Connection probabilities and RSW-type bounds for the two-dimensional FK Ising model

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Abstract

We prove Russo-Seymour-Welsh-type uniform bounds on crossing probabilities for the FK Ising (FK percolation with cluster weight $q = 2$) model at criticality, independent of the boundary conditions. Our proof relies mainly on Smirnov's fermionic observable for the FK Ising model [34], which allows us to get precise estimates on boundary connection probabilities. We stay in a discrete setting, in particular we do not make use of any continuum limit, and our result can be used to derive directly several noteworthy properties – including some new ones – among which the fact that there is no infinite cluster at criticality, tightness properties for the interfaces, and the existence of several critical exponents, in particular the half-plane one-arm exponent. Such crossing bounds are also instrumental for important applications such as constructing the scaling limit of the Ising spin field [7], and deriving polynomial bounds for the mixing time of the Glauber dynamics at criticality [26].

1 Introduction

It is fair to say that the two-dimensional Ising model has a very particular historical importance in statistical mechanics. This model of ferromagnetism has been the first natural model where the existence of a phase transition, a property common to many statistical mechanics models, has been proved, in Peierls' 1936 work [29]. In a series of seminal papers (particularly [28]), Onsager computed several macroscopic quantities associated with this model. Since then, the Ising model has attracted a lot of attention, and it has probably been one of the most studied models, giving birth to an extensive literature, both mathematical and physical.

A few decades later, in 1969, Fortuin and Kasteleyn introduced a dependent percolation model, for which the probability of a configuration is weighted by the number of clusters (connected components) that it contains. This percolation representation turned out to be extremely powerful to study the Ising model, and by now it has become known as the *random-cluster* model, or the *Fortuin-Kasteleyn percolation – FK percolation* for short. Recall that on a finite graph

G, the FK percolation process with parameters p, q is obtained by assigning to each configuration ω a probability proportional to

$$p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega)},$$

where $o(\omega)$, $c(\omega)$, and $k(\omega)$ denote respectively the number of open edges, closed edges, and connected components in ω. The definition of the model also involves *boundary conditions*, encoding connections taking place outside G. The boundary conditions can be seen as a set of additional edges between sites on the outer boundary, and they will play a central role in this article. The precise setup that we consider in this paper is presented in Section 2.

For the specific value $q = 2$, FK percolation provides a *geometric representation* of the Ising model [11]: there exists a coupling between the two models, whose general form is known as the *Edwards-Sokal coupling* [10]. In the present article, we restrict ourselves to this value $q = 2$, and we call this model the *FK Ising model*. We also stick to the square lattice $\mathbb{Z}^2$ – or subgraphs of it – though our arguments could possibly be carried out in the more general context of isoradial graphs, as in [9]. Note that our results are stated for the FK representation, but that the aforementioned coupling then allows one to translate them into results for the Ising model itself. For instance, as first noticed in [11], 2-point connection probabilities for the FK Ising model correspond via this coupling to 2-spin correlation functions for the Ising model.

For the value $q = 2$ and $\mathbb{Z}^2$ as an underlying graph, the model features a phase transition – in the infinite-volume limit – at the critical and self-dual point $p_c = p_{\mathrm{sd}} = \frac{\sqrt{2}}{1+\sqrt{2}}$: for $p < p_c$, there is a.s. no infinite open cluster, while for $p > p_c$, there is a.s. a unique one. These two regimes, known as sub-critical and super-critical, have totally different macroscopic behaviors. Between them lies a very interesting and rich regime, the critical regime, corresponding to the value $p = p_c$. Its behavior is intimately related to the behavior of the model through its phase transition, as indicated in particular by the scaling theory.

In this paper, we prove lower and upper bounds for crossing probabilities in rectangles of bounded aspect ratio. These bounds are uniform in the size of the rectangles and in the boundary conditions, and they are analogues for the FK Ising model to the celebrated Russo-Seymour-Welsh bounds for percolation [31, 32]. Formally, we consider *rectangles* R of the form $[\![0, n]\!] \times [\![0, m]\!]$ for $n, m > 0$, and translations of them – here and in the following, $[\![\cdot, \cdot]\!]$ is the integer interval between the two (real) end-points, *i.e.* the interval $[\cdot, \cdot] \cap \mathbb{Z}$. We denote by $\mathcal{C}_v(R)$ the event that there exists a *vertical crossing* in R, a path from the bottom side $[\![0, n]\!] \times \{0\}$ to the top side $[\![0, n]\!] \times \{m\}$ that consists only of open edges. Our main result is the following:

Theorem 1.1 (RSW-type crossing bounds) *Let $0 < \beta_1 < \beta_2$. There exist two constants $0 < c_- \leq c_+ < 1$ (depending only on β_1 and β_2) such that for any rectangle R with side lengths n and $m \in [\![\beta_1 n, \beta_2 n]\!]$ (*i.e.* with aspect ratio bounded away from 0 and ∞ by β_1 and β_2), one has*

$$c_- \leq \mathbb{P}^{\xi}_{p_{\mathrm{sd}},2,R}(\mathcal{C}_v(R)) \leq c_+$$

for any *boundary conditions* ξ*, where* $\mathbb{P}^{\xi}_{p_{\rm sd},2,R}$ *denotes the FK measure on* R *with parameters* $(p,q)=(p_{\rm sd},2)$ *and boundary conditions* ξ*.*

These bounds are in some sense a first glimpse of scale invariance. It was widely believed in the physics literature that the FK Ising model at criticality, *i.e.* for $p=p_c$, should possess a strong property of conformal invariance in the scaling limit [4, 5, 30]. A precise mathematical meaning was recently established by Smirnov in a groundbreaking paper [34]. One of the main tools there is the so-called *preholomorphic fermionic observable*, a complex observable that makes holomorphicity appear on the discrete level. This property can then be used to take continuum limits and describe the scaling limits so-obtained.

Our proof mostly relies on Smirnov's observable. More precisely, it is based on precise estimates on connection probabilities for boundary vertices, that allow us to use a second-moment method on the number of pairs of connected sites. For that, we use Smirnov's observable to reveal some harmonicity on the discrete level, which enables us to express macroscopic quantities such as connection probabilities in terms of discrete harmonic measures. Note in addition that other recent works (*e.g.* [3]) also suggest that this complex observable is a relevant way to look at FK percolation, both for $q=2$ and for other values of q. We would like to stress that our argument stays completely in a discrete setting, using essentially elementary combinatorial tools: in particular, we do not make use of any continuum limits [35].

Crossing bounds turned out to be instrumental to study the percolation model at and near its phase transition – for instance to derive Kesten's *scaling relations* [18], that link the main macroscopic observables, such as the density of the infinite cluster and the characteristic length. These bounds are also useful to study variations of percolation, in particular for models exhibiting a self-organized critical behavior. We thus expect Theorem 1.1 to be of particular interest to study the FK Ising model at and near criticality.

This theorem allows us to derive easily several noteworthy results. Among the consequences that we state, let us mention power law bounds for magnetization at criticality for the Ising model, first established by Onsager in [28], tightness results for the interfaces coming from the Aizenman-Burchard technology, and the value 1/2 of the one-arm half-plane exponent – that describes both the asymptotic probability of large-distance connections starting from a boundary point for the FK Ising model, and the decay of boundary magnetization in the Ising model. Moreover, Theorem 1.1 is used in [26] to establish a polynomial upper bound on the mixing time of the Glauber dynamics at criticality, and in [7], such crossing bounds allow the authors to construct subsequential scaling limits for the spin field of the critical Ising model.

Theorem 1.1 also appears to be useful in enabling to transfer properties of the scaling limit objects back to the discrete models. It is therefore expected to be helpful to prove the existence of critical exponents, in particular of the *arm exponents*. Connections between discrete models and their continuum counterparts usually involve decorrelation of different scales, and thus use spatial independence between regions which are far enough from each other. In the random cluster model, one usually addresses the lack of spatial independence by successive conditionings, using repeatedly the spatial (or domain) Markov property of FK percolation, by which what happens outside a given domain can be encoded by appropriate boundary conditions. For this reason, proving bounds that are *uniform in the boundary conditions* seems to be important. An example of application of this technique is given in Section 5.1.

We would also like to mention that other proofs of Russo-Seymour-Welsh-type bounds have already been proposed. In [9], Chelkak and Smirnov give a direct and elegant argument to explicitly compute certain crossing probabilities in the scaling limit, but their argument only applies for some specific boundary conditions (alternatively wired and free on the four sides). In [7], Camia and Newman also propose to obtain RSW as a corollary of a recently announced result [9]: the convergence of the full collection of interfaces for the Ising model to the conformal loop ensemble CLE(3). The interpretation of CLE(3) in terms of the Brownian loop soup [38] is also used. However, to the author's knowledge, the proofs of these two results are quite involved, and moreover, the reasoning proposed only applies for the infinite-volume measure. In these two cases, uniformity with respect to the boundary conditions is not addressed, and there does not seem to be an easy argument to avoid this difficulty. While weaker forms might be sufficient for some applications, it seems however that this stronger form is needed in many important cases, and that it considerably shortens several existing arguments.

The paper is organized as follows. In Section 2, we first remind the reader of the basic features of FK percolation, as well as properties of Smirnov's fermionic observable. In Section 3, we compare the observable to certain harmonic measures, and we establish some estimates on the latter. These estimates are central in the proof of Theorem 1.1, which we perform in Section 4. Then, Section 5 is devoted to presenting the consequences that we mentioned. In the last section, we state conjectures on crossing probabilities for FK models with general values of $q \geq 1$.

2 FK percolation background

2.1 Basic features of the model

In order to remain as self-contained as possible, we recall some basic features of the random-cluster models. Some of these properties, like the Fortuin-Kasteleyn-Ginibre (FKG) inequality, are common to many statistical mechanics models. The reader can consult the reference book [13] for more details, and proofs of the results stated.

Definition of the random-cluster measure

We define the *random-cluster* (or *FK percolation*) measure on arbitrary finite graphs, although in this paper, we will be mostly interested in finite subgraphs of the square lattice $\mathbb{Z}^2$.

Let $G = (V, E)$ be a finite graph. The *boundary* of G, denoted by ∂G, is a given subset of the set of vertices V. A *configuration* ω is a random subgraph of G given by the vertices of G, together with some subset of edges between them. An edge of G is called *open* if it belongs to ω, and *closed* otherwise. Two sites x and y are said to be *connected* if there is an *open path* – a path composed of open edges only – connecting them, an event which is denoted by $x \rightsquigarrow y$. Similarly, two sets of vertices X and Y are said to be connected if there exist two sites $x \in X$ and $y \in Y$ such that $x \rightsquigarrow y$; we use the notation $X \rightsquigarrow Y$. We also abbreviate $\{x\} \rightsquigarrow Y$ as $x \rightsquigarrow Y$. Sites can be grouped into (maximal) connected components, usually called *clusters*.

Contrary to usual independent percolation, the edges in the FK percolation model are dependent of each other, a fact which makes the notion of *boundary conditions* important. Formally, a set ξ of boundary conditions is a set of "abstract" edges, each connecting two boundary vertices, that encodes how these vertices are connected outside G. We denote by $\omega \cup \xi$ the graph obtained by adding the new edges in ξ to the configuration ω.

