RSW and Box-Crossing Property for Planar Percolation

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This article provides a brief summary on recent advances on the so-called Russo-Seymour-Welsh (RSW) Theory and its applications to the study of planar percolation models. In particular, we introduce a few properties of percolation models and discuss their connections.

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1. Percolation models

A percolation measure on a graph G is given by a measure $\mathbb{P}$ on subgraphs $\omega = (V_\omega, E_\omega)$ of $G = (V_G, E_G)$ with vertex set $V_\omega = V_G$ and edge set $E_\omega \subset E_G$. In this note, we will focus our attention on percolation measures on $\mathbb{Z}^2$ which are invariant under the isometries preserving $\mathbb{Z}^2$ (we call such a percolation measure *invariant*). We will also assume that the measures satisfy the FKG inequality:

(FKG) *For any two increasing events A and B (here increasing means that if $\omega \in A$ and $E_{\omega'}$ contains E_ω, then $\omega' \in A$),*

$$\mathbb{P}[A \cap B] \geq \mathbb{P}[A]\mathbb{P}[B].$$

The simplest such model is provided by Bernoulli percolation [5] on $\mathbb{Z}^2$ for which each edge of the square lattice $\mathbb{Z}^2$ is in ω with probability $p \in [0,1]$, independently of the other edges. In the last sixty years, more complicated percolation models emerged. Some percolation processes were introduced as direct generalizations of Bernoulli percolation intended to test the physical concept of universality. Namely, these models have different microscopic definitions (for example, they can be defined on different graphs) but their macroscopic behavior is expected to be the same as Bernoulli percolation. Yet one may imagine more general percolation models exhibiting very different behaviors. In particular, some dependent percolation models were introduced as geometric representations of quantum and classical lattice spin models.

In order to provide a second example of percolation model, let us define the Fortuin-Kasteleyn percolation (for details, see the book of Grimmett [13]). Fix $p \in [0,1]$ and $q > 0$. Let $\mathbb{P}_{p,q}$ be the weak limit of the measures $\mathbb{P}_{p,q,n}$ on $[-n,n]^2$ defined by

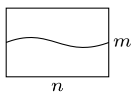
$$\mathbb{P}_{p,q,n}[\{\omega\}] = \frac{p^{|E_\omega|}(1-p)^{|E_G \setminus E_\omega|} q^{k(\omega)}}{Z^0(n,p,q)},$$

where $Z^0(n,p,q)$ is a normalizing constant and $k(\omega)$ is equal to the number of connected components in ω. For integer $q \geq 2$, the model is coupled to the q-state Potts model, which makes it a very interesting percolation model from the point of view of statistical physics. For $q = 1$, one recognizes Bernoulli percolation. Let us mention that for $q \geq 1$, $\mathbb{P}_{p,q}$ is an invariant percolation satisfying (FKG).

2. A few properties of percolation models

Percolation models exhibit a very rich behavior which is determined by the geometry of the large connected components of ω. To achieve a good understanding of these large connected components *for planar models*, one may study the so-called *crossing probabilities*. To this end, Russo[17] and Seymour-Welsh[18] proved a theorem – often referred to as the RSW theorem or simply (RSW) – for Bernoulli percolation which quickly became the main tool to study the *critical regime*. This result was later proved in more and more different and elegant ways[6–8,19], each one based on delicate properties of the model, including independence, translational invariance and some symmetries.

The zoo of new percolation models called for a more general RSW theorem applicable to a wider class of percolation models, but this more general version was lacking and the understanding of the critical phase remained limited to Bernoulli percolation. The object of this note is to briefly discuss some recent progress in generalizing (RSW) to percolation models with dependency (with the Fortuin-Kasteleyn percolation as an illustrating example in mind). Let us start by stating what is meant by the (RSW) property. For $n, m \geq 1$, consider the event $\mathcal{C}_h(n,m)$ (illustrated below) that the rectangle $[0,n] \times [0,m]$ contains a path of edges in ω from its left to its right side.

