Continuity of the phase transition for planar random-cluster and Potts models with $1 \leq q \leq 4$

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Abstract

This article studies the planar Potts model and its random-cluster representation. We show that the phase transition of the nearest-neighbor ferromagnetic q-state Potts model on $\mathbb{Z}^2$ is continuous for $q \in \{2, 3, 4\}$, in the sense that there exists a unique Gibbs state, or equivalently that there is no ordering for the critical Gibbs states with monochromatic boundary conditions.

The proof uses the random-cluster model with cluster-weight $q \geq 1$ (note that q is not necessarily an integer) and is based on two ingredients:

- The fact that the two-point function for the free state decays sub-exponentially fast for cluster-weights $1 \leq q \leq 4$, which is derived studying parafermionic observables on a discrete Riemann surface.
- A new result proving the equivalence of several properties of critical random-cluster models:
  - the absence of infinite-cluster for wired boundary conditions,
  - the uniqueness of infinite-volume measures,
  - the sub-exponential decay of the two-point function for free boundary conditions,
  - a Russo-Seymour-Welsh type result on crossing probabilities in rectangles with *arbitrary boundary conditions*.

The result leads to a number of consequences concerning the scaling limit of the random-cluster model with $q \in [1, 4]$. It shows that the family of interfaces (for instance for Dobrushin boundary conditions) are tight when taking the scaling limit and that any sub-sequential limit can be parametrized by a Loewner chain. We also study the effect of boundary conditions on these sub-sequential limits. Let us mention that the result should be instrumental in the study of critical exponents as well.

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1 Introduction

1.1 Motivation

The Potts model is a model of random coloring of $\mathbb{Z}^2$ introduced as a generalization of the Ising model to more-than-two components spin systems. In this model, each vertex of $\mathbb{Z}^2$ receives a spin among q possible colors. The energy of a configuration is proportional to the number of frustrated edges, meaning edges whose endpoints have different spins. Since its introduction by Potts [Pot52], the model has been a laboratory for testing new ideas and developing far-reaching tools. In two dimensions, it exhibits a reach panel of possible critical behaviors depending on the number of colors, and despite the fact that the model is exactly solvable (yet not rigorously for $q \neq 2$), the mathematical understanding of its phase transition remains restricted to a few cases (namely $q = 2$ and q large). We refer to [Wu82] for a review on this model.

The question of deciding whether a phase transition is continuous or discontinuous constitutes one of the most fundamental question in statistical physics, and an extensive literature has been devoted to this subject in the case of the Potts model. In the planar

case, Baxter [Bax71, Bax73, Bax89] used a mapping between the Potts model and solid-on-solid ice-models to compute the spontaneous magnetization at criticality for $q \geq 4$. He was able to predict that the phase transition was discontinuous for $q \geq 5$. While this computation gives a good insight on the behavior of the model, it relies on unproved assumptions which, forty years after their formulation, seem still very difficult to justify rigorously (and are related to the nature of the phase transition itself). Furthermore, Baxter outlined another heuristic prediction that the phase transition is continuous for $q \leq 4$, but, according to the author himself, the justification is not as solid for this case. Among other results, the present article proves that the phase transition is continuous for $q \in \{2, 3, 4\}$ without any reference to unproved assumptions.

Most of our article will be devoted to the study of the random-cluster model. This model is a probability measure on edge configurations, each edge being declared open or closed, such that the probability of a configuration is proportional to $p^{\#\text{ open edges}}(1-p)^{\#\text{ closed edges}}q^{\#\text{ clusters}}$, where clusters are maximal connected subgraphs of the configuration, and $(p, q) \in [0, 1] \times \mathbb{R}_+$. For $q = 1$, the model is Bernoulli percolation.

Since its introduction by Fortuin and Kasteleyn [FK72], the random-cluster model has become an important tool in the study of phase transitions. The spin correlations of Potts models are rephrased as cluster connectivity properties of their random-cluster representations. As a byproduct, properties of the random-cluster model can be transferred to the Potts model, and vice-versa. *The random-cluster model being an important model of its own, Theorems 3 and 4 below are of independent interest.*

While the understanding of critical Bernoulli percolation in dimensions two is relatively advanced, the case of the random-cluster model remains mysterious. The long range dependency makes the model challenging to study by probabilistic tools, and some of its most basic properties were not proved until recently. In this article, we derive several properties of the critical model, including a suitable generalization of the celebrated *Russo-Seymour-Welsh* approach available for Bernoulli percolation. This powerful tool enables us to prove several new results for the critical phase.

This article fits in the more general context of the study of conformally invariant planar lattice models. In the early eighties, physicists Belavin, Polyakov and Zamolodchikov postulated conformal invariance of critical planar statistical models [BPZ84b, BPZ84a]. This prediction enabled physicists to invoke Conformal Field Theory in order to formulate many conjectures on these models. From a mathematical perspective, proving rigorously the conformal invariance of a model (and properties following from it) constitutes a formidable challenge.

In recent years, the connection between discrete holomorphicity and planar statistical physics led to spectacular progress in this direction. Kenyon [Ken00, Ken01], Smirnov [Smi10] and Chelkak and Smirnov [CS12] exhibited discrete holomorphic observables in the dimer and Ising models and proved their convergence to conformal maps in the scaling limit. These results paved the way to the rigorous proof of conformal invariance for these two models. Other discrete observables have been proposed for a number of critical models, including self-avoiding walks and Potts models. While these observables are *not exactly discrete holomorphic*, their discrete contour integrals vanish, a property shared with discrete holomorphic functions. It is a priori unclear whether this property is of any relevance for the models. Nevertheless, in the case of the self-avoiding walk, it was proved to be sufficient to compute the connective constant of the hexagonal lattice

[DS12]. One can consider that this article is part of a program initiated informally in [DS12] and devoted to the study of the possible applications of the property mentioned above. In our case, we will use parafermionic observables introduced independently in [FK80, RC06, Smi06, Smi10] to prove our main theorem as well as several corollaries. (More precisely, we will use a corollary which may be deduced from the study of such observables.) Since such observables have been found in a large class of planar critical models, we believe that similar applications can be obtained for these models as well, and that the tools developed in this article should be instrumental there. Last but not least, these techniques improve the understanding of the scaling limit of these models, and we think that they will be useful for proving conformal invariance.

1.2 Definition of the models and main statements

1.2.1 Definition of Potts models and statement of the main theorem

Consider an integer $q \geq 2$ and a subgraph $G = (V_G, E_G)$ of the square lattice $\mathbb{Z}^2$. Here and below, V_G is the set of *vertices* of G and $E_G \subset V_G^2$ is the set of *edges*. For simplicity, the square lattice will be identified with its set of vertices, namely $\mathbb{Z}^2$. For two vertices $x, y \in V_G$, $x \sim y$ denotes the fact that $(x, y) \in E_G$.

Let $\tau \in \{1, \dots, q\}^{\mathbb{Z}^2}$. The *q-state Potts model on G with boundary conditions* τ is defined as follows. The space of configurations is $\Omega = \{1, \dots, q\}^{\mathbb{Z}^2}$. For a configuration $\sigma = (\sigma_x : x \in \mathbb{Z}^2) \in \Omega$, the quantity σ_x is called the *spin* at x (it is sometimes interpreted as being a color). The *energy* of a configuration $\sigma \in \Omega$ is given by the Hamiltonian

$$H_G^\tau(\sigma) \ := \begin{cases} -\displaystyle\sum_{\substack{x \sim y \\ \{x,y\} \cap G \neq \emptyset}} \delta_{\sigma_x, \sigma_y} & \text{if } \sigma_x = \tau_x \text{ for } x \notin V_G, \\ \infty & \text{otherwise.} \end{cases}$$

Above, $\delta_{a,b}$ denotes the Kronecker symbol equal to 1 if $a = b$ and 0 otherwise. The spin-configuration is sampled proportionally to its Boltzmann weight: at an inverse-temperature β, the probability $\mu_{G,\beta}^\tau$ of a configuration σ is defined by

$$\mu_{G,\beta}^\tau[\sigma] \ := \ \frac{\mathrm{e}^{-\beta H_G^\tau(\sigma)}}{Z_{G,\beta}^\tau} \quad \text{where} \quad Z_{G,\beta}^\tau \ := \ \sum_{\sigma \in \Omega} \mathrm{e}^{-\beta H_G^\tau(\sigma)}$$

is the *partition function* defined in such a way that the sum of the weights over all possible configurations equals 1. By construction, configurations that do not coincide with τ outside of G have probability 0.

Infinite-volume Gibbs measures can be defined by taking limits, as G tends to $\mathbb{Z}^2$, of finite-volume measures $\mu_{G,\beta}^\tau$. In particular, if (i) denotes the constant boundary configuration equal to $i \in \{1, \dots, q\}$, the sequence of measures $\mu_{G,\beta}^{(i)}$ converges, as G tends to infinity, to a Gibbs measure denoted by $\mu_{\mathbb{Z}^2,\beta}^{(i)}$. This measure is called the *infinite-volume Gibbs measure with monochromatic boundary conditions i*.

The Potts models undergo a phase transition in infinite volume at a certain *critical inverse-temperature* $\beta_c(q) \in (0, \infty)$ in the following sense

$$\mu_{\mathbb{Z}^2,\beta}^{(i)}[\sigma_0 = i] = \begin{cases} \frac{1}{q} & \text{if } \beta < \beta_c(q), \\ \frac{1}{q} + m_\beta > \frac{1}{q} & \text{if } \beta > \beta_c(q). \end{cases}$$

The value $\beta_c(q)$ is computed in [BD12a] and is equal to $\log(1+\sqrt{q})$ for any integer q (this value was previously known for $q = 2$ [Ons44] and $q \geq 26$ [LMR86]).

This article is devoted to the study of the phase transition at $\beta = \beta_c(q)$. The phase transition is said to be *continuous* if $\mu^{(i)}_{\mathbb{Z}^2,\beta_c(q)}[\sigma_0 = i] = \frac{1}{q}$ and *discontinuous* otherwise. The main result is the following.

Theorem 1 (Continuity of the phase transition for 2, 3 or 4 colors)**.** *Let $q \in \{2, 3, 4\}$. Then for any $i \in \{1, \ldots, q\}$, we have*

$$\mu^{(i)}_{\mathbb{Z}^2,\beta_c(q)}[\sigma_0 = i] = \tfrac{1}{q}.$$

This result was known in the $q = 2$ case. For two colors, the model is the Ising model. Onsager computed the free energy in [Ons44] and Yang obtained a formula for the magnetization in [Yan52]. In particular, this formula implies that the magnetization is zero at criticality. This results has been reproved in a number of papers since then. Let us mention a recent proof [Wer09] not using any exact integrability.

For q equal to 3 or 4, the result appears to be new. As mentioned in the previous section, exact yet non rigorous computations performed by Baxter strongly suggest that the phase transition is continuous for $q \leq 4$, and discontinuous for $q > 4$. This result therefore tackles the whole range of q for which the phase transition is continuous. Let us mention that the technology developed here is not restricted to the study of Potts models for $q \leq 4$: a property of Potts models with $q \geq 5$ colors witnessing some sort of ordering at criticality is also derived, see Proposition 5. Unfortunately, we were unable to show rigorously that the phase transition is discontinuous in the sense defined above.

In dimension $d \geq 3$, the phase transition is expected to be continuous if and only if $q = 2$. The best results in this direction are the following. On the one hand, the fact that the phase transition is continuous for the Ising model ($q = 2$) is known for any $d \geq 3$ [ADS15] (in fact, the critical exponents are known to be taking their mean-field value for $d \geq 4$ [AF86]). On the other hand, mean-field considerations together with Reflection Positivity enabled [BCC06] to prove that for any $q \geq 3$, the q-state Potts model undergoes a discontinuous phase transition above some dimension $d_c(q)$. Finally, Reflection Positivity can be harnessed to prove that for any $d \geq 2$, the phase transition is discontinuous provided q is large enough [KS82].

1.2.2 The random-cluster model

The proof of Theorem 1 is based on the study of a graphical representation of the Potts model, called the *random-cluster model*. Let us define it properly. Consider $p \in [0, 1]$, $q > 0$ and a subgraph $G = (V_G, E_G)$ of the square lattice. A configuration ω is an element of $\Omega' = \{0, 1\}^{E_G}$. An edge e with $\omega(e) = 1$ is said to be *open*, while an edge with $\omega(e) = 0$ is said to be *closed*. Two vertices x and y in V_G are said to be *connected* (this event is denoted by $x \longleftrightarrow y$) if there exists a sequence of vertices $x = v_1, v_2, \ldots, v_{r-1}, v_r = y$ such that $\{v_i, v_{i+1}\}$ is an open edge for every $i < r$. A *connected component* of ω is a maximal connected subgraph of ω. Let $o(\omega)$ and $c(\omega)$ be respectively the number of open and closed edges in ω.

The *random-cluster measure on E_G with edge-weight p, cluster-weight q, and* **free**

boundary conditions is defined by the formula

$$\phi^0_{G,p,q}[\omega] = \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k_0(\omega)}}{Z^0_{G,p,q}},$$

where $k_0(\omega)$ is the number of connected components of the graph ω, and $Z^0_{G,p,q}$ is defined in such a way that the sum of the weights over all possible configurations equals 1. We also define the *random-cluster measure on E_G with edge-weight p, cluster-weight q, and* **wired** *boundary conditions* by the formula

$$\phi^1_{G,p,q}[\omega] = \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k_1(\omega)}}{Z^1_{G,p,q}},$$

where $k_1(\omega)$ is the number of connected components of the graph ω, except that all the connected components of vertices in the vertex boundary ∂G, i.e. the set of vertices in V_G with less than four neighbors in G, are counted as being part of the same connected component. Again, $Z^1_{G,p,q}$ is defined in such a way that the sum of the weights over all possible configurations equals 1.

For $q \geq 1$, infinite-volume measures can be defined on $\mathbb{Z}^2$ by taking limits of finite-volume measures for graphs tending to $\mathbb{Z}^2$. In particular, the *infinite-volume random-cluster measure with free (resp. wired) boundary conditions* $\phi^0_{\mathbb{Z}^2,p,q}$ (resp. $\phi^1_{\mathbb{Z}^2,p,q}$) can be defined as the limit of the sequence of measures $\phi^0_{G,p,q}$ (resp. $\phi^1_{G,p,q}$) for $G \nearrow \mathbb{Z}^2$. We refer the reader to [Gri06] for more details on this construction.

The random-cluster model with $q \geq 1$ undergoes a phase transition in infinite volume in the following sense. There exists $p_c(q) \in (0,1)$ such that

$$\phi^1_{\mathbb{Z}^2,p,q}[0 \longleftrightarrow \infty] = \begin{cases} 0 & \text{if } p < p_c(q), \\ \theta^1(p,q) > 0 & \text{if } p > p_c(q), \end{cases}$$

where $\{0 \longleftrightarrow \infty\}$ denotes the event that 0 belongs to an infinite connected component. The value of $p_c(q)$ was recently proved to be equal to $\sqrt{q}/(1+\sqrt{q})$ for any $q \geq 1$ in [BD12a] (see also [DM15]). The result was previously proved in [Kes80] for Bernoulli percolation ($q = 1$), in [Ons44] for $q = 2$ using the connection with the Ising model and in [LMMS+91] for $q \geq 25.72$.

Similarly to the Potts model case, a notion of continuous/discontinuous phase transition can be defined: the phase transition is said to be *continuous* if $\phi^1_{\mathbb{Z}^2,p_c(q),q}[0 \longleftrightarrow \infty] = 0$ and discontinuous otherwise. The following theorem is the *alter ego* of Theorem 1.

Theorem 2 (Continuous phase transition for cluster-weight $1 \leq q \leq 4$)**.** *Let $q \in [1,4]$, then $\phi^1_{\mathbb{Z}^2,p_c(q),q}[0 \longleftrightarrow \infty] = 0$.*

Note that q may not be an integer in this theorem. Let us now describe briefly the coupling between Potts models and their random-cluster representation. Fix $q \geq 2$ integer. From a random-cluster configuration sampled according to $\phi^1_{G,p,q}$, color each component (meaning all the vertices in it) with one color chosen uniformly in $\{1,\dots,q\}$, except for the connected component containing the vertices in ∂G which receive color i. The law of the random coloring thus obtained is $\mu^{(i)}_{G,\beta}$, where $\beta = -\log(1-p)$. This

coupling between the random-cluster model with integer cluster-weight and the Potts models enables us to deduce Theorem 1 from Theorem 2 immediately: simply notice that

$$\mu^{(i)}_{G,\beta}[\sigma_0 = i] = \tfrac{1}{q} + \phi^1_{\mathbb{Z}^2,p,q}[0 \leftrightarrow \partial G],$$

and take the limit as G tends to $\mathbb{Z}^2$. For this reason, we now focus on Theorem 2.

1.2.3 An alternative for the behavior of critical random-cluster models

Proving Theorem 2 requires a much better understanding of the critical phase than the one available until now. Indeed, except for the $q = 1$, $q = 2$ and $q \geq 25.72$ cases, very little was known on critical random-cluster models. The following theorem provides new insight on the possible critical behavior of these models.

For an integer n, let Λ_n denote the *box* $[-n, n]^2$ *of size* n. An open path is a path of adjacent open edges (we refer to the next section for a formal definition). Let $0 \longleftrightarrow \partial\Lambda_n$ be the event that there exists an open path from the origin to the boundary of Λ_n. For a rectangle $R = [a, b] \times [c, d]$, let $\mathcal{C}_h(R)$ be the event that there exists an open path in R from $\{a\} \times [c, d]$ to $\{b\} \times [c, d]$.

Theorem 3. *Let $q \geq 1$. The following assertions are equivalent :*

P1 *(Absence of infinite cluster at criticality)* $\phi^1_{\mathbb{Z}^2,p_c,q}[0 \longleftrightarrow \infty] = 0$.

P2 $\phi^0_{\mathbb{Z}^2,p_c,q} = \phi^1_{\mathbb{Z}^2,p_c,q}$.

P3 *(Infinite susceptibility)* $\chi^0(p_c, q) := \sum_{x\in\mathbb{Z}^2} \phi^0_{\mathbb{Z}^2,p_c,q}[0 \longleftrightarrow x] = \infty$.

P4 *(Sub-exponential decay for free boundary conditions)*

$$\lim_{n\to\infty} \tfrac{1}{n} \log \phi^0_{\mathbb{Z}^2,p_c,q}[0 \longleftrightarrow \partial\Lambda_n] = 0.$$

P5 *(RSW) For any $\alpha > 0$, there exists $c = c(\alpha) > 0$ such that for all $n \geq 1$,*

$$\phi^0_{[-n,(\alpha+1)n]\times[-n,2n],p_c,q}[\mathcal{C}_h([0, \alpha n] \times [0, n])] \geq c.$$

The previous theorem does not show that these conditions are all satisfied, but that they are equivalent. In fact, whether the conditions are satisfied or not will depend on the value of q, see Section 1.2.4 for a more detailed discussion.

The previous result was previously known in a few cases:

- *Bernoulli percolation (random-cluster model with $q = 1$).* In such case **P2** is obviously satisfied. Furthermore, Russo [Rus78] proved that **P1**, **P3** and **P4** are all true (and therefore equivalent). Finally, **P5** was proved by Russo [Rus78] and Seymour-Welsh [SW78].
- *Random-cluster model with $q = 2$.* This model is directly related to the Ising via the coupling between Potts and random-cluster models mentioned above. Therefore, all of these properties can be proved to be true using the following results on the Ising model: Onsager proved that the critical Ising measure is unique and that

the phase transition is continuous in [Ons44]. Properties **P3** and **P4** follow from Simon's correlation inequality for the Ising model [Sim80]. Property **P5** was proved in [DHN11] using a proof specific to the Ising model. Interestingly, in the Ising case each property is derived independently and no direct equivalence was known previously.

- *Random-cluster model with* $q \geq 25.72$. In this case, none of the above properties are satisfied, as proved by using the Pirogov-Sinai technology [LMMS+91].

Except for these special cases, no general result was known, and Theorem 3 represents, to the best of our knowledge, the first formal proof of the equivalence between these conditions for a relatively large class of dependent percolation models. We expect that a similar result can be stated for a large class of models, and that some of the tools developed in this article may be extended to these models.

Remark. **P4**⇒**P1** implies that whenever there is an infinite-cluster for the wired boundary conditions, correlations decay exponentially fast at criticality for free boundary conditions.

Before proceeding further, let us discuss alternative conditions which could replace the conditions **P1**–**P5**. Once again, q is assume to be larger or equal to 1.

The condition **P1** has the following interpretation: it is equivalent to

P1' (Continuous phase transition) $\lim_{p \searrow p_c} \phi^1_{\mathbb{Z}^2,p,q}[0 \longleftrightarrow \infty] = 0$

(simply use [Gri06, (4.35)]). Note that the (almost sure) absence of an infinite-cluster for $\phi^0_{\mathbb{Z}^2,p_c,q}$ follows from Zhang's argument [Gri06, Theorem (6.17)(a)] and is true for any $q \geq 1$. Nevertheless, it does not imply the (almost sure) absence of an infinite-cluster for $\phi^1_{\mathbb{Z}^2,p_c,q}$ nor the continuity of the phase transition.

The property **P2** can be reinterpreted in terms of infinite-volume measures (see [Gri06] for a formal definition). Then, **P2** is equivalent to (see [Gri06, Theorem (4.34)])

P2' The infinite-volume measure on $\mathbb{Z}^2$ at p_c and q is unique.

Let us now turn to **P4** which can be understood in terms of the so-called *correlation length* defined for $p < p_c(q)$ by the formula

$$\xi(p,q) = \Big(- \lim_{n\to\infty} \tfrac{1}{n} \log \phi^0_{\mathbb{Z}^2,p,q}[(0,0) \longleftrightarrow (n,0)] \Big)^{-1}.$$

Now, **P4** is equivalent to

P4' (vanishing mass-gap) $\xi(p,q)$ tends to $+\infty$ as $p \nearrow p_c(q)$.

Condition **P1** together with **P3** have an interesting consequence in terms of the order of the phase transition for the Potts model. We do not enter in the details here but let us briefly mention that properties **P1** and **P3** are respectively equivalent to the continuity and the non-differentiability with respect to the magnetic field h of the free energy at $(\beta = \beta_c, h = 0)$. Therefore, these properties mean that the phase transition of the corresponding Potts model is of *second order*. The properties **P1**–**P4** (and their equivalent formulations) are classical definitions describing continuous phase transitions and are believed to be equivalent for many natural models, even though it is a priori unclear how this can be proved in a robust way.

Now that we have an interpretation for properties **P1**–**P4**, let us explain why **P5** is of particular interest: it provides an equivalent to the RSW theorem proved in [BD12a] which is *uniform in boundary conditions* (see Proposition 2 below). This uniformity with respect to boundary conditions is crucial for applications, especially when trying to decouple events, see e.g. Section 1.3.1. Let us also emphasize that the fact that **P5** can be derived from the other properties requires the development of a Russo-Seymour-Welsh theory for dependent percolation models. As mentioned above, such a theory existed for Bernoulli percolation [SW78, Rus78], and for $q = 2$ [DHN11], but in the latter case the proof was based on discrete holomorphicity, hence hiding the close connection between **P5** and the other properties. This Russo-Seymour-Welsh theory is expected to apply to a large class of planar models, and we insist on the fact that uniformity on boundary conditions is crucial.

Remark. The restriction on boundary conditions being at distance n from the rectangle can be relaxed in the following way: if **P5** holds, then for any $\alpha > 0$ and $\varepsilon > 0$, there exists $c = c(\alpha, \varepsilon) > 0$ such that for every $n \geq 1$,

$$\phi^{\xi}_{[-\varepsilon n,(\alpha+\varepsilon)n]\times[-\varepsilon n,(1+\varepsilon)n],p_c,q}\left[\mathcal{C}_h([0,\alpha n]\times[0,n])\right] \geq c.$$

It is natural to ask why boundary conditions are fixed at distance εn of the rectangle $[0, \alpha n] \times [0, n]$ and not simply on the boundary. The reason is the latter property is not equivalent to **P1**–**P5**. Indeed, it may in fact be the case that **P5** holds but that crossing probabilities of rectangles $[0, \alpha n] \times [0, n]$ with free boundary conditions on their boundary converge to zero as n tends to infinity. Such phenomenon does not occur for $1 \leq q < 4$ as shown in Theorem 7 below but is expected to occur for $q = 4$. For this reason, we will always work with boundary conditions at "macroscopic distance" from the boundary.

1.2.4 Random-cluster model with cluster weight $q \in [1, 4]$

The previous alternative provides us with a powerful tool to prove Theorem 2. Namely, it is sufficient to prove one of the properties **P2**–**P5** when $1 \leq q \leq 4$ to derive our result. We will therefore focus on property **P4**, which is the easiest to check.

In order to prove **P4**, we will use estimates on the probability of being connected by an open path which can be deduced from the fact that discrete contour integrals of the so-called *(edge) parafermionic observable* vanish. This observable was introduced in [Smi06, Smi10] for $q \in (0, 4)$ and then generalized to $q > 4$ in [BDS13] (the $q = 4$ case also requires the introduction of a slightly different observable). It satisfies local relations that can be understood as discretizations of the Cauchy-Riemann equations when the model is critical. These relations imply that discrete contour integrals vanish. We do not recall the definition of the parafermionic observable nor do we describe its principal properties, and simply mention an important corollary (see Equation (20)). For more details on the parafermionic observable, we refer to [Smi06] or [Dum13, Chapter 6].

In any case, the parafermionic observable can be used to show the following theorem dealing with random-cluster models with $1 \leq q \leq 4$.

Theorem 4 ([Dum12])**.** *Let* $1 \leq q \leq 4$*, then* $\lim_{n\to\infty} \frac{1}{n} \log \phi^0_{\mathbb{Z}^2,p_c(q),q}\left[0 \longleftrightarrow \partial\Lambda_n\right] = 0.$

Theorem 4 together with Theorem 3 implies that **P1–P5** are satisfied for $1 \leq q \leq 4$. In particular, these two theorems imply Theorem 2.

As mentioned above, the proof uses the fact that the discrete contour integrals of parafermionic observables vanish. The intrinsic difficulty of this theorem relies on the fact that, for our proof to work, the random-cluster model needs to be considered on the universal cover of $\mathbb{Z}^2$ minus a face; see Section 2.2 for more details. Bootstrapping the information from the universal cover to the plane is not straightforward.

Investigating more general lattice models on this universal cover seems an interesting direction of research. Indeed, lattice models on Riemann surfaces have been studied extensively. Nevertheless, the theory mostly deals with surfaces of higher genius, while in our case we are facing (a discretization of) a simply connected planar Riemann surface with a logarithmic singularity at the origin.

The proof of Theorem 4 can be found in [Dum12]. Nevertheless, other applications of parafermionic observables which are based on the same principle will be derived in this article, and most of the tools required for the proof of Theorem 4 will be used in other places as well. In addition, some of the ideas of the proof of Theorem 3 allows one to simplify drastically the proof exposed in [Dum12]. We therefore chose to include a streamlined version of the proof here (still, the details on parafermionic observables are omitted).

In order to conclude this section, let us mention that because the discrete contour integrals vanish, the parafermionic observable is therefore a discretization of a divergence-free differential form. For $q = 2$ (which corresponds to the Ising model), further information can be extracted from local integrability and the observable satisfies a strong notion of discrete holomorphicity, called *s-holomorphicity*. In this case, the observable can be used to understand many properties on the model, including conformal invariance of the observable [CS12, Smi10] and loops [CDH$^+$14, HK13], correlations [CI13, CHI15, Hon14, HS11] and crossing probabilities [BDH14, DHN11, CDH13]. It can also be extended away from criticality [BD12b, DGP14]. We do not discuss special features of the $q = 2$ case and we refer to the extensive literature for further information.

1.3 Applications to the study of the critical phase for $1 \leq q \leq 4$

The previous theorems have a large number of consequences regarding the understanding of the critical phase. We list some of them now.

1.3.1 Mixing properties at criticality

The bound **P5** on crossing probabilities enables us to study the spatial mixing properties at criticality. One may decouple events which are depending on edges in different areas of the space, and therefore compensate for the lack of independence. The next theorem illustrates this fact. It will be used in many occasions in the reminder of the paper.

Theorem 5 (Polynomial ratio weak mixing under condition **P5**). *Fix $q \geq 1$ such that Property* **P5** *of Theorem 3 is satisfied. There exists $\alpha > 0$ such that for any $2k \leq n$ and*

for any event A depending only on edges in Λ_k,

$$\left|\phi^{\xi}_{\Lambda_n,p_c,q}[A] - \phi^{\psi}_{\Lambda_n,p_c,q}[A]\right| \leq \left(\frac{k}{n}\right)^{\alpha} \phi^{\xi}_{\Lambda_n,p_c,q}[A]$$

uniformly in boundary conditions ξ and ψ.

Together with the Domain Markov property, this result implies the following inequality for $2k \leq m \leq n$ (with n possibly infinite),

$$\left|\phi^{\xi}_{\Lambda_n,p_c,q}[A \cap B] - \phi^{\xi}_{\Lambda_n,p_c,q}[A]\,\phi^{\xi}_{\Lambda_n,p_c,q}[B]\right| \leq \left(\frac{k}{m}\right)^{\alpha} \phi^{\xi}_{\Lambda_n,p_c,q}[A]\,\phi^{\xi}_{\Lambda_n,p_c,q}[B],$$

where the boundary conditions ξ are arbitrary, A is an event depending on edges of Λ_k only, and B is an event depending on edges of $\Lambda_n \setminus \Lambda_m$.

