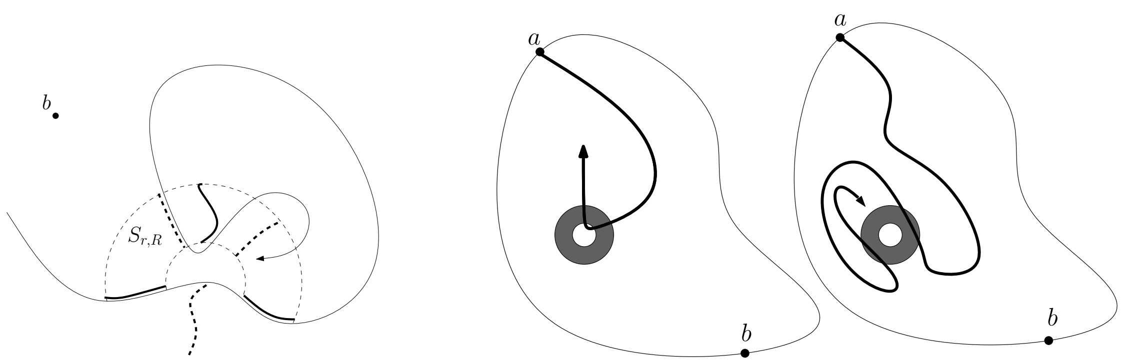

Figure 7: Left: An example of a fjord. Seen from b, the h-capacity (roughly speaking, the size) of the hull does not grow much while the curve is in the fjord. The event involves six alternating open and closed crossings of the annulus. **Right:** Conditionally on the beginning of the curve, the crossing of the annulus is unforced on the left, while it is forced on the right.

In the case of percolation, this corresponds to the six-arm events, and it boils down to proving that $\Delta_6 > 2$. We will prove this result in Proposition 3.15, and we show at the end of this subsection how it indeed implies that scaling limits are Loewner chains. Before that, we present a more general criterion characterizing Loewner chains.

A criterion for a random continuous curve to be a Loewner chain Recently, Kemppainen and Smirnov [KS10] proved a "structural theorem" characterizing random continuous curves that can be parametrized as Loewner chains. We describe it now. For a family of parametrized curves $(\gamma_\delta)_{\delta>0}$, define the following:

Definition 2.14 (Condition ($\star$), see Fig. 7)**.** *There exist $C > 1$ and $\Delta > 0$ such that for any $0 < \delta < r < R/C$, for any stopping time τ and for any annulus $S_{r,R}(x)$ not containing γ_τ, the probability that γ_δ crosses the annulus $S_{r,R}(x)$ (from the outside to the inside) after time τ* ***while it is not forced to enter*** *$S_{r,R}(x)$* ***again*** *is smaller than $C(r/R)^\Delta$.*

Roughly speaking, the previous condition is a uniform bound on unforced crossings. Note that it is necessary to precise the fact that the crossing is unforced.

Theorem 2.15. *If a family of curves (γ_δ) satisfies Condition $(\star)$, then it is tight. Moreover, any subsequential limit is a time-changed Loewner chain.*

Tightness is almost obvious, since Condition $(\star)$ implies the hypothesis in Aizenman-Burchard's theorem. The hard part is the proof that Condition $(\star)$ guarantees that the h-capacity of subsequential limits is strictly increasing and that they create Loewner chains generated by a curve. The reader is referred to [KS10] for a proof of this statement.

Proof of Theorem 2.13. Theorem 2.1 implies Condition $(\star)$ without difficulty. □

Alternative (sketched) proof of Theorem 2.13. Let us now sketch another way of proving Theorem 2.13. It does not require Theorem 2.15 and it harnesses Theorem 2.1 and Theorem 3.14 below. We refer to [Wer07] for additional details on this method.

We need to prove that the h-capacity is strictly increasing. Let us consider the discrete explorations directly in the upper half-plane, and already parametrized by their h-capacity. The idea is to proceed in three steps. Let $\sigma_\delta(z)$ (resp. $\sigma(z)$) be the time at which z is disconnected from infinity by the discrete curve γ_δ (resp. the continuous curve γ).

Step 1: simultaneously for every z, $\sigma_\delta(z)$ converges to $\sigma(z)$ almost surely. This is due to the fact that if one point z does not satisfy this property, the discrete model has to possess six arms of alternative colors (or three arms on the boundary of alternative colors). Yet, the six arm event has exponent larger than 2 and does not happen anywhere in the domain with probability going to 1.

Step 2: for any $u < u'$, there exists $v \in (u, u')$ such that $\gamma(v) \notin \gamma[0, u] \cup \partial\mathbb{H}$. Fix a dense family of points on $\gamma[0, u] \cup \partial\mathbb{H}$. Each of these points does not belong to the curve $\gamma[0, \infty]$ almost surely, thanks to Theorem 2.1. Therefore, none of these points belongs to $\gamma[0, \infty]$ almost surely. This implies that $\gamma[u, u']$ cannot be included in $\gamma[0, u] \cup \partial\mathbb{H}$.

Step 3: for every rational $u < u'$, $K_u \neq K'_u$. Recall that K_u is the hull created by $\gamma[0, u]$. It is thus sufficient to prove that there exists $v \in (u, u')$ such that $\gamma(v) \notin K_u \cup \partial\mathbb{H}$. We already know from the second step that there exists $\gamma(v) \notin \gamma[0, u] \cup \partial\mathbb{H}$. Thus $\gamma(v)$ is in one of the connected components of $\mathbb{H} \setminus \gamma[0, u]$. Assume it is not in the unbounded one. The first step implies that

$$\sigma_\delta[\gamma(v)] \leq \frac{v + \sigma[\gamma(v)]}{2}$$

with probability going to 1. It immediately implies that $\sigma_\delta[\gamma_\delta(v)] < v$ for δ small enough, which is impossible since discrete curves γ_δ do not have triple points. □

2.4.4 Convergence of exploration paths to SLE(6)

Fix a topological triangle $(\Omega_\delta, A_\delta, B_\delta, C_\delta)$ and $z_\delta \in \Omega_\delta^\star$. Define $E_{\Omega_\delta,A_\delta,B_\delta,C_\delta}(z_\delta)$ to be the event that there exists a non-self-intersecting path of open sites in Ω_δ, separating A_δ and z_δ from B_δ and C_δ, and let

$$H_{\Omega_\delta,A_\delta,B_\delta,C_\delta,z_\delta}(n) := P_\delta(E_{\Omega_\delta\setminus\gamma[0,n],\gamma_n,B_\delta,C_\delta}(z_\delta)).$$

Lemma 2.16. *$(H_{\Omega_\delta,A_\delta,B_\delta,C_\delta,z_\delta}(n))_{n\geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$ where $\mathcal{F}_n$ is the σ-algebra generated by the $\gamma[0, n]$.*

Proof. The slit domain created by "removing" the first n steps of the exploration path is again a topological triangle. Conditionally on $\gamma[0,n]$, the law of the configuration in the new domain is exactly percolation in $\Omega \setminus \gamma[0,n]$. This observation implies that $H_{\Omega,A,B,C,z}(n)$ is the random variable $1_{E_{\Omega_\delta,A_\delta,B_\delta,C_\delta}(z_\delta)}$ conditionally on $\mathcal{F}_n$, therefore it is automatically a martingale. □

Proposition 2.17. *Any subsequential limit of $(\gamma_\delta)_{\delta>0}$ which is a Loewner chain is (chordal) Schramm-Loewner evolution with parameter $\kappa = 6$.*

Proof. Consider a subsequential limit γ in the domain (Ω, a, b) which is a Loewner chain. Let ϕ be a map from (Ω, a, b) to $(\mathbb{H}, 0, \infty)$. Our goal is to prove that $\tilde{\gamma} := \phi(\gamma)$ is a chordal SLE(6) in the upper half-plane.

Since γ is assumed to be a Loewner chain, $\tilde{\gamma}$ is a growing hull from 0 to ∞; we can assume that it is parametrized by its h-capacity. Let W_t be its continuous driving process. Also define g_t to be the conformal map from $\mathbb{H} \setminus \tilde{\gamma}[0,t]$ to $\mathbb{H}$ such that $g_t(z) = z + 2t/z + O(1/z^2)$ when z goes to infinity.

Fix $c' \in \partial\Omega$ and $z' \in \Omega$. For $\delta > 0$, recall that $H_\delta(n) = H_{\Omega_\delta,A_\delta,B_\delta,C_\delta,z'_\delta}(n)$ is a martingale for γ_δ. Since the martingale is bounded, $H_\delta(\tau_t)$ is a martingale with respect to $\mathcal{F}_{\tau_t}$, where τ_t is the first time at which $\phi(\gamma_\delta)$ has a h-capacity larger than t. Since the convergence is uniform, $H_t(z') := \lim_{\delta\to 0} H_\delta(\tau_t)$ is a martingale with respect to $\mathcal{G}_t$, where $\mathcal{G}_t$ is the σ-algebra generated by the curve $\tilde{\gamma}$ up to the first time its h-capacity exceeds t. By definition, this time is t, and $\mathcal{G}_t$ is the σ-algebra generated by $\tilde{\gamma}[0,t]$ — in other words, it is the natural filtration associated to the process (W_t).