$\mathcal{C}_h(n,m) =$ 

We say that a percolation measure $\mathbb{P}$ on $\mathbb{Z}^2$ satisfies the property (RSW) if the following holds:

(RSW) *Let* $0 < \alpha < \beta < \infty$, *and* $x > 0$. *Then, there exists* $y = y(\alpha, \beta, x) > 0$ *such that for any* $n \geq 1$,

$$\mathbb{P}[\mathcal{C}_h(\alpha n, n)] \geq x \quad \Longrightarrow \quad \mathbb{P}[\mathcal{C}_h(\beta n, n)] \geq y.$$

In words, the property (RSW) can be interpreted as follows: a lower bound on the crossing probability for a rectangle of aspect ratio α implies a lower bound for a rectangle of larger aspect ratio β. For measures satisfying (FKG), the difficulty in proving (RSW) concerns the case $0 < \alpha \leq 1 < \beta$, the other cases being easy consequences of (FKG).

This seemingly tautological result has tremendous applications in percolation theory. Our goal is to discuss several versions of this property and to describe some of its applications. In particular, we wish to highlight the connection to the following two properties (which cannot hold at the same time):

(EXP) *There exists $c > 0$ such that for any $n \geq 1$,*

$$\mathbb{P}[0 \leftrightarrow \mathbb{Z}^2 \setminus [-n,n]^2, 0 \not\leftrightarrow \infty] \leq e^{-cn}.$$

(BCP) *For any $\rho > 0$, there exists $c = c(\rho) > 0$ such that for any $n \geq 1$,*

$$c \leq \mathbb{P}[\mathcal{C}_h(\rho n, n)] \leq 1 - c.$$

The first property, sometimes referred to as *exponential decay*, is typical of *non-critical* models. Many important results follow from (EXP): $\mathcal{C}^\infty$-regularity of the free energy, Ornstein-Zernike estimates, exponential decay of the volume of connected components, fast mixing, etc.

The second property, called the *box-crossing-property*, is typical of *critical* models. Alone, it brings little information on the model, but it can be combined with the following *mixing property*:

(MIX) *There exist $c, C \in (0,\infty)$ such that for any $n \geq 1$ and for any events A and B depending respectively on edges in $[-n,n]^2$ and outside $[-2n,2n]^2$,*

$$c\,\mathbb{P}[A]\mathbb{P}[B] \leq \mathbb{P}[A \cap B] \leq C\,\mathbb{P}[A]\mathbb{P}[B].$$

Note that Bernoulli percolation satisfies a much stronger statement than (MIX). Indeed, the independence between the status of the edges implies that $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$ if the events are chosen as above. Nevertheless, for many applications the property (MIX) is sufficient.

The combination of (BCP) and (MIX) leads to an anthology of results: polynomial bounds on connection probabilities, fractal properties of the critical phase, scaling relations[16], inequalities on critical exponents, existence of sub-sequential scaling limits when combined to Aizenman and Burchard's result[2,14]. We do not wish to spend too much time on the applications of these properties, but it is fair to say that *almost all* results on planar Bernoulli or dependent percolation models involve these properties – (EXP), (BCP) and (MIX) – one way or the other.

3. The original RSW theory for Bernoulli percolation

Russo[17] and Seymour-Welsh[18] proved the following result.

Theorem 3.1. *Let $p \in [0,1]$, the Bernoulli percolation measure $\mathbb{P}_p$ on $\mathbb{Z}^2$ with parameter p satisfies* (RSW).

In order to illustrate the importance of (RSW) and its connection to (EXP) and (BCP), let us describe one of the most impressive applications of the previous theorem. Let $p_c \in [0,1]$ be the critical parameter of Bernoulli percolation defined as the supremum of the p for which all connected components of ω are finite $\mathbb{P}_p$-almost surely. In 1980, Kesten[15] proved the following result.

Theorem 3.2. *For any $p \neq p_c$, the Bernoulli percolation measure $\mathbb{P}_p$ satisfies* (EXP) *while at p_c it satisfies* (BCP). *Furthermore, p_c is equal to* $1/2$.