Remark. For $p \neq p_c(q)$, estimates of this type (with an exponential speed of convergence instead of polynomial) can be established by using the rate of spatial decay for the influence of a single vertex [Ale98]. At criticality, the correlation between distant events does not boil down to correlations between points and a finer argument must be harnessed. Crossing-probability estimates which are *uniform in boundary conditions* are the key in order to prove such results.

Remark. We will see several specific applications of this theorem in the next chapters. The most striking consequence is the fact that the dependence on boundary conditions can be forgotten as long as the boundary conditions are sufficiently distant from the set of edges determining whether the events under consideration occur or not. For instance, it allows us to state several theorems in infinite volume, keeping in mind that most of these results possess natural counterparts in finite volume by using the fact that

$$c\phi_{p_c,q}[A] \leq \phi^{\xi}_{p_c,q,\Lambda_{2n}}[A] \leq C\phi_{p_c,q}[A]$$

for any event A depending on edges in Λ_n only, and any boundary conditions ξ (the constants c and C are universal).

1.3.2 Consequences for the scaling limit

The phase transition being continuous, the scaling limit of the model is expected to be conformally invariant. This work opens new perspectives in the study of this scaling limit and we now mention several possible directions of research. We refer the reader to Section 4 for the definition of Dobrushin domains and the exploration path.

Conjecture 1 (Schramm, [Sch07])**.** *Let $0 \leq q \leq 4$ and $p = p_c(q)$. Let (Ω, a, b) be a simply connected domain with two marked points on its boundary a and b. Let $(\Omega_\delta, a_\delta, b_\delta)$ be a sequence of Dobrushin domains converging in the Carathéodory sense towards (Ω, a, b). The law of the exploration path (γ_δ) for critical random-cluster model with cluster-weight q and Dobrushin boundary conditions in $(\Omega_\delta, a_\delta, b_\delta)$ converges, as the mesh size δ tends to zero, to the Schramm-Loewner Evolution between a and b in Ω with parameter $\kappa = \frac{4\pi}{\arccos(-\sqrt{q}/2)}$.*

Schramm-Loewner Evolutions are very-well studied objects; see e.g. [Law05]. This convergence would therefore lead to a deep understanding of the critical phase of the random-cluster models. For the $q = 0$ case corresponding to the perimeter curve of the uniform spanning tree, the conjecture was proved by Lawler, Schramm and Werner [LSW04]. The $q = 2$ case was formally proved in [CDH$^+$14] even though the fundamental contribution leading to this result was achieved in [Smi10]. Conjecture 1 is open for any other values of q (even for the $q = 1$ case corresponding to Bernoulli percolation).

Lawler, Schramm and Werner proposed a global strategy for proving Conjecture 1, which can be summarized as follows:

1. Prove compactness of the family of exploration paths $(\gamma_\delta)_{\delta>0}$ and show that any sub-sequential limits can be parametrized as a Loewner chain (with a continuous driving process denoted by W).

2. Prove the convergence of some discrete observables of the model.

3. Extract from the limit of these observables enough information to evaluate the conditional expectation and quadratic variation of W_t and use Lévy's theorem to prove that W_t is equal to $\sqrt{\kappa}B_t$, where B_t is the standard Brownian motion. As a consequence, any sub-sequential limits must be the Schramm-Loewner Evolution of parameter κ.

Step 1 of this program is provided by Theorem 6 below. We refer to [Law05] for details on Loewner chains.

Let X be the set of continuous parametrized curves and d be the distance on X defined by

$$d(\gamma_1, \gamma_2) = \min_{\varphi_1:[0,1]\to I, \varphi_2:[0,1]\to J \text{ increasing}} \sup_{t\in I} |\gamma_1(\varphi_1(t)) - \gamma_2(\varphi_2(t))|,$$

where $\gamma_1 : I \to \mathbb{C}$ and $\gamma_2 : J \to \mathbb{C}$. Note that I and J can be equal to $\mathbb{R}_+ \cup \{\infty\}$.

Theorem 6. *Fix $1 \le q \le 4$, $p = p_c(q)$ and a simply connected domain Ω with two marked points on its boundary a and b. Let $(\Omega_\delta, a_\delta, b_\delta)$ be a sequence of Dobrushin domains converging in the Carathéodory sense towards (Ω, a, b). Define γ_δ to be the exploration path in $(\Omega_\delta, a_\delta, b_\delta)$ with Dobrushin boundary conditions. Then, the family (γ_δ) is tight and any sub-sequential limit γ satisfies the following properties:*

R1 *γ is almost surely a continuous non-intersecting curve from a to b staying in Ω.*

R2 *For any parametrization $\gamma : [0, 1] \to \infty$, b is a simple point, in the sense that $\gamma(t) = b$ implies $t = 1$. Furthermore, almost surely $\gamma(t)$ is on the boundary of $\Omega \setminus \gamma[0, t]$ for any $t \in [0, 1]$.*

R3 *Let Φ be a conformal map from Ω to the upper half-plane $\mathbb{H}$ sending a to 0 and b to ∞. For any parametrization $\gamma : [0, 1] \to \mathbb{R}_+$, the h-capacity of the hull $\widehat{K}_s$ of $\Phi(\gamma[0, s])$ tends to ∞ when s approaches 1. Furthermore, if $(\widehat{K}_t)_{t\ge0}$ denotes $(\widehat{K}_s)_{s\in[0,1]}$ parametrized by h-capacity, then $(\widehat{K}_t)_{t\ge0}$ is a Loewner chain with a driving process $(W_t)_{t\ge0}$ which is α-Hölder for any $\alpha < 1/2$ almost surely. Furthermore, there exists $\varepsilon > 0$ such that for any $t > 0$, $\mathbb{E}[\exp(\varepsilon W_t/\sqrt{t})] < \infty$.*

R4 *There exists $\alpha > 0$ such that γ has Hausdorff dimension between $1 + \alpha$ and $2 - \alpha$ almost surely.*

Tightness criteria for random planar curves were first introduced in [AB99]. They were used as a key step in the proof of convergence of interfaces to the Schramm-Loewner Evolution for Bernoulli site percolation on the triangular lattice [CN07]. These criteria were improved in [KS12] to treat the case of random non-self-crossing planar curves parametrized as Loewner chains.

Step 2 represents the main challenge in the program outlined above (Step 3 is easy once Steps 1 and 2 have been achieved, see e.g. [Dum13, Section 13.2]). Smirnov succeeded to perform Step 2 for $q = 2$ using the fermionic observable [Smi10]. He also proposed to consider the parafermionic observables introduced in [Smi06] as a potential candidate for Step 2 in the case of general cluster-weights $q < 4$ (let us mention that the choice of the observables in Step 2 are not determined uniquely). For completeness, let us mention a conjecture which, together with the results of this paper, would imply Conjecture 1.

Conjecture 2 (Smirnov)**.** *Let $0 < q < 4$, $p = p_c(q)$ and (Ω, a, b) be a simply connected domain with two points on its boundary. For every $z \in \Omega$,*

$$\frac{1}{(2\delta)^\sigma} F_\delta(z) \;\to\; \phi'(z)^\sigma \quad \textit{when } \delta \to 0$$

where

- *for $\delta > 0$, F_δ is the vertex parafermionic observable of [Smi06] at $p_c(q)$ in $(\Omega_\delta, a_\delta, b_\delta)$.*
- $\sigma = 1 - \frac{2}{\pi}\arccos(\sqrt{q}/2)$,
- *ϕ is any conformal map from Ω to $\mathbb{R} \times (0,1)$ sending a to $-\infty$ and b to ∞.*

Let us mention that even though we are currently unable to prove Conjecture 2 (and therefore Conjecture 1), we are still able to obtain results on the scaling limit. Indeed, the geometry of the random curve γ can be easily related to the geometry of clusters boundaries at a discrete level. Keeping in mind that we are not able to prove that the scaling limit of cluster boundaries is well-defined, we may still extract sub-sequential limits and ask simple properties about these objects. For instance, property **R4** of the previous theorem implies that any sub-sequential scaling limit of cluster boundaries of random-cluster models with $1 \le q \le 4$ is a random fractal. The next theorem corresponds to another property of these sub-sequential scaling limits: it shows that macroscopic clusters touch the boundary of a smooth domain, for instance a rectangle, with free boundary conditions.

Theorem 7. *Fix $1 \le q < 4$ and $\alpha > 0$. There exists $c_1 > 0$ such that for any $n \ge 1$,*

$$\phi^0_{[0,\alpha n]\times[0,n],p_c(q),q}\left[\mathcal{C}_h([0,\alpha n]\times[0,n])\right] \ge c_1.$$

In the previous theorem, the free boundary conditions are directly on the boundary of the rectangle $[0, \alpha n] \times [0, n]$. This corresponds to the most direct generalization of the Russo-Seymour-Welsh theorem.

1.3.3 Consequences for critical exponents

Theorem 4 has several implications which are postponed to a future paper for so-called arm-events. Let us quickly mention that one can prove a priori bounds on the probability

of arm-events, the so-called quasi-multiplicativity and extendability of these probabilities, as well as universal exponents. These tools are crucial in order to compute critical exponents via the understanding of the scaling limit. Let us also mention that universal bounds can be deduced between different critical exponents.

Theorem 4 should also instrumental in the understanding of the near-critical regime, and in particular to derive the scaling relations between critical exponents (see [Dum13, Section 13.2.3] for more details).

In another direction, Theorem 3 provides the relevant criteria in order to prove that the critical exponents of random-cluster models are universal on isoradial graphs (see [GM14, CS12] for the case of percolation and Ising), see [DLM15].

1.3.4 Consequences for Potts model

The coupling between Potts and random-cluster models enables one to transfer properties from one model to the other. In order to illustrate this fact, let us state the following theorem (many other results could be proved, but this would make this article substantially longer) which is a direct consequence of the previous theorems.

Theorem 8. *For $q \in \{2,3,4\}$, there exists a unique Gibbs measure $\mu_{\mathbb{Z}^2,\beta_c(q),q}$ for the critical q-state Potts model. Furthermore, there exist $\eta_1, \eta_2 > 0$ such that for any $x \in \mathbb{Z}^2 \setminus \{0\}$,*

$$\frac{1}{|x|^{\eta_1}} \leq \mu_{\mathbb{Z}^2,\beta_c(q),q}[\sigma_x = \sigma_0] - \frac{1}{q} \leq \frac{1}{|x|^{\eta_2}}.$$

The main result should also have consequences for the Glauber dynamics of the Potts model. Recently, Lubetzky and Sly [LS12] used spatial mixing properties of the Ising model to establish a long standing conjecture on the mixing time of the Glauber dynamics of the Ising model at criticality. As a key step, they employ the equivalent of **P5** together with tools from the analysis of Markov chains, to provide polynomial upper bounds on the inverse spectral gap of the Glauber dynamics (and also on the total variation mixing time). We plan to prove similar results for 3 and 4 states Potts models in a subsequent paper.

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2 Preliminaries

The norm $|\cdot|$ will denote the Euclidean norm.

2.1 Graph notation

Primal and dual graphs. The *square lattice* $(\mathbb{Z}^2, \mathbb{E})$ is the graph with vertex set $\mathbb{Z}^2 = \{(n,m) : n,m \in \mathbb{Z}\}$ and edge set $\mathbb{E}$ given by edges between nearest neighbors. The square lattice will be identified with the set of vertices, i.e. $\mathbb{Z}^2$. The *dual square lattice* $(\mathbb{Z}^2)^*$ is the dual graph of $\mathbb{Z}^2$. The vertex set is $(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$ and the edges are given by

nearest neighbors. The vertices and edges of $(\mathbb{Z}^2)^*$ are called *dual-vertices* and *dual-edges*. In particular, every edge of $\mathbb{Z}^2$ is naturally associated to a dual-edge, denoted by e^*, that it crosses in its middle.

Except otherwise specified, we will only consider subgraphs of $\mathbb{Z}^2$, $(\mathbb{Z}^2)^*$, and use the following notations. For a graph G, we denote by V_G its vertex set and by E_G its edge set. Two vertices x and y are *neighbors (in G)* if $\{x, y\} \in E_G$. In such case, we write $x \sim y$. Furthermore, if x is an end-point of e, we say that e is *incident* to x. Finally, the *boundary* of G, denoted by ∂G, is the set of vertices of G with strictly fewer than four incident edges in E_G.

For a graph $G \subset \mathbb{Z}^2$, we define G^* to be the subgraph of $(\mathbb{Z}^2)^*$ with edge-set $E_{G^*} = \{e^* : e \in E_G\}$ and vertex set given by the end-points of these dual-edges.

Let Λ_n be the subgraph of $\mathbb{Z}^2$ induced by the vertex set $[-n, n]^2$.

Connectivity properties in graphs. A path in $\mathbb{Z}^2$ is a sequence of vertices $v_0, \dots, v_n$ in $\mathbb{Z}^2$ such that $v_i \sim v_{i+1}$ for any $0 \leq i < n$. The path will often be identified with the set of edges $(v_0, v_1), \{v_1, v_2\}, \dots, \{v_{n-1}, v_n\}$. The path is said *to start at x and to end at y* if $v_0 = x$ and $v_n = y$. If $x = y$, the path is called a *circuit*.

Two vertices x and y of G are *connected* if there exists a path of edges in E_G from x to y. A graph is said to be *connected* if any two vertices of G are connected. The *connected components* of G will be the maximal connected subgraphs of G. For three sets $X, Y \subset V_G$ and $F \subset E_G$, we say that X is connected to Y in F (we denote this fact by $X \overset{F}{\longleftrightarrow} Y$) if there exist two vertices $x \in X$ and $y \in Y$ and a path of edges in F starting at x and ending at y.

2.2 Background on the random-cluster model

This section is devoted to a very brief description of the tools we will use in the proofs of the next sections. The reader may consult [Gri06] or [Dum13] for more details, proofs and references.

Random-cluster model and critical point. Let $G = (V_G, E_G)$ be a finite subgraph of $\mathbb{Z}^2$ or $\mathbb{U}$. A *configuration* on G is an element $\omega = \{\omega(e) : e \in E_G\} \in \{0, 1\}^{E_G}$. An edge e is said to be *open* if $\omega(e) = 1$, and *closed* otherwise. A configuration ω can be seen as a subgraph of G, whose vertex set is V_G and edge set if the set of open edges $\{e \in E_G : \omega(e) = 1\}$. A path (resp. circuit) in ω will be called an *open path* (resp. *open circuit*). Two sets X and Y are *connected* in ω if there exists an open path from X to Y. The connected components of ω will be called *clusters*.

The boundary conditions on G are given by a partition $\xi = P_1 \sqcup \cdots \sqcup P_k$ of ∂G. From a configuration ω, define ω^ξ to be the graph with vertex set V_G and edge set given by edges of ω together with edges of the form $\{x, y\}$, where x and y belong to the same P_i. In such case, the vertices x and y are sometimes said to be *wired* together.

Definition. The *random-cluster measure on G with edge-weight p, cluster-weight q, and boundary conditions ξ* is defined by the formula

$$\phi^\xi_{G,p,q}[\omega] = \frac{p^{o(\omega)}(1-p)^{c(\omega)} q^{k(\omega^\xi)}}{Z^\xi_{G,p,q}},$$

where $k(\omega^\xi)$ is the number of connected components of the graph ω^ξ. As usual, $Z^\xi_{G,p,q}$ is defined in such a way that the sum of the weights over all possible configurations equals 1.

The following boundary conditions play a special role in this article. The partition ξ composed of singletons only is called the *free boundary conditions* and is denoted by $\xi = 0$. It corresponds to no additional connections. The partition $\xi = \{\partial G\}$ is called the *wired boundary conditions* and is denoted by $\xi = 1$. It corresponds to the fact that all the boundary vertices are connected by boundary conditions.

Infinite-volume measures and critical point. We do not aim for a complete description, or even a formal definition of random-cluster measures on $\mathbb{Z}^2$ and we refer to [Gri06, Chapter 4] for details. When $q \geq 1$, infinite-volume random-cluster measures can be defined by taking the limit of finite-volume measures. In particular, the sequence of measures $\phi^1_{\Lambda_n,p,q}$ (resp. $\phi^0_{\Lambda_n,p,q}$) converges to the *infinite-volume measure with wired (resp. free) boundary conditions* $\phi^1_{\mathbb{Z}^2,p,q}$ (resp. $\phi^0_{\mathbb{Z}^2,p,q}$). Furthermore, there exists $p_c = p_c(q) \in (0,1)$ such that for $p \neq p_c(q)$, the infinite-volume measure $\phi_{\mathbb{Z}^2,p,q}$ is unique and satisfies

$$\phi_{\mathbb{Z}^2,p,q}[0 \leftrightarrow \infty] = \begin{cases} 0 & \text{if } p < p_c(q), \\ \theta(p,q) > 0 & \text{if } p > p_c(q). \end{cases}$$

The critical parameter $p_c(q)$ was proved to be equal to $\sqrt{q}/(1+\sqrt{q})$ in [BD12a].

Positive association when $q \geq 1$. Denote the product ordering on $\{0,1\}^E$ by $\leq$. An event $\mathcal{A}$ depending on edges in E only is *increasing* if for any $\omega' \geq \omega$, $\omega \in \mathcal{A}$ implies $\omega' \in \mathcal{A}$.

For $q \geq 1$, the random-cluster model satisfies important properties regarding increasing events. The first such property is the FKG inequality [Gri06, Theorem 3.8]: for any boundary conditions ξ and for any increasing events $\mathcal{A}$ and $\mathcal{B}$,

$$\phi^\xi_{G,p,q}[\mathcal{A} \cap \mathcal{B}] \geq \phi^\xi_{G,p,q}[\mathcal{A}]\, \phi^\xi_{G,p,q}[\mathcal{B}].$$

The second important property is the comparison between boundary conditions [Gri06, Lemma 4.56]: for any increasing event $\mathcal{A}$ and for any $\xi \geq \psi$,

$$\phi^\xi_{G,p,q}[\mathcal{A}] \geq \phi^\psi_{G,p,q}[\mathcal{A}]. \tag{2}$$

Here, $\xi \geq \psi$ if the partition ψ is finer than the partition ξ (any wired vertices in ψ are wired in ξ). In such case, ξ is said to *dominate* ψ or equivalently ψ is said to be *dominated by* ξ. The free (resp. wired) boundary conditions are dominated by (resp. dominates) any other boundary conditions.

Domain Markov property and insertion tolerance. Consider a subgraph G' of G. The following proposition describes how the influence of the configuration outside G' on the measure within G' can be encoded using appropriate boundary conditions ξ.

Proposition 1 (Domain Markov Property)**.** *Let $p \in [0,1]$, $q > 0$ and ξ some boundary conditions. Fix $G' \subset G$. Let X be a random variable measurable which is measurable with respect to edges in $E_{G'}$. Then,*

$$\phi^{\xi}_{G,p,q}\left[X|\omega_{|E_G \setminus E_{G'}} = \psi\right] = \phi^{\psi^{\xi}}_{G',p,q}\left[X\right],$$

for any $\psi \in \{0,1\}^{E_G \setminus E_{G'}}$. Above, ψ^{ξ} is the partition on $\partial G'$ obtained as follows: two vertices $x, y \in \partial G'$ are in the same element of the partition if they are connected in ψ^{ξ}.

The previous proposition has the following corollary, called insertion tolerance. For $p, q > 0$, there exists $c_{\mathrm{IT}} > 0$ such that for any finite graph G, any $\omega \in \{0,1\}^{E_G}$, and any boundary conditions ξ, $\phi^{\xi}_{G,p,q}[\omega] \geq c_{\mathrm{IT}}^{|G|}$.

Dual representation A configuration ω on G can be uniquely associated to a *dual configuration* ω^* on the dual graph G^* defined as follows: set $\omega^*(e^*) = 1$ if $\omega(e) = 0$ and $\omega^*(e^*) = 0$ if $\omega(e) = 1$. A dual-edge e^* is said to be *dual-open* if $\omega^*(e^*) = 1$, it is *dual-closed* otherwise. A *dual-cluster* is a connected component of ω^*. We extend the notion of *dual-open* path in a trivial way. For two sets $X, Y \subset V_{G^*}$, we set $X \overset{*}{\longleftrightarrow} Y$ if there exists a dual-open path from X to Y.

If ω is distributed according to $\phi^{\xi}_{G,p,q}$, then ω^* is distributed according to $\phi^{\xi^*}_{G^*,p^*,q^*}$ with $q^* = q$ and $\frac{pp^*}{(1-p)(1-p^*)} = q$. In particular, for $p = p_c(q)$ we find $p^* = p = p_c(q)$. The boundary conditions ξ^* can be deduced from ξ in a case by case manner. We will mostly be interested in the case of $\xi = 0$ or 1, for which $\xi^* = 1$ and 0 respectively.

Russo-Seymour-Welsh estimate with wired boundary conditions. For a rectangle $R = [a,b] \times [c,d]$, let $\mathcal{C}_h(R)$ be the event that there exists an open path in R from $\{a\} \times [c,d]$ to $\{b\} \times [c,d] \subset \mathbb{Z}^2$. Such a crossing is called a *horizontal crossing* of R. Similarly, one defines $\mathcal{C}_v(R)$ to be the event that there exists an open path in R from $[a,b] \times \{c\}$ to $[a,b] \times \{d\}$ (such a path is called a *vertical crossing*). For a rectangle $R^* = [s,t] \times [u,v] \subset (\mathbb{Z}^2)^*$ (note that s,t,u,v are half-integers), let $\mathcal{C}^*_h(R^*)$ be the event that there exists a dual-open dual-path in R^* from $\{s\} \times [u,v]$ to $\{t\} \times [u,v]$ (such a path is called an *horizontal dual-crossing*), and similarly $\mathcal{C}^*_v(R^*)$ is the event that there exists a dual-open dual-path in R^* from $[s,t] \times \{u\}$ to $[s,t] \times \{v\}$ (such a path is called a *vertical dual-crossing*).

The following result will be important in the proof. Let us restate it here.

Theorem 9 ([BD12a, Corollary 9])**.** *For $\alpha > 1$ and $q \geq 1$, there exists $c_{\mathrm{RSW}} > 0$ such that for every $0 < m < \alpha n$,*

$$\phi^{1}_{\mathbb{Z}^2,p_c,q}\left[\mathcal{C}_h([0,m] \times [0,n])\right] \geq c_{\mathrm{RSW}}.$$

Note that this bound on crossing probabilities is much weaker than **P5**, since it holds only for wired boundary conditions. Let us mention a particularly interesting case of non wired boundary conditions. We refer the interested reader to [BD12a] for more details. Consider the random-cluster measure on Λ_n with wired boundary conditions on top and bottom, and free elsewhere (we call these boundary conditions mixed). Then

$$\phi^{\mathrm{mixed}}_{\Lambda_n,p_c,q}\left[\mathcal{C}_h(\Lambda_n)\right] \geq \frac{1}{1+q^2}. \tag{3}$$

From now on, we fix $q \geq 1$ and $p = p_c(q)$. In order to lighten the notation, we drop the reference to p and q and simply write ϕ_G^ξ instead of $\phi_{G,p_c(q),q}^\xi$.

3 Proof of Theorem 3

3.1 A preliminary result

Before diving into the proof, let us mention three useful equivalent formulations of **P5**. For $z \in \mathbb{R}^2$, define $\mathcal{A}_n(z)$ to be the event that there exists an open circuit in the annulus $z + (\Lambda_{2n} \setminus \Lambda_n)$ surrounding z. Also define $\mathcal{A}_n = \mathcal{A}_n(0)$.

Proposition 2. *Recall that we are at $q \geq 1$ and $p = p_c(q)$. The following assertions are equivalent;*

P5 *For any $\alpha > 0$, there exists $c_1 = c_1(\alpha) > 0$ such that for all $n \geq 1$, we have*

$$\phi^0_{[-n,(\alpha+1)n]\times[-n,2n]}\left[\mathcal{C}_h([0,\alpha n]\times[0,n])\right] \geq c_1.$$

P5a *There exists $c_2 > 0$ such that for all $n \geq 1$,*

$$\phi^0_{\Lambda_{2n}\setminus\Lambda_{n+1}}\left[\mathcal{A}_n\right] \geq c_2.$$

P5b *For any $R \geq 2$, there exists $c_3 = c_3(R) > 0$ such that for all $n \geq 1$,*

$$\phi^0_{\Lambda_{Rn}}\left[\mathcal{A}_n\right] \geq c_3.$$

P5c *For any $\alpha > 0$, there exists $c_4 = c_4(\alpha) > 0$ such that for all $n \geq 1$ and for all boundary conditions ξ on the boundary of $[-n,(\alpha+1)n]\times[-n,2n]$, we have*

$$c_4 \leq \phi^\xi_{[-n,(\alpha+1)n]\times[-n,2n]}\left[\mathcal{C}_h([0,\alpha n]\times[0,n])\right] \leq 1 - c_4.$$

The last condition justifies the fact that the result is uniform with respect to boundary conditions.

Proof. The proof of **P5a**⇒**P5b** and **P5c**⇒**P5** are obvious by comparison between boundary conditions. In order to prove **P5**⇒**P5a**, consider the four rectangles

$$\begin{aligned}
R_1 &:= [4n/3, 5n/3]\times[-5n/3, 5n/3],\\
R_2 &:= [-5n/3, -4n/3]\times[-5n/3, 5n/3],\\
R_3 &:= [-5n/3, 5n/3]\times[4n/3, 5n/3],\\
R_4 &:= [-5n/3, 5n/3]\times[-5n/3, -4n/3].
\end{aligned}$$

If the intersection of $\mathcal{C}_v(R_1)$, $\mathcal{C}_v(R_2)$, $\mathcal{C}_h(R_3)$ and $\mathcal{C}_h(R_4)$ occurs, then $\mathcal{A}_n$ occurs. In particular, the FKG inequality and the comparison between boundary conditions implies that c_2 can be chosen to be equal to $c_1(10)^4$.

Let us now turn to the proof of **P5b**⇒**P5**. We start by the lower bound. Fix some $R \geq 2$ as in **P5b** and the corresponding $c_3 > 0$. Let $\alpha > 0$. For $n \geq 4R$, the intersection of the events $\mathcal{A}_{n/(2R)}[(j\lfloor \frac{n}{R} \rfloor, \frac{n}{2})]$ for $j = 0, \dots, \lceil R\alpha \rceil$ is included in $\mathcal{C}_h([0, \alpha n] \times [0, n])$. The FKG inequality implies

$$\phi^0_{[-n,(\alpha+1)n]\times[-n,2n]} \left[\mathcal{C}_h([0, \alpha n] \times [0, n])\right] \geq c_3^{1+R\lceil \alpha \rceil}.$$

By comparison between boundary conditions, we obtain the lower bound for every ξ.

The upper bound may be obtained from this lower bound as follows. By comparison between boundary conditions once again, it is sufficient to prove the upper bound for the wired boundary conditions. In such case, the complement of $\mathcal{C}_h([0, \alpha n] \times [0, n])$ is $\mathcal{C}^*_v([\frac{1}{2}, \alpha n - \frac{1}{2}] \times [-\frac{1}{2}, n + \frac{1}{2}])$. Since the dual of the wired boundary conditions are the free ones, the boundary conditions for the dual measure are free. We can now harness **P5b** for the dual model to construct a dual path from top to bottom with probability bounded away from 0. This finishes the proof.

It remains to prove that **P5** implies **P5c** to conclude. Define

$$\begin{aligned} R &:= [0, \alpha n] \times [0, n], \\ \overline{R} &:= [-n, (\alpha + 1)n] \times [-n, 2n], \\ R^* &:= [\tfrac{1}{2}, \alpha n - \tfrac{1}{2}] \times [-\tfrac{1}{2}, n + \tfrac{1}{2}]. \end{aligned}$$

First, (2) implies that

$$c_1 \leq \phi^0_{\overline{R}} \left[\mathcal{C}_h(R)\right] \leq \phi^\xi_{\overline{R}} \left[\mathcal{C}_h(R)\right],$$

where the first inequality is due to **P5**. Now, we wish to prove the upper bound. Let us assume without loss of generality that $\alpha < 1$. As mentioned above, the complement of the event $\mathcal{C}_h(R)$ is the event $\mathcal{C}^*_v(R^*)$. We find that

$$c_1 \leq \phi^1_{\overline{R}} \left[\mathcal{C}_h(R)\right] = 1 - \phi^1_{\overline{R}} \left[\mathcal{C}^*_v(R^*)\right] \leq 1 - c_1(1/\alpha).$$

In the last inequality, we used the lower bound proved previously for $\alpha' = 1/\alpha$ and $n' = \alpha n$. To justify that we can do so, observe that the dual of wired boundary conditions are the free ones, and that since $\alpha < 1$, the dual graph of $\overline{R}$ contains a rotated version of $[\frac{1}{2} - \alpha n, \alpha n - \frac{1}{2} + \alpha n] \times [-\frac{1}{2} - \alpha n, n + \frac{1}{2} + \alpha n]$. □

3.2 Proof of Theorem 3: easy implications

In order to isolate the hard part of the proof, let us start by checking the four "simple" implications **P1**⇒**P2**, **P2**⇒**P3**, **P3**⇒**P4** and **P5**⇒**P1**.

Property **P1** *implies* **P2***:* This implication is classical, see e.g. [Dum13, Corollary 4.23].