We are now in a position to define the FK percolation measure itself, for any parameters $p \in [0, 1]$ and $q \geq 1$. Denoting by $o(\omega)$ (resp. $c(\omega)$) the number of open (resp. closed) edges of ω, and by $k(\omega, \xi)$ the number of connected components in $\omega \cup \xi$, the FK percolation process on G with parameters p, q and boundary conditions ξ is obtained by taking

$$\mathbb{P}^{\xi}_{p,q,G}(\{\omega\}) = \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega,\xi)}}{Z^{\xi}_{p,q,G}} \tag{1}$$

as a probability for any configuration ω, where $Z^{\xi}_{p,q,G}$ is an appropriate normalizing constant, called the *partition function*.

Among all the possible boundary conditions, two of them play a particular role. On the one hand, the *free* boundary conditions correspond to the case when there are no extra edges connecting boundary vertices; we denote by $\mathbb{P}^0_{p,q,G}$ the corresponding measure. On the other hand, the *wired* boundary conditions correspond to the case when all the boundary vertices are pair-wise connected, and the corresponding measure is denoted by $\mathbb{P}^1_{p,q,G}$.

Remark 2.1 *Note that for connections between sets (in particular for crossings and the definition of $\mathcal{C}_v(R)$), edges of ξ are* not *allowed to be used. Hence, even for the measure with wired boundary conditions, two points x and y on the boundary are not necessarily connected.*

Domain Markov property

The different edges of an FK percolation model being highly dependent, what happens in a given domain depends on the configuration outside the domain. However, the FK percolation model possesses a very convenient property known as the *domain Markov property*, which usually makes it possible to obtain some spatial independence. This property is used repeatedly in our proofs.

Consider a finite graph G, with E its set of edges. For a subset $F \subseteq E$, consider the graph G' having F as a set of edges, and the endpoints of F as a set of vertices. Then for any boundary conditions ϕ, $\mathbb{P}^{\phi}_{p,q,G}$ conditioned to match some configuration ω on $E \setminus F$ is equal to $\mathbb{P}^{\xi}_{p,q,G'}$, where ξ is the set of connections inherited from ω (one connects in ξ the boundary vertices that are connected in $G \setminus G'$ taking the boundary conditions into account). In other words, one can encode, using appropriate boundary conditions ξ, the influence of the configuration outside G'.

Strong positive association and infinite-volume measures

The random-cluster model with parameters $p \in [0,1]$ and $q \geq 1$ on a finite graph G has the *strong positive association property*. More precisely, it satisfies the so-called Holley criterion [13], a fact which has two important consequences. A first consequence is the well-known *FKG inequality*

$$\mathbb{P}^{\xi}_{p,q,G}(A \cap B) \geq \mathbb{P}^{\xi}_{p,q,G}(A)\, \mathbb{P}^{\xi}_{p,q,G}(B) \tag{2}$$

for any pair of *increasing* events A, B (increasing events are defined in the usual way [13]) and any boundary conditions ξ. This correlation inequality is fundamental to study FK percolation, for instance to combine several increasing events such as the existence of crossings in various rectangles.

A second property implied by strong positive association is the following monotonicity between boundary conditions, which is particularly useful when combined with the Domain Markov property. For any boundary conditions $\phi \leq \xi$ (all the connections present in ϕ belong to ξ as well), we have

$$\mathbb{P}^{\phi}_{p,q,G}(A) \leq \mathbb{P}^{\xi}_{p,q,G}(A) \tag{3}$$

for any increasing event A that depends only on G. We say that $\mathbb{P}^{\phi}_{p,q,G}$ is stochastically dominated by $\mathbb{P}^{\xi}_{p,q,G}$, denoted by $\mathbb{P}^{\phi}_{p,q,G} \leq_{\mathrm{st}} \mathbb{P}^{\xi}_{p,q,G}$.

In particular, this property directly implies that the free and wired boundary conditions are extremal in the sense of stochastic ordering: for any set of boundary conditions ξ, one has

$$\mathbb{P}^{0}_{p,q,G} \leq_{\mathrm{st}} \mathbb{P}^{\xi}_{p,q,G} \leq_{\mathrm{st}} \mathbb{P}^{1}_{p,q,G}. \tag{4}$$

An infinite-volume measure can be constructed as the increasing limit of FK percolation measures on the nested sequence of graphs $(\llbracket -n, n \rrbracket^2)_{n\geq 1}$ with free boundary conditions. For any fixed $q \geq 1$, classical arguments then show that there must exist a critical point $p_c = p_c(q)$ such that for any $p < p_c$, there is almost surely no infinite cluster of sites, while for $p > p_c$, there is almost surely one (see [13] for example).

Planar duality

In two dimensions, an FK measure on a subgraph G of $\mathbb{Z}^2$ with free boundary conditions can be associated with a dual measure in a natural way. First define the *dual lattice* $(\mathbb{Z}^2)^*$, obtained by putting a vertex at the center of each face of $\mathbb{Z}^2$, and by putting edges between nearest neighbors. The *dual graph* G^* of a finite graph G is given by the sites of $(\mathbb{Z}^2)^*$ associated with the faces adjacent to an edge of G. The edges of G^* are the edges of $(\mathbb{Z}^2)^*$ that connect two of its sites – note that any edge of G^* corresponds to an edge of G.

A dual model can be constructed on the dual graph as follows: for a percolation configuration ω, each edge of G^* is *dual-open* (or simply open), resp. *dual-closed*, if the corresponding edge of G is closed, resp. open. If the primal model is an FK percolation with parameters (p, q), then it follows from Euler's formula (relating the number of vertices, edges, faces, and components of a plane graph) that the dual model is again an FK percolation, with parameters (p^*, q^*) – in general, one must be careful about the boundary conditions. For instance, on a rectangle R, the FK percolation measure $\mathbb{P}^0_{p,q,R}$ is dual to the measure $\mathbb{P}^1_{p^*,q^*,R^*}$, where (p^*, q^*) satisfies

$$\frac{pp^*}{(1-p)(1-p^*)} = q \quad \text{and} \quad q^* = q. \tag{5}$$

The critical point $p_c(q)$ of the model is the self-dual point $p_{\mathrm{sd}}(q)$ for which $p = p^*$ (this has been recently proved in [2]), whose value can easily be derived:

$$p_{\mathrm{sd}}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}. \tag{6}$$

In the following, we need to consider connections in the dual model. Two sites x and y of G^* are said to be *dual-connected* if there exists a connected path of open dual-edges between them. Similarly to the primal model, we define *dual-clusters* as maximal connected components for dual-connectivity.

FK percolation with parameter $q = 2$: FK Ising model

For the value $q = 2$ of the parameter, the FK percolation model is related to the Ising model. More precisely, if starting from an FK percolation sample, one assigns uniformly at random a spin $+1$ or -1 to each cluster as a whole (sites in the same cluster get the same spin), independently, we get simply a sample of the Ising model. Conversely, one can get an FK percolation sample from an Ising sample by considering a percolation restricted to those edges that connect sites of the same spin. This coupling is called the *Edwards-Sokal coupling* [10], and it provides a link between correlations for the Ising model and connection probabilities for the FK Ising model.

In this case, the FK percolation model is now well-understood. The value $p_c = p_{\mathrm{sd}}$ is implied by the computation by Kaufman and Onsager of the partition function of the Ising model, and an alternative proof has been proposed recently by Beffara and Duminil-Copin [3]. Moreover, in [34], Smirnov proved conformal invariance of this model at the self-dual point p_{sd}.

Theorem 1.1 can be applied to the Ising model, using the previous coupling. For instance, one can deduce directly the following:

Corollary 2.2 *Consider the Ising model with* $(+)$ *or free boundary conditions in a rectangle* R *with dimensions* n *and* $m < \beta n$. *There exists a constant* $c_\beta > 0$ *such that*

$$\mathbb{P}_R^{free/+}(\mathcal{C}_v^+(R)) \geq c_\beta,$$

where $\mathcal{C}_v^+$ *denotes the existence of a vertical* $(+)$ *crossing.*

We could state this result for more general boundary conditions, for instance $(+)$ on one arc and free on the other arc. However, we have to be a little careful since not all boundary conditions can "go through this coupling". The corresponding result for $(-)$ boundary conditions is actually not expected to hold: one can notice for example that in any given smooth domain, a CLE(3) process – the object describing the scaling limit of cluster interfaces – a.s. does not touch the boundary.

In the following, we restrict ourselves to the FK percolation model with parameters $q = 2$ and $p = p_{\mathrm{sd}}(2) = \sqrt{2}/(1+\sqrt{2})$ (so that we forget the dependence on p and q), which is also known as the *critical FK Ising model* – we often call it the FK Ising model for short.

2.2 Smirnov's fermionic observable

In this part, we recall discrete holomorphicity and discrete harmonicity results for the FK Ising model, established by Smirnov in [34]. We do not include any proof, yet we remind the basic definitions and properties. These results are crucial in our proofs since they allow us to compare connection probabilities to harmonic measures. It should be noted that our proof only involves discrete arguments, the convergence results of [34] are not used. *Recall that from now,* $q = 2$ *and* $p = p_{\mathrm{sd}}(2)$.

Medial lattice of $\mathbb{Z}^2$

We first need to introduce the *medial lattice* associated with the square lattice $\mathbb{Z}^2$. The medial lattice $(\mathbb{Z}^2)_\diamond$, shown in Figure 1, has a site at the middle of each edge of $\mathbb{Z}^2$, and edges connecting nearest-neighbor sites. We obtain in this way a rotated copy of the square lattice (scaled by a factor $1/\sqrt{2}$).

The faces of the medial lattice correspond to sites of the primal or the dual lattice. We call a face *black* (resp. *white*) if it is associated with a site of $\mathbb{Z}^2$ (resp. $(\mathbb{Z}^2)^*$). We use extensively in the proof this correspondence between sites of the primal or dual lattices, and faces of the medial lattice. For instance, we say that two black faces are connected if the corresponding sites of the primal lattice are connected.

In addition to this, we put an orientation on $(\mathbb{Z}^2)_\diamond$: we orient the edges around each black face in counterclockwise direction.

Dobrushin domains and medial graphs

Informally speaking, a Dobrushin domain, as on Figure 1, is a domain with two points a and b dividing the boundary into two arcs (ab) and (ba), called the free

Figure 1: A domain $\mathcal{D}$ with Dobrushin boundary conditions: the vertices of the primal graph are black, the vertices of the dual graph $\mathcal{D}^*$ are white, and between them lies the medial lattice $\mathcal{D}_\diamond$. The arcs ∂_{ab} and ∂_{ba} are the two outermost medial paths (with arrows) from e_a to e_b. Note that ∂_{ab} and ∂_{ba} both have black faces to their left, and white faces to their right.

and the wired arcs.