The Cardy-Smirnov formula, or rather its extension to the inside of the domain (as introduced in the proof of Theorem 1.3 under the name h_A), gives the value of $H_t(z')$, or rather of its complexified version $\tilde{H}_t(z')$ like in the proof of the formula, in terms of conformal maps as

$$\tilde{H}_t(z') = f\left(\frac{g_t(z) - W_t}{g_t(c) - W_t}\right)$$

with an explicit, smooth function f (where we define $z := \phi(z')$ and $c := \phi(c')$). This is a martingale for every choice of z and c, so we get the family of identities

$$\mathbb{E}\left[f\left(\frac{g_t(z) - W_t}{g_t(c) - W_t}\right) \middle| \mathcal{G}_s \right] = f\left(\frac{g_s(z) - W_s}{g_s(c) - W_s}\right)$$

for all $z \in \mathbb{H}$, $c \in \mathbb{R}$ and $0 < s < t$ such that z and c are both within the domain of definition of g_t. We know the asymptotic expansion of g_s and g_t around infinity, so the above becomes

$$\mathbb{E}\left[f\left(\frac{z - W_t + 2t/z + O(1/c^2)}{c - W_t + 2t/c + O(1/z^2)}\right) \middle| \mathcal{G}_s \right] = f\left(\frac{z - W_s + 2s/z + O(1/z^2)}{c - W_s + 2s/c + O(1/c^2)}\right). \tag{2.12}$$

Letting z and c go to infinity with fixed ratio $z/c = \lambda \in \mathbb{H}$, we have

$$\begin{aligned}
f\left(\frac{z - W_s + 2s/z + O(1/z^2)}{c - W_s + 2s/c + O(1/c^2)}\right) &= f\left(\frac{\lambda - W_s/c + 2s/\lambda c^2 + O(1/c^2)}{1 - W_s/c + 2s/c^2 + O(1/c^3)}\right) \\
&= f\left(\lambda + \frac{(\lambda-1)W_s}{c} + \frac{(\lambda-1)W_s^2 + 2(1-\lambda^2)s/\lambda}{c^2} + O(c^{-3})\right). \\
&= f(\lambda) + \frac{(\lambda-1)f'(\lambda)W_s}{c} \\
&\qquad + \frac{(\lambda-1)W_s^2[f'(\lambda) + (\lambda-1)f''(\lambda)/2] + 2(1-\lambda^2)sf'(\lambda)/\lambda}{c^2} + O(c^{-3}).
\end{aligned}$$

Using this expansion on both sides of (2.12) and matching the terms, we obtain two identities for (W_t):

$$\mathbb{E}[W_t|\mathcal{G}_s] = W_s, \quad E[W_t^2|\mathcal{G}_s] = W_s^2 + \frac{4(1+\lambda)f'(\lambda)/\lambda}{2f'(\lambda) + (\lambda-1)f''(\lambda)}(t-s).$$

The function f is a conformal map from the upper-half plane to the equilateral triangle, sending 0, 1 and ∞ to the vertices of the triangle; up to additive and multiplicative constants, it can be written using the Schwarz-Christoffel formula as

$$f(\lambda) \propto \int^{\lambda} [z(1-z)]^{-2/3}\, \mathrm{d}z.$$

From this, one obtains $f'(\lambda) \propto [\lambda(1-\lambda)]^{-2/3}$ and

$$\frac{f''(\lambda)}{f'(\lambda)} = -\frac{2}{3}\left(\frac{1}{\lambda} - \frac{1}{1-\lambda}\right) = \frac{2(2\lambda-1)}{3\lambda(1-\lambda)}.$$

Plugging this into the previous expression shows that the factor of $(t-s)$ is identically equal to 6, and since we know that (W_t) is a continuous process, this implies that it is of the form $(\sqrt{6}B_t)$ where (B_t) is a standard real-valued Brownian motion, meaning that γ is exactly an SLE(6) process in (Ω, a, b). □

Proof of Theorem 1.4. By Theorem 2.11, the family of exploration processes is tight. Using Theorem 2.13, any subsequential limit is a time-changed Loewner chain. Consider such a subsequential limit and parametrize it by its h-capacity. Proposition 2.17 then implies that it is the Schramm-Loewner Evolution with parameter $\kappa = 6$. The possible limit being unique, the claim is proved. □

3 The critical point of percolation

We now arrive at a milestone of modern probability, Kesten's "$p_c = 1/2$" Theorem (Theorem 1.1). It was proved in the case of bond-percolation on the square lattice, but the same argument applies to site percolation on the triangular lattice. The rough philosophy of the proof is the following:

- First, exhibit a property at $p = 1/2$, which should be witness of the critical phase.
- Second, prove that the property holds only at $p = 1/2$, identifying $1/2$ to be the only possible value for the critical point.

One property "identifying" the critical phase is Theorem 2.1 or more generally crossing probabilities of rectangles with fixed aspect ratio. To illustrate that this result can hold only at criticality, let us prove the following result, which says that whenever there exists $L > 0$ such that the rectangle $[0, L] \times [0, 2L]$ is crossed horizontally with small enough probability, the probability of two points being connected decays exponentially fast (which implies that the model is subcritical and that its dual is supercritical). In the previous sentence, and from now on, $[0, n] \times [0, m]$ denotes the set of points of the form $k \cdot 1 + \ell \cdot e^{i\pi/3}$, with $0 \le k \le n$ and $0 \le \ell \le m$.

Proposition 3.1. *Fix $p \in (0, 1)$ and assume there exists $L \in \mathbb{N}$ such that*

$$\mathbb{P}_p([0, L] \times [0, 2L] \text{ is crossed horizontally}) < \frac{1}{e\binom{8}{2}}.$$

Then,

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n) \ \le \ ne^{-n/(2L)} \quad \text{for all } n \ge L.$$

In particular, $p \le 1 - p_c$.

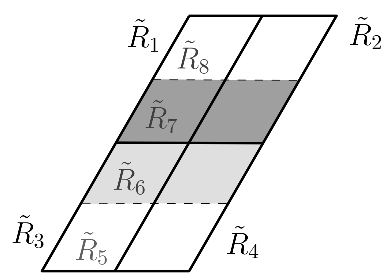

Figure 8: The rectangles $\tilde{R}_1, \ldots, \tilde{R}_6$.

Proof. The inequality $p \le 1 - p_c$ follows easily from the exponential decay of connectivity properties. Indeed, one can prove via Borel-Cantelli that there exists almost surely a finite number of open circuits surrounding the origin, thus proving that there exists an infinite closed-cluster. The definition of p_c then implies that $1 - p \ge p_c$.

Let $m > 0$ and consider the rectangles $\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_8$ defined as in Fig. 8. These rectangles have the property that whenever $[0, 2m] \times [0, 4m]$ is crossed horizontally, at least two of the rectangles $\tilde{R}_i$ are crossed in the easy direction by *disjoint* paths. We deduce, using the BK inequality, that

$$\mathbb{P}_p\big([0, 2m] \times [0, 4m] \text{ is crossed horizontally}\big) \le \binom{8}{2} \mathbb{P}_p\big([0, m] \times [0, 2m] \text{ is cros. hor.}\big)^2.$$

Iterating the construction, we easily obtain that for every $k \geq 0$,

$$\binom{8}{2}\mathbb{P}_p\big([0,2^k m]\times[0,2^{k+1}m] \text{ is cros. hor.}\big) \leq \left(\binom{8}{2}\mathbb{P}_p\big([0,m]\times[0,2m] \text{ is cros. hor.}\big)\right)^{2^k}.$$

In particular, if $m = L$, $\binom{8}{2}\mathbb{P}_p\big([0,n]\times[0,2n] \text{ is cros. hor.}\big) < 1/e$ and we deduce for $n = 2^k L$:

$$\mathbb{P}_p(0 \leftrightarrow \partial\Lambda_n) \ \leq \ ne^{-n/L}.$$

The claim follows for every n by monotonicity. □

The proposition can be reformulated in a nice way: let $\mathcal{C}_{\text{sub}}$ denote the statement that there exists $L > 0$ such that

$$\mathbb{P}_p([0,L]\times[0,2L] \text{ is crossed horizontally}) < \frac{1}{e\binom{8}{2}}.$$

The condition $\mathcal{C}_{\text{sub}}$ on p is a criterion of sub-criticality, and the fact that it holds for $1-p$ is a criterion of super-criticality.

The previous discussion implies that it suffices to prove that probabilities to cross a rectangle of aspect ratio 2 in the easy direction is going to 0 as $n \to \infty$ when $p < 1/2$, or equivalently that the probability to cross a rectangle of aspect ratio 2 in the hard direction is going to 1 as $n \to \infty$ when $p > 1/2$.

In order to prove this fact, we consider a more general question. We aim to understand the behavior of the function $p \mapsto \mathbb{P}_p(A)$ for a non-trivial increasing event A depending on sites of a subgraph of the triangular lattice (think of this event as being a crossing event). This increasing function is equal to 0 at $p = 0$ and to 1 at $p = 1$, and we are interested in the range of p for which its value is between ε and $1-\varepsilon$ for some positive ε (this range is usually referred to as a *window*). Under certain conditions on A, the window will be narrow for large graphs, and its width can be bounded above in terms of the size of the underlying graph. This kind of result is known as a *sharp threshold* behavior.

The study of $p \mapsto \mathbb{P}_p(A)$ harnesses a differential equality known as Russo's formula:

Proposition 3.2 (Russo [Rus78], Section 2.3 of [Gri99])**.** *Let $p \in (0,1)$ and A an increasing event depending on a* finite *set of sites V, then*

$$\frac{d}{dp}\mathbb{P}_p(A) = \sum_{v\in V}\mathbb{P}_p(v \textit{ pivotal for } A),$$

where v is pivotal for A if A occurs when v is open, and does not if v is closed.