The previous result is called sharpness of the phase transition. In physics terms, it means that the model has a finite correlation length except at the critical point. Note that it establishes a non trivial fact: that if (BCP) is not satisfied then it means that (EXP) is.

Let us highlight very briefly where (RSW) is used. Standard renormalization arguments show that there exists a constant $\epsilon > 0$ such that the following holds. If the probability $\mathbb{P}_p[\mathcal{C}_h(N/2, N)]$ of crossing a rectangle of aspect ratio 2 in the easy direction is smaller than ϵ for some value of N, then the probability of being connected to distance n decays exponentially fast in n. Therefore, for (EXP) not to be satisfied, one must have

$$\inf_{n\geq 2} \mathbb{P}_p[\mathcal{C}_h(n/2, n)] \geq \epsilon.$$

Then, (RSW) implies that $\inf_{n\geq 1} \mathbb{P}_p[\mathcal{C}_h(\rho n, n)] > 0$ for any $\rho > 0$.

On the other hand, if the probability $\mathbb{P}_p[\mathcal{C}_h(2N, N)]$ of crossing a rectangle of aspect ratio 2 in the hard direction is larger than $1 - \epsilon$ for a certain value of N, then the probability of being connected to distance n but not to infinity decays exponentially fast in n. Therefore, for (EXP) not to be satisfied, one must have

$$\sup_{n\geq 1} \mathbb{P}_p[\mathcal{C}_h(2n, n)] \leq 1 - \epsilon.$$

Then, (RSW) can in fact be used to show that $\sup_{n\geq 1} \mathbb{P}_p[\mathcal{C}_h(\rho n, n)] < 1$ for any $\rho > 0$. In conclusion, at each p, if (EXP) does not hold, then (BCP) is satisfied.

Furthermore, the two previous displayed equations implies that the set of values of p for which (BCP) holds is of the form $[p_1, p_2]$ with $p_1 \leq p_2$ (since it is the intersection on n of closed sets). Kesten's proof then consists in showing that p_1 must be equal to p_2, and that the only possible value for these two parameters is p_c. A byproduct of this last fact is that p_c coincides with the *self-dual point* of the model which is equal to $1/2$ for Bernoulli percolation. Many modern proofs of sharpness of the phase transition for more general models follow the same global strategy.

4. A weak RSW property and its connection to (RSW)

The property (RSW) is unfortunately not proved for all invariant percolation satisfying (FKG). In order to circumvent this difficulty, Bollobás and Riordan[6] introduced the following weaker property

$$(\mathbf{wRSW}) \qquad \inf_{n\geq 1} \mathbb{P}[\mathcal{C}_h(n, n)] > 0 \quad \Longrightarrow \quad \limsup_{n\geq 1} \mathbb{P}[\mathcal{C}_h(2n, n)] > 0$$

in the slightly different context of a Bernoulli percolation defined in the continuum (the model is called Voronoi percolation) to compute its critical point. The property is weaker than (RSW) for two reasons. First, it implies a lower bound for infinitely many scales n only. Second, the assumption on the left requires a uniform lower bound on the crossing probabilities *for squares* which is holding at *all* scales n instead of just one.

Recently, Tassion[19] proved that (wRSW) holds for a large class of percolation models.

Theorem 4.1. *An invariant percolation $\mathbb{P}$ satisfying* (FKG) *satisfies* (wRSW)*.*

For $q \geq 1$, $\mathbb{P}_{p,q}$ is an invariant percolation satisfying (FKG) and therefore by the previous theorem (wRSW). This property was used[10] to prove that $\mathbb{P}_{p,q}$ satisfies (EXP) except if $p = p_c(q)$, where $p_c(q)$ is defined as the supremum of the p for which all connected components of ω are finite $\mathbb{P}_{p,q}$-almost surely. While this result was already proved[4] a few years ago, this new approach seems more robust and could be applied to a large class of percolation models. Note that there is no mention of (BCP) at $p_c(q)$ and in this sense (wRSW) is weaker than (RSW).