More precisely, let e_a and e_b be two distinct edges of the medial lattice, a and b being their two adjacent black faces. Consider two self-avoiding paths ∂_{ab} and ∂_{ba} on the medial lattice, both starting at e_a and ending at e_b, that follow the orientation of the medial lattice and intersect only at e_a and e_b. We assume that the loop obtained by following $\partial_{ab} \setminus e_a \cup e_b$ (along its orientation) and then $\partial_{ba} \setminus e_a \cup e_b$ (in the reverse direction) is oriented counterclockwise. The *medial graph* $\mathcal{D}_\diamond = (V_\diamond, E_\diamond)$ associated with ∂_{ab} and ∂_{ba} consists of all the medial edges and vertices which are surrounded by the two arcs, as on Figure 1. The *boundary* of $V_\diamond$, denoted by $\partial V_\diamond$, is the set of vertices of $V_\diamond$ that belong to one of the two paths ∂_{ab} and ∂_{ba}.

Every such medial graph is naturally associated with a subgraph $\mathcal{D} = (V, E)$ of the primal lattice. The set V is composed of the sites in $\mathbb{Z}^2$ – black faces – adjacent to a medial edge of $E_\diamond$, and the set E consists of all the edges between sites of V that do not intersect ∂_{ab}. We define the *free arc* (ab) (resp. the *wired arc* (ba)) to be the set of sites of $\mathbb{Z}^2$ – black faces – adjacent to ∂_{ab} (resp. ∂_{ba}).

In the same manner, we can also define the dual graph $\mathcal{D}^*$ associated with $\mathcal{D}_\diamond$. We call *dual free arc* the set of white faces – on $\partial\mathcal{D}^*$ – adjacent to the arc ∂_{ab}. Note that these faces are a set of dual sites, contrary to the free arc itself, made of primal sites.

In most instances, the choice of arcs is natural and the correspondence between $\mathcal{D}_\diamond$ and $\mathcal{D}$ is straightforward. For this reason, we often specify Dobrushin domains as subgraphs of $\mathbb{Z}^2$ with two marked points a and b on the boundary. In this case, we denote them by $(\mathcal{D}, a, b)$.

FK Ising model and loop representation in Dobrushin domains

Let $(\mathcal{D}, a, b)$ be a Dobrushin domain. We consider a random cluster measure with wired boundary conditions on the wired arc – all the edges are pair-wise connected – and free boundary conditions on the free arc. These boundary conditions are called the *Dobrushin boundary conditions* on $(\mathcal{D}, a, b)$. We denote by $\mathbb{P}_{\mathcal{D},a,b}$ the associated random cluster measure with parameters $q = 2$ and $p = p_{\mathrm{sd}}(2)$.

For any FK percolation configuration in $\mathcal{D}$, we can consider the associated models on $\mathcal{D}$ and $\mathcal{D}^*$. The interfaces between the primal clusters and the dual clusters (if we follow the edges of the medial lattice) then form a family of loops, together with a path from e_a to e_b, called the *exploration path*, as shown

Figure 2: An FK percolation configuration in the Dobrushin domain $(\mathcal{D}, a, b)$, together with the corresponding interfaces on the medial lattice: the loops in grey, and the exploration path γ from e_a to e_b in black. Notice that the exploration path is the interface between the open cluster connected to the wired arc and the dual-open cluster connected to the dual free arc.

on Figure 2.

Remark 2.3 *The exploration path is the interface between the open cluster connected to the wired arc and the dual-open cluster connected to the dual free arc.*

A simple rearrangement of (1), using the duality property, shows that the probability of such a configuration is proportional to $(\sqrt{2})^{\#\text{loops}}$ – taking into account the fact that $q = 2$ and $p = p_{\text{sd}}(2) = p^*$. The orientation of the medial lattice naturally gives an orientation to the loops, so that we are now working with a model of oriented curves on the medial lattice.

Remark 2.4 *If we consider a Dobrushin domain $(\mathcal{D}, a, b)$, the slit domain created by "removing" the first T steps of the exploration path is again a Dobrushin domain (*i.e *we extend the arcs ∂_{ab} and ∂_{ba} by initially "bouncing" along the slit). We denote the new domain by $(\mathcal{D} \setminus \gamma[0, T], \gamma(T), b)$, where, with a slight abuse of notation, $\gamma(T)$ is used to denote the site of the primal lattice adjacent to the medial edge $\gamma(T)$. Then conditionally on γ, the law of the FK Ising model in this new domain is exactly $\mathbb{P}_{\mathcal{D}\setminus\gamma[0,T],\gamma(T),b}$. This observation will be central in our proof.*

Fermionic observable and local relations

Let $(\mathcal{D}, a, b)$ be a Dobrushin domain and γ the exploration path from e_a to e_b. The *winding* $W_\Gamma(z, z')$ of a curve Γ between two edges z and z' of the medial lattice is the overall angle variation (in radians) of the curve from the center of the edge z to the center of the edge z'. The *fermionic observable* F can now be defined by the formula (see [34], Section 2)

$$F(e) = \mathbb{E}_{\mathcal{D},a,b}[\mathrm{e}^{-\frac{1}{2}\cdot \mathrm{i} W_\gamma(e_a, e)} \mathbb{I}_{e\in\gamma}], \tag{7}$$

for any edge e of the medial lattice $\mathcal{D}_\diamond$. The constant $\sigma = 1/2$ appearing in front of the winding is called the *spin* (see [34], Section 2).

The quantity $F(e)$ is a complexified version of the probability that e belongs to the exploration path (note that it is defined on the medial graph $\mathcal{D}_\diamond$). The complex weight makes the link between F and probabilistic properties less explicit. Nevertheless, as we will see, the winding term can be controlled along the boundary. The observable F also satisfies the following local relation, from which Propositions 2.6 and 2.7 follow.

Figure 3: Indexation of the four medial edges around a vertex v.

Lemma 2.5 ([34], Lemma 4.5) *For any vertex $v \in V_\diamond \setminus \partial V_\diamond$, the relation*

$$F(e_1) + F(e_3) = F(e_2) + F(e_4) \tag{8}$$

is satisfied, where e_1, e_2, e_3 and e_4 are the four edges at v indexed in clockwise order, as on Figure 3.

We refer to [34] or [3] for the proof of this result. The key ingredient is a bijection between configurations that contribute to the values of F at the edges around v. Note that for other values of q, one can still define the fermionic observable in a way similar to Eq.(7): for an appropriate value $\sigma = \sigma(q)$ of the spin, the previous relation Eq.(8) still holds, see [3, 34].

Complex argument of the fermionic observable F and definition of H

Due to the specific value of the spin $\sigma = 1/2$, corresponding to the value $q = 2$, the complex argument modulo π of the fermionic observable F follows from its definition Eq.(7). For instance, if the edge e points in the same direction as the starting edge e_a, then the winding is a multiple of 2π, so that the term $\mathrm{e}^{-\frac{1}{2}\cdot \mathrm{i} W_\gamma(e_a,e)}$ is equal to ± 1, and $F(e)$ is purely real. The same reasoning can be applied to any edge to show that $F(e)$ belongs to the line $\mathrm{e}^{\mathrm{i}\pi/4}\mathbb{R}$, $\mathrm{e}^{-\mathrm{i}\pi/4}\mathbb{R}$ or $\mathrm{i}\mathbb{R}$, depending on the direction of e. Contrary to Lemma 2.5, this property is very specific to the FK Ising model.

For a vertex $v \in V_\diamond \setminus \partial V_\diamond$, keeping the same notations as for Lemma 2.5, $F(e_1)$ and $F(e_3)$ are always orthogonal (for the scalar product between complex numbers $(a, b) \mapsto \Re e(a\bar{b})$), as well as $F(e_2)$ and $F(e_4)$, so that Eq.(8) gives

$$|F(e_1)|^2 + |F(e_3)|^2 = |F(e_2)|^2 + |F(e_4)|^2 . \tag{9}$$

Consider now a vertex $v \in \partial V_\diamond$. It possesses two or four adjacent edges, depending on whether the corresponding boundary arc passes once or twice through this vertex. Assume that there are only two adjacent edges (the other case can be treated similarly), and denote by e_5 the “entering” edge, and by e_6 the “exiting” edge. For such a vertex on the boundary of the domain, e_5 belongs to the interface γ if and only if e_6 belongs to γ – indeed, by construction, the curve entering through e_5 must leave through e_6. Moreover, the windings of the curve $W_\gamma(e_a, e_5)$ and $W_\gamma(e_a, e_6)$ are constant since γ cannot wind around these edges. From these two facts, we deduce:

$$|F(e_5)|^2 = \left|\mathrm{e}^{-\frac{1}{2}\cdot \mathrm{i} W_\gamma(e_a,e_5)} \mathbb{P}_{\mathcal{D},a,b}(e_5 \in \gamma)\right|^2 = \mathbb{P}_{\mathcal{D},a,b}(e_5 \in \gamma)^2 = |F(e_6)|^2. \tag{10}$$

From Eqs.(9) and (10), one can easily prove the following proposition.

Proposition 2.6 ([34], Lemma 3.6) *There exists a* unique function H *defined on the faces of* $\mathcal{D}_\diamond$ *by the relation*

$$H(B) - H(W) = |F(e)|^2, \tag{11}$$

for any two neighboring faces B *and* W*, respectively black and white, separated by the edge* e*, and by fixing the value* 1 *on the black face corresponding to* a*. Moreover,* H *is then automatically equal to* 1 *on the black faces of the wired arc, and equal to* 0 *on the white faces of the dual free arc.*

This function H is a discrete analogue of the antiderivative of F^2, as explained in Remark 3.7 of [34].

Approximate Dirichlet problem for H

Let us denote by $H_\bullet$ and $H_\circ$ the restrictions of H respectively to the black faces and to the white faces. At a black site u of $\mathcal{D}$ which is not on the boundary, we can consider the usual *discrete Laplacian* Δ (on the graph $\mathcal{D}$): for a function f, $\Delta f(u)$ is the average of f on the four nearest black neighbors of u, minus $f(u)$. A similar definition holds for white sites of the graph $\mathcal{D}^*$.

The result below, proved in [34], is a key step to prove convergence of the observable as one scales the domain – but we will not discuss this question here. Its proof relies on an elementary yet quite lengthy computation.

Proposition 2.7 ([34], Lemma 3.8) *The function* $H_\bullet$ *(resp.* $H_\circ$*) is* subharmonic *(resp.* superharmonic*) inside the domain for the discrete Laplacian.*

Since we know that H is equal to 1 (resp. 0) on the black faces of the wired arc (resp. on the white faces of the dual free arc), the previous proposition can be seen as an approximate Dirichlet problem for the function H. In the next section, we make this statement rigorous by comparing H to harmonic functions corresponding to the same boundary problems (on the set of black faces, or on the set of white ones).

3 Comparison to harmonic measures

In this section, we obtain a comparison result for the boundary values of the fermionic observable F introduced in the previous section in terms of discrete harmonic measures. It will be used to obtain all the quantitative estimates on the observable that we need for the proof of Theorem 1.1.

3.1 Comparison principle

As in the previous section, let $(\mathcal{D}, a, b)$ be a discrete Dobrushin domain, with free boundary conditions on the arc (ab), and wired boundary conditions on the other arc (ba).