Property **P2** *implies* **P3***:* If **P2** holds,

$$\begin{aligned} (2n+1)\, \phi^0_{\mathbb{Z}^2} \left[0 \leftrightarrow \partial\Lambda_n\right] = (2n+1)\, \phi^1_{\mathbb{Z}^2} \left[0 \leftrightarrow \partial\Lambda_n\right] &\geq \sum_{x \in \{0\}\times[-n,n]} \phi^1_{\mathbb{Z}^2} \left[x \leftrightarrow (x + \partial\Lambda_n)\right] \\ &\geq \phi^1_{\mathbb{Z}^2} \left[\mathcal{C}_v([-n, n] \times [0, n])\right] \geq c, \end{aligned}$$

where $c > 0$ is a constant independent of n. The first equality is due to the uniqueness of the infinite-volume measure given by **P2** and the second inequality by Theorem 9. This leads to

$$\sum_{x\in\partial\Lambda_n} \phi^0_{\mathbb{Z}^2}[0 \leftrightarrow x] \geq \phi^0_{\mathbb{Z}^2}[0 \leftrightarrow \partial\Lambda_n] \geq \frac{c}{2n+1}.$$

As a consequence, $\sum_{x\in\mathbb{Z}^2} \phi^0_{\mathbb{Z}^2}[0 \leftrightarrow x] = \infty$ and **P3** holds true.

Property **P3** *implies* **P4***:* Assume that **P4** does not hold. In such case, the fact that

$$\begin{aligned}\phi^0_{\mathbb{Z}^2}[0 \longleftrightarrow (n+m)x] &\geq \phi^0_{\mathbb{Z}^2}[\{0 \longleftrightarrow nx\} \cap \{nx \longleftrightarrow (n+m)x\}] \\ &\geq \phi^0_{\mathbb{Z}^2}[0 \longleftrightarrow nx]\, \phi^0_{\mathbb{Z}^2}[0 \longleftrightarrow mx]\end{aligned}$$

(the second inequality follows from the FKG inequality) implies the existence of $M > 0$ such that for every $x = (x_1, x_2) \in \mathbb{Z}^2$,

$$\phi^0_{\mathbb{Z}^2}[0 \longleftrightarrow x] \leq \mathrm{e}^{-|x|/M}.$$

Summing over every $x \in \mathbb{Z}^2$ gives

$$\sum_{x\in\mathbb{Z}^2} \phi^0_{\mathbb{Z}^2}[0 \leftrightarrow x] < \infty$$

and thus **P3** does not hold.

Property **P5** *implies* **P1***:* Recall that **P5** implies **P5a**. We now prove a slightly stronger result which obviously implies **P1** and will be useful later in the proof.

Lemma 1. *Property* **P5a** *implies that there exists* $\varepsilon > 0$ *such that for any* $n \geq 1$,

$$\phi^1_{\mathbb{Z}^2}[0 \leftrightarrow \partial\Lambda_n] \leq n^{-\varepsilon}.$$

Proof. Let k be such that $2^k \leq n < 2^{k+1}$. Also define the annuli $A_j = \Lambda_{2^j} \setminus \Lambda_{2^{j-1}-1}$ for $j \geq 1$. We have

$$\begin{aligned}\phi^1_{\mathbb{Z}^2}[0 \longleftrightarrow \partial\Lambda_n] &\leq \prod_{j=1}^{k} \phi^1_{\mathbb{Z}^2}\left[\Lambda_{2^{j-1}} \overset{A_j}{\longleftrightarrow} \partial\Lambda_{2^j} \,\middle|\, \bigcap_{i>j} \left\{\Lambda_{2^{i-1}} \overset{A_i}{\longleftrightarrow} \partial\Lambda_{2^i}\right\}\right] \\ &\leq \prod_{j=1}^{k} \phi^1_{A_j}\left[\Lambda_{2^{j-1}} \overset{A_j}{\longleftrightarrow} \partial\Lambda_{2^j}\right].\end{aligned}$$

In the second line, we used the fact that the event upon which we condition depends only on edges outside of Λ_{2^j} or on $\partial\Lambda_{2^j}$ together with the comparison between boundary conditions.

Now, the complement of $\Lambda_{2^{j-1}} \overset{A_j}{\longleftrightarrow} \partial\Lambda_{2^j}$ is the event that there exists a dual-open circuit in A_j^* surrounding the origin. Property **P5a** implies that this dual-open circuit exists with probability larger than or equal to $c > 0$ independently of $n \geq 1$. This implies that

$$\phi^1_{\mathbb{Z}^2}[0 \longleftrightarrow \partial\Lambda_n] \leq \prod_{j=1}^{k}(1-c) = (1-c)^k \leq (1-c)^{\log n/\log 2}.$$

The proof follows by setting $\varepsilon = -\frac{\log(1-c)}{\log 2}$. □

Remark. The proof of the previous lemma illustrates the need for *bounds which are uniform with respect to boundary conditions.* Indeed, it could be the case that the ϕ^1-probability of an open path from the inner to the outer sides of A_j is bounded away from 1, but conditioning on the existence of paths in each annulus A_i (for $i < j$) could favor open edges drastically, and imply that the probability of the event under consideration is close to 1.

3.3 Proof of Theorem 3: P4 implies P5

Recall from Proposition 2 that **P5** is equivalent to **P5b** and we therefore choose to prove that **P4** implies **P5b** when $R = 8$. The proof follows two steps. First, we prove that either **P5b** holds or $\phi^0(0 \leftrightarrow \partial\Lambda_n)$ tends to 0 stretched-exponentially fast. Second, we prove that if the speed of convergence is stretched exponential, it is in fact exponential.

Proposition 3. *Exactly one of these two cases occurs :*

1. $\inf_{n\geq 1} \phi^0_{\Lambda_{8n}}[\mathcal{A}_n] > 0$.
2. *There exists $\alpha > 0$ such that for any $n \geq 1$,*

$$\phi^0_{\mathbb{Z}^2}[0 \longleftrightarrow \partial\Lambda_n] \leq \exp(-n^\alpha).$$

First, consider the strip $\mathbb{S} = \mathbb{Z} \times [-n, 3n]$, and the boundary conditions ξ defined to be wired on $\mathbb{Z} \times \{3n\}$, and free on $\mathbb{Z} \times \{-n\}$. Let $\phi^{1/0}_{\mathbb{S}}$ be the associated random-cluster measure. Recall that boundary conditions at infinity are not relevant since the strip is essentially one dimensional.

Lemma 2. *For all $k \geq 1$, there exists a constant $c = c(k) > 0$ such that, for all $n \geq 1$,*

$$\phi^{1/0}_{\mathbb{S}}[\mathcal{C}_h([-kn, kn] \times [0, 2n])] \geq c. \tag{4}$$

Proof. Fix $n, k \geq 1$. We will assume that n is divisible by 9 (one may adapt the argument for general values of n). By duality, the complement of $\mathcal{C}_h([-kn, kn]\times[0, 2n])$ is $\mathcal{C}^*_v([-kn+\frac{1}{2}, kn-\frac{1}{2}] \times [-\frac{1}{2}, 2n+\frac{1}{2}])$. Therefore, either (4) is true for $c = 1/2$, or

$$\phi^{1/0}_{\mathbb{S}}\left[\mathcal{C}^*_v([-kn+\tfrac{1}{2}, kn-\tfrac{1}{2}] \times [-\tfrac{1}{2}, 2n+\tfrac{1}{2}])\right] \geq 1/2.$$

We assume that we are in this second situation for the rest of the proof.

The dual of the measure on the strip with free boundary conditions on the bottom and wired on the top is the measure on the strip with free boundary conditions on the top and wired on the bottom. This measure is the image of $\phi^{1/0}_{\mathbb{S}}$ under the orthogonal reflection with respect to the horizontal line $\mathbb{R} \times \{n - \frac{1}{4}\}$ composed with a translation by the vector $(\frac{1}{2}, 0)$. We thus obtain that

$$\begin{aligned}\phi^{1/0}_{\mathbb{S}}[\mathcal{C}_v([-kn, kn] \times [0, 2n])] &\geq \phi^{1/0}_{\mathbb{S}}[\mathcal{C}_v([-kn, kn-1] \times [-1, 2n])] \\ &= \phi^{1/0}_{\mathbb{S}}\left[\mathcal{C}^*_v([-kn+\tfrac{1}{2}, kn-\tfrac{1}{2}] \times [-\tfrac{1}{2}, 2n+\tfrac{1}{2}])\right] \\ &\geq 1/2.\end{aligned}$$

Partitioning the segment $[-kn, kn]\times\{0\}$ into the union of $18k$ segments of length $\lambda := n/9$ (note that λ is an integer), the union bound gives us

$$\phi_{\mathbb{S}}^{1/0}\left[I \leftrightarrow \mathbb{Z}\times\{2n\}\right] \geq \frac{1}{36k} =: c_1, \tag{5}$$

where $I = [4\lambda, 5\lambda]\times\{0\}$. For future reference, let us also introduce the segment $J = [6\lambda, 7\lambda]\times\{0\}$.

Define the rectangle $R = [0, 9\lambda]\times[0, 2n]$. When the event estimated in Equation (5) is realized, there exists an open path in R connecting I to the union of the top, left and right boundaries of R. Using the reflection with respect to the vertical line $\{\frac{n}{2}\}\times\mathbb{R}$, we find that at least one of the two following inequalities occurs:

Case 1: $\phi_{\mathbb{S}}^{1/0}\left[I \overset{R}{\longleftrightarrow} [0,n]\times\{2n\}\right] \geq c_1/3$.

Case 2: $\phi_{\mathbb{S}}^{1/0}\left[I \overset{R}{\longleftrightarrow} \{0\}\times[0,2n]\right] \geq c_1/3$.

Figure 1: The construction in Case 1 with the two paths Γ_1 and Γ_2 and the domain Ω between the two paths. On the right, a combination of paths creating a long path from left to right.

Proof of (4) *in Case 1:* Consider the event that there exist

(i) an open path from I to the top of $[0, 2n]^2$ contained in $[0, 2n]^2$,
(ii) an open path from J to the top of $[0, 2n]^2$ contained in $[0, 2n]^2$,
(iii) an open path connecting these two paths in $[0, 2n]^2$.

Each path in (i) and (ii) exists with probability larger than $c_1/3$ (since R and $(2\lambda, 0)+R$ are included in $[0, 2n]^2$). Furthermore, let Γ_1 be the left-most path satisfying (i) and Γ_2 the right-most path satisfying (ii); see Fig. 1. The subgraph of $[0, 2n]^2$ between Γ_1 and Γ_2

is denoted by Ω. Conditioning on Γ_1 and Γ_2, the boundary conditions on Ω are wired on Γ_1 and Γ_2, and dominate the free boundary conditions on the rest of $\partial\Omega$. We deduce that boundary conditions on Ω dominate boundary conditions inherited by wired boundary conditions on the left and right sides of the box $[0,2n]^2$, and free on the top and bottom sides. As a consequence of (3), conditionally on Γ_1 and Γ_2, there exists an open path in Ω connecting Γ_1 to Γ_2 with probability larger than $1/(1+q^2)$. In conclusion,

$$\phi_{\mathbb{S}}^{1/0}\left[I \xleftrightarrow{[0,2n]^2} J\right] \geq \phi_{\mathbb{S}}^{1/0}\left[\text{(i), (ii) and (iii) occur}\right] \geq \left(\frac{c_1}{3}\right)^2 \times \frac{1}{(1+q^2)}. \tag{6}$$

For $x = j\lambda$, where $j \in \{-9k-5, \dots, 9k-6\}$, define the translate of the event considered in (6):

$$A_x := \left\{ (x+I) \xleftrightarrow{x+[0,2n]^2} (x+J) \right\}.$$

If A_x occurs for every such x, we obtain an open crossing from left to right in $[-kn, kn] \times [0, 2n]$. The FKG inequality implies that this happens with probability larger than $\left(\frac{c_1^2}{9(1+q^2)}\right)^{20k}$.

Proof of (4) *in Case 2:* Define the rectangle $R' = [4\lambda, 9\lambda] \times [0, 2n]$. Note that in Case 2, J is connected to one side of $[2\lambda, 11\lambda] \times [0, 2n]$ with probability bounded from below by $c_1/3$, hence the same is true for R' (since $[2\lambda, 11\lambda] \times [0, 2n]$ is wider than R'). Consider the event that there exist

(i) an open path from I to the right side of R contained in R,
(ii) an open path from J to the left side of R' contained in R',
(iii) an open path connecting these two paths in $[0, 2n]^2$.

The first path occurs with probability larger than $c_1/3$, and the second one with probability larger than $c_1/6$ (there exists a path to one of the sides with probability at least $c_1/3$, and therefore by symmetry in R' to the left side with probability larger than $c_1/6$). By the FKG inequality, the event that both (i) and (ii) occur has probability larger than $c_1^2/18$. We now wish to prove that conditionally on (i) and (ii) occurring, the event (iii) occurs with probability bounded from below uniformly in n.

Define the segments $K(y,z) = \{4\lambda\} \times [y,z]$ for $y \leq z \leq \infty$. They are all subsegments of the vertical line of first coordinate equal to 4λ.

Consider the right-most open path Γ_1 satisfying (ii). It intersects the segment $K(0,2n)$ at a unique point with second coordinate denoted by y. Also consider the left-most open path $\tilde{\Gamma}_2$ satisfying (i). Either Γ_1 and $\tilde{\Gamma}_2$ intersect, or they do not. In the first case, we are already done since (iii) automatically occurs. In the second, we consider the subpath Γ_2 of $\tilde{\Gamma}_2$ from I to the first intersection with $K(y,2n)$ (this intersection must exist since $\tilde{\Gamma}_2$ goes to the right side of R'). Let us now show that Γ_1 and Γ_2 are connected with good probability. Note the similarity with the construction in [BD12a] with symmetric domains, except that the lattice is not rotated here. The proof is therefore slightly more technical and we choose to isolate it from the rest of the argument.

Claim: There exists $c_2 > 0$ such that for any possible realizations γ_1 and γ_2 of Γ_1 and Γ_2,

$$\phi_{\mathbb{S}}^{1/0}\left[\gamma_1 \xleftrightarrow{R} \gamma_2 \,\middle|\, \Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2\right] \geq c_2. \tag{7}$$

Proof. Fig. 2 should be very helpful in order to follow this proof. Construct the subgraph Ω "between γ_1 and γ_2" formally delimited by:

- the arc γ_2,
- the segment $[0, n] \times \{0\}$,
- the arc γ_1,
- the segment $K(y+1, 2n)$ *excluded* (the vertices on this segment are not part of the domain).

We wish to compare Ω (left of Fig. 2) to a reference domain D (center of Fig. 2) defined as the upper half-plane minus the edges intersecting $\{4\lambda - \frac{1}{2}\} \times (y, \infty)$ and define the boundary conditions *mix* on D by:

- wired boundary conditions on $K(y, \infty)$ and $A := (-\infty, 4\lambda] \times \{0\}$;
- wired boundary conditions at infinity (by this we mean that we take the limit of measures on $D \cap \Lambda_n$, with wired boundary conditions on $\partial\Lambda_n$);
- free boundary conditions elsewhere.

The boundary conditions on Ω inherited by the conditioning $\Gamma_1 = \gamma_1$ and $\Gamma_2 = \gamma_2$ dominate wired on γ_1 and γ_2, and free elsewhere. Thus, we deduce that

$$\phi_{\mathbb{S}}^{1/0}\left[\gamma_1 \overset{\Omega}{\longleftrightarrow} \gamma_2 \,\middle|\, \Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2\right] \geq \phi_D^{\text{mix}}\left[K(y, \infty) \overset{D}{\longleftrightarrow} A\right]. \tag{8}$$

As mentioned above, the domain D is not exactly a symmetric domain but it is still very close to be one. Consider the domain $\tilde{D}$ (see on the right of Fig. 2) obtained from D by the reflection with respect to the vertical line $d = \{(4\lambda - \frac{1}{4}, y) : y \in \mathbb{R}\}$ and a translation by $(\frac{1}{2}, \frac{1}{2})$. Let $B = (-\infty, 4\lambda - 1] \times \{0\}$. Define the boundary conditions *mix* on $\tilde{D}$ as

- wired boundary conditions on $K(y+1, \infty) \cup B$ (it is very important that the two arcs are wired together);
- free boundary conditions at infinity;
- free boundary conditions elsewhere.

Using duality, we find that

$$\phi_D^{\text{mix}}\left[K(y, \infty) \overset{D}{\not\longleftrightarrow} A\right] = \phi_{\tilde{D}}^{\text{mix}}\left[K(y+1, \infty) \overset{\tilde{D}}{\longleftrightarrow} B\right]$$

and thus

$$\phi_D^{\text{mix}}\left[K(y, \infty) \overset{D}{\longleftrightarrow} A\right] + \phi_{\tilde{D}}^{\text{mix}}\left[K(y+1, \infty) \overset{\tilde{D}}{\longleftrightarrow} B\right] = 1. \tag{9}$$

Define the *mix′* boundary conditions on D as wired boundary conditions on $K(y+1, \infty) \cup B := (-\infty, 4\lambda - 1] \times \{0\}$ (the two arcs are once again wired together) and free elsewhere (they correspond to the boundary conditions *mix* on $\tilde{D}$). Since $\tilde{D} \subset D$,

$$\phi_{\tilde{D}}^{\text{mix}}\left[K(y+1, \infty) \overset{\tilde{D}}{\longleftrightarrow} B\right] \leq \phi_D^{\text{mix}'}\left[K(y, \infty) \overset{D}{\longleftrightarrow} A\right].$$

The boundary conditions for the term on the right can be compared to the boundary conditions *mix*. First, one may wire the vertices $(4\lambda, y)$ and $(4\lambda, y+1)$ together, and the vertices $(4\lambda - 1, 0)$ and $(4\lambda, 0)$ together, which increases the probability of an open path between $K(y, \infty)$ and A. Second, one may unwire the arcs B and $K(y+1, \infty)$, paying a

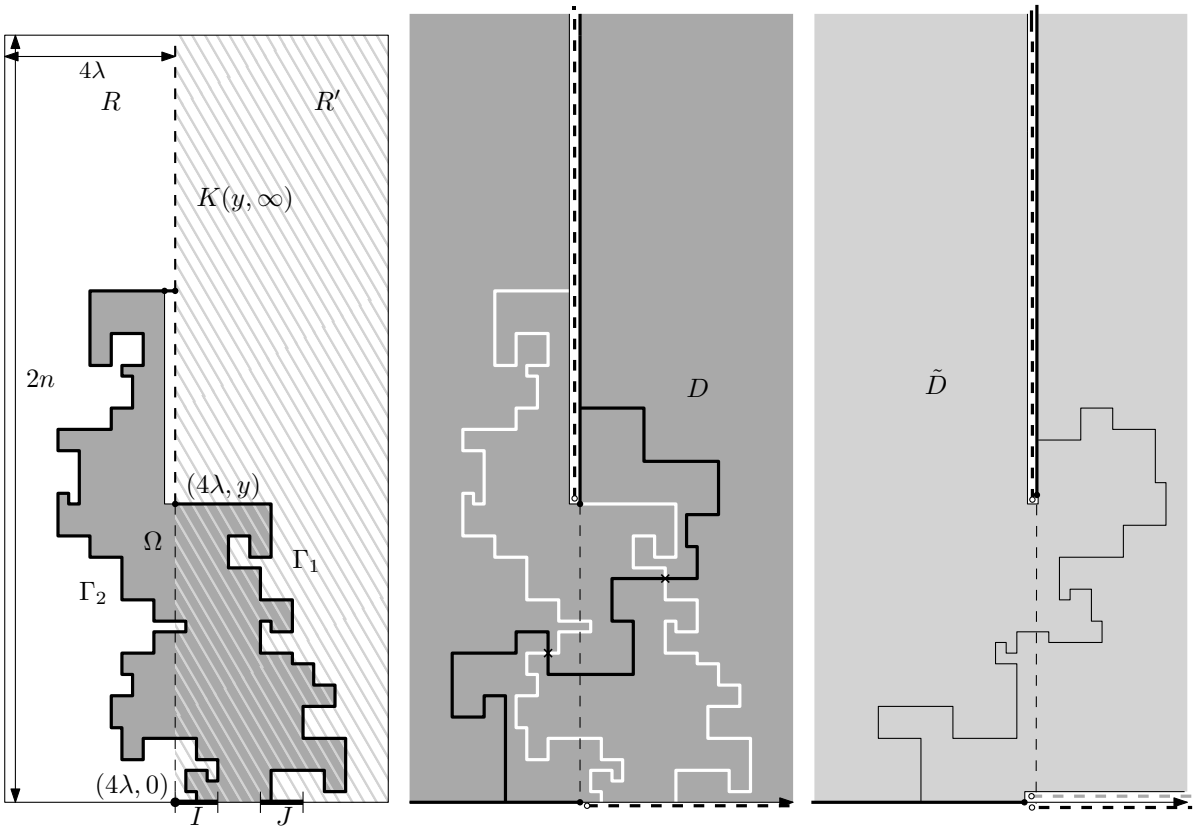

Figure 2: **Left.** The domain Ω. We depicted the part of the domain with free boundary conditions by putting dual wired boundary conditions on the associated dual arcs. The wired boundary conditions are depicted in bold. The rectangles R and R' are also specified (R' is in dashed). **Center.** The domain D. We depicted the domain Ω in white. The existence of an open path between $K(y,\infty)$ and A implies the existence of an open path between γ_1 and γ_2 in D (between the two crossings). **Right.** The domain $\tilde{D}$ with one path from $K(y+1,\infty)$ to B. The pre image of this path by the reflection mapping D onto $\tilde{D}$ is a dual-path in D preventing the existence of an open path from $K(y,\infty)$ to A.

multiplicative cost of q^2. Using the previous inequality and the comparison between the boundary conditions described in this paragraph, we deduce

$$\phi_{\tilde{D}}^{\text{mix}}\left[K(y+1,\infty) \overset{\tilde{D}}{\longleftrightarrow} B\right] \leq q^2 \phi_D^{\text{mix}}\left[K(y,\infty) \overset{D}{\longleftrightarrow} A\right].$$

Putting this inequality in (9) and then using (8), we find that

$$\phi_{\mathbb{S}}^{1/0}\left[\gamma_1 \overset{\Omega}{\longleftrightarrow} \gamma_2 \,\middle|\, \Gamma_1 = \gamma_1, \Gamma_2 = \gamma_2\right] \geq \phi_D^{\text{mix}}\left[K(y,\infty) \overset{D}{\longleftrightarrow} A\right] \geq \frac{1}{1+q^2}.$$

□

It follows from (7) and the probabilities of (i) and (ii) that

$$\phi_{\mathbb{S}}^{1/0}\left[I \overset{R}{\longleftrightarrow} J\right] \geq \frac{1}{1+q^2} \times \frac{c_1^2}{18}.$$

Here again, $20k$ translations of the event above guarantee the occurrence of an open crossing from left to right in $[-kn, kn] \times [0, 2n]$. This occurs with probability larger than $(\frac{c_1^2}{18(1+q^2)})^{20k}$ thanks to the FKG inequality again. □

In the next lemma, we consider horizontal crossings in rectangular shaped domains with free boundary conditions on the bottom and wired elsewhere.

Lemma 3. *For all $k > 0$ and $\ell \geq 4/3$, there exists a constant $c = c(k,\ell) > 0$ such that for all $n > 0$,*

$$\phi_D^{1/0}\left[\mathcal{C}_h\left([-kn, kn] \times [0, n]\right)\right] \geq c \tag{10}$$

with $D = [-kn, kn] \times [0, \ell n]$, and $\phi_D^{1/0}$ is the random-cluster measure with free boundary conditions on the bottom side, and wired on the three other sides.

Proof. For $\ell = 4/3$, the result follows directly from Lemma 2 since boundary conditions dominate boundary conditions in the strip $\mathbb{Z} \times [0, \frac{4n}{3}]$, and therefore there exists an horizontal crossing of the rectangle $[-kn, kn] \times [\frac{n}{3}, n]$ with probability bounded away from 0.

Now assume that the result holds for ℓ and let us prove it for $\ell + 1/3$. By comparison between boundary conditions in $[-kn, kn] \times [\frac{n}{3}, \ell n + \frac{n}{3}]$, we know that

$$\phi_D^{1/0}\left[\mathcal{C}_h([-kn, kn] \times [\tfrac{n}{3}, \tfrac{4n}{3}])\right] \geq c(k,\ell).$$

Conditioning on the highest such crossing, the boundary conditions below this crossing dominate the free boundary conditions on the bottom side of $[-kn, kn] \times [0, \frac{4n}{3}]$, and wired on the other three sides of $[-kn, kn] \times [0, \frac{4n}{3}]$. An application of the case $\ell = \frac{4}{3}$ enables us to set $c(k, \ell + \frac{1}{3}) = c(k,\ell)c(k, \frac{4}{3})$.

The proof follows from the fact that the probability in (10) is decreasing in ℓ. □

Lemma 4. *There exists a constant $C < \infty$ such that, for all $n \geq 1$,*

$$\phi_{\Lambda_{56n}}^0\left[\mathcal{A}_{7n}\right] \leq C\, \phi_{\Lambda_{8n}}^0\left[\mathcal{A}_n\right]^2.$$

Proof. Define $z_\pm = (\pm 5n, 0)$. If $\mathcal{A}_{7n}$ occurs, the boundary conditions on Λ_{7n} dominate the wired boundary conditions on Λ_{56n} due to the existence of the open circuit in $\Lambda_{14n} \setminus \Lambda_{7n}$. Theorem 9 thus implies the existence of a constant $c_1 > 0$ such that, for all n,

$$\phi^0_{\Lambda_{56n}}[\mathcal{A}_n(z_+) \cap \mathcal{A}_n(z_-) | \mathcal{A}_{7n}] \geq \phi^1_{\Lambda_{56n}}[\mathcal{A}_n(z_+) \cap \mathcal{A}_n(z_-)] \geq c_1.$$

It directly implies that for all n,

$$\phi^0_{\Lambda_{56n}}[\mathcal{A}_n(z_+) \cap \mathcal{A}_n(z_-)] \geq c_1 \phi^0_{\Lambda_{56n}}[\mathcal{A}_{7n}]. \tag{11}$$

Now, examine the domain $D = [-56n, 56n] \times [2n, 56n]$ and consider the measure $\phi^{1/0}_D$ with free boundary conditions on the bottom and wired boundary elsewhere. Also set

$$R^*_+ := [-56n - \tfrac{1}{2}, 56n + \tfrac{1}{2}] \times [2n + \tfrac{1}{2}, 3n - \tfrac{1}{2}].$$

Under $\phi^0_{56n}[\,\cdot\,|\mathcal{A}_n(z_+) \cap \mathcal{A}_n(z_-)]$, the boundary conditions on D are dominated by wired boundary conditions on the bottom and free boundary conditions on the other sides. As a consequence, Lemma 3 applied to $k = 56$ and $\ell = 54$ implies that

$$\phi^0_{\Lambda_{56n}}\left[\mathcal{C}^*_h\left(R^*_+\right) \middle| \mathcal{A}_n(z_+) \cap \mathcal{A}_n(z_-)\right] \geq \phi^{1/0}_D\left[\mathcal{C}^*_h\left(R^*_+\right)\right] \geq c_2 \tag{12}$$

for some universal constant $c_2 > 0$ independent of n. Similarly, with $D' = [-56n, 56n] \times [-56n, -2n]$ and

$$R^*_- := [-56n - \tfrac{1}{2}, 56n + \tfrac{1}{2}] \times [-3n + \tfrac{1}{2}, -2n - \tfrac{1}{2}],$$

we find

$$\phi^0_{\Lambda_{56n}}\left[\mathcal{C}^*_h\left(R^+_-\right) \middle| \mathcal{A}_n(z_+) \cap \mathcal{A}_n(z_-)\right] \geq c_2. \tag{13}$$

Define the event $\mathcal{B}_n$, illustrated on Fig. 3, which is the intersection of the events $\mathcal{A}_n(z_+)$, $\mathcal{A}_n(z_-)$, $\mathcal{C}^*_h\left(R^*_+\right)$ and $\mathcal{C}^*_h\left(R^*_-\right)$. Equations (11), (12) and (13) lead to the estimate

$$\phi^0_{\Lambda_{56n}}\left[\mathcal{B}_n\right] \geq c_3 \phi^0_{\Lambda_{56n}}\left[\mathcal{A}_{7n}\right], \tag{14}$$

where $c_3 > 0$ is a positive constant independent of n.