5. (RSW) for measures satisfying (MIX)

The property (wRSW) has limited applications to the study of $\mathbb{P}$. Nevertheless, the ideas developed by Tassion[19] to prove Theorem 4.1 can be combined[1] with (MIX) to show the following result.

Theorem 5.1. *If $\mathbb{P}$ is invariant and satisfies* (FKG) *and* (MIX)*, then*

$$\inf_{n\geq 2} \mathbb{P}[\mathcal{C}_h(n/2,n)] > 0 \quad \Longrightarrow \quad \inf_{n\geq 1} \mathbb{P}[\mathcal{C}_h(2n,n)] > 0.$$

We still have one constraint of (wRSW) remaining, namely that we need to start with an estimate at all scales, but this time one can prove a uniform bound on crossing probabilities which is true for *all* scales n.

The previous result should be taken with a grain of salt. For models with dependencies, proving (MIX) is usually based on a stronger version of (RSW) that we discuss below. Nevertheless, the understanding of some Bernoulli-type percolation models with very weak dependencies that automatically satisfy (MIX) – such as Voronoi or continuum percolation[1] – has progressed greatly thanks to this RSW-type results.

6. A strong (BCP) replacing (BCP) and (MIX)

General percolation measures do not satisfy *a priori* the mixing property (MIX) and may have long-range dependencies. For these measures, (MIX) is not a straightforward consequence of the definition (actually in general, it does not hold for all choices of parameters) and the property (BCP) alone is not really useful. For example, consider the percolation process which is empty with probability 1/2 and the full $\mathbb{Z}^2$ otherwise. This model satisfies (BCP) but has very different features than critical Bernoulli percolation.

To avoid such behaviors for dependent models, we consider the following property which is stronger than (BCP). For a set E, let $\mathcal{F}_E$ be the σ-algebra generated by the random variables $(\omega_e : e \in E)$ (in other words it is the set of events measurable in terms of edges in E only).

(sBCP) *For every $\rho > 0$, there exists $c = c(\rho) > 0$ such that for every $n \geq 1$,*

$$c \leq \mathbb{P}\big[\mathcal{C}_h(\rho n, n)\big|\mathcal{F}_{\mathbb{Z}^2\setminus[-n,(\rho+1)n]\times[-n,2n]}\big] \leq 1 - c \quad \mathbb{P} - \text{almost surely.}$$

In words, this condition yields that whatever the state of the edges at distance n of the rectangle, the conditional probability of the rectangle to be crossed is between c and $1 - c$. This property obviously implies (BCP). Actually, a model which satisfies (sBCB) shares similar features with critical Bernoulli percolation (existence of macroscopic clusters at every scale, polynomial bounds on connection probabilities, existence of interfaces, to name a few).

This property seems much more difficult to prove than the previous ones since it requires the understanding of the law of the edges in the rectangle *conditioned* on the process outside of the rectangle. Nevertheless, a large class of dependent percolation models satisfy the so-called *domain Markov property* (which is of big help here), and one may use this property to prove (sBCB) using a renormalization scheme. For instance, one may obtain [11] the following result (which was proved [9] before for $q = 2$):

Theorem 6.1. *For any $q \in [1, 4]$ and $p = p_c(q)$, the Fortuin-Kasteleyn measure $\mathbb{P}_{p,q}$ satisfies* (sBCB).

Note that Baxter conjectured [3] that the property is not satisfied for $q > 4$ and $p = p_c(q)$. Let us mention that the previous theorem implies (MIX) for the Fortuin-Kasteleyn percolation with parameters $q \in [1, 4]$ and $p_c(q)$, see [12].

7. Two conjectures

We would like to finish by mentioning two conjectures that we consider of interest. The first one concerns the assumption that the model defined on $\mathbb{Z}^2$ is invariant under the rotation by an angle of $\pi/2$. Currently, all proofs involve the use of rotational symmetry in a crucial way. We think that getting rid of this assumption is very important from the conceptual point of view.