For our estimates, we first extend the medial graph of our discrete domain by adding two extra layers of faces: one layer of white faces adjacent to the black faces of the wired arc, and one layer of black faces adjacent to the white faces of the dual free arc. We denote by $\bar{\mathcal{D}}_\diamond$ this extended domain.

Remark 3.1 *Note that a small technicality arises when adding a new layer of faces: some of these additional faces can overlap faces that were already here. For instance, if the domain has a slit, the free and the wired arc are adjacent along this slit, and the extra layer on the wired arc (resp. on the dual free arc) overlaps the dual free arc (resp. the wired arc). As we will see, $H_\bullet$ is equal to* 1 *on the wired arc, and to* 0 *on the additional layer along the dual free arc. One should thus remember in the following that the added faces are considered as different from the original ones – it will always be clear from the context which faces we are considering.*

For any given black face B, let us define $\left(X_{\bullet t}^{B}\right)_{t\geq 0}$ to be the continuous-time random walk on the black faces of $\bar{\mathcal{D}}_\diamond$ starting at B, that jumps with rate 1 on adjacent black faces, *except* for the black faces on the extra layer of black faces adjacent to the dual free arc onto which it jumps with rate $\rho := 2/(\sqrt{2}+1)$. Similarly, we denote by $\left(X_{\circ t}^{W}\right)_{t\geq 0}$ the continuous-time random walk on the white faces of $\bar{\mathcal{D}}_\diamond$ starting at a white face W that jumps with rate 1 on adjacent white faces, *except* for the white faces on the extra layer of white faces adjacent to the wired arc onto which it jumps with the same rate $\rho = 2/(\sqrt{2}+1)$ as previously.

For a black face B, we denote by $\mathbf{HM}_\bullet(B)$ the probability that the random walk $X_{\bullet t}^{B}$ hits the wired arc from b to a before hitting the extra layer adjacent to the free arc. Similarly, for W a white face, we denote by $\mathbf{HM}_\circ(W)$ the probability that the random walk $X_{\circ t}^{W}$ hits the additional layer adjacent to the wired arc before hitting the free arc. Note that there is no extra difficulty in defining these quantities for infinite discrete domains as well. We have the following result:

Proposition 3.2 (uniform comparability) *Let $(\mathcal{D}, a, b)$ be a discrete Dobrushin domain, and let e be a medial edge of ∂_{ab} (thus adjacent to the free arc). Let $B = B(e)$ be the black face bordered by e, and $W = W(e)$ be a white face adjacent to B that does not belong to the dual free arc. Then we have*

$$\sqrt{\mathbf{HM}_\circ(W)} \leq |F(e)| \leq \sqrt{\mathbf{HM}_\bullet(B)}. \tag{12}$$

Proof By (11) and the lines following (11), we have $|F(e)|^2 = H(B)$ and $H(W) = |F(e)|^2 - |F(e')|^2 \leq |F(e)|^2$, where e' is the medial edge between B and W: it is therefore sufficient to show that $H(B) \leq \mathbf{HM}_\bullet(B)$ and $H(W) \geq \mathbf{HM}_\circ(W)$. We only prove that $H(B) \leq \mathbf{HM}_\bullet(B)$, since the other case can be handled in the same way.

For this, we use a variation of a trick introduced in [9] and extend the function H to the extra layer of black faces – added as explained above – by setting H to be equal to 0 there. It is then sufficient to show that the restriction $H_\bullet$ of H

Figure 4: We extend $\mathcal{D}_\diamond$ by adding two extra layers of medial faces, and extend the functions $H_\bullet$ and $H_\circ$ there. Here is represented the extension along the dual free arc.

to the black faces of $\bar{\mathcal{D}}_\diamond$ is subharmonic for the Laplacian that is the generator of the random walk $X_\bullet$, since it has the same boundary values as $\mathbf{HM}_\bullet$ (which is harmonic for this Laplacian). Inside the domain, subharmonicity is given by Proposition 2.7, since there the Laplacian of $X_\bullet$ is the usual discrete Laplacian (associated with it is just a simple random walk). The only case to check is when a face involved in the computation of the Laplacian belongs to one of the extra layers. For the sake of simplicity, we study the case when only one face belongs to these extra layers.

Denote by B_W, B_N, B_E and B_S the black faces adjacent to B, and assume that B_S is on the extra layer (see Figure 4). The discrete Laplacian of $X_\bullet$ at face B is denoted by $\Delta_\bullet$. We claim that

$$\Delta_\bullet H_\bullet(B) = \frac{2+\sqrt{2}}{6+5\sqrt{2}}[H_\bullet(B_W)+H_\bullet(B_N)+H_\bullet(B_E)] + \frac{2\sqrt{2}}{6+5\sqrt{2}}H_\bullet(B_S) - H_\bullet(B) \geq 0. \tag{13}$$

For that, let us denote by e_1, e_2, e_3, e_4 the four medial edges at the bottom vertex v between B and B_S, in clockwise order, with e_1 and e_2 along B, and e_3 and e_4 along B_S (see Figure 4) – note that e_3 and e_4 are not edges of $\mathcal{D}_\diamond$, but of $(\mathbb{Z}^2)_\diamond$.

We extend F to e_3 and e_4 by requiring $F(e_3)$ and $F(e_1)$ to be orthogonal, as well as $F(e_4)$ and $F(e_2)$, and $F(e_1)+F(e_3) = F(e_2)+F(e_4)$ to hold true. This defines these two values uniquely: indeed, as noted before, we know that $F(e_2) = \mathrm{e}^{-\mathrm{i}\pi/4}F(e_1)$ on the boundary (since $W_\gamma(e_a, e_1)$ and $W_\gamma(e_a, e_2)$ are fixed, with $W_\gamma(e_a, e_2) = W_\gamma(e_a, e_1) + \pi/2$, and the curve cannot go through one of these edges without going through the other one), which implies, after a small calculation, that

$$|F(e_3)|^2 = \left|\Big(\tan\frac{\pi}{8}\Big)\mathrm{e}^{\mathrm{i}\pi/4}F(e_2)\right|^2 = \frac{2-\sqrt{2}}{2+\sqrt{2}}|F(e_2)|^2 = \frac{2-\sqrt{2}}{2+\sqrt{2}}H_\bullet(B).$$

If we denote by $\tilde{H}_\bullet$ the function defined by $\tilde{H}_\bullet = H_\bullet$ on B, B_W, B_N and B_E, and by

$$\tilde{H}_\bullet(B_S) = |F(e_3)|^2 = \frac{2-\sqrt{2}}{2+\sqrt{2}}H_\bullet(B), \tag{14}$$

then $\tilde{H}_\bullet$ satisfies the same relation Eq.(11) (definition of H) for e_3 and e_4, as inside the domain. Since the fermionic observable F verifies the same local equations, the computation performed in the Appendix C of [34] is valid, Proposition 2.7 applies at B (with $\tilde{H}$ instead of H), and we deduce

$$\Delta\tilde{H}_\bullet(B) = \frac{1}{4}[\tilde{H}_\bullet(B_W)+\tilde{H}_\bullet(B_N)+\tilde{H}_\bullet(B_E)+\tilde{H}_\bullet(B_S)] - \tilde{H}_\bullet(B) \geq 0. \tag{15}$$

Figure 5: Estimate of Lemma 3.3: the dashed line corresponds to the dual free arc.

Using the definition of $\tilde{H}_\bullet$, this inequality can be rewritten as

$$\frac{1}{4}[H_\bullet(B_W) + H_\bullet(B_N) + H_\bullet(B_E)] - \frac{6+5\sqrt{2}}{4(2+\sqrt{2})} H_\bullet(B) \geq 0. \tag{16}$$

Now using that $H_\bullet(B_S) = 0$, we get the claim, Eq.(13). □

3.2 Estimates on harmonic measures

In the previous subsection, we gave a comparison principle between the values of H near the boundary, and the harmonic measures associated with two (almost simple) random walks, on the two lattices composed of the black faces and of the white faces respectively. In this subsection, we give estimates for these two harmonic measures in different domains needed for the proof of Theorem 1.1. We start by giving a lower bound which is useful in the proof of the 1-point estimate.

Lemma 3.3 *For $\beta > 0$ and $n \geq 0$, let R_n^β be*

$$R_n^\beta = [\![-\beta n, \beta n]\!] \times [\![0, 2n]\!].$$

Then there exists $c_1(\beta) > 0$ such that for any $n \geq 1$,

$$\mathbf{HM}_\circ(W_x) \geq \frac{c_1(\beta)}{n^2} \tag{17}$$

*in the Dobrushin domain (R_n^β, u, u) (see Figure 5), for all $x = (x_1, 0)$ and $u = (u_1, 2n)$ such that $|x_1|, |u_1| \leq \beta n/2$ (*i.e.* far enough from the corners), W_x being any of the two white faces that are adjacent to x and not on the dual free arc.*

Proof This proposition follows from standard results on simple random walks (gambler's ruin type estimates). For the sake of conciseness, we do not provide a detailed proof. □

In the remaining part of this section, we consider *only* Dobrushin domains $(\mathcal{D}, a, b)$ that *contain the origin* on the free arc, and are *subsets* of the medial lattice $\mathbb{H}_\diamond$, where $\mathbb{H} = \{(x_1, x_2) \in \mathbb{Z}^2, x_2 \geq 0\}$ denotes the upper half plane – in this case, we say that $\mathcal{D}$ *is a Dobrushin* $\mathbb{H}$*-domain.* For the following estimates on harmonic measures, the Dobrushin domains that we consider can also be infinite. We are interested in the harmonic measure of the wired arc seen from

Figure 6: The two domains involved in the proof of Lemma 3.4.

a given point: without loss of generality, we can assume that this point is just the origin. Let B_0 be the corresponding black face of the medial lattice, and W_0 be an adjacent white face which is not on the free arc.

We first prove a lower bound on the harmonic measure. For that, we introduce, for $k \in \mathbb{Z}$ and $n \geq 0$, the segments

$$l_n(k) = \{k\} \times [\![0, n]\!] \quad (= \{(k, j) : 0 \leq j \leq n\}).$$

Lemma 3.4 *There exists a constant $c_2 > 0$ such that for any Dobrushin $\mathbb{H}$-domain $(\mathcal{D}, a, b)$, we have*

$$\mathbf{HM}_\circ(W_0) \geq \frac{c_2}{k}, \tag{18}$$

provided that, in $\mathcal{D}$, the segment $l_k(-k)$ disconnects from the origin the intersection of the free arc with the upper half-plane (see Figure 6).