If the typical number of pivotal sites is sufficiently large, for instance when the probability of A is not close to 0 nor 1, the window is necessarily narrow. There has been an extensive study of the largest (among all the sites) probability to be pivotal. We present one of the most striking results on the subject:

Theorem 3.3 (Kahn, Kalai, Linial [KKL88], see also [Fri04, FK96, KS06]). *Let $p_0 > 0$. There exists a constant $c = c(p_0) \in (0, \infty)$ such that the following holds. Consider a percolation model on a graph G with $|V|$ denoting the number of sites of G. For every $p \in [p_0, 1-p_0]$ and every increasing event A, there exists $v \in V$ such that*

$$\mathbb{P}_p(v \textit{ pivotal for } A) \ \geq c\,\mathbb{P}_p(A)\big(1-\mathbb{P}_p(A)\big)\frac{\log|V|}{|V|}.$$

This theorem does not imply that there always are many pivotal sites, since it deals only with the maximal probability over all sites. It could be that this maximum would be attained only at one site (for instance for the event that the origin is open). There is a particularly efficient way (first appeared in [BR06a, BR06c]) to avoid this problem. In the case of a *translation-invariant event* A on a torus with n vertices, sites play a symmetric role, so that the probability to be pivotal is the same for all of them. Proposition 3.2 together with Theorem 3.3 thus imply that in this case, for $p \in (p_0, 1-p_0)$,

$$\frac{d}{dp}\mathbb{P}_p(A) \geq c\big(\mathbb{P}_p(A)(1-\mathbb{P}_p(A)\big)\log n.$$

Integrating the previous inequality between two parameters $p_0 < p_1 < p_2 < 1-p_0$, we obtain

$$\frac{\mathbb{P}_{p_2}(A)}{1-\mathbb{P}_{p_2}(A)} \geq \frac{\mathbb{P}_{p_1}(A)}{1-\mathbb{P}_{p_1}(A)}n^{c(p_2-p_1)}.$$

If we further assume that $\mathbb{P}_{p_1}(A)$ stays bounded away from 0 uniformly in $n \geq 1$, we can find $c, C > 0$ such that

$$\mathbb{P}_{p_2}(A) \geq 1 - Cn^{-c(p_2-p_1)}. \tag{3.1}$$

Now that the theory is settled, we can prove the fundamental lemma which shows that Theorem 2.1 fails when $p \neq 1/2$ (in the sense that crossing probabilities of a rectangle of aspect ratio 2 go to 1 as $n \to \infty$ when $p > 1/2$). This result is true for every shape, we prove it for the shape of a rectangle $[0,1] \times [0,2]$.

Lemma 3.4. *Let $p < 1/2$, there exist $\varepsilon = \varepsilon(p) > 0$ and $c = c(p) > 0$ such that for every $n \geq 1$,*

$$\mathbb{P}_p\big([0,n]\times[0,2n] \textit{ is crossed horizontally}\big) \leq cn^{-\varepsilon}. \tag{3.2}$$

The proof uses Theorem 3.3: we consider a well-chosen translation-invariant event for which we can prove sharp threshold. Then, we bootstrap the result to our original event. Let us mention that Kesten proved in [Kes80] a sharp-threshold for the case needed using different arguments.

Proof. We work with the dual percolation. We need to prove that for $p > 1/2$, there exist $\varepsilon = \varepsilon(p)$ and $c = c(p)$ such that

$$\mathbb{P}_p\big([0,n]\times[0,2n] \text{ is crossed vertically}\big) \geq 1 - cn^{-\varepsilon}. \tag{3.3}$$

Consider the torus $\mathbb{T}_{4n}$ of size $4n$. Let B be the event that there exists a vertical crossing of a rectangle with dimensions $(n/2, 4n)$ in the torus of size $4n$. This event is invariant under translations and satisfies

$$\mathbb{P}_{1/2}(B) \geq \mathbb{P}_{1/2}\big([0,n/2]\times[0,4n] \text{ is crossed vertically}\big) \geq c > 0$$

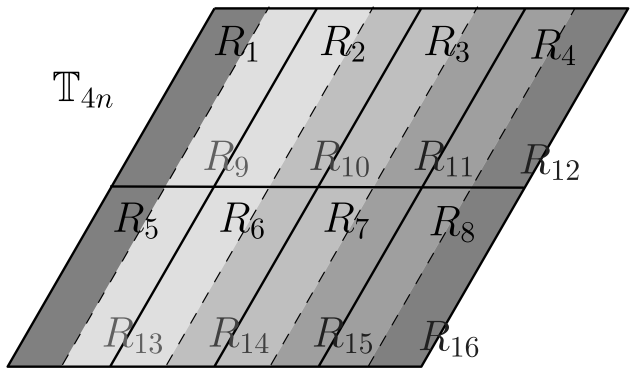

Figure 9: The rectangles $R_1, \ldots, R_{16}$. They are all translates of R_1.

uniformly in n. Since B is increasing, we can apply (3.1) to deduce that for $p > 1/2$, there exist $\varepsilon, c > 0$ such that

$$\mathbb{P}_p(B) \geq 1 - cn^{-\varepsilon}. \tag{3.4}$$

If B holds, one of the 16 rectangles $R_1, \ldots, R_{16}$ drawn in Fig. 9 must be crossed from top to bottom. We denote these events by $A_1, \ldots, A_{16}$ — they are translates of the event that $[0,n] \times [0,2n]$ is crossed horizontally. Using the Harris inequality in the second line, we find

$$\begin{aligned}\mathbb{P}_p(B) = 1 - \mathbb{P}_p(B^c) = 1 - \mathbb{P}_p\left(\bigcap_{i=1}^{16} A_i^c\right) \\ \leq 1 - \prod_{i=1}^{16} \mathbb{P}_p(A_i^c) = 1 - \big[1 - \mathbb{P}_p\big([0,n] \times [0,2n] \text{ is crossed vertically}\big)\big]^{16}.\end{aligned}$$

Plugging (3.4) into the previous inequality, we deduce

$$\mathbb{P}_p\left([0,n] \times [0,2n] \text{ is crossed vertically}\right) \geq 1 - (cn^{-\varepsilon})^{1/16}. \qquad \square$$

3.1 Definition of the correlation length

We have studied how probabilities of increasing events evolve as functions of p. If p is fixed and we consider larger and larger rectangles (of size n), crossing probabilities go to 1 whenever $p > 1/2$, or equivalently to 0 whenever $p < 1/2$. But what happens if $(p,n) \to (1/2, \infty)$ (this regime is called the *near-critical regime*)?

If one looks at two percolation pictures in boxes of size N, one at $p = 0.5$, and one at $p = 0.47$, it is not be necessarily possible to distinguish between them if N is not large enough. Yet, when the size of the picture gets bigger and bigger, connectivity properties start to differ drastically. The scale at which one starts to see that p is not critical is called the correlation length. Interestingly, it can naturally be expressed in terms of crossing probabilities

Definition 3.5. *For $\varepsilon > 0$ and $p < 1/2$, define the* correlation length *by*

$$L_p(\varepsilon) \ := \ \inf\big\{n > 0 \ : \ \mathbb{P}_p\big([0,n]^2 \textit{ is crossed horizontally}\big) \leq \varepsilon \ \big\}. \tag{3.5}$$

Extend the definition of the correlation length to every p by setting $L_p(\varepsilon) := L_{1-p}(\varepsilon)$.

Note that the definition itself of $L_p(\varepsilon)$ uses the fact that crossing probabilities converge to 0 when $p < 1/2$ to guarantee that the infimum is well-defined. Let us also mention that taking rhombi in the definition of the correlation length is not crucial. Indeed, the following result, called a Russo-Seymour-Welsh result, implies that one could equivalently define the correlation length with other aspect ratio, and that it would only change the value of the corresponding ε.

Theorem 3.6 (see *e.g.* [Gri99, Kes82])**.** *Let $p_0 > 0$, there exists a strictly increasing continuous function $\rho_{p_0} : [0,1] \to [0,1]$ such that $\rho_{p_0}(0) = 0$ and $\rho_{p_0}(1) = 1$ satisfying the following property: for every $p \in (p_0, 1-p_0)$ and every $n > 0$,*

$$\rho_{p_0}(\delta) \leq \mathbb{P}_p([0,2n] \times [0,n] \text{ crossed horizontally}) \leq \rho_{p_0}(1-\delta),$$

where

$$\delta := \mathbb{P}_p([0,n]^2 \text{ crossed horizontally}).$$

From now on, fix $p_0 > 0$. Let us mention that ε is not relevant as well, as long as it is taken small enough. More precisely, we want to argue that above the critical length, things look subcritical or supercritical depending on $p < 1/2$ or $p > 1/2$. To do so, we would like $\mathcal{C}_{\text{sub}}$ to be satisfied. In other words, we want $\mathbb{P}_p([0,2n] \times [0,n]$ crossed horizontally$) < 1/(e\binom{8}{2})$ for $n \geq L_p(\varepsilon)$. Fix $\varepsilon = \varepsilon(p_0)$ small enough so that $\rho(\varepsilon) < 1/(e\binom{8}{2})$ and drop it from the notations. Note that with this value of ε, the correlation length at criticality equals infinity, since probabilities to be connected at distance n do not decay exponentially fast for $p = 1/2$.