Assume $\mathcal{B}_n$ occurs and define Γ^*_1 to be the top-most horizontal dual-crossing of R^*_+ and Γ^*_2 to be the bottom-most horizontal dual-crossing of R^*_-. Note that these paths are dual paths. Let Ω^* be the set of dual-vertices in $R^* := [-3n + \frac{1}{2}, 3n - \frac{1}{2}]^2$ below Γ^*_1 and above Γ^*_2. Exactly as in the proof of Lemma 2, when conditioning on Γ^*_1, Γ^*_2 and everything outside Ω^*, the boundary conditions inside Ω^* are dual-wired on Γ^*_1 and Γ^*_2, and dual-free elsewhere. The dual measure inside Ω^* therefore dominates the restriction to Ω^* of the dual measure on R^* with dual-wired boundary conditions on the top and bottom, and dual-free boundary conditions on the left and right sides. Using (3), we find

$$\phi^0_{\Lambda_{56n}}\left[\mathcal{C}_n \mid \mathcal{B}_n\right] \geq \frac{1}{1+q^2},$$

where $\mathcal{C}_n = \{\Gamma_1 \overset{*}{\longleftrightarrow} \Gamma_2 \text{ in } R^*\}$. Similar inequalities hold for the events

$$\mathcal{D}_n = \left\{\Gamma_1 \overset{*}{\longleftrightarrow} \Gamma_2 \text{ in } (-10n, 0) + R^*\right\},$$
$$\mathcal{E}_n = \left\{\Gamma_1 \overset{*}{\longleftrightarrow} \Gamma_2 \text{ in } (10n, 0) + R^*\right\}.$$

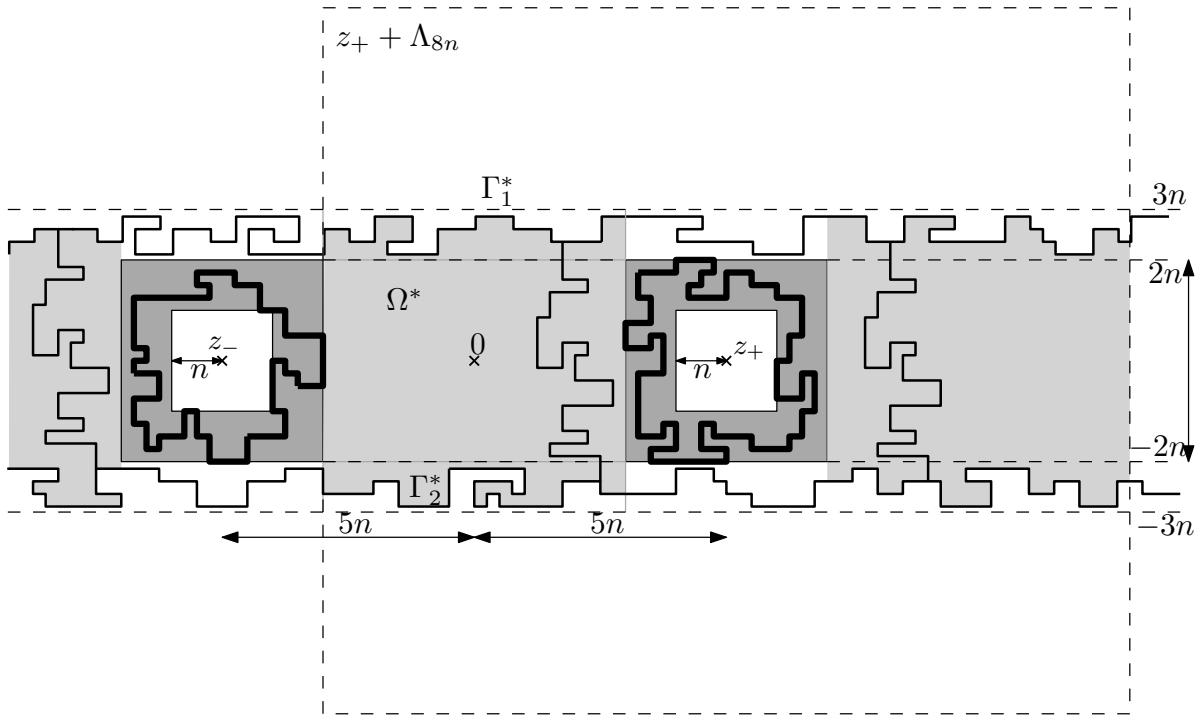

Figure 3: Primal open crossings are in bold, dual-open are in plain. The events $\mathcal{A}_n(z_+)$, $\mathcal{A}_n(z_-)$ and the existence of the dual horizontal crossings of R_+^* and R_-^* form $\mathcal{B}_n$. Conditionally on $\mathcal{B}_n$, Γ_1^* and Γ_2^* are connected in Ω^* by a dual-open path with probability larger than $1/(1+q^2)$.

The FKG inequality thus implies

$$\phi^0_{\Lambda_{56n}}\left[\mathcal{C}_n \cap \mathcal{D}_n \cap \mathcal{E}_n \mid \mathcal{B}_n\right] \geq \frac{1}{(1+q^2)^3}$$

which, together with (14), leads to

$$\phi^0_{\Lambda_{56n}}\left[\mathcal{B}_n \cap \mathcal{C}_n \cap \mathcal{D}_n \cap \mathcal{E}_n\right] \geq \frac{c_3}{(1+q^2)^3}\phi^0_{\Lambda_{56n}}\left[\mathcal{A}_{7n}\right]. \tag{15}$$

The event estimated in (15) implies in particular the existence of dual circuits in $z_+ + \Lambda^*_{8n}$ and $z_- + \Lambda^*_{8n}$ disconnecting $z_+ + \Lambda^*_{2n}$ from $z_- + \Lambda^*_{2n}$. Writing $\mathcal{F}_n$ for the event that such dual circuits exist and using the comparison between boundary conditions one last time (more precisely a "conditioning on the exterior-most circuit"-type argument), we obtain

$$\begin{aligned}
\phi^0_{\Lambda_{8n}}\left[\mathcal{A}_n\right]^2 &= \phi^0_{z_-+\Lambda_{8n}}\left[\mathcal{A}_n(z_-)\right]\phi^0_{z_++\Lambda_{8n}}\left[\mathcal{A}_n(z_+)\right] \\
&\geq \phi^0_{\Lambda_{56n}}\left[\mathcal{A}_n(z_-) \mid \mathcal{A}_n(z_+) \cap \mathcal{F}_n\right]\phi^0_{\Lambda_{56n}}\left[\mathcal{A}_n(z_+) \mid \mathcal{F}_n\right]\phi^0_{\Lambda_{56n}}\left[\mathcal{F}_n\right] \\
&= \phi^0_{\Lambda_{56n}}\left[\mathcal{A}_n(z_-) \cap \mathcal{A}_n(z_+) \cap \mathcal{F}_n\right] \\
&\geq \frac{c_3}{(1+q^2)^3}\phi^0_{\Lambda_{56n}}\left[\mathcal{A}_{7n}\right].
\end{aligned}$$

This inequality implies the claim. □

We need a last lemma before being able to prove Proposition 3.

Lemma 5. *Let* $1 \leq k \leq n$,

$$\phi^0_{\Lambda_n}\left[0 \longleftrightarrow \partial\Lambda_k\right] \leq \sum_{m \geq k} 72m^4 \max_{\substack{a \in \{0\}\times[0,m] \\ b \in \{m\}\times[0,m]}} \phi^0_{[0,m]^2}\left[a \longleftrightarrow b\right].$$

Proof. For $x = (x_1, x_2)$, define $\|x\|_\infty = \max\{|x_1|, |x_2|\}$.

Define $\mathcal{C}$ to be the connected component of the origin. Consider the event that a and b are two vertices in $\mathcal{C}$ maximizing the $\|\cdot\|_\infty$-distance between each other. Since these vertices are at maximal distance from each other, they can be placed on the two opposite sides of a square box Λ in such a way that $\mathcal{C}$ is included in this box. Let $\mathcal{A}_{\max}(a, b, \Lambda)$ be the event that a and b are connected in Λ and that their cluster is contained in Λ.

We now wish to estimate the probability of $\mathcal{A}_{\max}(a, b, \Lambda)$. Let Λ^* be the subgraph of $(\mathbb{Z}^2)^*$ composed of dual-edges whose end-points correspond to faces touching Λ. Let $\mathbf{C}$ be the set of dual self-avoiding circuits $\gamma = \{\gamma_0 \sim \gamma_1 \sim \cdots \sim \gamma_m \sim \gamma_0\}$ included in Λ^* surrounding a and b. As before, we denote by $\overline{\gamma}$ the interior of γ.

On the event $\mathcal{C}$, there exists $\gamma \in \mathbf{C}$ which is dual-open[1], and a and b are connected in $\overline{\gamma}$. As before, we may condition on the outermost dual-open circuit Γ in $\mathbf{C}$. We deduce as in the last proof that

$$\phi^\xi_{\Lambda_n}(a \longleftrightarrow b \text{ in } \overline{\gamma} \,|\, \Gamma = \gamma) = \phi^0_{\overline{\gamma}}(a \longleftrightarrow b \text{ in } \overline{\gamma}) \leq \phi^0_\Lambda(a \longleftrightarrow b \text{ in } \overline{\gamma}) \leq \phi^0_\Lambda(a \longleftrightarrow b).$$

[1]Note that this is true even when $\Lambda = \Lambda_n$ since free boundary conditions can be seen as dual-wired boundary conditions on Λ^*_n, and that therefore $\partial\Lambda^*_n$ provides us with a dual self-avoiding circuit in $\mathbf{C}$ which is dual-open. A similar reasoning applies when Λ only shares some sides with Λ_n.

We now partition $\mathcal{A}_{\max}(a,b,\Lambda)$ into the events $\{\Gamma=\gamma\}$ to find

$$\phi^{\xi}_{\Lambda_n}(\mathcal{A}_{\max}(a,b,\Lambda)) \leq \phi^0_{\Lambda}(a \longleftrightarrow b)$$

and therefore

$$\phi^{\xi}_{\Lambda_n}(\mathcal{A}_{\max}(a,b,\Lambda)) \leq \max_{\substack{a\in\{0\}\times[0,m]\\ b\in\{m\}\times[0,m]}} \phi^0_{[0,m]^2}\left[a \longleftrightarrow b\right], \tag{16}$$

where $m=\|a-b\|_\infty$.

We may now use the fact that if 0 is connected to distance k, there exist a and b at distance $m\geq k$ of each others and a box Λ having a and b on opposite sides such that $\mathcal{A}_{\max}(a,b,\Lambda)$ occurs. Let us bound the number of choices for a, b and Λ.

For a fixed $m\geq k$, there are $|\Lambda_m|=(2m+1)^2$ choices for a (since a must be at distance smaller or equal to m from the origin). Then a must be on the boundary of Λ and there are therefore $|\partial\Lambda|=4m$ choices for Λ. The number of choices for b is bounded by $m+1$ (it must be on the opposite sides of Λ). Therefore, for fixed m we can bound the number of possible triples (a,b,Λ) by $4m(2m+1)^2(m+1)\leq 72m^4$. We have been very wasteful in the previous reasoning and the bound on this number could be improved greatly but this will be irrelevant for applications.

Overall, (16) and a union bound gives

$$\phi^0_{\mathbb{Z}^2}\left[0\leftrightarrow\partial\Lambda_k\right] \leq \sum_{m\geq k} 72m^4 \max_{\substack{a\in\{0\}\times[0,m]\\ b\in\{m\}\times[0,m]}} \phi^0_{[0,m]^2}\left[a\leftrightarrow b\right].$$

□

Proof of Proposition 3. Obviously the cases 1 and 2 cannot occur simultaneously. Suppose that the first case does not occur and let us prove that the second does.

For all $n\geq 1$, set $u_n = C\phi^0_{\Lambda_{8n}}[\mathcal{A}_n]$, where C is defined as in Lemma 4. With this notation, Lemma 4 implies that $u_{7n}\leq u_n^2$ for any $n\geq 1$ which implies that for $0\leq \ell,k\leq n$,

$$u_{7^k n_0} \leq u_{n_0}^{2^k} \tag{17}$$

for any positive $k\geq 0$ and $n_0\geq 1$. Now, if $\liminf_{n\to\infty}\phi^0_{\Lambda_{8n}}[\mathcal{A}_n]=0$, then we may pick n_0 such that $u_{n_0}<1$. By (17), there exists $c_1>0$ such that for all n of the form $n=7^k n_0$,

$$u_n \leq \exp\left(-c_1 n^{\log 2/\log 7}\right).$$

Fix $n=7^k n_0$ and consider $\frac{n}{7}\leq m<n$. The FKG inequality and the comparison between boundary conditions imply that

$$\begin{aligned}
\phi^0_{[0,m]^2}\left[(0,k)\longleftrightarrow(m,\ell)\right] &\leq \left(\phi^0_{[-m,m]\times[0,m]}\left[(-m,\ell)\longleftrightarrow(m,\ell)\right]\right)^{1/2}\\
&\leq \left(\phi^0_{\Lambda_{8n}}\left[\mathcal{C}_h([-2n,2n]\times[0,m])\right]\right)^{1/14}\\
&\leq \left(\phi^0_{\Lambda_{8n}}\left[\mathcal{A}_n\right]\right)^{1/56} \leq \exp\left(-c_2 n^{\log 2/\log 7}\right).
\end{aligned}$$

In the first inequality, we used that if $(0,k)\longleftrightarrow(m,\ell)$ and $(-m,\ell)\longleftrightarrow(0,k)$, then $(-m,\ell)\longleftrightarrow(m,\ell)$. In the second inequality, we have used that if $(x,\ell)\longleftrightarrow(x+2m,\ell)$

occur for $x = 2mj$ with $j \in \{-7, \dots, 7\}$, then $\mathcal{C}_h([-2n, 2n] \times [0, m])$ occurs. Finally in the third inequality we combined four crossings as in the proof of **P5⇒P5a**. Lemma 5 implies the claim. □

Theorem 3 follows directly from Proposition 3 and the following proposition:

Proposition 4. *If there exists $\alpha > 0$ such that for all $n \geq 1$,*

$$\phi^0_{\mathbb{Z}^2}[0 \longleftrightarrow \partial\Lambda_n] \leq \exp(-n^\alpha),$$

then there exists $c > 0$ such that for all $n \geq 1$,

$$\phi^0_{\mathbb{Z}^2}[0 \longleftrightarrow \partial\Lambda_n] \leq \exp(-cn).$$

We start by a lemma. Let $n \geq 1$ and $\theta \in [-1, 1]$. Define the *tilted strip* in direction θ:

$$S(n, \theta) := \{(x, y) \in \mathbb{Z}^2 : 0 \leq y - \theta x \leq n\}.$$

Write $\phi^{1/0}_{S(n,\theta)}$ for the random-cluster measure on the tilted strip $S(n, \theta)$ with wired boundary conditions on the top side and free on the bottom side (the boundary conditions at infinity are irrelevant since the tilted strip is essentially a one-dimensional graph).

We will also consider a truncated version of the tilted strip $S(n, \theta)$. For $m \geq 0$, consider the *truncated tilted strip*

$$S(n, m, \theta) := S(n, \theta) \cap \Lambda_m.$$

We will always assume that $\theta m \in \mathbb{N}$ (the general case can be treated similarly). Write $\phi^{1/0}_{S(n,m,\theta)}$ for the random-cluster measure with free boundary conditions on the bottom side and wired on the other three sides.

For simplicity, we will call the bottom side of the strip or the truncated strip the *free arc*, and the rest of the boundary the *wired arc*.

Lemma 6. *For all $m \geq n \geq 1$ and $\theta \in [-1, 1]$,*

$$\phi^{1/0}_{S(n,m,\theta)}[0 \longleftrightarrow \text{wired arc}] \geq \frac{1}{5m^2n^2}.$$

Proof. Fix $n \geq 0$ and $\theta \in [-1, 1]$. Let us work in the strip $S(2n, \theta)$. From now on, we drop the dependence in n and θ and write for instance $S = S(n, \theta)$ and $S(m) = S(2n, m, \theta)$. Beware that there is a slightly confusing notation here: the height of the strip is $2n$ while the one of the truncated strip is n.

For $x \in S$, define the translate $S_x(m) := x + S(m)$ of $S(m)$. We extend the definition of wired and free arcs to this context. Let $\mathcal{A}(x)$ be the event that x is connected to the wired arc of $S_x(m)$ and every open path from a vertex $y \notin S_x(m)$ to x intersects the wired arc (of $S_x(m)$). In other words, no open path starting from x "exits" $S_x(m)$ through the free arc (i.e. the bottom side).

We consider the random function $F : \mathbb{N} \longrightarrow [0, 2n]$ defined by

$$F(k) := \min\{\ell : (k, \ell) \text{ is connected to the top side of } S\} - \theta k.$$

Recall that $\theta m \in \mathbb{N}$. Therefore, F can take only the $2nm+1$ following values:

$$\left\{0, \tfrac{1}{m}, \dots, \tfrac{2nm-1}{m}, 2n\right\}.$$

On the event $\{F(0) \le n\}$, there must exist $k \in \{-nm^2, \dots, nm^2\}$ such that $F(k) \le n$ and $F(k') \ge F(k)$ for every $|k'-k| \le m$. Otherwise, if there is no such k, then there exists a sequence $0 = k_0, \dots, k_{nm}$ with $|k_{i+1} - k_i| \le m$ and $0 < F(k_{i+1}) < F(k_i)$. But this provides $nm+1$ distinct values for F, all smaller or equal to n and strictly larger than 0, which is contradictory.

Now, for k satisfying $F(k) \le n$ and $F(k') \ge F(k)$ for every $|k'-k| \le m$, the event $\mathcal{A}((k, F(k)))$ is realized. In conclusion, if $F(0) \le n$, then there exists $x \in S(n, nm^2, \theta)$ such that $\mathcal{A}(x)$ is realized and the union bound shows the existence of $x \in S(n, \theta)$ (the lower half of S) such that

$$\phi_S^{1/0}\left[\mathcal{A}(x)\right] \ge \frac{\phi_S^{1/0}\left[F(0) \le n\right]}{|S(n, nm^2, \theta)|} = \frac{\phi_S^{1/0}\left[F(0) \le n\right]}{n(2nm^2+1)}.$$

Consider the interface between the open cluster connected to the top side of the box and the dual-open cluster dual-connected to the bottom side. By duality, this interface intersects $\{0\} \times [0, n]$ with probability larger or equal to $1/2$. Thus, $\phi_S^{1/0}\left[F(0) \le n\right] \ge \frac{1}{2}$ and therefore

$$\phi_S^{1/0}\left[\mathcal{A}(x)\right] \ge \frac{1}{5n^2m^2}.$$

In order to conclude, we simply need to prove that

$$\phi_{S(m)}^{1/0}\left[0 \longleftrightarrow \text{wired arc}\right] \ge \phi_S^{1/0}\left[\mathcal{A}(x)\right].$$

First, observe that since x is contained in the bottom half $S(n, \theta)$ of S, the set $S_x(m)$ is entirely included in S. Second, since there is no open path containing x and exiting $S_x(m)$ by the free arc, there exists a lowest dual-open path in $S_x(m)^*$, denoted Γ^*, preventing the existence of such a path, see Fig. 4. Let Ω be the set of vertices of $S_x(m)$ above Γ^*. The boundary conditions on Ω are dominated by free boundary conditions on the bottom side of $S_x(m)$ and wired on the three other sides of $S_x(m)$. If $A(x)$ occurs, then conditionally on Γ^*, x is connected to the wired arc of $S_x(m)$ by an open path contained in Ω. Thus,

$$\begin{aligned}\phi_S^{1/0}\left[x \longleftrightarrow \text{wired arc of } S_x(m) | \Gamma^*\right] &\le \phi_{S_x(m)}^{1/0}\left[x \longleftrightarrow \text{wired arc of } S_x(m)\right] \\ &= \phi_{S(m)}^{1/0}\left[0 \longleftrightarrow \text{wired arc of } S(m)\right].\end{aligned}$$

We omitted a few lines to get the first inequality since we already mentioned such an argument. The equality follows from invariance under translations. Since the previous bound is uniform in the possible realizations of Γ^*, we deduce

$$\phi_S^{1/0}\left[\mathcal{A}(x)\right] \le \phi_{S(m)}^{1/0}\left[0 \longleftrightarrow \text{wired arc of } S(m)\right].$$

The result follows readily. □

The next lemma will be used recursively in the proof of Proposition 4.

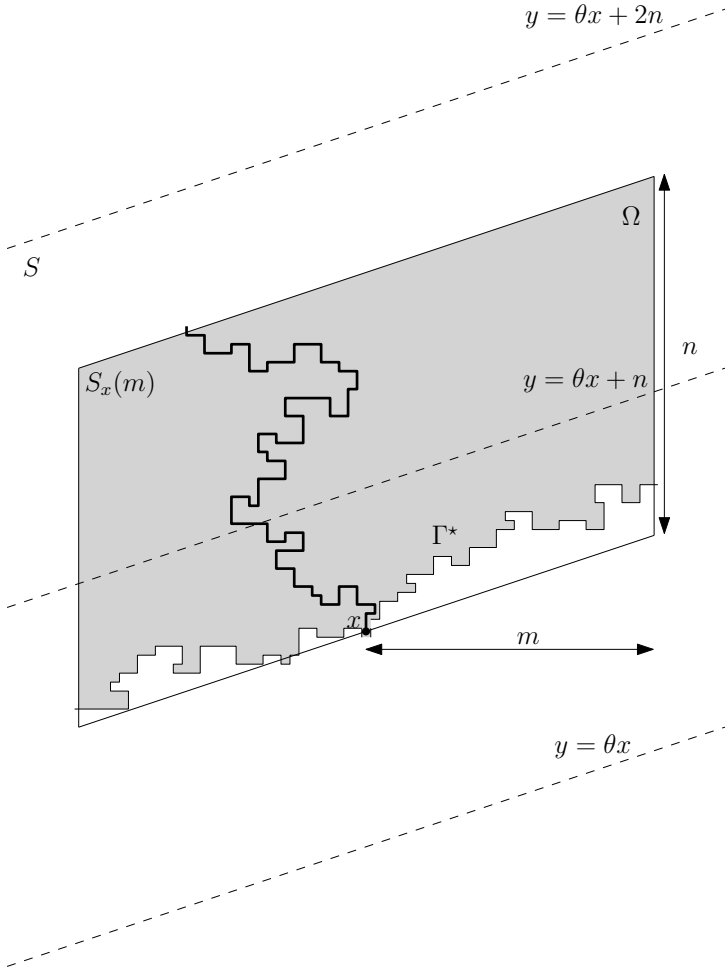

Figure 4: The event $\mathcal{A}(x)$. The bottom-most dual-path Γ^*.

Lemma 7. *Assume that there exists $\alpha > 0$ such that for all $n \geq 1$,*

$$\phi^0_{\mathbb{Z}^2}\left[0 \longleftrightarrow \partial\Lambda_n\right] \leq \exp(-n^\alpha).$$

Then for $\varepsilon > 0$ small enough, there exists a constant $C < \infty$ such that for any $n \geq 1$, any $u \in \{-n\} \times [-n, n]$ and any $v \in \{n\} \times [-n, n]$,

$$\phi^0_{\Lambda_n}\left[u \leftrightarrow v\right] \leq e^{Cn^\varepsilon} \phi^0_{\mathbb{Z}^2}\left[0 \leftrightarrow \partial\Lambda_n\right]^2 + Cn^6 \sum_{\substack{k,\ell \geq n^\varepsilon \\ k+\ell=2n}} \phi^0_{\mathbb{Z}^2}\left[0 \leftrightarrow \partial\Lambda_k\right] \phi^0_{\mathbb{Z}^2}\left[0 \leftrightarrow \partial\Lambda_\ell\right].$$

Proof. Fix $\varepsilon > 0$. Let us translate the box Λ_n in such a way that $u = -v$; the new box is denoted by $\tilde{\Lambda}_n$. Define the set

$$D = \left\{ z \in \tilde{\Lambda}_n : \mathrm{d}(z, [u, v]) < n^\varepsilon \right\}.$$

As illustrated in Fig. 5, we consider the sets D_- and D_+ of points $z \in \tilde{\Lambda}_n$ lying re-

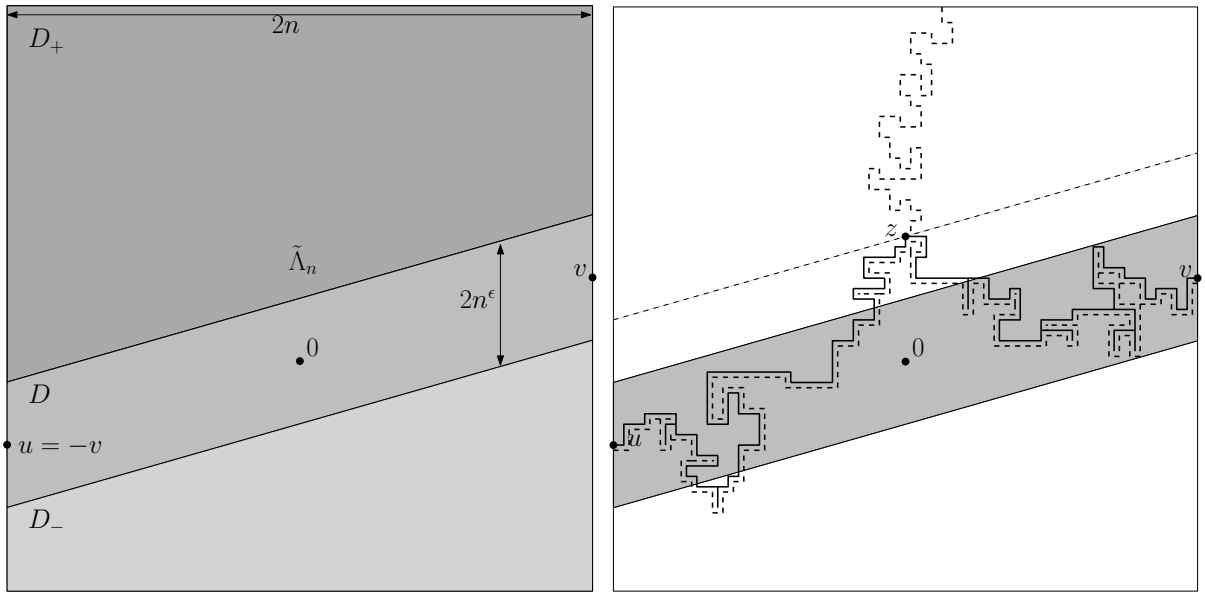

Figure 5: **Left.** The regions D, D_- and D_+. Note that 0 is not necessarily at the center of $\tilde{\Lambda}_n$. **Right.** The situation before closing the edges surrounding z when $\mathcal{G}_n(z)$ and $\{z + (\frac{1}{2}, \frac{1}{2}) \overset{*}{\longleftrightarrow} \partial\Lambda_n\}$ occurs. The dual-open paths are depicted in dash lines.

spectively below D and above D. On $\{u \overset{\tilde{\Lambda}_n}{\longleftrightarrow} v\}$, define Γ^- and Γ^+ to be respectively the lowest and highest open (non-necessarily self-avoiding) paths connecting u to v. The event $\{u \overset{\tilde{\Lambda}_n}{\longleftrightarrow} v\}$ is included in the union of the following three sub-events:

$$\begin{aligned}
\mathcal{E}_- &= \{u \overset{\tilde{\Lambda}_n}{\longleftrightarrow} v\} \cap \{\Gamma^- \cap D_+ \neq \emptyset\}, \\
\mathcal{E}_+ &= \{u \overset{\tilde{\Lambda}_n}{\longleftrightarrow} v\} \cap \{\Gamma^+ \cap D_- \neq \emptyset\}, \\
\mathcal{E} &= \{u \overset{\tilde{\Lambda}_n}{\longleftrightarrow} v\} \cap \{\Gamma^+ \subset D_+ \cup D\} \cap \{\Gamma^- \subset D_- \cup D\}.
\end{aligned} \tag{18}$$

In the rest of the proof, we will bound separately $\phi^0_{\tilde{\Lambda}_n}[\mathcal{E}_-]$ (and therefore $\phi^0_{\tilde{\Lambda}_n}[\mathcal{E}_+]$ by symmetry) and $\phi^0_{\tilde{\Lambda}_n}[\mathcal{E}]$, hence the two terms on the right-hand side of the inequality in the statement.

Estimation of $\phi^0_{\tilde\Lambda_n}[\mathcal{E}_-]$. For $z \in D_+ \cap \mathbb{Z}^2$, let $\mathcal{G}_n(z)$ be the event that:

- u is connected to v in $\tilde\Lambda_n$,
- $z \in \Gamma^-$ and $\mathrm{d}(z,[u,v]) = \max_{z' \in \Gamma^- \cap D_+} \mathrm{d}(z',[u,v])$.

Note that

$$\mathcal{E}_- = \bigcup_{z \in D_+} \mathcal{G}_n(z).$$

Conditionally on Γ^-, what is above Γ^- follows a random-cluster measure with wired boundary conditions on Γ^- and free on $\partial\tilde\Lambda_n$. Thus, by comparison between boundary conditions and Lemma 6 (with $m = n$ and $\theta = \frac{v_2-u_2}{v_1-u_1}$, where $u = (u_1,u_2)$ and $v = (v_1,v_2)$), we find that

$$\phi^0_{\tilde\Lambda_n}\left[z + (1/2,1/2) \overset{*}{\longleftrightarrow} \partial B^*_n \,\middle|\, \mathcal{G}_n(z)\right] \geq \frac{1}{5(2n)^4},$$

When both $\mathcal{G}_n(z)$ and $\{z + (1/2,1/2) \overset{*}{\longleftrightarrow} \partial B^*_n\}$ occur, closing the four dual edges surrounding the vertex z disconnects Γ^- into two paths separated by dual-open circuits (see Fig. 5). The respective end-to-end distances ℓ and k of these paths satisfy $k+\ell \geq 2n-2$.