Proof We know that $l_k(-k)$ disconnects the origin from the part of the free arc that lies in the upper half-plane, let us thus consider the connected component of $\mathcal{D} \setminus l_k(-k)$ that contains the origin. In this new domain $\mathcal{D}_0$, if we put free boundary conditions along $l_k(-k)$, the harmonic measure of the wired arc is smaller than the harmonic measure of the wired arc in the original domain $\mathcal{D}$. On the other hand, the harmonic measure of the wired arc in $\mathcal{D}_0$ is larger than the harmonic measure of the wired arc in the slit domain $(\mathbb{H} \backslash l_k(-k), (-k, k), \infty)$, which has respectively wired and free boundary conditions to the left and to the right of $(-k, k)$ (see Figure 6). Estimating this harmonic measure is straightforward, using the same arguments as before. □

We now derive upper bounds on the harmonic measures. We will need estimates of two different types. The first one takes into account the distance between the origin and the wired arc, while the second one requires the existence of a segment $l_n(k)$ disconnecting the wired arc from the origin (still inside the domain).

Lemma 3.5 *There exist constants $c_3, c_4 > 0$ such that for any Dobrushin $\mathbb{H}$-domain $(\mathcal{D}, a, b)$,*

(i) if $d_1(0)$ denotes the graph distance between the origin and the wired arc,

$$\mathbf{HM}_\bullet(B_0) \leq c_3 \frac{1}{d_1(0)}, \tag{19}$$

(ii) and if the segment $l_n(k)$ disconnects the wired arc from the origin inside $\mathcal{D}$,

$$\mathbf{HM}_\bullet(B_0) \leq c_4 \frac{n}{|k|^2}. \tag{20}$$

Figure 7: The two different upper bounds (i) and (ii) of Lemma 3.5.

Proof Let us first consider item (i). For $d = d_1(0)$, define the Dobrushin domain $(\mathcal{B}_d, (-d, 0), (d, 0))$, where $\mathcal{B}_d$ is the set of sites in $\mathbb{H}$ at a graph distance at most d from the origin (see Figure 7). The harmonic measure of the wired arc in $(\mathcal{D}, a, b)$ is smaller than the harmonic measure of the wired arc in this new domain $\mathcal{B}_d$, and, as before, this harmonic measure is easy to estimate.

Let us now turn to item (ii). Since $l_n(k)$ disconnects the wired arc from the origin, the harmonic measure of the wired arc is smaller than the harmonic measure of $l_n(k)$ inside $\mathcal{D}$, and this harmonic measure is smaller than it is in the domain $\mathbb{H} \setminus l_n(k)$ with wired boundary conditions on the left side of $l_n(k)$ – right side if $k < 0$ (see Figure 7). Once again, the estimates are easy to perform in this domain. □

4 Proof of Theorem 1.1

We now prove our result, Theorem 1.1. The main step is to prove the uniform lower bound for rectangles of bounded aspect ratio with free boundary conditions. We then use monotonicity to compare boundary conditions and obtain the desired result. In the case of free boundary conditions, the proof relies on a second moment estimate on the number N of pairs of vertices (x, u), on the top and bottom sides of the rectangle respectively, that are connected by an open path.

The organization of this section follows the second-moment estimate strategy. In Proposition 4.2, we first prove a lower bound on the probability of a connection from a given site on the bottom side of a rectangle to a given site on the top side. This estimate gives a lower bound on the expectation of N. Then, Proposition 4.3 provides an upper bound on the probability that two points on the bottom side of a rectangle are connected to the top side. This proposition is the core of the proof, and it provides the right bound for the second moment of N. It allows us to conclude the section by using the second moment estimate method, thus proving Theorem 1.1.

In this section, we use two main tools: the domain Markov property, and probability estimates for connections between the wired arc and sites on the free arc. We first explain how the previous estimates on harmonic measures can be used to derive estimates on connection probabilities. The following lemma is instrumental in this approach.

Lemma 4.1 *Let $(\mathcal{D}, a, b)$ be a Dobrushin domain. For any site x on the free arc of $\mathcal{D}$, we have*

$$\sqrt{\mathbf{HM}_\circ(W_x)} \leq \mathbb{P}_{\mathcal{D},a,b}(x \rightsquigarrow \textit{wired arc}) \leq \sqrt{\mathbf{HM}_\bullet(B_x)}, \tag{21}$$

where B_x is the black face corresponding to x, and W_x is any closest white face that is not *on the free arc.*

Proof Since x is on the free boundary of $\mathcal{D}$, there exists a white face on the free arc of $\mathcal{D}_\diamond$ which is adjacent to B_x: we denote by e the edge between these faces. As noted before, since the edge e is along the free arc, the winding $W_\gamma(e_a, e)$ of the exploration path γ at e is constant, and depends only on the direction of e. This implies that

$$\mathbb{P}_{\mathcal{D},a,b}(e \in \gamma) = |F(e)|.$$

In addition, e belongs to γ if and only if x is connected to the wired arc, which implies that $|F(e)|$ is exactly equal to $\mathbb{P}_{\mathcal{D},a,b}(x \rightsquigarrow \text{wired arc})$. Proposition 3.2 thus implies the claim. □

With this lemma at our disposal, we can prove the different estimates. Throughout the proof, we use the notation $c_i(\beta)$ for constants that depend neither on n nor on sites x, y or on boundary conditions. When they do not depend on β, we denote them by c_i (it is the case for the upper bounds). Recall the definition of R_n^β:

$$R_n^\beta = [\![-\beta n, \beta n]\!] \times [\![0, 2n]\!]. \tag{22}$$

Let $\partial_+ R_n^\beta$ (resp. $\partial_- R_n^\beta$) be the *top side* $[\![-\beta n, \beta n]\!] \times \{2n\}$ (resp. *bottom side* $[\![-\beta n, \beta n]\!] \times \{0\}$) of the rectangle R_n^β. We begin with a lower bound on connection probabilities.

Proposition 4.2 (connection probability for one point on the bottom side) *Let $\beta > 0$, there exists a constant $c(\beta) > 0$ such that for any $n \geq 1$,*

$$\mathbb{P}^0_{R_n^\beta}(x \rightsquigarrow u) \geq \frac{c(\beta)}{n} \tag{23}$$

for all $x = (x_1, 0) \in \partial_- R_n^\beta$, $u = (u_1, 2n) \in \partial_+ R_n^\beta$, satisfying $|x_1|, |u_1| \leq \beta n/2$.

Proof The probability that x and u are connected in the rectangle with free boundary conditions can be written as the probability that x is connected to the wired arc in (R_n^β, u, u) (where the wired arc consists of a single vertex). The previous lemma, together with the estimate of Lemma 3.3, concludes the proof. □

We now study the probability that two boundary points on the bottom edge of R_n^β are connected to the top edge, with boundary conditions wired on the top side and free on the other sides.

Proposition 4.3 (connection probability for two points on the bottom side) *There exists a constant $c > 0$ (uniform in β, n) such that for any rectangle R_n^β and any two points x, y on the bottom side $\partial_- R_n^\beta$,*

$$\mathbb{P}_{R_n^\beta, a_n, b_n}(x, y \rightsquigarrow \textit{wired arc}) \leq \frac{c}{\sqrt{|x-y|n}}, \tag{24}$$

Figure 8: The Dobrushin domain (R_n^β, c_n, d_n), together with the exploration path up to time T.

where a_n and b_n denote respectively the top-left and top-right corners of the rectangle R_n^β.

The proof is based on the following lemma, which is a strong form of the so-called half-plane one-arm probability estimate (see Subsection 5.1 for a further discussion of this result). For x on the bottom side of R_n^β and $k \geq 1$, we denote by $\mathcal{B}_k(x)$ the box centered at x with diameter k for the *graph distance.* We can now state the lemma needed:

Lemma 4.4 *There exists a constant $c_5 > 0$ (uniform in n, β and the choice of x) such that for all $k \geq 1$,*

$$\mathbb{P}_{R_n^\beta, a_n, b_n}(\mathcal{B}_k(x) \rightsquigarrow \textit{wired arc}) \leq c_5 \sqrt{\frac{k}{n}}. \tag{25}$$

Proof Consider $n, k, \beta > 0$, and the box R_n^β with one point $x \in \partial_- R_n^\beta$. Eq.(25) becomes trivial if $k \geq n$, so we can assume that $k \leq n$. For any choice of $\beta' \geq \beta$, the monotonicity between boundary conditions Eq.(4) implies that the probability that $\mathcal{B}_k(x)$ is connected to the wired arc $\partial_+ R_n^\beta$ in (R_n^β, a_n, b_n) is smaller than the probability that $\mathcal{B}_k(x)$ is connected to the wired arc in the Dobrushin domain $(R_n^{\beta'}, c_n, d_n)$, where c_n and d_n are the bottom-left and bottom-right corners of $R_n^{\beta'}$. From now on, we replace β by $\beta + 2$, and we work in the new domain (R_n^β, c_n, d_n). Notice that $\mathcal{B}_k(x)$ is then included in R_n^β and that the right-most site of $\mathcal{B}_k(x)$ is at a distance at least n from the wired arc.

We denote by T the hitting time – for the exploration path naturally parametrized by the number of steps – of the set of medial edges bordering (the black faces corresponding to) the sites of $\mathcal{B}_k(x)$; we set $T = \infty$ if the exploration path never reaches this set, so that $\mathcal{B}_k(x)$ is connected to the wired arc if and only if $T < \infty$.

Let z be the right-most site of the box $\mathcal{B}_k(x)$. Consider now the event $\{z \rightsquigarrow \text{wired arc}\}$. By conditioning on the curve up to time T (and on the event $\{\mathcal{B}_k(x) \rightsquigarrow \text{wired arc}\}$), we obtain

$$\begin{aligned}\mathbb{P}_{R_n^\beta, c_n, d_n}(z \rightsquigarrow \text{wired arc}) &= \mathbb{E}_{R_n^\beta, c_n, d_n}\left[\mathbb{I}_{T<\infty} \mathbb{P}_{R_n^\beta, c_n, d_n}(z \rightsquigarrow \text{wired arc} \mid \gamma[0,T])\right] \\ &= \mathbb{E}_{R_n^\beta, c_n, d_n}\left[\mathbb{I}_{T<\infty} \mathbb{P}_{R_n^\beta \setminus \gamma[0,T], \gamma(T), d_n}(z \rightsquigarrow \text{wired arc})\right],\end{aligned}$$

where in the second equality, we have used the domain Markov property, and the fact that it is sufficient for z to be connected to the wired arc in the new domain (since it is then automatically connected to the wired arc of the original domain).

Figure 9: This picture presents the different steps in the proof of Proposition 4.3: we first (1) condition on $\gamma[0, T_x]$ and use the uniform estimate (i) of Lemma 3.5, then (2) condition on $\gamma[0, T_{k+1}]$ and use the estimate (ii) of Lemma 3.5, in order to (3) conclude with Lemma 4.4.