In particular, we get that below the correlation length, crossing probabilities of topological rectangles are bounded from below: for any topological rectangle (Ω, A, B, C, D), there exists $c = c(\varepsilon, p_0) > 0$ such that for $p \in (p_0, 1-p_0)$ and $n < L_p(\varepsilon)$,

$$\mathbb{P}_p\big[\mathcal{C}_{1/n}(\Omega, A, B, C, D)\big] \;\geq\; c. \tag{3.6}$$

In this sense, the configuration looks critical. We will see in the next section that fractal properties are the same below the correlation length as at criticality.

We conclude this section by mentioning that usually, the correlation length is defined as the “inverse rate” of exponential decay of the two-point function. More precisely, since the quantity $\mathbb{P}_p(0 \leftrightarrow nx)$ is super-multiplicative, the quantity ξ_p can be defined by the formula

$$\frac{1}{\xi_p} \;=\; \lim_{n\to\infty} -\frac{1}{n}\log \mathbb{P}_p(0 \leftrightarrow nx).$$

Then, it is possible to prove that $L_p \asymp \xi_p$ when $p < 1/2$ (note that Proposition 3.1 gives one inequality, see Theorem 3.1 of [Nol08] *e.g.* for the other bound).

Remark 3.7. *Another way to understand the critical length is the following: when studying the super or subcritical percolation,* coarse-graining arguments *allow to relate properties of a percolation with parameter p to a percolation with new parameter p' much closer to 0 or 1. Usually, by taking N to be large enough, it is even possible to get p' in the* Peierls regime, *in which counting arguments are sufficient to estimate relevant quantities. Typically, the grain needs to be considered at parameter p is of order L_p.*

3.2 Percolation below the correlation length and arm exponents

Proposition 3.1 together with the definition of the correlation length imply that percolation in boxes much larger than the correlation length look subcritical or supercritical. The goal of this section is to describe percolation below the correlation length. In particular, we aim to prove that connectivity properties are basically the same as the connectivity properties at criticality.

To quantify connectivity properties, we introduce the notion of *arm-event*. Fix a sequence σ of j colors (”open” o or ”closed” c). For $n < N$, define $A_\sigma(n, N)$ to be the event that there are j *disjoint* paths from $\partial\Lambda_n$ to $\partial\Lambda_N$ with colors $\sigma_1, \ldots, \sigma_j$ where the paths are indexed in counter-clockwise order. We set $A_\sigma(N)$ to be $A_\sigma(k, N)$ where k is the smallest possible integer such that the event is non-empty. For instance, $A_o(n, N)$ is the one-arm event corresponding to the existence of a crossing from the inner to the outer boundary of $\Lambda_N \setminus \Lambda_n$.

An adaptation of Corollary 2.4 implies that there exists β_σ and β'_σ such that

$$(n/N)^{\beta_\sigma} \;\leq\; \mathbb{P}_{1/2}[A_\sigma(n, N)] \;\leq\; (n/N)^{\beta'_\sigma}.$$

It is therefore natural to predict that there exists a *critical exponent* $\alpha_\sigma \in (0, \infty)$ such that

$$\mathbb{P}_{1/2}[A_\sigma(n, N)] \;=\; (n/N)^{-\alpha_\sigma + o(1)},$$

where $o(1)$ is a quantity converging to 0 as n/N goes to 0. The quantity α_σ is called an *arm-exponent*.

We will compute critical exponents later in the survey. Before that, we get interested in the variation of $\mathbb{P}_p[A_\sigma(n, N)]$ as a function of p when $N < L_p$. We aim to show that the variation is not large, and that $\mathbb{P}_p[A_\sigma(n, N)]$ remains basically constant. This fact justifies the following claim: below L_p, percolation looks critical.

Theorem 3.8 (Kesten [Kes87]). *For a polychromatic sequence σ, we have $\mathbb{P}_p(A_\sigma(n)) \asymp \mathbb{P}_{1/2}(A_\sigma(n))$ for every p and $n \leq L_p$.*

3.2.1 Quasi-multiplicativity of arm exponents

Let us start with a technical yet fundamental result.

Theorem 3.9 (Quasi-multiplicativity). *Fix $p \in (p_0, 1 - p_0)$ and a polychromatic sequence σ. For every $n_1 < n_2 < n_3 < L_p$, we have*

$$\mathbb{P}_p\big[A_\sigma(n_1, n_3)\big] \asymp \mathbb{P}_p\big[A_\sigma(n_1, n_2)\big] \cdot \mathbb{P}_p\big[A_\sigma(n_2, n_3)\big].$$

The inequality

$$\mathbb{P}_p\big[A_\sigma(n_1, n_3)\big] \leq \mathbb{P}_p\big[A_\sigma(n_1, n_2)\big] \cdot \mathbb{P}_p\big[A_\sigma(n_2, n_3)\big].$$

is straightforward using independence. The other one is slightly more technical. Let us mention that in the case of one arm, the proof is fairly easy (see Fig. 10 for an illustration of the proof).

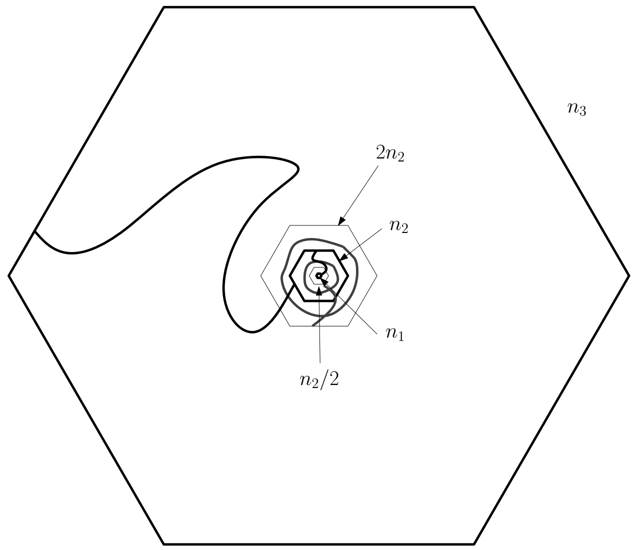

Figure 10: The paths in the annuli $\Lambda_{n_3}\setminus\Lambda_{n_2}$ and $\Lambda_{n_2}\setminus\Lambda_{n_1}$ are in black. A combination of two circuits connected by a path (in gray) connects the paths together. This figure occurs with probability bounded away from 0 thanks to crossing estimates.

For general σ, the proof requires the notion of well-separated arms. In words, well-separated arms extend slightly outside the boxes and their ends are at macroscopic distance of each others, see Fig. 11. More precisely, for $\delta > 0$, j arms $\gamma_1, \ldots, \gamma_j$ with end-points $x_k = \gamma_k \cap \partial\Lambda_n$, $y_k = \gamma_k \cap \partial\Lambda_N$ are said to be (δ-)*well-separated* if

- sites y_k are at distance larger than $2\delta N$ from each others.
- sites x_k are at distance larger than $2\delta n$ from each others.
- For every k, y_k is σ_k-connected to distance δN of $S_{n,N}$ in $\Lambda_{\delta N}(y_k)$,
- For every k, x_k is σ_k-connected to distance δn of $S_{n,N}$ in $\Lambda_{\delta n}(x_k)$.

Let $A_\sigma^{sep;\delta}(n,N) = A_\sigma^{sep}(n,N)$ be the event that $A_\sigma(n,N)$ holds true and there exist arms realizing $A_\sigma(n,N)$ which are δ-well-separated.

The previous definition has several convenient properties, such as:

Proposition 3.10. *Fix $p \in (p_0, 1-p_0)$ and $\delta < 1$ small enough. For every $n_1 \le n_2 \le \frac{n_3}{2} \le L_p$,*

$$\mathbb{P}_p[A_\sigma^{sep}(n_1,n_3)] \gtrsim \mathbb{P}_p(A_\sigma^{sep}(n_1,n_2)] \cdot \mathbb{P}_p[A_\sigma^{sep}(2n_2,n_3)].$$

From this, we deduce that for $p \in (0,1)$ and $\varepsilon, \delta < 1$, there exists $\alpha = \alpha(\delta,\varepsilon) > 0$ such that for every $n_1 \le n_2 \le n_3 < L_p(\varepsilon)$,

$$\mathbb{P}_p\big[A_\sigma^{sep}(n_1,n_2)\big] \lesssim \left(\frac{n_3}{n_2}\right)^\alpha \cdot \mathbb{P}_p\big[A_\sigma^{sep}(n_1,n_3)\big]. \tag{3.7}$$

To prove this inequalities, it suffices to see that $\mathbb{P}_p(A_\sigma^{sep}(n,N))$ is also bounded from below by a power of (n/N). This is an easy consequence of (3.6).