Using the comparison between boundary conditions once-again, we find

$$\begin{aligned}\phi^0_{\tilde\Lambda_n}[\mathcal{G}_n(z)] &\leq 5(2n)^4\, \phi^0_{\tilde\Lambda_n}\left[\mathcal{G}_n(z) \cap \{z + (1/2,1/2) \overset{*}{\longleftrightarrow} \partial B^*_n\}\right] \\ &\leq \frac{80}{c_{\mathrm{IT}}^4} n^4 \sum_{\substack{k,\ell \geq n^\varepsilon \\ k+\ell=2n-2}} \phi^0_{\mathbb{Z}^2}[u \leftrightarrow u + \partial\Lambda_k]\, \phi^0_{\mathbb{Z}^2}[v \leftrightarrow v + \partial\Lambda_\ell].\end{aligned}$$

The finite energy property is used in the second line to close the edges around z. Summing over all possible $z \in D_+$ gives

$$\phi^0_{\tilde\Lambda_n}[\mathcal{E}_-] \leq C_1 n^6 \sum_{\substack{k,\ell \geq n^\varepsilon \\ k+\ell=2n-2}} \phi^0_{\mathbb{Z}^2}[0 \leftrightarrow \partial\Lambda_k]\, \phi^0_{\mathbb{Z}^2}[0 \leftrightarrow \partial\Lambda_\ell].$$

The finite energy property once again implies that $\phi^0_{\mathbb{Z}^2}[0 \leftrightarrow \partial\Lambda_{r+1}] \geq c_{\mathrm{IT}}\phi^0_{\mathbb{Z}^2}[0 \leftrightarrow \partial\Lambda_r]$ for any $r \geq 0$ and thus

$$\phi^0_{\tilde\Lambda_n}[\mathcal{E}_-] \leq C_2 n^6 \sum_{\substack{k,\ell \geq n^\varepsilon \\ k+\ell=2n}} \phi^0_{\mathbb{Z}^2}[0 \leftrightarrow \partial\Lambda_k]\, \phi^0_{\mathbb{Z}^2}[0 \leftrightarrow \partial\Lambda_\ell].$$

Estimation of $\phi^0_{\tilde\Lambda_n}[\mathcal{E}]$. First, we wish to justify that conditionally on the occurrence of $\mathcal{E}$, there exists an open path between u and v which is staying in D with probability close to 1. To see this, remark that any open path between u and v must lie in the region Ω between Γ^- and Γ^+ (see Fig. 6). Furthermore, conditioning on Γ^+ and Γ^-, the boundary conditions on Ω are wired. In particular, the configuration in Ω dominates the restriction to Ω of a configuration $\tilde\omega$ sampled according to a random-cluster measure with wired boundary conditions at infinity. Since Γ^+ and Γ^- are already open, u and v are connected in D if there exists an open path in $\tilde\omega$ from left to right in D. The complement of this event is included in the event that a dual-vertex of D^* is dual-connected to distance n^ε of itself in $\tilde\omega$. The probability of this event can thus be bounded by $4n^{1+\varepsilon}\exp(-n^{\alpha\varepsilon})$

thanks to the assumption made on connection probabilities. We deduce

$$\phi^0_{\tilde{\Lambda}_n}\left[u \overset{D}{\longleftrightarrow} v\right] \geq \left(1 - 4n^{1+\varepsilon}\exp(-n^{\alpha\varepsilon})\right)\phi^0_{\tilde{\Lambda}_n}\left[\mathcal{E}\right].$$

Now, consider the set of edges E of D intersecting the line $\{\frac{1}{2}\} \times \mathbb{R}$. Also define w_- and w_+ to be respectively the highest point of D^*_- and the lowest point of D^*_+ with first coordinate equal to $\frac{1}{2}$. Let $\mathcal{F}$ be the event that

- all the edges of E are closed,
- w_- and w_+ are dual-connected to $\partial\tilde{\Lambda}^*_n$ in D^*_- and D^*_+ respectively.

Consider the event $u \overset{D}{\longleftrightarrow} v$ and modify the configuration by closing all edges in E. The finite energy property implies that

$$\begin{aligned}\phi^0_{\tilde{\Lambda}_n}&\left[\mathcal{F} \cap \{u \leftrightarrow u + \partial\Lambda_n\} \cap \{v \leftrightarrow v + \partial\Lambda_{n-1}\}\right] \\ &\geq \phi^0_{\tilde{\Lambda}_n}\left[u \overset{D}{\longleftrightarrow} v\right] \times c_{\mathrm{IT}}^{2\sqrt{2}n^\varepsilon} \times \left(\frac{1}{5(2n)^4}\right)^2,\end{aligned}$$

where the term $c_{\mathrm{IT}}^{2\sqrt{2}n^\varepsilon}$ is a uniform lower bound for the probability that all edges in E are closed, and $[5(2n)^4)]^{-2}$ comes from the fact that Lemma 6 gives

$$\begin{aligned}\phi^0_{\tilde{\Lambda}_n}&\left[w_- \overset{*}{\longleftrightarrow} \partial\tilde{\Lambda}^*_n \text{ in } D_- \,\middle|\, u \overset{D}{\longleftrightarrow} v\right] \geq \frac{1}{5(2n)^4} \quad \text{and} \\ \phi^0_{\tilde{\Lambda}_n}&\left[w_+ \overset{*}{\longleftrightarrow} \partial\tilde{\Lambda}^*_n \text{ in } D_+ \,\middle|\, u \overset{D}{\longleftrightarrow} v\right] \geq \frac{1}{5(2n)^4}.\end{aligned}$$

The event $\mathcal{F}$ forces the existence of a dual path disconnecting the cluster of u and the cluster of v (see Fig. 6). Conditioning on the cluster of u and its boundary, the boundary conditions in what remains are dominated by free boundary conditions at infinity, and we deduce that

$$\phi^0_{\tilde{\Lambda}_n}\left[\mathcal{E}\right] \leq e^{c_2 n^\varepsilon}\phi^0_{\mathbb{Z}^2}\left[0 \leftrightarrow \partial\Lambda_n\right]\phi^0_{\mathbb{Z}^2}\left[0 \leftrightarrow \partial\Lambda_{n-1}\right] \leq \frac{e^{c_2 n^\varepsilon}}{c_{\mathrm{IT}}}\phi^0_{\mathbb{Z}^2}\left[0 \leftrightarrow \partial\Lambda_n\right]^2,$$

where once again we used insertion tolerance in the last inequality. The claim follows readily. □

Remark. The previous lemma implies that $\phi^0_{[0,2n]^2}(u \leftrightarrow v)$ is bounded by the right hand-side of (18) for any u and v on two opposite sides of $[0, 2n]^2$. Let us argue that $\phi^0_{[0,2n-1]^2}(u' \leftrightarrow v')$ is also bounded by a universal constant C times the right-hand side of (18) uniformly on u' and v' on opposite sides of $[0, 2n-1]^2$. Indeed, the comparison between boundary conditions shows that

$$\phi^0_{[0,2n-1]^2}(u' \leftrightarrow v') \leq \phi^0_{[0,2n]^2}\big(u' \overset{[0,2n-1]^2}{\longleftrightarrow} v'\big).$$

Now let u and v be two neighbors of u' and v' on opposite sides of $[0, 2n]^2$. The finite energy property implies that

$$\phi^0_{[0,2n-1]^2}(u' \leftrightarrow v') \leq c_{\mathrm{IT}}\phi^0_{[0,2n]^2}(u \leftrightarrow v)$$

and we may apply the previous lemma.

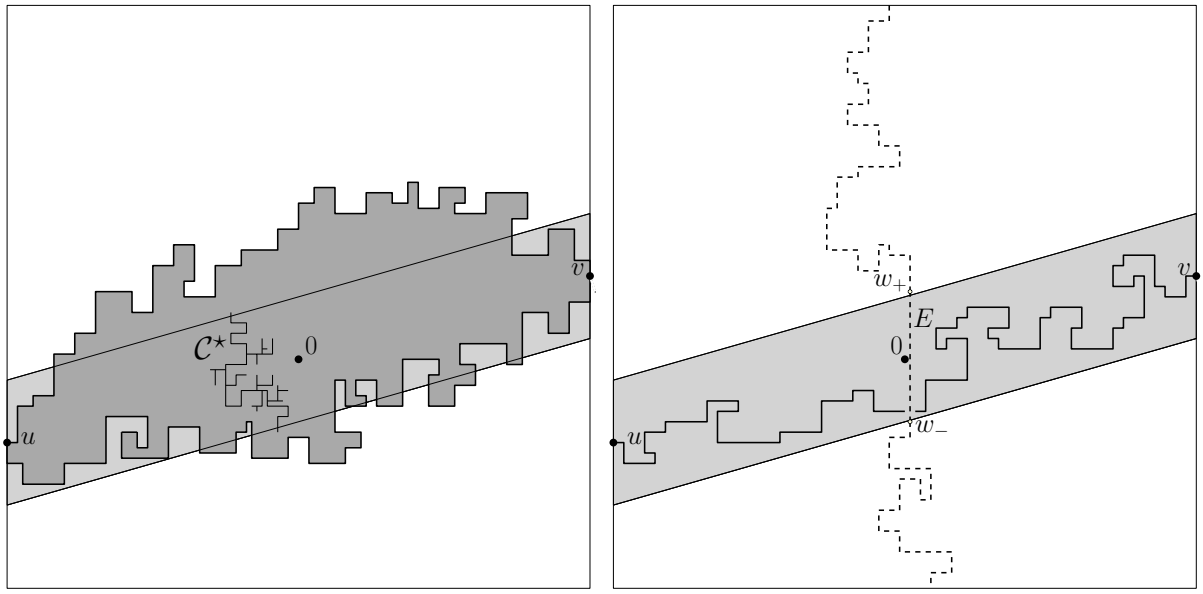

Figure 6: **Left.** The domain Ω between Γ_- and Γ_+. Inside, a dual cluster $\mathcal{C}^*$ preventing the existence of an open path from u to v in D. Since we assumed that connection probabilities decay as a stretched exponential, this cluster exists with very small probability. **Right.** Splitting the open path from u to v in two pieces.

Proof of Proposition 4. Assume that there exists $\alpha > 0$ such that

$$\phi^0_{\mathbb{Z}^2}\left[0 \longleftrightarrow \partial\Lambda_n\right] \leq \exp(-n^\alpha)$$

for any $n \geq 0$. Fix $\varepsilon < \beta < \alpha$ to be chosen later. Set

$$q_n = e^{n^\beta}\phi^0_{\mathbb{Z}^2}\left[0 \leftrightarrow \partial\Lambda_n\right].$$

Lemma 5 applied to $2n$ and Lemma 7 (together with the previous remark) imply that there exists $C_3 > 0$ such that

$$\begin{aligned}
q_{2n} &\leq e^{(2n)^\beta}\sum_{m\geq n} C_3 m^4\Big(e^{c_3 m^\varepsilon}\phi^0(0 \leftrightarrow \partial\Lambda_m)^2 \\
&\qquad\qquad + c_3 m^6 \sum_{\substack{k,\ell\geq m^\varepsilon\\ k+\ell=2m}} \phi^0(0 \leftrightarrow \partial\Lambda_k)\phi^0(0 \leftrightarrow \partial\Lambda_\ell)\Big) \\
&\leq e^{(2n)^\beta}\sum_{m\geq n} C_3 m^4\Big(e^{c_3 m^\varepsilon}e^{-2m^\beta}q_m^2 + c_3 m^6 \sum_{\substack{k,\ell\geq m^\varepsilon\\ k+\ell=2m}} e^{-(k^\beta+\ell^\beta)}q_k q_\ell\Big) \\
&\leq \Big(\max_{\substack{k,\ell\geq n^\varepsilon\\ k+\ell\geq 2n}} q_k q_\ell\Big)e^{(2n)^\beta}\sum_{m\geq n} C_3 m^4\Big(e^{c_3 m^\varepsilon}e^{-2m^\beta} + c_3 m^6 \sum_{\substack{k,\ell\geq m^\varepsilon\\ k+\ell=2m}} e^{-(k^\beta+\ell^\beta)}\Big) \\
&\leq C_4 \max_{\substack{k,\ell\geq n^\varepsilon\\ k+\ell\geq 2n}} q_k q_\ell,
\end{aligned}$$

where $C_4 < \infty$ is a constant independent of n. The existence of C_4 follows from a simple computation using $\varepsilon < \beta$ and the fact that $\beta < 1$ and $k, \ell \geq n^\varepsilon$ imply

$$e^{-(k^\beta+\ell^\beta)} \leq e^{-(k+\ell)^\beta}e^{-c_4 n^{\varepsilon\beta}}$$

for some constant $c_4 > 0$ [2].

Let us now come back to the proof. The finite energy property implies the existence of $c_5 > 0$) such that $c_5 q_k \leq q_{k+1} \leq q_k/c_5$ for any $k \geq 0$. Using this fact, the previous inequality immediately extends to odd integers and there exists $C_5 < \infty$ such that

$$q_n \leq C_5 \max_{\substack{k,\ell \geq n^\varepsilon \\ k+\ell \geq n}} q_k q_\ell.$$

We unfortunately need to include the following technical trick. We do not know a priori that (q_n) is decreasing. For this reason, we set $Q_n = C_5 \max\{q_m : m \geq n\}$. For this definition, we get

$$Q_n \leq \max_{\substack{k,\ell \geq n^\varepsilon \\ k+\ell \geq n}} Q_k Q_\ell.$$

We are now in a position to conclude. The assumption implies that (Q_n) tends to zero. Pick n_0 such that $Q_n < 1$ for $n \geq n_0^\varepsilon$. Since $(Q_n)_{n\geq 0}$ is decreasing, the maximum of $Q_k Q_\ell$ is not reached for $k \geq n$ or $\ell \geq n$ and we obtain that for $n \geq n_0$,

$$Q_n \leq \max_{\substack{n > k,\ell \geq n^\varepsilon \\ k+\ell \geq n}} Q_k Q_\ell.$$

We can now proceed by induction to prove that for $n \geq n_0$,

$$Q_n \leq \exp(-c_6 n) \quad \text{where} \quad c_6 := \max_{n_0^\varepsilon \leq n \leq n_0} -\tfrac{1}{n}\log(Q_n) > 0.$$

We therefore conclude that

$$\phi^0_{\mathbb{Z}^2}\left[0 \longleftrightarrow \partial\Lambda_n\right] \leq n \exp(n^\beta) \cdot \frac{1}{C_5} \exp(-c_6 n).$$

□

4 Proof of Theorem 4

4.1 An input coming from the theory of parafermionic observables

We will use the parafermionic observable. For a complete exposition of the current knowledge on parafermionic observables and a proof of this statement, we refer to [Dum13, Chapter 6].

[2] Let us make a small remark before proceeding forward with the proof. It was crucial to keep the division in the inequality of Lemma 7 between a term $k = \ell = m$ with a stretched exponential penalty $8m^3 e^{c_3 m^\varepsilon}$, and the general term $k + \ell = 2m$, for which we have only a polynomial penalty $8m^3 c_3 m^6$. If we would have replaced the polynomial bound by a stretched exponential one for every k and ℓ, the values of k or ℓ close to n^ε would have created difficulties since the correction would have been of the order of the largest of the two terms.

Dobrushin domains. In order to properly state and use the result, we first define the notion of Dobrushin domain.

Let us start by defining the *medial lattice* $(\mathbb{Z}^2)^\diamond$, which is the graph with the centers of edges of $\mathbb{Z}^2$ as vertex set, and edges connecting nearest vertices. The vertices and edges of the medial lattice are called medial-vertices and medial-edges. This lattice is a rotated and rescaled (by a factor $1/\sqrt{2}$) version of $\mathbb{Z}^2$. Edges of $(\mathbb{Z}^2)^\diamond$ are oriented counterclockwise around medial-faces having a vertex of $\mathbb{Z}^2$ at their center. Like that, the medial lattice can sometimes be seen as an oriented graph.

Let $a^\diamond$ and $b^\diamond$ be two distinct medial-vertices, and $\partial^\diamond_{ab} = \{v_0 \sim v_1 \sim \cdots \sim v_n\}$, $\partial^\diamond_{ba} = \{w_0 \sim w_1 \sim \cdots \sim w_m\}$ two paths of neighboring medial-vertices satisfying the following properties:

- The paths start from $a^\diamond$ and end at $b^\diamond$, i.e. $v_0 = w_0 = a^\diamond$ and $v_n = w_m = b^\diamond$.
- The paths follow the orientation of the medial lattice.
- The path $\partial^\diamond_{ab}$ goes counterclockwise (around the set enclosed by $\partial^\diamond_{ab} \cup \partial^\diamond_{ba}$), while $\partial^\diamond_{ba}$ goes clockwise.
- The paths are edge-avoiding.
- The paths intersect only at $a^\diamond$ and $b^\diamond$.

Note that $\partial^\diamond_{ab} \cup \partial^\diamond_{ba}$ is a non self-crossing edge-avoiding polygon. However, some vertices might be visited twice.

Definition (medial Dobrushin domains)**.** Let $\partial^\diamond_{ab}$ and $\partial^\diamond_{ba}$ be two paths as above, and let $\Omega^\diamond$ be the subgraph of $(\mathbb{Z}^2)^\diamond$ induced by the medial-vertices that are enclosed by or in the path $\partial^\diamond_{ab} \cup \partial^\diamond_{ba}$. Then, $(\Omega^\diamond, a^\diamond, b^\diamond)$ is called a *medial Dobrushin domain.* An example is given in Fig. 7.

As it stands, $a^\diamond$ and $b^\diamond$ have three incident medial-edges in $E_{\Omega^\diamond}$. Call e_a and e_b the fourth medial-edges incident to $a^\diamond$ and $b^\diamond$ respectively.

Definition (primal and dual Dobrushin domains with two marked points)**.** Let $(\Omega^\diamond, a^\diamond, b^\diamond)$ be a medial Dobrushin domain.

Let $\Omega \subset \mathbb{Z}^2$ be the graph with edge set composed of edges passing through end-points of medial-edges in $E_{\Omega^\diamond} \setminus \partial^\diamond_{ab}$ (if a medial-vertex is the end-point of a medial-edge in $E_{\Omega^\diamond} \setminus \partial^\diamond_{ab}$ and one in $\partial^\diamond_{ab}$, it is included) and vertex set given by the end-points of these edges. Let a and b be the two vertices of Ω bordered by e_a and e_b. The triplet (Ω, a, b) is called a *primal Dobrushin domain.* We denote by ∂_{ba} the set of edges corresponding to medial-vertices in $\partial\Omega^\diamond$ which are also end-points of medial-edges in $\partial^\diamond_{ba}$, and set $\partial_{ab} = \partial\Omega \setminus \partial_{ba}$.

Let $\Omega^* \subset \mathbb{Z}^*$ be the graph with edge set composed of dual-edges passing through medial-edges in $E_{\Omega^\diamond} \setminus \partial^\diamond_{ba}$ and vertex set given by the end-points of these dual-edges. Let a^* and b^* be the two dual-vertices of Ω^* bordered by e_a and e_b. The triplet (Ω^*, a^*, b^*) is called a *dual Dobrushin domain.* We denote by ∂^*_{ab} the set of dual-edges corresponding to medial-vertices in $\partial\Omega^\diamond$ which are also end-points of medial-edges in $\partial^\diamond_{ab}$, and set $\partial^*_{ba} = \partial\Omega^* \setminus \partial^*_{ab}$.

For a Dobrushin domain, let us define the Dobrushin boundary conditions on (Ω, a, b) to be wired on ∂_{ba}, and free on ∂_{ab}. The random-cluster measure with these boundary conditions and $p = p_c(q)$ is denoted by $\phi^{a,b}_\Omega$. We enforce the boundary conditions as follows: we open every edge of ∂_{ba} to create wired boundary conditions on this arc, and every dual-edge of ∂^*_{ab} to create dual-wired boundary conditions on this dual arc (and therefore free boundary conditions on the corresponding primal arc by duality).

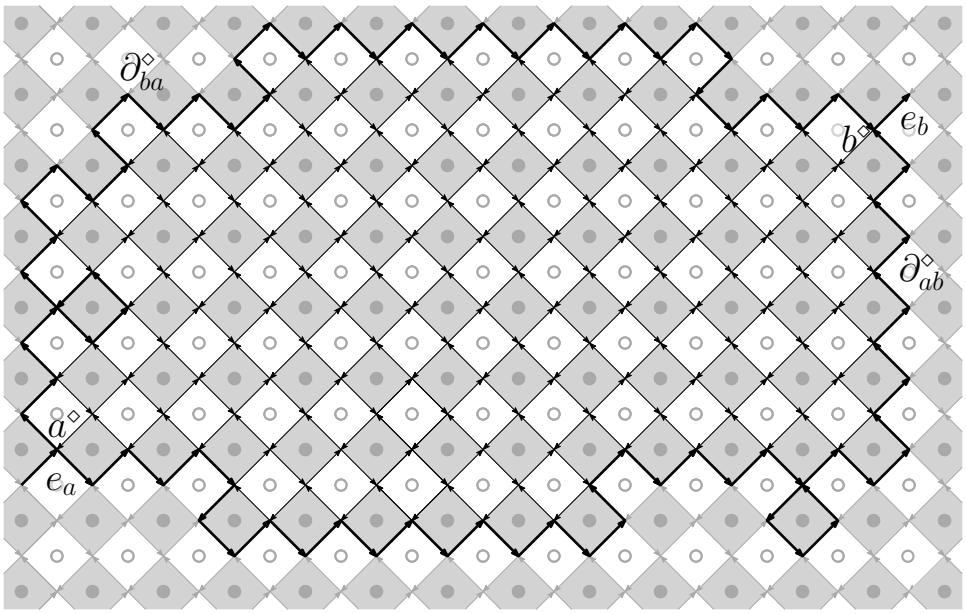

Figure 7: A medial Dobrushin domain. Note the position of e_a and e_b.

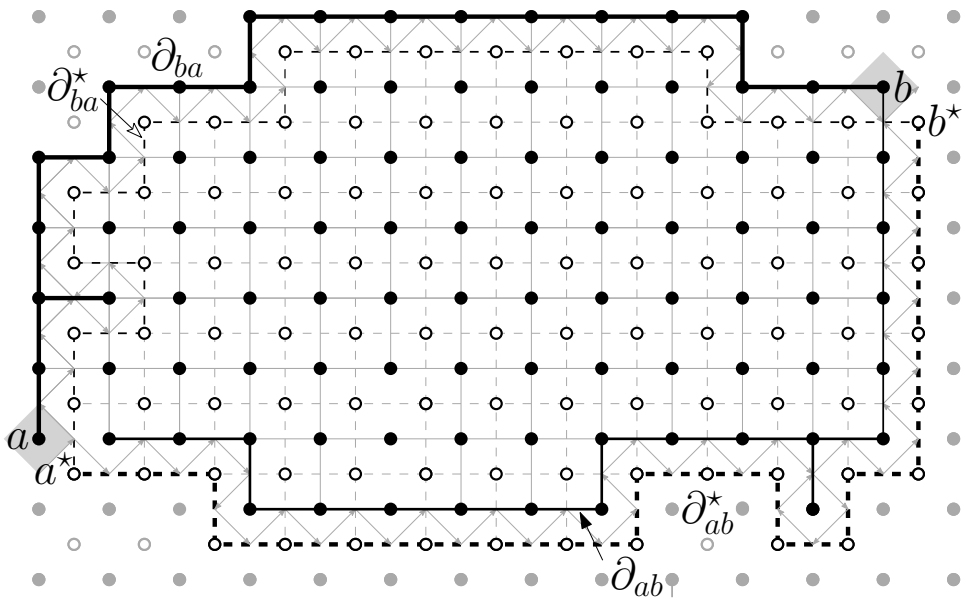

Figure 8: The primal and dual Dobrushin domains associated to a medial Dobrushin domain. Note the position of a, a^*, b and b^*.

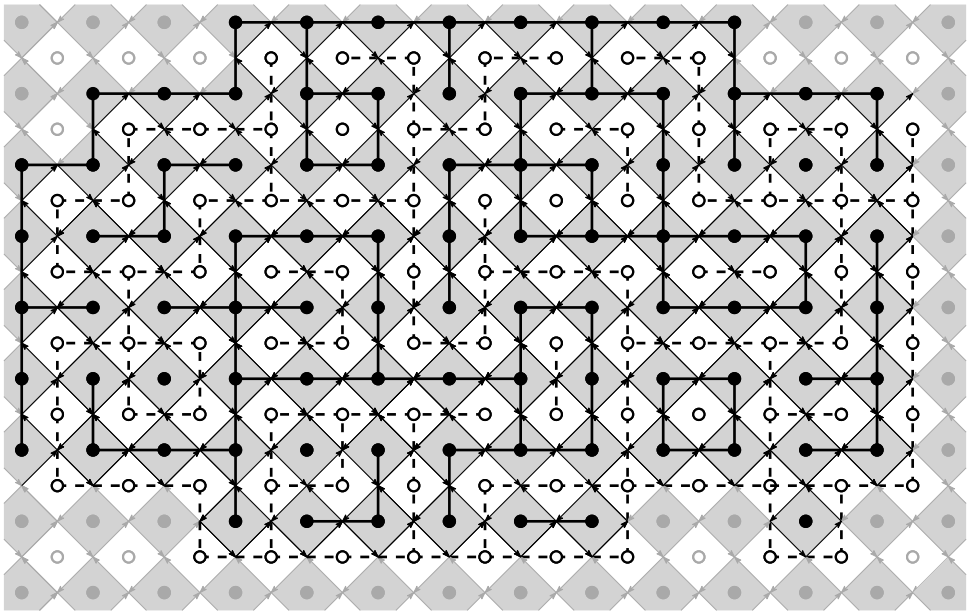

Figure 9: The configuration ω with its dual configuration ω^*.

The main statement. We start by defining the loop-configuration associated to a percolation-configuration. Fix a Dobrushin domain (Ω, a, b) and consider a configuration $\omega \in \{0,1\}^{E_\Omega}$ together with its dual-configuration $\omega^\star \in \{0,1\}^{E_{\Omega^\star}}$. Through every vertex of the medial graph $\Omega^\diamond$ of Ω passes either an open bond of Ω or a dual open bond of $\Omega^\star$. Draw self-avoiding loops on $\Omega^\diamond$ as follows: a loop arriving at a vertex of the medial lattice always makes a $\pm\pi/2$ turn so as not to cross the open or dual open bond through this vertex, see Fig. 10. The loop representation contains loops together with a self-avoiding path going from $a^\diamond$ to $b^\diamond$, see Fig. 10. This curve is called the *exploration path* and is denoted by γ.

Remark. The loops correspond to the interfaces separating clusters from dual clusters, and the exploration path corresponds to the interface between the cluster connected to ∂_{ba} and the dual cluster connected to ∂^*_{ab}.

We are now in a position to define the parafermionic observable.

Definition. The *winding* $\mathrm{W}_\Gamma(e, e')$ of a curve Γ (on the medial lattice) between two medial-edges e and e' of the medial graph is the total signed rotation in radians that the curve makes from the mid-point of the edge e to that of the edge e'. By convention, if Γ does not go through e', we set $\mathrm{W}_\Gamma(e, e') = 0$.

For a curve drawn on the medial lattice, the winding can be computed in a very simple way: it corresponds to $\frac{\pi}{2}$ times the number of $\frac{\pi}{2}$-turns on the left minus the number of $\frac{\pi}{2}$-turns on the right.

Definition (Smirnov [Smi10])**.** Consider a Dobrushin domain (Ω, a, b) and $q > 0$. The *edge parafermionic observable* $F = F(q, \Omega, a, b)$ is defined for any medial edge $e \in E_{\Omega^\diamond}$ by

$$F(e) \;:=\; \begin{cases} \phi^{a,b}_{\Omega,p_c(q),q}[\mathrm{e}^{\mathrm{i}\sigma \mathrm{W}_\gamma(e,e_b)}\mathbf{1}_{e\in\gamma}] & \text{if } q \neq 4, \\ \phi^{a,b}_{\Omega,p_c(4),4}[\mathrm{W}_\gamma(e,e_b)\mathrm{e}^{\mathrm{i}\mathrm{W}_\gamma(e,e_b)}\mathbf{1}_{e\in\gamma}] & \text{otherwise,} \end{cases}$$

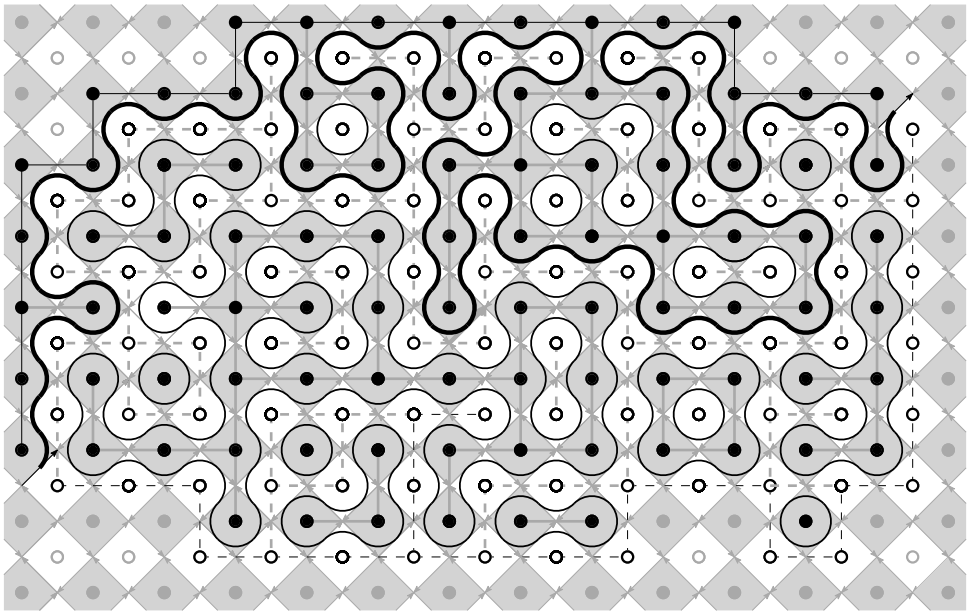

Figure 10: The loop representation associated to the primal and dual configurations in the previous picture. The exploration path is drawn in bold.

where γ is the exploration path and σ is given by the relation $\sin(\sigma\pi/2) = \sqrt{q}/2$.