On the one hand, since z is at a distance at least n from the wired arc (thanks to the new choice of β), we can combine Lemma 4.1 with item (i) of Lemma 3.5 to obtain

$$\mathbb{P}_{R_n^\beta, c_n, d_n}(z \rightsquigarrow \text{wired arc}) \leq \frac{c_3}{\sqrt{n}}. \tag{26}$$

On the other hand, if $\gamma(T)$ can be written as $\gamma(T) = z + (-r, r)$, with $0 \leq r \leq k$, then the arc $z + l_r(-r)$ disconnects the free arc from z in the domain $R_n^\beta \setminus \gamma[0, T]$, while if $\gamma(T) = z + (-r, 2k - r)$, with $k + 1 \leq r \leq 2k$, then the arc $z + l_r(-r)$ still disconnects the free arc from z. Using once again Lemma 4.1, this time with Lemma 3.4, we obtain that a.s.

$$\mathbb{P}_{R_n^\beta \setminus \gamma[0,T], \gamma(T), d_n}(z \rightsquigarrow \text{wired arc}) \geq \frac{c_4}{\sqrt{r}} \geq \frac{c_4}{\sqrt{2k}}. \tag{27}$$

This estimate being uniform in the realization of $\gamma[0, T]$, we obtain

$$\frac{c_4}{\sqrt{2k}} \mathbb{P}_{R_n^\beta, c_n, d_n}(T < \infty) \leq \mathbb{P}_{R_n^\beta, c_n, d_n}(z \rightsquigarrow \text{wired arc}) \leq \frac{c_3}{\sqrt{n}}, \tag{28}$$

which implies the desired claim, that is, Eq.(25). $\square$

Proof of Proposition 4.3 Let us take two sites x and y on $\partial_- R_n^\beta$. As in the previous proof, the larger the β, the larger the corresponding probability, we can thus assume that β has been chosen in such a way that there are no boundary effects. In order to prove the estimate, we express the event considered in terms of the exploration path γ. If x and y are connected to the wired arc, γ must go through two boundary edges which are adjacent to x and y, that we denote by e_x and e_y. Notice that e_x has to be discovered by γ before e_y is.

We now define T_x to be the hitting time of e_x, and T_k to be the hitting time of the set of medial edges bordering (the black faces associated with) the sites of $\mathcal{B}_{2^k}(y)$, for $k \leq k_0 = \lfloor \log_2 |x - y| \rfloor$ – where $\lfloor \cdot \rfloor$ is the integer part of a real number. If the exploration path does not cross this ball before hitting e_x, we set $T_k = \infty$. With these definitions, the probability that e_x and e_y are both on

γ can be expressed as

$$\mathbb{P}_{R_n^\beta,a_n,b_n}(x,y \rightsquigarrow \text{wired arc}) = \mathbb{P}_{R_n^\beta,a_n,b_n}(e_x,e_y \in \gamma) \tag{29}$$

$$= \sum_{k=0}^{k_0} \mathbb{P}_{R_n^\beta,a_n,b_n}(e_y \in \gamma, T_x < \infty, T_{k+1} < T_k = \infty) \tag{30}$$

$$= \sum_{k=0}^{k_0} \mathbb{E}_{R_n^\beta,a_n,b_n}\left[\mathbb{I}_{T_{k+1}<T_k=\infty}\mathbb{I}_{T_x<\infty}\mathbb{P}_{R_n^\beta,a_n,b_n}(e_y \in \gamma\,|\gamma[0,T_x]\,)\right], \tag{31}$$

where the third equality is obtained by conditioning on the exploration path up to time T_x. Recall that e_y belongs to γ if and only if y is connected to the wired arc. Moreover, if $\{T_k = \infty\}$, y is at a distance at least 2^k from the wired arc in $R_n^\beta \setminus \gamma[0,T_x]$. Hence, the domain Markov property, item (i) of Lemma 3.5 and Lemma 4.1 give that, on $\{T_k = \infty\}$,

$$\mathbb{P}_{R_n^\beta,a_n,b_n}(e_y \in \gamma\,|\gamma[0,T_x]\,) = \mathbb{P}_{R_n^\beta\setminus\gamma[0,T_x],x,b_n}(y \rightsquigarrow \text{wired arc}) \le \frac{c_3}{\sqrt{2^k}} \quad a.s. \tag{32}$$

By plugging this uniform estimate into (31), and removing the condition on $T_k = \infty$, we obtain

$$\mathbb{P}_{R_n^\beta,a_n,b_n}(e_x,e_y \in \gamma) \le \sum_{k=0}^{k_0} \frac{c_3}{\sqrt{2^k}}\mathbb{E}_{R_n^\beta,a_n,b_n}\left[\mathbb{I}_{T_{k+1}<\infty}\mathbb{P}_{R_n^\beta,a_n,b_n}(T_x<\infty\,|\gamma[0,T_{k+1}]\,)\right],$$

where we conditioned on the path up to time T_{k+1}. Now, e_x belongs to γ if and only if x is connected to the wired arc. Assuming $\{T_{k+1} < \infty\}$, the vertical segment connecting $\gamma(T_{k+1})$ to $\mathbb{Z}$ – of length at most 2^{k+1} – disconnects the wired arc from x in the domain $R_n^\beta \setminus \gamma[0,T_{k+1}]$. For $k+1 < k_0$, this vertical segment is at distance at least $\frac{1}{2}|x-y|$ from x. Applying the domain Markov property and item (ii) of Lemma 3.5, we deduce that, for $k+1 < k_0$, on $\{T_{k+1} < \infty\}$,

$$\mathbb{P}_{R_n^\beta,a_n,b_n}(e_x \in \gamma\,|\gamma[0,T_{k+1}]\,) = \mathbb{P}_{R_n^\beta\setminus\gamma[0,T_{k+1}],\gamma(T_{k+1}),b_n}(x \rightsquigarrow \text{wired arc}) \le 2c_4\frac{\sqrt{2^{k+1}}}{|x-y|} \; a.s..$$

Making use of this uniform bound, we obtain

$$\begin{aligned}
&\mathbb{P}_{R_n^\beta,a_n,b_n}(x,y \rightsquigarrow \text{wired arc})\\
&\le 2c_3c_4\sum_{k=0}^{k_0-2}\frac{\sqrt{2^{k+1}}}{\sqrt{2^k}|x-y|}\mathbb{P}_{R_n^\beta,a_n,b_n}(T_{k+1}<\infty) + 2c_3\frac{\mathbb{P}_{R_n^\beta,a_n,b_n}(T_x<\infty)}{\sqrt{2^{k_0-1}}}\\
&\le \frac{\sqrt{2}c_3c_4c_5}{|x-y|\sqrt{n}}\sum_{k=0}^{k_0-2}\sqrt{2^k} + \frac{2c_3c_5}{\sqrt{n2^{k_0-1}}}\\
&\le \frac{c}{\sqrt{n|x-y|}},
\end{aligned}$$

using also Lemma 4.4 (twice) for the second inequality. □

We are now in a position to prove our result.

Proof of Theorem 1.1 Let $\beta > 0$, $n > 0$, and also R_n^β defined as previously.

Step 1: lower bound for free boundary conditions. Let N_n be the number of connected pairs (x, u), with $x \in \partial_- R_n^\beta$, and $u \in \partial_+ R_n^\beta$. The expected value of this quantity is equal to

$$\mathbb{E}^0_{R_n^\beta}[N_n] = \sum_{\substack{u \in \partial_+ R_n^\beta \\ x \in \partial_- R_n^\beta}} \mathbb{P}^0_{R_n^\beta}(x \rightsquigarrow u). \tag{33}$$

Proposition 4.2 directly provides the following lower bound on the expectation by summing on the $(\beta n)^2$ pairs of points (x, u) far enough from the corners, *i.e.* satisfying the condition of the proposition:

$$\mathbb{E}^0_{R_n^\beta}[N_n] \geq c_6(\beta) n \tag{34}$$

for some $c_6(\beta) > 0$.

On the other hand, if x and u (resp. y and v) are pair-wise connected, then they are also connected to the horizontal line $\mathbb{Z} \times \{n\}$ which is (vertically) at the middle of R_n^β. Moreover, the domain Markov property implies that the probability – in R_n^β with free boundary conditions – that x and y are connected to this line is smaller than the probability of this event in the rectangle of half height with wired boundary conditions on the top side. In the following, we assume without loss of generality that n is even and we set $m = n/2$, so that the previous rectangle is $R_m^{2\beta}$, and we define a_m and b_m as before. Using the FKG inequality, and also the symmetry of the lattice, we get

$$\mathbb{P}^0_{R_n^\beta}(x \rightsquigarrow u, y \rightsquigarrow v) \quad \leq \mathbb{P}_{R_m^{2\beta}, a_m, b_m}(x, y \rightsquigarrow \text{wired arc})\, \mathbb{P}_{R_m^{2\beta}, a_m, b_m}(\bar{u}, \bar{v} \rightsquigarrow \text{wired arc}),$$

where $\bar{u}$ and $\bar{v}$ are the projections on the real axis of u and v. Summing the bound provided by Proposition 4.3 on all sites $x, y \in \partial_- R_n^\beta$ and $u, v \in \partial_+ R_n^\beta$, we obtain

$$\mathbb{E}^0_{R_n^\beta}[N_n^2] \leq c_7 m^2 \leq c_7 n^2 \tag{35}$$

for some constant $c_7 > 0$. Now, by the Cauchy-Schwarz inequality,

$$\mathbb{P}^0_{R_n^\beta}(\mathcal{C}_v(R_n^\beta)) = \mathbb{P}^0_{R_n^\beta}(N_n > 0) = \mathbb{E}^0_{R_n^\beta}[(\mathbb{I}_{N_n > 0})^2] \geq \frac{\mathbb{E}^0_{R_n^\beta}[N_n]^2}{\mathbb{E}^0_{R_n^\beta}[N_n^2]} \geq c_6(\beta)^2 / c_7, \tag{36}$$

since $\mathbb{E}^0_{R_n^\beta}[N_n] = \mathbb{E}^0_{R_n^\beta}[N_n \mathbb{I}_{N_n > 0}]$. We have thus reached the claim.

Step 2: lower and upper bounds for general boundary conditions. Using the ordering between boundary conditions Eq.(4), the lower bound that we have just proved for free boundary conditions actually implies the lower bound for any boundary conditions ξ.

For the upper bound, consider a rectangle R with dimensions $n \times m$ with $m \in [\![\beta_1 n, \beta_2 n]\!]$ and with boundary conditions ξ. Using once again Eq.(4), it is sufficient to address the case of wired boundary conditions, and in this case, the probability that there exists a dual crossing from the left side to the right side is at least $c_- = c_-(1/\beta_2, 1/\beta_1)$, since the dual model has free boundary conditions. We deduce, using the self-duality property, that

$$\mathbb{P}_R^\xi(\mathcal{C}_v(R)) \leq 1 - \mathbb{P}_R^1(\mathcal{C}_h^*(R)) = 1 - \mathbb{P}_{R^*}^0(\mathcal{C}_h(R^*)) \leq 1 - c_-, \tag{37}$$

where we use the notation $\mathcal{C}_h^*$ for the existence of a horizontal dual crossing, and R^* is as usual the dual graph of R (note that we have implicitly used the invariance by $\pi/2$-rotations). This concludes the proof of Theorem 1.1.

□

5 Consequences for the FK Ising and the (spin) Ising models

5.1 Critical exponents for the FK Ising and the Ising models

Power-law decay of the magnetization at criticality

We start by stating an easy consequence of Theorem 1.1. We consider the box $S_n = [\![-n, n]\!]^2$, its boundary being denoted as usual by ∂S_n. We also introduce the annulus $S_{m,n} = S_n \setminus \mathring{S}_m$ of radii $m < n$ centered on the origin, and we denote by $\mathcal{C}(S_{m,n})$ the event that there exists an open circuit surrounding S_m in this annulus.