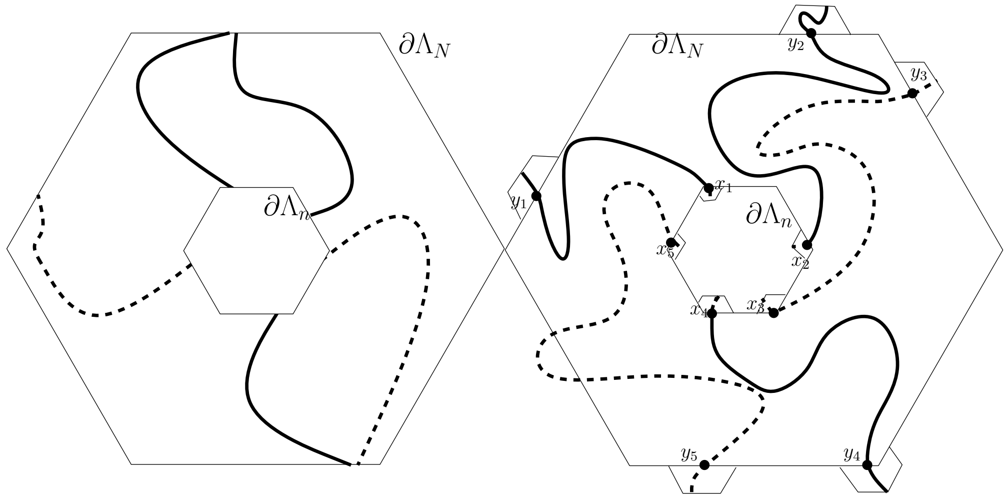

Figure 11: On the left, the five-arm event $A_{ocooc}(n, N)$. On the right, the event $A^{sep}_{ocooc}(n, N)$ with well-separated arms. Note that these arms are not at macroscopic distance of each others inside the domain, but only at their end-points.

Proof of Proposition 3.10. We have

$$\mathbb{P}_p\big[A^{sep}_\sigma(n_1, n_2) \cap A^{sep}_\sigma(2n_2, n_3)\big] \geq \mathbb{P}_p\big[A^{sep}_\sigma(n_1, n_2)\big] \cdot \mathbb{P}_p\big[A^{sep}_\sigma(2n_2, n_3)\big]$$

and it suffices to prove that $\mathbb{P}_p\big[A^{sep}_\sigma(n_1, n_2) \cap A^{sep}_\sigma(2n_2, n_3)\big]$ and $\mathbb{P}_p\big[A^{sep}_\sigma(n_1, n_3)\big]$ are comparable. To do so, condition on $A^{sep}_\sigma(n_1, n_2) \cap A^{sep}_\sigma(2n_2, n_3)$ and construct j disjoint tubes of width $\varepsilon = \varepsilon(\delta)$ connecting $(y_k + \Lambda_{\delta n_2}) \setminus \Lambda_{n_2}$ to $(y_k + \Lambda_{2\delta n_2}) \cap \Lambda_{2n_2}$ for every $k \leq j$. It is simple to show that this is topologically possible. Via (3.6), the σ_k-paths connecting x_k to $\partial\Lambda_{2\delta n_2}(x_k) \cap \Lambda_{n_2}$, and y_k to $\partial\Lambda_{\delta n_2}(y_k) \setminus \Lambda_{n_2}$ can be connected by a σ_k-path with positive probability $c = c(\delta, p_0)$. Therefore,

$$\mathbb{P}_p(A^{sep}_\sigma(n_1, n_3)) \geq c\mathbb{P}_p\big[A^{sep}_\sigma(n_1, n_2) \cap A^{sep}_\sigma(2n_2, n_3)\big],$$

thus concluding the proof. $\square$

If $A^{sep}_\sigma(n, N)$ and $A_\sigma(n, N)$ have uniformly comparable probabilities, Theorem 3.9 follows readily. Therefore, our main objective is now the following result:

Proposition 3.11. *Fix $p \in (p_0, 1 - p_0)$. For every $n < N \leq L_p$,*

$$\mathbb{P}_p\big[A^{sep}_\sigma(n, N)\big] \asymp \mathbb{P}_p\big[A_\sigma(n, N)\big].$$

Let us present how to conclude the proof of Theorem 3.9:

Proof of Theorem 3.9. As we said earlier, one inequality is straightforward. Let us deal with the other one. We have for $n_1 \leq n_2 \leq n_3$,

$$\begin{aligned}\mathbb{P}_p\big[A_\sigma(n_1, n_3)\big] &\geq \mathbb{P}_p\big[A^{sep}_\sigma(n_1, n_3)\big] \\ &\gtrsim \mathbb{P}_p\big[A^{sep}_\sigma(n_1, n_2)\big] \cdot \mathbb{P}_p\big[A^{sep}_\sigma(2n_2, n_3)\big] \\ &\asymp \mathbb{P}_p\big[A_\sigma(n_1, n_2)\big] \cdot \mathbb{P}_p\big[A_\sigma(2n_2, n_3)\big] \\ &\geq \mathbb{P}_p\big[A_\sigma(n_1, n_2)\big] \cdot \mathbb{P}_p\big[A_\sigma(n_2, n_3)\big],\end{aligned}$$

where in the third line, we used Proposition 3.11, in the second Proposition 3.10, and in the last, $A_\sigma(n_2, n_3) \subset A_\sigma(2n_2, n_3)$. $\square$

Let us now turn to the proof of Proposition 3.11. We start with the following lemma:

Lemma 3.12. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that for every $2n \leq N$,*

$$\mathbb{P}_p(\textit{any set of crossings of } S_{n,N} \textit{ can be made well separated}) \;\geq\; 1 - \varepsilon.$$

Proof. Fix $2n < N \leq L_p$. Consider T large enough so that there exist more than T disjoint crossings of $S_{n,2N}$ with probability less than ε. This is possible since the probability that there is one crossing is smaller than $c = c(\varepsilon) < 1$ via the definition of L_p, hence the Reimer inequality implies that the probability that there are T crossing is smaller than c^T.

Fix $\delta > 0$ such that in *any sub-domain* of the annulus $S_{\delta r,r}$, $\partial\Lambda_{\delta r}$ is not connected or dual connected to $\partial\Lambda_r$ with probability $1 - \varepsilon/T$, uniformly in the domain and the boundary conditions on $S_{\delta r,r}$. This fact can be proved easily using (3.6).

We can assume with probability $1 - 12\varepsilon$ that no crossing ends at distance less than δN of a corner of $S_{n,N}$. It is thus sufficient to work with vertical crossings in the trapeze shape $T_{n,N} := S_{n,N} \cap [-N, N] \times [n, N]$.

Now, condition on the left-most crossing γ_1 of $T_{n,N}$ and set y to be the ending point of γ_1 on the top. Construct the domain Ω to be the connected component of the right edge in $T_{n,N} \setminus \gamma_1$. We can assume with probability $1 - \varepsilon/T$ that no vertical crossing land at distance δN of y by ensuring that $\Omega \cap S_{\delta^2 N,\delta N}(y)$ contains open and closed circuits. Moreover, (3.6) allows to construct a path P in $\Lambda_{\delta^2 N}(y) \setminus (T_{n,N} \setminus \Omega)$ connecting γ_1 to the top of $\Lambda_{\delta^2 N}(y)$ with probability $c > 0$. This construction costs $c\varepsilon/T$ and γ_1 is guaranteed to be isolated from other crossings. Iterating the construction T times, we find the result.

The same reasoning applies to the interior side and we obtain the result. $\square$

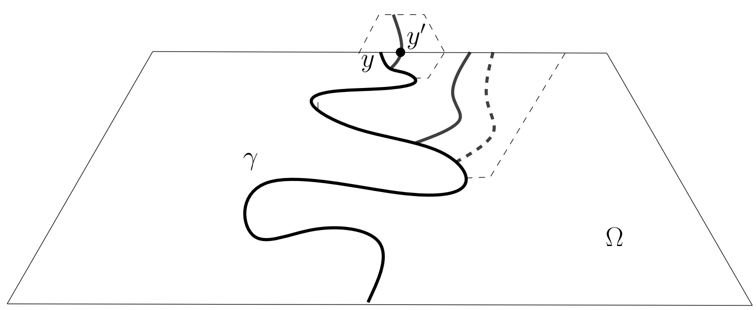

Figure 12: The construction of open and closed paths extending the crossing and preventing other crossings of finishing close to the path in the shape $T_{n,N}$.

Proof of Proposition 3.11. The lower bound $\mathbb{P}_p[A_\sigma^{sep}(n, N)] \leq \mathbb{P}_p[A_\sigma(n, N)]$ is straightforward. Let us prove the upper bound for $S_{2^n,2^N}$, first with only the separation on the exterior. Define $A_\sigma^{sep/ext}(2^n, 2^k)$ to be the event $A_\sigma(2^n, 2^k)$ with separation on the exterior only. Let

Λ_k be the event that crossings in $S_{2^{k-1},2^k}$ can be made separated. Lemma 3.12 ensures that $\mathbb{P}_p(B_k^c) \leq \varepsilon$. Note that $A_\sigma(2^n,2^k)\cap B_k \subset A_\sigma^{sep/ext}(2^n,2^k)$. We thus have

$$\begin{aligned}
\mathbb{P}_p\big[A_\sigma(2^n,2^N)\big] &\leq \sum_{k=n}^{N-1} \mathbb{P}_p\big[A_\sigma(2^n,2^k),B_k,B_{k+1}^c,\dots,B_{N-1}^c\big]\\
&\leq \sum_{k=n}^{N-1} \mathbb{P}_p\big[A_\sigma(2^n,2^k),B_k\big]\cdot\mathbb{P}_p(B_{k+2})\mathbb{P}_p(B_{k+4})\dots\\
&\leq \sum_{k=n}^{N-1} \mathbb{P}_p\big[A_\sigma^{sep/ext}(2^n,2^k)\big]\varepsilon^{N-n}\\
&\lesssim \left(\sum_{k=n}^{N-1} 2^{(\log[\varepsilon]-\alpha)(N-k)}\right)\cdot\mathbb{P}_p\big[A_\sigma^{sep/ext}(2^n,2^N)\big]
\end{aligned}$$

where we used Lemma 3.12 and the fact that Proposition 3.10 together with the a priori bound

$$\mathbb{P}_p[A_\sigma^{sep/ext}(r,R)] \;\geq\; (r/R)^\alpha$$

(which follows from the same techniques as the a priori bounds on $\mathbb{P}_p[A_\sigma(r,R)]$) implies

$$\mathbb{P}_p\big[A_\sigma^{sep/ext}(2^n,2^k)\big] \leq 2^{\alpha(k-N)}\mathbb{P}_p\big[A_\sigma^{sep/ext}(2^n,2^N)\big].$$