Remark. The observable $F_\delta(v)$ mentioned in Conjecture 2 is defined on a vertex v of $\Omega_\delta^\diamond = \delta(\mathbb{Z}^2)^\diamond \cap \Omega$ as half the sum of $F(e)$ over edges e incident to v.

Remark. Note that σ is real for $q \leq 4$, and belongs to $1 + i\mathbb{R}$ for $q > 4$. For $q \in [0, 4]$, σ has the physical interpretation of a spin, which is fractional in general, hence the name parafermionic (fermions have half-integer spins while bosons have integer spins, there are no particles with fractional spin, but the use of such fractional spins at a theoretical level has been very fruitful in physics). For $q > 4$, σ is not real anymore and does not have any physical interpretation.

The parafermionic observable satisfies a very specific property at criticality regarding contour integrals. One may define a dual $(\Omega^\diamond)^\star$ of $\Omega^\diamond$ in the following way: the vertex set of $(\Omega^\diamond)^\star$ is $V_\Omega \cup V_{\Omega^\star}$ and the edges of the dual connect nearest vertices together. We extend the definitions available for other graphs to this context. A *discrete contour* $\mathcal{C}$ is a finite sequence $z_0 \sim z_1 \sim \cdots \sim z_n = z_0$ in $(\Omega^\diamond)^\star$ such that the path $(z_0, \ldots, z_n)$ is edge-avoiding. The discrete contour integral of the parafermionic observable F along $\mathcal{C}$ is defined by

$$\oint_{\mathcal{C}} F(z)dz := \sum_{i=0}^{n-1} (z_{i+1} - z_i) F\left(\{z_i, z_{i+1}\}^\star\right),$$

where the z_i are considered as complex numbers and $\{z_i, z_{i+1}\}^\star$ denotes the edge of $\Omega^\diamond$ intersecting $\{z_i, z_{i+1}\}$ in its center.

Theorem 10 (Vanishing contour integrals)**.** *Let (Ω, a, b) be a Dobrushin domain, $q > 0$*

and $p = p_c$. For any discrete contour $\mathcal{C}$ of (Ω, a, b),

$$\oint_{\mathcal{C}} F(z)dz = 0.$$

Remark. The fact that discrete contour integrals vanish seems to be close to a well-known property of holomorphic functions: their contour integrals vanish. Nevertheless, one should be slightly careful when drawing such a parallel: the edge-observable is defined on edges, and should rather be understood as the discretization of a form than as the discretization of a function. As a form, the fact that these discrete contour integrals vanish should be interpreted as the discretization of the property of being closed.

For elementary contours surrounding a vertex of $\Omega^\diamond$ with four medial edges incident to it, the fact that discrete contour integrals vanish translate into the following relation:

$$F(A) - F(C) = \mathrm{i}(F(D) - F(B)), \tag{19}$$

where A, B, C and D are the four medial edges around the vertex, indexed in clockwise order. For clarity, we will use the following slightly modified version of the observable:

$$\widehat{F}(e) \ := \ \begin{cases} \phi^{a,b}_{\Omega,p_c(q),q}[\mathrm{e}^{\mathrm{i}\widehat{\sigma}\mathrm{W}_\gamma(e,e_b)}\mathbf{1}_{e\in\gamma}] & \text{if } q \neq 4, \\ \phi^{a,b}_{\Omega,p_c(4),4}[\mathrm{W}_\gamma(e,e_b)\mathbf{1}_{e\in\gamma}] & \text{otherwise,} \end{cases}$$

where $\widehat{\sigma} = 1 - \sigma$ satisfies $\cos(\pi\widehat{\sigma}/2) = \sqrt{q}/2$. With this definition, the relation that the discrete contour integrals vanish can be restated as follows: (19) becomes

$$\widehat{F}(A) + \widehat{F}(C) = \widehat{F}(B) + \widehat{F}(D),$$

and for any set V of vertices of $\Omega^\diamond$ having four incident edges in $E_{\Omega^\diamond} \cup \{e_a, e_b\}$, we can sum the previous relation to get

$$\sum_{e \text{ incident to exactly one vertex in } V} \eta(e)\widehat{F}(e) = 0, \tag{20}$$

where $\eta(e)$ equals 1 if e points towards a vertex in V, and -1 if it points away from a vertex in V. Notice that only edges with exactly one end-point in V contribute to this sum.

4.2 Proof of Theorem 4

The original proof of Theorem 4 can be found in [Dum12]. However, we choose to present a shorter proof here which is based on some of the new arguments of the previous section.

In this section, we fix $p = p_c(q)$. For simplicity, we start by treating the $1 \leq q \leq 2$ case in order to illustrate the strategy. Then, we focus on the general $1 \leq q \leq 4$ case.

Proof of Theorem 4 in the $1 \leq q \leq 2$ case. For $n \geq 1$, consider the set $R_n := [0, n] \times [-n, n]$ and recall that $\Lambda_n = [-n, n]^2$. See R_n as a Dobrushin domain with wired arc equal to the vertex $(0, 0)$ (see Fig 11). In such case, the exploration path is the loop going around $(0, 0)$, and e_a and e_b are the two edges of the boundary bordering (0,0).

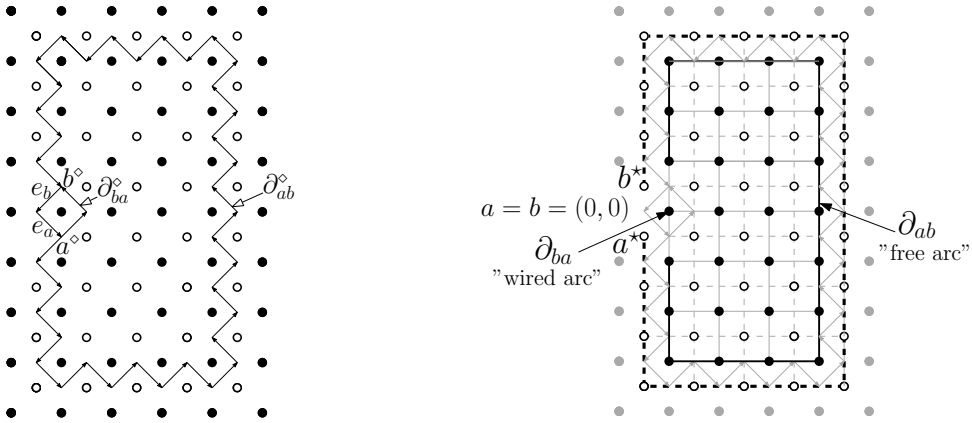

Figure 11: **Left.** The medial paths $\partial^\diamond_{ab}$ and $\partial^\diamond_{ba}$ allowing to define R_n as a Dobrushin domain. **Right.** The set R_n, seen as a Dobrushin domain.

Let V be the set of medial vertices of $R_n^\diamond$ with four incident medial edges in $E_{R_n^\diamond} \cup \{e_a, e_b\}$. Alternatively, V can be defined as the set of medial-vertices corresponding to the edges in R_n. Equation (20) implies

$$\sum_{e\in\partial_1} \eta(e)\widehat{F}(e) = -\sum_{e\in\partial_2} \eta(e)\widehat{F}(e),$$

where ∂_1 (resp. ∂_2) is the set of medial-edges of $R_n^\diamond$ incident to exactly one medial-vertex of V, and bordering a vertex of $\partial R_n \cap \Lambda_{n-1}$ (resp. $\partial R_n \cap \partial\Lambda_n$).

Let us bound the complex modulus of the sums on the left and right from below and above respectively. We start by the sum on the right. Observe that

$$|\widehat{F}(e)| \le \phi^0_{R_n}(e \in \gamma) = \phi^0_{R_n}(0 \longleftrightarrow x) \le \phi^0(0 \longleftrightarrow x),$$

where x is the vertex of $\partial R_n \cap \partial\Lambda_n$ bordered by the medial edge e. In the first inequality, we used that $\widehat{\sigma} \in \mathbb{R}$ and therefore $\mathrm{e}^{\mathrm{i}\widehat{\sigma}\mathrm{W}_\gamma(e,e_b)}$ is of modulus 1, in the equality, the fact that e is on the exploration path if and only if 0 and x are connected, and in the second inequality the comparison between the boundary conditions. Since every such vertex is bordered by two medial edges, we find that

$$\Big| -\sum_{e\in\partial_2} \eta(e)\widehat{F}(e)\Big| \le 2 \sum_{x\in\partial R_n\cap\partial\Lambda_n} \phi^0(0 \longleftrightarrow x) \le 2 \sum_{x\in\partial\Lambda_n} \phi^0(0 \longleftrightarrow x). \tag{21}$$

Let us now bound the left-hand side from below. First, consider the two medial edges e_a and e_b. Their contributions are given by

$$F(e_a) - F(e_b) := e^{i\widehat{\sigma}3\pi/2} - 1 = 2\mathrm{i}\mathrm{e}^{\mathrm{i}\widehat{\sigma}3\pi/4}\sin(\widehat{\sigma}3\pi/4).$$

Second, consider the medial edges e_1, e_2, e_3 and e_4 of ∂_1 bordering the vertices $x = (0, x_2)$ and $-x = (0, -x_2)$ where $1 \le x_2 < n$. Observe that for edges on the boundary, the winding of any possible realization of the exploration path going through an edge e is deterministic (we denote it by $\mathrm{W}(e)$). As a consequence,

$$F(e) = \mathrm{e}^{\mathrm{i}\widehat{\sigma}\mathrm{W}(e)}\phi^0_{R_n}(e \in \gamma) = \mathrm{e}^{\mathrm{i}\widehat{\sigma}\mathrm{W}(e)}\phi^0_{R_n}(0 \longleftrightarrow x).$$

Therefore, using the explicit values of $W(e_1)$, $W(e_2)$, $W(e_3)$ and $W(e_4)$, we obtain

$$\begin{aligned}\sum_{i=1}^{4}\eta(e_i)F(e_i) &= \Big(\sum_{i=1}^{4}\eta(e_i)\mathrm{e}^{\mathrm{i}\widehat{\sigma}\mathrm{W}(e_i)}\Big)\phi^0_{R_n}(0\longleftrightarrow x)\\ &= \Big(\mathrm{e}^{\mathrm{i}\widehat{\sigma}3\pi/2}-\mathrm{e}^{\mathrm{i}\widehat{\sigma}2\pi}+\mathrm{e}^{-\mathrm{i}\widehat{\sigma}\pi/2}-1\Big)\phi^0_{R_n}(0\longleftrightarrow x)\\ &= -4\mathrm{i}\mathrm{e}^{\mathrm{i}\widehat{\sigma}3\pi/4}\cos(\widehat{\sigma}\pi)\sin(\widehat{\sigma}\pi/4)\phi^0_{R_n}(0\longleftrightarrow x).\end{aligned}$$

Altogether, for $q\le 2$, $1/2\le\widehat{\sigma}\le 1$ and we deduce

$$\begin{aligned}\Big|\sum_{e\in\partial_1}\eta(e)F(e)\Big| &= 2\sin(\widehat{\sigma}3\pi/4)-4\cos(\widehat{\sigma}\pi)\sin(\widehat{\sigma}\pi/4)\sum_{x\in\partial R_n\cap\Lambda_{n-1}}\phi^0_{R_n}(0\longleftrightarrow x)\\ &\ge 2\sin(\widehat{\sigma}3\pi/4).\end{aligned}$$

Putting this relation together with (21), we find that for every $n\ge 1$,

$$\sum_{x\in\partial\Lambda_n}\phi^0(0\longleftrightarrow x)\ge\sin(\widehat{\sigma}3\pi/4). \tag{22}$$

Summing over every n shows **P3** of Theorem 3 and therefore Theorem 4. □

Remark. It is worth mentioning that the strategy used for $1\le q\le 2$ cannot work directly for $q>2$. Indeed, predictions using conformal invariance give that

$$\phi^0_{\mathbb{N}\times\mathbb{Z}}(0\longleftrightarrow\partial\Lambda_n)=n^{-\alpha(\pi,q)+o(1)}$$

where $\alpha(\pi,q)$ is a critical exponent. The value of the exponent increases from 1/3 to 1 when q goes from 1 to 4. In particular, for $q>2$, the value becomes larger or equal to 1/2, and it is natural to expect that

$$\phi^0_{R_n}(0\longleftrightarrow x)\approx\phi^0_{\mathbb{N}\times\mathbb{Z}}(0\longleftrightarrow\partial\Lambda_n/2)^2=n^{2\alpha(\pi,q)+o(1)}\ll 1/n$$

for n large enough. This is in contradiction with (22). For $q>2$, we will therefore extend our reasoning and work with domains of the form $R_n^{\theta_0}:=\{re^{i\theta}\in\mathbb{Z}^2:\theta\in[-\theta_0,\theta_0]\}$ with $\theta_0\ge\pi/2$. Indeed, in this case, conformal invariance predicts that $\phi^0_{R_n^{\theta_0}}(0\longleftrightarrow\partial\Lambda_n)=n^{-\alpha(\theta_0,q)}$, where $\alpha(\theta_0,q)=\alpha(\pi,q)\cdot\pi/\theta_0$, and therefore

$$\phi^0_{R_n^{\theta_0}}(0\longleftrightarrow x)\approx n^{\alpha(\pi,q)+\alpha(\theta_0,q)+o(1)}.$$

This gives hope that a strategy similar to the previous one could work as long as $\alpha(\pi,q)\le\theta_0/(\pi+\theta_0)$. Since $\alpha(\pi,q)$ tends to 1 as $q\le 4$, it would in fact be necessary to work with domains which are not subsets of $\mathbb{Z}^2$ (for instance, we will consider subsets of the graph $\mathbb{U}$, defined below).

We now focus on the proof of Theorem 4 in the $1\le q\le 4$ case.

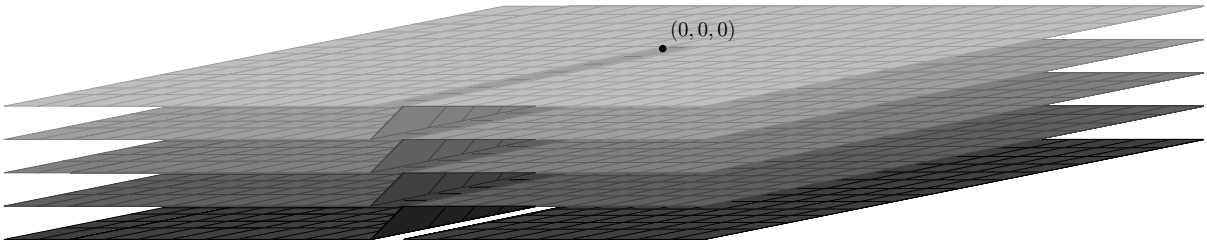

Figure 12: The graph $\mathbb{U}$.

Set C_n to be the slit domain obtained by removing from Λ_n the edges between the vertices of $\{(0,k) : 0 \leq k \leq n\}$. We define Dobrushin boundary conditions on C_n to be wired on $\{(0,k) : 0 \leq k \leq n\}$ and free elsewhere. For simplicity, we now refer to $\{(0,k) : 0 \leq k \leq n\}$ as the *wired arc*. The measure on C_n with these boundary conditions is denoted $\phi^{\text{dobr}}_{C_n}$. Equivalently, one may obtain $\phi^{\text{dobr}}_{C_n}$ by taking $\phi^0_{\Lambda_n}[\,\cdot\,|\omega(e) = 1$ for all e in wired arc$]$ and we therefore think of C_n as the box Λ_n with free boundary conditions and $\{(0,k) : 0 \leq k \leq n\}$ wired; see Fig. 13.

Lemma 8. *Let $q \in [1,4]$, there exists $c > 0$ such that for any $n \geq 1$,*

$$\phi^{\text{dobr}}_{C_n}[(0,-n) \longleftrightarrow \text{wired arc}] \geq \frac{c}{n^{16}}.$$

The main estimate used in the proof of this lemma will follow from (20) applied in a well-chosen domain. Then, we compare boundary conditions in this domain to Dobrushin boundary conditions in C_n. To exploit the whole power of (20), we will consider a domain which is non-planar. Namely, let us introduce the following graph $\mathbb{U}$ (see Fig. 12): the vertices are given by $\mathbb{Z}^3$ and the edges by

- $[(x_1,x_2,x_3),(x_1,x_2+1,x_3)]$ for every $x_1,x_2,x_3 \in \mathbb{Z}$,
- $[(x_1,x_2,x_3),(x_1+1,x_2,x_3)]$ for every $x_1,x_2,x_3 \in \mathbb{Z}$ such that $x_1 \neq 0$,
- $[(0,x_2,x_3),(1,x_2,x_3)]$ for every $x_2 \geq 0$ and $x_3 \in \mathbb{Z}$,
- $[(0,x_2,x_3),(1,x_2,x_3+1)]$ for every $x_2 < 0$ and $x_3 \in \mathbb{Z}$.

This graph is the universal cover of $\mathbb{Z}^2 \setminus F$, where F is the face centered at $(\frac{1}{2},-\frac{1}{2})$. It can also be seen at $\mathbb{Z}^2$ with a branching point at $(\frac{1}{2},-\frac{1}{2})$. All definitions of dual and medial graphs extend to this context, as well as (20).

Proof. For $n \geq 1$, define

$$U_n := \big\{(x_1,x_2,x_3) \in \mathbb{U} : |x_1|,|x_2| \leq n \text{ and } |x_3| \leq n^5\big\}.$$

We wish to apply (20) to the Dobrushin domain obtained from U_n by fixing the wired arc to be the vertex (0,0,0). Even if the domain is non-planar, one can easily define a medial graph. The medial edges e_a and e_b both correspond to the medial edge e of the boundary of this domain which is adjacent to 0, and the exploration path is the loop going through this edge. For simplicity, and in order to define the wiring of the curve between e_a and e_b correctly, we will consider e_a and e_b as begin two different edges, e_a (resp. e_b) being adjacent to only one medial vertex of the medial graph, namely the endpoint of e towards which (resp. from which) e is pointing. Let V be the set of medial-vertices incident to

four edges of $E_{U_n^\diamond} \cup \{e_a, e_b\}$. Equation (20) applied to V gives exactly as in the previous proof, that

$$F(e_a) - F(e_b) = -\sum_{e \in \partial} \eta(e) F(e),$$

where ∂ is the set of medial edges different from e_a and e_b incident to only one vertex of V. Now, let us focus for a moment on the $q < 4$ case. Then, $F(e_a) - F(e_b) = e^{i\widehat{\sigma} 2\pi} - 1$, and as before $|-\sum_{e \in \partial} \eta(e) F(e)| \leq 2 \sum_{x \in \partial U_n} \phi^0_{U_n}[x \leftrightarrow 0]$. We immediately deduce that

$$\sum_{x \in \partial U_n} \phi^0_{U_n}[0 \leftrightarrow x] \geq c_1 \tag{23}$$

for some constant $c_1 = c_1(q) > 0$ independent of n. The same computation may be performed for $q = 4$ and we also get (23): indeed in such case,

$$F(e_a) - F(e_b) = 2\pi - 1$$

and for two medial edges $e, e' \in \partial$ bordering the same primal vertex x on the boundary,

$$|\eta(e)F(e) + \eta(e')F(e')| = |W(e) - W(e')| \cdot \phi^0_{U_n}[0 \leftrightarrow x] \leq \frac{3\pi}{2} \phi^0_{U_n}[0 \leftrightarrow x],$$

where $W(e)$ (resp. $W(e')$) denotes the winding of any possible loop going from e (resp. e') to e_b.

We now wish to bootstrap this estimate to an estimate on C_n. Let us start by proving the following claim (observe that $|x_3| < n^5$ in the statement).

Claim: There exists $c_2 > 0$ (independent of n) such that there exists $x = (x_1, x_2, x_3) \in \partial U_n$ with $|x_3| < n^5$ and

$$\phi^0_{U_n}[0 \leftrightarrow x] \geq \frac{c_2}{n^6}.$$

We will prove this fact by showing that vertices x with $|x_3| = n^5$ have very small probability of being connected to the origin and therefore cannot account for much in (23).

Proof of the Claim. Let R_0^* be the dual graph of the subgraph of U_n with vertex set $R_0 := [-n, n] \times [0, n] \times \{0\}$, i.e. the graph with edge set $\{e^* : e \in E_{R_0}\}$ and vertex set given by the end-points of these edges. Note that uniformly in the state of edges outside R_0, the boundary conditions in R_0 are dominated by wired boundary conditions on the "bottom side" $[-n, n] \times \{0\} \times \{0\}$ of R_0, and free elsewhere. Passing to the dual model, we deduce that uniformly in the state of edges outside R_0,

$$\phi^0_{U_n}\left[(-\tfrac{1}{2}, -\tfrac{1}{2}, 0) \overset{*}{\longleftrightarrow} \partial U_n^* \text{ in } R_0^* \middle| \text{edges outside } R_0\right] \geq \phi^{1/0}_{R_0^*}\left[(-\tfrac{1}{2}, -\tfrac{1}{2}, 0) \overset{*}{\longleftrightarrow} \partial U_n^* \text{ in } R_0^*\right],$$

where $\phi^{1/0}_{R_0^*}$ is the (dual) random-cluster measure on R_0^* with free boundary conditions on the bottom and wired boundary conditions everywhere else. Lemma 6 (with $m = n$ and $\theta = 0$) thus implies that

$$\phi^0_{U_n}\left[(-\tfrac{1}{2}, -\tfrac{1}{2}, 0) \overset{*}{\longleftrightarrow} \partial U_n^* \text{ in } R_0^* \middle| \text{edges outside } R_0\right] \geq \frac{1}{5n^4}. \tag{24}$$

The same is also true for $R_k^* = (0,0,k) + R_0^*$ with $|k| \le n^5$.

If a vertex $x = (x_1, x_2, x_3) \in \partial U_n$ with $x_3 = n^5$ is connected to $(0,0,0)$, then none of the dual vertices $(-\frac{1}{2}, -\frac{1}{2}, k)$ are dual connected to ∂U_n in R_k^* for $0 < k < x_3$ (the symmetric claim holds for $x_3 = -n^5$). Equation (24) applied $|x_3| - 1$ times implies that

$$\phi^0_{U_n}\left[0 \longleftrightarrow x\right] \;\le\; \left(1 - \frac{1}{5n^4}\right)^{|x_3|-1}.$$

The probability is therefore exponentially small when $|x_3| = n^5$. Together with (23), the previous inequality implies that for n large enough,

$$\sum_{x \in \partial U_n : |x_3| < n^5} \phi^0_{U_n}\left[0 \leftrightarrow x\right] \ge \frac{c_1}{2}.$$

The claim follows directly from the union bound, provided that c_2 is chosen small enough.

$\diamond$

Fix x given by the claim and rotate and translate vertically[3] U_n in such a way that $x = (x_1, -n, 0)$ for some $-n \le x_1 \le n$. Consider C_n as a subgraph of U_n. The boundary conditions on C_n induced by the free boundary conditions on U_n are dominated by the Dobrushin boundary conditions on C_n defined above. Furthermore, the existence of an open path from x to the origin implies the existence of a path from x to the wired arc in C_n. Thus, the claim implies that

$$\phi^{\text{dobr}}_{C_n}\left[x \leftrightarrow \text{wired arc}\right] \;\ge\; \frac{c_2}{n^6}.$$

To conclude the proof, we need to obtain a lower bound for the probability that the vertex $(0, -n, 0)$ itself is connected to the wired arc. We use once again a "conditioning on the right-most and left-most paths type argument". Since we now work on a sub-domain of $\mathbb{Z}^2$, we drop the third coordinate from the notation.

We may assume that $x_1 \ge 0$ and that the two vertices $x = (x_1, -n)$ and $(-x_1, -n)$ are connected to the wired arc. The FKG inequality implies that this occurs with probability $(\frac{c_2}{n^6})^2$. Consider the right-most open path from $(x_1, -n)$ to the wired arc, and the left-most open path from $(-x_1, -n)$ to the wired arc. Let S be the part of C_n between these two paths, see Fig. 13. The boundary conditions in S dominate the free boundary conditions on the bottom of C_n, and wired elsewhere. We use a comparison between boundary conditions. The reasoning is the same as usual: we compare boundary conditions on S with the boundary conditions induced by boundary conditions on Λ_n with free boundary conditions on the bottom and wired boundary conditions on the other sides. Lemma 6 (applied to $2n$, $m = n$ and $\theta = 0$) thus implies that $(0, -n)$ is connected to the wired arc with conditional probability larger than $\frac{1}{20n^4}$, and we finally obtain

$$\phi^{\text{dobr}}_{C_n}\left[(0,-n) \longleftrightarrow \text{wired arc}\right] \ge \left(\frac{c_2}{n^6}\right)^2 \frac{1}{20n^4}.$$

$\square$

[3]Seen as a graph, U_n is invariant by rotation by $\pi/2$ since the line where x_3 "increases" is invisible from inside U_n.

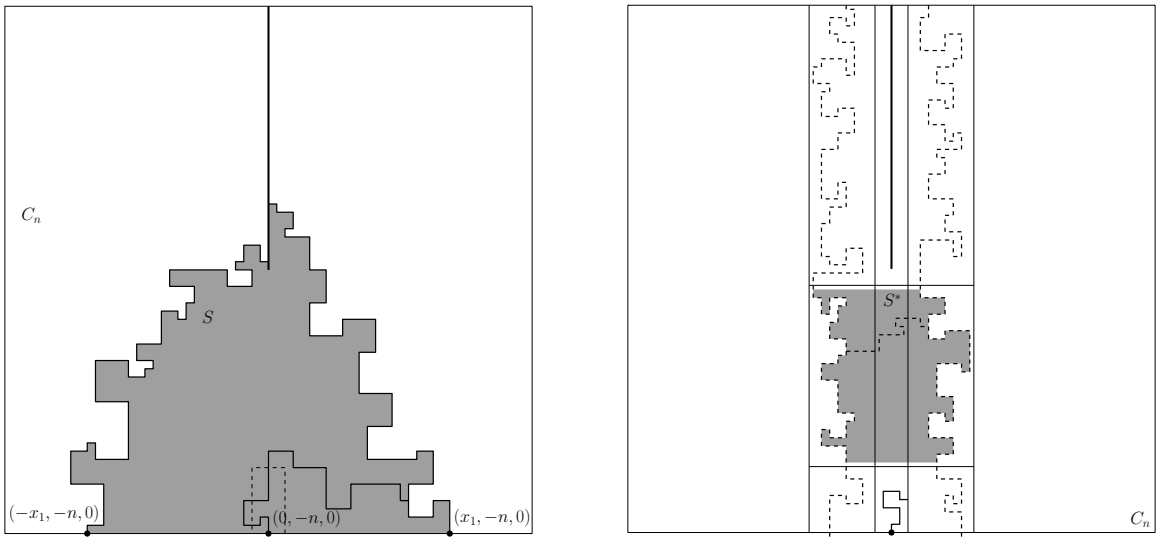

Figure 13: **Left.** The two paths connecting the wired arc to $(x_1, -n, 0)$ and $(-x_1, -n, 0)$ (or simply $(x_1, -n)$ and $(-x_1, -n)$) and the area S between them. **Right.** The two dual-open paths in the long rectangles R^*_{right} and R^*_{left}.

We are now in a position to prove Theorem 4. Let ∂_n be the set of vertices at distance $\frac{n}{16}$ of the vertex $(0, -n)$ in C_n. The reasoning is similar to the proof of Lemma 4 except that instead of isolating primal circuits around z_- and z_+ from each other, we will isolate the primal path from $(0, -n)$ to ∂_n from the wired arc.

Proof of Theorem 4 for $q \in (2, 4]$. Introduce the three rectangles

$$\begin{aligned}
R^*_{\text{right}} &:= \left[\tfrac{n}{16} + \tfrac{1}{2}, \tfrac{5n}{16} - \tfrac{1}{2}\right] \times [-(n + \tfrac{1}{2}), n + \tfrac{1}{2}], \\
R^*_{\text{left}} &:= \left[-\tfrac{5n}{16} + \tfrac{1}{2}, -\tfrac{n}{16} - \tfrac{1}{2}\right] \times [-(n + \tfrac{1}{2}), n + \tfrac{1}{2}], \\
R^* &:= \left[-\tfrac{5n}{16} + \tfrac{1}{2}, \tfrac{5n}{16} - \tfrac{1}{2}\right] \times \left[-\tfrac{3n}{4} + \tfrac{1}{2}, -\tfrac{n}{8} - \tfrac{1}{2}\right].
\end{aligned}$$

Define the three events $\mathcal{E} = \{(0, -n) \longleftrightarrow \partial_n \text{ in } C_n\}$, $\mathcal{F}_{\text{right}}$ and $\mathcal{F}_{\text{left}}$ that there exists a dual-open dual-path from bottom to top in R^*_{right} and R^*_{left} respectively. Let $\mathcal{C}$ be the event that there exists a dual-open dual-path in R^* connecting a dual open path crossing R^*_{left} from top to bottom to a dual open path crossing R^*_{right} from top to bottom.