Corollary 5.1 (circuits in annuli) *For every $\beta < 1$, there exists a constant $c_\beta > 0$ such that for all n and m, with $m \leq \beta n$,*

$$\mathbb{P}_{S_{m,n}}^0(\mathcal{C}(S_{m,n})) \geq c_\beta.$$

Proof This follows from Theorem 1.1 applied in the four rectangles $R_B = [\![-n, n]\!] \times [\![-n, -m]\!]$, $R_L = [\![-n, -m]\!] \times [\![-n, n]\!]$, $R_T = [\![-n, n]\!] \times [\![m, n]\!]$ and $R_R = [\![m, n]\!] \times [\![-n, n]\!]$. Indeed, if there exists a crossing in each of these rectangles in the "hard" direction, one can construct from them a circuit in $S_{m,n}$.

Now, consider any of these rectangles, R_B for instance. Its aspect ratio is bounded by $2/(1-\beta)$, so that Theorem 1.1 implies that there is a horizontal crossing with probability at least

$$\mathbb{P}_{R_B}^0(\mathcal{C}_H(R_B)) \geq c > 0.$$

Combined with the FKG inequality, this allows us to conclude: the desired probability is at least $c_\beta = c^4 > 0$. □

Proposition 5.2 (power-law decay of the magnetization) *For $p = p_{\rm sd}$, there exists a unique infinite-volume FK-Ising measure $\mathbb{P}_{\mathbb{Z}^2}$. For this measure, there is almost surely no infinite open cluster. Moreover, there exist constants $\alpha, c > 0$ such that for all $n \geq 0$,*

$$\mathbb{P}_{\mathbb{Z}^2}(0 \rightsquigarrow \partial S_n) \leq \frac{c}{n^\alpha}. \tag{38}$$

This result also applies to the Ising model: the magnetization at the origin decays at least as a power law.

Remark 5.3 *We would like to mention that an alternative proof of the fact that there is no spontaneous magnetization at criticality can be found in [14, 39]. Also, we actually know from Onsager's work that the connection probability follows a power law as $n \to \infty$, described by the one-arm plane exponent $\alpha_1 = 1/8$. It should be possible to prove the existence and the value of this exponent using conformal invariance, as well as the arm exponents for a larger number of arms. More precisely, one would need to consider the probability of crossing an annulus a certain (fixed) number of times in the scaling limit, and analyze the asymptotic behavior of this probability as the modulus tends to ∞. Theorem 1.1 then implies the so-called quasi-multiplicativity property, which allows one to deduce, using concentric annuli, the existence and the value of the arm exponents for the discrete model.*

Proof We first note that it is classical that the non-existence of infinite clusters implies the uniqueness of the infinite-volume measure: it is thus sufficient to prove Eq.(38). We consider the annuli $A_n = S_{2^n, 2^{n+1}}$ for $n \geq 1$, and $\mathcal{C}^*(A_n)$ the event that there is a dual circuit in A_n^*. We know from Corollary 5.1 that there exists a constant $c > 0$ such that

$$\mathbb{P}^1_{A_n}(\mathcal{C}^*(A_n)) \geq c \tag{39}$$

for all $n \geq 1$. By successive conditionings, we then obtain

$$\mathbb{P}_{\mathbb{Z}^2}(0 \rightsquigarrow \partial S_{2^N}) \leq \prod_{n=0}^{N-1} \mathbb{P}^1_{A_n}((\mathcal{C}^*(A_n))^c) \leq (1-c)^N, \tag{40}$$

and the desired result follows. □

n-point functions for the FK Ising and the Ising models

Since the work of Onsager [28], it is known that for the Ising model at criticality, the magnetization at the middle of a square of side length $2m$ with $(+)$ boundary conditions decays like $m^{-1/8}$. It is then tempting to say that the correlation of two spins at distance m in the plane (in the infinite-volume limit, say) decays like $m^{-1/4}$, and this is indeed what happens. To the knowledge of the authors, there is no straightforward generalization of Onsager's work that allows to derive this without difficult computations. However, this result can be made rigorous very easily with the help of Theorem 1.1. We give here only a result for two-point correlation functions, but exponents for n-spin correlations, for instance, can be obtained using exactly the same method.

Let us first use Theorem 1.1 to interpret Onsager's result in terms of the FK representation.

Lemma 5.4 *Let S_m be the square $[\![-m,m]\!]^2$ with arbitrary boundary conditions ξ. Then there exist two constants c_1 and c_2 (independent of m and ξ) such that we have*

$$c_1 m^{-1/8} \leq \mathbb{P}^{\xi}_{S_m}(0 \leadsto \partial S_m) \leq c_2 m^{-1/8}. \tag{41}$$

Proof This is a consequence of Onsager's result for wired boundary conditions (since it is derived in terms of the Ising model with (+) boundary conditions), which provides the upper bound by monotonicity. Using Theorem 1.1, we can obtain a lower bound independent of the boundary conditions by enforcing the existence of a circuit in the annulus $S_{m/2,m}$, and using the FKG inequality. For that, we just need to lower the constant, using monotonicity: the connection probability conditionally on the fact that there is a wired annulus around the origin is indeed larger than the connection probability with wired boundary conditions on ∂S_m. □

We can now state the result for two-point correlation functions in the infinite-volume Ising model.

Proposition 5.5 *Consider the Ising model on $\mathbb{Z}^2$ at critical temperature. There exist two positive constants C_1 and C_2 such that we have*

$$C_1|x-y|^{-1/4} \leq \mathbb{E}_{\beta_c}[\sigma_x\sigma_y] \leq C_2|x-y|^{-1/4}, \tag{42}$$

where for any $x, y \in \mathbb{Z}^2$, we denote by σ_x and σ_y the spins at x and y, and $\mathbb{P}_{\beta_c}$ is the infinite-volume Ising measure at β_c.

Proof The 2-spin correlation $\mathbb{E}_{\beta_c}[\sigma_x\sigma_y]$ can be expressed, in the corresponding FK representation, as the probability of the event $\{x \leadsto y\}$. Let now m be the integer part of $|x-y|/4$. The upper bound is easy and does not rely on Theorem 1.1: the event that x is connected to y implies that x is connected to $x+\partial S_m$ and that y is connected to $y+\partial S_m$. Using the domain Markov property, these two events are independent conditionally on the boundaries of the boxes being open: together with the previous lemma, this provides the upper bound.

Let us turn now to the lower bound. We can enforce the existence of a connected "8" in

$$[(x+S_{2m+2}) \cup (y+S_{2m+2})] \setminus [(x+S_m) \cup (y+S_m)]$$

that surrounds both x and y and separates them: this costs only a positive constant α, independent of m, using Theorem 1.1 in well-chosen rectangles and the FKG inequality. Using once again the FKG inequality, we get that

$$\mathbb{P}_{\mathbb{Z}^2}(x \rightsquigarrow y) \geq \alpha \mathbb{P}_{\mathbb{Z}^2}(x \rightsquigarrow x + \partial S_{2m+2}) \cdot \mathbb{P}_{\mathbb{Z}^2}(y \rightsquigarrow y + \partial S_{2m+2}), \tag{43}$$

and combined with the previous lemma, this yields the desired result. □

Half-plane one-arm exponent for the FK Ising model and boundary magnetization for the Ising model

As a by-product of our proofs, in particular of the estimates of Section 3, one can also obtain the value of the critical exponent for the boundary magnetization in the Ising model, near a free boundary arc (assuming it is smooth), and the corresponding one-arm half-plane exponent for the FK Ising model.

Let us first consider the one-point magnetization $\mathbb{E}_{\mathcal{D},a,b}[\sigma_x]$ for the Ising model at criticality in a discrete domain $(\mathcal{D}, a, b)$ with free boundary conditions on the counterclockwise arc (ab), and $(+)$ boundary conditions on the other arc (ba).

Proposition 5.6 *There exist positive constants c_1 and c_2 such that for any discrete domain $(\mathcal{D}, a, b)$ with $a = (-n, 0)$ and $b = (n, 0)$ $(n \geq 0)$, containing the rectangle $R_n = [\![-n, n]\!] \times [\![0, n]\!]$ and such that its boundary contains the lower arc $[\![-n, n]\!] \times \{0\}$, we have*

$$c_1 n^{-1/2} \leq \mathbb{E}_{\mathcal{D},a,b}[\sigma_0] \leq c_2 n^{-1/2}, \tag{44}$$

uniformly in n.

Proof The magnetization at the origin can be expressed, in the corresponding FK representation, as the probability that the origin is connected to the wired counterclockwise arc (ba). By Lemma 4.1, we can compare this probability to the harmonic measures $\mathbf{HM}_\circ$ and $\mathbf{HM}_\bullet$, for which estimates similar to the estimates in Lemmas 3.4 and 3.5 hold. □

This result can be equivalently stated for the one-arm half-plane probability for FK percolation:

Proposition 5.7 *Consider the rectangle $R_n = [\![-n, n]\!] \times [\![0, n]\!]$. There exist positive constants c_1 and c_2 such that for any boundary conditions ξ such that the bottom side $\partial^- R_n$ is free, one has*

$$c_1 n^{-1/2} \leq \mathbb{P}^{\xi}_{R_n}(0 \rightsquigarrow \partial^+ R_n) \leq c_2 n^{-1/2}, \tag{45}$$

uniformly over all n.

Proof We get the upper bound using monotonicity and the previous proposition, since (+) boundary conditions in the Ising model correspond to wired boundary conditions in the corresponding FK representation. For the lower bound, by Theorem 1.1 and the FKG inequality, we can enforce the existence of a crossing in the half-annulus $R_n \setminus R_{n/2}$ that disconnects 0 from $\partial R_n \setminus \partial^- R_n$ to the price of a constant independent of ξ. Using monotonicity and FKG, the probability that 0 is connected by an open path to this crossing (conditionally on its existence) is larger than the probability that 0 is connected to the boundary with wired boundary conditions on $\partial R_n \setminus \partial^- R_n$, without conditioning. Hence, the lower bound of the previous proposition gives the desired result. $\square$

Remark 5.8 *Note that contrary to the power laws established using the* SLE *technology, there are no potential logarithmic corrections here – as is the case with the "universal" arm exponents for percolation (corresponding to 2 and 3 arms in the half-plane, and 5 arms in the plane). Furthermore, one can follow the same standard reasoning as for percolation, based on the RSW lower bound, to prove that the two- and three-arm half-plane exponents, with alternating "types" (primal or dual), have values* 1 *and* 2 *respectively.*

5.2 Regularity of interfaces and tightness

Theorem 1.1 can be used to apply the technology developed by Aizenman and Burchard [1], to prove regularity of the collection of interfaces, which implies tightness using a variant of the Arzelà-Ascoli theorem.