Choosing ε small enough, we obtain δ and $c>0$ such that

$$\mathbb{P}_p\big[A_\sigma(2^n,2^N)\big] \leq c\cdot\mathbb{P}_p\big[A_\sigma^{sep/ext}(2^n,2^N)\big]$$

One can then obtain the separation on the interior in the same way. Now, fix $n<N$ arbitrary. Define s,r by the formulas $2^{s-1}<n\leq 2^s$ and $2^r\leq N<2^{r+1}$. We have

$$\mathbb{P}_p\big[A_\sigma(n,N)\big] \leq \mathbb{P}_p\big[A_\sigma(2^s,2^r)\big] \asymp \mathbb{P}_p\big[A_\sigma^{sep}(2^s,2^r)\big] \asymp \mathbb{P}_p\big[A_\sigma^{sep}(n,N)\big]. \qquad \square$$

We mention a classical corollary of the comparison between well-separated arms and usual arms: one can choose a landing sequence $I=(I_k)_{k\leq j}$ of disjoint areas of size δ on the boundary of the square $Q=[-1,1]^2$, found in counter-clockwise order following ∂Q.

Let $A_{j,\sigma}^I(n,N)$ be the event that there exist arms from the interior to the exterior of $S_{n,N}$, and such that γ_k ends on NI_k.

Corollary 3.13. *Fix $j>0$. For any choice of I, σ, $n<N$, we have*

$$\mathbb{P}_p\big[A_\sigma^I(n,N)\big] \;\asymp\; \mathbb{P}_p\big[A_\sigma(n,N)\big].$$

3.2.2 Some specific critical arm exponents

The quasi-multiplicativity allows for the derivation of so-called *universal exponents.*

Theorem 3.14. *For every $0<k<n\leq L_p$,*

$$\mathbb{P}_p\big[A_{ocooc}(k,n)\big]\asymp(k/n)^2, \qquad \mathbb{P}_p\big[A_{oc}^{HP}(k,n)\big]\asymp k/n, \qquad \mathbb{P}_p\big[A_{oco}^{HP}(k,n)\big]\asymp(k/n)^2.$$

where $A_\sigma^{HP}(n,N)$ is the existence of j paths in $[-N,N]\times[0,N]\setminus[-n,n]\times[0,n]$ form $[-n,n]\times[0,n]$ to $([-N,N]\times[0,N])^c$.

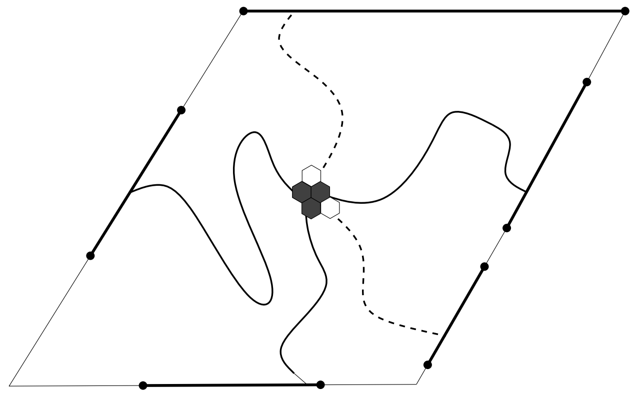

Figure 13: Only one site per rectangle can satisfy the following topological picture.

Proof. We treat the first case only, since the others are similar and actually technically easier. We only need to look at the case $k = 1$ via quasi-multiplicativity.

Let us first prove the lower bound. Fix $n < L_p$, we work in the box $[0, n]^2$ for simplicity. The same reasoning extends to the case of the box. Consider the following construction: assume there exist a horizontal crossing of $[-n, n] \times [-n/4, 0]$ and a dual horizontal crossing of $[-n, n] \times [0, n/4]$. This happens with probability bounded from below by $c > 0$ not depending on n. By conditioning on the lowest interface Γ between an open and a closed crossing of $[-n, n] \times [-n/4, n/4]$, the configuration above it is a random-cluster configuration with free boundary conditions. Let Ω be the connected component of $\Lambda_n \setminus \Gamma$ containing $[-n, n] \times \{n\}$. Assume that $[-n/4, 0] \times [-n, n] \cap \Omega$ is dual crossed horizontally, and that $[0, n/4] \times [-n, n] \cap \Omega$ is crossed horizontally. The probability of this event is once again bounded from below uniformly in n, thanks to (3.6). Note that we need a strong form of crossing probabilities in order to guarantee the existence of the last crossing since the boundary of Ω can be very rough.

Summarizing, all these events occur with probability larger than $c' > 0$. Moreover, the existence of all these crossings implies the existence of a site in $\Lambda_{n/4}$ with five arms emanating from it. The union bound implies

$$(n/4)^2 \mathbb{P}_p[A_{ocooc}(n/4)] \geq c'.$$

In order to prove an upper bound for $\mathbb{P}_p[A_{ocooc}(n)]$, recall that it suffices to show it for well-separated arms for which we choose landing sequences. Consider the event described in Fig. 13. Topologically, no two sites in Λ_n can satisfy this event simultaneously, which implies the claim. $\square$

This result has an interesting corollary:

Corollary 3.15. *Fix $p \in (0, 1)$. There exists $\alpha > 0$ such that for every $0 < k < n \leq L_p$,*

$$\mathbb{P}_p\big[A_{ocococ}(k, n)\big] \lesssim (k/n)^{2+\alpha}$$
$$\mathbb{P}_p\big[A_{ococ}(k, n)\big] \gtrsim (k/n)^{2-\alpha}.$$

The first inequality is useful since it relates to convergence to SLE(6), as mentioned before. The second one is also interesting, especially when $k = 1$, since it implies the existence of at least n^α *pivotal points*. This fact is crucial in the study of the near-critical regime, as well as the dynamical percolation, see [Gar10] and references therein.

Proof. Note that for any sequence σ, Reimer's inequality implies

$$\mathbb{P}_p\big[A_{\sigma o}(k,n)\big] \le \mathbb{P}_p\big[A_\sigma(k,n)\big] \cdot \mathbb{P}_p\big[A_o(k,n)\big].$$

The result also holds with $\{o\}$ replace by $\{c\}$. Since $\mathbb{P}_p[A_o(k,n)] \ge (k/n)^\alpha$ for some constant α, we deduce the result from the fact that $\mathbb{P}_p\big[A_{ocooc}\big] \asymp (k/n)^2$. $\square$

3.2.3 Stability of arm probabilities below the critical length

We now prove Theorem 3.8. The idea is to estimate the logarithmic derivative of arm-event probabilities in terms of the derivative of the crossing probabilities. In order to do so, we relate the probability to be pivotal for arm-events with the probability to be pivotal for crossing events.

Proof of Theorem 3.8. We treat the case of $\mathbb{P}_p(A_o(n))$ when $p > 1/2$. Recall that n is assumed to be smaller than L_p, so that RSW holds at every scale smaller than n. We will be using the fact that crossing probabilities are bounded away from 0 and 1 uniformly in $n \le L_p$. We cannot stress enough the fact that it holds as long as (and roughly speaking if and only if) $n < L_p$.

Russo's formula implies

$$\frac{d}{dp}\mathbb{P}_p\big[A_o(n)\big] \;=\; \sum_{v\in\Lambda_n} \mathbb{P}_p\big[v \text{ pivotal for } A_o(n)\big]. \tag{3.8}$$

The site v is pivotal if and only if there are four arms of alternating colors emanating from it, one of the open arm going to the origin, the other to the boundary of the box, and the two closed arms forming a circuit around the origin (see Fig 14). The event that a site v (at distance $|v|$ of the origin) is pivotal is thus included in the intersection of events $A_o(|v|/2)$, $A_o(2|v|,n)$ and the translate of $A_{ococ}(|v|/2)$ by v (see Fig 14 again). We deduce, using independence, that

$$\begin{aligned}\mathbb{P}_p\big[v \text{ pivotal for } A_o(n)\big] &\le \mathbb{P}_p\big[A_o(|v|/2)\big] \cdot \mathbb{P}_p\big[A_o(2|v|,n)\big] \cdot \mathbb{P}_p\big[A_{ococ}(|v|/2)\big] \\ &\lesssim \mathbb{P}_p\big[A_o(n)\big] \cdot \mathbb{P}_p\big[A_{ococ}(|v|/2)\big]\end{aligned}$$

where in the second line we have used Theorem 3.9 and the fact that $\mathbb{P}_p\big[A_o(|v|/2, 2|v|)\big]$ is of order 1 (use crossing estimates). Plugging this inequality into (3.8), we find

$$\frac{d}{dp}\mathbb{P}_p\big[A_o(n)\big] \;\lesssim\; \mathbb{P}_p\big[A_o(n)\big] \cdot \sum_{v\in\Lambda_n} \mathbb{P}_p\big[A_{ococ}(|v|/2)\big] \tag{3.9}$$

which integrates into

$$\log \mathbb{P}_p\big[A_o(n)\big] - \log \mathbb{P}_{1/2}\big[A_o(n)\big] \;\lesssim\; \int_{1/2}^{p} \sum_{v\in\Lambda_n} \mathbb{P}_{p'}\big[A_{ococ}(|v|/2)\big]\, dp'. \tag{3.10}$$