Conditioning on $\mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}} \cap \mathcal{C}$, boundary conditions on R_n are dominated by free boundary conditions in the plane. Therefore

$$\phi^0_{\mathbb{Z}^2}\left[0 \leftrightarrow \partial\Lambda_{n/16}\right] \geq \phi^{\text{dobr}}_{C_n}\left[\mathcal{E} | \mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}} \cap \mathcal{C}\right] \geq \phi^{\text{dobr}}_{C_n}\left[\mathcal{E} \cap \mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}} \cap \mathcal{C}\right].$$

We now prove a lower bound on the term on the right:

$$\phi^{\text{dobr}}_{C_n}\left[\mathcal{E} \cap \mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}} \cap \mathcal{C}\right] = \phi^{\text{dobr}}_{C_n}\left[\mathcal{E}\right] \cdot \phi^{\text{dobr}}_{C_n}\left[\mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}} | \mathcal{E}\right] \cdot \phi^{\text{dobr}}_{C_n}\left[\mathcal{C} | \mathcal{E} \cap \mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}}\right].$$

First, Lemma 8 implies that $\phi^{\text{dobr}}_{C_n}(\mathcal{E}) \geq \frac{c}{n^{16}}$. Second, conditioned on everything on the left of $\{\frac{n}{16}\} \times [-n, n]$, boundary conditions on $[\frac{n}{16}, n] \times [-n, n]$ are dominated by wired boundary conditions on the left side and free elsewhere. In particular, boundary

conditions for the dual model dominate free boundary conditions on the left side and wired elsewhere. Lemma 3 implies that $\phi^{\text{dobr}}_{C_n}[\mathcal{F}_{\text{right}}|\mathcal{E}] \geq c_2$ and the same lower bound holds true for $\phi^{\text{dobr}}_{C_n}[\mathcal{F}_{\text{left}}|\mathcal{F}_{\text{right}} \cap \mathcal{E}]$. We obtain

$$\phi^{\text{dobr}}_{C_n}[\mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}}|\mathcal{E}] \geq c_2^2.$$

Third, we turn to $\phi^{\text{dobr}}_{C_n}[\mathcal{C}|\mathcal{E} \cap \mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}}]$. Let S^* be the area of the dual graph in R^* between the right-most dual-open path from top to bottom in R^*_{right}, and the left-most dual-open path crossing from top to bottom in R^*_{left}, see Fig. 13. The boundary conditions for the dual model on S^* dominate free boundary conditions on top and bottom, and wired elsewhere. The domain Markov property and the comparison between boundary conditions imply that boundary conditions for the dual model on S^* dominate free boundary conditions on top and bottom sides of R^*, and wired on the two other sides. Therefore, the probability of having a dual-open path in S^* crossing from left to right is larger than $1/(1+q^2)$ thanks to (3). In particular,

$$\phi^{\text{dobr}}_{C_n}[\mathcal{C}|\mathcal{E} \cap \mathcal{F}_{\text{left}} \cap \mathcal{F}_{\text{right}}] \geq \frac{1}{1+q^2}.$$

Putting everything together, we find that

$$\phi^0_{\mathbb{Z}^2}\left[0 \leftrightarrow \partial\Lambda_{n/16}\right] \geq \frac{c}{n^{16}} \cdot c_2^2 \cdot \frac{1}{1+q^2}$$

and indeed

$$\lim_{n\to\infty} -\tfrac{1}{n}\log\left(\phi^0_{\mathbb{Z}^2}[0 \leftrightarrow \partial\Lambda_n]\right) = 0.$$

□

Remark. We worked in the specific domain U_n but one may apply these techniques in more general subdomains of $\mathbb{U}$ to obtain exponential decay of correlations using the parafermionic observable and ideas from [DT15a, DT15b]. We refer to [Dum15] for an exposition of a proof based on this idea in the $q \leq 3$ case (the proof could be extended using the techniques of Section 5.3 to the $q \in [3,4]$ case, but the proof has not been written down). Note that for $q > 4$, a proof of exponential decay of correlations using the observable only can be found in [BDS13].

4.3 Ordering for $q > 4$

As mentioned in the introduction, the phase transition of the random-cluster model (or equivalently of the Potts model) with $q > 4$ is expected to be discontinuous. In particular, this would mean that there exists an infinite cluster almost surely for the critical measure with wired boundary conditions. We are currently unable to prove this result. Nevertheless, we are able to prove the following (much) weaker result.

The graph $\mathbb{U}$ is planar and we can define its dual graph, denoted $\widetilde{\mathbb{U}}$. Notice that all the vertices of $\widetilde{\mathbb{U}}$ have degree 4, except one vertex, denoted $\mathbf{b}$, which has infinite degree (it corresponds to the vertex at the center of the spiral). Given a finite subgraph $\widetilde{U}$ of $\widetilde{\mathbb{U}}$, containing the vertex $\mathbf{b}$, we define the random cluster measure in $\widetilde{U}$ with wired boundary conditions: the boundary of $\widetilde{U}$ is given by $\mathbf{b}$ together with all the vertices of $\widetilde{U}$ with

degree strictly smaller than 4 in $\widetilde{U}$. We can then define the random-cluster measure on $\widetilde{\mathbb{U}}$ with wired boundary condition, denoted by $\phi^1_{\widetilde{\mathbb{U}}}$, by taking the limit as $\widetilde{U}$ converges to $\widetilde{\mathbb{U}}$. We fix a vertex $\mathbf{v}$ of $\widetilde{\mathbb{U}}$, disjoint from $\mathbf{b}$, and write $\mathbf{v} \longleftrightarrow \infty$ for the event that there exists an infinite open path from $\mathbf{v}$ which is *disjoint from* $\mathbf{b}$.

Proposition 5. *For $q > 4$, $\phi^1_{\widetilde{\mathbb{U}}}(\mathbf{v} \longleftrightarrow \infty) > 0$.*

It would be very interesting to improve this result by showing that it implies $\phi^1_{\mathbb{Z}^2}[0 \leftrightarrow \infty] > 0$. As for today, we did not manage to do so. Let us mention a question whose understanding may help solving this problem. Consider the upper half-plane $\mathbb{H} = \mathbb{Z} \times \mathbb{N}$. Assume that there exists an infinite cluster in this plane (once again not using any boundary edge) for the random-cluster measure with wired boundary conditions on the boundary of $\mathbb{H}$. Can one show that there exists an infinite cluster for the random-cluster measure on $\mathbb{Z}^2$ with wired boundary conditions. Obviously, this is true for Bernoulli percolation since $\mathbb{H} \subset \mathbb{Z}^2$. Yet this fact is not sufficient to prove the claim for general cluster-weights $q > 1$ since the wiring on $\mathbb{Z}^2 \times \{0\}$ may influence the measure inside the upper half-plane by favoring open edges.

Let us now prove the proposition.

Proof. Consider the random-cluster measure on $\mathbb{U}$ with free boundary conditions. Let us prove that $\mathbf{0} := (0,0,0)$ and $(0,0,-k)$ are connected to each other with probability decaying exponentially fast in $k \geq 0$. The Borel-Cantelli lemma would then imply that finitely many pairs of integers $k \geq 0$ and $\ell > 0$ are such that $(0,0,-k)$ and $(0,0,\ell)$ are connected to each other. This immediately shows the existence of an infinite dual-open cluster in the dual model, which is the random cluster on $\widetilde{\mathbb{U}}$ with wired boundary conditions.

To prove this exponential decay, we invoke (20) again.

Let $n > k+1 \geq 2$. Let U_n be the subdomain of $\mathbb{U}$ generated by $\mathbb{U} \cap [-n,n]^3$, except that the edge $f_k := \{(0,0,-k),(0,-1,-k)\}$ is removed. U_n can be seen as the Dobrushin domain with $a = b = (0,0,0)$. Let V be the set of all medial vertices in U_n with four incident medial edges. Medial vertices in V corresponds to primal edges in $\mathbb{U} \cap [-n,n]^3$ except the edge f_k. Let e_1, e_2, e_3, e_4 be the four medial edges incident to the medial vertex corresponding to f_k. These four medial edges are incident to exactly one vertex in V. Two of them, say e_1 and e_2, border the vertex $(0,0,-k)$. Let E be the set of all medial edges adjacent to exactly one vertex in V, distinct from e_1, e_2, e_a, e_b. Equation (20) implies

$$-\eta(e_1)F(e_1) - \eta(e_2)F(e_2) - \sum_{e \in E} \eta(e)F(e) = F(e_a) - F(e_b). \tag{25}$$

The Dobrushin boundary conditions are simply the free boundary conditions in this case. Set $\tilde{\sigma} = \mathrm{i}\hat{\sigma}$ which is a real number *since we assume that* $q > 4$. Without loss of generality, we can assume $\tilde{\sigma} > 0$. We find

$$-\eta(e_1)F(e_1) - \eta(e_2)F(e_2) = e^{\tilde{\sigma} 2\pi(k+1)}(e^{\pi\tilde{\sigma}/2} - 1)\phi^0_{U_n}\left[\mathbf{0} \leftrightarrow (0,0,-k)\right], \tag{26}$$

and

$$F(e_a) - F(e_b) = e^{\tilde{\sigma} 2\pi} - 1. \tag{27}$$

One can arrange the edges of E in pairs, by putting together two medial edges when they border the same (primal) vertex. Using this pairing, one sees that

$$\sum_{e\in E}\eta(e)F(e)\leq 0. \tag{28}$$

Plugging Equations (26),(27) and (28) in (25), we deduce that there exists $C=C(q)>0$ not depending on k or n so that

$$\phi^0_{U_n}\left[\mathbf{0}\leftrightarrow(0,0,-k)\right]\leq Ce^{-2\pi\tilde{\sigma}k}.$$

Letting n tend to infinity, and using the finite energy property (recall that the edge f_k is closed in U_n, but not necessarily in $\mathbb{U}$), we conclude that

$$\phi^0_{\mathbb{U}}\left[\mathbf{0}\leftrightarrow(0,0,-k)\right]\leq Ce^{-2\pi\tilde{\sigma}k}.$$

□

Remark. The fact that $\sigma\in[0,1]$ for $q\leq 4$ and $\sigma=1-\mathrm{i}\mathbb{R}_+$ for $q>4$ explains the difference of behavior between $q\leq 4$ and $q>4$ random-cluster models.

5 Proofs of other theorems

5.1 Proof of Theorem 5

This section contains the proof of Theorem 5.

Lemma 9. *Let $k\leq n$ and ξ some arbitrary boundary conditions on $\partial\Lambda_n$. There exist two couplings $\mathbf{P}$ and $\mathbf{Q}$ on configurations (ω_ξ,ω_1) with the following properties:*

- *ω_ξ and ω_1 have respective laws $\phi^\xi_{\Lambda_n}$ and $\phi^1_{\Lambda_n}$.*
- *$\mathbf{P}$-almost surely, if ω_1^* contains a dual-open dual-circuit in $\Lambda_n^*\setminus\Lambda_k^*$ and Γ^* is the outermost such circuit, then Γ^* is also closed in ω_ξ, and furthermore ω_1 and ω_ξ coincide inside Γ^*.*
- *$\mathbf{Q}$-almost surely, if ω_ξ contains an open circuit in $\Lambda_n\setminus\Lambda_k$ and $\tilde{\Gamma}$ is the outermost such circuit, then $\tilde{\Gamma}$ is also open in ω_1, and furthermore ω_1 and ω_ξ coincide inside $\tilde{\Gamma}$.*

Proof. We start by explaining how to sample $\phi^\xi_{\Lambda_n}$. The Domain Markov property enables us to construct a configuration as follows. Consider uniform random variables U_e on $[0,1]$ for every edge e. Choose an edge e_1 and declare it open if U_{e_1} is smaller or equal to $\phi^\xi_{\Lambda_n}[\omega(e_1)=1]$. Choose another edge e_2 and declare it open if $U_{e_2}\leq\phi^\xi_{\Lambda_n}[\omega(e_2)=1|\omega(e_1)]$. We iterate this procedure for every edge. Also note that we can stop the procedure after a certain number of edges and sample the rest of the edges according to the right conditional law. The domain Markov property guarantees that the measure thus obtained is $\phi^\xi_{\Lambda_n}$. Note that the choice of the next edge can be random, as long as it depends only on the state of edges discovered so far.

Of course, the previous construction is useless for one measure, but it becomes interesting if we consider two measures: one may sample both configurations based on the same random variables U_e with a specific way of choosing the next edges. Let us now describe the way we are choosing the edges:

- *Construction of* $\mathbf{P}$: After t steps, the edge $e_{t+1} \in E_{\Lambda_n} \setminus E_{\Lambda_k}$ is chosen in such a way that it has one end-point connected to $\partial\Lambda_n$ by an open path in ω_1, until it is not possible anymore. Then sample all remaining edges at once according to the correct conditional law. If there is a closed circuit surrounding Λ_k in ω_1, then there is a time t such that at time $t+1$, no undiscovered edges has an end-point which is connected to the boundary in ω_1. Since this procedure guarantees that $\omega_1 \geq \omega_\xi$, no such edges is connected to the boundary in ω_ξ as well. Therefore, the configuration sampled inside the remaining domain is a random-cluster model with free boundary conditions in both cases. In particular, both configurations coincide in Λ_k.

- *Construction of* $\mathbf{Q}$: After t steps, the edge $e_{t+1} \in E_{\Lambda_n} \setminus E_{\Lambda_k}$ is chosen in such a way that one end-point of e^*_{t+1} is dual-connected to $\partial\Lambda^*_n$ by a dual-open path in ω^*_ξ, until it is not possible anymore. Then sample all remaining edges according to the correct conditional law. If there is an open circuit surrounding Λ_k in ω_ξ, then there is a first time t such that the open circuit is discovered exactly at time t. Once again, $\omega_1 \geq \omega_\xi$ and this circuit is also open in ω_1. Then, the configuration inside the connected component of Λ_k in $\Lambda_n \setminus \{e_1, e_2, \ldots, e_t\}$ will be sampled according to a random-cluster configuration with wired boundary conditions. In particular, both configurations coincide in Λ_k.

□

Proof of Theorem 5. It is clearly sufficient to prove that there exists $\alpha > 0$ such that

$$\left|\phi^\xi_{\Lambda_n}[A] - \phi^1_{\Lambda_n}[A]\right| \leq \left(\frac{k}{n}\right)^\alpha \phi^\xi_{\Lambda_n}[A]$$

for any event A depending on edges in Λ_k. Let E be the event that there exists a dual-open dual-circuit in ω^*_ξ included in $\Lambda^*_n \setminus \Lambda^*_k$. We deduce

$$\begin{aligned}\phi^\xi_{\Lambda_n}[A] &\geq \phi^\xi_{\Lambda_n}[A \cap E] = \mathbf{Q}[\omega_\xi \in A \cap E] \geq \mathbf{Q}[\omega_1 \in A \cap E] \\ &= \phi^1_{\Lambda_n}[A \cap E] \geq (1 - (k/n)^\alpha)\phi^1_{\Lambda_n}[A]\end{aligned}$$

where in the third inequality, we used the fact that if ω_1 belongs to $A \cap E$, then ω_1 and therefore ω_ξ belong to E. Since on E, ω_1 and ω_ξ coincide in Λ_k, then $\omega_\xi \in A$. The existence of α in the last inequality follows exactly as in the proof of Lemma 1 from Property **P5a** applied in concentric annuli $\Lambda_{k2^{i+1}} \setminus \Lambda_{k2^i+1}$ with $0 \leq i \leq \log_2(n/k)$.

Reciprocally, if F denotes the event that there is an open circuit in $\Lambda_n \setminus \Lambda_k$, we find

$$\begin{aligned}\phi^1_{\Lambda_n}[A] &\geq \phi^1_{\Lambda_n}[A \cap F] = \mathbf{P}[\omega_1 \in A \cap F] \geq \mathbf{P}[\omega_\xi \in A \cap F] \\ &= \phi^\xi_{\Lambda_n}[A \cap F] \geq (1 - (k/n)^\alpha)\phi^\xi_{\Lambda_n}[A]\end{aligned}$$

where once again, we used in the third inequality that if $\omega_\xi \in A \cap F$, then ω_1 is in F, and since ω_1 and ω_ξ then coincide on Λ_k, we get that $\omega_1 \in A$. The last inequality is due to **P5a** once again. □

5.2 Proof of Theorem 6

In the statement of Theorem 6, we consider an approximation of a simply connected domain Ω with two points a and b on its boundary. More precisely, $(\Omega_\delta, a_\delta, b_\delta)$ denotes a Dobrushin domain defined on the square lattice of mesh size δ, i.e. $\delta\mathbb{Z}^2$. All the definitions and result extend to this context in a direct fashion.

A family of Dobrushin domain *approximates* a continuous domain (Ω, a, b) if $(\Omega_\delta, a_\delta, b_\delta)$ converges in the Carathéodory sense as δ tends to 0. This convergence is classical, we refer to [Dum13, Chapter 3] for details. Let us simply say that for smooth domains, it corresponds to the convergence in the Hausdorff sense.

In what follows, γ_δ denotes the exploration path in the Dobrushin domain $(\Omega_\delta, a_\delta, b_\delta)$.

In order to prove Theorem 6, [KS12] shows that we only need to check the condition **G2** defined now. Consider a fixed simply connected domain (Ω, a, b) and a parametrized continuous curve Γ from a to b in Ω. A connected set C is said to disconnect $\Gamma(t)$ from b if it disconnects a neighborhood of $\Gamma(t)$ from a neighborhood of b in $\Omega \setminus \Gamma[0,t]$.

Fig. 14 will help the reader here. For any annulus $A = A(z, r, R) := z + (\Lambda_R \setminus \Lambda_r)$, let A_t be the subset of Ω satisfying $A_t := \emptyset$ if $\partial(z + \Lambda_r) \cap \partial(\Omega \setminus \Gamma[0,t]) = \emptyset$, and otherwise

$$A_t := \left\{ \begin{array}{l} z \in A \setminus \Gamma[0,t] \text{ such that the connected component of } z \\ \text{in } A \setminus \Gamma[0,t] \text{ does not disconnect } \Gamma(t) \text{ from } b \text{ in } \Omega \setminus \Gamma[0,t] \end{array} \right\}.$$

Consider the exploration path γ_δ as a continuous curve from $a_\delta^\diamond$ to $b_\delta^\diamond$ parametrized in such a way that it goes along one medial vertex in time 1 (in particular, after time n the path explored n medial-vertices). For simplicity, once the path reaches $b_\delta^\diamond$, it remains at $b_\delta^\diamond$ for any subsequent time.

Condition G2 *There exists $C > 1$ such that for any (γ_δ) in $(\Omega_\delta, a_\delta, b_\delta)$, for any stopping time τ and any annulus $A = A(z, r, R)$ with $0 < Cr < R$,*

$$\phi_{\Omega_\delta}^{a_\delta, b_\delta}\Big(\gamma_\delta[\tau, \infty] \text{ makes a crossing of } A \text{ contained in } A_\tau \Big| \gamma_\delta[0, \tau]\Big) < \tfrac{1}{2}.$$

Above, "$\gamma_\delta[\tau, \infty]$ makes a crossing of A contained in A_τ" means that there exists a sub-path $\gamma_\delta[t_1, t_2]$ of the continuous path $\gamma_\delta[\tau, \infty]$ that intersects both the boundary of $z + \Lambda_r$ and the boundary of $z + \Lambda_R$, and such that $\gamma_\delta[t_1, t_2] \subset A_\tau$.

Before proving Condition **G2** (and therefore Theorem 6), let us introduce the notion of slit Dobrushin domain. Fig. 14 provides an example of what it may be.

Fix a Dobrushin domain $(\Omega_\delta, a_\delta, b_\delta)$ and consider the exploration path γ_δ in the loop representation on Ω_δ. The path γ_δ can be seen as a random parametrized curve (the parametrization being simply given by the number of steps along the curve between a medial-vertex in γ_δ and $a_\delta^\diamond$).

Definition. The *slit domain* $\Omega_\delta \setminus \gamma_\delta[0, n]$ is defined as the subdomain of Ω_δ constructed by removing all the primal edges crossed by $\gamma_\delta[0, n]$ and by keeping only the connected component of the remaining graph containing b_δ. It is seen as a Dobrushin domain by fixing the points c_δ and b_δ, where c_δ is the vertex of $\delta\mathbb{Z}^2$ bordered by the last medial edge of $\gamma_\delta[0, n]$.

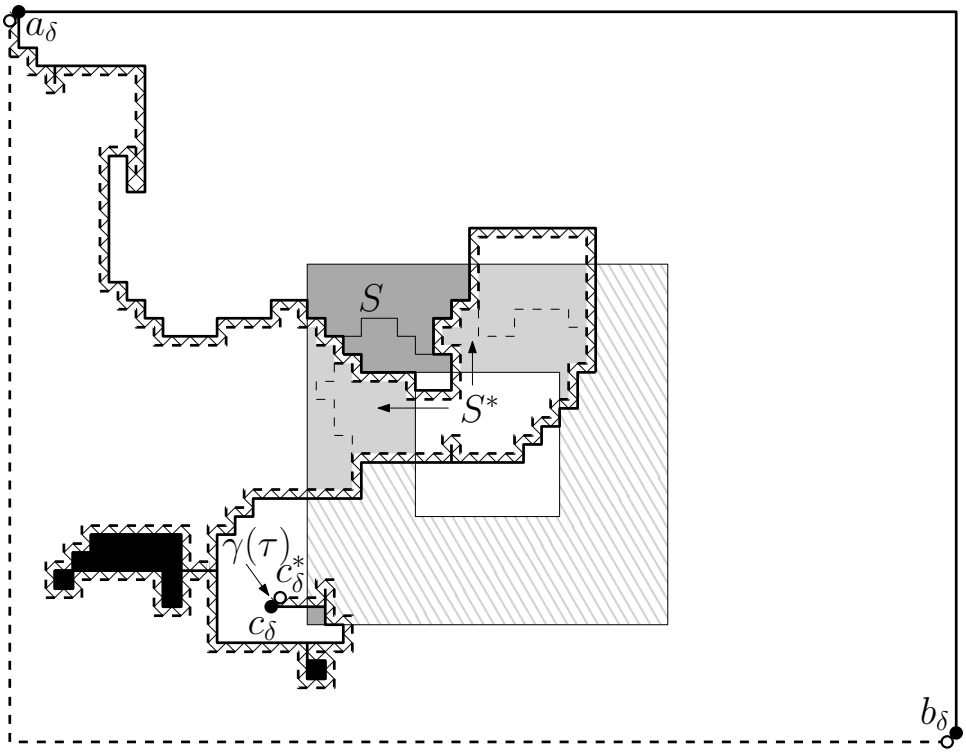

Figure 14: The dashed area is a connected component of $A \backslash \gamma_\delta[0,\tau]$ which is disconnecting $\gamma_\delta(\tau)$ from $b_\delta^\diamond$, or equivalently c_δ from b_δ, and which is therefore not in A_τ. The black parts are not included in the slit domain since they correspond to connected components that are not containing $b_\delta^\diamond$. Conditioning on $\gamma_\delta[0,\tau]$ induces Dobrushin boundary conditions in the new domain. The dark grey area is S and the light-gray S^*. We depicted a blocking open path in S and a dual-open path in each connected component of S^*.

Similarly, one may define the dual Dobrushin domain. The marked point is then c_δ^*, where c_δ^* is the dual-vertex of $(\delta\mathbb{Z}^2)^*$ bordered by the last medial edge of $\gamma_\delta[0,n]$. It is worth mentioning that the construction is symmetric for the dual Dobrushin domain: the dual of the slit domain $\Omega_\delta \setminus \gamma[0,n]$ is simply the connected component of b_δ^* in the subgraph of Ω_δ^* obtained by removing the dual-edges crossed by the curve.

Remark. The notation $\Omega_\delta \setminus \gamma_\delta[0,n]$ could somewhat be misleading, since Ω_δ is a subset of $\delta\mathbb{Z}^2$ and $\gamma_\delta[0,n]$ is a path of medial edges. Nevertheless, we allow ourselves some latitude here since we find this notation both concise and intuitive.

If one starts with Dobrushin boundary conditions on $(\Omega_\delta, a_\delta, b_\delta)$, then conditionally on $\gamma_\delta[0,n]$ the law of the configuration inside $\Omega_\delta \setminus \gamma_\delta[0,n]$ is a random-cluster model with Dobrushin boundary conditions wired on $\partial_{b_\delta c_\delta}$ and free elsewhere. This comes from the fact that the exploration path $\gamma_\delta[0,n]$ "slides between open edges and dual-open dual-edges" and therefore the edges on its left must be open and the dual-edges on its right dual-open. This implies that the arc $\partial_{a_\delta c_\delta}$ must be wired (and therefore $\partial_{b_\delta c_\delta}$ is since $\partial_{b_\delta a_\delta}$ was already wired to start with) and the dual arc $\partial_{c_\delta^* a_\delta^*}$ is dual-wired.

We are now in a position to prove Condition **G2**.

Proof of Condition **G2**. Let $A(z,r,R)$ and A_τ as defined above. We can fix a realization of $\gamma_\delta[0,\tau]$, and work in the slit Dobrushin domain $(\Omega_\delta \setminus \gamma_\delta[0,\tau], c_\delta, b_\delta)$.

See A_τ as the union of connected components of the Dobrushin domain $\Omega_\delta^\diamond$ seen as an open domain of $\mathbb{R}^2$ minus the path $\gamma_\delta[0,\tau]$. We denote generically a connected component by $\mathcal{C}$ (we see it as a subset of $\mathbb{R}^2$).

The connected components can be divided into three classes:

- $\partial\mathcal{C}$ intersects both $\partial^\diamond_{c_\delta b_\delta}$ and $\partial^\diamond_{b_\delta c_\delta}$;
- $\partial\mathcal{C}$ intersects $\partial^\diamond_{b_\delta c_\delta}$ but not $\partial^\diamond_{c_\delta b_\delta}$.
- $\partial\mathcal{C}$ intersects $\partial^\diamond_{c_\delta b_\delta}$ but not $\partial^\diamond_{b_\delta c_\delta}$;

In fact, there cannot be any connected component of the first type. Indeed, let us assume that such a connected component $\mathcal{C}$ does exist. Let γ be a self-avoiding path in $\mathcal{C}$ going from $\partial^\diamond_{b_\delta c_\delta}$ to $\partial^\diamond_{c_\delta b_\delta}$. Topologically, $c_\delta^\diamond$ and $b_\delta^\diamond$ must be on two different sides of Γ in $(\Omega_\delta \setminus \gamma_\delta[0,\tau])^\diamond \setminus \Gamma$. But this means that $\mathcal{C}$ disconnects c_δ from b_δ, and therefore that $\mathcal{C}$ is not part of A_τ, which is contradictory.

We can therefore safely assume that the connected components are either of the second or third types. We now come back to the interpretation in terms of graphs.

Let S be the subgraph of $\Omega_\delta \setminus \gamma_\delta[0,\tau]$ given by the union of the connected components (seen as primal graphs this time) of the second type (see Fig. 14). This set is a subset of A_τ. Furthermore, the boundary conditions induced by the conditioning on $\gamma_\delta[0,\tau]$ are wired on $\partial S \setminus \partial A_\tau$. Therefore, conditioned on $\gamma_\delta[0,\tau]$ and the configuration outside $A(z,r,R)$, the configuration ω in S dominates $\omega'_{|S}$, where ω' follows the law of a random-cluster model in $A(z,r,R)$ with free boundary conditions. In particular, if there exists an open circuit in ω' surrounding $z+\Lambda_r$ in $A(z,r,R)$, then the restriction of this path to S is also open in ω and it disconnects $z+\partial\Lambda_r$ from $z+\partial\Lambda_R$ in S. In particular, the exploration path $\gamma_\delta[\tau,\infty]$ cannot cross A_τ inside S since this would require the existence of a dual-open dual-path from the outer to the inner part of A_τ^*.

Property **P5a** implies that this open circuit exists in ω' with probability larger than a constant $c>0$ not depending on δ, and that therefore $\gamma_\delta[\tau,\infty]$ cannot cross A_τ inside S with probability larger than c uniformly on the configuration outside A_τ.

Let now S^* be the subgraph of $\Omega_\delta^* \setminus \gamma_\delta[0,\tau]$ given by the union of the connected components (seen as dual graphs) of the third type. The same reasoning for the dual model implies that with probability $c > 0$, the exploration path $\gamma_\delta[\tau,\infty]$ cannot cross A_τ^* inside S^*.

Altogether, $\gamma_\delta[\tau,\infty]$ cannot cross A_τ with probability c^2. Now, Proposition 2 shows that c can be taken to be equal to $1-(1-c_2)^{\lfloor \log_2(R/r)\rfloor}$. Since $R/r \geq C$, we can guarantee that $c^2 \geq 1/2$ by choosing C large enough.

5.3 Proof of Theorem 7

We omit certain details in the proof of Theorem 7 (for instance when we use **P5**, since we already did it several times in this article). We start by a lemma. Define the upper half-plane $\mathbb{H} = \mathbb{Z} \times \mathbb{N}$.