This compactness property for the set of interfaces is important to construct the scaling limits of discrete interfaces, once we have a way to identify their limit uniquely (using for instance the so-called martingale technique, detailed in [33]). Here, the fermionic observable provides a conformally invariant martingale, and its convergence to a holomorphic function has been proved in [34], leading to the following important theorem:

Theorem 5.9 (Smirnov [35]) *For any Dobrushin domain $(\mathcal{D}, a, b)$, with discrete lattice approximations $(\mathcal{D}_\epsilon, a_\epsilon, b_\epsilon)$, the $\mathbb{P}_{\mathcal{D}_\epsilon, a_\epsilon, b_\epsilon}$-law of the exploration path γ_ϵ from a_ϵ to b_ϵ converges weakly to the law of a chordal* SLE(16/3) *path in $\mathcal{D}$, from a to b.*

We briefly explain how one can use the crossing bounds to obtain the compactness of the interfaces. Note that this result has also been proved, in a different way, in [15] and in the forthcoming article [16].

As usual, curves are defined as continuous functions from $[0,1]$ into a bounded domain $\mathcal{D}$ – more precisely, as equivalence classes up to strictly increasing reparametrization. The *curve distance* is given by

$$d(\gamma_1, \gamma_2) \;=\; \inf_{\phi} \sup_{u \in [0,1]} |\gamma_1(u) - \gamma_2(\phi(u))|, \tag{46}$$

where the infimum is taken over all strictly increasing bijections $\phi : [0,1] \to [0,1]$.

Let $S_{n,N}(x) = x + S_{n,N}$ be the annulus of radii $n < N$ centered at x. We denote by $\mathcal{A}_k(x; r, R)$ the event that there are $2k$ pairwise disjoint crossings of the curve in $S_{n,N}(x)$ (from its inner boundary to its outer boundary).

Theorem 5.10 (Aizenman-Burchard [1]) *Let $\mathcal{D}$ be a compact domain and denote by $\mathbb{P}_\epsilon$ the law of a random curve $\tilde{\gamma}_\epsilon$ with short-distance cut-off $\epsilon > 0$. If for any $k > 0$, there exists $C_k < \infty$ and $\lambda_k > 0$ such that for all $\epsilon < r < R$ and $x \in \mathcal{D}$,*

$$\mathbb{P}_\epsilon(\mathcal{A}_k(x; r, R)) \le C_k \left(\frac{r}{R}\right)^{\lambda_k}, \tag{47}$$

and $\lambda_k \to \infty$, then the curves $(\tilde{\gamma}_\epsilon)$ are precompact for the weak convergence associated with the curve distance.

This theorem can be applied to the family (γ_ϵ) of exploration paths defined in Theorem 5.9, using the following argument. If $\mathcal{A}_k(x; r, R)$ holds, then there are k open paths, alternating with k dual paths, connecting the inner boundary of the annulus to its outer boundary. Moreover, one can decompose the annulus $S_{r,R}(x)$ into roughly $\log_2(R/r)$ annuli of the form $S_{r,2r}(x)$, so that it is actually sufficient to prove that

$$\mathbb{P}_{\mathcal{D}_\epsilon, a_\epsilon, b_\epsilon}(\mathcal{A}_k(x; r, 2r)) \le c^k \tag{48}$$

for some constant $c < 1$. Since the paths are alternating, one can deduce that there are k open crossings, each one being surrounded by two dual paths. Hence, using successive conditionings and the domain Markov property, the probability for each crossing is smaller than the probability that there is a crossing in the annulus, which is less than some constant $c < 1$ by Corollary 5.1 (note that this reasoning also holds on the boundary).

Hence, Theorem 5.10 implies that the family (γ_ϵ) is precompact for the weak convergence.

5.3 Spatial mixing at criticality

Theorem 1.1 also provides estimates on spatial mixing for both the FK Ising and the Ising models. In the following proposition, we give an example of decorrelation between events for the FK Ising model.

Proposition 5.11 *There exist $c, \alpha > 0$ such that for any $k \le n$,*

$$\left|\mathbb{P}_{\mathbb{Z}^2}(A \cap B) - \mathbb{P}_{\mathbb{Z}^2}(A)\mathbb{P}_{\mathbb{Z}^2}(B)\right| \le c\left(\frac{k}{n}\right)^{\alpha} \mathbb{P}_{\mathbb{Z}^2}(A)\mathbb{P}_{\mathbb{Z}^2}(B) \tag{49}$$

for any event A (resp. B) depending only on the edges in the box S_k (resp. outside S_n), the measure $\mathbb{P}_{\mathbb{Z}^2}$ being the (unique) infinite-volume FK percolation measure for $q = 2$ and $p = p_{\mathrm{sd}}$.

Proof First, it is sufficient to prove

$$\left|\mathbb{P}^{\xi}_{\Lambda_n}(A) - \mathbb{P}^{1}_{\Lambda_n}(A)\right| \le c\left(\frac{k}{n}\right)^{\alpha} \mathbb{P}^{\xi}_{\Lambda_n}(A) \tag{50}$$

for any event A depending on edges in Λ_k.

Claim: There exists a coupling P on configurations (ω_ξ, ω_1) with the following properties:

- ω_ξ (resp. ω_1) has law $\mathbb{P}^{\xi}_{\Lambda_n}$ (resp. $\mathbb{P}^{1}_{\Lambda_n}$).
- if ω_1 contains a closed circuit in $\Lambda_n \setminus \Lambda_k$, let Γ be the exterior most such circuit. Then Γ is also closed in ω_ξ and ω_1 and ω_ξ coincide inside Γ.
- if ω_ξ contains an open circuit in $\Lambda_n \setminus \Lambda_k$, let $\tilde{\Gamma}$ be the exterior most such circuit. Then $\tilde{\Gamma}$ is also open in ω_1 and ω_1 and ω_ξ coincide inside $\tilde{\Gamma}$.

Proof of the claim Consider uniform random variables U_e for every edge e. Sample both configurations based on the same random variables U_e from the exterior, meaning that after k steps, you look at one edge with one endpoint connected to the boundary of Λ_n by an open path, until it is not possible anymore (meaning that you discovered a closed circuit). Note that ω_1 is larger than ω_ξ by comparison between boundary conditions. Therefore, the circuit will also be closed in ω_ξ. Then the configurations inside this circuit will be the same since boundary conditions are free in this new domain. Similarly, the last condition also holds.

□

Now, since A depends only on the edges in Λ_k, we can prove that conditionally to A, there exists a dual circuit in $\phi^1_{\Lambda_n}$ and $\phi^{\xi}_{\Lambda_n}$ with probability $1-c(k/n)^{\alpha}$. Let E be this event. We deduce

$$\begin{aligned}
\mathbb{P}^{\xi}_{\Lambda_n}(A) &\ge \mathbb{P}^{\xi}_{\Lambda_n}(A \cap E) \\
&= P(\omega_\xi \in A \cap E) \\
&\ge P(\omega_1 \in A \cap E) \\
&= \phi^1(A \cap E) \\
&\ge (1 - c(k/n)^{\alpha})\phi^1(A)
\end{aligned}$$

where in the third line, we used the fact that if ω_ξ belongs to E, then ω_1 belongs to E, and they coincide in Λ_k, so that $\omega_\xi \in A$ if $\omega_1 \in A$.

Reciprocally, if F denotes the event that there is an open circuit in $\Lambda_n \setminus \Lambda_k$, we find

$$\begin{aligned}
\mathbb{P}^{1}_{\Lambda_n}(A) &\ge \mathbb{P}^{1}_{\Lambda_n}(A \cap F) \\
&= P(\omega_1 \in A \cap F) \\
&\ge P(\omega_\xi \in A \cap F) \\
&= \phi^{\xi}(A \cap F) \\
&\ge (1 - c(k/n)^{\alpha})\phi^{\xi}(A)
\end{aligned}$$

where once again, we used in the third line that if $\omega_\xi \in F$, then ω_1 is in F, and in this case, they coincide on Λ_k so that $\omega_\xi \in A$ implies that $\omega_1 \in A$. $\square$

More generally, Theorem 1.1 would lead to ratio mixing properties, with an explicit polynomial estimate. Away from criticality, estimates of this type can be established by using the rate of spatial decay for the influence of a single site. At criticality, the correlation between distant events does not boil down to correlations between points and a finer argument must be found. Crossing-probability estimates which are *uniform in boundary conditions* are perfectly suited for these problems.

Recently, Lubetzky and Sly [26] used spatial mixing properties of the Ising model in order to derive an important conjecture on the mixing time of the Glauber dynamics of the Ising model at criticality. As a key step, they harness Theorem 1.1 in order to prove a suitable analogue of the previous proposition. Together with tools from the analysis of Markov chains, the spatial mixing property provides polynomial upper bounds on the inverse spectral gap of the Glauber dynamics (and also on the total variation mixing time).

6 Conjecture for general values of q

We conclude this article by stating a conjecture on FK models for other values $q \geq 1$. As we have seen, crossing estimates at criticality are useful for many purposes, proving such bounds should thus be fundamental for studying two-dimensional FK percolation models.

For $1 \leq q < 4$, the FK model at $p = p_{\mathrm{sd}}(q)$ is conjectured to be conformally invariant in the scaling limit. More precisely, the collection of interfaces in a domain with free boundary conditions should converge to the so-called $\mathrm{CLE}(\kappa(q))$ process, with $\kappa(q) = 4\pi/\arccos(-\sqrt{q}/2)$. The following conjecture is thus natural:

Conjecture 6.1 *Consider the FK percolation model of parameter* $(p_{\mathrm{sd}}(q), q)$ *with* $1 \leq q < 4$ *and let* $0 < \beta_1 < \beta_2$. *There exist two constants* $0 < c_-(q) \leq c_+(q) < 1$ *such that for any rectangle* R *with side lengths* n *and* $m \in [\![\beta_1 n, \beta_2 n]\!]$, *one has*

$$c_-(q) \leq \mathbb{P}^{\xi}_{p_{\mathrm{sd}}(q),q,R}(\mathcal{C}_v(R)) \leq c_+(q)$$

for any *boundary conditions* ξ.

At $q > 4$, the random-cluster model (conjecturally) undergoes a first order phase transition at $p_{\mathrm{sd}}(q) = \sqrt{q}/(1+\sqrt{q})$, in the following sense (this result has been proved for $q \geq 25.72$, see [13] and references therein): at criticality, there exist different infinite-volume measures. If one considers the infinite-volume measure with wired boundary conditions, the probability of having an infinite cluster is 1, while if one considers the infinite-volume measure with free boundary conditions, the probability of having an infinite cluster is 0 and the two-point functions decay exponentially fast. Therefore, the probability of having a crossing goes to 1 (resp. to 0) with wired boundary conditions (resp. free boundary conditions). A result analogue to Theorem 1.1 does not thus hold in this setting.

At $q = 4$, the picture should be slightly different. It is conjectured that the family of interfaces converges to the CLE(4) process, which would imply that the probability of having crossings between two opposite sides with free boundary conditions converges to 0. Nevertheless, a slight modification of the previous conjecture is expected to hold true: the probability of having a circuit surrounding the origin in an annulus of fixed modulus, with free boundary conditions, stays bounded away from 0 and 1 uniformly in the size of the annulus.

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