It remains to prove that the right-hand side is of order 1. Theorem 3.9 and Corollary 3.15 imply

$$\mathbb{P}_{p'}\big[A_{ococ}(k)\big] \lesssim (n/k)^{2-\alpha}\,\mathbb{P}_{p'}\big[A_{ococ}(n)\big],$$

for $n \leq L_p \leq L_{p'}$ (since $1/2 < p' < p$). Put into (3.10), it gives

$$\begin{aligned}\log \mathbb{P}_p\big[A_o(n)\big] - \log \mathbb{P}_{1/2}\big[A_o(n)\big] &\lesssim \int_{1/2}^{p} \left(\sum_{v\in\Lambda_n} (2n/|v|)^{2-\alpha}\,\mathbb{P}_{p'}\big[A_{ococ}(n)\big]\right) dp' \\ &\lesssim \int_{1/2}^{p} n^2 \mathbb{P}_{p'}\big[A_{ococ}(n)\big]\ dp'.\end{aligned}$$

To conclude, Russo's formula implies

$$\begin{aligned}1 &\geq \mathbb{P}_p([0,n/2]^2 \text{ is crossed}) - \mathbb{P}_{1/2}([0,n/2]^2 \text{ is crossed}) \\ &= \int_{1/2}^{p} \sum_{v\in\Lambda_{n/2}} \mathbb{P}_{p'}\big[v \text{ pivotal for } [0,n/2]^2 \text{ being crossed}\big]\ dp' \\ &\geq \int_{1/2}^{p} \frac{3n^2}{4}\mathbb{P}_{p'}\big[A_{ococ}(n)\big]\ dp',\end{aligned}$$

where we have used the fact that v is pivotal for the event $\{[0,n/2]^2$ is crossed$\}$ if there are four arms of alternating colors going to the boundary of $[0,n/2]^2$. In particular, we find the required bound

$$\log \mathbb{P}_p\big[A_o(n)\big] - \log \mathbb{P}_{1/2}\big[A_o(n)\big] \ \lesssim\ 4/3.$$

The same reasoning can be applied for any sequence σ. The main step is to get (3.9) with $\{o\}$ replaced by σ, the end of the proof being the same. In order to obtain this inequality, one harnesses a generalization of Russo's formula; we refer to Theorem 26 of [Nol08] for a complete exposition. □

4 Critical exponents for percolation exponents.

4.1 Critical arm exponents

The fact that SLE paths can be "encoded" via Brownian motions paves the way to the use of standard techniques such as stochastic calculus in order to study the properties of SLE curves. Consequently, SLEs are now fairly well understood: path properties have been derived in [RS05], their Hausdorff dimension can be computed [Bef04, Bef08a], etc. In addition to this, several critical exponents can be related to properties of the interfaces, and thus be computed using SLE.

It is easy to show, using a color-switching argument very similar to the one harnessed in Lemma 2.6, that α_σ depends only on the length of the sequence, as long as we consider bi-chromatic sequences. From now on, we set α_j to be the exponent for bi-chromatic sequences of length j and $A_j(n,m)$ for the corresponding event. By extension, we set α_1 to be the exponent of the one-arm event.

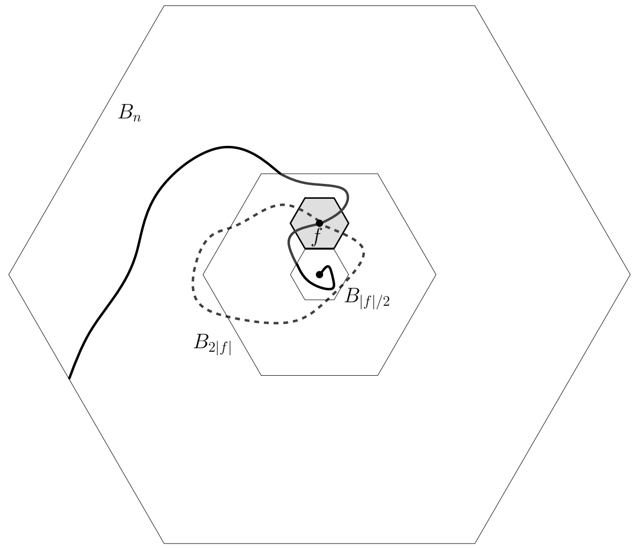

Figure 14: The event that f is pivotal for $A_o(n)$. The dotted line corresponds to a closed circuit.

Theorem 4.1 ([LSW02, SW01]). *$\alpha_1 = \dfrac{5}{48}$ and $\alpha_j = \dfrac{j^2-1}{12}$ for $j > 1$.*

The proof of this is heavily based on the use of Schramm-Loewner Evolution. We sketch the proof and we refer the reader to existing literature on the topic for details [LSW02, SW01]. The argument is two-fold. First, arm-exponents can be related to the corresponding exponents for SLE. And second, these exponents can be computed using stochastic and conformal invariance techniques. We will not describe the second step, since the computation can be found in many places in the literature already, and that it would bring us far from our main subject of interest in this review.

Lemma 4.2. *Assume that for any $R > 0$, $\mathbb{P}_{1/2}[A_j(m, Rm)]$ converges as m goes to ∞. For any $\varepsilon > 0$, there exists $R > 0$ such that for N large enough,*

$$\left| \frac{\log \mathbb{P}_{1/2}[A_j(N)]}{\log N} - \lim_{m\to\infty} \frac{\log \mathbb{P}_{1/2}[A_j(m, Rm)]}{\log R} \right| \leq \varepsilon.$$

Proof. Fix $\varepsilon > 0$. First note that it is sufficient to prove the result for N of the form R^n. Using Theorem 3.9 iteratively, there exists a universal constant $C > 1$ such that for any n,

$$\left| \log \mathbb{P}_{1/2}[A_j(R^n)] - \sum_{k=0}^{n-1} \log \mathbb{P}_{1/2}[A_j(R^k, R^{k+1})] \right| \leq n \log C. \tag{4.1}$$

Now, if $\mathbb{P}_{1/2}[A_j(m, Rm)]$ converges as m goes to ∞, then

$$\frac{1}{n} \sum_{k=0}^{n-1} \log \mathbb{P}_{1/2}[A_j(R^k, R^{k+1})] \longrightarrow \lim_{m\to\infty} \log \mathbb{P}_{1/2}[A_j(m, Rm)] \qquad \text{as } n \text{ goes to } \infty.$$

Now, let R large enough that $\log C / \log R \leq \varepsilon/2$. The statement follows readily by dividing (4.1) by $n \log R$ and plugging the previous limit into it. $\square$

Proof of Theorem 4.1. The previous lemma implies that in order to compute arm-exponents, it suffices to show that $\lim_{m\to\infty} \mathbb{P}_{1/2}[A_j(m, Rm)]$ exists and to compute its limit when R goes to ∞.

Let us first deal with $j = 1$. Let Λ be the box centered at the origin with hexagonal shape and edge-length 1. Consider a exploration process in the discrete domain $(R\Lambda)_\delta$ defined as follows:

- It starts from the corner R.
- Inside the domain, the exploration γ takes on the left when it faces an open hexagonal, and on the right otherwise.
- On the boundary of $(R\Lambda)_\delta \setminus \gamma$, γ carries on in the connected component of $(R\Lambda) \setminus \gamma$ containing the origin (it always bumps in such a way that it can reach the origin eventually).

The existence of one open path from $\partial\Lambda$ to $\partial(R\Lambda)$ corresponds to the fact that the exploration does not close any counterclockwise loop before reaching Λ.

It can be shown that the exploration γ converges to a so-called radial SLE_6 [LSW02], so that the probability of $\mathbb{P}_{1/2}[A_1(m, Rm)]$ converges to the probability that a SLE_6 does not close counterclockwise loops before reaching Λ (denote this probability by $P[A_1^{\mathrm{SLE}_6}(1, R)]$). This quantity has been computed in [LSW02] and it was proved that

$$\frac{\log P[A_1^{\mathrm{SLE}_6}(1, R)]}{\log R} \to -5/48 \qquad \text{as } R \text{ goes to } \infty,$$

thus concluding the proof in this case.

Let us now deal with α_j for $j > 1$ even. Let us consider the case of $\sigma = ococ..c$ with length j, since all the polychromatic exponents with the same number of colors are equal. The corresponding event for the exploration process is that it does $j + 1$ crossings of the annulus $(R\Lambda) \setminus \Lambda$. The probability of this event $A_j^{\mathrm{SLE}_6}(1, R)$ for SLE_6 was also estimated in [LSW01a, LSW01b] and it was proved that

$$\frac{\log P[A_j^{\mathrm{SLE}_6}(1, R)]}{\log R} \to -(j^2 - 1)/12 \qquad \text{as } R \text{ goes to } \infty,$$

thus concluding the proof int his case. The case of j odd can also be handled. Let us mention that the previous paragraphs constitute a sketch of proof only, and the real story is fairly more complicated, we refer to [LSW02, SW01] (or [Wer40, Wer07]) and the references therein for a full proof. $\square$

Fractal properties of critical percolation. These arm exponents can be used to measure the size (Hausdorff dimension) of various sets describing percolation clusters at criticality. A set S is said to be fractal of dimension d_S if the density of points in S within a box of size n decays as n^{-x_S}, with $x_S = 2 - d_S$ in two dimensions. The codimension x_S is related to arm exponents in many cases:

- The 1-arm exponent is related to the existence of long connections, from the center of a box to its boundary. It measures the size of big clusters, like the incipient infinite cluster (IIC) as defined by Kesten [Kes86], which scales as $n^{2-5/48} = n^{91/48}$.