Lemma 10. *Assume that there exists $c > 0$ such that for any $n \geq 1$,*

$$\phi^0_{\mathbb{H}}\left[[-n,n]\times\{0\} \longleftrightarrow \mathbb{Z}\times\{n\}\right] \geq c.$$

Then, for any $\alpha > 0$ there exists $c(\alpha) > 0$ such that for every $n \geq 1$,

$$\phi^0_{[-n,n]\times[0,\alpha n]}\left[\mathcal{C}_v([-n,n]\times[0,\alpha n])\right] \geq c(\alpha).$$

Proof. First, Property **P5** allows us to choose $\beta > 1$ so that for any $m \geq 1$,

$$\phi^1_{\mathbb{Z}^2}\left[\mathcal{C}_h([0,(\beta-1)m]\times[0,m])\right] \leq \tfrac{c}{3}.$$

Let

$$\mathcal{A}(m) := \left\{[-m,m]\times\{0\} \xleftrightarrow{[-\beta m,\beta m]\times[0,m]} [-\beta m,\beta m]\times\{m\}\right\}.$$

With this choice of β, and because of the comparison between boundary conditions, we find that

$$\phi^0_{\mathbb{H}}\left[\mathcal{A}(m)\right] \geq \tfrac{c}{3}.$$

Using Property **P5** for the dual model, we deduce that there exists $c_1 > 0$ such that for any $m \geq 1$,

$$\phi^0_{[-2\beta m,2\beta m]\times[0,2m]}\left[\mathcal{F}(m)\cap\mathcal{A}(m)\right] \geq c_1,$$

where $\mathcal{F}(m)$ is the existence of a dual-path in the half-annulus

$$[-2\beta m,2\beta m]\times[0,2m]\setminus[-\beta m,\beta m]\times[0,m]$$

from $[\beta m,2\beta m]\times\{0\}$ to $[-2\beta m,-\beta m]\times\{0\}$. (The occurrence of $\mathcal{F}_n$ and $\mathcal{A}_n$ is illustrated on Fig. 15.) After conditioning on the exterior most such dual path, the domain Markov property and the comparison between boundary conditions imply that

$$\phi^0_{[-2\beta m,2\beta m]\times[0,2m]}\left[\mathcal{A}(m)\right] \geq c_1. \tag{29}$$

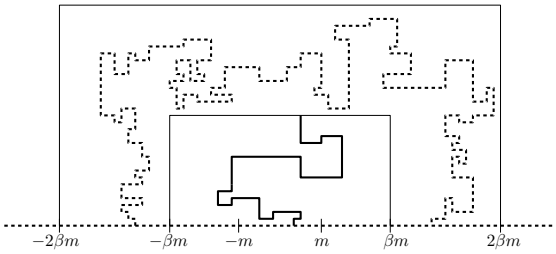

Figure 15: The occurrence of $\mathcal{A}$ and $\mathcal{F}$ in $\mathbb{H}$ with free boundary conditions on $\mathbb{Z} \times \{0\}$

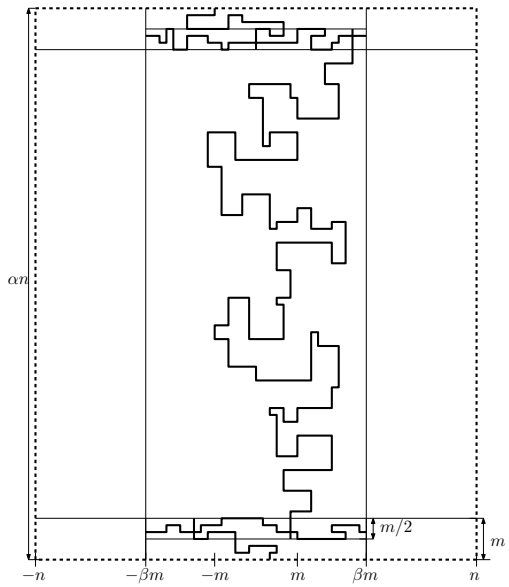

Figure 16: The occurrence of $\mathcal{A}, \ldots, \mathcal{E}$ in $[-n, n] \times [0, \alpha n]$ with free boundary conditions

We now fix $\alpha \geq 2$ and $n \geq 1$. We assume without loss of generality that αn is an integer. Set $m = \lfloor n/(2\beta) \rfloor$. Define the following four events:

$$\begin{aligned}
\mathcal{B}(m) &= \big\{[-\beta m, \beta m] \times \{\alpha n - m\} \xleftrightarrow{[-\beta m, \beta m] \times [\alpha n - m, \alpha n]} [-m, m] \times \{\alpha n\}\big\}, \\
\mathcal{C}(m) &= \mathcal{C}_h\big([-\beta m, \beta m] \times [\tfrac{m}{2}, m]\big), \\
\mathcal{D}(m) &= \mathcal{C}_h\big([-\beta m, \beta m] \times [\alpha n - m, \alpha n - \tfrac{m}{2}]\big), \\
\mathcal{E}(m) &= \mathcal{C}_v\big([-\beta m, \beta m] \times [\tfrac{m}{2}, \alpha n - \tfrac{m}{2}]\big).
\end{aligned}$$

Note that the event $\mathcal{B}(m)$ is the equivalent of the event $\mathcal{A}(m)$, but "at the top of the rectangle". By (29) and comparison between boundary conditions, $\phi^0_{[-n,n]\times[0,\alpha n]}[\mathcal{A}(m)]$ and $\phi^0_{[-n,n]\times[0,\alpha n]}[\mathcal{B}(m)]$ are larger or equal to c_1. Furthermore, Property **P5** also implies that $\phi^0_{[-n,n]\times[0,\alpha n]}[\mathcal{C}(m)]$ and $\phi^0_{[-n,n]\times[0,\alpha n]}[\mathcal{D}(m)]$ are larger than $c_2 > 0$, and that $\phi^0_{[-n,n]\times[0,\alpha n]}[\mathcal{E}(m)]$ is larger than $c_3(\alpha) > 0$.

Now observe that if the five events $\mathcal{A}(m), \ldots, \mathcal{E}(m)$ occur simultaneously, then the rectangle $[-n, n] \times [0, \alpha n]$ is crossed vertically (see Fig. 16). The FKG inequality thus implies that

$$\begin{aligned}
\phi^0_{[-n,n]\times[0,\alpha n]}[\mathcal{C}_v([-n, n] \times [0, \alpha n])] &\geq \phi^0_{[-n,n]\times[0,\alpha n]}[\mathcal{A}(m) \cap \mathcal{B}(m) \cap \mathcal{C}(m) \cap \mathcal{D}(m) \cap \mathcal{E}(m)] \\
&\geq c_1^2 \cdot c_2^3.
\end{aligned}$$

We just proved the result for $\alpha \geq 2$ with $c(\alpha) = c_1^2 c_2^2 c_3(\alpha) > 0$. For $\alpha \in (0, 2)$, simply apply the comparison between boundary conditions in the rectangle $[-\frac{\alpha n}{4}, \frac{\alpha n}{4}] \times [0, \alpha n]$ which has aspect ratio 2 and is included in $[-n, n] \times [0, \alpha n]$. $\square$

The previous lemma shows that in order to prove Theorem 7, we may focus on showing

that there exists a constant $c > 0$ such that for any $n \geq 1$,

$$\phi^0_{\mathbb{H}}\left[[-n,n] \times \{0\} \longleftrightarrow \mathbb{Z} \times \{n\}\right] \geq c.$$

In order to do so, we use the second-moment method. Let $p_n = \phi^0_{\mathbb{H}}\left[0 \leftrightarrow \mathbb{Z} \times \{n\}\right]$ and define

$$\mathsf{N} := \sum_{x \in [-n,n] \times \{0\}} \mathbf{1}_{x \leftrightarrow \mathbb{Z} \times \{n\}}.$$

By definition $\phi^0_{\mathbb{H}}\left[\mathsf{N}\right] = (2n+1)p_n$. Furthermore,

$$\begin{aligned}
\phi^0_{\mathbb{H}}\left[\mathsf{N}^2\right] &\leq (2n+1) \sum_{x \in [-2n,2n] \times \{0\}} \phi^0_{\mathbb{H}}\left[0, x \leftrightarrow \mathbb{Z} \times \{n\}\right] \\
&\leq (2n+1) \cdot C_6 \sum_{x \in [-2n,2n] \times \{0\}} \phi^0_{\mathbb{H}}\left[\Lambda_{2|x|} \leftrightarrow \mathbb{Z} \times \{n\}\right] \phi^0_{\mathbb{H}}\left[0 \leftrightarrow \partial\Lambda_{|x|/4}\right]^2 \\
&\leq (2n+1) \cdot C_7 \sum_{x \in [-2n,2n] \times \{0\}} \phi^0_{\mathbb{H}}\left[0 \leftrightarrow \mathbb{Z} \times \{n\}\right] \phi^0_{\mathbb{H}}\left[0 \leftrightarrow \mathbb{Z} \times \{|x|\}\right] \\
&= (2n+1) \cdot C_7 \cdot p_n \cdot 2 \sum_{k=0}^{2n} p_k. \qquad (30)
\end{aligned}$$

In the second, we used that

$$\left\{0, x \leftrightarrow \mathbb{Z} \times \{n\}\right\} \subset \left\{\Lambda_{2|x|} \leftrightarrow \mathbb{Z} \times \{n\}\right\} \cap \left\{0 \leftrightarrow \partial\Lambda_{|x|/4}\right\} \cap \left\{x \leftrightarrow x + \partial\Lambda_{|x|/4}\right\}$$

and Theorem 5 (more precisely a direct adaptation to the upper half-plane) to decouple the different events on the right side. In the third line, we used Property **P5** as follows. Let $\mathcal{A}$ be the event that

- $\partial\Lambda_{|x|/8} \leftrightarrow \partial\Lambda_{4|x|}$,
- there exists an open path in $\Lambda_{|x|/4} \setminus \Lambda_{|x|/8}$ disconnecting 0 from infinity in $\mathbb{H}$,
- there exists an open path in $\Lambda_{4|x|} \setminus \Lambda_{2|x|}$ disconnecting 0 from infinity in $\mathbb{H}$.

With this definition, we see that if $0 \leftrightarrow \partial\Lambda_{|x|/4}$, $\Lambda_{2|x|} \leftrightarrow \mathbb{Z} \times \{n\}$ and $\mathcal{A}$ occur, then 0 is connected to $\mathbb{Z} \times \{n\}$. Now, the event $\mathcal{A}$ has probability bounded away from 0 uniformly in x thanks to **P5**, so that the FKG inequality directly implies the third inequality of (30). Similarly, one can use **P5** to prove that $\phi^0_{\mathbb{H}}\left[0 \leftrightarrow \partial\Lambda_{|x|/4}\right] \leq C_8 \phi^0_{\mathbb{H}}\left[0 \leftrightarrow \mathbb{Z} \times \{|x|\}\right]$.

Let us now state the following claim.

Claim: There exists $C_9 > 0$ such that for any $n \geq 0$,

$$\sum_{k=0}^{n} p_k \leq C_9 n p_n.$$

Before proving the claim, let us finish the proof of Theorem 7. The claim and (30) imply that

$$\phi^0_{\mathbb{H}}\left[\mathsf{N}^2\right] \leq (2n+1)C_7 p_n 2\Big(C_9 n p_n + n p_n\Big) \leq C_7(C_9+1)(2n+1)2np_n^2 \leq C_7(C_9+1)\phi^0_{\mathbb{H}}\left[\mathsf{N}\right]^2.$$

(We also used that $p_k \leq p_n$ for $k \geq n$.) By applying the Cauchy-Schwarz inequality, we find

$$\phi^0_{\mathbb{H}}\left[[-n,n]\times\{0\} \longleftrightarrow \mathbb{Z}\times\{n\}\right] = \phi^0_{\mathbb{H}}\left[\mathsf{N}>0\right] \geq \frac{\phi^0_{\mathbb{H}}\left[\mathsf{N}\right]^2}{\phi^0_{\mathbb{H}}\left[\mathsf{N}^2\right]} \geq \frac{1}{C_7(C_9+1)} > 0.$$

Proof of the Claim. We use (20) one more time. Recall that $q<4$. Let $\mathbb{V}$ be the subgraph of $\mathbb{Z}^3$ defined as follows: the vertices are given by $\mathbb{Z}^3$ and the edges by

- $[(x_1,x_2,x_3),(x_1,x_2+1,x_3)]$ for every $x_1,x_2,x_3\in\mathbb{Z}$,
- $[(x_1,x_2,x_3),(x_1+1,x_2,x_3)]$ for every $x_1,x_2,x_3\in\mathbb{Z}$ such that $x_1\neq 0$,
- $[(0,x_2,x_3),(1,x_2,x_3)]$ for every $x_2\geq -n$ and $x_3<0$,
- $[(0,x_2,x_3),(1,x_2,x_3+1)]$ for every $x_2<-n$ and $x_3<0$,
- $[(0,x_2,x_3),(1,x_2,x_3)]$ for every $x_2>n$ and $x_3\geq 0$,
- $[(0,x_2,x_3),(1,x_2,x_3+1)]$ for every $x_2\leq n$ and $x_3\geq 0$.

For $\theta>0$ and $n\geq 0$, let

$$V_{n,\theta} := \left\{(x_1,x_2,x_3)\in\mathbb{V} : |x_1|\leq n, |x_2|\leq 2n \text{ and } |x_3|\leq\theta\right\}.$$

We also set $a=(1,-n,0)$ and $b=(1,n,0)$. One can see $V_{n,\theta}$ as a Dobrushin domain constructed from the two medial vertices $a^\diamond=(1,-n-\frac{1}{2},0)$ and $b^\diamond=(1,n+\frac{1}{2},0)$. (The definitions need to be slightly adapted to this framework). The boundary conditions are wired on the segment ∂_{ba} between a and b, and free elsewhere. Let ∂ be the part of ∂_{ab} composed of vertices such that $|x_1|=n$ or $|x_2|=2n$. One can choose $\theta=\theta(q)$ large enough such that the following holds: for every vertex x on $\partial_{ab}\setminus\partial$, the two medial edges e_1,e_2 of $\partial^\diamond_{ab}$ bordering x satisfies

$$\mathrm{Im}\left(\eta(e_1)\widehat{F}(e_1)+\eta(e_2)\widehat{F}(e_2)\right)\geq 0. \tag{31}$$

Let V be the set of medial vertices with four incident medial edges in $V^\diamond_{n,\theta}$. In order to apply Equation (20), we wish to compute the sum

$$S := \sum_{\substack{e \text{ incident to exactly}\\ \text{one vertex in } V}} \mathrm{Im}\left(\eta(e)\widehat{F}(e)\right).$$

- The contribution of the medial edges e_a,e_b to the sum S is given by

$$\mathrm{Im}(e^{\mathrm{i}\hat{\sigma}3\pi/2}-1) = \sin(\hat{\sigma}3\pi/2)\geq 0$$

- The contribution of the medial edges bordering a dual-vertex u on the arc $\partial^\star_{ba} = \{\frac{3}{2}\}\times[-n+\frac{1}{2},n-\frac{1}{2}]$ is given by

$$2\cos\left(\hat{\sigma}\tfrac{3\pi}{4}\right)\sin\left(\hat{\sigma}\tfrac{\pi}{4}\right)\sum_{u\in\partial^*_{ba}}\phi^{a,b}_{V_{n,\theta}}\left[u\overset{*}{\longleftrightarrow}\partial^*_{ab}\right].$$

- Using Equation (31), we find that the contribution $S(\partial_{ab})$ of all the medial edges bordering a vertex on the arc ∂_{ab} satisfies

$$S(\partial_{ab})\geq -2\sum_{x\in\partial}\phi^{a,b}_{V_{n,\theta}}\left[x\longleftrightarrow\partial_{ba}\right].$$

Using that $S = 0$ together with the three equations above, we obtain

$$2\cos\left(\hat{\sigma}\tfrac{3\pi}{4}\right)\sin\left(\hat{\sigma}\tfrac{\pi}{4}\right)\sum_{u\in\partial^*_{ba}}\phi^{a,b}_{V_{n,\theta}}\left[u \overset{*}{\longleftrightarrow} \partial^*_{ab}\right] \leq 2\sum_{x\in\partial}\phi^{a,b}_{V_{n,\theta}}\left[x \longleftrightarrow \partial_{ba}\right]. \tag{32}$$

Notice that the condition $1 \leq q < 4$ implies that $2\cos\left(\hat{\sigma}\frac{3\pi}{4}\right)\sin\left(\hat{\sigma}\frac{\pi}{4}\right) > 0$. Then, using the comparison between boundary conditions and Property **P5**, we find that for every x on ∂, $\phi^{a,b}_{V_{n,\theta}}\left[x \longleftrightarrow \partial_{ba}\right] \leq C_{10}p_n$. Plugging this in Equation (32), we obtain

$$\sum_{u\in\partial^*_{ab}}\phi^{a,b}_{V_{n,\theta}}\left[u \overset{*}{\longleftrightarrow} \partial^*_{ba}\right] \leq C_{11}np_n.$$

Now, for a dual-vertex u of ∂^*_{ab} at distance k from a or b (let us assume without loss of generality that u is at distance k from a). Let $\Lambda^+_k = [-k,k]\times[0,k]$. If the following three events occur simultaneously:

- $u \overset{*}{\longleftrightarrow} u + \partial\Lambda^+_k$,
- the half-annulus $u + \partial\Lambda^+_k \setminus \Lambda^+_{k/2}$ contains a dual-open dual path disconnecting u from the ∂^*_{ba} in $V_{n,\theta}$,
- that there exists a dual-open dual path disconnecting a from ∂ in

  $$\{u \in V_{n,\theta} : k/2 \leq |(u_1,u_2)-(0,n)| \leq k\}$$

  (this set is a topological rectangle winding around the singularity a at distance $k/2$ of it),

then u is connected by a dual-open path to ∂^*_{ba}. Now, the second and third events occur with probability bounded away from 0 uniformly in n thanks to **P5** and the fact that $\theta(q) < \infty$ does not depend on n. By conditioning on these two events and using the comparison between boundary conditions, we find that

$$\phi^{a,b}_{V_{n,\theta}}\left[u \overset{*}{\longleftrightarrow} \partial^*_{ba}\right] \geq c_7\phi^1_{u+\Lambda^+_k}\left[u \overset{*}{\longleftrightarrow} u+\partial\Lambda^+_k\right] \geq c_7p_k,$$

where in the last inequality we used duality and the comparison between boundary conditions. In conclusion, we find

$$2c_7\sum_{k=0}^{n}p_k \leq C_{11}np_n$$

and the conclusion follows immediately.

◇

Remark. Note that $\theta = \theta(q) < \infty$ for any $q < 4$, but that $\theta(q)$ tends to infinity as q tends to 4 and there is no such $\theta(q)$ for $q = 4$. This fact explains why this proof does not work for $q = 4$ (and should not work since the statement is expected to be false).

5.4 Proof of Theorem 8

The coupling between Potts and random-cluster models shows that

$$\mu_{\mathbb{Z}^2,\beta_c(q),q}[\sigma_x = \sigma_0] - \frac{1}{q} = \phi^1_{\mathbb{Z}^2,p_c(q),q}\left[0 \leftrightarrow x\right].$$

Lemma 1 implies the existence of $\eta_2 > 0$ such that

$$\phi^1_{\mathbb{Z}^2,p_c(q),q}\left[0 \leftrightarrow x\right] \leq \frac{1}{|x|^{\eta_2}},$$

thus implying the second inequality of Theorem 8.

In order to obtain the first, we do not need **P5** but just Theorem 9. Indeed, if $\mathcal{C}_h([0,n] \times [0,n])$ occurs, then one of the vertices of $[0,n] \times \{0\}$ is connected to distance n. As a consequence, the union bounded implies that

$$\phi^1_{\mathbb{Z}^2,p_c(q),q}\left[0 \leftrightarrow \partial\Lambda_n\right] \geq \frac{c}{n}.$$

Let $\mathcal{A}(n)$ be the event that there exists an open circuit in $\Lambda_{2n} \setminus \Lambda_n$ surrounding the origin. The FKG inequality and Theorem 9 one more time show that there exists $c' > 0$ so that

$$\begin{aligned}\phi^1_{\mathbb{Z}^2,p_c(q),q}\left[0 \leftrightarrow x\right] &\geq \phi^1_{\mathbb{Z}^2,p_c(q),q}\left[0 \leftrightarrow \partial\Lambda_{3|x|}\right] \phi^1_{\mathbb{Z}^2,p_c(q),q}\left[x \leftrightarrow \partial\Lambda_{3|x|}\right] \phi^1_{\mathbb{Z}^2,p_c(q),q}\left[\mathcal{A}(|x|)\right] \\ &\geq \frac{c'}{|x|^2}\end{aligned}$$

and the proof follows by choosing η_1 small enough.

References

[AB99] M. Aizenman and A. Burchard. Hölder regularity and dimension bounds for random curves. *Duke Math. J.*, 99(3):419–453, 1999.

[ADS15] M. Aizenman, H. Duminil-Copin, and V. Sidoravicius. Random Currents and Continuity of Ising Model's Spontaneous Magnetization. *Communications in Mathematical Physics*, 334:719–742, 2015.

[AF86] M. Aizenman and R. Fernández. On the critical behavior of the magnetization in high-dimensional Ising models. *J. Statist. Phys.*, 44(3-4):393–454, 1986.

[Ale98] Kenneth S. Alexander. On weak mixing in lattice models. *Probab. Theory Related Fields*, 110(4):441–471, 1998.

[Bax71] R. J. Baxter. Generalized ferroelectric model on a square lattice. *Studies in Appl. Math.*, 50:51–69, 1971.

[Bax73] R. J. Baxter. Potts model at the critical temperature. *Journal of Physics C: Solid State Physics*, 6(23):L445, 1973.

[Bax89] Rodney J. Baxter. *Exactly solved models in statistical mechanics.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1989. Reprint of the 1982 original.

[BD12a] V. Beffara and H. Duminil-Copin. Critical point and duality in planar lattice models. 2012.

[BD12b] V. Beffara and H. Duminil-Copin. Smirnov's fermionic observable away from criticality. *Ann. Probab.*, 40(6):2667–2689, 2012.

[BDS13] V. Beffara, H. Duminil-Copin, and S. Smirnov. On the critical parameters of the $q \geq 4$ random-cluster model on isoradial graphs. preprint, 2013.

[BPZ84a] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Phys. B*, 241(2):333–380, 1984.

[BPZ84b] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov. Infinite conformal symmetry of critical fluctuations in two dimensions. *J. Statist. Phys.*, 34(5-6):763–774, 1984.

[BDH14] S. Benoist, H. Duminil-Copin, and C. Hongler. Conformal invariance of crossing probabilities for the Ising model with free boundary conditions. arXiv:1410.3715, 2014.

[BCC06] Marek Biskup, Lincoln Chayes, and Nicholas Crawford. Mean-field driven first-order phase transitions in systems with long-range interactions. *J. Stat. Phys.*, 122(6):1139–1193, 2006.

[CN07] F. Camia and C.M. Newman. Critical percolation exploration path and SLE_6: a proof of convergence. *Probab. Theory Related Fields*, 139(3-4):473–519, 2007.

[CDH13] D. Chelkak, H. Duminil-Copin, and C. Hongler. Crossing probabilities in topological rectangles for the critical planar FK-Ising model. arXiv:1312.7785, 2013.

[CDH$^+$14] D. Chelkak, H. Duminil-Copin, C. Hongler, A. Kemppainen, and S. Smirnov. Convergence of Ising interfaces to Schramm's SLE curves. *C. R. Acad. Sci. Paris Math.*, 352(2):157–161, 2014.

[CHI15] Dmitry Chelkak, Clément Hongler, and Konstantin Izyurov. Conformal invariance of spin correlations in the planar Ising model. *Ann. of Math. (2)*, 181(3):1087–1138, 2015.

[CI13] Dmitry Chelkak and Konstantin Izyurov. Holomorphic spinor observables in the critical Ising model. *Comm. Math. Phys.*, 322(2):303–332, 2013.

[CS12] Dmitry Chelkak and Stanislav Smirnov. Universality in the 2D Ising model and conformal invariance of fermionic observables. *Invent. Math.*, 189(3):515–580, 2012.

[Dum12] H. Duminil-Copin. Divergence of the correlation length for critical planar FK percolation with $1 \leq q \leq 4$ via parafermionic observables. *Journal of Physics A: Mathematical and Theoretical*, 45(49):494013, 2012.

[Dum13] H. Duminil-Copin. *Parafermionic observables and their applications to planar statistical physics models*, volume 25 of *Ensaios Matematicos*. Brazilian Mathematical Society, 2013.

[Dum15] H. Duminil-Copin. Geometric representations of lattice spin models. book, Edition Spartacus, 2015.

[DGP14] H. Duminil-Copin, C. Garban, and G. Pete. The near-critical planar FK-Ising model. *Comm. Math. Phys.*, 326(1):1–35, 2014.

[DHN11] H. Duminil-Copin, C. Hongler, and P. Nolin. Connection probabilities and RSW-type bounds for the two-dimensional FK Ising model. *Comm. Pure Appl. Math.*, 64(9):1165–1198, 2011.

[DLM15] H. Duminil-Copin, J.-H. Li, and I. Manolescu. Universality for random-cluster models on isoradial graphs. preprint, 2015.

[DM15] H. Duminil-Copin and I. Manolescu. Random-cluster model with critical weights on isoradial graphs. *to appear in Probability Theory and Related Fields*, page 40 pages, 2015.

[DS12] H. Duminil-Copin and S. Smirnov. The connective constant of the honeycomb lattice equals $\sqrt{2+\sqrt{2}}$. *Ann. of Math. (2)*, 175(3):1653–1665, 2012.

[DT15a] H. Duminil-Copin and V. Tassion. A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. arXiv:1502.03050, 2015.

[DT15b] H. Duminil-Copin and V. Tassion. A new proof of the sharpness of the phase transition for Bernoulli percolation on $\mathbb{Z}^d$. arXiv:1502.03051, 2015.

[FK72] C. M. Fortuin and P. W. Kasteleyn. On the random-cluster model. I. Introduction and relation to other models. *Physica*, 57:536–564, 1972.

[FK80] Eduardo Fradkin and Leo P Kadanoff. Disorder variables and para-fermions in two-dimensional statistical mechanics. *Nuclear Physics B*, 170(1):1–15, 1980.

[Gri06] Geoffrey Grimmett. *The random-cluster model*, volume 333 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[GM14] Geoffrey R. Grimmett and Ioan Manolescu. Bond percolation on isoradial graphs: criticality and universality. *Probab. Theory Related Fields*, 159(1-2):273–327, 2014.

[Hon14] C. Hongler. Conformal invariance of Ising model correlations. In *XVIIth International Congress on Mathematical Physics*, pages 326–335. World Sci. Publ., Hackensack, NJ, 2014.

[HK13] Clément Hongler and Kalle Kytölä. Ising interfaces and free boundary conditions. *J. Amer. Math. Soc.*, 26(4):1107–1189, 2013.

[HS11] Clément Hongler and Stanislav Smirnov. Critical percolation: the expected number of clusters in a rectangle. *Probab. Theory Related Fields*, 151(3-4):735–756, 2011.

[KS12] Antti Kemppainen and Stanislav Smirnov. Random curves, scaling limits and loewner evolutions. arXiv:1212.6215, 2012.

[Ken00] Richard Kenyon. Conformal invariance of domino tiling. *Ann. Probab.*, 28(2):759–795, 2000.

[Ken01] Richard Kenyon. Dominos and the Gaussian free field. *Ann. Probab.*, 29(3):1128–1137, 2001.

[Kes80] Harry Kesten. The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Comm. Math. Phys.*, 74(1):41–59, 1980.

[KS82] R. Kotecký and S. B. Shlosman. First-order phase transitions in large entropy lattice models. *Comm. Math. Phys.*, 83(4):493–515, 1982.

[LMR86] L. Laanait, A. Messager, and J. Ruiz. Phases coexistence and surface tensions for the Potts model. *Comm. Math. Phys.*, 105(4):527–545, 1986.

[LMMS+91] Lahoussine Laanait, Alain Messager, Salvador Miracle-Solé, Jean Ruiz, and Senya Shlosman. Interfaces in the Potts model. I. Pirogov-Sinai theory of the Fortuin-Kasteleyn representation. *Comm. Math. Phys.*, 140(1):81–91, 1991.

[Law05] Gregory F. Lawler. *Conformally invariant processes in the plane*, volume 114 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

[LSW04] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.*, 32(1B):939–995, 2004.

[LS12] Eyal Lubetzky and Allan Sly. Critical Ising on the square lattice mixes in polynomial time. *Comm. Math. Phys.*, 313(3):815–836, 2012.

[Ons44] Lars Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev. (2)*, 65:117–149, 1944.

[Pot52] Renfrey Burnard Potts. Some generalized order-disorder transformations. In *Proceedings of the Cambridge Philosophical Society*, volume 48, pages 106–109. Cambridge Univ Press, 1952.

[RC06] V. Riva and J. Cardy. Holomorphic parafermions in the Potts model and stochastic Loewner evolution. *J. Stat. Mech. Theory Exp.*, (12):P12001, 19 pp. (electronic), 2006.

[Rus78] Lucio Russo. A note on percolation. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 43(1):39–48, 1978.

[Sch07] Oded Schramm. Conformally invariant scaling limits: an overview and a collection of problems. In *International Congress of Mathematicians. Vol. I*, pages 513–543. Eur. Math. Soc., Zürich, 2007.

[SW78] P. D. Seymour and D. J. A. Welsh. Percolation probabilities on the square lattice. *Ann. Discrete Math.*, 3:227–245, 1978. Advances in graph theory (Cambridge Combinatorial Conf., Trinity College, Cambridge, 1977).

[Sim80] Barry Simon. Correlation inequalities and the decay of correlations in ferromagnets. *Comm. Math. Phys.*, 77(2):111–126, 1980.

[Smi06] Stanislav Smirnov. Towards conformal invariance of 2D lattice models. In *International Congress of Mathematicians. Vol. II*, pages 1421–1451. Eur. Math. Soc., Zürich, 2006.

[Smi10] Stanislav Smirnov. Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model. *Ann. of Math. (2)*, 172(2):1435–1467, 2010.

[Wer09] Wendelin Werner. *Percolation et modèle d'Ising*, volume 16 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2009.

[Wu82] F. Y. Wu. The Potts model. *Rev. Modern Phys.*, 54(1):235–268, 1982.

[Yan52] C.N. Yang. The spontaneous magnetization of a two-dimensional Ising model. *Phys. Rev. (2)*, 85:808–816, 1952.