Conjecture 7.1. *Property* (RSW) *holds for any Bernoulli-percolation measure $\mathbb{P}_p$ of parameter p on a quasi-periodic planar lattice $\mathbb{L}$.*

The constraint that the measure satisfies (FKG) is very natural. On the other hand, one would ideally prefer not to use (MIX) to prove (RSW). Therefore, we propose the following conjecture.

Conjecture 7.2. *If $\mathbb{P}$ is an invariant percolation on $\mathbb{Z}^2$ satisfying* (FKG)*, then $\mathbb{P}$ satisfies* (RSW).

References

1. D. Ahlberg, V. Tassion, A. Teixeira, *Sharpness of the phase transition for continuum percolation*, In preparation (2016).
2. M. Aizenman, and A. Burchard, *Hölder regularity and dimension bounds for random curves*, Duke Mathematical Journal **99** (1999), no. 3, 419–453.
3. R. Baxter, *Potts model at the critical temperature*, Journal of Physics C: Solid State Physics **6** (1973), no. 23, L445.
4. V. Beffara and H. Duminil-Copin, *The self-dual point of the two-dimensional random-cluster model is critical for* $q \geq 1$, Probab. Theory Related Fields **153** (2012), no. 3-4, 511–542.
5. S. R. Broadbent and J. M. Hammersley, *Percolation processes. I. Crystals and mazes*, Proc. Cambridge Philos. Soc. **53** (1957), 629–641.
6. B. Bollobás and O. Riordan, *The critical probability for random Voronoi percolation in the plane is 1/2*, Probab. Theory Related Fields **136** (2006), no. 3, 417–468.
7. B. Bollobás and O. Riordan, *A short proof of the Harris-Kesten theorem*, Bull. London Math. Soc. **38** (2006), no. 3, 470–484.
8. B. Bollobás and O. Riordan, *Percolation on self-dual polygon configurations*, An irregular mind, Bolyai Soc. Math. Stud., vol. 21, János Bolyai Math. Soc., Budapest, 2010, pp. 131–217.
9. H. Duminil-Copin, C. Hongler, and P. Nolin, *Connection probabilities and RSW-type bounds for the two-dimensional FK Ising model*, Comm. Pure Appl. Math. **64** (2011), no. 9, 1165–1198.
10. H. Duminil-Copin, A. Raoufi, and V. Tassion, *A new computation of the critical point for the planar random-cluster model with* $q \geq 1$, in preparation, 2016.
11. H. Duminil-Copin, V. Sidoravicius, and V. Tassion, *Continuity of the phase transition for planar random-cluster and Potts models with* $1 \leq q \leq 4$, arXiv:1505.04159, 2015.
12. H. Duminil-Copin, *Parafermionic observables and their applications to planar statistical physics models*, Ensaios Matematicos, vol. 25, Brazilian Mathematical Society, 2013.
13. G. Grimmett, *The random-cluster model*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 333, Springer-Verlag, Berlin, 2006.
14. A. Kemppainen, and S. Smirnov, *Random curves, scaling limits and Loewner evolutions*, arXiv:1212.6215.
15. H. Kesten, *The critical probability of bond percolation on the square lattice equals* $\frac{1}{2}$, Comm. Math. Phys. **74** (1980), no. 1, 41–59.
16. H. Kesten, *Scaling relations for* $2D$*-percolation*, Comm. Math. Phys. **109** (1987), no. 1, 109–156.
17. L. Russo, *A note on percolation*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **43** (1978), no. 1, 39–48.
18. P. D. Seymour and D. J. A. Welsh, *Percolation probabilities on the square lattice*, Ann. Discrete Math. **3** (1978), 227–245, Advances in graph theory (Cambridge Combinatorial Conf., Trinity College, Cambridge, 1977).
19. V. Tassion, *Crossing probabilities for Voronoi percolation*, To appear in Annals of Probability.