- The monochromatic 2-arm exponent describes the size of the backbone of a cluster. The fact that this *backbone* is much thinner than the cluster itself was used by Kesten [Kes86] to prove that the random walk on the IIC is sub-diffusive (while it has been proved to converge toward a Brownian Motion on a supercritical infinite cluster, see [BB07, MP07] for instance).

- The polychromatic 2-arm exponent is related to the boundary points of big clusters, which are thus of fractal dimension $2 - \alpha_2 = 7/4$.

- The 3-arm exponent concerns the external (accessible) perimeter of a cluster, which is the accessible part of the boundary: one excludes fjords which are connected to the exterior only by 1-site wide passages. The dimension of this frontier is $2 - \alpha_3 = 4/3$. These last two exponents can be observed on random interfaces, numerically and in real-life experiments as well (see [DSB03, SRG85] for instance).

- The 4-arm exponent with alternating colors counts the pivotal sites (see the next section for more information). Its dimension is $2 - \alpha_4 = 3/4$. This exponent is crucial is the study of noise-sensitivity of percolation.

4.2 Near-critical exponents

It is now time to relate arm-exponents to near-critical ones. The goal of this section is to prove the following:

Theorem 4.3 (Kesten [Kes87])**.** *For every $p > 1/2$, we have*

$$(p - 1/2)\ L_p^2\ \mathbb{P}_{1/2}[A_{ococ}(L_p)] \asymp 1 \quad \textit{and} \quad \theta(p) \asymp \mathbb{P}_{1/2}[A_o(L_p)].$$

Theorems 4.1 and 4.3 imply Theorem 1.5 easily:

Proof of Theorem 1.5. Since $\mathbb{P}_{1/2}[A_{ococ}(n)] = n^{-5/4+o(1)}$ and $\mathbb{P}_{1/2}[A_o(n)] = n^{-5/48+o(1)}$, we deduce that $\theta(p) = (p - 1/2)^{5/36+o(1)}$, which is exactly the claim of Theorem 1.5. □

More generally, if we only assume the existence of α_1 and α_4 such that $\mathbb{P}_{1/2}[A_{ococ}(n)] = n^{-\alpha_4+o(1)}$ and $\mathbb{P}_{1/2}[A_o(n)] = n^{-\alpha_1+o(1)}$, the previous statement implies the existence of ν and β such that $L_p = (p - 1/2)^{-\nu+o(1)}$ and $\theta(p) = (p - 1/2)^{\beta+o(1)}$ and moreover

$$(2 - \alpha_4)\nu = 1 \qquad \text{and} \qquad \beta = \alpha_1\nu.$$

A connection between different critical exponents is called a scaling relation. In this case, they are called Kesten's relations.

Proof. Let us deal with the first equality. We aim to apply Russo's formula to the event A that $[0, L_p]^2$ is crossed. On the one hand, with the definition of L_p, $\mathbb{P}_p(A)$ equals $1 - \varepsilon$. On the other hand, one can check that $\mathbb{P}_{1/2}(A) = 1/2$. Moreover, a site is pivotal for A if and only if there are four alternating arms starting from it and going to the boundary of $[0, L_p]^2$. Except for points near the boundary, this occurs with $\mathbb{P}_{p'}$-probability of order $\mathbb{P}_{p'}[A_{ococ}(L_p)]$ for every $p' \in (1/2, p)$. Therefore, if we neglect the effect of the boundary, we obtain

$$1 \asymp \mathbb{P}_p(A) - \mathbb{P}_{1/2}(A) \asymp \int_{1/2}^{p} L_p^2 \, \mathbb{P}_{p'}[A_{ococ}(L_p)] dp'.$$

There are several ways to deal with the boundary effect. One can control the probability to be pivotal for boundary points separately, or one can do the following: for the lower bound, it is sufficient to count points far from the boundary like we did in the previous section, for the upper bound, one can work with the event that the torus of size n contains a circuit with non-trivial homotopy. There, the probability to be pivotal is the same for every site, and is smaller than $\mathbb{P}_p[A_{ococ}(n)]$.

Theorem 3.8 implies $\mathbb{P}_{p'}[A_{ococ}(L_p)] \asymp \mathbb{P}_{1/2}[A_{ococ}(L_p)]$ for every $p' \in (1/2, p)$, so that

$$1 \ \asymp \ (p - 1/2) \ L_p^2 \ \mathbb{P}_{1/2}[A_{ococ}(L_p)].$$

We now turn to the second relation. On the one hand, it is straightforward that $\theta(p) = \mathbb{P}_p(0 \leftrightarrow \infty) \leq \mathbb{P}_p(0 \leftrightarrow \partial[0, L_p]^2) = \mathbb{P}_p[A_o(L_p)]$. On the other hand, Proposition 3.1 implies that $\mathbb{P}_p[A_o(L_p, n)] \geq c$ where $c > 0$ is uniform in n and p. Using the quasi-multiplicativity, we can deduce

$$\mathbb{P}_p[A_o(n)] \ \geq \ \mathbb{P}_p[A_o(L_p)]\mathbb{P}_p[A_o(L_p, n)] \ \geq \ c \cdot \mathbb{P}_p[A_o(L_p)]$$

uniformly in n and p. Once again, Theorem 3.8 implies $\mathbb{P}_p[A_o(L_p)] \asymp \mathbb{P}_{1/2}[A_o(L_p)]$, which in turn implies the theorem by noticing that $\theta(p) = \lim_{n\to\infty} \mathbb{P}_p[A_o(n)]$. □

5 A few open questions

Percolation on the triangular lattice Percolation on the triangular lattice is now very well understood. Nevertheless, several questions remain open. We selected two of them.

We know the behavior of most thermodynamical quantities (the cluster density θ, the truncated mean-cluster size $\xi(p) = (p - 1/2)^{-\nu + o(1)}$ as $p \to p_c$, the two-point functions $\mathbb{P}_{1/2}(0 \leftrightarrow x) = |x|^{2-d-\eta+o(1)}$ as $x \to \infty$ and many others). Nevertheless, the behavior of the following fundamental quantity remains unproved:

Question 1. *Prove that the mean number of clusters per site $\kappa(p) = \mathbb{E}_p(|C|^{-1})$ behaves like $|1/2 - p|^{2+\alpha+o(1)}$, where C is the cluster at the origin and $\alpha = -2/3$.*

Interestingly, the critical exponent for $j \neq 1$ disjoint arms of the same color is not equal to the polychromatic arms exponent [BN10]. A natural open question would be to compute these exponents:

Question 2. *Compute the monochromatic exponents.*

Percolation on other graphs Conformal invariance of percolation has been proved only on the triangular lattice. In physics, it is conjectured that the scaling limit of percolation should be universal, meaning that it should not depend on the lattice. For instance, interfaces of bond-percolation on the square lattice at criticality (when the bond-parameter is 1/2) should also converge to SLE(6).

Question 3. *Prove conformal invariance for critical percolation on another planar lattice.*

For general graphs, the question of embedding the graph becomes crucial. Indeed, if one embeds the square lattice by gluing long rectangles, then the model will not be rotationally invariant. We refer to [Bef08b] for further details on the subject.

Question 4. *For a general lattice, how to construct a natural embedding on which percolation is conformally invariant?*

In order to understand universality, a natural class of lattices to start with is the class of lattices where box crossings can be proved. Note that proofs of RSW often invoke some symmetry (rotational invariance for instance). A proof valid for lattices without any symmetry would be of great importance:

Question 5. *Prove RSW for critical percolation on all planar lattices.*

Let us mention that an important step towards the case of general lattices was accomplished in [GM11a, GM11b], where box crossings are proved to exist with positive probability for critical anisotropic percolation models on the hexagonal, triangular and square lattices.

Percolation in high dimension is well understood (see *e.g.* [HS94]), thanks to the so-called triangular condition and lace-expansion techniques associated to it. In intermediate dimensions, the critical phase is not understood. Of course, one of the main conjectures in probability is to prove that $\theta(p_c) = 0$ for bond-percolation on $\mathbb{Z}^3$. Even weakening of this conjecture seems to be very hard. For instance, the same question on the "sandwich" $\mathbb{Z}^2 \times \{0, 1\}$ is still open:

Question 6. *Prove that $\theta(p_c) = 0$ on $\mathbb{Z}^2 \times \{0, 1\}$.*

Other two-dimensional models of statistical physics Conformal invariance (for instance of crossing probabilities) is not restricted to percolation (see [Smi06, Smi10] and references therein). It should hold for a wide class of two-dimensional lattice models at criticality. Among natural generalizations of percolation, we mention the class of random-cluster models and of loop $O(n)$-models (including the Ising model and the self-avoiding walk). The only three models in this family for which conformal invariance has been proved are the Ising model, the $q = 2$-random cluster model (which is a geometric representation of the Ising model), and the uniform spanning tree (the '$q = 0$'-random cluster model).

Question 7. *Prove conformal invariance of another two-dimensional critical lattice model.*

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